数理解析研究所講究録
1491 巻 2006 年 25-35

Uniqueness of solutions for Schrödinger maps and related estimates for the product of functions

京都大学大学院理学研究科　加藤 淳 (Jun Kato)*
Department of Mathematics, Kyoto University

1 Introduction

In this note, we consider the uniqueness of solutions of the Cauchy problem for the system of the nonlinear Schrödinger equations arising form the Schrödinger map, which is called the modified Schrödinger map.

We briefly explain the Schrödingre map and the derivation of the modified Schrödinger map. The Schrödinger map from $\mathbb{R}\times\mathbb{R}^n$ to S^2 is described by the map $s:\mathbb{R}\times\mathbb{R}^n\to S^2\subset\mathbb{R}^3$ satisfying

$$\partial_t s = s\times\Delta s, \tag{1.1}$$

where $\times$ denotes the exterior product in $\mathbb{R}^3$. In general, the Schrödinger map is formulated as the Schrödingerlike evolution of the harmonic map. In the case where the target manifold is S^2 have a special importance, because they naturally arise from Landau-Lifshitz equations governing the static as well as dynamical properties of magnetization.

Applying $(s\times\partial_t s)\cdot$ to each term of the equation (1.1), we have

$$\partial_t s\cdot\Delta s = 0, \tag{1.2}$$

since

$$(s\times\partial_t s)\cdot\partial_t s = \det(s\ \partial_t s\ \partial_t s) = 0,$$
$$(s\times\partial_t s)\cdot(s\times\Delta s) = |s|^2\partial_t s\cdot\Delta s - (s\cdot\Delta s)(s\cdot\partial_t s) = \partial_t s\cdot\Delta s.$$

* 現在の所属：名古屋大学大学院多元数理科学研究科
Department of Mathematics, Nagoya University

Thus, integrating (1.2) over $\mathbb{R}^n$, we obtain

$$\partial_t \int_{\mathbb{R}^n} |\nabla s(t,x)|^2 dx = 0, \tag{1.3}$$

which implies that the solution of (1.1) conserves the $\dot{H}^1$-norm. So, it is natural to expect the well-posedness of the Cauchy problem for (1.1) in such a energy class. We also notice that the equation (1.1) is invariant with respect to the scale transformation

$$s(t,x) \mapsto s(\lambda^2 t, \lambda x)$$

for all $\lambda > 0$. Observing the relation between the size of $\dot{H}^r$-norm of the initial data and life span of the time-local solution by using the scale transformation above, it is considered that the Cauchy problem of (1.1) is well-posed in H^r only if $r \geq n/2$. So, the energy class H^1 is critical in two space dimensions, and this critical case provides interesting problems similar to the wave maps [7].

In what follows, we consider the local well-posedness of the Cauchy problem of (1.1) in two space dimensions for the data in H^r with the small r as long as possible. To begin with, we rewrite the equation as follows. By using the stereographic projection

$$\mathbb{C} \ni z \mapsto \Big(\frac{2\,\mathrm{Re}\, z}{1+|z|^2}, \frac{2\,\mathrm{Im}\, z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\Big) \in S^2,$$

the equation (1.1) is rewritten as the nonlinear Schrödinger equation

$$\partial_t z = i \sum_{j=1}^{2} \Big(\partial_j - \frac{2\,\overline{z}\,\partial_j z}{1+|z|^2}\Big)\partial_j z. \tag{1.4}$$

Then, in [5], Nahmod, Stefanov and Uhlenbeck used the $U(1)$ gauge invariance of (1.4) to transform it into a system of nonlinear Schrödinger equations, called the modified Schrödinger map:

$$\begin{aligned} i\,\partial_t u_1 + \Delta u_1 &= -2i\,\boldsymbol{A}\cdot\nabla u_1 + A_0 u_1 + |\boldsymbol{A}|^2 u_1 + 4i\,\mathrm{Im}(u_2\overline{u}_1)u_2, \\ i\,\partial_t u_2 + \Delta u_2 &= -2i\,\boldsymbol{A}\cdot\nabla u_2 + A_0 u_2 + |\boldsymbol{A}|^2 u_2 + 4i\,\mathrm{Im}(u_1\overline{u}_2)u_1, \end{aligned} \tag{1.5}$$

where

$$u_j = e^{i\psi}\frac{\partial_j z}{1+|z|^2}, \quad j = 1, 2, \tag{1.6}$$

and the Coulomb gauge has been chosen. Then, $\boldsymbol{A} = \boldsymbol{A}[u] = (A_1[u], A_2[u])$ and $A_0 = A_0[u]$ are determined from

$$\begin{aligned} \mathrm{div}\boldsymbol{A} &= 0, \\ \partial_j A_k - \partial_k A_j &= -4\,\mathrm{Im}(u_j \overline{u}_k), \\ -\Delta A_0 &= 4 \sum_{j,k=1}^{n} \partial_j \partial_k \mathrm{Re}(u_j \overline{u}_k) - 2\Delta |u|^2, \end{aligned}$$

where we denoted $u = (u_1, u_2)$. For the explicit representation of $\boldsymbol{A}$ and A_0, see (2.1), (2.2). In the rest of this note we devote to the local well-posedness of the Cauchy problem for the modified Schrödinger map.

Remark 1.1. (1) Although in general it is quite intricate to work out the equivalence between the Schrödinger map problem (1.1) and the modified Schrödinger map problem (1.5), for the sphere and in cases which cover the range of solutions considered in this paper such equivalence has been discussed in Kenig and Nahmod [3].
(2) It is important to notice that due to the relation (1.6) the well-posedness of the modified Schrödinger map problem (1.5) in H^s corresponds to the well-posedness of the Schrödinger map problem (1.1) in H^{s+1}. Thus, the energy class of the Schrödinger map problem (1.1) corresponds to the L^2 for the modified Schrödinger map problem (1.5). In fact, we can compute $\partial_t \|u(t)\|_{L^2}^2 = 0$ directly from (1.5).

2 Main Result

In the following we consider the well-posedness of the Cauchy problem for the modified Schrödinger map in two space dimensions,

$$\text{(MS)} \begin{cases} i\,\partial_t u_1 + \Delta u_1 = -2i\,\boldsymbol{A}[u] \cdot \nabla u_1 + A_0[u]u_1 + |\boldsymbol{A}[u]|^2 u_1 + 4i\,\mathrm{Im}(u_2 \overline{u}_1) u_2, \\ i\,\partial_t u_2 + \Delta u_2 = -2i\,\boldsymbol{A}[u] \cdot \nabla u_2 + A_0[u]u_2 + |\boldsymbol{A}[u]|^2 u_2 + 4i\,\mathrm{Im}(u_1 \overline{u}_2) u_1, \\ u_1(0,x) = u_0^1(x), \quad u_2(0,x) = u_0^2(x), \end{cases}$$

where u_1, u_2 are complex valued functions, and $A[u] = (A_1[u], A_2[u])$, $A_0[u]$ are determined by

$$A_j[u] = 4\,G_j * \mathrm{Im}(u_1\overline{u}_2), \quad j = 1, 2, \tag{2.1}$$

$$G_1(x) = \frac{1}{2\pi}\frac{x_2}{|x|^2}, \quad G_2(x) = -\frac{1}{2\pi}\frac{x_1}{|x|^2},$$

$$A_0[u] = -4\sum_{j,k=1}^{2} R_j R_k \mathrm{Re}(u_j\overline{u}_k) - 2|u|^2. \tag{2.2}$$

Here, $R_j = \partial_j(-\Delta)^{-1/2}$ denotes the Riesz transforms.

One of the advantage to consider (MS) instead of (1.1) is that (MS) have the nice structure in the nonlinear term, which enable us to construct energy estimate for the solution to (MS). In fact, Nahmod, Stefanov, and Uhlenbeck [6] showed the local well-posedness for the data in H^s when $s > 1$ by using the energy method. Then, independently, the author [1], and Kenig and Nahmod [3], showed the existence of the solution for the data in H^s when $s > 1/2$.

Theorem 2.1 ([1], [3]). *Let $u_0 \in H^s(\mathbb{R}^2)$ with $s > 1/2$. Then, there exist $T > 0$ and at least one solution $u \in L^\infty(0,T;H^s) \cap C_w([0,T];H^s)$ to (MS) satisfying*

$$J^{s-1/2-\varepsilon}u \in L^2(0,T;L^\infty) \tag{2.3}$$

where $0 < \varepsilon < s - 1/2$, $J = (I-\Delta)^{1/2}$. The solution is unique when $s \geq 1$.

The improvement of regularity comes form the use of a variant of the Strichartz estimates which was first introduced by Koch and Tzvetkov [4] in the context of the Benjamin-Ono equation. However, the uniqueness of solutions could not be proved in the same class due to the lack of the good structure such as (MS) on the equation satisfied by the difference of the two solutions.

The purpose of this note is to show the idea of the proof of our following recent result on the uniqueness of solutions to (MS). In the following we use the notation $L^p_T X$ to denote $L^p(0,T;X)$ for a Banach space X.

Theorem 2.2 ([2]). *Let u and v be smooth solutions to (MS) with the same smooth data satisfying*

$$u, v \in L^\infty(0,T;H^{1/2}) \cap L^p(0,T;B^{1/2}_{q,2}) \tag{2.4}$$

for some $q > 4$ with $1/p = 1/2 - 1/q$. Then, $u \equiv v$ holds. Moreover, the estimate

$$\|u(t) - v(t)\|_{H^{-1/2}} \le C\|u(t') - v(t')\|_{H^{-1/2}} \tag{2.5}$$

holds when $t > t'$, where the constant C depends on $\|u\|_{L^\infty_T H^{1/2}}$, $\|v\|_{L^\infty_T H^{1/2}}$, $\|u\|_{L^p_T B^{1/2}_{q,2}}$, and $\|v\|_{L^p_T B^{1/2}_{q,2}}$, and $B^s_{p,q}$ is the Besov space.

Theorem 2.1 was proved by using the compactness argument based on a priori estimates of the solution to (MS):

$$\|J^{s-1/2-\varepsilon}u\|_{L^2_T L^\infty} \le C\|u_0\|_{H^s}, \tag{2.6}$$

$$\|u\|_{L^\infty_T H^s} \le C(\|u_0\|_{H^{1/2+\varepsilon'}})\|u_0\|_{H^s}, \tag{2.7}$$

for $s > 1/2$, ε, $\varepsilon' \in (0, s - 1/2)$. When $s > 3/4$, we observe that the solution to (MS) satisfy the condition (2.4) by interpolating a priori estimates (2.6), (2.7). Indeed, if we set $s = 3/4 + 2\varepsilon$, we have

$$\|u\|_{L^2_T B^{1/4}_{\infty,2}} \lesssim \|J^{1/4+\varepsilon}u\|_{L^2_T L^\infty} \le M, \quad \|u\|_{L^\infty_T B^{3/4+2\varepsilon}_{2,2}} \lesssim \|u\|_{L^\infty_T H^{3/4+2\varepsilon}} \le M,$$

for some constant $M > 0$, thus we obtain

$$\|u\|_{L^p_T B^{1/2}_{q,2}} \le \left\| \|u\|^{1-2/q}_{B^{1/4}_{\infty,2}} \|u\|^{2/q}_{B^{3/4+2\varepsilon}_{2,2}} \right\|_{L^p_T} \le \|u\|^{2/p}_{L^2_T B^{1/4}_{\infty,2}} \|u\|^{1-2/p}_{L^\infty_T B^{3/4+2\varepsilon}_{2,2}} \le M,$$

where $1/q = 1/(4 + 16\varepsilon)$ and $1/p = 1/2 - 1/q$. Therefore, in the case $s > 3/4$ we are able to apply Theorem 2.2 in the proof of Theorem 2.1, and obtain the following corollary.

Corollary 2.3. *Let $u_0 \in H^s(\mathbb{R}^2)$ with $s > 3/4$. Then, there exist $T > 0$ and a unique solution $u \in C([0,T]; H^s)$ to (MS) satisfying*

$$J^{s-1/2-\varepsilon}u \in L^2(0, T; L^\infty)$$

where $0 < \varepsilon < s - 1/2$.

Remark 2.4. In Corollary 2.3 we could improve the condition of the regularity on the initial data which was already known, $s = 1$. There is still a gap from the condition $s > 1/2$ which the existence of a solution is known. There is also a gap from the critical space L^2.

3 Idea of Proof of Theorem 2.1

In this section we describe the idea of the proof of Theorem 2.1. For simplicity, we consider the more simple problem which contains the essential part of (MS),

$$\text{(P)}\begin{cases} i\,\partial_t u + \Delta u = i\boldsymbol{A}[u]\cdot\nabla u, \quad (t,x)\in(0,T)\times\mathbb{R}^2, \\ u(0,x)=u_0, \quad x\in\mathbb{R}^2, \end{cases}$$

where $\boldsymbol{A}[u] = \boldsymbol{G} * |u|^2$. Here, we denoted $\boldsymbol{G} = (G_1, G_2)$. We notice that

$$\boldsymbol{A}[u] \sim |x|^{-1} * |u|^2 \sim D^{-1}|u|^2,$$

where $D^s = (-\Delta)^{s/2}$.

Let u, v be the solutions to (P), then $w \equiv u - v$ satisfy

$$i\,\partial_t w + \Delta w = i\boldsymbol{A}[u]\cdot\nabla w + i\big(\boldsymbol{A}[u]-\boldsymbol{A}[v]\big)\cdot\nabla v. \tag{3.1}$$

The usual way to show the uniqueness is to estimate the L^2-norm of w. In fact, multiplying $\overline{w}$ to both sides of the equation (3.1), taking the imaginary part, and then integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\,\partial_t\|w(t)\|_{L^2}^2 = \operatorname{Re}\int_{\mathbb{R}^2}\big(\boldsymbol{A}[u]-\boldsymbol{A}[v]\big)\cdot\nabla v\,\overline{w}\,dx.$$

If we consider the solutions in the class

$$u, v \in C([0,T]; H^s)$$

with $s > 1$, then the uniqueness of solutions is easily obtained as follows. Let $1 < s_0 < \min(s,2)$, and set $1/p = 1 - s_0/2$, $1/2 = 1/p + 1/q$, and $1/r = 1/q + 1/2$. Then, applying the Hölder inequality and the Sobolev embedding we obtain

$$\begin{aligned}
\frac{1}{2}\,\partial_t\|w(t)\|_{L^2}^2 &\le \Big|\int_{\mathbb{R}^2}\big(\boldsymbol{A}[u]-\boldsymbol{A}[v]\big)\cdot\nabla v\,\overline{w}\,dx\Big| \\
&\le \|D^{-1}(|u|^2-|v|^2)\|_{L^q}\|\nabla v\|_{L^p}\|w\|_{L^2} \\
&\lesssim \||u|^2-|v|^2\|_{L^r}\|v\|_{\dot H^{s_0}}\|w\|_{L^2} \\
&\lesssim \big(\|u\|_{L^q}+\|v\|_{L^q}\big)\|v\|_{\dot H^{s_0}}\|w\|_{L^2}^2 \\
&\lesssim \big(\|u\|_{\dot H^{2-s_0}}+\|v\|_{\dot H^{2-s_0}}\big)\|v\|_{\dot H^{s_0}}\|w\|_{L^2}^2.
\end{aligned}$$

Since $H^s \hookrightarrow \dot H^{2-s_0}$, $H^s \hookrightarrow \dot H^{s_0}$, by using the Gronwall inequality we obtain

$$\|w(t)\|_{L^2} \le C\|w(0)\|_{L^2},$$

which implies the uniqueness of solutions.

To show the uniqueness of less regular solutions, we consider the estimate of w in $H^{-1/2}$ instead of L^2 to overcome the loss of the derivative the nonlinearity. We use the following energy estimate.

Lemma 3.1. *Let w be a solution to*

$$i\partial_t w + \Delta w - i\boldsymbol{a}\cdot\nabla w = F, \tag{3.2}$$

where $\boldsymbol{a}$ is $\mathbb{R}^2$-valued function. Then, for $0<s<1$, $0<t<T$, we have

$$\|w(t)\|_{H^{-s}} \le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\Big(\|w(0)\|_{H^{-s}} + \int_0^t \|F(t')\|_{H^{-s}}dt'\Big). \tag{3.3}$$

Idea of Proof of Lemma 3.1. For $0\le \tau < T$, we denote by $S(t,\tau)f$ the solution to

$$\begin{cases} i\,\partial_t v + \Delta v - i\boldsymbol{a}\cdot\nabla v = 0, & (t,x)\in(\tau,T)\times\mathbb{R}^2,\\ v(\tau,x) = f(x), & x\in\mathbb{R}^2. \end{cases}$$

Then, the solution to (3.2) is written as

$$w(t) = S(t,0)w(0) - i\int_0^t S(t,\tau)F(\tau)d\tau.$$

Thus, to prove (3.3) it suffices to show

$$\|S(t,\tau)f\|_{H^{-s}} \le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\|f\|_{H^{-s}}. \tag{3.4}$$

To prove (3.4) we consider the dual problem for fixed $t\in(0,T]$,

$$\begin{cases} i\,\partial_\tau \tilde v + \Delta\tilde v - i\nabla\cdot(\boldsymbol{a}\,\tilde v) = 0, & (\tau,x)\in(0,t)\times\mathbb{R}^2,\\ \tilde v(t,x) = g(x), & x\in\mathbb{R}^2. \end{cases}$$

We denote by $\tilde S(\tau,t)g$ the solution to the problem above. Then, $\tilde S(\tau,t)$ is dual operator to $S(t,\tau)$. In fact, the simple calculation shows that

$$\partial_{t'}\langle S(t',\tau)f, \tilde S(t',t)g\rangle = 0$$

by using the equation, and integrating this from τ to t we derive

$$\langle S(t,\tau)f, g\rangle = \langle f, \tilde{S}(\tau,t)g\rangle.$$

Meanwhile, from the equation we have

$$\partial_\tau \|\tilde{v}(\tau)\|_{L^2}^2 \le \int \boldsymbol{a}\cdot\nabla|\tilde{v}|^2 dx \le \|\nabla \boldsymbol{a}\|_{L^\infty}\|\tilde{v}(\tau)\|_{L^2}^2.$$

Similarly, we have

$$\partial_\tau \|\nabla\tilde{v}(\tau)\|_{L^2} \le C\|\nabla \boldsymbol{a}\|_{L^\infty}\|\nabla\tilde{v}(\tau)\|_{L^2}.$$

Thus, interpolating them we obtain

$$\|\tilde{S}(\tau,t)g\|_{H^s} \le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\|g\|_{H^s}, \tag{3.5}$$

for $0\le s\le 1$. Therefore, by using the duality we obtain

$$\begin{aligned}
\|S(t,\tau)f\|_{H^{-s}} &= \sup_{\|\varphi\|_{H^s}=1}\Big|\int S(t,\tau)f\,\varphi\,dx\Big| \\
&= \sup_{\|\varphi\|_{H^s}=1}\Big|\int f\,\tilde{S}(\tau,t)\varphi\,dx\Big| \\
&\le \sup_{\|\varphi\|_{H^s}=1}\|f\|_{H^{-s}}\|\tilde{S}(\tau,t)\varphi\|_{H^s} \\
&\le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(\tau)\|_{L^\infty}d\tau\Big\}\|f\|_{H^{-s}}.
\end{aligned}$$

Thus we obtain (3.4). □

Applying Lemma 3.1 to (3.1) with $s=1/2$ we obtain

$$\begin{aligned}
\|w(t)\|_{H^{-1/2}} \le \exp&\Big\{C\int_0^T \|\nabla \boldsymbol{A}[u](t')\|_{L^\infty}dt'\Big\} \\
&\times\Big(\|w(0)\|_{H^{-1/2}} + \int_0^t \|(\boldsymbol{A}[u]-\boldsymbol{A}[v])\cdot\nabla v\|_{H^{-1/2}}dt'\Big).
\end{aligned}\tag{3.6}$$

Since $\nabla \boldsymbol{A}[u] \sim R_jR_k|u|^2$, for sufficiently small $\delta>0$ and $\delta > 2/\tilde{q}$, we have

$$\|\nabla \boldsymbol{A}[u]\|_{L^\infty} \lesssim \|J^\delta R_jR_k|u|^2\|_{L^{\tilde{q}}} \lesssim \|J^\delta|u|^2\|_{L^{\tilde{q}}} \lesssim \|J^\delta u\|_{L^{\tilde{q}}}^2 \lesssim \|u\|_{B^{1/2}_{q,2}}^2.$$

So, the problem is to estimate the product of functions in the Sobolev spaces of negative order which appears in the last term in (3.6).

Remark 3.2. One might think there would be another possibility to apply Lemma 3.1 instead of $H^{-1/2}$. However, from the general version of the lemma below, and from the structure of the nonlinear term, the space $H^{-1/2}$ provides the best result in our method.

Lemma 3.3. *Suppose $n = 2$ and $q > 4$. Then the following estimates hold.*

$$\|fg\|_{H^{-1/2}} \lesssim \|g\|_{B^{1/2}_{q,2}} \|f\|_{H^{-1/2}}, \tag{3.7}$$

$$\|(G * (fg))\nabla h\|_{H^{-1/2}} \lesssim \big(\|g\|_{H^{1/2}}\|h\|_{H^{1/2}} + \|g\|_{B^{1/2}_{q,2}}\|h\|_{B^{1/2}_{q,2}}\big)\|f\|_{H^{-1/2}}. \tag{3.8}$$

If we apply (3.8) to estimate the last term of (3.6), then we obtain

$$\|w(t)\|_{H^{-1/2}} \le C\Big(\|w(0)\|_{H^{-1/2}} + \int_0^t (\|u(\tau)\|_X^2 + \|v(\tau)\|_X^2)\|w(\tau)\|_{H^{-1/2}} d\tau\Big),$$

where we denoted $X = H^{1/2} \cap B^{1/2}_{q,2}$. Thus, by the Gronwall inequality we obtain

$$\|w(t)\|_{H^{-1/2}} \le C\|w(t)\|_{H^{-1/2}}.$$

Thus, Theorem 2.2, the uniqueness of the solution, follows.

Finally we describe the idea of the proof of Lemma 3.3.

Idea of Proof of Lemma 3.3. To prove (3.8) we first show that

$$\|fg\|_{H^{1/2}} \lesssim \|g\|_{B^{1/2}_{q,2}} \|f\|_{H^{1/2}} \tag{3.9}$$

holds. In fact, by using fractional Leibniz rule we have

$$\|fg\|_{H^{1/2}} \lesssim \|f\|_{H^{1/2}}\|g\|_{B^0_{\infty,2}} + \|f\|_{B^0_{r,2}}\|g\|_{B^{1/2}_{q,2}},$$

where $1/2 = 1/q + 1/r$. Then, the embeddings $B^{1/2}_{q,2} \hookrightarrow B^0_{\infty,2}$ and $H^{1/2} \hookrightarrow H^{2/q} \hookrightarrow B^0_{r,2}$ give (3.9). Thus, by using the duality we obtain

$$\begin{aligned}\|fg\|_{H^{-1/2}} &= \sup_{\|\varphi\|_{H^{1/2}}=1} \Big|\int fg\varphi\, dx\Big| \\ &\le \sup_{\|\varphi\|_{H^{1/2}}=1} \|f\|_{H^{-1/2}}\|g\varphi\|_{H^{1/2}} \\ &\lesssim \|f\|_{H^{-1/2}}\|g\|_{B^{1/2}_{q,2}}.\end{aligned}$$

Now we turn to the proof of (3.8). Since $\operatorname{div} G * (fg) = 0$, we have

$$\|(G * (fg))\nabla h\|_{H^{-1/2}} = \|\operatorname{div}\{(G * (fg))h\}\|_{H^{-1/2}} \lesssim \|(G * (fg))h\|_{H^{1/2}}. \tag{3.10}$$

To estimate the right hand side of (3.10) we divide $G*(fg)$ into the high frequency part and the low frequency part,

$$G*(fg) = S_0\big(G*(fg)\big) + (1-S_0)\big(G*(fg)\big). \tag{3.11}$$

Here, S_0 is defined as the Fourier multiplier by φ, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ near the origin.

As for the high frequency part, the second term on the right hand side of (3.11), we easily obtain

$$\begin{aligned}
\|\{(1-S_0)\big(G*(fg)\big)\}h\|_{H^{1/2}} &\lesssim \|h\|_{B^{1/2}_{q,2}}\|(1-S_0)\big(G*(fg)\big)\|_{H^{1/2}} \\
&\lesssim \|h\|_{B^{1/2}_{q,2}}\|fg\|_{H^{-1/2}} \\
&\lesssim \|h\|_{B^{1/2}_{q,2}}\|g\|_{B^{1/2}_{q,2}}\|f\|_{H^{-1/2}},
\end{aligned}$$

by using (3.9), (3.7).

As for the low frequency part, the first term on the right hand side of (3.11), we estimate

$$\|\{S_0\big(G*(fg)\big)\}h\|_{H^{1/2}} \lesssim \|S_0\big(G*(fg)\big)\|_{W^{1,\infty}}\|h\|_{H^{1/2}}. \tag{3.12}$$

To complete the proof we have to estimate $S_0\big(G*(fg)\big)$ and its gradient. By translation invariance it suffices to do this at the origin. The argument for $S_0\big(G*(fg)\big)$ and for its gradient is the same. We observe that

$$\begin{aligned}
S_0\big(G*(fg)\big)(0) &= \{\Phi*(fg)\}(0) \\
&= \int_{\mathbb{R}^2}\Phi(y)f(y)g(y)dx,
\end{aligned}$$

where we set $\Phi = \mathcal{F}^{-1}[\varphi]*G$. Note that $\Phi \in L^r(\mathbb{R}^2)$ for $2<r<\infty$. Thus,

$$\begin{aligned}
|S_0\big(G*(fg)\big)(0)| &= \left|\int_{\mathbb{R}^2}\Phi(y)f(y)g(y)dy\right| \\
&\le \|\Phi g\|_{H^{1/2}}\|f\|_{H^{-1/2}} \\
&\lesssim \|\Phi\|_{B^{1/2}_{q,2}}\|g\|_{H^{1/2}}\|f\|_{H^{-1/2}}.
\end{aligned}$$

Finally, we notice that

$$\|\Phi\|_{B^{1/2}_{q,2}} \lesssim \|\Phi\|_{B^{1/2+\epsilon}_{q,q}} \lesssim \|\Phi\|_{B^0_{q,q}} \sim \|\Phi\|_{L^q} < \infty,$$

since Φ is supported in the low frequency part in the Fourier space, and $\Phi \in L^r(\mathbb{R}^2)$ for $2<r<\infty$. This completes the proof of (3.8). $\square$

参考文献

[1] J. Kato, Existence and uniqueness of the solution to the modified Schrödinger map, Math. Res. Lett. **12** (2005), 171–186.

[2] J. Kato, H. Koch, Uniqueness of modified Schrödinger maps in $H^{3/4+\varepsilon}(\mathbb{R}^2)$, to appear in Comm. Partial Differential Equations.

[3] C. E. Kenig, A. Nahmod, The Cauchy problem for the hyperbolic-elliptic Ishimori system and Schrödinger maps, Nonlinearity **18** (2005), 1987–2009.

[4] H. Koch, N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, Int. Math. Res. Not. **26** (2003), 1449–1464.

[5] A. Nahmod, A. Stefanov, K. Uhlenbeck, On Schrödinger maps, Comm. Pure Appl. Math. **56** (2003), 114–151.

[6] A. Nahmod, A. Stefanov, K. Uhlenbeck, Erratum: On Schrödinger maps [Comm. Pure Appl. Math. **56** (2003), 114–151], Comm. Pure Appl. Math. **57** (2004), 833–839.

[7] D. Tataru, The wave maps equation, Bull. Amer. Math. Soc. (N.S.) **41** (2004), 185–204.