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NORMAL SUBSPACES OF κ^2

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Let F and H be subsets of a space X. F and H are *separated* if there are disjoint open sets U and V with $F \subset U$ and $H \subset V$. Moreover let $\mathcal{U}$ be an open cover of a space X. A collection $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ of subsets of X is *srinking* of $\mathcal{U}$ if $F(U) \subset U$ for each $U \in \mathcal{U}$. Here we do not require $\mathcal{F}$ covers X. A space is *normal* if each pair of disjoint closed sets are separated. A space X is *shrinking* if each open cover of X has a closed shrinking, i.e. a shrinking by closed sets, which covers X. By these definitions, shrinking spaces are normal, and collectionwise normal spaces are normal. It is well known that all subspaces of an ordinal space, more generally all GO-spaces, are shrinking, (collectionwise) normal and countably paracompact. It is also well known the product space $\omega_1 \times (\omega_1 + 1)$ is not normal, but it is countably paracompact. In [KOT], the normality of $A \times B$, where A and B are subspaces of an ordinal, was characterized and it was shown that normality, shrinking and collectionwise normality of $A \times B$ are equivalent. In particular:

Theorem 1. [KOT] *Let A and B be subspaces of ω_1. Then the followifng are equivalent:*

(1) *$A \times B$ is (collectionwise) normal.*

(2) *$A \times B$ is shrinking.*

(3) *A is not stationary in ω_1, B is not stationary in ω_1 or $A \cap B$ is stationary.*

(4) *$A \times B$ is countably paracomapct.*

Take disjoint stationary sets A and B in ω_1. Then by this theorem, $A \times B$ is neither normal nor countably paracompact.

Question in [KOT].

(a) If A and B are subspaces of an ordinal, then is $A \times B$ countably meta-comapct?

(b) For any subspace X of the square of an ordinal, are normality, collectionwise normality and shrinking property equivalent?

Recently an affimative answer of (a) is given by N. Kemoto and K. D. Smith as follows.

Theorem 2. [KS] *All subspaces of the square of an ordinal are countably meta-compact.*

In the proof of Theorem 2, they used a set-theoretical technic "the diagonal intersection". We thought this technic would be applicable for solving (b). We have gotten a complete affirmative answer of (b). For brevity, we will show the equivalence of normality and shrinking property of subspaces of ω_1^2.

Note that, if A is a countable subspace of ω_1, then, since A is non-stationary, by Theorem 1, $A \times B$ is normal for each $B \subset \omega_1$. In particular, as is well known, $(\omega+1) \times \omega_1$ is normal. But as is shown in the next example, there is a non-normal subspace of $(\omega + 1) \times \omega_1$.

Example 1. Put $X = \omega \times \omega_1 \cup \{\omega\} \times (\omega_1 \setminus \mathrm{Lim}(\omega_1))$, where $\mathrm{Lim}(A) = \{\alpha < \omega_1 : \sup(A \cap \alpha) = \alpha\}$. Note that $\mathrm{Lim}(A)$ is the set of all cluster point of A in ω_1, hence it is closed in ω_1. Put $F = \omega \times \mathrm{Lim}(\omega_1)$ and $H = \{\omega\} \times (\omega_1 \setminus \mathrm{Lim}(\omega_1))$. Then they are disjoint closed sets in X. Let U be an open set containing H. For each $\alpha \in \omega_1 \setminus \mathrm{Lim}(\omega_1)$, pick $n(\alpha) \in \omega$ such that $[n(\alpha), \omega] \times \{\alpha\} \subset U$. Since $\omega_1 \setminus \mathrm{Lim}(\omega_1)$ is uncountable, there are uncountable subset $C \subset \omega_1 \setminus \mathrm{Lim}(\omega_1)$ and $n \in \omega$ such that $n(\alpha) = n$ for each $\alpha \in C$. Observe that $[n, \omega] \times C \subset U$. Pick $\alpha \in \mathrm{Lim}(C)$. Noting $\mathrm{Lim}(C) \subset \mathrm{Lim}(\omega_1)$, we have $\langle n, \alpha \rangle \in [n, \omega] \times \mathrm{Lim}(C) \cap F \subset \mathrm{Cl}\, U \cap F$. This argument shows X is not normal.

We use the following notation: Let $X \subset \omega_1^2$, $\alpha < \omega_1$ and $\beta < \omega_1$. Put $V_\alpha(X) = \{\beta < \omega_1 : \langle \alpha, \beta \rangle \in X\}$, $H_\beta(X) = \{\alpha < \omega_1 : \langle \alpha, \beta \rangle \in X\}$ and $\triangle(X) = \{\alpha < \omega_1 : \langle \alpha, \alpha \rangle \in X\}$. Moreover put $A = \{\alpha < \omega_1 : V_\alpha(X)$ is stationary in $\omega_1\}$ and $B = \{\beta < \omega_1 : H_\beta(X)$ is stationary in $\omega_1\}$. Finally, for subsets C and D of ω_1, put $X_C = X \cap C \times \omega_1$, $X^D = X \cap \omega_1 \times D$ and $X_C^D = X \cap C \times D$.

We will show:

Theorem. *Let $X \subset \omega_1^2$. Then the following are equivalent.*

(1) *X is shrinking.*

(2) *X is normal.*

(3) (3-1a) *If α is a limit ordinal in ω_1 and $V_\alpha(X)$ is not staionary in ω_1, then there is a cub (=closed unbounded) set $D \subset \omega_1$ such that $X_{\{\alpha\}}$ and X^D are separated.*

(3-1b) *If β is a limit ordinal in ω_1 and $H_\beta(X)$ is not staionary in ω_1, then there is a cub set $C \subset \omega_1$ such that $X^{\{\beta\}}$ and X_C are separated.*

(3-2) *If $\triangle(X)$ is not stationary in ω_1, then there is a cub set $C \subset \omega_1$ such that X_C and X^C are separated.*

Intuitively, we may consider (3-1a) is a condition which guarantees the normality (shrinking) of $X_{\alpha+1}$ for each $\alpha < \omega_1$, and (3-1b) the normality (shrinking) of $X^{\beta+1}$ for each $\beta < \omega_1$. After knowing $X_{\alpha+1}$ and $X^{\beta+1}$ are normal (shrinking) for each $\alpha, \beta < \omega_1$, (3-2) is a condition which guarantees the normality (shrinking) of X.

Before proving this theorem, we prepare some lemmas.

Lemma 0.

(1) *If C is a cub set in ω_1, then $\omega_1 \backslash C$ is represented as a free union of bounded open intervals of ω_1, and covered by a disjoint collection of bounded closed and open intervals in ω_1.*

(2) *If $X \subset \omega_1^2$, C is a cub set in ω_1 and $X_{\alpha+1}$ is normal (shrinking) for each $\alpha < \omega_1$, then $X_{\omega_1 \backslash C}$ is normal (shrinking).*

Proof. (1): Put $h(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Then

$$\omega_1 \backslash C = \bigoplus_{\alpha \in C \backslash \operatorname{Lim}(C)} (h(\alpha), \alpha) \subset \bigoplus_{\alpha \in C \backslash \operatorname{Lim}(C)} (h(\alpha), \alpha].$$

(2): Assume $X_{\alpha+1}$ is normal (shrinking) for each $\alpha < \omega_1$. Let $\alpha < \omega_1$ be a limit ordinal. Take a strictly increasing cofinal sequenc $\{\alpha_n : n \in \omega\}$ in α. Then $X_\alpha = \bigoplus_{n \in \omega} X_{(\alpha_{n-1}, \alpha_n]}$, where $\alpha_{-1} = -1$, is normal (shrinking), because $X_{(\alpha_{n-1}, \alpha_n]}$ is a closed and open subspace of X_{α_n+1}. Therefore X_α is normal (shrinking) for each $\alpha < \omega_1$. Since, by (1), $X_{\omega_1 \backslash C} = \bigoplus_{\alpha \in C \backslash \operatorname{Lim}(C)} X_{(h(\alpha), \alpha)}$, it is normal (shrinking).

It is easy to show:

Lemma 1. *Assume X is the finite union of closed subspaces X_i's, $i \in n$. If $\mathcal{U}$ is an open cover such that, for each $i \in n$, $\mathcal{U}$ has a closed shrinking $\mathcal{F}_i$ which covers X_i, then $\mathcal{U}$ has a closed shrinking which covers X.*

This shows the following:

Lemma 2. *Assume X is the union of two shrinking open subspaces Y and Z. If $X\setminus Y$ and $X\setminus Z$ are separated, then X is shrinking.*

Lemma 3. *If X is a normal subspace of ω_1^2 such that $\triangle(X)$ is not stationary in ω_1, then there is a cub set C in ω_1 such that $X \cap C^2 = \emptyset$.*

Proof. First we show the following claim.

Claim. $A = \{\alpha < \omega_1 : V_\alpha(X)$ *is stationary in* $\omega_1\}$ *is not stationary in* ω_1.

Proof of Claim. Assume A is staionary in ω_1. For each $\alpha \in A$, fix $h(\alpha) < \omega_1$ with $\alpha < h(\alpha) \in V_\alpha(X) \cap \bigcap_{\alpha' \in A\cap\alpha} \mathrm{Lim}(V_{\alpha'}(X))$. For each $\alpha \in \omega_1\setminus A$, define $h(\alpha) = 0$. Take a cub set C' in ω_1 disjoint from $\triangle(X)$ and put $C = \{\alpha < \omega_1 : \forall\alpha' < \alpha(h(\alpha') < \alpha)\} \cap C'$. Then C is cub in ω_1, therefore $A' = A \cap C$ is staionary in ω_1. For each $\alpha \in A'$, put $x_\alpha = \langle\alpha, h(\alpha)\rangle$, then, by $h(\alpha) \in V_\alpha(X)$, we have $x_\alpha \in X$. We shall show $F = \{x_\alpha : \alpha \in A'\}$ is closed discrete in X. To show this, let $\langle\gamma,\delta\rangle \in X$. First assume $\gamma \in \omega_1\setminus C$. Then, by the closedness of C, there is $\gamma' < \gamma$ such that $(\gamma',\gamma] \cap C = \emptyset$. Then $U = (\gamma',\gamma] \times \omega_1 \cap X$ is an neighborhood of $\langle\gamma,\delta\rangle$ missing F. Next assume $\gamma \in C$. If $\gamma > \delta$, then $U = (\delta,\gamma] \times [0,\delta] \cap X$ is also a neighborhood of $\langle\gamma,\delta\rangle$ missing F. So assume $\gamma \le \delta$. Since C' is disjoint from $\triangle(X)$ and $\gamma \in C \subset C'$, we have $\gamma \ne \delta$. Then $U = [0,\gamma] \times (\gamma,\delta] \cap X$ is an neighborhood of $\langle\gamma,\delta\rangle$ which intersects F with at most one point. This argument shows F is closed discrete in X.

Since A' is staionary in ω_1, we can decompose A' into two disjoint stationary sets T_0 and T_1 in ω_1. Put $F_i = \{x_\alpha : \alpha \in T_i\}$ for each $i \in 2 = \{0,1\}$. Let U_i be an open set containing F_i for each $i \in 2$. For each $\alpha \in T_i$, by $x_\alpha = \langle\alpha, h(\alpha)\rangle \in F_i \subset U_i$, we can fix $f(\alpha) < \alpha$ and $g(\alpha) < h(\alpha)$ such that $(f(\alpha), \alpha] \times (g(\alpha), h(\alpha)] \cap X \subset U_i$. By the PDL(=Pressing Down Lemma), we find $\gamma_i < \omega_1$ and a staionary set $T_i' \subset T_i$ such that $f(\alpha) = \gamma_i$ for each $\alpha \in T_i'$. Put $\gamma = \max\{\gamma_0, \gamma_1\}$. Then we have $(\gamma, \alpha] \times (g(\alpha), h(\alpha)] \cap X \subset U_i$ for each $\alpha \in T_i'$ with $i \in 2$. Fix $\alpha_0 \in A$ with $\gamma < \alpha_0$. Moreover fix $\beta_0 \in \bigcap_{i\in 2} \mathrm{Lim}(T_i') \cap V_{\alpha_0}(X)$ with $\alpha_0 < \beta_0$. We shall show $\langle \alpha_0, \beta_0 \rangle \in \mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U_1$. To show this, let V be an open neighborhood of $\langle \alpha_0, \beta_0 \rangle$. Then we can find $\beta < \beta_0$ such that $\alpha_0 \leq \beta$ and $\{\alpha_0\} \times (\beta, \beta_0] \cap X \subset V$. By $\beta_0 \in \mathrm{Lim}(T_0')$, we can find $\delta, \delta' \in T_0'$ with $\beta < \delta < \delta' < \beta_0$. Since $T_0' \subset C$ and $\delta < \delta'$, we have $h(\delta) < \delta'$. On the other hand, by $\beta < h(\delta)$, $g(\delta) < h(\delta)$ and $h(\delta) \in \mathrm{Lim}(V_{\alpha_0}(X))$, there is $\nu_0 \in V_{\alpha_0}(X)$ such that $\max\{\beta, g(\delta)\} < \nu_0 < h(\delta)$. Then

$$\langle \alpha_0, \nu_0 \rangle \in \{\alpha_0\} \times (\beta, \beta_0] \cap (\gamma, \delta] \times (g(\delta), h(\delta)] \cap X \subset V \cap U_0.$$

Thus $\langle \alpha_0, \beta_0 \rangle \in \mathrm{Cl}\, U_0$. Similarly we have $\langle \alpha_0, \beta_0 \rangle \in \mathrm{Cl}\, U_1$. But this contradicts the normality of X. This completes the proof of the Claim.

Similarly we can prove B is not stationary in ω_1.

Take a cub set D in ω_1 disjoint from $A \cup B \cup \triangle(X)$. For each $\gamma \in D$, by $\gamma \notin A \cup B$, we can fix a cub set C_γ in ω_1 disjoint from $V_\gamma(X) \cup H_\gamma(X)$. Then, by a similar argument of [Ku, II, Lemma 6.14], $E = \{\alpha \in D : \forall \gamma \in D \cap \alpha (\alpha \in C_\gamma)\}$ is a cub set in ω_1. Assume $\langle \gamma, \alpha \rangle \in X \cap E^2$. Since D is disjoint from $\triangle(X)$ and $E \subset D$, we have $\gamma \neq \alpha$. We may assume $\gamma < \alpha$. Since $\alpha \in E$ and $\gamma \in E \cap \alpha \subset D \cap \alpha$, we have $\alpha \in C_\gamma$. Thus $\alpha \notin V_\gamma(X)$. This shows $\langle \gamma, \alpha \rangle \notin X$, a contradiction. This completes the proof of Lemma 3.

Proof of the Theorem. (1) $\rightarrow$ (2) is evident.

(2) $\to$ (3): Let X be a normal subspace of ω_1^2.

(3-1a): Assume α is a limit ordinal in ω_1 and $V_\alpha(X)$ is not stationary in ω_1. Take a cub set D in ω_1 disjoint from $V_\alpha(X)$. Since $X_{\{\alpha\}}$ and X^D are disjoint closed sets of the normal space X, they are separated.

(3-1b): Similar.

(3-2): By Lemma 3.

(3) $\to$ (1): Assume the clause (3). First we show the following Lemma.

Lemma 4. *$X_{\alpha+1}$ is shrinking for each $\alpha < \omega_1$.*

Proof. We prove this Lemma by induction. The cases of $\alpha = 0$ and $\alpha = \alpha' + 1$ are almost trivial. So assume α is a limit ordinal in ω_1 and $X_{\alpha'+1}$ is shrinking for each $\alpha' < \alpha$.

First assume $V_\alpha(X)$ is not staionary in ω_1. By (3-1a), take a cub set D in ω_1 such that $X_{\{\alpha\}}$ and X^D are separated, therefore $X_{\{\alpha\}}$ and $X^D_{\alpha+1}$ are separated. The argument in the proof of (2) of Lemma 0 shows X_α is shrinking open subspace of $X_{\alpha+1}$. Since $X^{\omega_1\backslash D}_{\alpha+1}$ is a free union of countable subspaces, it is shrinking open subspace of $X_{\alpha+1}$. Since $X_{\alpha+1} = X_\alpha \cup X^{\omega_1\backslash D}_{\alpha+1}$, $X_{\alpha+1}\backslash X_\alpha = X_{\{\alpha\}}$ and $X_{\alpha+1}\backslash X^{\omega_1\backslash D}_{\alpha+1} = X^D_{\alpha+1}$, by Lemma 2, $X_{\alpha+1}$ is shrinking.

Next assume $V_\alpha(X)$ is stationary in ω_1. Let $\mathcal{U}$ be an open cover of $X_{\alpha+1}$. For each $\beta \in V_\alpha(X)$, fix $f(\beta) < \alpha$, $g(\beta) < \beta$ and $U(\beta) \in \mathcal{U}$ such that $(f(\beta), \alpha] \times (g(\beta), \beta] \cap X \subset U(\beta)$. By the PDL and $|\alpha| < \omega_1$, we find $\alpha_0 < \alpha$, $\beta_0 < \beta$ and a stationary set $S \subset V_\alpha(X)$ such that $f(\beta) = \alpha_0$ and $g(\beta) = \beta_0$ for each $\beta \in S$. Put $Z = (\alpha_0, \alpha] \times (\beta_0, \omega_1) \cap X$. We show:

Claim. *There is a closed shrinking which covers Z*

Proof of the Claim. For each pair β and β' in S, define $\beta \sim \beta'$ by $U(\beta) = U(\beta')$. Then $\sim$ is an equivalence relation on S. For each $E \in {}^{S}/_{\sim}$, put $U_E = U(\beta)$ for some (eqivalently, any) $\beta \in E$.

Case 1. There is $E \in {}^{S}/_{\sim}$ which is unbounded in ω_1.

In this case, for each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} Z, & \text{if } U = U_E, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is a desired one.

Case 2. Each $E \in {}^{S}/_{\sim}$ is bounded in ω_1.

In this case, by induction, take a strictly increasing sequence $\{\beta(\delta) : \delta < \omega_1\}$ in ω_1 and a sequence $\{E(\delta) : \delta < \omega_1\} \subset {}^{S}/_{\sim}$ satisfying $\sup(\bigcup_{\delta' < \delta} E(\delta')) < \beta(\delta) \in E(\delta)$ for each $\delta < \omega_1$. Note that elements of $\{E(\delta) : \delta < \omega_1\}$ are all distinct. For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\alpha_0, \alpha] \times (\beta_0, \beta(\delta)] \cap X, & \text{if } U = U_{E(\delta)} \text{ for some } \delta < \omega_1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is a desired one. This completes the proof of the Claim.

By the inductive assumption, X_{α_0+1} is shrinking. Moreover, by countability of $X_{\alpha+1}^{\beta_0+1}$, it is shrinking. Since $X_{\alpha+1}$ is the union of closed subspaces X_{α_0+1}, $X_{\alpha+1}^{\beta_0+1}$ and Z, by the above claim and Lemma 1, we can find a closed shrinking of $\mathcal{U}$ which covers $X_{\alpha+1}$. This completes the proof of Lemma 4.

In a similar way, using (3-1b), we can show $X^{\beta+1}$ is shrinking for each $\beta < \omega_1$.

To show X is shrinking, first assume $\triangle(X)$ is not stationary. By (3-2), there is a cub set C in ω_1 such that X_C and X^C are separated. Therefore, by Lemma 0 and Lemma 2, $X = X_{\omega_1 \setminus C} \cup X^{\omega_1 \setminus C}$ is shrinking.

Finally assume $\triangle(X)$ is stationary. Let $\mathcal{U}$ be an open cover of X. For each $\alpha \in \triangle(X)$, fix $f(\alpha) < \alpha$ and $U(\alpha) \in \mathcal{U}$ such that $(f(\alpha), \alpha]^2 \cap X \subset U(\alpha)$. Then, by the PDL, we find $\alpha_0 < \omega_1$ and a stationary set $S \subset \triangle(X)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Put $Z = (\alpha_0, \omega_1)^2 \cap X$. Then, by a similar argument of Lemma 4, we can get a closed shrinking of $\mathcal{U}$ which covers $X = X_{\alpha_0+1} \cup X^{\alpha_0+1} \cup Z$. Thus X is shrinking. This completes the proof of the Theorem.

Hereafter we give some examples and related problems.

Consider $X = \omega_1^2$. Since $V_\alpha(X)$ and $H_\beta(X)$ are the stationary set ω_1 for each $\alpha, \beta < \omega_1$ and $\triangle(X)$ is also the staionary set ω_1, the clause (3) of the Theorem is satisfied. So X is normal.

Example 2. Let A and B be disjoint stationary sets in ω_1 and put $X = A \times B$. Let α be a limit ordinal in ω_1. Then we have

$$V_\alpha(X) = \begin{cases} B, & \text{if } \alpha \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, if $V_\alpha(X)$ is not stationary, it must be $\alpha \notin A$ and $V_\alpha(X) = \emptyset$, so $X_{\{\alpha\}} = \emptyset$. Therefore $X_{\{\alpha\}}$ and X^{ω_1} are separated. This argument witnesses (3-1a). Similarly we have (3-1b). Therefore $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$.

Note that $\triangle(X) = \emptyset$. Let C be a cub set in ω_1. Then $X \cap C^2 = (A \cap C) \times (B \cap C) \neq \emptyset$, equivalently $X_C \cap X^C \neq \emptyset$. Thus X_C and X^C can not be separated. Therefore X is not normal, because the clause (3-2) is not satisfied.

Example 3. Let $X = \{\langle \alpha, \beta \rangle \in \omega_1^2 : \alpha \leq \beta\}$ and $Y = \{\langle \alpha, \beta \rangle \in \omega_1^2 : \alpha < \beta\}$. Checking (3-1a) and (3-1b), we can show $X_{\alpha+1}$, $X^{\beta+1}$, $Y_{\alpha+1}$ and $Y^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$.

Since $\triangle(X) = \omega_1$ is staionary, (3-2) for X is stisafied. Thus X is normal(but this is obvious, because X is a closed subspace of ω_1^2). On the other hand, note that $\triangle(Y) = \emptyset$. For each cub set C in ω_1, pick α and β in C with $\alpha < \beta$. Then $\langle \alpha, \beta \rangle \in Y \cap C^2$. Therefore (3-2) for Y is not satisfied. Thus Y is not normal.

Let $X = \omega_1 \times (\omega_1 + 1)$. Observe that $X \cap \omega_1^2 = \omega_1^2$ is normal, and $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$. Since $\{\langle \alpha, \alpha \rangle : \alpha \in \omega_1\}$ and $X^{\{\omega_1\}}$ can not be separated, X is not normal. Note that both $\triangle(X)$ and $H_{\omega_1}(X)$ are the stationary set ω_1. Next we give such an example $X \subset \omega_1 \times (\omega_1 + 1)$, but $\triangle(X)$ and $H_{\omega_1}(X)$ are not stationary.

Example 4. Let

$$X = [\omega_1 \setminus \mathrm{Lim}(\omega_1)] \times [(\omega_1 + 1) \setminus \mathrm{Lim}(\omega_1)] \cup \{\langle \alpha, \alpha + 1 \rangle : \alpha \in \mathrm{Lim}(\omega_1)\}.$$

Observe that $X \cap \omega_1^2$ is normal, $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$ and both $\triangle(X)$ and $H_{\omega_1}(X)$ are the non-stationary set $\omega_1 \setminus \mathrm{Lim}(\omega_1)$. By a similar argument in Lemma 3, we can see $F = \{\langle \alpha, \alpha + 1 \rangle : \alpha \in \mathrm{Lim}(\omega_1)\}$ is closed (discrete). We shall show F and $X^{\{\omega_1\}}$ can not be separated. To show this, let U be an open set containing F. For each $\alpha \in \mathrm{Lim}(\omega_1)$, by $\langle \alpha, \alpha + 1 \rangle \in F \subset U$, take $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \times \{\alpha + 1\} \cap X \subset U$. By the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \mathrm{Lim}(\omega_1)$ such that $f(\alpha) = \alpha_0$ for eachf $\alpha \in S$. Take $\beta \in \omega_1 \setminus \mathrm{Lim}(\omega_1)$ with $\alpha_0 < \beta$. Noting $\langle \beta, \alpha + 1 \rangle \in X$ for each $\alpha \in S$ with $\alpha > \beta$, we have

$$\langle \beta, \omega_1 \rangle \in \mathrm{Cl}\{\langle \beta, \alpha + 1 \rangle : \alpha \in S, \alpha > \beta\} \cap X^{\{\omega_1\}} \subset \mathrm{Cl}\, U \cap X^{\{\omega_1\}}.$$

Thus F and $X^{\{\omega_1\}}$ can not be separated.

In these connections, we have the next question.

Question 1. Does there exsist a non-normal subspace X of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$.

In this connection, we show:

Proposition. *If $X = A \times B$ is a subspace of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$, then X is normal.*

Proof. If A is not staionary in ω_1, then take a cub set C in ω_1 disjoint from A. Then, by (2) of Lemma 0, $X = X_{\omega_1 \setminus C}$ is normal. Similarly X is normal if B is not stationary in ω_2. So we may assume A and B are stationary in respectively ω_1 and ω_2. Let $\mathcal{U} = \{U_i : i \in 2\}$ be an open cover of X. Fix $\alpha \in A$. For each $\beta \in B$, fix $f(\alpha, \beta) < \alpha$, $g(\alpha, \beta) < \beta$ and $i(\alpha, \beta) \in 2$ such that $(f(\alpha, \beta), \alpha] \times (g(\alpha, \beta), \beta] \cap X \subset U_{i(\alpha,\beta)}$. Applying the PDL to B, we find $f(\alpha) < \alpha$, $g(\alpha) < \omega_2$, $i(\alpha) \in 2$ and a stationarty set $B(\alpha) \subset B$ in ω_2 such that $f(\alpha, \beta) = f(\alpha)$, $g(\alpha, \beta) = g(\alpha)$ and $i(\alpha, \beta) = i(\alpha)$ for each $\beta \in B(\alpha)$. Then, applying the PDL to A, we find $\alpha_0 < \omega_1$, $i_0 \in 2$ and a stationary set $A' \subset A$ in ω_1 such that $f(\alpha) = \alpha_0$ and $i(\alpha) = i_0$ for each $\alpha \in A'$. Put $\beta_0 = \sup\{g(\alpha) : \alpha \in A'\}$. Then we have $Z = (\alpha_0, \omega_1) \times (\beta_0, \omega_2) \cap X \subset U_{i_0}$. Since X is the union of closed subspaces, X_{α_0+1}, X^{β_0+1} and $Z, \mathcal{U}$ has a closed shrinking which covers X. Therefore $X = A \times B$ is normal.

By Theorem 1, normality and countable paracompactness of $A \times B \subset \omega_1^2$ are equivalent. In this connection, it is natural to ask:

Question 2. For any $X \subset \omega_1^2$, are normality and countable paracompactness equivalent?

Finally we restate a question from [KOT]

Question 3. For any subspace X of the square of an ordinal, are countable paracompactness, expandability, strong D-property and weak D(ω)-property equivalent?

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