数理解析研究所講究録
958 巻 1996 年 166-177



# The Riemann Zeta-Function and the Hecke Congruence Subgroups

YOICHI MOTOHASHI (本橋洋一.
日大理工)

## 1. - Introduction

The aim of our talk is to show an explicit relation between the Riemann zeta-function $\zeta(s)$ and the Hecke congruence subgroups $\Gamma_0(q)$ with variable level $q$.

Extending our investigation [10] on the following version of the fourth power mean of $\zeta(s)$

$$\frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+i(T+t))|^4 e^{-(t/G)^2}dt,$$

where $T$, $G$ are arbitrary positive numbers, we already observed such a relation in our former talk [11] delivered two years ago at an occasion similar to this meeting. There we reported an explicit formula for the integral

$$\frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+i(T+t))L(\tfrac{1}{2}+i(T+t),\chi)|^2 e^{-(t/G)^2}dt,$$

where $\chi$ is a primitive Dirichlet character mod $q$. It contains a contribution of the discrete spectrum of the hyperbolic Laplacian $-y^2\Delta$ acting on the Hilbert space $L^2(\Gamma_0(q)\setminus \mathrm{H})^{\dagger}$ in the form of a sum of the values at $\frac{1}{2}$ of Hecke $L$-functions twisted by $\chi$.

This time we extend our investigation to another direction. We consider the following version of the Deshouillers-Iwaniec mean value problem [4][5]:

$$I_2(T,G;\mathcal{A})=\frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+i(T+t))|^4|\mathcal{A}(\tfrac{1}{2}+i(T+t))|^2 e^{-(t/G)^2}dt, \tag{1.1}$$

where

$$\mathcal{A}(s)=\sum_n \alpha_n n^{-s}$$

with an arbitrary finite complex vector $\{\alpha_n\}$. We are going to show an explicit formula for $I_2(T,G;\mathcal{A})$, which exhibits the relation mentioned at the beginning.

Our argument is essentially the same as that of $[10]^{\ddagger}$, and may also be regarded as latter's combination with that of Deshouillers-Iwaniec [3]. There are, however, some interesting subtleties

Talk at the conference *Analytic Number Theory* held at RIMS, Kyoto University, October 3-7, 1994.

† Here and in what follows we use obvious conventions without defining them explicitly.

‡ But there is a notable difference, too; see the concluding remark.

and ramifications induced by the requirement of the explicitness in the end result. They are mostly related to the Kuznetsov type of trace formulas (Theorems 1 and 2 below), on which the success of our argument is dependent. Technically speaking, what concerns us most is the choice of a representative set of inequivalent cusps of each $\Gamma_0(q)$. It becomes in fact a delicate task if, for instance, it is required to have certain arithmetical *transparence* in the contribution of the continuous spectrum. In the third section we shall develop an argument having this aim in mind.

But we are not going to deal with the problem in its full generality. Mainly for the sake of simplicity, we assume that the coefficients $\alpha_n$ of $\mathcal{A}$ are supported by the set of square-free integers; i.e., in what follows the condition

$$\alpha_n = 0 \text{ whenever } \mu(n) = 0 \tag{1.2}$$

is always imposed.

## 2. - Reduction to sums of Kloosterman sums

Now, expanding out the factor $|\mathcal{A}(\frac{1}{2}+i(T+t))|^2$ in (1.1) we get

$$I_2(T,G;\mathcal{A}) = \sum_{\substack{a,b,r\\(a,b)=1}} \frac{\alpha_{ar}\overline{\alpha_{br}}}{r\sqrt{ab}}(b/a)^{iT} J(T,G;b/a),$$

where

$$J(T,G;b/a) = \frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+i(T+t))|^4 (b/a)^{it} e^{-(t/G)^2} dt.$$

Then we introduce

$$Y(u,v,w,z;G;b/a) = \frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty} \zeta(u+it)\zeta(v-it)\zeta(w+it)\zeta(z-it)(b/a)^{it}e^{-(t/G)^2}dt,$$

where $u,v,w,z$ are complex variables, all of which have real parts larger than 1. Shifting the contour far right, $Y(u,v,w,z;G;b/a)$ is meromorphically continued to the entire $\mathbb{C}^4$. The specialization $(u,v,w,z) = (\frac{1}{2}+iT, \frac{1}{2}-iT, \frac{1}{2}+iT, \frac{1}{2}-iT) = P_T$, say, in the result of this analytic continuation shows that $Y(P_T;G;b/a)$ is equal to $J(T,G;b/a)$ plus a negligibly small term as $T\to\infty$, provided

$$0 < G \le T(\log T)^{-1},$$

which we shall assume henceforth.

On the other hand we have, in the region of absolute convergence,

$$Y(u,v,w,z;G;b/a) = \sum_{k,l,m,n=1}^{\infty} \frac{1}{k^u l^v m^w n^z}\exp\Big(-\big(\frac{G}{2}\log\frac{bln}{akm}\big)^2\Big).$$

We apply the dissection argument of Atkinson [2] to this quadruple sum. So it is divided into three parts according to the cases: $akm = bln$, $akm > bln$ and $akm < bln$. The first part is expressible in terms of the zeta-function and does not have much special to note. The second and the third parts are, in a sense, conjugate to each other, and so it is enough to consider the second part. It is equal to

$$\zeta(u+v)a^{-v}b^{-u}\sum_{\substack{c|a\\ d|b}} c^v d^u \mathcal{Y}(u,v,w,z;G;d/c), \tag{2.1}$$

where

$$\mathcal{Y}(u,v,w,z;G;d/c) = \sum_{(ck,dl)=1} k^{-u}l^{-v} \sum_{ckm>dln} m^{-w}n^{-z} \exp\left(-\left(\frac{G}{2}\log\frac{dln}{ckm}\right)^2\right).$$

Here it should be noted that the assumption (1.2) implies that

$$\mu(cd) \neq 0. \tag{2.2}$$

In the inner sum we perform change of variables by putting $ckm = dln + f$, so that $n \equiv -\overline{dl}f \bmod ck$. The value of the variable $l$ is classified according to $\bmod\, ck$, too. Also we introduce the Mellin transform

$$W(s,w) = \int_0^\infty x^{s-1}(1+x)^{-w}\exp\left(-\left(\frac{G}{2}\log(1+x)\right)^2\right)dx,$$

as we did in [10]. Then we get, after a rearrangement,

$$\begin{aligned} \mathcal{Y}(u,v,w,z;G;d/c) &= \frac{1}{c^{v+w+z}d^w}\sum_{f=1}^\infty \sum_{\substack{k=1\\ (k,d)=1}}^\infty k^{-u-v-w-z} \\ &\times \sum_{\substack{h=1\\ (h,ck)=1}}^{ck} \frac{1}{2\pi i}\int_{(\eta_0)} \zeta\left(v+w-s,\frac{h}{ck}\right)\zeta\left(w+z-s,-\frac{\overline{dh}}{ck}f\right)W(s,w)\left(\frac{1}{ck}\sqrt{\frac{f}{d}}\right)^{-2s} ds, \end{aligned} \tag{2.3}$$

where $\zeta(s,\omega)$ is the Hurwitz zeta-function, i.e., the meromorphic continuation of

$$\sum_{n+\omega>0} (n+\omega)^{-s}.$$

The right side of (2.3) is absolutely convergent if we have, for instance,

$$\eta_0 > 1,\ \mathrm{Re}(u) > \mathrm{Re}(w)+1,\ \mathrm{Re}(v+w) > \eta_0+1,\ \mathrm{Re}(w+z) > \eta_0+1.$$

The contour of the last integral is to be shifted to the right appropriately. For this sake we introduce the condition

$$\mathrm{Re}(v+w) < \eta,\ \mathrm{Re}(w+z) < \eta,\ \mathrm{Re}(u+v+w+z) > 2(\eta+1). \tag{2.4}$$

The positive parameter $\eta$ is to be taken sufficiently large. Then we shift the contour $(\eta_0)$ to $(\eta)$. Two poles are encountered; they are at $w+z-1$ and $v+w-1$. Their contribution can be easily computed in terms of the zeta-function. So, let us concentrate on the part containing the integral along the contour $(\eta)$, which we denote as $\mathcal{Y}_0(u,v,w,z;G;d/c)$.

Invoking the functional equation

$$\zeta(s,\omega) = 2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty}\sin\left(\frac{\pi s}{2}+2\pi n\omega\right)n^{s-1}, \qquad \mathrm{Re}(s)<0,$$

we get

$$\begin{aligned}\mathcal{Y}_0(u,v,w,z;G;d/c) = \frac{2c^u d^{\frac{1}{2}(u+v-w+z)}}{(2\pi)^{u-w+1}} \sum_{m,n=1}^{\infty} & m^{\frac{1}{2}(v+w-u-z-1)} n^{-\frac{1}{2}(u+v+w+z-1)}\sigma_{w+z-1}(n) \\ & \times (\mathcal{X}_+ + \mathcal{X}_-)(m,n;u,v,w,z;G;d/c),\end{aligned} \tag{2.5}$$

where $\sigma$ is the usual sum of powered divisor function and

$$\mathcal{X}_\pm(m,n;u,v,w,z;G;d/c) = \sum_{\substack{k=1\\(k,d)=1}}^{\infty} \frac{1}{ck\sqrt{d}} S(m,\pm\bar{d}n;ck)\phi_\pm\left(\frac{4\pi\sqrt{mn}}{ck\sqrt{d}};u,v,w,z\right).$$

Here $S(m,n;l)$ is the Kloosterman sum

$$\sum_{\substack{h=1\\(h,l)=1}}^{l} e((mh+n\bar{h})/l), \qquad h\bar{h}\equiv 1 \bmod l,$$

and

$$\phi_+(x;u,v,w,z) = \frac{1}{2\pi i}\cos\left(\frac{\pi}{2}(v-z)\right)\int_{(\eta)} \Gamma(1+s-v-w)\Gamma(1+s-w-z)W(s,w)\left(\frac{x}{2}\right)^{u+v+w+z-2s-1}ds,$$

$$\begin{aligned}\phi_-(x;u,v,w,z) = \frac{1}{2\pi i}\int_{(\eta)} & \cos\left(\frac{\pi}{2}(v+2w+z-2s)\right)\Gamma(1+s-v-w)\Gamma(1+s-w-z) \\ & \times W(s,w)\left(\frac{x}{2}\right)^{u+v+w+z-2s-1}ds.\end{aligned}$$

The double sum at (2.5) and the last integrals are all absolutely convergent in the domain (2.4). Hence the expression (2.5) yields a meromorphic continuation of $Y(u,v,w,z;G;b/a)$. However, the point $P_T$ is not contained in (2.4). Thus an analytic continuation of $\mathcal{Y}_0(u,v,w,z;G;d/c)$ to a neighbourhood of $P_T$ is required. This is accomplished after expanding $\mathcal{X}_\pm(m,n;u,v,w,z;G;d/c)$ spectrally by means of a version of Kuznetsov's trace formulas for the Hecke congruence subgroups.

## 3. - Trace formulas

We are now going to exhibit the trace formulas that we use. We stress that throughout this section the parameter $q$ is assumed to be square-free.

By the general theory we have

$$\{ \text{ cusps } \} \equiv \{ \frac{1}{w};\ w|q \} \bmod \Gamma_0(q).$$

Deshouillers and Iwaniec constructed their important theory [3] on this choice of the representative set of cusps, though here the situation is simplified by the condition $\mu(q) \neq 0$. Our choice is different from theirs. We map each point $1/w$ to a point equivalent mod $\Gamma_0(q)$ in a special way. For this sake we write $q = wv$, and also $q = w_i v_i$ in the sequel. Then we consider the congruence (cf. Hejhal [6, p.534])

$$\xi_w := \begin{pmatrix} k & l \\ q & s \end{pmatrix} \begin{pmatrix} 1 & \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ & 1 \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \bmod w, \\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} & \bmod v, \end{cases}$$

where

$$\begin{pmatrix} k & l \\ q & s \end{pmatrix} \in \Gamma_0(q).$$

This has a solution such that

$$k = 1 + w - w\overline{w}, \quad w\overline{w} \equiv 1 \bmod v,$$

$$l \equiv -1 \bmod v, \quad l \equiv -f \bmod w,$$

$$f \equiv -\overline{w} \bmod v.$$

Hereafter let $\xi_w$ stand for such a solution. Then we write

$$[w] = \xi_w(\infty).$$

The points $[w]$, $w|q$, constitute obviously a representative set of inequivalent cusps of $\Gamma_0(q)$. Further, we introduce

$$\varpi_w = \xi_w \begin{pmatrix} \sqrt{v} & \\ & \frac{1}{\sqrt{v}} \end{pmatrix}. \tag{3.1}$$

so that we have

$$\varpi_w(\infty) = [w], \quad \text{and} \quad \varpi_w^{-1}\Gamma_{[w]}\varpi_w = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix},\ n \in \mathbb{Z} \right\},$$

where $\Gamma_{[w]}$ is the stabilizer of $[w]$ in $\Gamma_0(q)$.

Now we consider the Poincaré series

$$U_m(z,[w];s)=\sum_{g\in\Gamma_{[w]}\backslash\Gamma_0(q)}(\operatorname{Im}\varpi_w^{-1}g(z))^s e(m\varpi_w^{-1}g(z)).\qquad (m\in\mathbb{Z})$$

The Fourier expansion of $U_m(z,[w_1];s)$ around the cusp $[w_2]$ is as follows:

$$\begin{aligned}&U_m(\varpi_{w_2}(z),[w_1];s)=\delta_{w_1,w_2}y^s e(m(z+b_{w_1,w_2}))\\&+y^{1-s}\sum_{n=-\infty}^{\infty}e(nx)\sum_{((v_1,w_2)(v_2,w_1),r)=1}\frac{S(\overline{(v_1,w_2)}m,\overline{(w_1,v_2)}n;(v_1,v_2)(w_1,w_2)r)}{((w_1,w_2)r\sqrt{v_1v_2})^{2s}}\\&\qquad\times\int_{-\infty}^{\infty}\exp\Big(-2\pi iny\xi-\frac{2\pi m}{((w_1,w_2)r\sqrt{v_1v_2})^2y(1-i\xi)}\Big)(1+\xi^2)^{-s}d\xi.\end{aligned}\tag{3.2}$$

Here $\delta$ is the Kronecker delta, $b_{w_1,w_2}$ a certain real number, $S$ the Kloosterman sum; and the bars denote congruence inverses mod $(v_1,v_2)(w_1,w_2)r$. Thus, in particular, $U_m(\varpi_{w_2}(z),[w_1];s)$ is regular for $\mathrm{Re}(s)>3/4$ and in $L^2(\Gamma_0(q)\setminus \mathrm{H})$ whenever $m>0$.

The last identity yields, in particular, the following Fourier expansion of the Eisenstein series $E(z,[w];s)=U_0(z,[w];s)$: For any combination of cusps $[w_1]$ and $[w_2]$

$$\begin{aligned}E(\varpi_{w_2}(z),[w_1];s)&=\delta_{w_1,w_2}y^s+\sqrt{\pi}y^{1-s}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\phi_0(s;w_1,w_2)\\&+2y^{\frac12}\frac{\pi^s}{\Gamma(s)}\sum_{n\neq0}|n|^{s-\frac12}\phi_n(s,w_1,w_2)K_{s-\frac12}(2\pi|n|y)e(nx)\end{aligned}\tag{3.3}$$

with

$$\phi_n(s;w_1,w_2)=\frac{\sigma_{1-2s}(n,\chi_q)}{L(2s,\chi_q)((v_1,w_2)(v_2,w_1))^s}\prod_{p|(v_1,v_2)(w_1,w_2)}\Big(\sigma_{1-2s}(n_p)(1-p^{-2s})-1\Big),$$

where $\chi_q$ is the principal character mod $q$, $\sigma_\alpha(n,\chi)=\sum_{d|n}d^\alpha\chi(d)$, and $n_p=(n,p^\infty)$. We note

$$\phi_0(s;w_1,w_2)=q^{-2s}\varphi((v_1,v_2)(w_1,w_2))\frac{\zeta(2s-1)}{L(2s,\chi_q)}\prod_{p|(v_1,w_2)(v_2,w_1)}(p^s-p^{1-s}),\tag{3.4}$$

(cf. Hejhal [6, p.535]).

A version of the Kuznetsov trace formulas for $\Gamma_0(q)$ follows from (3.2) and (3.3) (cf. [7] [8]). But, before stating them we need to introduce some notation from the theory of automorphic forms: Thus, let $\{\lambda_j=\frac14+\kappa_j^2>0\}$ be the set of all eigenvalues of $-y^2\Delta$ on the Hilbert space $L^2(\Gamma_0(q)\setminus\mathrm{H})$.

Let $\psi_j$ be the Maass wave corresponding to $\lambda_j$ so that the set $\{\psi_j\}$ forms an orthonormal system. The Fourier expansion of $\psi_j(z)$ around the cusp $[w]$ is denoted by

$$\psi_j(\varpi_w(z)) = \sqrt{y}\sum_{n\neq 0}\rho_j(n,[w])K_{i\kappa_j}(2\pi|n|y)e(nx).$$

Also let $\{\psi_{k,j};\ 1 \le j \le \vartheta_q(k)\}$ be an orthonormal base of the space of holomorphic cusp forms of weight $k$ with respect to $\Gamma_0(q)$. The Fourier expansion of $\psi_{k,j}(z)$ around the cusp $[w]$ takes the form

$$\psi_{k,j}(\varpi_w(z)) = \sum_{n>0} a_{k,j}(n,[w])e(nz).$$

Now the trace formulas that are to be applied to $\mathcal{X}_\pm(m,n;u,v,w,z;G;d/c)$ are embodied in the following two theorems:

**Theorem 1.**
*Let $m, n > 0$. Let $\phi(x)$ be sufficiently smooth for $x \ge 0$ and of rapid decay as $x$ tends either to $+0$ or to $+\infty$. Then we have*

$$\sum_{\substack{r=1\\((v_1,w_2)(w_1,v_2),r)=1}}^{\infty} \frac{1}{(w_1,w_2)r\sqrt{v_1v_2}} S(\overline{(v_1,w_2)}m, \overline{(w_1,v_2)}n; (v_1,v_2)(w_1,w_2)r)\phi\Big(\frac{4\pi\sqrt{mn}}{(w_1,w_2)r\sqrt{v_1v_2}}\Big)$$

$$= \sum_{j=1}^{\infty} \frac{\overline{\rho_j(m,[w_1])}\rho_j(n,[w_2])}{\cosh \pi\kappa_j}\hat{\phi}(\kappa_j) + \sum_{\substack{k=2\\2|k}}^{\infty} \frac{(k-1)!}{\pi^{k+1}4^{k-1}} \sum_{j=1}^{\vartheta_q(k)} \frac{\overline{a_{k,j}(m,[w_1])}a_{k,j}(n,[w_2])}{(mn)^{(k-1)/2}}\hat{\phi}(\tfrac{1}{2}(1-k)i)$$

$$+\frac{1}{\pi}\sum_{q=wv}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m,\chi_q)\sigma_{-2ir}(n,\chi_q)(n/m)^{ir}}{|L(1+2ir,\chi_q)|^2((v,w_1)(w,v_1))^{\frac{1}{2}-ir}((v,w_2)(w,v_2))^{\frac{1}{2}+ir}}$$

$$\times \prod_{p|(v,v_1)(w,w_1)} (\sigma_{2ir}(m_p)(1-p^{-1+2ir})-1) \prod_{p|(v,v_2)(w,w_2)} (\sigma_{-2ir}(n_p)(1-p^{-1-2ir})-1)\hat{\phi}(r)dr,$$

*where*

$$\hat{\phi}(r) = \frac{\pi i}{2\sinh \pi r}\int_0^{\infty} (J_{2ir}(x) - J_{-2ir}(x))\frac{\phi(x)}{x}dx.$$

**Theorem 2.**
*If $n$ is replaced by $-n$ on both sides of the last identity, then the equality still holds, provided $\check{\phi}$ plays the rôle of $\hat{\phi}$ but the contribution of holomorphic cusp forms is deleted, where*

$$\check{\phi}(r) = 2\cosh(\pi r)\int_0^{\infty} K_{2ir}(x)\frac{\phi(x)}{x}dx.$$

The condition on $\phi$ can be relaxed considerably, though we are not going to give the details (see the relevant part of [8]). Also our formulas should be compared with the corresponding formulas of Deshouillers and Iwaniec [3, Theorem 1].

### 4. - Spectral decomposition

Now we specialize the above discussion by setting

$$\begin{aligned} w_1 = c, \quad v_1 = d, \\ w_2 = q, \quad v_2 = 1, \end{aligned}$$

so that $cd = q$. Then we get the sum

$$\sum_{\substack{k=1 \\ (k,d)=1}}^{\infty} \frac{1}{ck\sqrt{d}} S(\bar{d}m, \pm n; ck)\phi(\frac{4\pi\sqrt{mn}}{ck\sqrt{d}}).$$

Hence Theorems 1 and 2 can be applied to $\mathcal{X}_{\pm}(m, n; u, v, w, z; G; d/c)$.

Here we should remark that the transforms $\hat{\phi}_+(r; u, v, w, z)$ and $\check{\phi}_-(r; u, v, w, z)$, which appear in this procedure, have been already studied in [10] (with a slightly different notation). We know in particular that they can be continued to functions that are meromorphic over the entire $\mathbb{C}^5$ and of rapid decay with respect to $r$ uniformly for any finite $(u, v, w, z)$. Thus, as far as the condition (2.4) is satisfied, we may insert the resulting spectral decomposition of $\mathcal{X}_{\pm}$ into the formula (2.5) and exchange the order of sums freely. This implies immediately that our problem has been reduced to the study of the functions

$$\begin{aligned} L_j(s, [c]) &= \sum_{m>0} \rho_j(m, [c]) m^{-s}, \\ L_{k,j}(s, [c]) &= \sum_{m>0} a_{k,j}(m, [c]) m^{-s-\frac{1}{2}(k-1)}, \\ \mathcal{D}_j(s, \alpha; [q]) &= \sum_{m>0} \rho_j(m, [q]) \sigma_\alpha(m) m^{-s}, \\ \mathcal{D}_{k,j}(s, \alpha; [q]) &= \sum_{m>0} a_{k,j}(m, [q]) \sigma_\alpha(m) m^{-s-\frac{1}{2}(k-1)}. \end{aligned}$$

We give some details on $L_j$ and $\mathcal{D}_j$ only, for those related to holomorphic forms are analogous and in fact easier.

For this sake we have to refine our definition relating to cusp forms slightly: We may assume that the system $\{\psi_j\}$ has been chosen in such a way that we have $\psi_j(-\bar{z}) = \varepsilon_j \psi_j(z)$, $\varepsilon_j = \pm 1$. Then we observe that the parity of $\psi_j(z)$ is inherited by $\psi_j(\varpi_c(z))$; i.e.,

$$\psi_j(\varpi_c(-\bar{z})) = \varepsilon_j \psi_j(\varpi_c(z)).$$

This is a consequence of the definition (3.1) of $\varpi_c$. Also let $\psi_j^*(z)$ stand for $\psi_j(-1/(qz))$. Then $\psi_j^*$ is a cusp form of the unit length with respect to $\Gamma_0(q)$. Again by virtue of (3.1) we have the relations

$$\psi_j(\varpi_c(-1/(qz))) = \psi_j^*(\varpi_c(z)),$$

$$\psi_j^*(\varpi_c(-\bar{z})) = \varepsilon_j \psi_j^*(\varpi_c(z)).$$

From these we can conclude that $L_j(s,[c])$ is entire and satisfies the functional equation

$$L_j(s,[c]) = \frac{1}{\pi}\left(\frac{\sqrt{q}}{2\pi}\right)^{1-2s}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)(\varepsilon_j\cosh\pi\kappa_j - \cos\pi s)L_j^*(1-s,[c]),$$

where $L_j^*(s,[c])$ is related to $\psi_j^*$ in the same way as $L_j(s,[c])$ does to $\psi_j$. A consequence of this equation is the bound

$$L_j(s,[c]) \ll_q \kappa_j^A e^{\pi\kappa_j/2} \tag{4.1}$$

that holds uniformly for bounded $s$, where $A$ depends only on $\mathrm{Re}(s)$.

Next let us consider $\mathcal{D}_j(s,\alpha;[q])$. We may assume without loss of generality that $\psi_j$ is even, or $\varepsilon_j = +1$. We then introduce

$$\mathcal{D}_j(s,\alpha;[w]) = v^{-s+(\alpha+1)/2}\sum_{n>0}\rho_j(vn,[w])\sigma_\alpha(n)n^{-s},$$

where $q = wv$ as before, and put

$$\mathcal{D}_j^*(s,\alpha;[w]) = L(2s-\alpha,\chi_q)\mathcal{D}_j(s,\alpha;[w]).$$

It should be remarked that this factor $L(2s-\alpha,\chi_q)$ essentially cancels out the factor $\zeta(u+v)$ in (2.1) when we apply the result of this section to our original problem. Further let $E(z,s)$ be the Eisenstein series for the full modular group, and put

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s).$$

Similarly we put

$$E^*(z,[w];s) = \pi^{-s}\Gamma(s)L(2s,\chi_q)E(z,[w];s).$$

Then we have, under an appropriate condition on $(s,\alpha)$ to secure absolute convergence,

$$\mathcal{D}_j^*(s,\alpha;[w]) = \frac{2}{\Gamma(s,\alpha,i\kappa_j)}\int_{\Gamma_0(q)\backslash\mathrm{H}}\psi_j(z)E^*(z,(1-\alpha)/2)E^*(z,[w];s-\alpha/2)\frac{dxdy}{y^2}, \tag{4.2}$$

where

$$\Gamma(s,t,u) = \Gamma(\tfrac{1}{2}(s+u))\Gamma(\tfrac{1}{2}(s-u))\Gamma(\tfrac{1}{2}(s-t+u))\Gamma(\tfrac{1}{2}(s-t-u)).$$

The relation (4.2) and the expansion (3.3) yield immediately that

$$(\alpha^2-1)((2s-\alpha-1)^2-1)\mathcal{D}_j^*(s,\alpha;[w])$$

is entire over $\mathbb{C}^2$. Also (4.2) implies the functional equation for $\mathcal{D}_j^*(s,\alpha;[w])$: Since (3.4) gives the scattering matrix of $\Gamma_0(q)$, we have, for any $w_1|q$,

$$\mathcal{D}_j^*(s,\alpha;[w_1]) = q^{-2s+\alpha}\prod_{p|q}(1-p^{2s-\alpha-2})^{-1}$$
$$\times\sum_{w_2|q}\varphi((v_1,v_2)(w_1,w_2))\prod_{p|(v_1,w_2)(v_2,w_1)}(p^{s-\alpha/2}-p^{1+\alpha/2-s})\mathcal{D}_j^*(1-s,-\alpha;[w_2]).$$

From this we may deduce that, if $(s,\alpha)$ is well-off the polar set and remains in an arbitrary compact set, then

$$\mathcal{D}_j^*(s,\alpha;[w]) \ll_q \kappa_j^A e^{\pi\kappa_j/2}, \tag{4.3}$$

where $A$ depends only on the real parts of $s$ and $\alpha$.

**5. - The explicit formula**

What remains is now straightforward. The results of the preceding section (especially (4.1) and (4.3)) and their obvious counterpart for holomorphic forms imply that we have a meromorphic continuation of $\zeta(u+v)\mathcal{Y}_0(u,v,w,z;G;d/c)$ to the entire $\mathbb{C}^4$. Also it is easy to see that the contribution to it of the holomorphic and non-holomorphic cusp forms is regular at the point $P_T$. Thus the specialization $(u,v,w,z)=P_T$ causes no trouble in that part. The contribution of the continuous spectrum is quite involved; however we may deal with it in much the same way as we did in the corresponding part of [10].

Then, after a somewhat tedious rearrangement, we obtain our main result:

**Theorem 3.**
*Let*

$$\mathcal{A}(s)=\sum_n \alpha_n n^{-s},$$

*be a Dirichlet polynomial, where $\alpha_n=0$ unless $n$ is square-free. Then we have, for any sufficiently large $T$ and $G$ such that $G\le T(\log T)^{-1}$,*

$$\frac{1}{G\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\tfrac12+i(T+t))|^4|\mathcal{A}(\tfrac12+i(T+t))|^2e^{-(t/G)^2}dt$$
$$= \textit{Main term}(T,G;\mathcal{A})$$
$$+\sum_{\substack{a,b,r\\(a,b)=1}}\frac{\alpha_{ar}\overline{\alpha}_{br}}{abr}\sum_{\substack{c|a\\d|b}}\frac{(cd)^2}{\varphi(cd)}\{\mathcal{K}(c,d;T,G)+\mathcal{H}(c,d;T,G)\}$$
$$+\frac{1}{\pi}\sum_{\substack{a,b,r\\(a,b)=1}}\frac{\alpha_{ar}\overline{\alpha}_{br}}{\varphi(ab)r}\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+ir)|^6}{|\zeta(1+2ir)|^2}\prod_{p|ab}\left\{4|1+\frac{1}{p^{\frac12+ir}}|^{-2}-\frac1p\right\}\Theta_0(r;T,G)dr.$$

*Here*

$$\mathcal{K}(c,d;T,G)=\sum_{\substack{j=1\\ \frac14+\kappa_j^2\in Sp(\Gamma_0(cd))}}^{\infty}\frac{1}{\cosh\pi\kappa_j}\overline{L_j(\tfrac12;[c])\mathcal{D}_j^*(\tfrac12,0;[cd])}\Theta_j(\kappa_j;T,G),$$

$$\mathcal{H}(c,d;T,G)=16\sum_{\substack{j=2\\ 2|k}}^{\infty}(-1)^{k/2}\frac{(k-1)!}{(4\pi)^{k+1}}\sum_{j=1}^{\vartheta_{cd}(k)}\overline{L_{k,j}(\tfrac12;[c])\mathcal{D}_{k,j}^*(\tfrac12,0;[cd])}\Xi(\tfrac12(k-1);T,G);$$

*and*

$$\Theta_j(r;T,G)=\mathrm{Re}\left\{\left(1+\frac{i\varepsilon_j}{\sinh\pi r}\right)\Xi(ir;T,G)\right\},$$

$$\Xi(\xi;T,G)=\frac{\Gamma(\frac12+\xi)^2}{\Gamma(1+2\xi)}\int_0^\infty x^{-\frac12+\xi}(x+1)^{-\frac12}\cos(T\log(1+x))$$
$$\times F(\tfrac12+\xi,\tfrac12+\xi;1+2\xi;-x)\exp(-(\frac{G}{2}(\log(1+x))^2)dx,$$

*where* $\varepsilon_0=1$ *and* $F$ *is the hypergeometric function.*

This should be compared with the theorem of [10]. The main term is essentially a biquadratic polynomial of $\log T$; it is possible to make it explicit in terms of $\{\alpha_n\}$, $T$, $G$. Also the function $\mathcal{D}_{k,j}^*$ is defined analogously as $\mathcal{D}_j^*$. All sums and integrals in the above are absolutely convergent.

*Concluding Remarks*:

1. This is related to the second footnote. The argument in the above is an extended version of our original (unpublished) proof of the theorem of [10]. As it is apparent, we have exploited the inner structure of the divisor function $\sigma_\alpha(n)$. But in [10] we used the fact that $\sigma_\alpha(n)$ appears as the Fourier coefficient of the Eisenstein series for the full modular group, and thus it is more akin to the theory of automorphic forms. The present argument would not extend immediately to a similar problem involving Hecke series, say, in place of the zeta-function. It has, however, the advantage that it can be applied to Dirichlet $L$-functions as well, whereas the argument of [10] has some difficulty to extend to such a direction.

2. The functions $\mathcal{D}_j^*(s,\alpha;[q])$ and $\mathcal{D}_{k,j}^*(s,\alpha;[q])$ can be related to products of two Hecke series (note that $\varpi_q\in\Gamma_0(q)$). This can be proved by appealing to Atkin-Lehner's theory [1] on *new forms*. Then the appearance of Theorem 3 would become closer to that of the theorem of [10], where we have objects like $|\rho_j(1)|^2H_j(\frac12)^3$ instead of the crude $\overline{L_j(\frac12,[c])\mathcal{D}_j^*(\frac12,0;[cd])}$.

3. The relation between the zeta-function and the Hecke congruence subgroups can be made more explicit than the way in which Theorem 3 exhibits it. Following the argument developed in our recent paper [12] we consider the function

$$Z_2(\xi;\mathcal{A})=\int_1^\infty|\zeta(\tfrac12+it)|^4|\mathcal{A}(\tfrac12+it)|^2t^{-\xi}dt.$$

We can show, by a modification of the above argument, that $Z_2(\xi;\mathcal{A})$ is meromorphic over the entire $\mathbb{C}$. There is a trivial pole of the fifth order at $\xi = 1$. It is possible to have several simple poles on the segment $(\frac{1}{2},\ \frac{3}{4})$, which correspond to exceptional eigenvalues of the non-Euclidean Laplacian. All other poles are located in the half plane $\mathrm{Re}(\xi) \le \frac{1}{2}$; and on the line $\mathrm{Re}(\xi) = \frac{1}{2}$ we may have infinitely many simple poles of the form $\frac{1}{2} \pm i\kappa_j$. But, in order to make the last statement rigorous, we have to prove a certain *non-vanishing* theorem for sums involving $\overline{L_j(\frac{1}{2},[c])}\mathcal{D}_j^*(\frac{1}{2},0;[cd])$. In the case of $\mathcal{A} \equiv 1$ we have proved such a non-vanishing result in [9]. It should be mentioned that if there are exceptional eigenvalues, then it would imply that for certain $\mathcal{A}$ the asymptotic formula for the mean value

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 |\mathcal{A}(\tfrac{1}{2}+it)|^2 dt$$

had to have the *second* main term of the order $T^\theta$ with $\frac{1}{2} < \theta < \frac{3}{4}$. This appears to be very unlike.

## REFERENCES


[1] A. O. ATKIN - J. LEHNER. *Hecke operators on* $\Gamma_0(m)$. Math. Ann., **185**, 134-160(1970).

[2] F. V. ATKINSON. *The mean value of the Riemann zeta-function.* Acta Math., **81**, 353-376 (1949).

[3] J.-M. DESHOUILLERS - H. IWANIEC. *Kloosterman sums and Fourier coefficients of cusp forms.* Invent. math., **70**, 219-288(1982).

[4] J.-M. DESHOUILLERS - H. IWANIEC. *Power mean-values for Dirichlet's polynomials and the Riemann zeta-function.* Mathematika, **29**, 202-212(1982).

[5] J.-M. DESHOUILLERS - H. IWANIEC. *Power mean-values for Dirichlet's polynomials and the Riemann zeta-function. II.* Acta Arith., **43**, 305-312(1984).

[6] D. A. HEJHAL. *The Selberg trace formula for* $PSL(2,\mathbb{R})$. *II.* Lect. Notes in Math., vol.**1001**, Springer, Berlin, 1983.

[7] N. V. KUZNETSOV. *Petersson hypothesis for parabolic forms of weight zero and Linnik hypothesis. Sums of Kloosterman sums.* Mat. Sbornik, **111**, 334-383(1980). (Russian).

[8] Y. MOTOHASHI. *On Kuznetsov's trace formulae.* Manuscript, 1995.

[9] Y. MOTOHASHI. *Spectral mean values of Maass wave form L-functions.* J. Number Theory, **42**, 258-284(1992).

[10] Y. MOTOHASHI. *An explicit formula for the fourth power mean of the Riemann zeta-function.* Acta Math., **170**, 181-220(1993).

[11] Y. MOTOHASHI. *On the mean square of the product of the zeta- and L-functions.* Kōkyūroku RIMS, **837**, 57-62(1993).

[12] Y. MOTOHASHI. *A relation between the Riemann zeta-function and the hyperbolic Laplacian.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (to appear).