数理解析研究所講究録
1060巻 1998年 220-223

Mitsui's Dirichlet series for 2×2 symmetric matrices

Shigeki Egami
Faculty of Engneering, Toyama University
3190 Gofuku, Toyama city, 930 JAPAN
e-mail:megami@eng.toyama-u.ac.jp

April 6, 1998

1 Introduciton.

In 1982, in a conference of MSJ, late professor T. Mitsui proposed the following problem:

Problem 1 *Let L_n be the lattice of integral symmetric matrices of order n. Set*

$$F_n(s) = \sum_{T \in L_n, T>0} trace(T)^{-s}, \qquad \Re(s) > \frac{n(n+1)}{2} = \nu.$$

Is it possible to continue $F_n(s)$ meromorphically to the whole s-plane?

It was pointed out by him(and is not difficult to show) that $F_n(s)$ can be continued to $Re(s) > \nu - 1$ and holomorphic except $s = \nu$ which is a pole of order 1 with residue

$$c_\nu = Vol\{x \in Sym_n(\mathbf{R}); x > 0, trace(x) \leq 1\}$$

In this paper we report an affirmative answer for n=2.

2 Source of the idea.

The following two ideas are fundamental for our solution. The first that was proposed by F. Sato in a discussion with the author about this problem is to consider in the framework in the theory of automorphic forms on Lie groups. That is to study

$$F_n(x, s) = \sum_{T \in L_n, T>0} trace(xTx^T)^{-s}$$

as an automorphic function on $SO_n(\mathbf{R})\backslash SL_n(\mathbf{R})/SL_n(\mathbf{Z})$.

The second is contained in an old result of Hecke. Let K be a real quadratic field. He studied the Dirichlet series

$$\Phi_K(s) = \sum_{\alpha:\text{totally positive integer}\in K} trace_K(\alpha)^{-s}.$$

and

$$\Phi_K(x,s) = \sum_{\alpha:\text{totally positive integer}\in K} (\alpha\varepsilon^x + \alpha'\varepsilon^{-x})^{-s}$$

, where ε denotes the totally positive fundamental unit in K. Since the latter is a periodic (automorphic on $\mathbf{R}/\mathbf{Z}$!)function we can apply fourier series expansion. The resulting formula calculated by Hecke is

$$\Phi_K(x,s) = \sum_m h(m,s)\zeta(\lambda_m,s),$$

where $h(m,\cdot)$ is a certain "gamma factor" and $\zeta(\lambda,s)$ is the zeta function associated to the "Grössencharakter" λ. Since the each terms of the infinite series is meromorphic and the convergence is sufficiently rapid Hecke was able to show moromorphy of $\Phi_K(x,s)$ (hence $\Phi_K(s)$ also).

The similarity to our problem may be clear. However in applying Hecke's method we encounter the following questions:

1. What is "fourier series" in our case?

2. What is "zeta function with Grössencharacter" in our case?

Fortunatery, the above two problems were already studied extensively. The first is Selberg-Langlands spectral theory and the second is Siegel-Maass- Shintani-Hejhal-Sato zeta function associated to symmetric matrices. So that we can expect the full solution of Mitsui's problem along these theories. In fact the author gave a solution of an analogue of Mitsui's problem for n=2 in which the lattice L_2 is replaced by $L_{p,q}$ associated to the indefinete division quoternion $Q_{p,q}$([1]). In this case almost direct combination of existing results on 1. and 2. is sufficient. But original problem treated in this paper proposes additional difficulties concerning continuous spectrum already in the case n=2. For larger n, though we belive the effectiveness of the above method we should resolve much harder analytical problems in the actual implimentation.

3 Notation.

First we fix notations on automorphic spectral theory. Let H denotes the upper half plane and $\Gamma = SL_2(\mathbf{Z})$ are considerd to act on H as an a linear fractional transformation . A measurable function f on H is called automorphic if f

is invariant under Γ. For automorphic sunctions f and g we define the inner product

$$< f,g >= \int_F f(x+iy)\overline{g(x+iy)}\frac{dxdy}{y^2}.$$

where F denotes a fundamental domain of Γ. We denote $L^2(F)$ the Hilbert space of functions consists of $< f,f >< \infty$. Let $D = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ denotes the non-Euclidean Laplacian. Smooth automorphic function f such that $Df = -\lambda f$ is called automorphic eigenfunction for eigenvalue λ. Such functions are classified in the following three classes:

1. Eisenstein series

$$E(z,s) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}, \qquad \Re s > 1.$$

   and its analytic continuation with respect to s. It is easily seen that the corresponding eigenvalue is $s(s-1)$.

2. Constants.

3. Maass cusp forms $\{\phi_j(z)\}_{j=1}^{\infty}$ which forms an orthogonal bases of $L_0^2(F) = \{f \in L^2(F); f$ vanishes at infinity in $F\}$. We express the eigenvalue of ϕ_j as $\frac{1}{4}+r_j^2$.

Next we explain the zeta functions. For a positive matrix

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

we denote $z_T = x_T + iy_T$, where $x_T = \frac{b}{c}$ and $y_T = \frac{\sqrt{ac-b^2}}{c}$. Let $\phi(z)$ be an automorphic eigenfunction. Then we define the zeta function of 2×2 integral matrices with ϕ by

$$\zeta_+(\phi,u) = \sum_{\Gamma\backslash L+} \frac{\phi(z_T)}{E(T)det(T)^u} \quad ,\Re u > \frac{3}{2}$$

,where L^+ denotes the set of the positive elements of L_2. The summation is taken over all the equivalence classes defined by the action $x \mapsto \gamma x \gamma^T$, and $E(T) = |\{\gamma \in \Gamma; \gamma T \gamma^T = T\}|$. The result Siegel-Maass-Shintani-Hejhal-Sato([3])says that $\zeta_+(\phi,u)$ can be extended to a moremorphic function in the u-plane which is holomorphic except some explicitly located poles and satisfies certain functional equations. In the case of Eisenstein series the analyticity with parameter s is also established.

4 Outline of the proof.

Let $z = x + iy \in H$. Define

$$f(z,s) = \sum_{a,b,c \in \mathbf{Z}, a>0, ac-b^2>0} \frac{y^s}{(c(x^2+y^2) - 2bx + a)^s}$$

for $\Re s > 3$. It is not difficult to show that $f(z,s)$ is bounded for these s and invariant under Γ, so that an element of $L^2(F)$. Moreover we have

$$F_2(x^{-1}, s) = f(z_{(x^T x)}, s).$$

We apply the spectral theory on $L^2(F)$.

Proposition 1 *For $\Re s > 3$ we have*

$$\begin{aligned} f(z,s) &= \sum_{j=1}^{\infty} h(r_j, s)\zeta_+(\phi_j, \frac{s}{2}) + \frac{3}{\pi} h(\frac{i}{2}, s)\zeta_+(\mathbf{1}, \frac{s}{2}) \\ &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r,s)\zeta_+(E(\cdot, \frac{1}{2} + ir), \frac{s}{2})E(z, \frac{1}{2} - ir)dr \end{aligned}$$

, *where*

$$h(r,s) = \frac{\sqrt{\pi}\Gamma(\frac{1}{2}(s - \frac{1}{2} - ir))\Gamma(\frac{1}{2}(s - \frac{1}{2} + ir))}{4\Gamma(s)}.$$

The analytic continuation of the first(discrete) term is similar to [1]. The second term is trivial. As for the third term we must use an argument of Barnes(see [4]). Hence finally we have

Theorem 1 *f(z,s) can be continued to an meromorphic function on the whole s-plane.*

Since $f(i,s) = F_2(s)$ we have an affiamative solution of **Problem** for n=2.
Detailed proof will appear in [2].

References

[1] Shigeki Egami, RIMS Proceeding 886(1994).

[2] Shigeki Egami and Fumihiro Sato, in preparation

[3] Fumihiro Sato, preprint 1993.

[4] Don Zagier, Eisenstein seris and the Selberg trace formula I, in Proceeding of Bombay Colloquium 1979, Springer 1981.