STUDIES ON SOLITONS AND EVOLUTION EQUATIONS OF NONLINEAR WAVE SYSTEMS

Yuji ISHIMORI

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ABSTRACT

Nonlinear waves in one-dimensional dispersive systems and related evolution equations are studied from the view points of soliton physics.

First, nonlinear waves in a lattice with $(2n, n)$ Lennard-Jones potential are investigated in small-amplitude and long-wavelength approximations. Equations derived are classified into three types according to the value of the force-range parameter n. For $n = 2$ and $\geqq 4$, we get the Benjamin-Ono equation and Korteweg-de Vries equation, respectively. Furthermore, an exact solution describing a multiple collision of periodic waves is obtained for the B-O equation. It is shown that the solution reduces to the algebraic multi-soliton solution in a long wave limit.

Secondly, discreteness effects on dynamics of a Sine-Gordon kink in a lattice system are studied with use of a perturbation formalism due to MaLaughlin and Scott. It is shown from the zeroth order condition that a kink moves in a periodic (Peierls) potential field which causes wobbling or pinning of the kink. The first order correction for the kink consists of two parts, that is, a dressing part and a radiation one. The dressed kink is steeper in shape than the continuum S-G kink and the amplitude of the backward radiation is larger than that of the forward radiation. These results are in accord with a existing numerical work.

Thirdly, relationship among some schemes of the inverse scattering transform are discussed. It is shown that two inverse scattering formalisms by Ablowitz, Kaup, Newell and Segur and by Wdati, Konno and Ichikawa are connected through a gauge transformation and a transformation of the space and time coordinates depending on a dependent variable. One-soliton solutions associated with the W-K-I scheme are also examined.

Finally, an integrable spin model on the one-dimensional lattice is obtained from the differential-difference nonlinear Schrödinger equation by introducing the concept of gauge equivalence. The spin model is the differential-difference analogue of the continuous isotropic Heisenberg spin chain. The inverse scattering method associated with it is discussed and the canonical action angle variables are constructed.

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6. Y.Ishimori and T.Munakata, "Kink Dynamics in the Discrete Sine-Gordon System –A Perturbational Approach–", J.Phys.Soc.Japan **51** (1982) 3367-3374 (Chapter III).
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CONTENTS

CHAPTER I

INTRODUCTION

The study of wave motion is one of the most important subjects in physics.

In general, waves having small amplitudes are described by linear wave equation. Such linear waves are resolved into independent components, normal modes, and various physical problems associated with them can be easily solved. As the mathematical tool to treat it, the Fourier transform method is usually used.

For nonlinear waves with finite amplitudes, the situation is quite different. It is hardly possible to treat the effects of nonlinearity except to take them as a perturbation into the basis solutions of the linearized theory. However, in recent years, the nonlinear wave theory has been developed considerably outside the framework of perturbation theory and a large number of exactly solvable nonlinear wave systems have been found. In these systems, the concept of "solitons" played an important role and gave the viewpoint that nonlinearity can result in qualitatively new phenomena which cannot be constructed via perturbation theory from linearized equations[1),2)]. Soliton theory has been applied to various problems in the fields such as hydrodynamics, plasma physics, solid state physics, nonlinear optics and field theory. In this article, some problems associated with the soliton theory are studied.

Before mentioning the contents of the article, we explain the meaning of the name "soliton". This name was coined by Zabusky and Kruskal[3)] in 1965, who carried out computer experiments for the Korteweg-de Vries (K-dV) equation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \tag{1.1.1}$$

where ∂_t and ∂_x denote partial defferentiation with respect to time t and space-coordinate x, respectively. At that time it had been already known that the K-dV equation has a special solution with a pulselike shape, a solitary wave solution. Z-K showed how solitary waves would scatter upon collision. The result indicated that in spite of their nonlinear interaction, solitary waves would emerge from the collision having the same shapes and velocities with which they entered. To indicate this remarkable property, Z-K named the solitary wave of the K-dV equation "soliton" which means a solitary wave particle. The character of solitons which preserve their identities gave a suggestion that solitons are a kind of normal modes. This was supported by the inverse scattering method discovered by Gardner, Green, Kruskal and Miura (G-G-K-M)[4)] in 1967. They showed that the inverse scattering method enables us to solve the initial value problem for the K-dV equation through a succession of linear computations and that any solution can be resolved into independent components called scattering data which are composed of solitons and continuous radiation. The work by Zakharov and Faddeev[5)] in 1971 revealed more clearly that solitons are normal modes. They showed that the inverse scattering method for the K-dV equation may be considered as a canonical transformation connecting the canonical variables ($q = u, p = \int^x udx$) and the new ones (Q, P) constructed from the scattering data. The Hamiltonian for the K-dV equation written by the variables (Q, P) takes the form

$$H = 8\int_{-\infty}^{\infty} k^3 P(k)dk - \frac{16}{5}\sum_{n=1}^{N} P_n^{\frac{5}{2}}, \tag{1.1.2}$$

where the first and second terms represent continuous radiation and solitons, respectively. It is interesting to see that the continuous part of the Hamiltonian is essentially the same as the Hamiltonian for the linearized K-dV equation. Since this Hamiltonian is only a function of the canonical momentum, the variables (Q, P) are of the action-angle type and the equations of motion can be easily integrated, that is, the K-dV equation is a completely integrable Hamiltonian system. We note here that in (Q, P) space solitons have no interaction between them but in the original (q, p) space they do interaction which causes only a phase shift.

Subjects in the soliton theory are to find the solvable models, to develope methods which present the exact solutions of nonlinear wave equations, the research on mathematical structures of the solvable models, applications to physical problems (for example calculation of physical functions such as the partition and correlation functions, reduction of real systems to solvable ones and construction of a perturbation theory based on solvable nonlinear wave equations), the extention of the concept of solitons, and so on. The treatment in this article is confined to problems concerned with one-dimensional and classical systems.

In chapter II, we investigate the propagation of nonlinear waves in a continuum model of a lattice. In general, if we consider the nearest-neighbor interacting monoatomic lattice, the nonlinear waves with small but finite amplitude in a continuum model of it are described by the K-dV equation whatever types of the interatomic potentials are.[6)] In some systems such as metals, however, the interatomic forces are long range ones, and the interaction from far-neighboring atoms may affect the nonlinear wave propagation considerably. To study this problem, we take the $(2n, n)$ Lennard-Jones potential as interatomic potential and consider fully effects of the long-range interactions. We obtain the Benjamin-Ono (B-O) equation[7)] for $n = 2$ and the K-dV equation for $n \geqq 4$. As mentioned before, the K-dV equation is the equation which caused the discovery of solitons and its mathematical properties have been investigated in detail. Though for the B-O equation some problems have been left unsolved, the exact N-soliton solution (a solution describing a multiple collision of N solitons) has been obtained by various methods. We obtain the exact N-soliton solution of the B-O equation through the so-called Hirota's method[2)]. The key point of the method is that through a dependent variable transformation an original nonlinear wave equation can be rewritten in a bilinear form. For example, the K-dV equation (1.1.1) is transformed into the bilinear form

$$D_x(D_t + D_x^3) f \cdot f = 0, \tag{1.1.3}$$

through the transformation

$$u = -2\partial_x^2 \log f, \tag{1.1.4}$$

where D_x and D_t are defined by

$$D_x^m D_t^n f \cdot f = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) f(x',t')|_{x=x',t=t'}. \tag{1.1.5}$$

We can solve eq.(1.1.3) exactly using a kind of perturbational approach and obtain the N-soliton solutions. The context of chapter II is taken from published papers 8) and 9).

In chapter III, we study the effects of discreteness on the soliton (or kink) in the Sine-Gordon (S-G) system[10)]. The S-G equation has almost become ubiquitous in the theory of condensed matter, since it is a simple wave equation in a periodic medium. In many cases, the equation is derived from a lattice system by a continuum approximation. The aim of our study is to clarify the dynamical behavior of S-G soliton in a discrete media. For this purpose, we treat the effects of discreteness as a perturbation. Construction of a perturbation theory based on the integrable nonlinear wave equations has been done by many authors. Here we use the Green's function approach developed by McLaughlin and Scott[11)] because its formalism is suitable for our problem. The context of chapter III is taken from a published paper 12).

In chapter IV, we discuss relationships among some schemes of the inverse scattering transform. The inverse scattering method is one of the most important discovery in the soliton theory. After the work[4)] of G-G-K-M for the K-dV equation, Lax[13)] formulated the method in an elegant and general form, that greatly influenced subsequent developments. Several authors showed that this method is applicable to other equations, for example, the nonlinear Schrödinger equation by Zakharov and Shabat[14], the modified K-dV equation by Wadati[15] and Tanaka[16], and the S-G equation by Ablowitz, Kaup, Newell and Segur (A-K-N-S)[17]. Especially, A-K-N-S set up a general framework of the inverse scattering method including these examples[18]. Afterward there has been a continuous rise in research to the inverse scattering method, and at present the number of nonlinear wave equations solvable by the method has reached two figures[2].

Here we explain the framework of the inverse scattering method. In the method, we solve the initial value problem for a nonlinear wave equation by considering the auxiliary equatons:

$$\partial_\mu \Phi(x,t;\lambda) = Q_\mu(x,t;\lambda)\Phi(x,t;\lambda), \quad (\mu = x,t), \tag{1.1.6}$$

where in general Φ is a n-component vector and Q_μ are $n \times n$ matrices which depend on the wave variable $u(x,t)$ and the eigenvalue λ. By appropriate choice of the matrices Q_μ, we interpret the original nonlinear wave equation as the compatibility condition of eq.(1.1.6), which gives

$$\partial_\nu Q_\mu - \partial_\mu Q_\nu + Q_\mu Q_\nu - Q_\nu Q_\mu = 0. \tag{1.1.7}$$

For example, with the particular choice

$$Q_x = \begin{bmatrix} -i\lambda & u \\ 1 & i\lambda \end{bmatrix}, \tag{1.1.8a}$$

$$Q_t = \begin{bmatrix} -4i\lambda^3 - 2iu\lambda + (\partial_x u) & 4u\lambda^2 + 2i(\partial_x u)\lambda + 2u^2 - (\partial_x^2 u) \\ 4\lambda^2 + 2u & 4i\lambda^3 + 2iu\lambda - (\partial_x u) \end{bmatrix}, \tag{1.1.8b}$$

then eq.(1.1.7) becomes the K-dV equation (1.1.1). If we think of the spatial component of eq.(1.1.6) as a time independent scattering problem, the wave variable $u(x)$ plays the role of a scattering potential, and eq.(1.1.6) gives a connection between the variable $u(x)$ at a fixed time t and the scattering data associated with the linear eigenvalue problem. It is also shown that the scattering data $a(\lambda), b(\lambda)$ have a trivial time dependence

$$a(\lambda,t) = a(\lambda,0), \tag{1.1.9a}$$

$$b(\lambda,t) = e^{i\omega(\lambda)t} b(\lambda,0). \tag{1.1.9b}$$

The initial value problem is solved much like Fourier transform method. The direct transform maps $u(x) \to a(\lambda), b(\lambda)$ at time $t = 0$. The time evolution of a and b from $t = 0$ to some later time t is given by eq.(1.1.9). At time t we must perform an inverse transform which maps $a(\lambda,t), b(\lambda,t)$ back into $u(x,t)$. This last step is accomplished by the so-called Gel'fand-Levitan equation.

Our discussion in this chapter concerns with the choice of the matrices Q_μ. We focus our attention on Q_μ presented by A-K-N-S and similar matrices by other authors, and clarify the relationships among them. It is shown that some nonlinear wave equations are equivalent. The context of chapter IV is taken from published papers 19) and 20).

In chapter V, we present an integrable spin system on the one-dimensional lattice. The details of the inverse scattering approach to this spin model are given and canonical action-angle variables are constructed. We note here that in chapter IV and V the concept of gauge equivalence plays an important role. It is based on the property that eqs.(1.1.6) and (1.1.7) are form-invariant under the gauge transformation[21)]

$$\Phi' = g^{-1}\Phi, \tag{1.1.10a}$$

$$Q'_\mu = g^{-1}Q_\mu g - g^{-1}\partial_\mu g, \tag{1.1.10b}$$

where g is an arbitrary matrix. A part of contents of chapter V was published in a paper 22).

Finally, in chapter VI, we state concluding remarks and mention the future problems of our studies.

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CHAPTER II

SOLITONS IN A ONE-DIMENSIONAL LENNARD-JONES LATTICE

2.1 Introduction

Since the discovery of solitons by Zabusky and Kruskal[1)], many studies have been made on the nonlinear wave propagation in one-dimensional anharmonic lattices[2)]. Equations important in this problem are the Zabusky equation (or the Boussinesq equation), the Korteweg-de Vries (K-dV) equation, the modified K-dV equation, the nonlinear Schrödinger equation, the Sine-Gordon equation, the Toda lattice equation and so on[2),3)]. They all have been investigated in detail both numerically and analytically and are known to have N-soliton solutions[3)]. In these lattices, solitons play an important role for the physical properties such as heat conduction[4)].

The equations mentioned above are derived for a lattice with the nearest-neighbor interaction. In some lattices such as metals[5)], however, interatomic forces may extend further than to nearest neighbors. A lattice with the long-range interaction has, as is well known, the dispersion relation different from that of a lattice with the nearest-neighbor interaction, and may have soliton solutions not observed before.

In this chapter, we investigate this problem. As a model of the nonlinear lattice, we take a one-dimensional lattice with $(2n, n)$ Lennard-Jones (L-J) potential expressed as

$$U(r) = 4U_0 \left[\left(\frac{\sigma}{r}\right)^{2n} - \left(\frac{\sigma}{r}\right)^{n} \right], \tag{2.1.1}$$

where U_0 is the potential depth, 2σ the diameter of constituent particle and n is a positive integer. The smaller the value of parameter n, the longer is the range of force. Under the nearest-neighbor approximation, formerly Visscher et al.[6)] studied the (12,6) L-J lattice in connection with thermal conductivity in the nonlinear lattice and recently Yoshida and Sakuma[7)] presented the Boussinesq-like equation for the (2,1) L-J lattice. We shall investigate the general $(2n, n)$ L-J lattice with effects of the long-range interactions fully taken into account.

The plan of this chapter is as follows. In section 2.2, we present the general equations of motion for small vibration. In section 2.3, introducing the continuum approximation, we derive three types of nonlinear wave equations according to the value of the parameter n. For $n = 2$ and $\geqq 4$, we get the Benjamin-Ono (B-O) equation and the Korteweg-de Vries (K-dV) equation, respectively. In section 2.4, N-soliton solution of the B-O equation is examined. Concluding remarks are given in section 2.5.

2.2 Equations of motion for small vibration

We consider a lattice consisting of an infinite number of equally spaced identical particles of mass M, lying along a straight line. Let the equilibrium spacing between the particles be a and the longitudinal displacement of the pth particle from its equilibrium position be u_p (Fig.2.1). Then the total potential energy of the lattice, V, is given by

$$V = \sum_p \sum_{m>0} U(x_{p+m} - x_p), \tag{2.2.1}$$

where x_p is the position of the pth particle and given by

$$x_p = pa + u_p. \tag{2.2.2}$$

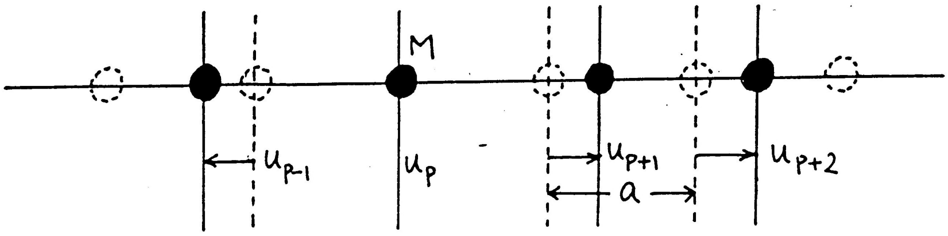

Fig.2.1. One-dimensional L-J lattice.
(◌)positions of particles when in equilibrium.
(●)positions of particles when displaced as for a longitudinal wave.

We assume that the particle displacement is very small compared to the interparticle distance. Expanding $U(x_{p+m} - x_p)$ in the displacement u_p and neglecting terms higher than $O(u_p^3)$, we obtain from eqs.(2.2.1) and (2.2.2)

$$V = V_0 + \frac{1}{2}\sum_p \sum_{m>0} U''(ma)(u_{p+m} - u_p)^2 + \frac{1}{6}\sum_p \sum_{m>0} U'''(ma)(u_{p+m} - u_p)^3, \tag{2.2.3}$$

where V_0 is the potential energy of the lattice corresponding to the equilibrium configuration,

$$V_0 = \sum_p \sum_{m>0} U(ma). \tag{2.2.4}$$

We have also used the fact that the terms linear in u_p vanish because the lattice is in equilibrium when $u_p = 0$ for all p. Requiring that V_0 be a minimum with respect to variations in the lattice spacing a [5)], we have from eqs.(2.1.1) and (2.2.4)

$$a = \left[\frac{2\zeta(2n)}{\zeta(n)}\right]^{\frac{1}{n}} \sigma, \tag{2.2.5}$$

where $\zeta(n)$ is the Riemann zeta function. We observe that for $n = 1$ the lattice spacing is zero. From now on we will assume $n \geqq 2$.

Let F_p denote the total force acting on the pth particle. Then it is given by $-\dfrac{\partial V}{\partial u_p}$, so that the equation of motion of the pth particle is

$$M\frac{d^2 u_p}{dt^2} = F_p, \tag{2.2.6}$$

with

$$F_p = \sum_{m>0} U''(ma)(u_{p+m} + u_{p-m} - 2u_p) + \frac{1}{2}\sum_{m>0} U'''(ma)[(u_{p+m} - u_p)^2 - (u_{p-m} - u_p)^2]. \tag{2.2.7}$$

If interactions only among nearest-neighbors are taken into account, then eq.(2.2.6) with eq.(2.2.7) reduces to

$$M\frac{d^2 u_p}{dt^2} = U''(ma)(u_{p+1} + u_{p-1} - 2u_p) + \frac{1}{2}U'''(ma)[(u_{p+1} - u_p)^2 - (u_{p-1} - u_p)^2]. \tag{2.2.8}$$

As is well known, a continuum limit of this equation yields the Boussinesq or the K-dV equation which has sech^2-type soliton solution[2)].

2.3 Nonlinear waves with long-wavelengths

In this section we study the nonlinear waves described by eq.(2.2.6) with eq.(2.2.7) which includes long-range force components. For the $(2n, n)$ L-J potential, this equation is rather complicated to study analytically. Here we consider smooth waves with wavelengths long compared with the lattice spacing, so that we adopt a continuum approximation.

For this purpose, it is convenient to introduce the Fourier expansion for u_p,

$$u_p = \sum_k Q_k \exp(ikx), \tag{2.3.1}$$

with

$$|k| < \frac{\pi}{a}, \tag{2.3.2}$$

where x is the equilibrium position of the pth particle, $x = pa$. Because u_p is real, we have $Q_{-k} = Q_k^*$. With use of eq.(2.3.1), the expression for F_p is written as

$$F_p = \sum_k [-2I(k)] Q_k e^{ikx} + \sum_k \sum_{k'} [iJ(k+k') - 2iJ(k)] Q_k Q_{k'} e^{i(k+k')x}, \tag{2.3.3}$$

where

$$I(k) = \sum_{m=1}^{\infty} U''(ma)[1 - \cos(mka)], \tag{2.3.4}$$

and

$$J(k) = \sum_{m=1}^{\infty} U'''(ma) \sin(mka). \tag{2.3.5}$$

For the $(2n, n)$ L-J potential (2.1.1), $I(k)$ and $J(k)$ are given by

$$I(k) = \frac{2n\zeta(n)U_0}{\zeta(2n)a^2} \left[\frac{(2n+1)\zeta(n)}{\zeta(2n)} A_{2n+2}(ka) - (n+1)A_{n+2}(ka) \right], \tag{2.3.6}$$

and

$$J(k) = \frac{2n(n+1)\zeta(n)U_0}{\zeta(2n)a^3} \left[-\frac{(4n+2)\zeta(n)}{\zeta(2n)} B_{2n+3}(ka) + (n+2)B_{n+3}(ka) \right], \tag{2.3.7}$$

where $A_n(ka)$ and $B_n(ka)$ are defined by eqs.(2.A.1) and (2.A.2) in Appendix 2.1. We note that the exact dispersion relation of the linear wave is expressed from the linearized version of eq.(2.3.3) as

$$\omega_k^2 = \frac{2}{M} I(k), \tag{2.3.8}$$

where ω_k is the frequency of the wave with wavevector k.

Let us take a continuum limit of eqs.(2.3.3), (2.3.6) and (2.3.7). Assuming that $|ka| << 1$, keeping the leading terms of $A_n(ka)$ and $B_n(ka)$ and neglecting the higher order terms i ka, we find that F_p has three types of expressions for $n = 2, 3, \cdots$ (see Appendix 2.1):

$$F_p = M \sum_k (-\omega_k^2) Q_k e^{ikx} - 3(n+1)Mc^2 \sum_k \sum_{k'} (ik)(ik') Q_k Q_{k'} e^{i(k+k')x}, \tag{2.3.9}$$

where c is the sound speed given by

$$c^2 = \frac{2n^2[\zeta(n)]^2 U_0}{\zeta(2n)M}, \tag{2.3.10}$$

and ω_k^2 is written as

$$\left.\begin{aligned} \omega_k^2 &= c^2(k^2 + \delta|k|^3) \\ \delta &= \frac{\pi a}{4\zeta(2)} \end{aligned}\right\} \quad \text{for} \quad n = 2, \tag{2.3.11}$$

$$\left.\begin{aligned} \omega_k^2 &= c^2(k^2 - \delta k^4 \log|ka|) \\ \delta &= \frac{a^2}{9\zeta(3)} \end{aligned}\right\} \quad \text{for} \quad n = 3, \tag{2.3.12}$$

and

$$\left.\begin{aligned} \omega_k^2 &= c^2(k^2 - \delta k^4) \\ \delta &= \frac{a^2}{12n}\left[(2n+1)\frac{\zeta(2n-2)}{\zeta(2n)} - (n+1)\frac{\zeta(n-2)}{\zeta(n)}\right] \end{aligned}\right\} \quad \text{for} \quad n \geqq 4. \tag{2.3.13}$$

Three equations show that the value of the force-range parameter n mainly contributes to the form of the dispersion relation.

We will derive from these expressions the equations govering $u_p = u(x,t)$ and study solitary wave solutions of them.

Case $n = 2$. Substituting eq.(2.3.1) into eqs.(2.3.9) and (2.3.11), we have from eq.(2.2.6)

$$\partial_t^2 u = c^2[\partial_x^2 u - \partial_x^3 \delta \mathcal{H}(u) - 9(\partial_x u)(\partial_x^2 u)], \tag{2.3.14}$$

where $\mathcal{H}$ is the Hilbert transform operator defined by

$$\mathcal{H}[f(x)] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x')}{x'-x}dx'. \tag{2.3.15}$$

We have also used the identity

$$\mathcal{H}(e^{ikx}) = i(\text{sgn}k)e^{ikx}. \tag{2.3.16}$$

A solitary wave solution of eq.(2.3.14) is written as

$$u = \frac{4}{9}\delta \tan^{-1}\left(\frac{x - \lambda t}{\Delta}\right), \tag{2.3.17a}$$

$$\lambda^2 = c^2\left(1 - \frac{\delta}{\Delta}\right), \tag{2.3.17b}$$

$$\Delta > \delta, \tag{2.3.17c}$$

where we have used the identity

$$\mathcal{H}\left(\frac{1}{x^2+\Delta^2}\right) = \frac{-x}{\Delta(x^2+\Delta^2)}. \tag{2.3.18}$$

It is well known that a solitary wave solution of the Zabusky equaiotn which is derived as a continuum limit of eq.(2.2.8) is compressive and supersonic. However, in the above solution, the propagation speed is smaller than the sound speed and the lattice is expanded around a solitary wave. Equation (2.3.14) can be reduced to the equation which describes the waves moving in one direction in the rest frame, by using the reductive perturbation method[8),9)]. Let us introduce the stretched coordinates

$$\xi = \epsilon(x - ct), \tag{2.3.19a}$$

$$\tau = \epsilon^2 t, \tag{2.3.19b}$$

and expand $\partial_x u$ as

$$\partial_x u = \epsilon v + \epsilon^2 w + \cdots . \tag{2.3.20}$$

Substituting eqs.(2.3.19) and (2.3.20) into eq.(2.3.14) and collecting terms of order ϵ^3, we obtain

$$\partial_\tau v - \frac{\delta c}{2}\partial_\xi^2 \mathcal{H}(v) - \frac{9c}{2} v \partial_\xi v = 0. \tag{2.3.21}$$

This equation is equivalent to the B-O equation which describes internal waves in stratified fluids of great depth[10),11)]. A soliton solution of eq.(2.3.21) is

$$v = \frac{A}{1 + \dfrac{(\xi - \lambda\tau)^2}{\Delta^2}}, \tag{2.3.22a}$$

$$A = -\frac{4\delta}{9|\Delta|}, \tag{2.3.22b}$$

$$\lambda = -\frac{c\delta}{2|\Delta|}, \tag{2.3.22c}$$

which has the Lorentzian profile vanishing algebraically as $|x| \to \infty$.

Case $n = 3$. Substituting eq.(2.3.1) into eqs.(2.3.9) and (2.3.12), we have from eq.(2.2.6)

$$\partial_t^2 u = c^2[\partial_x^2 u - \delta \partial_x^5 \mathcal{T}(u) - 12(\partial_x u)(\partial_x^2 u)], \tag{2.3.23}$$

where $\mathcal{T}$ is the integral transform operator defined by

$$\mathcal{T}[f(x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{sgn}(x' - x)\left(\log\left|\frac{x' - x}{a}\right| + \gamma\right) f(x')dx', \tag{2.3.24}$$

and γ is Euler's constant. We have also used the identity

$$\mathcal{T}(e^{ikx}) = \frac{\log|ak|}{ik} e^{ikx}. \tag{2.3.25}$$

At present, analytic solutions of eq.(2.3.23) have not been found.

Case $n \geqq 4$. Substituting eq.(2.3.1) into eqs.(2.3.9) and (2.3.13), we get from eq.(2.2.6)

$$\partial_t^2 u = c^2[\partial_x^2 u + \delta \partial_x^4 u - 3(n+1)(\partial_x u)(\partial_x^2 u)], \tag{2.3.26}$$

which is essentially the same as a long-wave equation of nearest-neighbor system (2.2.8), namely the Zabusky equation. A solitary wave solution of eq.(2.3.26) is expressed as

$$u = -\frac{4\delta}{(n+1)\Delta}\tanh\left(\frac{x-\lambda t}{\Delta}\right), \tag{2.3.27a}$$

$$\lambda^2 = c^2\left(1+\frac{4\delta}{\Delta^2}\right), \tag{2.3.27b}$$

where Δ is an arbitrary constant. Unlike the case $n = 2$, this solution describes a compressed wave with supersonic speed. If instead of eq.(2.3.19) we introduce the sttetched coordinates

$$\xi = \epsilon^{\frac{1}{2}}(x-ct), \tag{2.3.28a}$$

$$\tau = \epsilon^{\frac{3}{2}}t, \tag{2.3.28b}$$

then we can reduce eq.(2.3.26) to the K-dV equation. It follows that

$$\partial_\tau v + \frac{\delta c}{2}\partial_\xi^3 v - \frac{3(n+1)c}{2}v\partial_\xi v = 0. \tag{2.3.29}$$

A soliton solution of eq.(2.3.29) is

$$v = A\mathrm{sech}^2\left(\frac{\xi-\lambda\tau}{\Delta}\right), \tag{2.3.30a}$$

$$A = -\frac{4\delta}{(n+1)\Delta^2}, \tag{2.3.30b}$$

$$\lambda = \frac{2\delta c}{\Delta^2}, \tag{2.3.30c}$$

where Δ is an arbitrary constant.

We note here that the total compression by a K-dV soliton takes various values depending on the amplitude of the soliton but the total expansion by a B-O soliton is determined only by δ which depends on the lattice constant a and the potential parameter n.

2.4 N-perodic wave and N-soliton solutions of the Benjamin-Ono equation

In the preceding section, we have derived the B-O equation for the (4,2) L-J lattice. For the equation, Benjamin[10)] and later Ono[11] presented a periodic wave solution and a one-soliton solution. Recently, Joseph found an exact solution which describes a collision of two solitons[12)]. Motivated by the work of Joseph, Chen et al[13)] and, independently Case[14)], obtained a solution describing a multiple collision of N solitons (N-soliton solution), applying a pole expansion method. Matsuno obtained the N-soliton solution in a matrix form applying Hirota's method[15)]. He also discussed the initial value problem[16),17)]. The inverse scattering method for the B-O equation was developed by Kodama et al.[18),19)].

Ablowitz and Satsuma studied a relationship between soliton and algebraic solutions of a certain class of nonlinear wave equations and developed a method to get algebraic solutions by taking a long wave limit on soliton solutions[20]. As for the B-O equation, Benjamin has already suggested that the Lorentzian pulse (algebraic one-soliton solution) is obtained as a long wave limit of a periodic wave solution. Thus it is likely that the equation admits a series of periodic wave solution, each of which has the corresponding algebraic soliton solutions as the limit.

In this section, we shall show that a solution describing a multiple collision of periodic waves with different periods (N-periodic wave solution) is obtained for the B-O equation by Hirota's method and that the solution is reduced to the algebraic N-soliton solution as the long wave limit. In subsection 2.4.1, we transform the equation into a bilinear form and discuss about one-periodic wave solution and algebraic one-soliton solution. In subsection 2.4.2, we study the case of two-periodic wave solution. Finally, in sebsection 2.4.3, we extend the results in 2.4.1 and 2.4.2 to a general N-periodic wave solution.

2.4.1 One-periodic wave solution

We rewrite eq.(2.3.21) as

$$\partial_t u + 2u\partial_x u + \partial_x^2 \mathcal{H}(u) = 0, \tag{2.4.1}$$

rescaling v, ξ and τ. We introduce a dependent variable transformation

$$u(x,t) = i\partial_x \log \frac{f'(x,t)}{f(x,t)}, \tag{2.4.2}$$

and assume that f (f') can be written as an infinite or finite product of $x - z_n$ ($x - z'_n$) for z_n (z'_n) in the upper (lower)-half complex plane (the zeroes of f, f' should not necessarily be simple). It is then easy to see that

$$\mathcal{H}\left(i\partial_x \log \frac{f'}{f}\right) = -\partial_x \log(ff'). \tag{2.4.3}$$

Substituting eq.(2.4.2) into eq.(2.4.1), using eq.(2.4.3) and integrating once with respect to x, we have a bilinear form of the B-O equation,

$$(iD_t - D_x^2)f' \cdot f = 0, \tag{2.4.4}$$

where we have taken the integration constant to be zero and D_t and D_x^2 are defined by eq.(1.1.5) (see for example ref. 21 as for the properties of these operators).

We can construct some special solutions of eq.(2.4.1) by applying a kind of perturbational technique on eq.(2.4.4)[21]. The simplest solution of eq.(2.4.4) is written as

$$f = 1 + \exp(i\xi_1 + \phi_1), \tag{2.4.5a}$$

$$f' = 1 + \exp(i\xi_1 - \phi_1), \tag{2.4.5b}$$

where

$$\xi_1 = k_1(x - c_1 t) + \xi_1^{(0)}, \tag{2.4.6}$$

$$c_1 = k_1 \coth \phi_1, \tag{2.4.7}$$

and k_1, ϕ_1 are real parameters and $\xi_1^{(0)}$ is an arbitrary phase constant. We see from eqs.(2.4.5) that for real $\xi_1^{(0)}$ the zeroes of f (f') are in the upper (lower)-half complex plane if $\dfrac{\phi_1}{k_1} > 0$. In this case, the assumption for deriving eq.(2.4.3) is satisfied and substitution of eq.(2.4.5) into eq.(2.4.2) yields

$$u = k_1 \frac{\tanh \phi_1}{1 + \mathrm{sech}\phi_1 \cos \xi_1}. \tag{2.4.8}$$

This solution is essentially the same as the periodic wave solution presented by Benjamin[10] and Ono[11]. As they have already mentioned, a Lorentzian pulse is obtained by taking a long wave limit on eq.(2.4.8): We consider $k_1 << 1$ and $c_1 = O(1)$. Then choosing $\xi_1^{(0)} = \pi$ and using

$$\tanh \phi_1 = \frac{k_1}{c_1}, \tag{2.4.9}$$

$$\mathrm{sech}\phi_1 = 1 - \frac{k_1^2}{2c_1^2} + O(k_1^4), \tag{2.4.10}$$

$$\cos \xi_1 = -\left[1 - \frac{k_1^2 \theta_1^2}{2} + O(k_1^4)\right], \tag{2.4.11}$$

we have from eq.(2.4.8)

$$u = \frac{\dfrac{2}{c_1}}{\theta_1^2 + \dfrac{1}{c_1^2}} + O(k_1^2), \tag{2.4.12}$$

where

$$\theta_1 = x - c_1 t, \tag{2.4.13}$$

(we may add an arbitrary phase constant to θ_1). Thus we recover the algebraic one-soliton solution

$$u = \frac{2c_1}{c_1^2 \theta_1^2 + 1}, \tag{2.4.14}$$

as the limit, $k_1 \to 0$, of eq.(2.4.8). Compared with the one-soliton solution of the K-dV equation, the periodic wave solution, eq.(2.4.8), has an additional arbitrary parameter and so it yields the algebraic solution with one parameter.

2.4.2 Two-periodic wave solution

A two-periodic wave solution is obtained by choosing

$$f = 1 + \exp(i\xi_1 + \phi_1) + \exp(i\xi_2 + \phi_2) + \exp(i\xi_1 + i\xi_2 + \phi_1 + \phi_2 + A_{12}), \tag{2.4.15a}$$

$$f' = 1 + \exp(i\xi_1 - \phi_1) + \exp(i\xi_2 - \phi_2) + \exp(i\xi_1 + i\xi_2 - \phi_1 - \phi_2 + A_{12}), \tag{2.4.15a}$$

where

$$\xi_j = k_j(x - c_j t) + \xi_j^{(0)}, \tag{2.4.16}$$

$$c_j = k_j \coth \phi_j, \tag{2.4.17}$$

and k_j, ϕ_j are real parameters satisfying

$$\frac{\phi_j}{k_j} > 0, \tag{2.4.18}$$

and $\xi_j^{(0)}$ are arbitrary phase constants. Substituting eq.(2.4.15) into eq.(2.4.4), we find the solution satisfies eq.(2.4.4) if

$$\exp A_{12} = \frac{(c_1 - c_2)^2 - (k_1 - k_2)^2}{(c_1 - c_2)^2 - (k_1 + k_2)^2}. \tag{2.4.19}$$

Generally, eq.(2.4.15) gives a complex u. But, if we choose the arbitrary phase constants adequately, we can get a real solution: Taking the imaginary part of $\xi_j^{(0)}$ equal to be $\frac{A_{12}}{2}$, we have

$$f = 1 + \exp\left(i\xi_1 + \phi_1 - \frac{A_{12}}{2}\right) + \exp\left(i\xi_2 + \phi_2 - \frac{A_{12}}{2}\right) + \exp(i\xi_1 + i\xi_2 + \phi_1 + \phi_2), \tag{2.4.20a}$$

$$f' = \exp(i\xi_1 + i\xi_2 - \phi_1 - \phi_2) \cdot f^*, \tag{2.4.20b}$$

where asterisk denotes complex conjugate. Substituting eq.(2.4.20) into eq.(2.4.2), we obtain

$$u = -(k_1 + k_2) + i\partial_x \log\frac{f^*}{f}, \tag{2.4.21}$$

which is a real solution.

We show that the solution, eq.(2.4.20), satisfies the assumption necessary for deriving eq.(2.4.3). For $k_1 >> k_2$ or $k_2 >> k_1$, we have from eq.(2.4.19) $\exp(A_{12}) \to 1$ and eq.(2.4.20a) may be written as

$$f = [1 + \exp(i\xi_1 + \phi_1)][1 + \exp(i\xi_2 + \phi_2)], \tag{2.4.22}$$

which has zeroes only in the upper-half plane under the condition (2.4.18). Then, for arbitrary k_1 and k_2, the zeroes always remain in the upper-half plane unless they cross the real axis, i.e., eq.(2.4.20) vanish for real ξ_1 and ξ_2. We now prove that eq.(2.4.20) do not become zero under the condition

$$(c_1 - c_2)^2 > (|k_1| + |k_2|)^2. \tag{2.4.23}$$

If f would vanish for real ξ_1 and ξ_2, we have from eq.(2.4.20a)

$$\begin{aligned} &\left[\cos\frac{\xi_1}{2}\cosh\frac{\phi_1+\phi_2}{2} + \exp\left(-\frac{A_{12}}{2}\right)\cos\frac{\xi_1}{2}\cosh\frac{\phi_1-\phi_2}{2}\right]\cos\frac{\xi_2}{2} \\ &+ \left[-\sin\frac{\xi_1}{2}\cosh\frac{\phi_1+\phi_2}{2} + \exp\left(-\frac{A_{12}}{2}\right)\sin\frac{\xi_1}{2}\cosh\frac{\phi_1-\phi_2}{2}\right]\sin\frac{\xi_2}{2} = 0, \end{aligned} \tag{2.4.24a}$$

$$\begin{aligned} &\left[\sin\frac{\xi_1}{2}\sinh\frac{\phi_1+\phi_2}{2} + \exp\left(-\frac{A_{12}}{2}\right)\sin\frac{\xi_1}{2}\sinh\frac{\phi_1-\phi_2}{2}\right]\cos\frac{\xi_2}{2} \\ &+ \left[\cos\frac{\xi_1}{2}\sinh\frac{\phi_1+\phi_2}{2} - \exp\left(-\frac{A_{12}}{2}\right)\cos\frac{\xi_1}{2}\sinh\frac{\phi_1-\phi_2}{2}\right]\sin\frac{\xi_2}{2} = 0. \end{aligned} \tag{2.4.24b}$$

The determinant of the coefficients of eq.(2.4.24) is given by

$$\begin{aligned} &\frac{1}{2}[\sinh(\phi_1+\phi_2) - \exp(-A_{12})\sinh(\phi_1-\phi_2) + \exp\left(-\frac{A_{12}}{2}\right)\sinh\phi_2\cos\xi_1] \\ &= \frac{1}{2}[\sinh\phi_1\cosh\phi_2\{1 - \exp(-A_{12})\} \\ &+ \sinh\phi_2\{\cosh\phi_1 + \exp(-A_{12})\cosh\phi_1 + \exp\left(-\frac{A_{12}}{2}\right)\cos\xi_1\}], \end{aligned}$$

which does not vanish supposing $\phi_1\phi_2 A_{12} > 0$. Then f has no zero on the real axis. This condition is consistent with eq.(2.4.18) if eq.(2.4.23) holds. The similar argument is possible f'. Hence we have verified that the solution (2.4.20) satisfies eq.(2.4.3) under the conditions (2.4.18) and (2.4.23). This result also guarantees that u does not become singular. Substituting eq.(2.4.20) into eq.(2.4.2), we have the explicit form of two-periodic wave solution,

$$u = \frac{U_1}{U_2}, \tag{2.4.25a}$$

where

$$\begin{aligned} U_1 = &\exp\left(\frac{A_{12}}{2}\right)(k_1+k_2)\sinh(\phi_1+\phi_2) \\ &+\exp\left(-\frac{A_{12}}{2}\right)(k_1-k_2)\sinh(\phi_1-\phi_2) \\ &+2(k_1\sinh\phi_1\cos\xi_2 + k_2\sinh\phi_2\cos\xi_1), \end{aligned} \tag{2.4.25b}$$

$$\begin{aligned} U_2 = &\exp\left(\frac{A_{12}}{2}\right)[\cosh(\phi_1+\phi_2)+\cos(\xi_1+\xi_2)] \\ &+\exp\left(-\frac{A_{12}}{2}\right)[\cosh(\phi_1-\phi_2)+\cos(\xi_1-\xi_2)] \\ &+2(\cosh\phi_2\cos\xi_1+\cosh\phi_1\cos\xi_2). \end{aligned} \tag{2.4.25c}$$

In principle we can get an algebraic solution as a long wave limit of eq.2.4.25). It is easier, however, to study the f and f' themselves. Choosing the real part of phase constants in eq.(2.4.20a) equal to be π, we have

$$\begin{aligned} f = &1 - \exp\left(ik_1\theta_1+\phi_1-\frac{A_{12}}{2}\right) - \exp\left(ik_2\theta_2+\phi_2-\frac{A_{12}}{2}\right) \\ &+\exp(ik_1\theta_1+ik_2\theta_2+\phi_1+\phi_2), \end{aligned} \tag{2.4.26}$$

where

$$\theta_j = x - c_j t. \tag{2.4.27}$$

We consider $k_1,\ k_2 << 1$ and $c_1,\ c_2 = O(1)$ with $\dfrac{k_1}{k_2} = O(1)$. Then using

$$\exp(ik_j\theta_j+\phi_j) = 1 + k_j\left(i\theta_j+\frac{1}{c_j}\right) + O(k_j^2), \tag{2.4.28}$$

$$\exp(A_{ij}) = 1 + \frac{4k_ik_j}{(c_i-c_j)^2} + O(k^4), \tag{2.4.29}$$

we find

$$f = k_1k_2\left[\left(i\theta_1+\frac{1}{c_1}\right)\left(i\theta_2+\frac{1}{c_2}\right)+\frac{4}{(c_1-c_2)^2}+O(k)\right]. \tag{2.4.30}$$

Similarly we have from eq.(2.4.20b)

$$f' = k_1 k_2 \left[\left(i\theta_1 - \frac{1}{c_1} \right) \left(i\theta_2 - \frac{1}{c_2} \right) + \frac{4}{(c_1 - c_2)^2} + O(k) \right]. \tag{2.4.31}$$

Thus, in the limit of $k_1,\ k_2 \to 0$, we obtain

$$\begin{aligned} u &= i\partial_x \log \frac{\left(i\theta_1 - \dfrac{1}{c_1} \right) \left(i\theta_2 - \dfrac{1}{c_2} \right) + \dfrac{4}{(c_1 - c_2)^2}}{\left(i\theta_1 + \dfrac{1}{c_1} \right) \left(i\theta_2 + \dfrac{1}{c_2} \right) + \dfrac{4}{(c_1 - c_2)^2}} \\ &= \frac{2c_1 c_2 \left[c_1\theta_1^2 + c_2\theta_2^2 + \dfrac{(c_1 + c_2)^3}{c_1 c_2 (c_1 - c_2)^2} \right]}{\left[c_1 c_2 \theta_1 \theta_2 - \dfrac{(c_1 + c_2)^2}{(c_1 - c_2)^2} \right]^2 + (c_1\theta_1 + c_2\theta_2)^2}. \end{aligned} \tag{2.4.32}$$

This solution describes a collision of two algebraic solitons.

2.4.3 N-periodic wave solution

The form of two-periodic wave solution suggests that of N-periodic wave solution. We can prove by mathematical induction that the following solution satisfies eq.(2.4.4) (see Appendix 2.2);

$$f = \sum_{\mu=0,1} \exp \left[\sum_{j=1}^{N} \mu_j (i\xi_j + \phi_j) + \sum_{i<j}^{(N)} \mu_i \mu_j A_{ij} \right], \tag{2.4.33a}$$

$$f' = \sum_{\mu=0,1} \exp \left[\sum_{j=1}^{N} \mu_j (i\xi_j - \phi_j) + \sum_{i<j}^{(N)} \mu_i \mu_j A_{ij} \right], \tag{2.4.33b}$$

where

$$\xi_j = k_j (x - c_j t) + \xi_j^{(0)}, \tag{2.4.34}$$

$$c_j = k_j \coth \phi_j, \tag{2.4.35}$$

$$\frac{\phi_j}{k_j} > 0, \tag{2.4.36}$$

$$\exp A_{ij} = \frac{(c_i - c_j)^2 - (k_i - k_j)^2}{(c_i - c_j)^2 - (k_i + k_j)^2}, \tag{2.4.37}$$

and the notation $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1 = 0, 1,\ \mu_2 = 0, 1, \cdots, \mu_N = 0, 1$ and $\sum_{i<j}^{(N)}$ the summation over all possible combinations of the N elements with the specific condition $i < j$.

In order to get a real u, we should choose the phase constants in eq.(2.4.33) correctly. we have from eq.(2.4.33b)

$$f' = \exp\left[\sum_{j=1}^{N}(i\xi_j - \phi_j) + \sum_{i<j}^{(N)} A_{ij}\right]$$
$$\times \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N}(\mu_j - 1)(i\xi_j - \phi_j) + \sum_{i<j}^{(N)}(\mu_i\mu_j - 1)A_{ij}\right]$$
$$= \exp\left[\sum_{j=1}^{N}(i\xi_j - \phi_j) + \sum_{i<j}^{(N)} A_{ij}\right] \tag{2.4.38}$$
$$\times \sum_{\nu=0,1} \exp\left[\sum_{j=1}^{N}\nu_j\left(-i\xi_j + \phi_j - \sum_{i\neq j}^{N} A_{ij}\right) + \sum_{i<j}^{(N)}\nu_i\nu_j A_{ij}\right].$$

Thus, if we choose the imaginary part of $\xi_j^{(0)}$ as $\sum_{i\neq j}^{N}\frac{A_{ij}}{2}$ for $j = 1, 2, \cdots, N$, we obtain

$$f' = \exp\left[\sum_{j=1}^{N}(i\xi_j - \phi_j) + \sum_{i<j}^{(N)} A_{ij}\right] \cdot f^*. \tag{2.4.39}$$

In the same manner as the two-periodic wave solution, we can show that, in the case of $N = 3$, the zeroes of $f(f')$ remain in the upper (lower)-half plane for the choice of the imaginary part of phase constants and under the conditions, $\frac{\phi_j}{k_j} > 0$ and $(c_i - c_j)^2 > (|k_i| + |k_j|)^2$ for arbitrary i, j. It may be possible to show the above result for $N = 4, 5, \cdots$, though we have not gotten the regorous proof. However, we can say that at least for $A_{ij} \approx 0$ (i, j are arbitrary) the zeroes of f and f' do not cross the real axis. Then eq.(2.4.33) satisfies the assumption necessary to derive eq.(2.4.3) and we have from eq.(2.4.2) a real and non-singular solution,

$$u = -\sum_{j=1}^{N} k_j + i\partial_x \log\frac{f^*}{f}. \tag{2.4.40}$$

We now show that the N-periodic wave solution is reduced to an algebraic N-soliton solution in a long wave limit. Choosing the phase constants as

$$\xi_j^{(0)} = \pi + i\sum_{i\neq j}^{N}\frac{A_{ij}}{2}, \tag{2.4.41}$$

for $j = 1, 2, \cdots, N$, eq.(2.4.33) is written by

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N}\mu_j(ik_j\theta_j + i\pi + \phi_j) + \sum_{i<j}^{(N)}\left(\mu_i\mu_j - \frac{\mu_i + \mu_j}{2}\right)A_{ij}\right], \tag{2.4.42}$$

where $theta_j$ is defined by eq.(2.4.27). If we take $k_m = 0$ in eq.(2.4.42), then $\phi_m = A_{mj} = 0$ and f vanishes, which indicates that f is factorized by $\prod_{j=1}^{N} k_j$. Therefore, if we expand f in terms of k_j, the leading terms of eq.(2.4.42) are in the order of $\prod_{j=1}^{N} k_j$. We consider $k_j \to 0$ with the same asymptotic order and $c_j = O(1)$ for $j = 1, 2, \cdots, N$. Then, using eqs(2.4.28) and (2.4.29), we obtain

$$f \approx \sum_{\mu=0,1} \prod_{j=1}^{N} (-1)^{\mu_j} \left[1 + \mu_j k_j \left(i\theta_j + \frac{1}{c_j} \right) \right] \prod_{i<j}^{(N)} \left[1 + \left(\mu_i \mu_j - \frac{\mu_i + \mu_j}{2} \right) k_i k_j B_{ij} \right], \tag{2.4.43}$$

where

$$B_{ij} = \frac{4}{(c_i - c_j)^2}. \tag{2.4.44}$$

The leading terms of eq.(2.4.43) are given by those in the order of $\prod_{j=1}^{N} k_j$ of

$$\prod_{j=1}^{N} \left[1 + k_j \left(i\theta_j + \frac{1}{c_j} \right) \right] \prod_{i<j}^{(N)} (1 + k_i k_j B_{ij}).$$

Thus as the limit $k_j \to 0$ of the N-periodic wave solution, we obtain an algebraic solution in the following from;

$$u = i\partial_x \log \frac{f^*}{f}, \tag{2.4.45}$$

where

$$\begin{aligned} f = & \prod_{j=1}^{N} \left(i\theta_j + \frac{1}{c_j} \right) + \frac{1}{2} \sum_{i,j}^{N} B_{ij} \prod_{l \neq, j}^{N} \left(i\theta_l + \frac{1}{c_l} \right) + \cdots \\ & + \frac{1}{M! 2^M} \sum_{i,j,\cdots,m,n}^{(N)} \underbrace{B_{ij} B_{kl} \cdots B_{mn}}_{M} \prod_{p \neq i,j,\cdots,m,n}^{N} \left(i\theta_p + \frac{1}{c_p} \right) + \cdots . \end{aligned} \tag{2.4.46}$$

The notation $\sum_{i,j,\cdots,m,n}^{(N)}$ means the summation over all possible combinations of $i, j, \cdots, m, n$ which are taken from $1, 2, \cdots, N$ and all different. For $N = 1, 2, 3$, eq.(2.4.46) is written as

$$f|_{N=1} = i\theta_1 + \frac{1}{c_1}, \tag{2.4.47a}$$

$$f|_{N=2} = \left(i\theta_1 + \frac{1}{c_1} \right) \left(i\theta_2 + \frac{1}{c_2} \right), \tag{2.4.47b}$$

$$\begin{aligned} f|_{N=3} = & \left(i\theta_1 + \tfrac{1}{c_1} \right) \left(i\theta_2 + \tfrac{1}{c_2} \right) \left(i\theta_3 + \tfrac{1}{c_3} \right) \\ & + B_{12} \left(i\theta_3 + \frac{1}{c_3} \right) + B_{23} \left(i\theta_1 + \frac{1}{c_1} \right) + B_{31} \left(i\theta_2 + \frac{1}{c_2} \right). \end{aligned} \tag{2.4.47c}$$

We may express eq.(2.4.46) in a determinant form,

$$f = \begin{vmatrix} i\theta_1 + \dfrac{1}{c_1} & \sqrt{B_{12}} & \cdots & \sqrt{B_{1N}} \\ -\sqrt{B_{12}} & i\theta_1 + \dfrac{1}{c_1} & \cdots & \sqrt{B_{2N}} \\ \vdots & \vdots & & \vdots \\ -\sqrt{B_{1N}} & -\sqrt{B_{2N}} & \cdots & i\theta_N + \dfrac{1}{c_N} \end{vmatrix}, \tag{2.4.48}$$

which is essentially the same as the algebraic N-soliton solution presented by Matsuno[15)].

Finally, we study the asymptotic behavior of algebraic N-soliton solution. Without loss of generality, we may assume in the limit $t \to \infty$,

$$\begin{aligned} &\theta_1,\ \theta_2, \cdots,\ \theta_{M-1} = \infty, \\ &\theta_M = \text{finite}, \\ &\theta_{M+1},\ \theta_{M+2}, \cdots,\ \theta_N = -\infty. \end{aligned}$$

Then it is easily seen from from eqs.(2.4.45) and (2.4.46) that u has the following asymptotic form,

$$u = \frac{2c_M}{c_M^2\theta_M^2 + 1}, \tag{2.4.49}$$

which is an algebraic solution with phase θ_M. Similarly we obtain the same asymptotic form of u for $t \to -\infty$. Hence we find that the B-O solitons have no phase shift after the collisions of them unlike those which take place between K-dV solitons.

2.5 Concluding remarks

In this chapter, we have investigated nonlinear wave propagations in the one-dimensional lattice with the $(2n,\ n)$ L-J potential. Introducing the approximations of small amplitude and long wavelength, we have obtained eq.(2.3.14) or the B-O equation for $n = 2$, eq.(2.3.23) for $n = 3$ and the Zabusky equation or the K-dV equation for $n \geqq 4$. The results show that the value of the force-range parameter n contributes not to the nonlinear terms but to the dispersion terms of the equations. It is well known that both the B-O and K-dV equations have soliton solutions formed by balancing of the nonlinearity and dispersion effects of the systems. The reason why the B-O soliton is algebraic is that the B-O equation is more dispersive than the K-dV equation. It is interesting to study whether eq.(2.3.23) having an intermediate dispersion term between the B-O and the K-dV equations gives soliton solutions or not, though the problem is still open.

Appendix 2.1 Formulas of Fourier series

We give some formulas of the Fourier series which are used in the text.

We define the functions $A_n(ka)$ and $B_n(ka)$ as

$$A_n(ka) = \sum_{l=1}^{\infty} \frac{1}{l^n}[1 - \cos(lka)], \tag{2.A.1}$$

$$B_n(ka) = \sum_{l=1}^{\infty} \frac{1}{l^n} \sin(lka), \tag{2.A.2}$$

where $n \geqq 2$. Then, for $|ka| \leqq \pi$, we have the recurrence formulas for them:

$$A_2(ka) = \frac{\pi}{2}|ka| - \frac{1}{4}(ka)^2, \tag{2.A.3}$$

$$\begin{aligned} A_3(ka) &= -\frac{1}{2}\log 2 \cdot (ka)^2 + \int_0^{|ka|} (t_1 - |ka|) \log\left(\sin\frac{t_1}{2}\right) dt_1 \\ &= -\frac{1}{2}(ka)^2 \log|ka| + \frac{3}{4}(ka)^2 + \frac{1}{288}(ka)^4 + \cdots, \end{aligned} \tag{2.A.4}$$

$$A_{n+2}(ka) = \frac{1}{2}\zeta(n)(ka)^2 - \int_0^{ka}\int_0^{t_2} A_n(t_1)dt_1dt_2, \tag{2.A.5}$$

$$B_n(ka) = A'_{n+1}(ka). \tag{2.A.6}$$

From these equations we find that if $|ka| << 1$ we obtain

$$A_4 = \frac{1}{2}\zeta(2)(ka)^2 - \frac{\pi}{12}|ka|^3 + O[(ka)^4], \tag{2.A.7}$$

$$A_5 = \frac{1}{2}\zeta(3)(ka)^2 + \frac{1}{24}(ka)^4 \log|ka| + O[(ka)^4], \tag{2.A.8}$$

$$A_6 = \frac{1}{2}\zeta(4)(ka)^2 - \frac{1}{24}\zeta(2)(ka)^4 + O[(ka)^5], \tag{2.A.9}$$

$$A_7 = \frac{1}{2}\zeta(5)(ka)^2 - \frac{1}{24}\zeta(3)(ka)^4 + O[(ka)^6 \log|ka|], \tag{2.A.10}$$

$$B_5 = \zeta(4)(ka) - \frac{1}{6}\zeta(2)(ka)^3 + O[(ka)^4], \tag{2.A.11}$$

$$B_6 = \zeta(5)(ka) - \frac{1}{6}\zeta(3)(ka)^3 + O[(ka)^5 \log|ka|], \tag{2.A.12}$$

and for $n \geqq 8$

$$A_n = \frac{1}{2}\zeta(n-2)(ka)^2 - \frac{1}{24}\zeta(n-4)(ka)^4 + O[(ka)^6], \tag{2.A.13}$$

$$B_n = \zeta(n-2)(ka) - \frac{1}{6}\zeta(n-4)(ka)^3 + O[(ka)^5]. \tag{2.A.14}$$

Substitution of these equations into eqs.(2.3.3), (2.3.6) and (2.3.7) gives eqs.(2.3.9-13).

Appendix 2.2 N-periodic wave solution of the B-O equation

We show that eq.(2.4.33) satisfies eq.(2.4.4). Substituting eq.(2.4.33) into eq.(2.4.4), we have

$$\begin{aligned} &\sum_{\mu=0,1}\sum_{\nu=0,1}\left[\sum_{j=1}^{N}(\mu_j - \nu_j)k_jc_j + \left\{\sum_{j=1}^{N}(\mu_j - \nu_j)k_j\right\}^2\right] \\ &\times \exp\left[\sum_{j=1}^{N} i(\mu_j + \nu_j)\xi_j - \sum_{j=1}^{N}(\mu_j - \nu_j)\phi_j + \sum_{i<j}^{(N)}(\mu_i\mu_j + \nu_i\nu_j)A_{ij}\right] = 0. \end{aligned} \tag{2.B.1}$$

Let the coefficients of the terms $\exp\left(\sum_{j=1}^{n} i\xi_j + \sum_{j=n+1}^{m} 2i\xi_j\right)$ in the left hand side of eq.(2.B.1) be $F(1,2,\cdots,n;n+1,n+2,\cdots,m)$. F may be expressed as

$$F = \sum_{\mu=0,1}\sum_{\nu=0,1} \text{cond.}(\mu,\nu)\left[\sum_{j=1}^{N}(\mu_j-\nu_j)k_jc_j + \left\{\sum_{j=1}^{N}(\mu_j-\nu_j)k_j\right\}^2\right]$$
$$\times\exp\left[-\sum_{j=1}^{N}(\mu_j-\nu_j)\phi_j + \sum_{i<j}^{(N)}(\mu_i\mu_j+\nu_i\nu_j)A_{ij}\right], \tag{2.B.2}$$

where cond.(μ,ν) implies the summation over μ and ν are performed under the conditions

$$\begin{aligned}
&\mu_j+\nu_j=1 \quad \text{for } j=1,2,\cdots,n,\\
&\mu_j=\nu_j=1 \quad \text{for } j=n+1,n+2,\cdots,m,\\
&\mu_j=\nu_j=0 \quad \text{for } j=m+1,m+2,\cdots,N.
\end{aligned}$$

Introducing notations $\sigma_j=\mu_j-\nu_j$ for $j=1,2,\cdots,n$ and using

$$\exp\left[(1+\sigma_i\sigma_j)\frac{A_{ij}}{2}\right] = \frac{(c_i-c_j)^2-(\sigma_ik_i-\sigma_jk_j)^2}{(c_i-c_j)^2-(k_i+k_j)^2}, \tag{2.B.3}$$

$$\exp[(1-\sigma_j)\phi_j] = \frac{c_j-\sigma_jk_j}{c_j-k_j}, \tag{2.B.4}$$

eq.(2.B.2) is reduced to

$$F = \text{const.}\widehat{F}(\sigma_1k_1,\sigma_2k_2,\cdots,\sigma_nk_n;c_1,c_2,\cdots,c_n), \tag{2.B.5}$$

where

$$\widehat{F} = \sum_{\sigma=\pm1}\left[\sum_{j=1}^{N}c_j\sigma_jk_j + \left(\sum_{j=1}^{n}\sigma_jk_j\right)^2\right]\prod_{j=1}^{n}(c_j-\sigma_jk_j)\prod_{i<j}^{(n)}[(c_i-c_j)^2-(\sigma_ik_i-\sigma_jk_j)^2], \tag{2.B.6}$$

and the constant does not depend on σ_j. Thus if

$$\widehat{F}(\sigma_1k_1,\sigma_2k_2,\cdots,\sigma_nk_n;c_1,c_2,\cdots,c_n)=0, \tag{2.B.7}$$

holds for $n=1,2,\cdots,N$, eq.(2.4.33) satisfies eq.(2.4.4).

Equation (2.B.7) can be proved by mathematical induction. It is easily verified that eq.(2.B.7) holds for $n = 1, 2$. $\widehat{F}$ has the following properties;

A) $\widehat{F}(\sigma_1 k_1, \sigma_2 k_2, \cdots, \sigma_n k_n; c_1, c_2, \cdots, c_n)|_{k_1=0}$

$$= c_1 \prod_{j=2}^{n} [(c_1 - c_j)^2 - k_j^2] \widehat{F}(\sigma_2 k_2, \sigma_3 k_3, \cdots, \sigma_n k_n; c_2, c_3, \cdots, c_n).$$

B) $\widehat{F}(\sigma_1 k_1, \sigma_2 k_2, \cdots, \sigma_n k_n; c_1, c_2, \cdots, c_n)|_{k_1=k_2, c_1=c_2}$

$$= -8(c_1^2 + k_1^2) k_1^2 \prod_{j=3}^{n} [(c_1 - c_j)^2 - (k_1 - k_j)^2][(c_1 - c_j)^2 - (k_1 + k_j)^2]$$

$$\times \widehat{F}(\sigma_3 k_3, \sigma_4 k_4, \cdots, \sigma_n k_n; c_3, c_4, \cdots, c_n).$$

C) $\widehat{F}$ is unchanged by the replacement k_i and c_i with k_j and c_j for arbitrary i, j.

D) $\widehat{F}$ is an even function of $k_1, k_2, \cdots, k_n$.

The properties A), C), and D) imply $\widehat{F}$ is factorized by $\prod_{j=1}^{n} k_j^2$, and B), C), and D) show that $\widehat{F}$ is written as

$$\widehat{F} = \prod_{i<j}^{(n)} (c_i - c_j)^2 G_1 + {\prod_{i<j}^{(n)}}' (c_i - c_j)^2 (c_1 - c_2)(k_1^2 - k_2^2) G_2 + \cdots + \prod_{i<j}^{(n)} (k_i^2 - k_j^2)^2 G_M,$$

where $G_1, G_2, \cdots, G_M$ are certain polynomials of $k_1^2, k_2^2, \cdots, k_n^2, c_1, c_2, \cdots, c_n$ and the prime attached to $\prod_{i<j}^{(n)}$ denotes product over all i, j except $i = 1$ and $j = 2$. The above argument shows that the degree of $\widehat{F}$ with respect to $k_1, k_2, \cdots, k_n, c_1, c_2, \cdots, c_n$ is at least $n^2 + n$, if $\widehat{F}$ would not be identically zero. On the other hand, eq.(2.B.6) implies $\widehat{F}$ is at most of degree $n^2 + 2$. Therefore $\widehat{F}$ must be zero for $n > 2$ and eq.(2.B.7) is proved.

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CHAPTER III

KINK DYNAMICS IN THE DISCRETE SINE-GORDON SYSTEM

A PERTURBATIONAL APPROACH

3.1 Introduction

Recently there has been growing interest in condensed matter systems capable of supporting "soliton" excitations. Among them is the Sine-Gordon (S-G) chain which models dislocations[1)], twin boundaries[2),3)], charge-density waves[4)], superionic conductors[5)] and so on[6)]. Most of the previous studies on the S-G system are concerned with its continuum limit, that is, the S-G equation, (3.2.10).

In physical applications, however, there occurs a minimum distance scale, e.g. a lattice constant and discreteness of the system gives rise to the Peierls barrier[2),3)] which prohibits a dislocation (a kink) from moving freely without any external stresses. Similar effect is known in crystal growth for which a finite potential (free energy) gap is necessary between a crystal and a gas phase[7)]. The discreteness plays an important role also in dynamical problems such as kink propagation in a discrete media. Numerical simulation by Currie et al[8)] shows that an initial travelling kink (the time $t = 0$ configuration of the exact one-kink solution of the S-G equation), in the course of time, changes its shape a little by shrinking and radiates phonons resulting in spontaneous damping of kink motion. All of these phenomena are lost under the continuum approximation.

So far as we know there has been no systematic studies on discreteness effects in the S-G chain except for static ones to calculate the Peierls force[2),3)]. As a suitable first step we apply a perturbational formalism due to McLaughlin and Scott[9)] (M-S) to investigation of discreteness effects. The smallness parameter is the ratio h of the lattice constant to the kink width.

As the zeroth order approximation the formalism gives equation of motion for the center of a kink. A kink is shown to propagate wobbling or to be pinned in the Peierls field. The first order approximation consists of dressing of the "bare" kink and radiation.

In section 3.2, we present our system, a discrete S-G chain and rewrite equation of motion in such a form suitable for the application of M-S formalism[9)]. In section 3.3, we discuss the zeroth order approximation and the first order one is treated in section 3.4. Section 3.5 containes summary of this chapter. In Appendix 3.1, we summarize M-S formalism in order to achieve more transparency in sections 3.2-4.

3.2 Discrete Sine-Gordon system

We consider the S-G system which is described by the following Hamiltonian

$$H = \frac{m}{2}\sum_n \dot{x}_n^2 + \frac{k}{2}\sum_n (x_{n+1} - x_n)^2 + A_0 \sum_n \left(1 - \cos\frac{2\pi x_n}{a}\right), \tag{3.2.1}$$

where x_n ($\dot{x}_n$) is the displacement (velocity) of the nth particle with mass m, k the elastic constanst and a the lattice constant. $2A_0$ denotes the height of the substrate potential. Using characteristic velocity c_0 and frequency ω_0 defined by $c_0^2 = \dfrac{ka^2}{m}$ and $\omega_0^2 = \dfrac{(2\pi)^2 A_0}{ma^2}$, we define a dimensionless parameter

$$h = \frac{a\omega_0}{c_0}, \tag{3.2.2}$$

which gives a measure of discreteness because the width of a kink is of order $\dfrac{c_0}{\omega_0}$[6)]. Putting $\phi(nh) = \dfrac{2\pi x_n}{a}$, $\widetilde{t} = \omega_0 t$, $\widetilde{x} = \dfrac{xh}{a}$ and $\widetilde{H} = \dfrac{hH}{A_0}$ (hereafter we omit the tilder on $x,\ t,\ H$) and using the Poisson sum formula

$$\sum_{n=-\infty}^{\infty} f(nh)h = \int_{-\infty}^{\infty} dx f(x) \left(1 + 2\sum_{s=1}^{\infty} \cos\frac{2\pi sx}{h}\right), \tag{3.2.3}$$

we rewrite eq.(3.2.1) as

$$H = H_0 + \epsilon H_1, \tag{3.2.4}$$

with

$$H_0 = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + (1-\cos\phi)\right], \tag{3.2.5a}$$

$$\begin{aligned}\epsilon H_1 = &\int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + (1-\cos\phi)\right]\left(2\sum_{s=1}^{\infty}\cos\frac{2\pi sx}{h}\right)\\ &+\int_{-\infty}^{\infty} dx \left[\frac{1}{2h^2}\sum_{m,n}{}^{*}\frac{h^{m+1}}{m!n!}(\partial_x^m\phi)(\partial_x^n\phi)\right]\left(1+2\sum_{s=1}^{\infty}\cos\frac{2\pi sx}{h}\right),\end{aligned} \tag{3.2.5b}$$

where the asterisk on $\sum$ means that we omit the term $(m,\ n) = (1,1)$ in the summation. Hamiltonian ϵH_1, which vanishes in the continuum limit, $h \to 0$, represents effects of discreteness. Equation of motion for $\phi(x,t)$ is obtained from the Lagrangian density as Euler's equation. It follows that (see Appendix 3.2)

$$\partial_t^2\phi - \partial_x^2\phi + \sin\phi = \epsilon f(\phi), \tag{3.2.6}$$

$$\begin{aligned}\epsilon f(\phi) = &-(\partial_t^2\phi - \partial_x^2\phi + \sin\phi)\left(2\sum_{s=1}^{\infty}\cos\frac{2\pi sx}{h}\right)\\ &+\frac{2}{h^2}\sum_{n=2}^{\infty}\frac{h^{2n}}{(2n)!}(\partial_x^{2n}\phi)\left(1+2\sum_{s=1}^{\infty}\cos\frac{2\pi sx}{h}\right).\end{aligned} \tag{3.2.7}$$

It is readily seen from the identity

$$\sum_{n=-\infty}^{\infty}\delta(x-nh) = \frac{1}{h}\left(1+2\sum_{s=1}^{\infty}\cos\frac{2\pi sx}{h}\right), \tag{3.2.8}$$

that eqs.(3.2.6) and (3.2.7) can be transformed to

$$\sum_{n=-\infty}^{\infty}\delta(x-nh)\left[\partial_t^2\phi - \frac{\phi(x+h,t)+\phi(x-h,t)-2\phi(x,t)}{h^2} + \sin\phi\right] = 0, \tag{3.2.9}$$

which is equivalent to the discrete equation of motion derivable from Hamiltonian (3.2.1) and shows that the lattice points exist at $x = nh$.

In the continuum limit, Hamiltonian (3.2.4) reduces to eq.(3.2.5a) which generates the S-G equation given by

$$\partial_t^2\phi_0 - \partial_x^2\phi_0 + \sin\phi_0 = 0. \tag{3.2.10}$$

The one-kink solution of this equation[6)] is written as

$$\phi_0 = 4\tan^{-1}\exp\theta, \tag{3.2.11a}$$

$$\theta = \frac{x - ut - x_0}{\sqrt{1-u^2}}, \tag{3.2.11b}$$

and its energy is given by

$$H_0 = \frac{8}{\sqrt{1-u^2}}. \tag{3.2.12}$$

The width of the kink D_0 is $\pi\sqrt{1-u^2}$ if the kink is approximated by the tangent at its center, hence Hamiltonian ϵH_1 must not be neglected unless h is much smaller than D_0. In later sections, we study dynamical properties of the system, (3.2.4) or (3.2.6) by treating ϵH_1 or $\epsilon f(\phi)$ as perturbation.

3.3 Modulation of the kink parameters

In this section, we consider the zeroth order solution $\overrightarrow{\phi}_0$ as given by eqs.(3.A.6) and (3.A.7). Temporal evolution of the parameters u and x_0 under the structural perturbation is calculated by eqs.(3.A.16) and (3.A.17). In our case, the generic term $\epsilon f(\phi_0)$ is given by

$$\epsilon f(\phi_0) = \frac{2}{h^2}\sum_{n=2}^{\infty}\frac{h^{2n}}{(2n)!}(\partial_x^{2n}\phi_0)\left(1 + 2\sum_{s=1}^{\infty}\cos\frac{2\pi s x}{h}\right). \tag{3.3.1}$$

Since $\partial_x^{2n}\phi_0$ is an odd function of θ, eqs.(3.A.16) and (3.A.17) reduce to

$$\frac{du}{dt} = \frac{1}{h^2}(1-u^2)^{\frac{3}{2}}\sum_{s=1}^{\infty}\sum_{n=2}^{\infty}I_n(s)\sin\frac{2\pi s X}{h}, \tag{3.3.2}$$

$$\frac{dx_0}{dt} = -\frac{u}{2h^2}(1-u^2)\sum_{n=2}^{\infty}J_n(0) - \frac{u}{h^2}(1-u^2)\sum_{s=1}^{\infty}\sum_{n=2}^{\infty}J_n(s)\cos\frac{2\pi s X}{h}, \tag{3.3.3}$$

where

$$I_n(s) = \frac{h^{2n}}{(2n)!(1-u^2)^n}\int_{-\infty}^{\infty}(\partial_\theta^{2n}\phi_0)\mathrm{sech}\theta\sin\frac{2\pi s\sqrt{1-u^2}\theta}{h}d\theta, \tag{3.3.4a}$$

$$J_n(s) = \frac{h^{2n}}{(2n)!(1-u^2)^n}\int_{-\infty}^{\infty}(\partial_\theta^{2n}\phi_0)\theta\mathrm{sech}\theta\cos\frac{2\pi s\sqrt{1-u^2}\theta}{h}d\theta, \tag{3.3.4b}$$

and

$$X = \int_0^t u(t')dt' + x_0(t), \tag{3.3.5}$$

which is the position of the kink center. Noticing that

$$\partial_\theta\phi_0 = 2\mathrm{sech}\theta = 2\sum_{n=0}^{\infty}\left[\frac{i(-1)^n}{\theta + i(n+\frac{1}{2})\pi} - \frac{i(-1)^n}{\theta - i(n+\frac{1}{2})\pi}\right], \tag{3.3.6}$$

and using the residue theorem[10], we can calculate the integrals (3.3.4a) and (3.3.4b). Then the leading contribution (We neglect terms of order $\exp\left(-\frac{s}{h}\right)$, $s \geqq 2$, retaining terms of order $\exp\left(-\frac{1}{h}\right)$) of eqs.(3.3.2) and (3.3.3) are obtained as

$$\frac{du}{dt} = \frac{\alpha_0}{h^2}(1-u^2)^{\frac{3}{2}}\exp\left(-\frac{\pi^2\sqrt{1-u^2}}{h}\right)\sin\frac{2\pi X}{h}, \tag{3.3.7}$$

$$\frac{dx_0}{dt} = -\frac{h^2 u}{24(1-u^2)} + \frac{\alpha_0 \pi}{2h^2} u(1-u^2) \exp\left(-\frac{\pi^2\sqrt{1-u^2}}{h}\right) \cos\frac{2\pi X}{h}, \tag{3.3.8}$$

with

$$\alpha_0 = 4\pi \sum_{n=2}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)(2n)!} = 4\pi[\pi^2 + C_i(2\pi) - \gamma - \log(2\pi)] \fallingdotseq 30\pi, \tag{3.3.9}$$

where $C_i(x)$ is the cosine integral[11] and γ the Euler constant. Here it is to be noted that every term with higher order derivatives with respect to θ in eq.(3.3.4) with $s = 1$ gives contribution of order $\frac{1}{h^2}\exp\left(-\frac{1}{h}\right)$. From eqs.(3.3.3) and (3.3.5), we have

$$U = \frac{dX}{dt} = u\left[1 - \frac{h^2}{24(1-u^2)} + \frac{\alpha_0\pi}{2h^2}(1-u^2)\exp\left(-\frac{\pi^2\sqrt{1-u^2}}{h}\right)\cos\frac{2\pi X}{h}\right]. \tag{3.3.10}$$

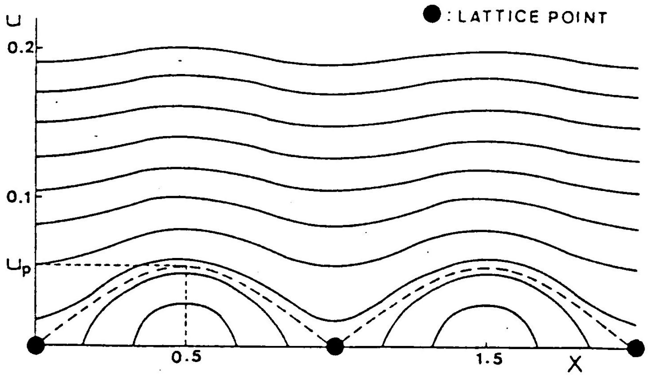

Fig.3.1. The locus $(u,\ X)$ of the kink parameter obtained from eq.(3.3.11) for case $h = 1$. It is not difficult to see that the locus $(u,\ X)$ which passes $(u_p, X = \frac{h}{2})$ runs through $(u = 0, X = 0)$ and $(u = 0, X = h)$. This critical locus is shown by a dashed curve.

Integrating of eqs.(3.3.7) and (3.3.10) can be easily performed to obtain

$$\frac{8}{\sqrt{1-u^2}}\left[1 - \frac{h^2}{72(1-u^2)}\right] + \frac{4\alpha_0}{h}\exp\left(-\frac{\pi^2\sqrt{1-u^2}}{h}\right)\cos\frac{2\pi X}{h} = \text{const.}, \tag{3.3.11}$$

which gives kink trajectories in the $(u,\ X)$ phase plane and shows that the kink behaves as a single particle moving in an effective sinusoidal periodic potential (Peierls potential). The trajectory in case $h = 1$ is shown in Fig.3.1. From Fig.3.1, we see that the kink is pinned between two adjacent lattice points for $u \leqq u_p$, the critical pinning velocity and excutes wobbling motion of frequency

$$\omega_w = \frac{2\pi u}{h}, \tag{3.3.12}$$

for $u > u_p$. When h is small, u_p is also small and it is given by

$$u_p = \sqrt{\frac{2\alpha_0}{\pi h}}\exp\left(-\frac{\pi^2}{2h}\right). \tag{3.3.13}$$

For $h = 1$, we get from eq.(3.3.13) $u_p \approx 0.056$ and this agrees well with u_p (≈ 0.053) obtained from numerical calculation of eq.(3.3.11) (see Fig.3.1). Numerical simulation[8] reports that a kink is pinned for $u(t=0)=0.5$ in case $h=2$. Although $h=2$ seems to be too large to apply our perturbational results, (3.3.11), a tentative estimate of u_p for case $h=2$ based on eq.(3.3.11) gives $u_p=0.51$, thus the simulation result conforming to our pinning condition. A pinning frequency with which a pinned kink oscillates near the bottom of the Peierls potential can be estimated, based on eqs.(3.3.7) and (3.3.10) under the approximation that the Peierls potential is harmonic near the bottom of the potential, to be

$$\omega_p = \sqrt{\frac{2\pi\alpha_0}{h^3}} \exp\left(-\frac{\pi^2}{2h}\right). \tag{3.3.14}$$

For $h=2$, the ω_p obtained by simulation[8)] is 0.77 and from eq.(3.3.14) we get $\omega_p = 0.73$.

From the definition of U, eqs.(3.3.5) and (3.3.10), we see that the velocity of the modified kink is not u but U. The (time) averaged velocity U_m of the kink can be read off from Fig.3 of ref.8 for cases ($h=0.25,\ u(t=0)=0.5$) and ($h=0.5,\ u(t=0)=0.5$) to be 0.4988 and 0.491, respectively. From eq.(3.3.10), we get $U_m=0.4983$ and 0.493 for each case. Thus our perturbational approach achieves rather good agreement with simulation data[8)]. Now we turn to the first order approximation $\epsilon\vec{\phi}_1$.

3.4 First order corrections

In calculating the first order correction $\epsilon\vec{\phi}_1$, we will make use of the Green's function given explicitly in the Appendix 3.1. In our case, $\epsilon\vec{\phi}_1$, eq.(3.A.19) consists of two components, the dressing component ϕ_d and the radiation one ϕ_r. From eqs.(3.3.1) and (3.A.19), we have

$$\phi_d = \int_0^t dt' \int_{-\infty}^{\infty} dx' g_c(x,t|x',t') \frac{2}{h^2} \sum_{n=2}^{\infty} \frac{h^{2n}}{(2n)!} \partial_{x'}^{2n} \phi_0(x',t'), \tag{3.4.1}$$

$$\phi_r = \int_0^t dt' \int_{-\infty}^{\infty} dx' g_c(x,t|x',t') \frac{4}{h^2} \sum_{s=1}^{\infty}\sum_{n=2}^{\infty} \frac{h^{2n}}{(2n)!} \cos\frac{2\pi s x'}{h} \partial_{x'}^{2n} \phi_0(x',t'), \tag{3.4.2}$$

where $g_c(x,t|x',t')$ is given by eq.(3.A.14a). In eqs.(3.4.1) and (3.4.2) we use eq.(3.2.11) for $\phi_0(x,t)$ neglecting time dependence of u and x_0 since it gives higher order corrections to ϕ_d and ϕ_r. Here we will consider the steady state behavior of these equations[9)]. Let $t\to\infty$, then

$$\begin{aligned}\phi_d = &\sum_\omega \frac{1}{4\pi}\int_{-\infty}^{\infty} dk \frac{e^{-ik(x-X)}}{(\omega-uk)^3\omega}(k-\omega u - i\sqrt{1-u^2}\tanh\theta)\frac{2\sqrt{1-u^2}}{h^2} \\ &\times\left[\sum_{n=2}^{\infty}\frac{h^{2n}}{(2n)!(1-u^2)^n}\int_{-\infty}^{\infty} d\theta' e^{ik\sqrt{1-u^2}\theta'}(k-\omega u + i\sqrt{1-u^2}\tanh\theta')\partial_{\theta'}^{2n}\phi_0(\theta')\right],\end{aligned} \tag{3.4.3}$$

$$\begin{aligned}\phi_r = &\sum_\omega \sum_{\substack{s=-\infty \\ (\neq 0)}}^{\infty} \frac{1}{4\pi}\int_{-\infty}^{\infty} dk \frac{e^{-i\left(kx-kx_0-\omega t-\frac{2\pi s x_0}{h}\right)}}{s^2\omega_w^2 i\omega} \\ &\times(k-\omega u - i\sqrt{1-u^2}\tanh\theta)\pi\delta(\omega-ku-s\omega_w)\frac{2\sqrt{1-u^2}}{h^2} \\ &\times\left[\sum_{n=2}^{\infty}\frac{h^{2n}}{(2n)!(1-u^2)^n}\int_{-\infty}^{\infty} d\theta' e^{\frac{i\sqrt{1-u^2}\omega\theta'}{u}}(k-\omega u + i\sqrt{1-u^2}\tanh\theta')\partial_{\theta'}^{2n}\phi_0(\theta')\right],\end{aligned} \tag{3.4.4}$$

where we have integrated out over t' and $\delta(x)$ is the Dirac delta function. The wobbling frequency ω_w is given by eq.(3.3.12). It is easily verified that the leading contribution to ϕ_d is of order h^2 and written as

$$\phi_d = \frac{h^2}{12(1-u^2)^2}(3\tanh\theta - \theta)\mathrm{sech}\theta. \tag{3.4.5}$$

Since the frequency $s\omega_w \pm uk$ is of order $\frac{1}{h}$, we can calculate the leading contribution to ϕ_r from the integral over θ' in a similar way to the previous section. After some algebra it follows that

$$\phi_r = R^+\cos(k^+x - \omega^+t + \gamma^+) - R^+\cos(k^-x + \omega^-t + \gamma^-), \tag{3.4.6}$$

with

$$R^\pm = \frac{8\pi e^{-\frac{\sqrt{1-u^2}\omega^\pm}{u}}}{h^2\omega_w^2\sqrt{\omega_w^2 - 1 + u^2}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{h\omega^\pm}{u}\right)^{2n}\left[\frac{u(k^\pm \mp u\omega^\pm)}{\omega^\pm} \mp \frac{1-u^2}{2n}\right], \tag{3.4.7a}$$

$$\gamma^\pm = \mp\frac{\omega^\pm x_0}{u} + \tan^{-1}\frac{\sqrt{1-u^2}\tanh\theta}{\sqrt{\omega_w^2 - 1 + u^2}}, \tag{3.4.7b}$$

$$k^\pm = \frac{\sqrt{\omega_w^2 - 1 + u^2} \pm u\omega_w}{1-u^2}, \tag{3.4.7c}$$

$$\omega^\pm = \omega_w \pm uk^\pm, \tag{3.4.7d}$$

under the condition $\omega_w > \sqrt{1-u^2}$.

As is mentioned by M-S[9]. eqs.(3.4.5) and (3.4.6) express the phonon dressing of the kink and radiation with Doppler-Shifted wobbling frequencies $\omega_w \pm uk$, respectively. This dressing makes the shape of the kink steeper, modifying the width of the kink D_0 to

$$D \approx D_0\left[1 - \frac{h^2}{12(1-u^2)^{\frac{3}{2}}}\right]. \tag{3.4.8}$$

This contraction was observed by Currie et al.[8]. For case ($h = 0.5,\ u(t=0) = 0.5,\ D_0 = 2.72$), D from simulation is 2.55, while our result, eq.(3.4.8) gives $D = 2.62$. The loss of energy of a kink due to radiation ϕ_r, eq.(3.4.6), can be estimated as follows: Since the emitted energy propagates at the group velocity $\frac{d\omega}{dk} = \frac{k^\pm}{\omega^\pm}$ and the phonon energy density H_p is given by $\frac{1}{2}\left[(\partial_t\phi_r)^2 + (\partial_x\phi_r)^2 + \phi_r^2\right]$ from eq.(3.2.5a), we see that the radiation power P_r is given by

$$P_r = \frac{1}{2}(\omega^+k^+R^{+2} + \omega^-k^-R^{-2}), \tag{3.4.9}$$

where the facter $\frac{1}{2}$ results from time average. From eqs.(3.4.7a) and (3.4.7d) we see that radiation to the backward direction is dominant. This explains the report that phonon are generated mainly in the wake of the kink[8].

3.5 Summary

In this chapter, we have studied effects on kink dynamics of discreteness in the S-G system. In deriving the Hamiltonian (3.2.4), (3.2.5) and (3.2.6), we borrowed an idea due to Cahn[7)] who studied discreteness effects in crystal growth. All our results stem from the perturbation method based on a Green's function formalism[9)]. We showed that a kink behaves as if it were put in a periodic potential field (3.3.7) and as to the pinning we gave two formulas (3.3.13) and (3.3.14) for the critical pinning velocity and the pinning frequency, respectively. As the first order corrections, the dressing part ϕ_d, eq.(3.4.5), and the radiation part ϕ_r, eq.(3.4.6), were obtained. The change in shape of a kink and the radiation power loss are explicitly given by eqs.(3.4.8) and (3.4.9), respectively.

Appendix 3.1 Summary of the perturbation scheme for one-kink

Here we consider the structurally perturbed S-G equation

$$\partial_t^2\phi - \partial_x^2\phi + \sin\phi = \epsilon f(\phi). \tag{3.A.1}$$

Following M-S we write eq.(3.A.1) as

$$\begin{bmatrix} \partial_t & -1 \\ -\partial_x^2 + \sin(\cdot) & \partial_t \end{bmatrix} \overrightarrow{\phi} = \epsilon \overrightarrow{f}(\phi), \tag{3.A.2}$$

with

$$\overrightarrow{\phi} = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}, \quad \overrightarrow{f}(\phi) = \begin{bmatrix} 0 \\ f(\phi) \end{bmatrix}, \tag{3.A.3}$$

and expand $\overrightarrow{\phi}$ as follows:

$$\overrightarrow{\phi} = \overrightarrow{\phi}_0 + \epsilon\overrightarrow{\phi}_1 + \cdots, \tag{3.A.4}$$

where

$$\begin{bmatrix} \partial_t & -1 \\ -\partial_x^2 + \sin(\cdot) & \partial_t \end{bmatrix} \overrightarrow{\phi}_0 = 0. \tag{3.A.5}$$

The parameters in $\overrightarrow{\phi}_0$, the velocity u and the initial phase x_0, are considered to be time-dependent and $\overrightarrow{\phi}_0$ is expressed as

$$\overrightarrow{\phi}_0 = \begin{bmatrix} 4\tan^{-1}\exp\theta \\ \dfrac{-2u(t)}{\sqrt{1-u(t)^2}}\operatorname{sech}\theta \end{bmatrix}, \tag{3.A.6}$$

$$\theta = \frac{x - \displaystyle\int_0^t u(t')dt' - x_0(t)}{\sqrt{1-u(t)^2}}. \tag{3.A.7}$$

Then eq.(3.A.5) is satisfied to order ϵ^0 and $\epsilon\overrightarrow{\phi}_1$ is governed by the linear equation[9)]

$$L\epsilon\overrightarrow{\phi}_1 = \begin{bmatrix} \partial_t & -1 \\ -\partial_x^2 + \cos\phi_0 & \partial_t \end{bmatrix} \epsilon\overrightarrow{\phi}_1 = \epsilon\overrightarrow{F}(\phi_0), \tag{3.A.8a}$$

$$\overrightarrow{\phi}_1(t=0) = \overrightarrow{0}, \tag{3.A.8b}$$

where

$$\epsilon\overrightarrow{F}(\phi_0) = \epsilon\overrightarrow{f}(\phi_0) - \dot{x}_0\partial_{x_0}\overrightarrow{\phi}_0 - \dot{u}\partial_u\overrightarrow{\phi}_0, \tag{3.A.9}$$

and the time derivatives, $\dot{x}_0$ and $\dot{u}$, are considered to be of order ϵ. The first order correction $\epsilon\overrightarrow{\phi}_1$ can be calculated with use of the Green's function $G(x,t|x',t')$ as

$$\epsilon\overrightarrow{\phi}_1 = \int_0^t dt' \int_{-\infty}^{\infty} dx' G(x,t|x',t')\epsilon\overrightarrow{F}(x',t'), \tag{3.A.10}$$

where the matrix kernel $G(x,t|x',t')$ is defined by

$$L(x,t)G(x,t|x',t') = 0 \quad \text{for } t > t' \geqq 0, \quad \lim_{t\to t'} G(x,t|x',t') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \delta(x-x'). \tag{3.A.11}$$

The Green's function consists of two parts,

$$G(x,t|x',t') = G_d(x,t|x',t') + G_c(x,t|x',t'), \tag{3.A.12}$$

where

$$G_c = \begin{bmatrix} -\partial_{t'}g_c & g_c \\ -\partial_{t'}\partial_t g_c & \partial_t g_c \end{bmatrix}, \quad G_d = \begin{bmatrix} -\partial_{t'}g_d & g_d \\ -\partial_{t'}\partial_t g_d & \partial_t g_d \end{bmatrix}, \tag{3.A.13}$$

and

$$g_c = \sum_\omega \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dk}{(\omega - uk)^2 i\omega} \exp[-ik(x-x') + i\omega(t-t')] \tag{3.A.14a}$$

$$\times(k - u\omega - i\sqrt{1-u^2}\tanh\theta)(k - u\omega + i\sqrt{1-u^2}\tanh\theta'),$$

$$g_d = \frac{1}{2\sqrt{1-u^2}}[(t-t') - u(x-x')]\mathrm{sech}\theta\,\mathrm{sech}\theta'. \tag{3.A.14b}$$

The notation $\sum_\omega$ indicates the summation over two branches, $\omega = \pm\sqrt{1+k^2}$. We note that when we use eq.(3.A.10) to calculate the first order correction $\epsilon\overrightarrow{\phi}_1$, we can neglect time-dependence of u and x_0 since it gives higher order corrections. Thus in eq.(3.A.14) we use θ as given in eq.(3.2.11). Similarly θ' is given by eq.(3.2.11) with t and x replaced by t' and x', respectively. As stressed by M-S[9)], $\epsilon\overrightarrow{\phi}_1$ will exhibit linear temporal growth unless

$$\int_{-\infty}^{\infty} dx' G_d(x,t|x',t')\overrightarrow{F}(x',t') = 0. \tag{3.A.15}$$

The non-secularity condition (3.A.15) together with eqs.(3.A.9), (3.A.13) and (3.A.14b) leads to equations of motion for $u(t)$ and $x_0(t)$,

$$\frac{du}{dt} = -\frac{1}{4}(1-u^2)\int_{-\infty}^{\infty} \epsilon f(\phi_0)\mathrm{sech}\theta dx, \tag{3.A.16}$$

$$\frac{dx_0}{dt} = -\frac{u}{4}\sqrt{1-u^2}\int_{-\infty}^{\infty} \epsilon f(\phi_0)\theta\,\mathrm{sech}\theta dx. \tag{3.A.17}$$

On the other hand, from eqs.(3.A.10) and (3.A.15) and also from the identity

$$\int_{-\infty}^{\infty} G_c(x,t|x',t')[\dot{x}_0\partial_{x_0}\vec{\phi}_0(x',t') + \dot{u}\partial_u\vec{\phi}_0(x',t')] = 0, \tag{3.A.18}$$

$\epsilon\vec{\phi}_1$ is expressed simply as

$$\epsilon\vec{\phi}_1 = \int_0^t dt' \int_{-\infty}^{\infty} dx' G_c(x,t|x',t')\epsilon\vec{f}(\phi_0(x',t')). \tag{3.A.19}$$

Equation (3.A.18) is an important property of the continuum part of G, G_c which was not explicitly noted by M-S[9)]. This follows from the orthogonality of the eigenfunction of the operator L, (3.A.8).

Appendix 3.2 Derivation of eqs.(3.2.6) and (3.2.7)

A Lagrangian density $\mathcal{L}$ for eqs(3.2.4) and (3.2.5) is

$$\mathcal{L} = \left[\frac{1}{2}(\partial_t\phi)^2 - (1-\cos\phi) - \frac{1}{2h^2}\sum_{m,n=1}^{\infty}\frac{h^{m+n}}{m!n!}(\partial_x^m\phi)(\partial_x^n\phi)\right]\sum_{s=-\infty}^{\infty} e^{i\frac{2\pi sx}{h}}. \tag{3.B.1}$$

The corresponding Euler's equation

$$\partial_t\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} - \sum_{l=1}^{\infty}(-1)^l\partial_x^l\left[\frac{\partial\mathcal{L}}{\partial(\partial_x^l\phi)}\right] = 0, \tag{3.B.2}$$

gives

$$(\partial_t^2\phi + \sin\phi)\left(\sum_{s=-\infty}^{\infty} e^{i\frac{2\pi sx}{h}}\right) + \frac{1}{h^2}\sum_{s=-\infty}^{\infty}\sum_{l=1}^{\infty}(-1)^l\partial_x^l\left[\sum_{n=1}^{\infty}\frac{h^{l+n}}{l!n!}(\partial_x^n\phi)e^{i\frac{2\pi sx}{h}}\right] = 0. \tag{3.B.3}$$

Noticing

$$\begin{aligned}
&\sum_{l=1}^{\infty}(-1)^l\partial_x^l\left[\sum_{n=1}^{\infty}\frac{h^{l+n}}{l!n!}(\partial_x^n\phi)e^{i\frac{2\pi sx}{h}}\right] \\
&= \sum_{m=1}^{\infty}\left[\sum_{n=1}^{m}\sum_{l\geqq m-n}^{\infty}(-1)^l\frac{h^{l+n}}{l!n!}\frac{l!}{(m-n)!(l-m+n)!}\left(\frac{i2\pi s}{h}\right)^{l-m+n} - \frac{h^m}{m!}\right](\partial_x^m\phi)e^{i\frac{2\pi sx}{h}} \\
&= \sum_{m=1}^{\infty}\left[\sum_{n=1}^{m}(-1)^{m-n}\frac{h^{m-n}}{n!(m-n)!}\sum_{p=0}^{\infty}\frac{(-i2\pi s)^p}{p!} - \frac{h^m}{m!}\right](\partial_x^m\phi)e^{i\frac{2\pi sx}{h}} \\
&= \sum_{m=1}^{\infty} -[(-1)^m+1]\frac{h^m}{m!}(\partial_x^m\phi)e^{i\frac{2\pi sx}{h}},
\end{aligned} \tag{3.B.4}$$

we have

$$\left[\partial_t^2\phi + \sin\phi - \frac{2}{h^2}\sum_{n=1}^{\infty}\frac{h^{2n}}{(2n)!}(\partial_x^{2n}\phi)\right]\left(\sum_{s=-\infty}^{\infty} e^{i\frac{2\pi s x}{h}}\right) = 0. \tag{3.B.5}$$

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CHAPTER IV

RELATIONSHIPS AMONG SOME SCHEMES OF THE INVERSE SCATTERING TRANSFORM

4.1 Introduction

In the last decade or so, a number of nonlinear wave equations which can be integrated by the inverse scattering method have been found[1),2)]. The equations are expressed as the consistency condition

$$\partial_t U - \partial_x V + [U, V] = 0, \tag{4.1.1}$$

for a system of linear equations

$$\partial_x \Phi = U(x, t; \lambda)\Phi, \tag{4.1.2a}$$

$$\partial_t \Phi = V(x, t; \lambda)\Phi, \tag{4.1.2b}$$

where Φ is a N component vector and U and V are $N \times N$ matrices that are usually rational functions of the spectral parameter λ[2)]. In the simplest applications, U and V are 2×2 matrices. It is known that typical 2×2 matrices, U and V, presented by Ablowitz, Kaup, Newell and Segur (A-K-N-S)[3)] lead to a wide class of nonlinear wave equations. In order to cover more integrable systems, Wadati, Konno and Ichikawa (W-K-I)[4)] proposed a generalization of the inverse scattering formalism, especially U (note that U is 2×2 matrix), and found a new series of integrable nonlinear wave equations[5)].

As is well known, U and V are not unique for given nonlinear equation because eqs.(4.1.1) and (4.1.2) are form-invariant under the gauge transformation

$$\widetilde{\Phi} = g^{-1}\Phi, \tag{4.1.3a}$$

$$\widetilde{U} = g^{-1}Ug - g^{-1}\partial_x g, \tag{4.1.3b}$$

$$\widetilde{V} = g^{-1}Ug - g^{-1}\partial_t g, \tag{4.1.3b}$$

where g is an arbitrary matrix function of x and t. For instance, with use of this property, Zakharov and Takhtadzhyan (Z-T) showed that the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet are equivalent[10)]. Also, Orfanidis developed a systematic method of constructing the σ-model associated with any given nonlinear equation solvable by the inverse scattering method[11)].

In this chapter, we investigate a connection between two inverse scattering formalisms by A-K-N-S and by W-K-I, and show that in addition to the gauge transformation there is a coordinate-transformation under which eqs.(4.1.1) and (4.1.2) are form-invariant. The transformation depends on a dependent variable.

First, in section 4.2, we present a inverse scattering transform which is gauge equivalent to the A-K-N-S scheme. Second, applying a coordinate-transformation to this inverse scattering transform, we obtain the W-K-I scheme in section 4.3. Thus, through two transformations, the A-K-N-S and W-K-I schemes are connected to each other. In section 4.5, we consider the loop soliton which is a solution of a new equation presented by W-K-I and present a physical interpretation of the coordinate-transformation. Concluding remarks are given in section 4.6.

4.2 Gauge transformation

An example of U and V given by A-K-N-S[3)] is

$$U = -i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \lambda + \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}, \tag{4.2.1a}$$

$$V=-4\alpha i\begin{bmatrix}1&0\\0&-1\end{bmatrix}\lambda^3+4\alpha\begin{bmatrix}0&u\\v&0\end{bmatrix}\lambda^2-2\beta i\begin{bmatrix}1&0\\0&-1\end{bmatrix}\lambda^2-2\alpha i\begin{bmatrix}uv&-\partial_x u\\\partial_x v&-uv\end{bmatrix}\lambda,$$
$$+2\beta\begin{bmatrix}0&u\\v&0\end{bmatrix}\lambda+\alpha\begin{bmatrix}v\partial_x u-u\partial_x v&2u^2v-\partial_x^2 u\\2uv^2-\partial_x^2 v&u\partial_x v-v\partial_x u\end{bmatrix}-\beta i\begin{bmatrix}uv&-\partial_x u\\\partial_x v&-uv\end{bmatrix}, \tag{4.2.1b}$$

where α and β are real constants. Equation (4.1.1) for these U and V yields the set of nonlinear wave equations

$$\partial_t u+\alpha(\partial_x^3 u-6uv\partial_x u)-\beta i(\partial_x^2 u-2u^2v)=0, \tag{4.2.2a}$$

$$\partial_t v+\alpha(\partial_x^3 v-6uv\partial_x v)+\beta i(\partial_x^2 v-2uv^2)=0. \tag{4.2.2b}$$

If we take $v=\mp u^*$, these equations are reduced to

$$\partial_t u+\alpha(\partial_x^3 u\pm 6|u|^2\partial_x u)-\beta i(\partial_x^2 u\pm 2|u|^2u)=0, \tag{4.2.3}$$

which is the generalized nonlinear equation presented by Hirota[12)].

Following Z-T, we define g as a solution of eq.(4.1.2) with eq.(4.2.1) for $\lambda=0$, that is,

$$\partial_x g=\begin{bmatrix}0&u\\v&0\end{bmatrix}g, \tag{4.2.4a}$$

$$\partial_t g=\alpha\begin{bmatrix}v\partial_x u-u\partial_x v&2u^2v-\partial_x^2 u\\2uv^2-\partial_x^2 v&u\partial_x v-v\partial_x u\end{bmatrix}g-\beta i\begin{bmatrix}uv&-\partial_x u\\\partial_x v&-uv\end{bmatrix}g. \tag{4.2.4b}$$

Then it is easy to verify that if we let

$$S=g^{-1}\begin{bmatrix}1&0\\0&-1\end{bmatrix}g, \tag{4.2.5}$$

we have

$$S\partial_x S=2g^{-1}\begin{bmatrix}0&u\\v&0\end{bmatrix}g, \tag{4.2.6a}$$

$$\partial_x^2 S=2g^{-1}\begin{bmatrix}2uv&-\partial_x u\\-\partial_x v&-2uv\end{bmatrix}g, \tag{4.2.6b}$$

$$S(\partial_x^2 S)S=2g^{-1}\begin{bmatrix}2uv&-\partial_x u\\\partial_x v&-2uv\end{bmatrix}g. \tag{4.2.6c}$$

Thus, from eqs.(4.1.3) and (4.2.1), we obtain a gauge equivalent scheme to the A-K-N-S one:

$$\partial_x\widetilde{\phi}=\widetilde{U}\widetilde{\Phi}, \tag{4.2.7a}$$

$$\partial_t\widetilde{\phi}=\widetilde{V}\widetilde{\Phi}, \tag{4.2.7b}$$

with

$$\widetilde{U}=-iS\lambda, \tag{4.2.8a}$$

$$\widetilde{V}=-4\alpha iS\lambda^3+(2\alpha S\partial_x S-2\beta iS)\lambda^2+\left[\frac{1}{4}\alpha i(\partial_x^2 S-3S(\partial_x^2 S)S)+\beta S\partial_x S\right]\lambda. \tag{4.2.8b}$$

Because of eq.(4.2.5), we may take

$$S = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad a^2 + bc = 1. \tag{4.2.9}$$

We note here that when $v = -u^*$, $g^{-1} = g^\dagger$ and $S^\dagger = S$.

The compatibility equation for eq.(4.2.7) with eq.(4.2.8) is

$$\partial_t S + \alpha \left[\partial_x^3 S + \frac{3}{2}\partial_x\{(\partial_x S)^2 S\} \right] - \frac{i}{2}\beta[S, \partial_x^2 S] = 0. \tag{4.2.10}$$

If $\alpha = 1$ and $\beta = 0$, eq.(4.2.10) gives a spin chain reducible to the K-dV or modified K-dV equation, which was presented by Orfanidis[11)] (From what he refers, a general spin chain reducible to the Hirota equation was given by N.Papanicolau). In this case, the choice

$$a = \cos\theta, \quad b = -c = i\sin\theta, \tag{4.2.11}$$

or

$$a = \cosh\phi, \quad b = c = i\sinh\phi, \tag{4.2.12}$$

yields the modified K-dV equation

$$\partial_t\theta + \frac{1}{2}(\partial_x\theta)^3 + \partial_x^3\theta = 0, \tag{4.2.13}$$

or

$$\partial_t\phi - \frac{1}{2}(\partial_x\phi)^3 + \partial_x^3\phi = 0, \tag{4.2.14}$$

respectively. In addition to eq.(4.2.12), by choosing

$$a = e^\phi, \quad b = i(e^{-\phi} - e^\phi), \quad c = -ie^\phi, \tag{4.2.15}$$

we have eq.(4.2.14) again. For $\alpha = 0$ and $\beta = 1$, taking

$$a = S_3, \quad b = i(S_1 + iS_2), \quad c = -i(S_1 - iS_2), \tag{4.2.16a}$$

$$\overrightarrow{S} = (S_1, S_2, S_3), \quad S_1^2 + S_2^2 + S_3^2 = 1, \tag{4.2.16b}$$

we get from eq.(4.2.10)

$$\partial_t \overrightarrow{S} = \overrightarrow{S} \times (\partial_x^2 \overrightarrow{S}), \tag{4.2.17}$$

which is the Heisenberg ferromagnet equation[14),15)]. Also letting

$$a = T_3, \quad b = i(T_1 + iT_2), \quad c = i(T_1 - iT_2), \tag{4.2.18a}$$

$$\overrightarrow{T} = (T_1, T_2, T_3), \quad T_3^2 - T_1^2 - T_2^2 = 1, \tag{4.2.18b}$$

we obtain

$$\partial_t \overrightarrow{T} = \left[\overrightarrow{T} \times (\partial_x^2 \overrightarrow{T}) \right], \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{4.2.19}$$

which expresses a pseudo spin chain.

4.3 Transformation of the space and time coordinates

Now we transform the space and time coordinates into new ones,

$$\xi = \int^x a(x,t)dx, \tag{4.3.1a}$$

$$\tau = t, \tag{4.3.1b}$$

under the boundary condition $\xi \to x$ as $x \to \infty$. Then, with the use of eq.(4.2.10), especially

$$\partial_t a = -\alpha\partial_x \left[\partial_x^2 a + \frac{3}{2}\{(\partial_x a)^2 + (\partial_x b)(\partial_x c)\}a\right] + \frac{i}{2}\beta\partial_x(b\partial_x c - c\partial_x b), \tag{4.3.2}$$

we have from eq.(4.3.1)

$$\partial_x = a\partial_\xi, \tag{4.3.3a}$$

$$\partial_t = \partial_\tau + \left(-\alpha\left[\partial_x^2 a + \frac{3}{2}\{(\partial_x a)^2 + (\partial_x b)(\partial_x c)\}a\right] + \frac{i}{2}\beta(b\partial_x c - c\partial_x b)\right)\partial_\xi. \tag{4.3.3b}$$

Substituting eq.(4.3.3) into eq.(4.2.7), we obtain

$$\partial_\xi\widetilde{\phi} = U'\widetilde{\Phi}, \tag{4.3.4a}$$

$$\partial_\tau\widetilde{\phi} = V'\widetilde{\Phi}, \tag{4.3.4b}$$

with

$$U' = -i\begin{bmatrix} 1 & \dfrac{b}{a} \\ \dfrac{c}{a} & -1 \end{bmatrix}\lambda, \tag{4.3.5a}$$

$$\begin{aligned} V' = &-4\alpha i\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\lambda^3 + 2\alpha\begin{bmatrix} a^2\partial_\xi a + ab\partial_\xi c & a^2\partial_\xi b - ab\partial_\xi a \\ ac\partial_\xi a - a^2\partial_\xi c & a^2\partial_\xi a + ac\partial_\xi b \end{bmatrix}\lambda^2 - 2\beta i\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\lambda^2 \\ &+\alpha i\begin{bmatrix} 0 & \partial_\xi\{a(a\partial_\xi b - b\partial_\xi a)\} \\ \partial_\xi\{a(a\partial_\xi c - c\partial_\xi a)\} & 0 \end{bmatrix}\lambda + \beta\begin{bmatrix} 0 & \partial_\xi b \\ -\partial_\xi c & 0 \end{bmatrix}\lambda. \end{aligned} \tag{4.3.5b}$$

This system is equivalent to the W-K-I scheme. Putting

$$iq = \frac{b}{a}, \quad ir = \frac{c}{a}, \quad \frac{1}{\sqrt{1-rq}} = a, \tag{4.3.6}$$

we have the same U' and V' as ones presented by W-K-I[5)]:

$$U' = -i\begin{bmatrix} 1 & iq \\ ir & -1 \end{bmatrix}\lambda, \tag{4.3.7a}$$

$$V' = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}, \tag{4.3.7b}$$

where

$$A = \frac{-4\alpha i}{\sqrt{1-rq}}\lambda^3 + \frac{\alpha(r\partial_\xi q - q\partial_\xi r)}{(1-rq)^{\frac{3}{2}}}\lambda^2 - \frac{2\beta i}{\sqrt{1-rq}}\lambda^2, \tag{4.3.8a}$$

$$B = \frac{4\alpha q}{\sqrt{1-rq}}\lambda^3 + \frac{2\alpha i\partial_\xi q}{(1-rq)^{\frac{3}{2}}}\lambda^2 + \frac{2\beta q}{\sqrt{1-rq}}\lambda^2 - \partial_\xi\left[\frac{\alpha\partial_\xi q}{(1-rq)^{\frac{3}{2}}}\right]\lambda + \partial_\xi\left[\frac{\beta i q}{\sqrt{1-rq}}\right]\lambda, \tag{4.3.8b}$$

$$C = \frac{4\alpha r}{\sqrt{1-rq}}\lambda^3 + \frac{2\alpha i\partial_\xi r}{(1-rq)^{\frac{3}{2}}}\lambda^2 + \frac{2\beta r}{\sqrt{1-rq}}\lambda^2 + \partial_\xi\left[\frac{\alpha\partial_\xi r}{(1-rq)^{\frac{3}{2}}}\right]\lambda - \partial_\xi\left[\frac{\beta i r}{\sqrt{1-rq}}\right]\lambda. \tag{4.3.8c}$$

The compatibility condition for these system gives the set of nonlinear wave equations,

$$\partial_\tau q + \alpha\partial_\xi^2\left[\frac{\partial_\xi q}{(1-rq)^{\frac{3}{2}}}\right] - \beta i\partial_\xi^2\left[\frac{q}{\sqrt{1-rq}}\right] = 0, \tag{4.3.9a}$$

$$\partial_\tau r + \alpha\partial_\xi^2\left[\frac{\partial_\xi r}{(1-rq)^{\frac{3}{2}}}\right] + \beta i\partial_\xi^2\left[\frac{r}{\sqrt{1-rq}}\right] = 0. \tag{4.3.9b}$$

If we take $\alpha = 1$ and $\beta = 0$, the set of eq.(4.3.9) is reduced to

$$\partial_\tau q + \partial_\xi^2\left[\frac{\partial_\xi q}{(1+q^2)^{\frac{3}{2}}}\right] = 0 \quad \text{for } r = -q, \tag{4.3.10}$$

$$\partial_\tau q + \partial_\xi^2\left[\frac{\partial_\xi q}{(1-q^2)^{\frac{3}{2}}}\right] = 0 \quad \text{for } r = q, \tag{4.3.11}$$

and

$$\partial_\tau p = 2\partial_\xi^3 p^{-\frac{1}{2}} \quad \text{for } r = -1 \quad \text{and } q = p-1. \tag{4.3.12}$$

Equation (4.3.10) describes the nonlinear oscillation of elastic beams under tension as shown by W-K-I[6]. Also eq.(4.3.12) is Harry-Dym (H-D) equation[15]. If we let $\alpha = 0$ and $\beta = 1$, eq.(4.3.9) is reduced to

$$i\partial_\tau q + \partial_\xi^2\left[\frac{q}{\sqrt{1+|q|^2}}\right] = 0 \quad \text{for } r = -q^*, \tag{4.3.13}$$

$$i\partial_\tau q + \partial_\xi^2\left[\frac{q}{\sqrt{1-|q|^2}}\right] = 0 \quad \text{for } r = q^*. \tag{4.3.14}$$

4.4 One-soliton solutions

Wadati et al. obtained one-soliton solutions of eqs.(4.3.10)[7], (4.3.12)[8] and (4.3.13)[9] with use of the inverse scattering method for system (4.3.7) and observed that the solutions can not be expressed in a closed form unlike the usual soliton solutions. Here, with help of known one-soliton solutions of eq.(4.2.13), (4.2.14), (4.2.17) and (4.2.19) which are directly connected to eqs.(4.3.10-14) by eqs.(4.3.1) and (4.3.6), we construct solutions of eqs.(4.3.10-14).

A one-soliton solution of eq.(4.2.13)[16)] is given by

$$\theta = 4\tan^{-1}\exp\delta, \tag{4.4.1a}$$

$$\delta = k(x-k^2t)+\delta_0, \tag{4.4.1b}$$

where k is a real parameter and δ_0 is an arbitrary phase constant. Since eq.(4.2.13) is transformed into eq.(4.3.10) through

$$\xi = \int^x \cos\theta\, dx, \quad \tau = t, \tag{4.4.2a}$$

$$q = \tan\theta, \tag{4.4.2b}$$

the one-soliton solution of eq.(4.3.13) is written as

$$q = \frac{2\mathrm{sech}\delta\cdot\tanh\delta}{2\mathrm{sech}^2\delta - 1}, \tag{4.4.3a}$$

$$k(\xi - k^2\tau) + \delta_0 = \delta - 2\tanh\delta + 2. \tag{4.4.3b}$$

Because of eq.(4.3.6), we have the condition that $\cos\theta = 1 - 2\mathrm{sech}^2\delta \geqq 0$. Hence this solution is a discrete solitary wave. As shown by W-K-I[7)], however, if this condition is removed, the solution becomes meaningful as a physical solution (see the following section).

A one-soliton solution of eq.(4.2.14) is given by

$$\phi = \pm\log\tanh^2\frac{\delta}{2}, \tag{4.4.4a}$$

$$\delta = k(x-k^2t)+\delta_0. \tag{4.4.4b}$$

Since eqs.(4.3.11) and (4.3.12) are derived from eq.(4.2.14) through the transformations

$$\xi = \int^x \cosh\phi\, dx, \quad \tau = t, \tag{4.4.5a}$$

$$q = \tanh\phi, \tag{4.4.5b}$$

and

$$\xi = \int^x \exp\phi\, dx, \quad \tau = t, \tag{4.4.6a}$$

$$q = \exp(-2\phi), \tag{4.4.6b}$$

respectively, the one-soliton solutions of them are written as

$$q = \mp\frac{2\mathrm{cosech}\delta\cdot\coth\delta}{1+2\mathrm{cosech}^2\delta}, \tag{4.4.7a}$$

$$k(\xi - k^2\tau) + \delta_0 = \delta - 2\coth\delta + 2, \tag{4.4.7b}$$

and

$$p = \tanh^{\mp4}\frac{\delta}{2}, \tag{4.4.8a}$$

$$k(\xi - k^2\tau) + \delta_0 = \delta - 2\tanh^{\pm1}\frac{\delta}{2} + 2, \tag{4.4.8b}$$

respectively. Equation (4.4.8) with the upper sign is the same as the cusp soliton presented by Wadati et al.[8]. It is interesting to note that H-D equation, eq.(4.3.12), have both divergent and nondivergent soliton solutions like the K-dV equation.

In the same way that we got the one-soliton solutions of eqs.(4.3.10-12), we obtain one-soliton solutions of eqs.(4.3.13) and (4.3.14). It follows that

$$\left.\begin{aligned} q &= \frac{-\dfrac{2P}{P^2+k^2}\mathrm{sech}\Delta(P\tanh\Delta+ik)}{1-\dfrac{2P^2}{P^2+k^2}\mathrm{sech}^2\Delta}e^{-i\delta} \\ \xi &= x-\frac{P}{P^2+k^2}(\tanh\Delta-1) \end{aligned}\right\} \quad \text{for eq.(4.3.13)}, \tag{4.4.9}$$

and

$$\left.\begin{aligned} q &= \frac{-\dfrac{2P}{P^2+k^2}\mathrm{cosech}\Delta(P\coth\Delta+ik)}{1+\dfrac{2P^2}{P^2+k^2}\mathrm{cosech}^2\Delta}e^{-i\delta} \\ \xi &= x-\frac{P}{P^2+k^2}(\coth\Delta-1) \end{aligned}\right\} \quad \text{for eq.(4.3.14)}, \tag{4.4.10}$$

with

$$\Delta = 2Px+8kPt+\Delta_0, \tag{4.4.11a}$$

$$\delta = 2kx+4(k^2-P^2)t+\delta_0, \tag{4.4.11b}$$

where P and k are real parameters and Δ_0 and δ_0 are arbitrary phase constants. We have also used the one-soliton solutions of eqs.(4.2.17)[14] and (4.2.19),

$$S_1+iS_2 = -\frac{2P^2}{P^2+k^2}\mathrm{sech}\Delta(P\tanh\Delta+ik)e^{-i\delta}, \tag{4.4.12a}$$

$$S_3 = 1-\frac{2P^2}{P^2+k^2}\mathrm{sech}^2\Delta, \tag{4.4.12b}$$

and

$$T_1+iT_2 = -\frac{2P^2}{P^2+k^2}\mathrm{cosech}\Delta(P\coth\Delta+ik)e^{-i\delta}, \tag{4.4.13a}$$

$$T_3 = 1+\frac{2P^2}{P^2+k^2}\mathrm{cosech}^2\Delta, \tag{4.4.13b}$$

and the transformations,

$$\xi = \int^x S_3\,dx, \quad \tau = t, \tag{4.4.14a}$$

$$q = \frac{S_1+iS_2}{S_3}, \tag{4.4.14b}$$

and

$$\xi = \int^x T_3\,dx, \quad \tau = t, \tag{4.4.15a}$$

$$q = \frac{T_1+iT_2}{T_3}. \tag{4.4.15b}$$

Equation (4.4.9) is the same as the solution presented by Shimizu and Wdati[9].

4.5 The loop soliton

The one-soliton solution of the modified K-dV equation, eq.(4.4.1), gives

$$|\theta(x=\infty)-\theta(x=-\infty)|=2\pi, \tag{4.5.1}$$

for any value of k. This property is analogous to that of the Sine-Gordon (S-G) equation of a mechanical model[17] in which θ describes an angle of rotation of the pendula. Hence similar topological properties to the S-G equation[17),18] may be expected in the modified K-dV equation. In this section, we show that this is true, using the results in the previous section.

The modified K-dV equation, eq.(4.2.13), is expressed in the $(\xi,\ \tau)$ space as

$$\partial_\tau\theta+\cos^2\theta\partial_\xi^3\sin\theta=0, \tag{4.5.2}$$

where we have used eq.(4.4.2a) and removed the condition $\cos\theta\geqq 0$. This equation is the same as the equation derived by W-K-I, which describes waves propagating along a stretched rope if we let $y_\xi=\tan\theta$ in eq.(5) of ref.7. The variable θ is a tangential angle along the stretched rope. W-K-I obtained the one-soliton solution of eq.(4,5,2),

$$y=\frac{2}{k}\mathrm{sech}\delta, \tag{4.5.3a}$$

$$k(\xi-k^2\tau)+\delta_0=\delta-2\tanh\delta+2, \tag{4.5.3b}$$

which has a shape of loop[7]. This solution can be derived from eq.(4.4.1) with eq.(4.4.2).

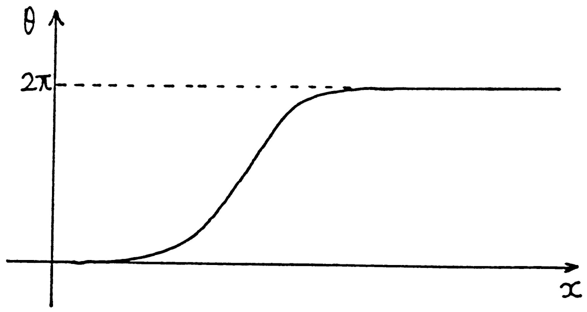

Fig.4.1. The curve of the one-soliton solution for $k>0$ in the x space.

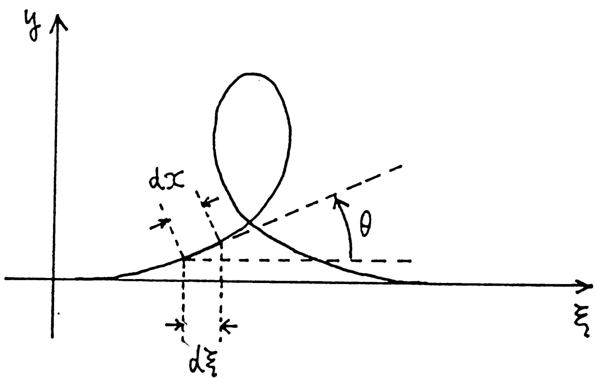

Fig.4.2. The curve of the one-soliton solution for $k>0$ in the ξ space (loop soliton).

In Figs.4.1 and 4.2, we have sketched the one-soliton solution of the modified K-dV equation in the $(x,\ t)$ space and the $(\xi,\ \tau)$ space, respectively. Since we have from eq.(4.4.2a)

$$\frac{d\xi}{dx} = \cos\theta, \tag{4.5.4}$$

dx is an increment of the length of arc on the stretched rope (see Fig.4.2). 2π in eq.(4.5.1) corresponds to the total increment of the tangential angle along the stretched rope, from $x = -\infty$ to $x = \infty$. Thus we see that the modified K-dV soliton is essentially the same as the loop soliton. Therefore, from the fact that the difference between the number of loop solitons with $k > 0$ and of loop solitons with $k < 0$ is conserved, the difference between the numbers of solitons and of antisolitons for the modified K-dV equation must be conserved in any collision. The situation is similar to that of the S-G equation[18].

4.6 Concluding remarks

In this chapter, we have shown that the A-K-N-S and W-K-I schemes of the inverse scattering transform are connected through the gauge transformation and the transformation of the space and time coordinates. The common property which each scheme has is that a nonlinear wave equation generated by it has the same linear dispersion relation. Recently, Wdati and Sogo[19] found that a scheme by Kaup and Newell (K-N)[20] is also gauge equivalent to the A-K-N-S one. The matrix U in K-N scheme is written as

$$U = -i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\lambda + \begin{bmatrix}0 & q\\ r & 0\end{bmatrix}\sqrt{\lambda}. \tag{4.6.1}$$

K-N scheme generates the derivative nonlinear Schrödinger equation,

$$i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0, \tag{4.6.2}$$

whose linear dispersion relation is the same as that of the nonlinear Schrödinger equation (eq.(4.2.3) with $\alpha = 0$). The gauge transformation is done by g defined as

$$g = \begin{bmatrix} f & 0\\ \dfrac{irf}{2\sqrt{\lambda}} & \dfrac{f^{-1}}{2\sqrt{\lambda}}\end{bmatrix}, \tag{4.6.3a}$$

$$f = \exp\left(\frac{i}{2}\int^x rq\, dx\right), \tag{4.6.3b}$$

$$u = \frac{qf^{-2}}{2}, \quad v = \left(-i\partial_x r + \frac{r^2 q}{2}\right) f^2. \tag{4.6.3c}$$

It is to be noted that this gauge transformation depends on the spectral parameter λ. In Fig.4.3, we have shown the interrelation among the various schemes of the inverse scattering transform. Those results suggest that all nonlinear equations having the same linear dispersion relation, which are integrable by the inverse scattreing method, are equivalent.

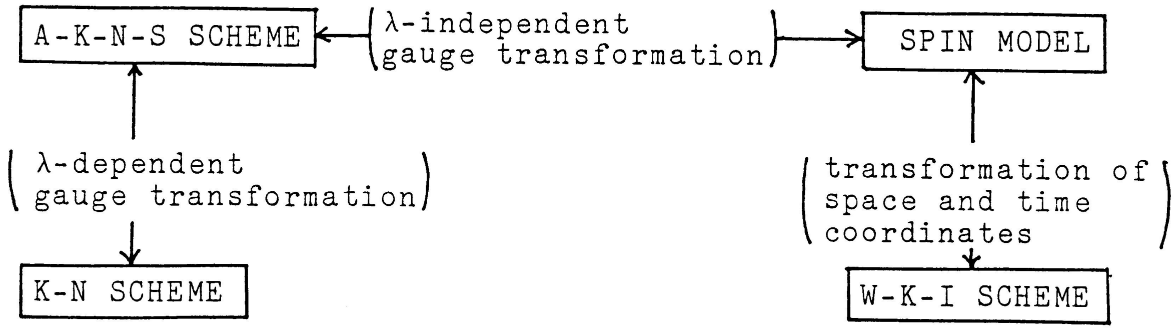

Fig.4.3. Relationships among some schemes of the inverse scattering transform

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CHAPTER V

A COMPLETELY INTEGRABLE CLASSICAL SPIN CHAIN

5.1 Introduction

In recent years, it has been found that some classical spin chains, for example, the continuous isotropic Heisenberg spin chain[1),2)] and the general spin chain described by the Landau-Lifshitz equation[3)], belong to the class of completely integrable Hamiltonian systems[4)]. These systems can be integrated by the inverse scattering method and their excitations are completely composed of the continuum (magnons) and discrete (solitons) modes. Apart from the quantum spin chain such as spin $\frac{1}{2}$ Heisenberg XYZ model[5)], the examples of the completely integrable classical spin chains are so far limited to the case of the continuum models.

As mentioned in the previous chapter, the continuous isotropic Heisenberg spin chain is equivalent to the nonlinear Schrödinger equation and the schemes of the inverse scattering transform for them are connected to each other through the gauge transformation[6)]. For the nonlinear Schrödinger equation, Ablowitz and Ladik have presented its lattice model (the differential-difference nonlinear Schrödinger equation)[7),8)]. These facts imply that by considering a differential-difference analogue of the continuous gauge transformation an integrable classical spin model on the one-dimensional lattice is obtained from the differential-difference nonlinear Schrödinger equation.

In this chapter, we realize the above idea and present a lattice spin model whose Hamiltonian is given by

$$H = -2\sum_n \log\left[\frac{1}{2}\left(1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}\right)\right] - h\sum_n (S_n^z - 1), \tag{5.1.1}$$

where the magnitude of spins $\overrightarrow{S}_n = (S_n^x, S_n^y, S_n^z)$ is assumed to be unity and h is a constant. Details of the inverse scattering approach to this spin model is also studied. Although our spin model (5.1.1) is only a mathematical one, for low-energy excitations, the results we obtain seem to be available for the Heisenberg spin chain because the approximate expression for eq.(5.1.1) when all the angles between the nearest-neighbor spins are small is $H = -\sum_n \left(\overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1} - 1\right) - h\sum_n (S_n^z - 1)$, which is the Heisenberg Hamiltonian[9)]. In our discussion, we impose fixed boundary conditions at infinity for $\overrightarrow{S}_n$, i.e.,

$$\overrightarrow{S}_n \to (0, 0, 1) \quad \text{for } n \to \pm\infty. \tag{5.1.2}$$

This chapter is organized as follows. In section 5.2, our model (5.1.1) is derived from the differential-difference nonlinear Schrödinger equation. In section 5.3, the inverse scattering method associated with it is discussed. We study the problems of direct and inverse scattering and derive the Gel'fand-Levitan equation. A special example of the initial value problem is also examined. The procedure follows essentially the same line as one for known discrete systems[7),8)]. In section 5.4, we construct canonical action angle variables[10)] to show that our spin model belongs to the class of completely integrable Hamiltonian systems. Then these variables are related to an infinite set of constants of motion.

5.2 Model

In this section, we apply the concept of gauge equivalence to the differential-difference nonlinear Schrödinger equation and construct an integrable spin model.

As is well known, the nonlinear differential-difference equations integrable by the inverse scattering method are expressed as the compatibility condition

$$\dot{L}_n(z) = M_{n+1}(z)L_n(z) - L_n(z)M_n(z), \tag{5.2.1}$$

for a set of two equations

$$\Phi_{n+1} = L_n(z)\Phi_n, \tag{5.2.2a}$$

$$\dot{\Phi}n = M_n(z)\Phi_n, \tag{5.2.2b}$$

where $L_n,\ M_n$ and Φ_n are $N \times N$ matrices. If we take

$$L_n(z) = \begin{bmatrix} z & q_n z^{-1} \\ -q_n^* z & z^{-1} \end{bmatrix}, \tag{5.2.3a}$$

$$M_n(z) = i\begin{bmatrix} 1 - z^2 - q_n q_{n-1}^* & -q_n + q_{n-1}z^{-2} \\ -q_n^* + q_{n-1}^* z^2 & -1 + z^{-2} + q_n^* q_{n-1} \end{bmatrix}, \tag{5.2.3b}$$

then the compatibility condition (5.2.1) gives the differential-difference nonlinear Schrödinger equation,

$$i\dot{q}_n = q_{n+1} + q_{n-1} - 2q_n + |q_n|^2(q_{n+1} + q_{n-1}). \tag{5.2.4}$$

We note here that the eigenvalues z are assumed to be invariant ($\dot{z} = 0$). The form of the matrices $L_n(z)$ and $M_n(z)$, eq.(5.2.3), is different from that presented Ablowitz and Ladik[7),8)]. But if we let $L_n'(z) = g^{-1}L_n(z)g,\ M_n'(z) = g^{-1}M_n(z)g$ with

$$g = \begin{bmatrix} z^{-\frac{1}{2}} & 0 \\ 0 & z^{\frac{1}{2}} \end{bmatrix},$$

the matrices $L_n'(z)$ and $M_n'(z)$ coincide with ones of Ablowitz and Ladik.

The gauge transformation for the differential-difference equations (5.2.1) and (5.2.2) is

$$\widetilde{\Phi}n = g_n^{-1}\Phi_n, \tag{5.2.5a}$$

$$\widetilde{L}_n(z) = g_{n+1}^{-1}L_n(z)g_n, \tag{5.2.5b}$$

$$\widetilde{M}_n(z) = g_n^{-1}M_n(z)g_n - g_n^{-1}\dot{g}_n, \tag{5.2.5c}$$

where g_n is an arbitrary matrix. Here we define g_n as a solution of eq.(5.2.2) with eq.(5.2.3) for $z = 1$, that is,

$$g_{n+1} = \begin{bmatrix} 1 & q_n \\ -q_n^* & 1 \end{bmatrix} g_n, \tag{5.2.6a}$$

$$\dot{g}_n = i\begin{bmatrix} -q_n q_{n-1}^* & -q_n + q_{n-1} \\ -q_n^* + q_{n-1}^* & q_n^* q_{n-1} \end{bmatrix} g_n. \tag{5.2.6b}$$

Then it is easy to verify that

$$g_{n+1}^{-1}L_n(z)g_n = \frac{z + z^{-1}}{2}I + \frac{z - z^{-1}}{2}S_n, \tag{5.2.7a}$$

$$g_n^{-1}M_n(z)g_n - g_n^{-1}\dot{g}_n = i\left(1 - \frac{z^2 + z^{-2}}{2}\right)g_{n-1}^{-1}g_n S_n - i\frac{z^2 - z^{-2}}{2}g_{n-1}^{-1}g_n, \tag{5.2.7b}$$

where

$$S_n = g_n^{-1}\sigma^z g_n, \quad S_n^2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.2.8a}$$

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.2.8b}$$

Since the Hermitian conjugate of eq.(5.2.6) yields (see Appendix 5.1)

$$S_n = \prod_{j=n}^{\infty} \left(1 + |q_j|^2\right) g_n^\dagger \sigma^z g_n, \quad S_n^\dagger = S_n, \tag{5.2.9}$$

we may take

$$S_n = \begin{bmatrix} S_n^z & S_n^x - iS_n^y \\ S_n^x + iS_n^y & -S_n^z \end{bmatrix}, \quad (S_n^x)^2 + (S_n^y)^2 + (S_n^z)^2 = 1. \tag{5.2.10}$$

From eqs.(5.2.6a), (5.2.8a) and (5.2.10), we have

$$g_{n-1}^{-1} g_n = 2S_{n-1}(S_n + S_{n-1})^{-1} = \frac{S_{n-1}(S_n + S_{n-1})}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n-1}}, \tag{5.2.11a}$$

$$\overrightarrow{S}_n = (S_n^x,\ S_n^y,\ S_n^z). \tag{5.2.11b}$$

The new matrices $L_n(z)$ and $M_n(z)$ (we omit the tilder on them) now read

$$L_n(z) = \frac{z + z^{-1}}{2} I + \frac{z - z^{-1}}{2} S_n, \tag{5.2.12a}$$

$$M_n(z) = i\left(1 - \frac{z^2 + z^{-2}}{2}\right) \frac{S_n + S_{n-1}}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n-1}} - i\frac{z^2 - z^{-2}}{2} \frac{I + S_{n-1}S_n}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n-1}}. \tag{5.2.12b}$$

The compatibility condition for these matrices gives (see Appendix 5.1)

$$\dot{\overrightarrow{S}}_n = \frac{2}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}} \overrightarrow{S}_n \times \overrightarrow{S}_{n+1} + \frac{2}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n-1}} \overrightarrow{S}_n \times \overrightarrow{S}_{n-1}. \tag{5.2.13}$$

This is the integrable spin model which is the differential-difference analogue of the continuous isotropic Heisenberg spin chain (recently Date et al. presented a difference-difference analogue of the continuous isotropic Heisenberg spin chain using a general method of discretizing soliton equations[11)]). Using eqs.(5.2.6) and (5.2.11), we can find the relation between q_n and $\overrightarrow{S}_n$ (see Appendix 5.1). It follows that

$$1 + |q_n|^2 = \frac{2}{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}}, \tag{5.2.14a}$$

$$i(q_n q_{n-1}^* - q_n^* q_{n-1}) = \frac{2\overrightarrow{S}_n \cdot (\overrightarrow{S}_{n+1} + \overrightarrow{S}_{n-1})}{(1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1})(1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n-1})}. \tag{5.2.14b}$$

If the spins $\overrightarrow{S}_n$ satisfy the following classical equations

$$\dot{\overrightarrow{S}}_n = \{\overrightarrow{S}_n,\ H\}, \tag{5.2.15}$$

with the Poisson bracket $\{A,\ B\}$ defined by[9)]

$$\{A,\ B\} = \epsilon_{\alpha\beta\gamma} \sum_n \frac{\partial A}{\partial S_n^\alpha} \frac{\partial B}{\partial S_n^\beta} S_n^\gamma, \tag{5.2.16}$$

then the Hamiltonian H which generates eq.(5.2.13) is written as

$$H = -2 \sum_n \log(1 + \vec{S}_n \cdot \vec{S}_{n+1}) + \text{const.}. \tag{5.2.17}$$

Unlike the Heisenberg Hamiltonian, eq.(5.2.17) is singular when the nearest-neighbor spins are antiparallel (Fig.5.1).

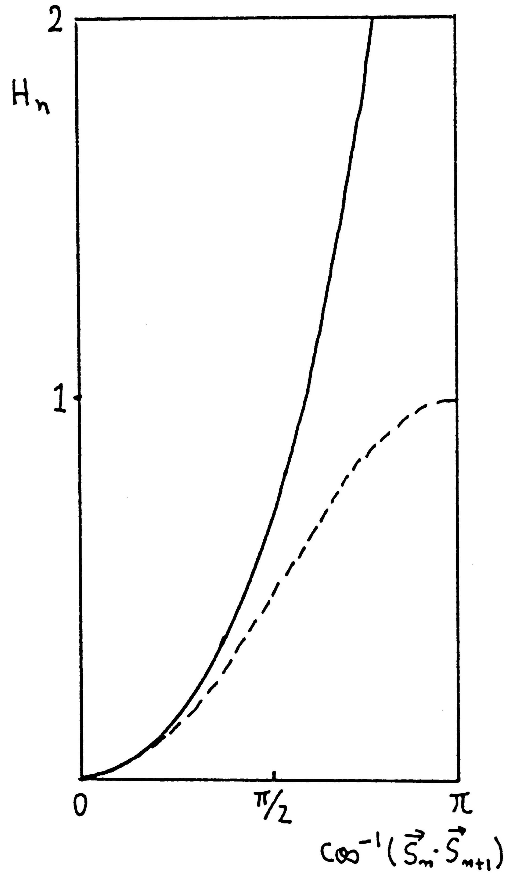

Fig.5.1. The dashed and solid curves represent the energy of interaction H_n between two spins $\vec{S}_n$ and $\vec{S}_{n+1}$ for the Heisenberg model ($H_n = -(\vec{S}_n \cdot \vec{S}_{n+1} - 1)$) and our model ($H_n = -2\log\{(1 + \vec{S}_n \cdot \vec{S}_{n+1})/2\}$), respectively.

The Hamiltonian (5.2.17) does not have the second term in eq.(5.1.1). But this term is not essential because the transformation $\widetilde{S}_n^x + i\widetilde{S}_n^y = (S_n^x + iS_n^y)e^{iht}$ enables us to eliminate the third term in the equation of motion

$$\dot{\vec{S}}_n = \frac{2}{1 + \vec{S}_n \cdot \vec{S}_{n+1}} \vec{S}_n \times \vec{S}_{n+1} + \frac{2}{1 + \vec{S}_n \cdot \vec{S}_{n-1}} \vec{S}_n \times \vec{S}_{n-1} + \vec{S}_n \times \vec{h}, \tag{5.2.18a}$$

$$\vec{h} = (0, 0, h), \tag{5.2.18b}$$

which is derived from the Hamiltonian (5.1.1). For eq.(5.2.18), the matrix $M_n(z)$ must be take the form

$$M_n(z) = i\left(1 - \frac{z^2+z^{-2}}{2}\right)\frac{S_n + S_{n-1}}{1+\overrightarrow{S}_n\cdot\overrightarrow{S}_{n-1}} - i\frac{z^2-z^{-2}}{2}\frac{I+S_{n-1}S_n}{1+\overrightarrow{S}_n\cdot\overrightarrow{S}_{n-1}} + i\frac{h}{2}\sigma^z. \tag{5.2.19}$$

In the following sections, we consider eq.(5.2.18).

5.3 Inverse scattering method

In this section, we study the inverse scattering method for the system (5.2.2) with eqs(5.2.12a) and (5.2.19). By the method, the initial value problem for our spin model is solved. Schematically, we can illustrate this approach by means of a diagram (Fig.5.2).

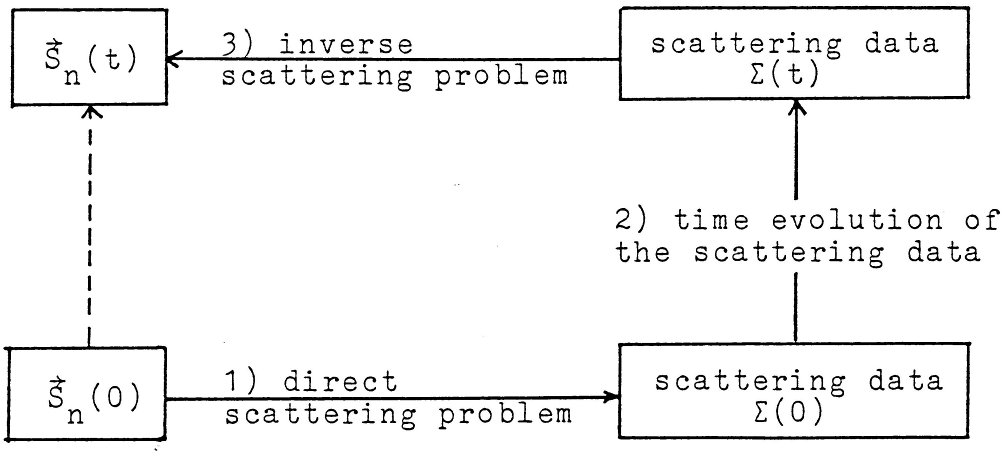

Fig.5.2. Diagram of the inverse scattering method.

Here the dashed line indicates the direct but in general intractable route to the solution. As shown in Fig.5.2, the inverse scattering method consists of the following procedure:

1) Map the initial data $\{\overrightarrow{S}_n(t=0)\}$ into certain scattering data
$\Sigma\{b(z,t=0)/a(z,t=0), z_j(t=0), c_j(t=0); j=1,\cdots,M\}$.
2) Calculate the time evolution of the scattering data
$\Sigma\{b(z,t)/a(z,t), z_j(t), c_j(t); j=1,\cdots,M\}$.
3) Construct $\overrightarrow{S}_n(t)$ from the time-dependent scattering data $\Sigma(t)$
through the Gel'fand-Levitan equation.

These steps are discussed in the following subsections.

5.3.1 Scattering problem

From the boundary condition (5.1.2), we have

$$\left.\begin{array}{l} L_n(z) \to E(z) \\ \\ M_n(z) \to i\Omega(z) \end{array}\right\} \quad \text{for } n \to \pm\infty, \tag{5.3.1}$$

where

$$E(z) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}, \tag{5.3.2a}$$

$$\Omega(z) = \sigma^z \left[I - E^2(z) + \frac{h}{2} I \right]. \tag{5.3.2b}$$

In order to study the problems of direct and inverse scattering, we introduce the eigenfunctions $\phi(n,z)$ defined by

$$\Phi_n = \phi(n,z) \exp[i\Omega(z)t]. \tag{5.3.3}$$

Then, eqs.(5.2.2) become

$$\phi(n+1,z) = L_n(z)\phi(n,z), \tag{5.3.4}$$

$$\dot{\phi}(n,z) = M_n(z)\phi(n,z) - i\phi(n,z)\Omega(z). \tag{5.3.5}$$

Since $\phi(n+1,z) = E(z)\phi(n,z)$ for $n \to \pm\infty$, the solutions of eq.(5.3.4) for $n \to \pm\infty$ have the form $\phi(n,z) = E^n(z)\phi_0$, where ϕ_0 is a matrix which does not depend on n. Here we introduce the following Jost functions $\phi(n,z)$ and $\psi(n,z)$ with boundary conditions:

$$\phi(n,z) \to E^n(z) \quad \text{as } n \to -\infty, \tag{5.3.6}$$

$$\psi(n,z) \to E^n(z) \quad \text{as } n \to +\infty, \tag{5.3.7}$$

which are consistent with eq.(5.3.5). From eqs.(5.3.4), (5.3.6) and (5.3.7), these Jost functions are written as

$$\phi(n,z) = \lim_{N\to\infty} L_{n-1}(z)L_{n-2}(z)\cdots L_{-N}E^{-N}(z), \tag{5.3.8}$$

$$\psi(n,z) = \lim_{N\to\infty} L_n^{-1}(z)L_{n+1}^{-1}(z)\cdots L_N^{-1}E^{N+1}(z), \tag{5.3.9}$$

where

$$L_n^{-1}(z) = \frac{z+z^{-1}}{2} I - \frac{z-z^{-1}}{2} S_n, \tag{5.3.10}$$

and the transition matrix $T(z)$ defined by

$$\phi(n,z) = \psi(n,z)T(z), \tag{5.3.11a}$$

$$T(z) = \begin{bmatrix} a(z) & -\bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix}, \tag{5.3.11b}$$

is expressed as

$$T(z) = \lim_{N\to\infty} E^{-N-1}(z)T_N(z)E^{-N}(z), \tag{5.3.12a}$$

$$T_N(z) = L_N(z)L_{N-1}(z)\cdots L_{-N}(z). \tag{5.3.12b}$$

Since $\det E(z) = \det L_n(z) = 1$, we have

$$\det T(z) = a(z)\bar{a}(z) + b(z)\bar{b}(z) = 1. \tag{5.3.13}$$

We note that $T(z)$ depends parametrically on time through the S_n. The explicit dependence on time is found from eq.(5.3.5) for $n \to \pm\infty$ as

$$\dot{T}(z) = i[\Omega(z), T(z)], \tag{5.3.14}$$

where $[\Omega(z), T(z)] = \Omega(z)T(z) - T(z)\Omega(z)$. In terms of the matrix elements, we have

$$\dot{a}(z) = \dot{\bar{a}}(z) = 0, \tag{5.3.15a}$$

$$\dot{b}(z) = i\omega(z)b(z), \quad \dot{\bar{b}} = -i\omega(z)\bar{b}(z), \tag{5.3.15b}$$

$$\omega(z) = z^2 + z^{-2} - 2 - h. \tag{5.3.15c}$$

Finally, we study the analytic properties of $a(z)$, $\bar{a}(z)$ and the Jost functions. The results we obtain help to derive the Gel'fand-Levitan equation in the next subsection.

From eq.(5.3.8), the columns of $\phi(n, z)$ are written as

$$\begin{aligned}\begin{bmatrix}\phi_{11}\\ \phi_{21}\end{bmatrix} z^{-n} = \lim_{N\to\infty} &\left(\frac{I+S_{n-1}}{2} + \frac{I-S_{n-1}}{2}z^{-2}\right)\left(\frac{I+S_{n-2}}{2} + \frac{I-S_{n-2}}{2}z^{-2}\right) \\ &\cdots\left(\frac{I+S_{-N}}{2} + \frac{I-S_{-N}}{2}z^{-2}\right)\begin{bmatrix}1\\ 0\end{bmatrix},\end{aligned} \tag{5.3.16a}$$

$$\begin{aligned}\begin{bmatrix}\phi_{12}\\ \phi_{22}\end{bmatrix} z^{n} = \lim_{N\to\infty} &\left(\frac{I-S_{n-1}}{2} + \frac{I+S_{n-1}}{2}z^{2}\right)\left(\frac{I-S_{n-2}}{2} + \frac{I+S_{n-2}}{2}z^{2}\right) \\ &\cdots\left(\frac{I-S_{-N}}{2} + \frac{I+S_{-N}}{2}z^{2}\right)\begin{bmatrix}0\\ 1\end{bmatrix}.\end{aligned} \tag{5.3.16b}$$

These equations show that for the $S_n - \sigma^z$ decaying sufficiently rapidly as $n \to \pm\infty$, the columns $(\phi_{11}, \phi_{21})z^{-n}$ and $(\phi_{12}, \phi_{22})z^n$ are analytic for $|z| > 1$ and $|z| < 1$, respectively. Similar analysis is possible for $\psi(n, z)$, that is, the column $(\psi_{12}, \psi_{22})z^n$ is analytic for $|z| > 1$ and the column $(\psi_{11}, \psi_{21})z^{-n}$ is analytic for $|z| < 1$. In terms of the Jost functions, $a(z)$ and $\bar{a}(z)$ are expressed as

$$a(z) = (\phi_{11}z^{-n})(\psi_{22}z^n) - (\phi_{21}z^{-n})(\psi_{12}z^n), \tag{5.3.17a}$$

$$\bar{a}(z) = (\phi_{22}z^{n})(\psi_{11}z^{-n}) - (\phi_{12}z^{n})(\psi_{21}z^{-n}), \tag{5.3.17b}$$

and we infer from the analytic properties of $\phi(n, z)$ and $\psi(n, z)$ that $a(z)$ and $\bar{a}(z)$ are analytic for $|z| > 1$ and $|z| < 1$, respectively.

5.3.2 Gel'fand-Levitan equation

In this subsection, we derive the Del'fand-Levitan equation for our system.

For $\psi(n, z)$, from eq.(5.3.9), the following triangular representation is suggested:

$$\psi(n, z) = E^n(z) + \sum_{n'=n}^{\infty} K(n, n')E^{n'}(z)[I - E^2(z)], \tag{5.3.18}$$

where the matrix kernel $K(n, n')$ depends functionally on the S_n but is independent of the eigenvalue z. By virtue of the above representation one can derive the linear summation equation (i.e. a discrete analogue of the Gel'fand-Levitan equation). For this purpose, it is convenient to rewrite eq.(5.3.11) as

$$\frac{1}{a(z)}\begin{bmatrix}\phi_{11}\\ \phi_{21}\end{bmatrix} = \begin{bmatrix}\psi_{11}\\ \psi_{21}\end{bmatrix} + \frac{b(z)}{a(z)}\begin{bmatrix}\psi_{12}\\ \psi_{22}\end{bmatrix}, \tag{5.3.19a}$$

$$\frac{1}{\bar{a}(z)}\begin{bmatrix}\phi_{12}\\ \phi_{22}\end{bmatrix} = -\frac{\bar{b}(z)}{\bar{a}(z)}\begin{bmatrix}\psi_{11}\\ \psi_{21}\end{bmatrix} + \begin{bmatrix}\psi_{12}\\ \psi_{22}\end{bmatrix}. \tag{5.3.19b}$$

Substitute eq.(5.3.18) into eq.(5.3.19a) and operate with

$$\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz \frac{z^{-m-3}}{1-z^{-2}} \quad (m \geqq n), \tag{5.3.20}$$

to find

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz \frac{z^{-n}}{a(z)}\begin{bmatrix}\phi_{11}(n,z)\\ \phi_{21}(n,z)\end{bmatrix}\frac{z^{n-m-3}}{1-z^{-2}} \\
&= -\begin{bmatrix}K_{11}(n,m)\\ K_{21}(n,m)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz \frac{b(z)}{a(z)}\frac{z^{-n-m-3}}{1-z^{-2}} \\
&+ \sum_{n'=n}^{\infty}\begin{bmatrix}K_{12}(n,n')\\ K_{22}(n,n')\end{bmatrix}\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz \frac{b(z)}{a(z)} z^{-n'-m-3}.
\end{aligned} \tag{5.3.21}$$

Here we have used the identities

$$\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz\ z^{n'-m-1} = \delta_{n'm}, \tag{5.3.22a}$$

$$\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz \frac{z^{-n-m-3}}{1-z^{2}} = 0 \quad \text{for } m \geqq n, \tag{5.3.22b}$$

where $\oint_{|z|=1+\epsilon}$ is the contour integral on the circle $|z| = 1+\epsilon$ ($\epsilon \to +0$) and $\delta_{n'm}$ is the Kronecker delta function. We now evaluate the left hand side of eq.(5.3.21). Since the column $(\phi_{11}, \phi_{21})z^{-n}$ and $a(z)$ are analytic in the region $|z| > 1$, this integral is decomposed into the contour integral on the circle $|z| = \infty$ and the residues at poles of $1/a(z)$ (zeroes of $a(z)$) (Fig.5.3).

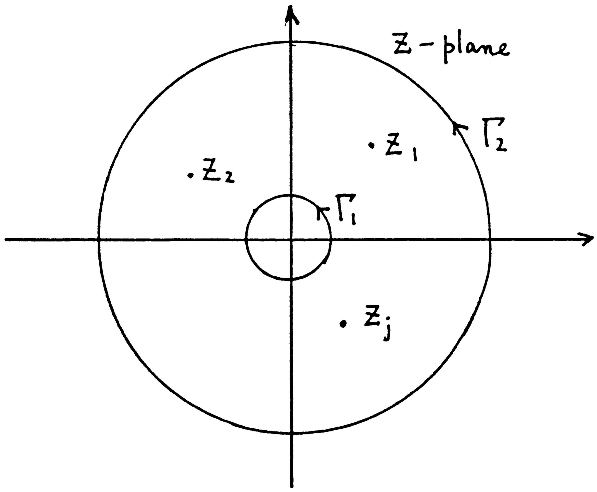

Fig.5.3. Path of the integrations. Γ_1 and Γ_2 denote the circles $|z| = 1+\epsilon$ and $|z| = \infty$, respectively. z_j are zeroes of $a(z)$.

Noticing that

$$\frac{1}{2\pi i}\oint_{|z|=\infty} dz \frac{z^{-n}}{a(z)}\begin{bmatrix}\phi_{11}(n,z)\\ \phi_{21}(n,z)\end{bmatrix}\frac{z^{n-m-3}}{1-z^{-2}} = \delta_{nm+2}\lim_{z\to\infty}\frac{z^{-n}}{a(z)}\begin{bmatrix}\phi_{11}(n,z)\\ \phi_{21}(n,z)\end{bmatrix} = 0, \tag{5.3.23}$$

and assuming $a(z)$ has $2M$ simple zeroes, $a(z_j)=0$, we have

$$\begin{aligned}\text{L.H.S. of eq.(5.3.21)} &= -\sum_{j=1}^{2M}\frac{1}{a'(z_j)}\begin{bmatrix}\phi_{11}(n,z_j)\\ \phi_{21}(n,z_j)\end{bmatrix}\frac{z_j^{-m-3}}{1-z_j^{-2}}\\ &= -\begin{bmatrix}0\\1\end{bmatrix}\sum_{j=1}^{2M}c_j\frac{z_j^{-n-m-3}}{1-z_j^{-2}}-\sum_{n'=n}^{\infty}\begin{bmatrix}K_{12}(n,n')\\ K_{22}(n,n')\end{bmatrix}\sum_{j=1}^{2M}c_j z_j^{-n'-m-3},\end{aligned} \tag{5.3.24}$$

where prime on $a(z)$ indicates the differentiation with respect to z and c_j is given by

$$c_j=\frac{b(z_j)}{a'(z_j)}. \tag{5.3.25}$$

Equation (5.3.21) now reads

$$\begin{bmatrix}K_{11}(n,m)\\ K_{21}(n,m)\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}f(n+m)-\sum_{n'=n}^{\infty}\begin{bmatrix}K_{12}(n,n')\\ K_{22}(n,n')\end{bmatrix}g(n'+m)=0, \tag{5.3.26}$$

where

$$f(n+m)=\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}dz\frac{b(z)}{a(z)}\frac{z^{-n-m-3}}{1-z^{-2}}+\sum_{j=1}^{2M}c_j\frac{z_j^{-n-m-3}}{1-z_j^{-2}}, \tag{5.3.27a}$$

$$g(n'+m)=\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}dz\frac{b(z)}{a(z)}z^{-n'-m-3}+\sum_{j=1}^{2M}c_j z_j^{-n'-m-3}. \tag{5.3.27b}$$

In a similar manner, another set of summation equations can be derived from eq.(5.3.19b). Thus, we obtain the Gel'fand-Levitan equation for the matrix kernel $K(n,m)$,

$$K(n,m)+F(n+m)+\sum_{n'=n}^{\infty}K(n,n')G(n'+m)=0, \tag{5.3.28}$$

where $F(n+m)$ and $G(n'+m)$ are given by

$$F(n+m)=\begin{bmatrix}0 & \bar{f}(n+m)\\ -f(n+m) & 0\end{bmatrix}, \tag{5.3.29a}$$

$$G(n'+m)=\begin{bmatrix}0 & \bar{g}(n'+m)\\ -g(n'+m) & 0\end{bmatrix}, \tag{5.3.29b}$$

with

$$\bar{f}(n+m)=\frac{1}{2\pi i}\oint_{|z|=1-\epsilon}dz\frac{\bar{b}(z)}{\bar{a}(z)}\frac{z^{n+m+1}}{1-z^{2}}-\sum_{j=1}^{2M}\bar{c}_j\frac{\bar{z}_j^{n+m+1}}{1-\bar{z}_j^{2}}, \tag{5.3.30a}$$

$$\bar{g}(n'+m)=\frac{1}{2\pi i}\oint_{|z|=1-\epsilon}dz\frac{\bar{b}(z)}{\bar{a}(z)}z^{n'+m+1}-\sum_{j=1}^{2M}\bar{c}_j\bar{z}_j^{n'+m+1}, \tag{5.3.30b}$$

$$\bar{c}_j=\frac{\bar{b}(\bar{z}_j)}{\bar{a}'(\bar{z}_j)},\quad \bar{a}(\bar{z}_j)=0. \tag{5.3.30c}$$

We must notice that eqs.(5.3.9) and (5.3.12) give the symmetry properties as

$$\psi(n,z) = \sigma^y \psi^*(n, z^{*-1})\sigma^y, \tag{5.3.31a}$$

$$T(z) = \sigma^y T^*(z^{*-1})\sigma^y, \tag{5.3.31b}$$

from which we have relations

$$\bar{a}(z) = a^*(z^{*-1}), \quad \bar{b}(z) = b^*(z^{*-1}), \tag{5.3.32a}$$

$$\bar{z}_j = z_j^{*-1}, \quad \bar{c}_j = -z_j^{*-2} c_j^*, \tag{5.3.32b}$$

$$\bar{f}(n) = f*(n), \quad \bar{g}(n) = g^*(n), \tag{5.3.32c}$$

and

$$K_{12}(n,m) = -K_{21}^*(n,m), \quad K_{21}(n,m) = K_{11}^*(n,m). \tag{5.3.32d}$$

Furthermore, since eq.(5.3.10) show that $a(z)$ and $b(z)$ are even in z, the zeroes of $a(z)$ comes in $\pm$ pairs (z_j^+ and z_j^-), and $c_j(z_j^+) = -c_j(z_j^-)$. Considering these properties, the Gel'fand-Levitan equation is rewritten as

$$K_{11}(n,m) + 2\sum_{n'=n}^{\infty} K_{21}^*(n,n')\tilde{g}(n'+m) = 0, \tag{5.3.33a}$$

$$K_{21}(n,m) - 2\tilde{f}(n+m) - 2\sum_{n'=n}^{\infty} K_{11}^*(n,n')\tilde{g}(n'+m) = 0, \tag{5.3.33b}$$

with

$$\tilde{f}(n+m) = \frac{1}{2\pi i}\int_{\tilde{C}} dz \frac{b(z)}{a(z)} \frac{z^{-n-m-3}}{1-z^{-2}} + \sum_{j=1}^{M} c_j \frac{z_j^{-n-m-3}}{1-z_j^{-2}}, \tag{5.3.34a}$$

$$\tilde{g}(n'+m) = \frac{1}{2\pi i}\int_{\tilde{C}} dz \frac{b(z)}{a(z)} z^{-n'-m-3} + \sum_{j=1}^{M} c_j z_j^{-n'-m-3}, \tag{5.3.34b}$$

$$m = n+2s, \quad n' = n+2s', \tag{5.3.34c}$$

$$s = 0,1,2,\cdots, \quad s' = 0,1,2,\cdots, \tag{5.3.34d}$$

where $\tilde{C}$ is the contour along the upper half of the circle $|z| = 1+\epsilon$ and $\mathrm{Im} z_j > 0$.

To close the subsection, we relate $K(n,m)$ to S_n. Substituting eq.(5.3.18) into the eigenvalue problem (5.3.4) and comparing the coefficients of $E^n(z)$, one finds that the kernel $K(n,m)$ satisfies a partial difference equation

$$\begin{aligned}&[K(n,m) + K(n,m+2) - 2K(n+1,m+1)]\sigma^z \\ &+ S_n[K(n,m) - K(n,m+2)] = 0 \quad \text{for } m \geqq n,\end{aligned} \tag{5.3.35}$$

with the boundary condition

$$S_n = [I + K(n,n)]\sigma^z [I + K(n,n)]^{-1}. \tag{5.3.36}$$

In terms of matrix elements, eq.(5.3.36) is expressed as

$$S_n^z = \frac{|1+K_{11}(n,n)|^2 - |K_{21}(n,n)|^2}{|1+K_{11}(n,n)|^2 + |K_{21}(n,n)|^2}, \tag{5.3.37a}$$

$$S_n^x + iS_n^y = \frac{2K_{21}(n,n)[1+K_{11}^*(n,n)}{|1+K_{11}(n,n)|^2 + |K_{21}(n,n)|^2}, \tag{5.3.37b}$$

which is the stereographic transformation.

5.3.3 Initial value problem

Now we are in a position to perform the procedure in Fig.5.2. Steps 1)-3) are carried out with use of the following equations;

1) Equation (5.3.12) and $a(z_j) = 0$.
2) Equation (5.3.15) and $\dot{z} = 0$.
3) Equation (5.3.33) with eqs.(5.3.34) and (5.3.25) and eq.(5.3.37).

Particularly, when $b(z,t) = 0$ for $|z| = 1 + \epsilon$, the Gel'fand-Levitan equation is reduced to a system of linear algebraic equations and can be solved explicitly. With use of this property, we can construct the multi-soliton solution which is an exact solution of eq.(5.2.18). Here we consider the case that $a(z)$ has only one simple zero $z_1 = e^{\alpha+i\beta}$ with $0 < \alpha$ and $0 < \beta < \pi$. Then, eq.(5.3.34) becomes

$$\tilde{f}(n+m) = \frac{c_1 z_1^{-n-m-3}}{1-z_1^{-2}}, \tag{5.3.38a}$$

$$\tilde{g}(n'+m) = c_1 z_1^{-n'-m-3}, \tag{5.3.38b}$$

and the Gel'fand-Levitan equation (5.3.33) reduces to

$$K_{11}(n,m) + 2c_1 \sum_{n'=n}^{\infty} K_{21}^*(n,n') z_1^{-n'-m-3} = 0, \tag{5.3.39a}$$

$$K_{21}(n,m) - \frac{2c_1 z_1^{-n-m-3}}{1-z_1^{-2}} - 2c_1 \sum_{n'=n}^{\infty} K_{11}^*(n,n') z_1^{-n'-m-3} = 0, \tag{5.3.39b}$$

where from eqs.(5.3.15) and (5.3.25), c_1 is given by

$$c_1 = c_1(t=0)\exp[-(2\sinh 2\alpha \sin 2\beta)t - 2i(1-\cosh 2\alpha\cos 2\beta + h)t]. \tag{5.3.40}$$

It is easy to find that

$$K_{11}(n,n) = -\frac{1-e^{-4\alpha}}{2(1-e^{-2\alpha-2i\beta})}\mathrm{sech}\xi_n e^{-\xi_n}, \tag{5.3.41a}$$

$$K_{21}(n,n) = \frac{1-e^{-4\alpha}}{2(1-e^{-2\alpha-2i\beta})}\mathrm{sech}\xi_n e^{-i\eta_n}, \tag{5.3.41b}$$

where

$$\xi_n = 2\alpha n + 2(\sinh 2\alpha \sin 2\beta)t + \xi^{(0)}, \tag{5.3.42a}$$

$$\eta_n = 2\beta n + 2(1-\cosh 2\alpha\cos 2\beta + h)t + \eta^{(0)}, \tag{5.3.42b}$$

$$\xi^{(0)} = \alpha + \log \frac{\sinh 2\alpha}{|c_1(t=0)|}, \tag{5.3.42c}$$

$$\eta^{(0)} = 3\beta + \arg c_1^*(t=0). \tag{5.3.42d}$$

Substituting eq.(5.3.41) into eq.(5.3.37), we obtain a one-soliton solution

$$S_n^z = 1 - \frac{\sinh^2 2\alpha}{\cosh 2\alpha - \cos 2\beta} \mathrm{sech}\xi_{n+1} \mathrm{sech}\xi_n, \tag{5.3.43a}$$

$$S_n^x + iS_n^y = \frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta} \mathrm{sech}\xi_{n+1} (\cosh 2\alpha - e^{-2i\beta} + \sinh 2\alpha \tanh \xi_n) e^{-i\eta_n}. \tag{5.3.43b}$$

From eqs.(5.1.1) and (5.3.43), the energy of a soliton is given by

$$E_s = 8\alpha + h \frac{2\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta}. \tag{5.3.44}$$

As an example of the initial data, we consider the special case

$$\overrightarrow{S}_n(t=0) = \begin{cases} (\sin\theta\cos\delta, \sin\theta\sin\delta, \cos\theta) & \text{for } n = 0 \\ 0 & \text{for } n \neq 0, \end{cases} \tag{5.3.45}$$

with $0 < \theta < \pi$ and $0 \leqq \delta < 2\pi$. The transition matrix (5.3.12) at time $t = 0$ is immediately obtained as $E^{-1}(z)L_0(z)$, namely

$$a(z) = \frac{1}{2}[1 - \cos\theta + (1 + \cos\theta)z^2], \tag{5.3.46a}$$

$$b(z) = \frac{1}{2} \sin\theta e^{i\delta}(1 - z^{-2}). \tag{5.3.46b}$$

The zero of $a(z)$ is calculated from eq.(5.3.46a) as

$$z_1 = \pm i \tan \frac{\theta}{2}. \tag{5.3.47}$$

From the conditions $|z_1| > 1$ and Im $z_1 > 0$, we see that a single soliton emerges from such initial data when $\frac{\pi}{2} < \theta < \pi$.

5.4 Canonical action angle variables

In this section, we show that the inverse scattering transform can be interpreted as a canonical transformation from the spin variable to the scattering data. Our procedure follows closely that of Fogedby[2)] for the continuous isotropic Heisenberg spin chain.

5.4.1 Poisson bracket relations

The canonical variables $(q_n,\ p_n)$ for our spin model are given by

$$q_n = \tan^{-1} \frac{S_n^y}{S_n^x}, \tag{5.4.1a}$$

$$p_n = S_n^z, \tag{5.4.1b}$$

because q_n and p_n satisfy the Poisson bracket

$$\{q_n, p_m\} = \delta_{nm}, \tag{5.4.2a}$$

$$\{q_n, q_m\} = \{p_n, p_m\} = 0. \tag{5.4.2b}$$

In order to present new canonical variables constructed by the scattering data, we must derive the Poisson brackets for the scattering data. First we show that the following Poisson brackets for the elements of transition matrix are expressed as eq.(4.5.8):

$$\{T_{ij}(z), T_{kl}(z')\} = \epsilon_{\alpha\beta\gamma} \sum_n \frac{\partial T_{ij}(z)}{\partial S_n^\alpha} \frac{\partial T_{kl}(z')}{\partial S_n^\beta} S_n^\gamma. \tag{5.4.3}$$

The definition of the transition matrix $T(z)$ yields

$$\frac{\delta T(z)}{\delta S_n^\alpha} = \psi^{-1}(n', z) \frac{\delta \phi(n', z)}{\delta S_n^\alpha} - \psi^{-1}(n', z) \frac{\delta \psi(n', z)}{\delta S_n^\alpha} T(z). \tag{5.4.4}$$

Since we have from eqs.(5.3.4) and (5.3.9)

$$\frac{\delta \psi(n+1, z)}{\delta S_n^\alpha} = 0, \tag{5.4.5a}$$

$$\frac{\delta \phi(n+1, z)}{\delta S_n^\alpha} = \frac{z - z^{-1}}{2} \sigma^\alpha \phi(n, z), \tag{5.4.5b}$$

the choice $n' = n + 1$ reduces eq.(5.4.4) to

$$\frac{\delta T(z)}{\delta S_n^\alpha} = \frac{z - z^{-1}}{2} \psi^{-1}(n+1, z) \sigma^\alpha \phi(n, z). \tag{5.4.6}$$

Using eq.(5.3.4) and $[\sigma^\alpha,\ \sigma^\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma^\gamma$, we can derive the identity

$$\begin{aligned}
&\sum_\alpha [\psi^{-1}(n+1, z)\sigma^\alpha \phi(n+1, z)]_{ij} [\psi^{-1}(n+1, z')\sigma^\alpha \phi(n+1, z')]_{kl} \\
&- \sum_\alpha [\psi^{-1}(n, z)\sigma^\alpha \phi(n, z)]_{ij} [\psi^{-1}(n, z')\sigma^\alpha \phi(n, z')]_{kl} \\
&= i(zz'^{-1} - z^{-1}z') \\
&\times \epsilon_{\alpha\beta\gamma} [\psi^{-1}(n+1, z)\sigma^\alpha \phi(n, z)]_{ij} [\psi^{-1}(n+1, z')\sigma^\beta \phi(n, z')]_{kl} S_n^\gamma.
\end{aligned} \tag{5.4.7}$$

From eqs.(5.4.6) and (5.4.7), the Poisson bracket (5.4.3) is written as

$$\begin{aligned}
&\{T_{ij}(z), T_{kl}(z')\} = \frac{(z - z^{-1})(z' - z'^{-1})}{4(zz'^{-1} - z^{-1}z'} \\
&\times \lim_{N\to\infty} \sum_\alpha \big([E^{-N}(z)\sigma^\alpha E^N(z)T(z)]_{ij} [E^{-N}(z')\sigma^\alpha E^N(z')T(z')]_{kl} \\
&\qquad - [T(z)E^N(z)\sigma^\alpha E^{-N}(z)]_{ij} [T(z')E^N(z')\sigma^\alpha E^{-N}(z')]_{kl} \big),
\end{aligned} \tag{5.4.8}$$

where we have used the boundary conditions for Jost functions and the definition of the transition matrix.

If we take $z = e^{\epsilon+ik}$ with $0 < k < \pi$, the non-vanishing Poisson brackets for the matrix elements $a(e^{ik})$ and $b(e^{ik})$ are given by

$$\{a(e^{ik}), b(e^{ik'})\} = -\sin k \sin k' a(e^{ik})b(e^{ik'}) \left[\mathcal{P}\frac{1}{\sin(k-k')} + i\pi\delta(k-k')\right], \tag{5.4.9a}$$

$$\{a(e^{ik}), b^*(e^{ik'})\} = \sin k \sin k' a(e^{ik})b^*(e^{ik'}) \left[\mathcal{P}\frac{1}{\sin(k-k')} + i\pi\delta(k-k')\right], \tag{5.4.9b}$$

$$\{b(e^{ik}), b^*(e^{ik'})\} = -2i\pi(\sin k)^2 |a(e^{ik})|^2 \delta(k-k'), \tag{5.4.9c}$$

where we have used the identities

$$\frac{1}{\sin(k-k'-i\epsilon)} = \mathcal{P}\frac{1}{\sin(k-k')} + i\pi\delta(k-k'), \tag{5.4.10a}$$

$$\lim_{N\to\infty} \frac{\sin[N(k-k')]}{\sin(k-k')} = \pi\delta(k-k'). \tag{5.4.10b}$$

To compute differentiation for the eigenvalue z_j, we make use of the implicit equation $a(z_j(S_n), S_n) = 0$. By differentiation with respect to S_n^α we obtain

$$\frac{\delta z_j}{\delta S_n^\alpha} = -\frac{1}{a'(z_j)} \left(\frac{\delta a(z)}{\delta S_n^\alpha}\right)_{z=z_j}. \tag{5.4.11}$$

Noticing that

$$\frac{\delta b(z_j)}{\delta S_n^\alpha} = \frac{\delta z_j}{\delta S_n^\alpha} b'(z_j) + \left(\frac{\delta b(z)}{\delta S_n^\alpha}\right)_{z=z_j} = -\frac{b'(z_j)}{a'(z_j)} \left(\frac{\delta a(z)}{\delta S_n^\alpha}\right)_{z=z_j} + \left(\frac{\delta b(z)}{\delta S_n^\alpha}\right)_{z=z_j}, \tag{5.4.12}$$

we have

$$\{z_i, b(z_j)\} = -\frac{1}{a'(z_i)}\{a(z), b(z')\}_{z=z_i, z'=z_j} + \frac{b'(z_j)}{a'(z_i)a'(z_j)}\{a(z), a(z')\}_{z=z_i, z'=z_j}. \tag{5.4.13}$$

Substitution eq.(5.4.8) into eq.(5.4.13) gives

$$\{z_i, b(z_j)\} = -\frac{i}{4}(z_i - z_i^{-1})^2 z_i b(z_i)\delta_{ij}. \tag{5.4.14}$$

In the same manner, it is verified that the Poisson brackets for other pairs vanish.

Now we are in a position to construct new canonical variables for our spin chain. The variables associated with the continuous scattering data are given by

$$Q(k) = -\arg b(e^{ik}), \tag{5.4.15a}$$

$$P(k) = -\frac{1}{\pi \sin^2 k} \log |a(e^{ik})|, \tag{5.4.15b}$$

and satisfy the Poisson brackets

$$\{Q(k), P(k')\} = \delta(k-k'), \tag{5.4.16a}$$

$$\{Q(k), Q(k')\} = \{P(k), P(k')\} = 0. \tag{5.4.16b}$$

For the discrete variables, we define the following relations:

$$Q_j = -i \log b(z_j), \tag{5.4.17a}$$

$$P_j = \frac{z_j + z_j^{-1}}{z_j - z_j^{-1}}, \tag{5.4.17b}$$

which obey the Poisson brackets

$$\{Q_i, P_j\} = \delta_{ij}, \tag{5.4.18a}$$

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0. \tag{5.4.18b}$$

Note that Q_j and P_j are complex variables. The transformation $(q_n,\ p_n) \to (Q(k),\ P(k);\ Q_j,\ P_j)$ is a canonical transformation because it has the property of preserving the Poisson brackets.

The time dependence of the new canonical variables are determined from eq.(5.3.15). The result is

$$\dot{Q}(k) = 2(1 - \cos 2k) + h, \tag{5.4.19a}$$

$$\dot{Q}_j = -\frac{4}{P_j^2 - 1} + h, \tag{5.4.19b}$$

$$\dot{P}(k) = \dot{P}_j = 0. \tag{5.4.19c}$$

Thus, the canonical variables $Q(k),\ P(k),\ Q_j$ and P_j are of the action angle type[10)], and our spin model is the completely integrable Hamiltonian system. The real Hamiltonian which generates eq.(5.4.19) is given by

$$H = \int_0^{\pi} \omega(k) P(k) dk + \sum_{j=1}^{M} 4 \log \left| \frac{P_j + 1}{P_j - 1} \right| + h \sum_{j=1}^{M} (P_j + P_j^*), \tag{5.4.20}$$

where

$$\omega(k) = 2(1 - \cos 2k) + h. \tag{5.4.21}$$

In the next subsection, we demonstrate that the Hamiltonian (5.4.20) coincides with the original Hamiltonian (5.1.1).

We finally comment on the discrete (soliton) part of the Hamiltonian (5.4.20), denoted by H_s. Taking $z_j = e^{\alpha_j + i\beta_j}$, we have from eqs.(5.4.17b) and (5.4.20)

$$H_s = \sum_{j=1}^{M} \left(8\alpha_j + h \frac{2 \sinh 2\alpha_j}{\cosh 2\alpha_j - \cos 2\beta_j} \right). \tag{5.4.22}$$

When $h = 0$, the energy depends only on α_j, and it is possible to excite the different soliton modes (different β_j) having the same energy (same α_j). Thus we may say that for the soliton modes a kind of degeneracy occurs. However, if an external field h is applied, the different modes have different energies and the degeneracy is resolved.

5.4.2 Conservation laws

As well known, the nonlinear wave equations solvable by the inverse scattering method have an infinite number of constants of motion. These quantities are derived from asymptotic expansions of diagonal elements of the transition matrix. In our case, this technique is applied to $\bar{a}(z)$ defined by eq.(5.3.11).

We consider the asymptotic expansions of $\log \bar{a}$ in powers of z^2 when $z \to 0$ and in powers of $z^2 - 1$ when $z \to 1$, that is,

$$\log \bar{a}(z) = \begin{cases} \sum_{j=0}^{\infty} C_j z^{2j} & \text{for } z \to 0, \\ \\ \sum_{j=0}^{\infty} D_j (z^2 - 1)^j & \text{for } z \to 1. \end{cases} \tag{5.4.23}$$

Since $\bar{a}(z)$ is time independent, the $\{C_j\}$ and $\{D_j\}$ are constants of motion. They are expressed in terms of both the spin variables and action angle variables.

We begin by deriving the following identity:

$$\begin{aligned} \log \bar{a}(z) = i\text{arg } \bar{a}(0) + \int_0^{\pi} \sin^2 k P(k) \frac{z^2 + e^{2ik}}{z^2 - e^{2ik}} dk \\ + \sum_{j=1}^{M} \log \left| \frac{P_j + 1}{P_j - 1} \right| + \sum_{j=1}^{M} \log \frac{z^2 - \dfrac{P_j^* - 1}{P_j^* + 1}}{z^2 - \dfrac{P_j + 1}{P_j - 1}}, \end{aligned} \tag{5.4.24}$$

which holds for $|z| < 1$. From the analytic properties of $\bar{a}(z)$, the function

$$\log \left[\bar{a}(z) \prod_{j=1}^{M} |\bar{z}_j|^4 \left(\frac{z^2 - \bar{z}_j^{*-2}}{z^2 - \bar{z}_j^2} \right) \right], \tag{5.4.25}$$

is analytic for $|z| < 1$. With use of Poisson integral formula we have from eq.(5.4.25)

$$\log \bar{a}(z) = i\text{arg } \bar{a}(0) + \frac{1}{2\pi} \int_0^{2\pi} \log |\bar{a}(e^{ik})| \frac{e^{ik} + z}{e^{ik} - z} dk + \sum_{j=1}^{M} \log \left[\frac{1}{|\bar{z}_j|^2} \left(\frac{z^2 - \bar{z}_j^2}{z^2 - \bar{z}_j^{*-2}} \right) \right]. \tag{5.4.26}$$

By means of the definition of the canonical variables, eqs.(5.4.15) and (5.4.17), we obtain eq.(5.4.24). From the asymptotic expansion of eq.(5.4.24), we can easily find the $\{C_j\}$ and $\{D_j\}$. The first two C_j are

$$C_0 = -\int_0^{\pi} \sin^2 k P(k) dk - \sum_{j=1}^{M} \log \left| \frac{P_j + 1}{P_j - 1} \right| + i\text{arg } \bar{a}(0), \tag{5.4.27a}$$

$$C_1 = -2 \int_0^{\pi} \sin^2 k e^{-2ik} P(k) dk + \sum_{j=1}^{M} \left(\frac{P_j - 1}{P_j + 1} - \frac{P_j^* + 1}{P_j^* - 1} \right), \tag{5.4.27b}$$

and the first three D_j are

$$D_0 = i \int_0^{\pi} \sin k \cos k P(k) dk + i \sum_{j=1}^{M} \text{arg } \frac{1 - P_j}{1 + P_j^*} + i\text{arg } \bar{a}(0), \tag{5.4.28a}$$

$$D_1 = \frac{1}{2}\int_0^\pi P(k)dk + \frac{1}{2}\sum_{j=1}^{M}(P_j + P_j^*), \tag{5.4.28b}$$

$$D_2 = -\frac{i}{4}\int_0^\pi \cot kP(k)dk + \frac{1}{8}\sum_{j=1}^{M}(P_j^2 - P_j^{*2}) - \frac{1}{2}D_1. \tag{5.4.28c}$$

The unknown constant $\bar{a}(0)$ can be determined by the fact that from eq.(5.3.12) we have $\bar{a}(z=1)=1$, i.e. $\log\bar{a}(z=1) = D_0 = 0$. The result is

$$\bar{a}(0) = -\int_0^\pi \sin k \cos kP(k)dk - \sum_{j=1}^{M}\arg\frac{1-P_j}{1+P_j^*}. \tag{5.4.29}$$

In order to relate $\log\bar{a}(z)$ to spin variables, we adopt recursive techniques. There are two methods to determine the $\{C_j\}$ and $\{D_j\}$, one of which is useful for $\{C_j\}$, the other for the $\{D_j\}$. We first calculate the $\{C_j\}$. From eqs.(5.3.7) and (5.3.15b), we have a relation

$$\log\bar{a}(z) = \lim_{n\to\infty}\log z^n\phi_{22}(n,z). \tag{5.4.30}$$

The eigenvalue problem (5.3.4) reduces to a difference equation for the expression $z^n\phi_{22}(n,z)$, which is

$$\begin{aligned}
&2S_n^-[z^{n+2}\phi_{22}(n+2,z)]\\
&-[(S_n^- + S_{n+1}^-)(z^2+1) + (S_{n+1}^-S_n^z - S_n^-S_{n+1}^z)(z^2-1)][z^{n+1}\phi_{22}(n+1,z)]\\
&+2z^2S_{n+1}^-[z^n\phi_{22}(n,z)] = 0,
\end{aligned} \tag{5.4.31}$$

where $S_n^- = S_n^x - iS_n^y$. Define

$$z^n\phi_{22}(n,z) = \prod_{l=-\infty}^{n} f_l, \tag{5.4.32}$$

to rewrite eq.(5.4.31) as

$$\begin{aligned}
&2S_n^- f_{n+2}f_{n+1} - [(S_n^- + S_{n+1}^-)(z^2+1) + (S_{n+1}^-S_n^z - S_n^-S_{n+1}^z)(z^2-1)f_{n+1}\\
&+2z^2S_{n+1}^- f_n = 0.
\end{aligned} \tag{5.4.33}$$

With help of a recursive manner, we verify that f_n has the expansion

$$f_n = f_n^{(0)} + z^2 f_n^{(1)} + \cdots, \tag{5.4.34a}$$

$$f_n^{(0)} = \frac{S_{n-1}^- + S_{n-2}^- + S_{n-1}^zS_{n-2}^- - S_{n-1}^-S_{n-2}^z}{2S_{n-2}^-}, \tag{5.4.34b}$$

$$f_n^{(1)} = \frac{S_{n-1}^- + S_{n-2}^- + S_{n-2}^zS_{n-1}^- - S_{n-2}^-S_{n-1}^z}{2S_{n-2}^-} - \frac{S_{n-3}^- + S_{n-4}^- + S_{n-3}^zS_{n-4}^- - S_{n-3}^-S_{n-4}^z}{S_{n-3}^- + S_{n-2}^- + S_{n-3}^-S_{n-2}^z - S_{n-2}^-S_{n-3}^z}. \tag{5.4.34c}$$

From eqs.(5.4.30), (5.4.32) and (5.4.34), C_0 and C_1 are given by $C_0 = \sum_{n=-\infty}^{\infty} \log f_n^{(0)}$ and $C_1 = \sum_{n=-\infty}^{\infty} \frac{f_n^{(1)}}{f_n^{(0)}}$, respectively. Particularly, Re C_0 is written as

$$\mathrm{Re}\, C_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \log \frac{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}}{2}. \tag{5.4.35}$$

Comparing eq.(5.4.35) with eq.(5.4.27a), we obtain

$$4 \int_0^{\pi} \sin^2 k P(k) dk + 4 \sum_{j=1}^{M} \log \left| \frac{P_j + 1}{P_j - 1} \right| = -2 \sum_{n=-\infty}^{\infty} \log \frac{1 + \overrightarrow{S}_n \cdot \overrightarrow{S}_{n+1}}{2}. \tag{5.4.36}$$

We next calculate the $\{D_j\}$. The boundary condition (5.3.6) gives

$$\log \bar{a}(z) = \sum_{n=-\infty}^{\infty} \log \frac{\phi_{22}(n+1, z) z}{\phi_{22}(n, z)}. \tag{5.4.37}$$

By introducing

$$u(n, z) = \frac{\phi_{12}(n, z)}{\phi_{22}(n, z)}, \tag{5.4.38}$$

eq.(5.3.4) is written as

$$\begin{aligned} &(z^2 + 1)[u(n+1, z) - u(n, z)] \\ &+(z^2 - 1)[S_n^- u(n+1, z) u(n, z) - S_n^z \{u(n+1, z) + u(n, z)\} - S_n^{-*}] = 0, \end{aligned} \tag{5.4.39}$$

and eq.(5.4.37) becomes

$$\log \bar{a}(z) = \sum_{n=-\infty}^{\infty} \log \left[1 + \frac{1}{2}(z^2 - 1)\{1 - S_n^z + S_n^- u(n, z)\} \right]. \tag{5.4.40}$$

From eq.(5.4.39), the expansion for $u(n, z)$ in powers of $(z^2 - 1)$ is

$$u(n, z) = (z^2 - 1) g_n^{(1)} + (z^2 - 1)^2 g_n^{(2)} + \cdots , \tag{5.4.41a}$$

$$g_n^{(1)} = \frac{1}{2} \sum_{l=-\infty}^{n-1} (S_l^x + i S_l^y). \tag{5.4.41b}$$

Therefore, D_1 and D_2 are found to be

$$D_1 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (1 - S_n^z), \tag{5.4.42a}$$

$$D_2 = \frac{1}{8} \sum_{n=-\infty}^{\infty} S_n^- \sum_{l=-\infty}^{n} S_l^{-*} - \frac{1}{2} D_1, \tag{5.4.42b}$$

where $-2D_1$ is the total spin. Comparing eq.(5.4.42a) with eq.(5.4.28a), we have a relation

$$\int_0^{\pi} P(k)dk + \sum_{j=1}^{M}(P_j + P_j^*) = -\sum_{n=-\infty}^{\infty}(S_n^z - 1). \tag{5.4.43}$$

From eqs.(5.4.36) and (5.4.43), we see that the Hamiltonian (5.4.20) coincides with the Hamiltonian (5.1.1).

Appendix 5.1 Derivation of equations (5.2.9), (5.2.13) and (5.2.14)

First we derive eq.(5.2.9). We assume

$$g_n^{-1} = \alpha_n g_n^{\dagger}. \tag{5.A.1}$$

From eq.(5.2.6a), we have

$$g_{n+1}^{-1} = g_n^{-1}\begin{bmatrix} 1 & -q_n \\ q_n^* & 1 \end{bmatrix}\frac{1}{1+|q_n|^2}, \tag{5.A.2}$$

$$g_{n+1}^{\dagger} = g_n^{\dagger}\begin{bmatrix} 1 & -q_n \\ q_n^* & 1 \end{bmatrix}. \tag{5.A.3}$$

These equations suggest

$$\alpha_n = \prod_{j=n}^{\infty}(1+|q_j|^2). \tag{5.A.4}$$

On the other hand, from eq.(5.2.6b) we have

$$\frac{d}{dt}g_n^{-1} = -ig_n^{-1}\begin{bmatrix} -q_n q_{n-1}^* & -q_n + q_{n-1} \\ -q_n^* + q_{n-1}^* & q_n^* q_{n-1} \end{bmatrix}, \tag{5.A.5}$$

$$\frac{d}{dt}g_n^{\dagger} = -ig_n^{\dagger}\begin{bmatrix} -q_n^* q_{n-1} & -q_n + q_{n-1} \\ -q_n^* + q_{n-1}^* & q_n q_{n-1}^* \end{bmatrix}, \tag{5.A.6}$$

which yield

$$\dot{\alpha}_n = i\alpha_n(q_n q_{n-1}^* - q_n^* q_{n-1}). \tag{5.A.7}$$

This equation is consistent with eq.(5.A.4), that is,

$$\begin{aligned} i\frac{d}{dt}\log\alpha_n &= i\sum_{j=n}^{\infty}\frac{\dot{q}_j^* q_j + q_j^*\dot{q}_j}{1+|q_j|^2} \\ &= \sum_{j=n}^{\infty}[\{-q_{j+1}^* - q_{j-1}^* + 2q_j^* - |q_j|^2(q_{j+1}^* + q_{j-1}^*)\}q_j - \text{c.c.}]\frac{1}{1+|q_j|^2} \\ &= \sum_{j=n}^{\infty}[q_j^*(q_{j+1} + q_{j-1}) - (q_{j+1}^* + q_{j-1}^*)q_j] = q_n^* q_{n-1} - q_n q_{n-1}^*. \end{aligned} \tag{5.A.8}$$

Hence we obtain eq.(5.2.9).

We next derive eq.(5.2.13). Substitute eq.(5.2.12) into eq.(5.2.1) to find

$$
\begin{aligned}
&\frac{z-z^{-1}}{2}\dot{S}_n \\
&= -i[(z-z^{-1})^2(S_n+S_{n+1})^{-1} + (z^2-z^{-2})S_n(S_n+S_{n+1})^{-1}] \\
&\times\left(\frac{z+z^{-1}}{2}I + \frac{z-z^{-1}}{2}S_n\right) \\
&+i\left(\frac{z+z^{-1}}{2}I + \frac{z-z^{-1}}{2}S_n\right) \\
&\times[(z-z^{-1})^2(S_n+S_{n-1})^{-1} + (z^2-z^{-2})S_{n-1}(S_n+S_{n-1})^{-1}] \qquad (5.A.9) \\
&= -i\frac{(z-z^{-1})^3}{2}[(S_n+S_{n+1})^{-1}S_n + S_n(S_n+S_{n+1})^{-1} \\
&\qquad\qquad -S_{n-1}(S_n+S_{n-1})^{-1} - S_n(S_n+S_{n-1})^{-1}] \\
&-i\frac{(z-z^{-1})^2(z+z^{-1})}{2}[(S_n+S_{n+1})^{-1} + S_n(S_n+S_{n+1})^{-1}S_n \\
&\qquad\qquad -(S_n+S_{n-1})^{-1} - S_nS_{n-1}(S_n+S_{n-1})^{-1}] \\
&-2i(z-z^{-1})[S_n(S_n+S_{n+1})^{-1} - S_{n-1}(S_n+S_{n-1})^{-1}].
\end{aligned}
$$

Noticing that

$$(S_n+S_{n+1})^{-1}S_n = S_{n+1}(S_n+S_{n+1})^{-1}, \qquad (5.A.10)$$

we obtain

$$
\begin{aligned}
i\dot{S}_n &= 2[S_n, (S_n+S_{n-1})^{-1} + (S_n+S_{n+1})^{-1}] \\
&= \frac{1}{1+\vec{S}_n\cdot\vec{S}_{n+1}}[S_n, S_{n+1}] + \frac{1}{1+\vec{S}_n\cdot\vec{S}_{n-1}}[S_n, S_{n-1}],
\end{aligned} \qquad (5.A.11)
$$

which is equivalent to eq.(5.2.13).

We finally derive eq.(5.2.14). From eqs.(5.2.6a) and (5.2.11a), we have

$$g_n^{-1}\begin{bmatrix} 1 & -q_{n-1} \\ q_{n-1}^* & 1 \end{bmatrix} g_n = 2S_{n-1}(S_n+S_{n-1})^{-1}. \qquad (5.A.12)$$

Determinant of eq.(5.A.12) gives

$$1+|q_n|^2 = \frac{2}{1+\vec{S}_n\cdot\vec{S}_{n+1}}, \qquad (5.A.13)$$

which is eq.(5.2.14a). Equation (5.2.14b) is derived from eqs.(5.A.8) and (5.A.11), that is,

$$i(q_n q_{n-1}^* - q_n^* q_{n-1}) = \frac{d}{dt}\sum_{j=n}^{\infty}\log(1+|q_j|^2)$$

$$= -\frac{d}{dt}\sum_{j=n}^{\infty}\log(1+\overrightarrow{S}_j\cdot\overrightarrow{S}_{j+1}) = -\sum_{j=n}^{\infty}\frac{\dot{\overrightarrow{S}}_j\cdot\overrightarrow{S}_{j+1}+\overrightarrow{S}_j\cdot\dot{\overrightarrow{S}}_{j+1}}{1+\overrightarrow{S}_j\cdot\overrightarrow{S}_{j+1}}$$

$$= \sum_{j=n}^{\infty}\left[\frac{2\overrightarrow{S}_{j-1}\cdot(\overrightarrow{S}_j\times\overrightarrow{S}_{j+1})}{(1+\overrightarrow{S}_j\cdot\overrightarrow{S}_{j-1})(1+\overrightarrow{S}_j\cdot\overrightarrow{S}_{j+1})} - \frac{2\overrightarrow{S}_j\cdot(\overrightarrow{S}_{j+1}\times\overrightarrow{S}_{j+2})}{(1+\overrightarrow{S}_{j+1}\cdot\overrightarrow{S}_j)(1+\overrightarrow{S}_{j+1}\cdot\overrightarrow{S}_{j+2})}\right]$$

$$= \frac{2\overrightarrow{S}_{n-1}\cdot(\overrightarrow{S}_n\times\overrightarrow{S}_{n+1})}{(1+\overrightarrow{S}_n\cdot\overrightarrow{S}_{n-1})(1+\overrightarrow{S}_n\cdot\overrightarrow{S}_{n+1})}.$$

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CHAPTER VI

CONCLUDING REMARKS

Here we summarize the results obtained in the preceding chapters and give some remarks.

In chapter II, introducing the small amplitude and long-wavelength approximations, we have derive the equations describing the nonlinear waves in the $(2n, n)$ Lennard-Jones lattice and examined the soliton solutions of them. To see effects of the long-range interactions, we have not used the nearest-neighbor approximation. For $n \geqq 4$ we have obtained the Korteweg-de Vries equation, and found that the lattice is essentially the same as the system with the nearest-neighbor interaction. For $n = 2$ we have gotten the Benjamin-Ono equation which is more dispersive than the K-dV equation and whose soliton solution is algebraic. Although we have considered the L-J lattice, the typical example of a lattice with long-range interaction is a Coulomb system. To treat the system, we must study a diatomic lattice because of charge neutrality. It may be worth-while to study a Coulomb system.

In chapter III, we have investigated the discreteness effects on the Sine-Gordon kink with use of the perturbation theory developed by McLaughlin and Scott. We have seen that the propagation of the S-G kink in the discrete medium is very different from that in the continuum medium, that is, the kink is pinned between two adjacent lattice points when the kink velocity u is smaller than the critical pinning velocity u_p and excutes wobbling motion with radiation loss when $u > u_p$. We have not discussed multi-soliton and breather dynamics which contain important problems such as the creation or annihilation of kink-antikink pairs. This is the subject for future study.

In chapter IV, equivalence of the nonlinear wave equations generated by the A-K-N-S and W-K-I schemes of the inverse scattering method was shown by introducing the gauge transformation (namely the transformation of wave variables) and the transformation of space and time coordinates. Our discussions were confined to the case that the dispersion relation of the linearized equations is $\omega = \alpha k^3 + \beta k^2$. The results indicate a possibility that all nonlinear wave equations having the same dispersion relation, which are integrable by the inverse scattering method, are equivalent.

In chapter V, we have constructed the classical spin model which is differential-difference analogue of the continuous isotropic Heisenberg spin chain. As is well known, the classical spin is derived in the case $s >> 1$ where s is the magnitude of a spin operator $\overrightarrow{S}$, namely $\overrightarrow{S}^2 = s(s+1)$. As mentioned in the section 5.1, for $s = \dfrac{1}{2}$ the Heisenberg model is the solvable spin model. These facts suggest that a solvable spin model with arbitrary spin exists. We hope that the problem will be solved in the near future.