

Energy-based analysis of frequency entrainment described by van der Pol and phase-locked loop equations

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This paper analyzes frequency entrainment described by van der Pol and phase-locked loop (PLL) equations. The PLL equation represents the dynamics of a PLL circuit that appear in typical phase-locking phenomena. These two equations describe frequency entrainment by a periodic force. The entrainment originates from two different types of limit cycles: libration for the van der Pol equation and rotation for the PLL one. To explore the relationship between the geometry of limit cycles and the mechanism of entrainment, we investigate the entrainment using an energy balance relation. This relation is equivalent to the energy conservation law of dynamical systems with dissipation and input terms. We show response curves for the dc component, harmonic amplitude, phase difference, and energy supplied by a periodic force. The obtained curves indicate that the entrainments for the two equations have different features of supplied energy, and that the entrainment for the PLL equation possibly has the same mechanism as does the regulation of the phase difference for the van der Pol equation. © 2007 American Institute of Physics.

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Frequency entrainment appears in populations of interacting oscillators. These oscillators exchange their stored energy with each other and thereby show entrained states. The relationship between the exchange of energy and the mechanisms of entrainment is well known in pendulum clocks hanging on a common wall. Understanding the energy aspect of entrainment is important for linking its functionality and engineering applications. The novel point of this paper is that it examines frequency entrainment via the energy conservation law of dynamical systems. The energy balance relation applied in this paper makes it possible to determine what energy the oscillators exchange at entrained states.

I. INTRODUCTION

This paper analyzes frequency entrainment from the viewpoint of energy balance in dynamical systems. Frequency entrainment has been studied extensively for nonlinear dynamical systems of interacting self-sustained oscillators.¹⁻⁴ In such systems, the oscillators exchange their stored energy through interaction and thereby synchronize to a common frequency. For example, in pendulum clocks,³ the exchange of energy by trigger mechanism is necessary for maintaining steady oscillations. In mechanical viexciters,⁵ the exchange of energy between imbalance rotators is also important for their engineering practice. Thus, energy balance in dynamical systems is crucial for understanding frequency entrainment. However, the term “energy balance” has not been fully used in the context of frequency entrainment, except in some previous studies, as mentioned below.

We analyze frequency entrainment described by the van der Pol and phase-locked loop (PLL) equations. The van der Pol equation⁶ is a well-known mathematical model of nonlinear dynamics. Section I begins with an introduction of the PLL equation and explains the reason for using these two equations.

The PLL equation is derived to represent the dynamics of a PLL circuit as a FM demodulator.⁷ PLL is a well-known electronic device in electrical and communication systems,^{8,9} and has also attracted a lot of interest in nonlinear oscillations, e.g., pull-in characteristics^{8,9} and chaotic oscillations.^{7,10,11} The PLL equation has a stable limit cycle resulting from the periodic nature of phase space, called the stable limit cycle of the second kind¹ or stable rotation.¹² The rotation is conceptually shown in Fig. 1. It represents a desynchronized steady state in a PLL circuit and is regarded as another type of self-excited oscillation with natural rotation frequencies. Such rotating steady states in the PLL circuit are studied¹³ to determine its pull-in range using the Galerkin method, and those in the Josephson junction circuit are also examined¹² to clarify their current-voltage characteristics.

Van der Pol and PLL equations describe frequency entrainment by periodic forcing. The entrainment for these two equations originates from different types of self-sustained oscillations. The van der Pol equation has a stable limit cycle of the first kind¹ or stable libration,¹² which is conceptually shown in Fig. 1. We then investigate entrained librations at the driving frequency of the periodic force. On the other hand, the rotation frequency in the PLL equation can be also entrained for the driving frequency of the modulation signal. The entrained state is discussed in the mathematical models of the Josephson junction circuit¹⁴⁻¹⁶ and electric power system.¹⁷ The entrainment is of significance for estimating pull-in characteristics under modulation signals. Unfortu-

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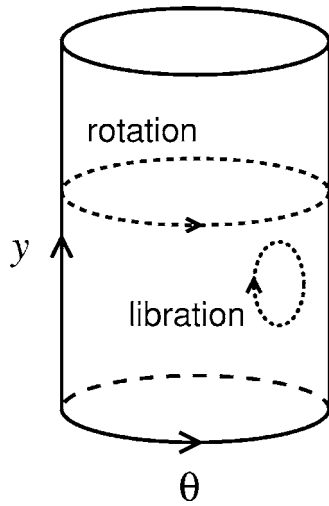


FIG. 1. Two kinds of limit cycles on cylindrical phase space. A libration on cylindrical phase space is isomorphic to limit cycles in phase plane. It represents a self-excited oscillation in the autonomous system described by the van der Pol equation. A rotation appears because of the periodic nature of cylindrical phase space and is regarded as another type of self-excited oscillation. This rotation is observed in the autonomous system described by the PLL equation. This paper analyzes entrained oscillations with the stable librations and rotations at frequencies of driving oscillatory force.

nately, entrainment has not been fully understood—in particular, its response characteristics and mechanism. In addition, the difference in entrainment described by the two equations has not been clarified.

The reason that we analyze the van der Pol and PLL equations is to explore the difference in frequency entrainment. The two main points of this paper are to derive response curves for harmonic entrainment described by the PLL equation and to investigate its mechanism. Response curves show characteristic features of frequency entrainment: They are particularly reported in the van der Pol equation.^{6,1,2} The main contribution of this paper is to show response curves for energy supplied by the driving oscillatory force in the van der Pol and PLL equations. The energy balance relation used here is equivalent to the energy conservation law of dynamical systems with dissipation and input terms. This relation is well known in classical mechanics¹⁸ and is related to passivity in circuits and systems.^{19,20} The obtained response curves show that entrainments for the two equations have different features of supplied energy, and that they possibly have the same mechanism as do the regulation of the phase difference.^{21,22} These results provide an important clue for exploring the unsolved difference in frequency entrainment for librations and rotations.

Investigating frequency entrainment based on energy balance and response curves is a basic approach. Cartwright²³ studied the van der Pol equation and clarified the behavior of dissipated energy in drift regions. Kuramitsu *et al.*^{24,25} investigated multimode oscillations and synchronization in nonlinear circuits using averaged potential and provided a physical description of synchronization using dissipation in the circuits. For the van der Pol equation, response curves of amplitude and phase difference were presented.^{6,1,2} Amplitude response curves for the PLL equation were also

derived.²⁶ The derivation of response curves is the starting point of the present study.

This paper is organized as follows. Section II introduces model systems and some mathematical preliminaries, including definition of energy balance relation and its physical meaning. The derivation of response curves based on the averaging method is also presented for van der Pol^{6,1,2} and PLL equations.²⁶ Section III is devoted to energy-based analysis of frequency entrainment. Response curves in the van der Pol equation are shown for harmonic amplitude, phase difference, and energy supplied by driving oscillatory force. The curves for supplied energy in this paper are original. Response curves for the PLL equation are derived for the dc component, harmonic amplitude, phase difference, and energy supplied by driving oscillatory force. They are also original except for the amplitude.²⁶ Section IV offers several physical and mathematical interpretations of frequency entrainment. Mechanisms of entrainment are considered for van der Pol and PLL equations. Section V concludes this paper with a summary.

II. MODEL EQUATIONS AND PRELIMINARIES

The purpose of this section is to introduce the two model equations and some mathematical preliminaries. The contents include mathematical models, definition of energy balance relation and its physical meaning, and derivation of response curves for harmonic entrainment.

A. Van der Pol equation

The dynamics of van der Pol’s oscillator with periodic force are represented by the following system:²

$$\frac{du}{dt} = v, \tag{1}$$

$$\frac{dv}{dt} = \mu(1 - \beta u - \gamma u^2)v - u + B \cos \nu t,$$

where μ is the small positive parameter, and β and γ are the fixed parameters. $B \cos \nu t$ represents the driving oscillatory term.

An energy balance relation for the system (1) is now introduced. A smooth function $S_v(u, v)$ is defined by

$$S_v(u, v) \triangleq \frac{1}{2}v^2 + \frac{1}{2}u^2. \tag{2}$$

S_v is called storage function and implies the physical interpretation of stored energy in the system (1). For any solution $[u(t), v(t)]$ of the system (1), the following equality is obtained at times t_1 and $t_2 (\geq t_1)$:

$$\begin{aligned} & S_v(u(t_2), v(t_2)) - S_v(u(t_1), v(t_1)) \\ &= \mu \int_{t_1}^{t_2} [1 - \beta u(\tau) - \gamma \{u(\tau)\}^2] \{v(\tau)\}^2 d\tau \\ &+ B \int_{t_1}^{t_2} v(\tau) \cos \nu \tau d\tau. \end{aligned} \tag{3}$$

The left-hand side of the equality (3) represents the change in stored energy during the time interval $[t_1, t_2]$. On the right-hand side, the first term stands for the total dissipated energy during $[t_1, t_2]$, and the second term is the total externally supplied energy by driving oscillatory force during $[t_1, t_2]$. The positiveness of supplied energy implies that the total work done by the driving force is positive. Its negativness represents positive total work done by the system (1). The equality (3) hence denotes the energy conservation law of the system (1) with dissipation and driving terms. This equality is called the *energy balance relation* for the system (1).

Next, the well-known derivation of response curves is introduced for the van der Pol equation. The following derivation is slightly different from that in the literature.^{1,2,6} The system (1) under $B=0$ has one stable libration with natural angular frequency ν_0 . When the driving frequency ν is in the neighborhood of ν_0 , we can observe an entrained oscillation at the frequency ν . This type of entrainment is called harmonic entrainment. Here, it is assumed that the solution $u(t)$ is approximated as follows:

$$u(t) = A_{\nu 1}(t)\cos\{\nu t + \varphi_{\nu 1}(t)\}, \tag{4}$$

where $A_{\nu 1}$ and $\varphi_{\nu 1}$ are constant for entrained oscillations. $\varphi_{\nu 1}$ represents the phase difference between $u(t)$ and the driving force $B \cos \nu t$. By substituting the above solution $u(t)$ and $v(t) = -\nu A_{\nu 1}(t)\sin\{\nu t + \varphi_{\nu 1}(t)\}$ in the system (1) and using the averaging method,²⁷ the following averaged system is obtained:

$$\frac{dA_{\nu 1}}{dt} = \mu \frac{A_{\nu 1}}{2} \left(1 - \gamma \frac{A_{\nu 1}^2}{4} \right) - \frac{B}{2\nu} \sin \varphi_{\nu 1}, \tag{5}$$

$$A_{\nu 1} \frac{d\varphi_{\nu 1}}{dt} = \frac{1 - \nu^2}{2\nu} A_{\nu 1} - \frac{B}{2\nu} \cos \varphi_{\nu 1},$$

where similar averaged systems are derived.^{5,23} Equilibrium points of the averaged system (5) possibly represent entrained oscillations in the original system (1). The response curves for amplitude $A_{\nu 1}$ and phase difference $\varphi_{\nu 1}$ are obtained by evaluating the equilibrium points for the parameters (ν, B) of the driving oscillatory force.

B. PLL equation

The dynamics of the PLL circuit as an FM demodulator are represented by the following system:⁷

$$\begin{aligned} \frac{d\phi}{dt} &= y, \\ \frac{dy}{dt} &= -ky - \sin \phi + k\sigma + m\sqrt{k^2 + \Omega^2} \cos \Omega t, \end{aligned} \tag{6}$$

where k and σ are the fixed parameters. $k\sigma + m\sqrt{k^2 + \Omega^2} \cos \Omega t$ is the driving dc term with oscillatory force. The driving term is slightly modified from the original one⁷ for simplicity. The two variables (ϕ, y) belong to cylindrical phase space $S^1 \times \mathbb{R}$ because of the periodic restoring term $-\sin \phi$.

An energy balance relation for system (6) is also introduced. A smooth function $S_p(\phi, y)$ is now defined as

$$S_p(\phi, y) \triangleq \frac{1}{2}y^2 - \cos \phi. \tag{7}$$

S_p implies a physical interpretation of stored energy in the system (6). For any solution $[\phi(t), y(t)]$, the following equality is obtained at times t_1 and $t_2 (\geq t_1)$:

$$\begin{aligned} S_p(\phi(t_2), y(t_2)) - S_p(\phi(t_1), y(t_1)) &= -k \int_{t_1}^{t_2} \{y(\tau)\}^2 d\tau + k\sigma \int_{t_1}^{t_2} y(\tau) d\tau \\ &\quad + m\sqrt{k^2 + \Omega^2} \int_{t_1}^{t_2} y(\tau) \cos \Omega \tau d\tau. \end{aligned} \tag{8}$$

The left-hand side of the equality (8) denotes the change in stored energy during the time interval $[t_1, t_2]$. On the right-hand side, the first term stands for the total dissipated energy during $[t_1, t_2]$. The second term denotes the total externally supplied energy by the driving dc force during $[t_1, t_2]$, and the third term represents the total externally supplied energy by the driving oscillatory force. The equality (8) is an energy balance relation for the system (6).

Next, response curves are introduced for the PLL equation: The following derivation was given.²⁶ The system (6) with $m=0$ and $|k\sigma| > 4k/\pi$ has one stable rotation with angular frequency Ω_0 . The condition of k and σ is obtained via Melnikov's method.²⁸ The corresponding solution $\phi_0(t)$ is represented using the periodic function $x_0(t)$ as

$$\phi_0(t) = \Omega_0 t + x_0(t), \quad x_0(t) \equiv x_0\left(t + \frac{2\pi}{\Omega_0}\right). \tag{9}$$

In the case that the driving frequency Ω is close to Ω_0 , we can expect an entrained oscillation at the frequency Ω . At the harmonic entrainment, the solution $\phi(t)$ of the system (6) is expressed with the periodic function $x(t)$:

$$\phi(t) = \Omega t + x(t), \quad x(t) \equiv x\left(t + \frac{2\pi}{\Omega}\right). \tag{10}$$

Here, it is assumed that the solution $x(t)$ is approximately given by

$$x(t) = \frac{A_{p0}}{2} + A_{p1}(t)\cos\{\Omega t + \varphi_{p1}(t)\}, \tag{11}$$

where A_{p1} and φ_{p1} are constant for entrained oscillations. A_{p0} is a function of parameters (Ω, m) and is also constant for entrained oscillations. $\varphi_{p1}(t)$ represents the phase difference between the solution $x(t)$ and driving force $m\sqrt{k^2 + \Omega^2} \cos \Omega t$. By substituting the above solution $\phi(t)$ and $y(t) = \Omega - \Omega A_{p1}(t)\sin\{\Omega t + \varphi_{p1}(t)\}$ in the system (6) and using the averaging method, the averaged system is obtained as

$$\begin{aligned} \frac{dA_{p1}}{dt} &= \frac{1}{2\Omega} \left[\{J_0(A_{p1}) + J_2(A_{p1})\} \cos\left(\frac{A_{p0}}{2} - \varphi_{p1}\right) \right. \\ &\quad \left. - m\sqrt{k^2 + \Omega^2} \sin \varphi_{p1} - k\Omega A_{p1} \right], \\ A_{p1} \frac{d\varphi_{p1}}{dt} &= \frac{1}{2\Omega} \left[\{J_0(A_{p1}) - J_2(A_{p1})\} \sin\left(\frac{A_{p0}}{2} - \varphi_{p1}\right) \right. \\ &\quad \left. - m\sqrt{k^2 + \Omega^2} \cos \varphi_{p1} - \Omega^2 A_{p1} \right], \end{aligned} \tag{12}$$

where J_n for $n=0,2$ are Bessel functions of the first kind. In parallel with the derivation for the van der Pol equation, equilibrium points of the averaged system (12) possibly represent entrained oscillations in the original system (6). Here, the three variables A_{p0} , A_{p1} , and φ_{p1} are required for the derivation of response curves. Thus, we need to obtain three independent equations. Two of the required equations are induced by equating the right-hand side of the averaged system (12) to zero. The remaining equation is obtained from the energy balance relation (8). If we suppose that A_{p1} and φ_{p1} are constant, then the following equality is derived:

$$0 = -\pi k\Omega(2 + A_{p1}^2) + 2\pi k\sigma - \pi m\sqrt{k^2 + \Omega^2} A_{p1} \sin \varphi_{p1}. \tag{13}$$

The three determining equations are well defined if they are linearly independent. The response curves for the dc component $A_{p0}/2$, amplitude A_{p1} , and phase difference φ_{p1} are obtained by solving the determining equations for the parameters (Ω, m) of the driving oscillatory force.

III. ENERGY-BASED ANALYSIS

This section is devoted to the energy-based analysis of frequency entrainment. First, response curves for the dc component, harmonic amplitude, and phase difference are presented. Next, response curves for total externally supplied energy by the driving oscillatory force are provided. The main focus here is on the relations between response curves for the dc component/harmonic amplitude/phase difference and energy supplied by the driving oscillatory force.

A. Van der Pol equation

Response curves of amplitude and phase difference were shown for harmonic entrainment.^{1,2,6,23} Figure 2 shows response curves of harmonic amplitude A_{v1} and phase difference φ_{v1} . The parameter setting is the same as that in Ref. 2:

$$\mu = 0.15, \quad \beta = \gamma = \frac{4}{3}. \tag{14}$$

The natural frequency ν_0 is nearly equal to unity. The response curves are obtained with the averaged system in Sec. II A. Solid lines denote the response curves for stable solutions, and broken lines denote those for unstable solutions. The criterion of stability is based on the Routh-Hurwitz method.²

The features of response curves in Fig. 2 are as follows. In part (a) of harmonic amplitude, every curve for the stable

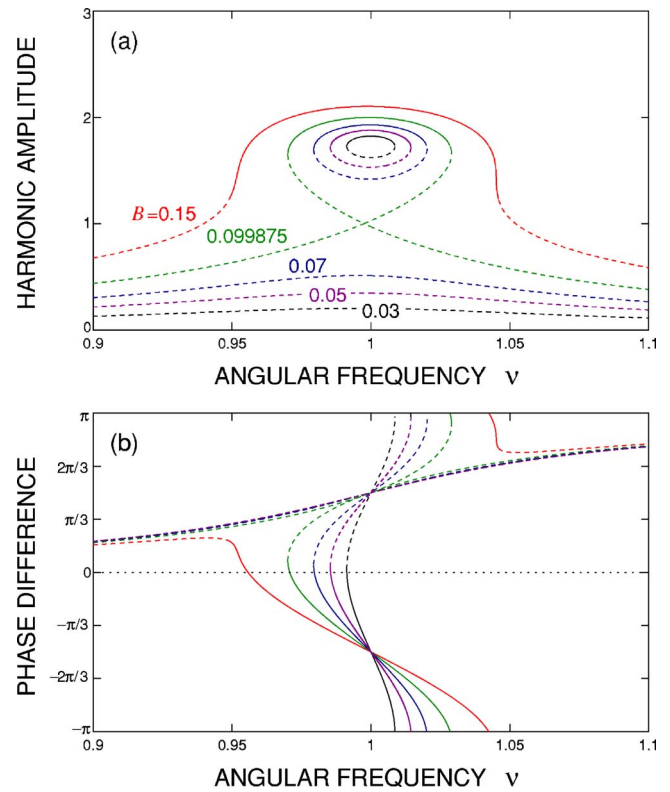


FIG. 2. Response curves for harmonic entrainment described by the van der Pol equation. The figure shows the change in the harmonic amplitude A_{v1} and phase difference φ_{v1} for driving frequency ν . Solid lines denote the response curves for stable solutions, and broken lines denote those for unstable solutions.

solutions has a maximum value near the natural frequency ν_0 . In part (b) of phase difference, every curve for the stable solutions has the value $-\pi/2$ at $\nu = \nu_0$. Every curve for the unstable solutions has the value $\pi/2$ at $\nu = \nu_0$. The phase difference of the stable solutions decreases monotonically as the driving frequency ν increases and vice versa. In Figs. 2(a) and 2(b), all the stable solutions have the maximum amplitude at the phase difference $-\pi/2$.

The energy balance relation (3) for harmonic entrainment is examined next. The amplitude A_{v1} and phase difference ϕ_{v1} are supposed to be constant. By substituting $u(\tau) = A_{v1} \cos(\nu\tau + \varphi_{v1})$ and $v(\tau) = -\nu A_{v1} \sin(\nu\tau + \varphi_{v1})$ in the equality (3) with $t_1=0$ and $t_2=2\pi/\nu$, the following equality is derived:

$$0 = \mu\pi\nu A_{v1}^2 \left(1 - \gamma \frac{A_{v1}^2}{4} \right) - \pi A_{v1} B \sin \varphi_{v1}. \tag{15}$$

On the right-hand side of the equality (15), the first term represents the total dissipated energy during $[0, 2\pi/\nu]$, and the second represents the total externally supplied energy by driving oscillatory force. The equality (15) implies that during one period, the total supplied energy is equal to the total dissipated energy. The total supplied energy is positive and increases monotonically as the square A_{v1}^2 of harmonic amplitude increases under the condition $A_{v1}^2 > 4/\gamma$.²⁹

Figure 3 shows response curves for total supplied energy under the parameters (14). Solid lines denote the supplied energy for stable solutions, and broken lines denote that for

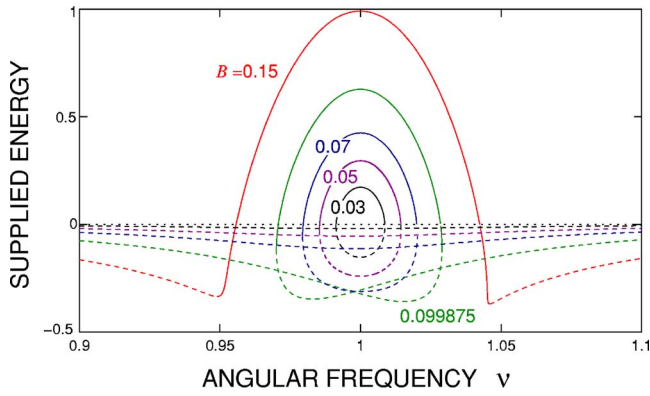


FIG. 3. Response curves for total externally supplied energy $-\pi A_{v1}B \sin \varphi_{v1}$ by driving oscillatory force in the van der Pol equation. Solid lines denote the supplied energy for stable solutions, and broken lines for unstable solutions.

unstable solutions. All the supplied energy for the stable solutions reaches maximum values near the natural frequency ν_0 . All the supplied energy for the unstable solutions is negative in the present range of driving frequency ν .

We summarize the relationship between response curves for the stable solutions in Figs. 2 and 3. Figures 2(a) and 3 show that the response curves for harmonic amplitude and supplied energy have common features. Figures 2(b) and 3 demonstrate that each response curve has a maximum value of the supplied energy at the phase difference $-\pi/2$. The supplied energy increases, attains maximum values, and decreases as the phase difference decreases monotonically.

B. PLL equation

Amplitude response curves for harmonic entrainment were presented in Ref. 26. Figure 4 shows response curves for the dc component $A_{p0}/2$, harmonic amplitude A_{p1} , and phase difference φ_{p1} under the parameters

$$k = 0.56, \quad \sigma = 1.7. \tag{16}$$

They are the same parameters as in Ref. 26. The natural rotation frequency Ω_0 is obtained as 1.6. The response curves are obtained with the averaged method in Sec. II B.³⁰ At each green circle in Fig. 4, the values of $(A_{p0}/2, A_{p1}, \varphi_{p1})$ are not uniquely defined because of the lack of determining equations.

Figure 4 does not include any stability condition. To investigate the stability, we analyze the response curves in Fig. 5 with numerical integration of the original system (6). Solid lines describe the response curves for stable solutions, and broken lines describe those for saddle-type solutions. The criterion of stability is based on linearization of the original system (6). The symbol “ \times ” in Fig. 5(b) denotes the point of parameter set $(\Omega, m) = (\Omega_0, 0)$. The corresponding value of A_{p1} is the harmonic amplitude of $x_0(t)$ in stable rotation. Figures 4 and 5, under the parameters (16), imply that the response curves with averaging method and numerical integration are almost consistent.

The features of response curves in Fig. 5 are as follows. Some of the features are discussed in Ref. 26. In part (a) of the dc component, every curve for the stable solutions has

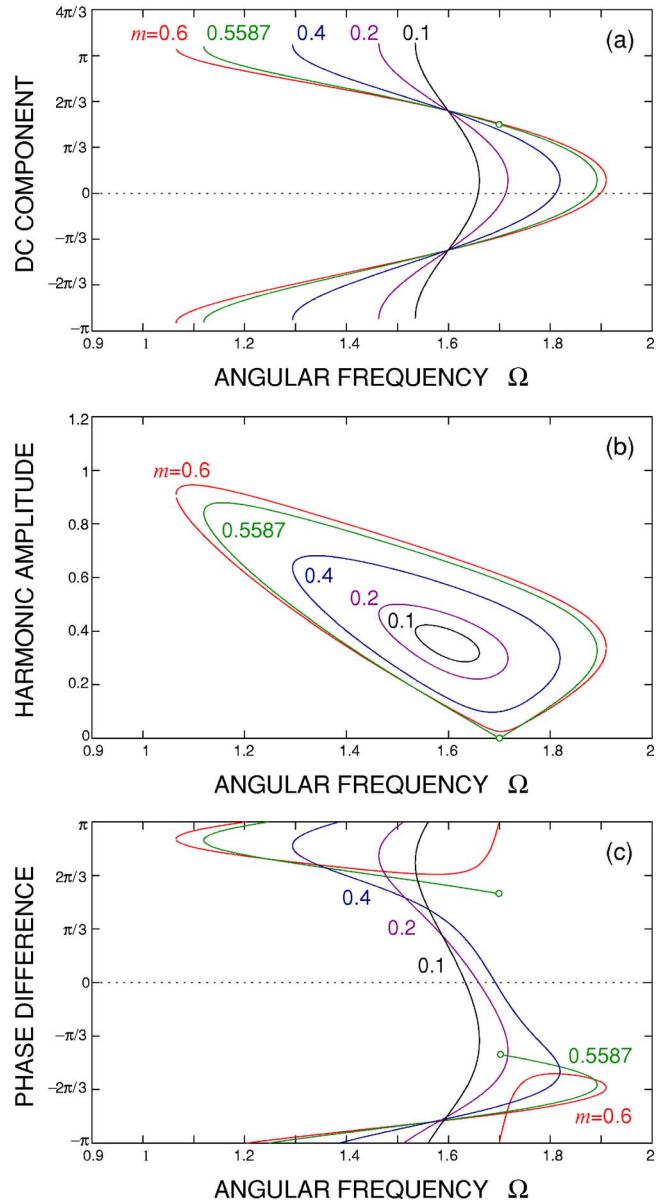


FIG. 4. Response curves for harmonic entrainment described by the PLL equation [1]. This figure is obtained using the averaging method described in Sec. II B and shows the change of (a) dc component $A_{p0}/2$, (b) harmonic amplitude A_{p1} , and (c) phase difference φ_{p1} for driving frequency Ω . The response curves of the harmonic amplitude A_{p1} are also reproduced from Ref. 26. The closed lines in the figure contain both the response curves of stable and unstable solutions.

the value 1.9 at $\Omega = \Omega_0$. Every curve for the saddle-type solutions has the value -1.2 at $\Omega = \Omega_0$. The dc component of the stable (or saddle-type) solutions decreases (or increases) monotonically as the driving frequency Ω increases.

In Fig. 5(b) of harmonic amplitude, every curve for the stable solutions attains a minimum value near Ω_0 . The amplitude of $x(t)$ is precisely zero at the parameter set $(\Omega, m) = (1.7, 0.5587)$. Note that all harmonics of $x(t)$ have zero amplitude at the parameter set.

In Fig. 5(c) of phase difference, every response curve for the stable solutions has zero value near Ω_0 except $m=0.6$. At $m=0.6$, the phase difference of the stable solutions becomes π near Ω_0 . At $m=0.5587$, the curves for the phase difference

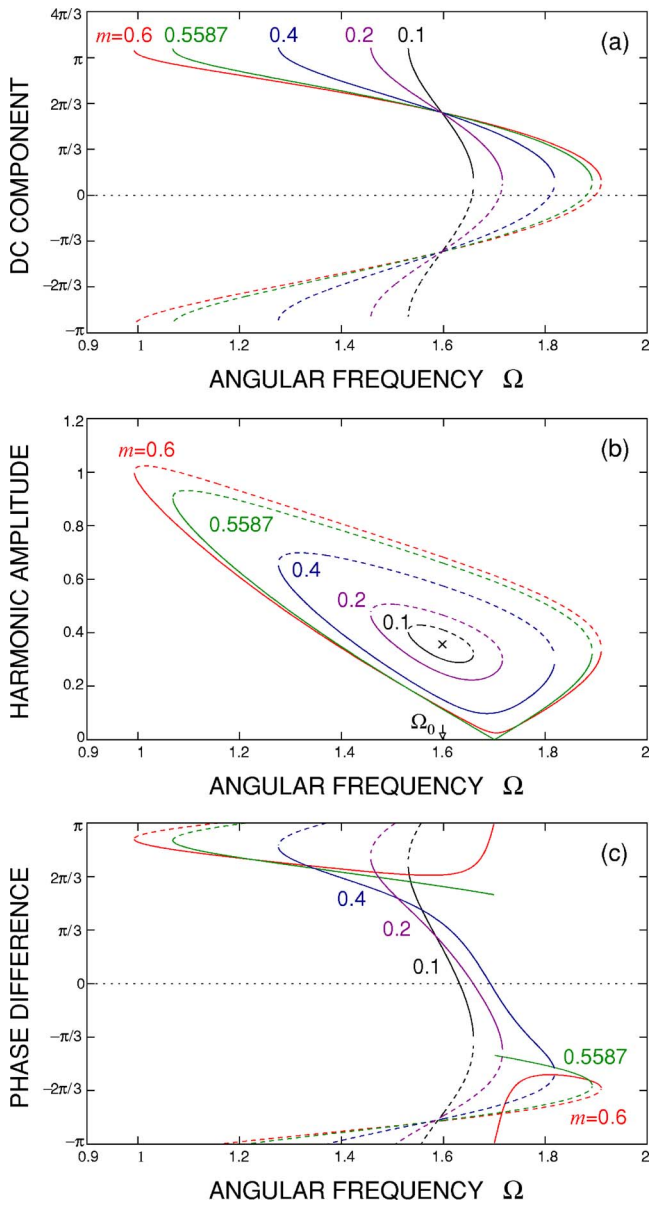


FIG. 5. Response curves for harmonic entrainment described by the PLL equation [11]. This figure is obtained by numerical integration of the original system (6). The response curves of harmonic amplitude are also reproduced from Ref. 26. Solid lines describe the response curves of stable solutions, and broken lines describe those of saddle-type solutions.

possess one discontinuous point at $\Omega=1.7$. The corresponding phase difference is not defined because of the zero harmonic amplitude. In Figs. 5(b) and 5(c), all stable solutions have minimum amplitude at zero phase difference except $m=0.6$ and at the value π under $m=0.6$.

Next, the energy balance relations (8) and (13) are examined for harmonic entrainment. The total energy supplied by the driving dc force is constant. Figure 6 shows the response curves for the total energy supplied by the driving oscillatory force under the parameters (16). Part (a) is obtained by the averaging method in Sec. II B, and part (b) by numerical integration of the original system (6). Figures 6(a) and 6(b) are almost consistent under the parameters (16). In part (b) solid lines denote the supplied energy for stable solutions, and broken lines denote that for saddle-type solu-

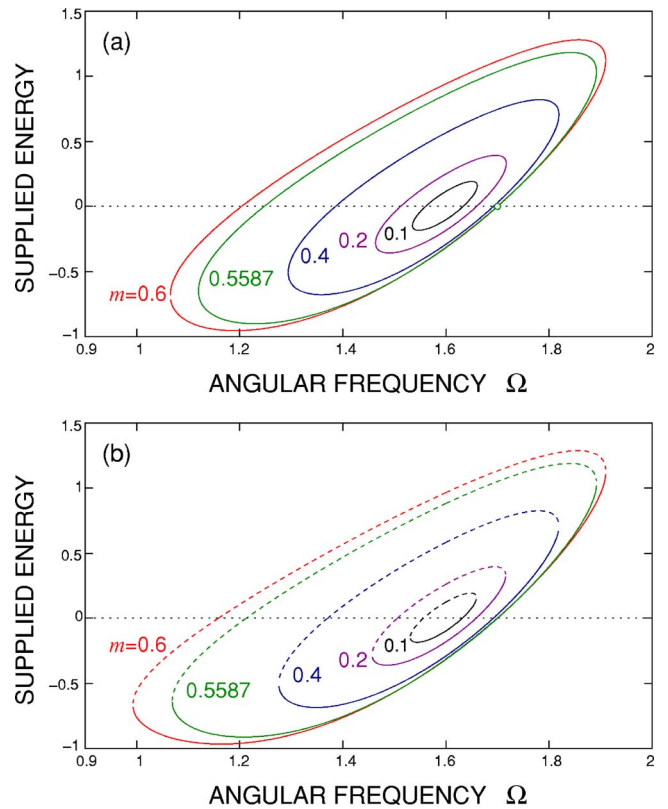


FIG. 6. Response curves of total externally supplied energy by driving oscillatory force in the PLL equation. Part (a) is obtained using the averaging method described in Sec. II B, and part (b) is obtained using numerical integration of the original system (6). In part (b), solid lines denote the total supplied energy for stable solutions, and broken lines denote that for saddle-type solutions.

tions. The supplied energy for all the solutions increases monotonically as the driving frequency Ω increases around the natural frequency Ω_0 .

We describe the relationship between response curves for the stable solutions in Figs. 5 and 6(b). In Figs. 5 and 6(b), the supplied energy increases monotonically around its zero value as the dc component decreases. Figures 5(b) and 6(b) show that the supplied energy has zero value at the minimum harmonic amplitude. In Figs. 5(c) and 6(b), each response curve has a maximum value of the supplied energy at zero phase difference except for $m=0.6$ and at the value π under $m=0.6$. The supplied energy increases monotonically around its zero value as the phase difference decreases. These figures indicate that the response curves for the supplied energy show a different shape from those of the dc component, harmonic amplitude, and phase difference.

IV. DISCUSSION

The previous section provided the response curves for total externally supplied energy by driving oscillatory force and described the relationships between response curves of dc component/harmonic amplitude/phase difference and supplied energy. This section offers several physical and mathematical interpretations of frequency entrainment. The contents include the relationship between response curves and

resonance phenomena, physical interpretations of conventional averaging method via the energy balance relation, and mechanisms of the entrainment.

A. Response curves and resonance phenomena

This section first introduces the origin of response curves for the van der Pol equation. The obtained relation between response curves in Figs. 2 and 3 is observed as a resonance phenomenon in forced oscillators.¹ This fact is not remarkable because the van der Pol's oscillator uses the resonance phenomenon of an LC circuit for generating self-sustained oscillations.⁶

We next consider the case of the PLL equation: The present question is whether the response curves in Figs. 4–6 have any feature that includes resonance phenomena. The frequency entrainment here is observed in the range of driving frequency Ω , far from the resonance frequency of the system (6).³¹ At $m=0$, the occurrence of stable rotations does not result from the resonance phenomenon. The rotations occur when the supplied energy by driving dc force per cycle is equal to the average of dissipated energy by damping.¹ The response curves for the PLL equation are not related to the resonance phenomenon.

B. Averaged systems and energy balance relation

This section focuses on the derivation of the response curves in Sec. II. The derivation for the van der Pol equation did not explicitly consider the energy balance relation (3). As mentioned previously, the determining equations for the response curves are obtained by equating the right-hand side of the averaged system (5) to zero. The zero assumption for the amplitude equation, which represents the averaged dynamics of A_{v1} , is equivalent to the energy balance relation (15). In other words, the determining equations satisfy the energy balance relation (15). On the other hand, in the PLL equation, the energy balance relation (13) is crucial for the derivation of the response curves. The energy balance relations are therefore necessary for investigating the response curves for the van der Pol and PLL equations.

Next, consider the conventional averaging method applied to the van der Pol equation. The zero assumption for the amplitude equation is equivalent to the energy balance relation (15). This implies that the dynamics of amplitude A_{v1} play a central role in adjusting the energy balance in the system (1). The physical description of amplitude and energy balance has been already provided by Kuramitsu *et al.*^{24,25} using averaged potential. This description is numerically shown in Figs. 2(a) and 3.

We further consider the averaging method for the PLL equation. The averaged system (12) represents the dynamics of harmonic amplitude and phase difference with three redundant variables ($A_{p0}/2, A_{p1}, \varphi_{p1}$). The redundancy can be avoided in the derivation of response curves in Sec. II B, because we consider the energy balance relation (13). The obtained response curves in Fig. 4 indicate that the three determining equations are linearly independent. This demonstrates that the averaged system (12) is not related to the energy balance relation (13). This relation also does not con-

tain the dc component $A_{p0}/2$. Thus, it can be said that the dynamics of amplitude A_{p1} and phase difference φ_{p1} play a role in adjusting the energy balance in the system (6).

C. Mechanism of frequency entrainment

The energy-based analysis sheds light on the mechanisms of frequency entrainment. A mechanism for the van der Pol equation is the regulation of phase difference.^{21,22} Entrainment is achieved if the difference of frequencies of external force ν and self-sustained oscillation ν_0 is covered by the shift of phase difference φ_{v1} in forced oscillation. It is also stated in Sec. IV B that the energy balance in van der Pol equation is adjusted by the dynamics of harmonic amplitude A_{p1} .

We next consider the mechanism for PLL equation. It is stated above that the energy balance in PLL equation is adjusted by the dynamics of amplitude A_{p1} and phase difference φ_{p1} . The stable rotation $\phi_0(t)$ and entrained oscillation $\phi(t)$ have been represented as

$$\begin{aligned}\phi_0(t) &= \Omega_0 t + A_{p1}^* \cos \Omega_0 t, \\ \phi(t) &= \Omega t + \frac{A_{p0}}{2} + A_{p1} \cos(\Omega t + \varphi_{p1}),\end{aligned}\tag{17}$$

where the stable rotation is assumed to be on the point (A_{p1}^*, Ω_0) . On the basis of the above solutions, one possible variable that plays a central role in achieving the entrainment is the dc component $A_{p0}/2$. Figures 2(b) and 5(a) show that the response curves for the phase difference φ_{v1} and dc component $A_{p0}/2$ have common features. Thus, it can be said that entrainment is achieved if the difference in frequencies of Ω_0 and Ω is covered by the shift of the dc component $A_{p0}/2$ in forced oscillation. This is valid if a dynamical model with the dc component variable $A_{p0}/2$ is derived under proper assumptions including the energy balance relation (13), like the case of the van der Pol equation. This is a forthcoming work of our analytical studies.

V. CONCLUSIONS

This paper used the concept of an energy balance relation to analyze the frequency entrainment described by the van der Pol and PLL equations. The main contribution here is to show that the entrainments have different features with respect to energy supplied by the driving oscillatory force. The energy balance relations determine what energy the self-oscillatory systems and external sources exchange. In other words, we have discussed energy conversion by the systems at the entrained states. This leads to the characterization of frequency entrainment based on energy conversion by nonlinear oscillators.

We also pointed out that entrainment for the PLL equation was possibly achieved by the shift of dc component $A_{p0}/2$. The dc component is regarded as a phase difference, because the variable ϕ represents the phase error of input signals to the phase comparator in the PLL circuit.⁷ If the above statement is true, the mechanisms of entrainment for the two equations regulate the phase difference. This sug-

gests that the entrainments with the same mechanism have different features for supplied energy by driving oscillatory force.

This paper is limited to the analysis of frequency entrainment. A similar energy-based analysis of other types of synchronization³² is a challenging issue. The main point for future studies is how we define the energy balance relation for each type of synchronization. This paper used the concept of an energy balance relation, which originates from the well-known energy conservation law of the systems (1) and (6). This concept is closely related to passivity of circuits and systems.^{19,20} The analysis performed using the driving oscillatory force also has the common basis to frequency response in control engineering.³³ Both passivity and frequency response are widely used in system engineering. We believe that the present analysis is applicable to physical and engineering systems with frequency entrainment and synchronization.

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- ³⁰In Fig. 4, the lower side of the response curve for A_{p1} is described by considering the absolute value $|A_{p1}|$. The corresponding upper side of the response curve for φ_{v1} is also described by considering the value $\varphi_{p1} - \pi$. ($-A_{p1}, \varphi_{p1} - \pi$) is also a solution of the determining equations because the equations have a symmetrical property of (A_{p1}, φ_{p1}) .
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