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Innovation approach to random fields

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§0. Introduction.

The purpose of this note is to clarify the notion of "**innovation**" for a stochastic process $X(t)$, having been suggested by P. Lévy's idea of stochastic infinitesimal equation. Then, we proceed to the case of a random field $X(C)$ depending on a manifold C, where we also see the role of the innovation for the study of $X(C)$ in question.

Let $X(t)$, $t \in R$, be a stochastic process. We are interested in the case where the variation $\delta X(t)$ over an infinitesimal interval $[t, t+dt)$ can be expressed in the form

(1) $\delta X(t) = \Phi(X(s), s \le t, Y(t), t, dt)$.

This is the so-called Lévy's stochastic infinitesimal equation (see [1]). In the above expression Φ is a non-random function and the $Y(t)$ is the *innovation* which is a random variable independent of the $X(s)$, $s \le t$, namely $Y(t)$ stands for the new information gained by $X(t)$ in the time interval $[t. t+dt)$.

Although the equation (1) has only formal significance, it can well describes the probabilistic structure of the process $X(t)$. The randomness that is contained in the $\{X(t)\}$ is entirely involved in the system $\{Y(t)\}$, which is a system of elementary random variables. Having been motivated by the idea that comes from equation (1) above, we are naturally led to a stochastic analysis for $X(t)$ based on the innovation; in particular, we are led to white noise analysis, where the $Y(t)$ is taken to be white noise $\dot{B}(t)$.

Before we come to a setup of our analysis, we have to mention an important remark. The innovation should not be understood as a system of continuously many independent random variables, but the Y(t)'s are independent random variables, each of which is to be associated to an infinitesimal time intervals with length *dt*. Thus, their sum or the integral like $\int^t Y(s)\sqrt{ds} = Z(t)$ gives an additive process. The choice of $\sqrt{ds}$ in the integral is easily acceptable if one thinks of $\sqrt{n}$ -law for the sum of independent identically distributed random variables. It is noted that the given X(t) and the Z(t) defined above have the same innovation Y(t) at any infinitesimal interval [t, t+dt). The elementary random variable, which is idealized, may be either $dZ(t) = Y(t)\sqrt{dt}$ or $\dot{Z}(t) = \frac{dZ(t)}{dt} = \frac{Y(t)}{\sqrt{dt}}$.

Now recall the Lévy-Itô decomposition of an additive process, in fact, that of Lévy process that satisfies some additional assumptions involving stationary increments property. The decomposition theorem says that Z(t) is a sum of a non-random term $m \cdot t$ and two independent processes $\sigma B(t)$ and P(t):

(2) $\quad Z(t) = m\cdot t + \sigma\cdot B(t) + P(t),$

where B(t) is a Brownian motion and where P(t) is a compound Poisson process.

We are particularly interested in the Gaussian part, and by taking $\sigma = 1$ in (2), we shall form a Gaussian system of *idealized elementary random variables* $\{\dot{B}(t)\}$ with $\dot{B}(t) = \frac{dB(t)}{dt}$. Such a choice of the system makes a milestone of our stochastic analysis. More precisely, we take the $\dot{\mathbf{B}} = \{\dot{B}(t)\}$ to be a system of variables of a random function $\varphi(\dot{B})$ and we shall carry on stochastic analysis, namely differential and integral calculus in the variables $\dot{B}(t)$'s. The calculus, called the *white noise analysis*, has extensively been developed so far in this line.

We shall then propose a next step, namely white noise analysis of random fields X(C) depending on a manifold C by introducing the innovation. For this purpose we generalize the Lévy's stochastic infinitesimal equation (1). Namely, we propose an equation

(3) $\delta X(C) = \Phi(X(C'), C' \subset C, Y(s), s \in C, C, \delta C)$,

which still has only formal significance, but it does suggest our approach. The equation (3) may be called a *stochastic variational equation*. The analysis that will be developed can still be in our calculus by taking the innovation to be white noise.

§1. Background and representation of Gaussian processes.

Since we shall discuss in line with white noise analysis, every random phenomenon is assumed to be expressed as a generalized white noise functional.

Start with a white noise (E^*, μ) with R^d-parameter (i.e. E^* is an extension of $L^2(R^d)$), and let $(S)^*$ be the space of generalized white noise functionals. The infinite dimensional rotation group $O^*_\infty = O^*_\infty(E^*)$ acts on E^* and keeps the white noise measure μ invariant.

One of the motivations of our approach is the canonical representation theory for Gaussian processedthat is originated by P. Lévy (see e.g. [2]). Given a Gaussian process X(t) with E(X(t)) = 0. If X(t) is expressed in the form

(4) $X(t) = \int^t F(t,u)\, \dot{B}(u)du,$

then it is called a representation of $X(t)$ in terms of white noise $\dot{B}(u)$. Or we may write (4) in the form

(4') $$X(t) = \int^{t} F(t,u)\ x(u)du,$$

where $x \in E^{*}$ is a sample function of $\dot{B}(u)$ and $X(t)$ itself is often written as $X(t,x)$. Note that there d is taken to be 1.

Among various representations is a *canonical representation* that satisfies the relation : for any t and s with $t > s$ the equality

$$E(X(t)/B_{s}(X)) = \int^{s} F(t,u)\ \dot{B}(u)du,$$

holds, where $B_{s}(X)$ is the smallest σ-field generated by the random variables $X(u)$, $u \le s$.

For a canonical representation one can prove

(5) $$B_{t}(X) = B_{t}(\dot{B}), \quad \text{for every } t.$$

In addition, the kernel $F(t,u)$, being viewed as an integral operator, defines its inverse $F(u,t)^{-1}$ which plays a role of the *whitening*, since $F(t.u)$ is surjective operator acting on the space spanned by the $X(t)$'s. Through such properties we understand the role of innovation. With this spirit we can consider innovation for some general stochastic processes without the assumption that $X(t)$ is Gaussian.

We further expect such a relation between process and innovation for the case of random fields. This will be discussed in the next section.

§2. Innovation for Gaussian random fields.

Let $X(C)$ be a Gaussian random field which lives in $(S)^{*}$, and be indexed by a contour $C \subset R^{2}$. Now suppose it is expressed in the form

(6) $$X(C) = \int_{(C)} F(C,u)x(u)\,du, \qquad x \in E^*,$$
(C) : the domain enclosed by C,

where x is a sample function of R^2-parameter white noise and $F(C,u)$ is a non-random kernel function.

Obviously $X(C)$ is a Gaussian random variable with $E(X(C)) = 0$ and $\mathrm{Var}(X(C)) = \int_{(C)} F(C,u)^2 du$. Assume that C runs through a certain class of contours, say $C \in \mathbf{C} = \{C \ ; \ C \text{ is diffeomorphic to } S^1\}$. Then we are given a Gaussian random field indexed by C and X(C) may be viewed as a generalization of a Gaussian process with parameter $t \in R$.

Now we come to the stochastic variational equation (3) for X(C) and can discuss its innovation. The variation $\delta X(C)$ has already been discussed in [4] formula (4.3). It is known that, by taking n = 1 and by taking $C + \delta C$ outside of C, δC being represented by $\delta n(s)$, we have

(7) $$\delta X(C) = \int_C \{F(C,s)x(s) + \int_{(C)} \delta F(C,u)(s)x(u)du\}\delta n(s)ds$$

and that the two terms of the righthand side can be discriminated. Also it is known that x(s) is obtained from the first term. In fact, the proof needs the help by the rotation group O_∞; good reference [6].

What we claim now in this note is that $\{x(s),\ s \in C\}$ is defined to be the *innovation* and we understand that it is associated with the *infinitesimal domain between* C *and* $C + \delta C$. If one is permitted to use a formal expression, then the accumulated innovation is

(8) $$Z(C) = \int_{(C)} x(s)\delta n(s)ds,$$

and it has the same innovation as X(C). $\{x(s),\ s \in C\}$ is a sample of the elementary random variables. Thus, a good interpretation is given to the reason why white noise is useful for calculus of random fields.

§3. Concluding remarks.

1) Innovation for non Gaussian random fields.

We follow the results in [4] to discuss the case where X(C) is not a Gaussian variable but a generalized white noise functional, and it is easy to come to the notion of innovation to discuss its roles.

2) The case of a random field $X(t)$, $t \in R^d$.

As is seen in [7], normal derivatives along a curve or a surface may not be a generalized stochastic process. In order to form innovation-like process, the calculus is not always straight forward, in reality some regularization is necessary.

[References]

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