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The automorphism groups of certain commutant subalgebras of lattice vertex operator algebras

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1 Introduction

An element e of weight 2 of a vertex operator algebra V is called an Ising vector if the vertex subalgebra generated by e is isomorphic to the simple Virasoro VOA $L(\frac{1}{2},0)$ with central charge $\frac{1}{2}$. Any Ising vector e defines an automorphism τ_e of V with $\tau_e^2=1$ by using representation of $L(\frac{1}{2},0)$. In the case of the Moonshine VOA $V^{\natural}$, τ_e gives a $2A$-involution of the Monster simple group $\mathbb{M}=\mathrm{Aut}(V^{\natural})$. An Ising vector e is called σ-type if $\tau_e=1$. An Ising vector e of σ-type defines an automorphism σ_e of V with $\sigma_e^2=1$. It is known that if a set E of Ising vectors of σ-type such that $\sigma_e(f)\in E$ for any $e,f\in E$, the subgroup of $\mathrm{Aut}(V)$ generated by $\{\sigma_e|e\in E\}$ is 3-transposition group. Matsuo classified all 3-transposition groups defined by such a set E of Ising vectors of σ-type.

Let R be a root lattice. Let $V_{\sqrt{2}R}$ be the lattice vertex operator algebras associated to the lattice whose norm is twice of R's and $V_{\sqrt{2}R}^+$ the fixed point subalgebra of the lattice VOA $V_{\sqrt{2}R}$ by the lift of (-1)-isometry on R. There are a lot of Ising vectors (of σ-type) and conformal vectors in $V_{\sqrt{2}R}^+$. We consider the commutant subalgebra M_R of a conformal vector $\tilde{\omega}_R$ fixed by $\mathrm{Aut}(R)$ in $V_{\sqrt{2}R}^+$. Then $\mathrm{Aut}(R)/\langle -1\rangle$ acts on M_R faithfully.

This talk is about the result obtained by a joint work with Ching Hung Lam of National Cheng Kung University in Taiwan and Hiroshi Yamauchi of the University of Tokyo. We study the classification of Ising vectors of $V_{\sqrt{2}R}^+$.

Then we apply our results to study commutant subalgebras M_R related to root lattice R. We completely classify all Ising vectors in such commutant subalgebras. Moreover, we show that M_R is generated by Ising vectors and determine their full automorphism groups.

2 Ising vectors and σ-involutions

An element $e \in V_2$ is a conformal vector with central charge $c \in \mathbb{C}$ if $L_{(n)} := e_{(n+1)}, n \in \mathbb{Z}$ satisfy the Virasoro relation

$$[L_{(m)}, L_{(n)}] = (m+n)L_{(m-n)} + \delta_{m+n,0}\frac{m^3-m}{12}c$$

for $m, n \in \mathbb{Z}$. A conformal vector e of a VOA V with central charge $\frac{1}{2}$ is called an *Ising vector* if the subalgebra $\mathrm{Vir}(e)$ generated by e is isomorphic to the simple Virasoro VOA $L(\frac{1}{2},0)$ with central charge $\frac{1}{2}$. It is well-known that the Virasoro VOA $L(\frac{1}{2},0)$ is rational and has exactly three irreducible modules $L(\frac{1}{2},0), L(\frac{1}{2},\frac{1}{2}), L(\frac{1}{2},\frac{1}{16})$.

Let e be an Ising vector of a VOA V. Since $\mathrm{Vir}(e)$ is rational, V is a semisimple $\mathrm{Vir}(e)$-module. For $h = 0, 1/2, 1/16$, denote by $V_e(h)$ the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L(\frac{1}{2},h)$. Then we have the isotypical decomposition:

$$V = V_e(0) \oplus V_e(\frac{1}{2}) \oplus V_e(\frac{1}{16})$$

Define a linear automorphism τ_e on V by

$$\tau_e = \begin{cases} 1 & on \quad V_e(0) \oplus V_e(\frac{1}{2}) \\ -1 & on \quad V_e(\frac{1}{16}). \end{cases}$$

Then, τ_e is an automorphism of V with $\tau_e^2 = 1$. On the $\langle\tau_e\rangle$-fixed point subalgebra $V^{\langle\tau_e\rangle} = V_e(0) \oplus V_e(\frac{1}{2})$, define a linear automorphism σ_e by

$$\sigma_e = \begin{cases} 1 & on \quad V_e(0) \\ -1 & on \quad V_e(\frac{1}{2}). \end{cases}$$

Then, σ_e is an automorphism of $V^{\langle\tau_e\rangle}$ with $\sigma_e^2 = 1$. We will refer $\tau_e \in \mathrm{Aut}(V)$ (resp. $\sigma_e \in \mathrm{Aut}(V^{\langle\sigma_e\rangle})$) to as the *$\tau$-involution* (resp. *$\sigma$-involution*). An Ising vector e of V is called *of σ-type* if τ_e defines identity on V i.e. $V_e(\frac{1}{16}) = 0$.

We consider a VOA $V = \oplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = 0$. Then the weight two subspace V_2 equipped with the product

$$a \cdot b := a_{(1)}b, \ \ a, b \in V_2$$

forms a commutative algebra with an symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$a_{(3)}b = \langle a, b \rangle \mathbf{1}, \ \ a, b \in V_2,$$

and satisfying

$$\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle, \ \ a, b, c \in V_2.$$

This algebra is called the *Griess algebra* of V. If $e \in V_2$ is a conformal vector with central charge c, $\frac{1}{2}e$ is an idempotent of the Griess algebra V_2 and $\langle e, e \rangle = \dfrac{c}{2}$.

About σ-involutions, the following is known.

Theorem 2.1 (Miyamoto). *Assume that $V_0 = \mathbb{C}\mathbf{1}$, $V_1 = 0$ and $\langle \cdot, \cdot \rangle$ is positive-definite. If $e, f \in V_2$ are Ising vectors of σ-thpe and $e \neq f$, then the order of $\sigma_e \sigma_f$ is 2 or 3, and*
(1) If $|\sigma_e \sigma_f| = 2$, then $\langle e, f \rangle = 0$ and $e \cdot f = 0$.
(2) If $|\sigma_e \sigma_f| = 3$, then $\langle e, f \rangle = \frac{1}{32}$ and $e \cdot f = \frac{1}{4}(e + f - e^{\sigma_f})$.

3 Ising vectors of $V_{\sqrt{2}R}^{+}$

Let R be a root lattice with root system $\Phi(R)$. Let ℓ be the rank of R and h the Coxeter number of R. We denote by $\sqrt{2}R$ the lattice whose norm is twice of R's. Let $V_{\sqrt{2}R}$ be a lattice VOA associated to the lattice $\sqrt{2}R$. For any isometry g on R, g is extended to a linear automorphism of $V_{\sqrt{2}R}$ by setting

$$\tilde{g}(\alpha^1_{(-n_1)} \cdots \alpha^k_{(-n_k)} e^{\sqrt{2}\alpha}) = g(\alpha^1)_{(-n_1)} \cdots g(\alpha^k)_{(-n_k)} e^{\sqrt{2}g(\alpha)}$$

for $\alpha^1, \ldots, \alpha^k, \alpha \in R$. This extension gives an automorphism of the VOA $V_{\sqrt{2}R}$ and $\tilde{g}$ is called a *lift* of g. We consider the lift θ of (-1)-isometry on R and the fixed point subalgebra

$$V_{\sqrt{2}R}^{+} = \{v \in V | \theta(v) = v\}$$

of the lattice VOA $V_{\sqrt{2}R}$. It is clear that $V_{\sqrt{2}R}^+$ has a grading $V_{\sqrt{2}R}^+ = \oplus_{n\geq 0}(V_{\sqrt{2}R}^+)_n$ such that $(V_{\sqrt{2}R}^+)_0 = \mathbb{C}\mathbf{1}$ and $(V_{\sqrt{2}R}^+)_1 = 0$, and

$$\omega = \frac{1}{4h} \sum_{\alpha\in\Phi(R)} \alpha_{(-1)}{}^2\mathbf{1}$$

is the Virasoro vector of $V_{\sqrt{2}R}^+$.

We give a classification of Ising vectors of $V_{\sqrt{2}R}^+$. For $\alpha \in \Phi(R)$ we set

$$\omega^{\pm}(\alpha) = \frac{1}{8}\alpha_{(-1)}{}^2\mathbf{1} \pm \frac{1}{4}\left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}\right).$$

It is easy to show that $\omega^{\pm}(\alpha)$, $\alpha \in \Phi(R)$, are Ising vectors of σ-type. of $V_{\sqrt{2}R}^+$. Set

$$\begin{aligned} s_R &= \frac{2}{h+2} \sum_{\alpha\in\Phi(R)} \omega^-(\alpha) \\ &= \frac{1}{4(h+2)} \sum_{\alpha\in\Phi(R)} \alpha_{(-1)}{}^2\mathbf{1} - \frac{1}{h+2} \sum_{\alpha\in\Phi(R)} e^{\sqrt{2}\alpha} \end{aligned}$$

and

$$\begin{aligned} \tilde{\omega}_R &= \omega - s_R \\ &= \frac{2}{h+2}\omega + \frac{1}{h+2} \sum_{\alpha\in\Phi(R)} e^{\sqrt{2}\alpha}. \end{aligned}$$

Then s_R and $\tilde{\omega}_R$ are mutually orthogonal Ising vectors which are fixed under the action of $\mathrm{Aut}(R)$. The central charge $\tilde{c}_R$ of $\tilde{\omega}_R$ is given by the following:

R	A_n	D_n	E_6	E_7	E_8
$\tilde{c}_R$	$2n/(n+2)$	1	6/7	7/10	1/2

In particular, $\tilde{\omega}_{E_8}$ is also an Ising vector of σ-type.of $V_{\sqrt{2}R}^+$.

For $x \in R$, define

$$\varphi_x = exp\left(\frac{\pi\sqrt{-2}}{2}x_{(0)}\right).$$

Then φ_x is an automorphism of $V_{\sqrt{2}R}^+$ with $\varphi_{2x} = 1$. We set

$$\begin{aligned} I_R &= \{\omega^{\pm}(\alpha) \,|\, \alpha \in \Phi(R)\}, \\ \tilde{I}_R &= \{\varphi_x\tilde{\omega}_R \,|\, x \in R\}. \end{aligned}$$

The inner products of these elements is given by

$$(*) \quad \begin{aligned} \langle \omega^+(\alpha), \omega^-(\alpha)\rangle &= 0, \\ \langle \omega^\pm(\alpha), \omega^\pm(\beta)\rangle &= \langle \omega^\pm(\alpha), \omega^\mp(\beta)\rangle = \frac{1}{32}\langle \alpha, \beta\rangle^2, \\ \langle \omega^\pm(\alpha), \varphi_x\tilde{\omega}_R\rangle &= \frac{1 \pm (-1)^{\langle x, \alpha\rangle}}{2(h+2)}, \\ \langle \tilde{\omega}_R, \varphi_x\tilde{\omega}_R\rangle &= \begin{cases} 0 & if \quad \langle x, x\rangle = 4 \\ \frac{1}{32} & if \quad \langle x, x\rangle = 2 \\ \frac{1}{4} & if \quad x \in 2E_8 \end{cases}. \end{aligned}$$

for distinct $\alpha, \beta \in \Phi(R)$ and $x \in R$.

It is known that $V^+_{\sqrt{2}D_{2n}}$ and $V^+_{\sqrt{2}E_8}$ are code VOAs and Lam classified Ising vectors of σ-type of a code VOA. We denote by $I(V)$ the set of Ising vectors of a VOA V. Then, the following hold.

Theorem 3.1. *we have*
(1) $I(V^+_{\sqrt{2}D_{2n}}) = I_{D_{2n}}$
(2) $I(V^+_{\sqrt{2}E_8}) = I_{E_8} \cup \tilde{I}_{E_8}$

Since a root lattice of ADE type is contained in E_8 or D_{2n} for sufficient large n, by using the above theorem, the Ising vectors of $V^+_{\sqrt{2}R}$ are given by the following.

Theorem 3.2. *For any root lattice R, $I(V^+_{\sqrt{2}R}) = I_R \cup \left(\bigcup_{K \subset R, K \simeq E_8} \tilde{I}_K \right)$*

4 Commutant subalgebras M_R

For a VOA V and a conformal vector e of V, we define the commutant subalgebra $\mathrm{Com}_V(e)$ by

$$\mathrm{Com}_V(e) = \{v \in V | e_{(0)}v = 0\}.$$

Let R be a root lattice and let us fix $\gamma \in \Phi(E_8)$. We set

$$M_R = \mathrm{Com}_{V^+_{\sqrt{2}R}}(\tilde{\omega}_R)$$

and

$$M'_{E_8} = \mathrm{Com}_{V^+_{\sqrt{2}E_8}}(\tilde{\omega}_{E_8}) \cap \mathrm{Com}_{V^+_{\sqrt{2}E_8}}(\omega^+(\gamma)).$$

We have $M_R \cap E = \{e \in E \,|\, \langle \tilde{\omega}_R, e\rangle = 0\}$ for a set E of Ising vectors.. By Theorem 3.2 and $(*)$, the Ising vectors of $V^+_{\sqrt{2}R}$ are given by the following.

Theorem 4.1. (1) $I(M_R) = M_R \cap I(V_{\sqrt{2}R})$ *and*

$$\begin{aligned} M_R \cap I_R &= \{\omega^-(\alpha) \,|\, \alpha \in \Phi(R)\}, \\ M_{E_8} \cap \tilde{I}_{E_8} &= \{\varphi_x(\tilde{\omega}_{E_8}) \,|\, x \in E_8,\ \langle x, x\rangle = 4\}. \end{aligned}$$

(2) $I(M'_{E_8}) = (M'_{E_8} \cap I_{E_8}) \cup (M'_{E_8} \cap \tilde{I}_{E_8})$ *and*

$$\begin{aligned} M'_{E_8} \cap I_{E_8} &= \{\omega^-(\alpha) \,|\, \alpha \in \Phi(E_8),\ \langle \alpha, \gamma\rangle \in 2\mathbb{Z}\}, \\ M'_{E_8} \cap \tilde{I}_{E_8} &= \{\varphi_x \tilde{\omega}_{E_8} \,|\, x \in E_8,\ \langle x, x\rangle = 4,\ \langle x, \gamma\rangle \in 1 + 2\mathbb{Z}\}. \end{aligned}$$

For $E \subset I(V)$ satisfying $\sigma_e(f) \in E$ for any $e, f \in E$, we define

$$\mathrm{Aut}(E, \langle,\rangle) = \{g \in \mathrm{Sym}_E \,|\, \langle g(e), g(f)\rangle = \langle e, f\rangle, e, f \in E\}.$$

Set

$$I_R^- = \{\omega^-(\alpha) \,|\, \alpha \in \Phi(R)\}.$$

Then the following hold.

Proposition 4.2. *The map* $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(I_R^-, \langle,\rangle)$, $g \mapsto \tilde{g}|_{I_R^-}$ *is a surjective group homomorphism with* $\mathrm{Ker}\phi = \langle -1\rangle$. *Therefore,*

$$\mathrm{Aut}(I_R^-, \langle,\rangle) \simeq \mathrm{Aut}(R)/\langle -1\rangle.$$

On the other hand, we proved

Theorem 4.3. *If R is a root lattice of ADE type and VOA V is M_R or M'_{E_8},*
(1) *V is generated by the weight 2 subspace V_2, in paticular, by $I(V)$.*
(2) *The map* $\mathrm{Aut}(V) \to \mathrm{Aut}(I(V), \langle,\rangle), \rho \mapsto \rho|_{I(V)}$ *is an injective homomorphism.*

By Proposition 4.2 and Theorem 4.3,

Theorem 4.4. *If $R \neq E_8$, then* $\mathrm{Aut}(M_R) \simeq \mathrm{Aut}(R)/\langle -1\rangle$.

In the case that $R = E_8$, the following hold.

Theorem 4.5.

$$\begin{aligned} \mathrm{Aut}(M_{E_8}) &\simeq \mathrm{Aut}(I(M_{E_8}), \langle,\rangle) \simeq \mathrm{Sp}_8(2) \\ \mathrm{Aut}(M'_{E_8}) &\simeq \mathrm{Aut}(I(M'_{E_8}), \langle,\rangle) \simeq \mathrm{O}_8^-(2) \end{aligned}$$