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Inverse Bifurcation Problem and Multiplicative Wiener-Hopf Equation

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We illustrate a role that a class of Wiener-Hopf equations plays when one solves inverse problems by means of the implicit function theorem, through a study of an inverse problem in bifurcation theory. A general theory of the class of Wiener-Hopf equations is developed and applied to the inverse problem.

1. Introduction

In a common abstract framework, the relationship between a cause x and an effect y can be stated in terms of the equation

$$F(x,y)=0, \tag{1.1}$$

where F is a smooth map of a neighborhood of $(0,0)$ in $X\times Y$ into Z; X,Y,Z are Banach spaces. Then the direct problem is no other than to solve (1.1) for y. The classical implicit function theorem tells us that if

$$F(0,0)=0 \tag{1.2}$$

and if

$$F_y(0,0) \text{ is an isomorphism of } Y \text{ onto } Z, \tag{1.3}$$

where $F_y(0,0)$ denotes the Fréchet derivative of F with respect to y at $(0,0)$, then for x close to 0 there is a unique solution $\pi(x)$ with $\pi(0)=0$. This can be interpreted as that the direct problem is solved locally, in other words, that we can establish the correspondence $\pi(x)$ which assigns the effect y to the cause x near 0. In this situation we can pose the inverse problem, which is formulated as: to solve $\pi(x)=y$.

When one tends to apply the implicit function theorem again to (1.1) in order to solve this inverse problem, one needs to investigate the operator $F_x(0,0)$ from X to Z. The simplest case is: $F_x(0,0)$ is an isomorphism of X onto Y. In this case the direct map $\pi(x)$ is a diffeomorphism of a neighborhood of 0 in X to a neighborhood of 0 in Y, and therefore the inverse problem $\pi(x)=y$ has a unique solution x in a neighborhood of 0 for each y close to 0 and this unique solution depends smoothly on y. The next simplest case is:

$$\text{Range } F_x(0,0)=Z, \quad \text{Ker } F_x(0,0)=: X_1 \text{ is of finite dimension } n, \tag{1.4}$$

that is, $F_x(0,0)$ is surjective and the kernel is finite dimensional. In this case, by the implicit function theorem, one can see (referring to Lemma 3.1) that there are a neighborhood $U_1 \times V$ of $(0,0)$ in $X_1 \times Y$, a neighborhood U of 0 in X, and a diffeomorphism G of $U_1 \times V$ onto U such that the diagram

(1.5)
$$\begin{tikzcd} X_1 \times Y \supset U_1 \times V \arrow[r, "G", "\approx"'] \arrow[dr, "p_Y"'] & U \subset X \arrow[d, "\pi"] \\ & V \subset Y \end{tikzcd}$$

commutes, where p_Y denotes the projection: $X \times Y \to Y$. This means that the inverse problem $\pi(x) = y$ has solutions $G(x_1, y)$ parametrized by x_1 in n dimensinal parameter set U_1 for each $y \in Y$ close to 0. Thus, under the assumptions (1.2), (1.3), (1.4), the cause x are determined from the effect y with n dimensinal *ambiguity.*

In practical inverse problems, what sort of structures enables us to carry out the scheme mentioned above in the abstract framework? In many inverse problems in the fields of mathematical sciences, the cause-effect relationships (1.1) are stated in the form of nonlinear integral equations or such kind of equations, and hence $F_x(0,0)$ in (1.4) become integral operators. As is well known, if they are the Wiener-Hopf operators with non-positive indices then the condition (1.4) is fulfilled. This is just the situation in which the scheme becomes vivid.

In the present article we illustrate how the scheme can be carried out through a study of an inverse problem in bifurcation theory, which we have treated in the work [1]. Our purpose is two-fold: first, to explain a practical role that the theory of the Wiener-Hopf equation plays; and secondly, to obtain a more general result for the inverse problem than in [1].

We now pose the problem in bifurcation theory. Let λ be a real parameter, q be a real continuous function, g be a C^1-function with $g(0) = g'(0) = 0$, and consider the nonlinear Sturm-Liouville problem

$$\begin{cases} v'' + [\lambda - q(x)]v = g(v), & x \in [0,1]; \\ v'(0) = v(1) = 0. \end{cases} \tag{1.6}$$

By the assumption on g, the linearized problem of (1.6) near $v \equiv 0$ becomes:

$$\begin{cases} v'' + [\lambda - q(x)]v = 0, & x \in [0,1]; \\ v'(0) = v(1) = 0. \end{cases} \tag{1.7}$$

Let λ_1 be the first eigenvalue of this problem and define the first bifurcating branch $\Gamma(g)$ to be the set of the points (λ, h) for which there exists a solution v of (1.6) such that $v(0) = h$, $v(x) \neq 0$ for $0 \leq x < 1$, and the point $(\lambda_1, 0)$. Let us denote by $v(h, x; g, \lambda)$ the solution of the the initial value problem

$$\begin{cases} v'' + [\lambda - q(x)]v = g(v), & x \in [0,1]; \\ v(0) = h, \ v'(0) = 0. \end{cases} \tag{1.8}$$

Then, for $h \neq 0$, a point (λ, h) belongs to $\Gamma(g)$ if and only if $v(h, 1; g, \lambda) = 0$, provided that g and $\mu := \lambda - \lambda_1$ are sufficiently small. Hence, introducing a map F for small functions g, μ

by

$$F(g,\mu)(h) := \begin{cases} h^{-1}v(h,1;g,\lambda_1+\mu(h)) & (h \neq 0), \\ 0 & (h = 0), \end{cases} \tag{1.9}$$

we see that $(\lambda_1 + \mu(h), h) \in \Gamma(g)$ if and only if $F(g, \mu) = 0$. On the other hand it follows from an elementary argument (see the proof of [1, Theorem 2.3]) that if g is small then λ for which $(\lambda, h) \in \Gamma(g)$ is unique for each h. So we arrive at

$$\Gamma(g) = \{(\lambda_1 + \mu(h), h)\} \Longleftrightarrow F(g,\mu) = 0, \tag{1.10}$$

provided that g, μ are small.

In this way the relationship between the nonlinearity (cause) and the first bifurcating branch (effect) can be written in the form (1.1). Clearly (1.2) is satisfied. In this problem one can easily compute the Fréchet derivative $F_\mu(0,0)$ formally (leaving the function spaces setting till §3) as follows:

$$F_\mu(0,0)\mu = v_1'(1)^{-1} \int_0^1 v_1(x)^2 dx \cdot \mu, \tag{1.11}$$

where $v_1(x)$ is the eigenfunction of (1.7) corresponding to λ_1, normalized by the condition $v_1(0) = 1$. Since this operator is just a multiplication with nonzero constant, (1.3) is satisfied. Hence the direct map $\pi\colon g \mapsto \mu$ is established. This result is naturally corresponding to the well-known fact that solutions of (1.6) with no zeros in $[0, 1)$ bifurcate at the first eigenvalue of the linearized problem (1.7).

We can now formulate the inverse problem (Inverse Bifurcation Problem): *given* μ, *find* g *satisfying* $\pi(g) = \mu$. In terms of the first eigenfunction $v_1(x)$ the Fréchet derivative $F_g(0,0)$ is computed as

$$F_g(0,0)g(h) = h^{-1}(-v_1'(1))^{-1} \int_0^1 v_1(x) g(hv_1(x)) dx. \tag{1.12}$$

Assuming that $v_1(x)$ is monotonically decreasing : $v_1'(x) < 0$ $(0 < x \leq 1)$ and setting

$$\Psi(t) := \frac{t}{v_1'(1)v_1'(v_1^{-1}(t))} \tag{1.13}$$

with the inverse function v_1^{-1} of v_1, we rewrite (1.12) in the form

$$F_g(0,0)g(h) = h^{-1} \int_0^1 \Psi(t) g(ht) dt. \tag{1.14}$$

This operator can be regarded as a kind of the Wiener-Hopf operators. In fact, for $0 \leq h \leq b$, changing variables via $h = be^{-x}$, $t = e^{x-y}$ and setting $\varphi(x) := g(be^{-x})$, $k(x) := \Psi(e^x)e^x\chi_{(-\infty,0]}(x)$ (χ denotes the characteristic function) yield

$$\int_0^1 \Psi(t) g(ht) dt = \int_0^\infty k(x-y)\varphi(y) dy. \tag{1.15}$$

This is a Wiener-Hopf operator, which has the pecularities: (i) of the first kind; (ii) with the kernel $k(x)$ being singular at $x = 0$; (iii) with the kernel $k(x)$ vanishing for $x > 0$. Because of (iii), condition (1.4) in our scheme is expected to be realized.

In §2, a general analysis of integral operators J_Φ defined by $J_\Phi u(x) = \int_0^1 \Phi(t)u(xt)dt$ is developed. Under some assumptions on Φ and in an appropriate framework, it is shown that J_Φ *is surjective if* $D_\Phi(z) := \int_0^1 \Phi(t)t^z dt$ *has no zeros on the imaginary axis* $\mathrm{Re}\, z = 0$. The precise formulation is given in Theorem 2.2, which is an extension of the result in our earlier work [2]. In §3, Theorem 2.2 is applied to the inverse bifurcation problem. Under some assumptions on the potential q and in an appropriate function spaces setting, it is shown that *given small function* $\mu(h)$ *on the interval* $a \le h \le b$ *with* $a < 0 < b$, *there exist nonlinear terms* g *such that the section in* $a \le h \le b$ *of the first bifurcating branch* $\Gamma(g)$ *coincides with* $\{(\lambda_1 + \mu(h), h) \,|\, a \le h \le b\}$. *For each* μ, *the set of the nonlinear terms is a* $2N$*-dimensional submanifold in a neighborhood of the origin in a function space, where* N *is the number of zeros of* $\zeta_q(z) := \int_0^1 v_1(x)^z dx$ *in the half plane* $\mathrm{Re}\, z > \alpha + 2$, *with some* $\alpha \in (0, 1/2)$. The precise formulation is given in Theorem 3.4, which gives a generalization of the result in our earlier work [1].

2. Multiplicative Wiener-Hopf Equation

This section is concerned with integral operators J_Φ defined by

$$J_\Phi u(x) := \int_0^1 \Phi(t)u(xt)dt, \tag{2.1}$$

where $\Phi(t)$ are given functions in $L^1[0,1]$. We use the notation

$$D_\Phi(z) := \int_0^1 \Phi(t)t^z dt. \tag{2.2}$$

If $\Phi \in L^1[0,1]$ then $D_\Phi(z)$ is holomorphic in the region $\mathrm{Re}\, z > 0$, and continuous in the right half plane $\mathrm{Re}\ z \ge 0$. As a matter of convenience we adopt the following:

Definition 2.1. A function $\Phi \in L^1[0,1]$ is said to be non-resonant if $D_\Phi(z)$ has no zeros on the imaginary axis.

Let $I := [a,b]$ be a bounded interval with $a \le 0 \le b$ and set, for functions $u(x)$ on I,

$$||u||_0 := \sup_{x \in I \setminus \{0\}} |u(x)|, \quad ||u||_\alpha := \sup_{x,y \in I\setminus\{0\}, x \ne y} \frac{||y|^\alpha u(y) - |x|^\alpha u(x)|}{|y-x|^\alpha},$$

where $0 < \alpha < 1$. By using the notation $\theta := x\frac{d}{dx}$ we define, for $0 < \alpha < 1$, $n = 1, 2, ...$,

$$C^{n,\alpha}(I) := \{u \in C^n(I \setminus \{0\}) \,|\, \sum_{i=0}^n ||\theta^i u||_0 + ||\theta^n u||_\alpha < \infty\}, \tag{2.3}$$

where $C^n(G)$ denotes the space of functions having continuous derivatives up to the order n on G. The space $C^{n,\alpha}(I)$ become Banach spaces with the norms

$$||u||_{n,\alpha} := \sum_{i=0}^n ||\theta^i u||_0 + ||\theta^n u||_\alpha.$$

Our main goal here is to establish the following:

Theorem 2.2. *Let n be a nonnegative integer. Let $\Phi(t) \in C^{n+2}(0,1)$ and assume that*
$(\mathrm{I})_n$ $\Phi, \theta\Phi, \cdots, \theta^n\Phi \in L^1[0,1]$, *and moreover* $\theta^n\Phi(t)$ *is written as*

$$\theta^n\Phi(t) = At^{\epsilon-1}(1-t^\beta)^{\delta-1} + R(t), \quad \beta, \epsilon > 0,\ 0 < \delta < 1,$$

where $A \neq 0$ and $R(t) \in C(0,1] \cap C^2(0,1)$ satisfies the conditions

$$|R(t)| \leq Mt^{\nu-1},\ |R'(t)| \leq Mt^{\nu-2}(1-t)^{\rho-1},\ |R''(t)| \leq Mt^{\nu-3}(1-t)^{\rho-2} \tag{2.4}$$

with some $M, \nu, \rho > 0$;

(II)$_n$ $\Phi(1) = \theta\Phi(1) = \cdots = \theta^{n-1}\Phi(1) = 0$.

(For $n = 0$, (II)$_n$ are dropped.) Let $0 < \alpha < 1 - \delta$. If Φ is non-resonant then, for each nonnegative integer k, the operator $J_\Phi : \mathcal{C}^{k,\alpha}(I) \to \mathcal{C}^{k+n,\alpha+\delta}(I)$ is surjective. The kernel Ker J_Φ *is of finite dimension N in the case $ab = 0$ and $2N$ in the case $ab < 0$. Here N is the number of zeros of $D_\Phi(z)$ (counted by multiplicity) in* Re $z > 0$. *The kernel is spanned by the functions*

$$\chi_{[a,0]}|x|^{b_p}(\log|x|)^\ell, \quad \chi_{[0,b]}|x|^{b_p}(\log|x|)^\ell \quad (\ell = 0, 1, ..., N_p - 1;\ p = 1, 2, ..., m),$$

where $b_1, ..., b_m$ denote the mutually distinct zeros of $D_\Phi(z)$ and $N_1, ..., N_m$ are the multiplicities of them.

The proof of Theorem 2.2 is divided into several steps and will be completed after Lemma 2.9. In our argument the differential operators

$$\theta_a := \theta + a = x\frac{d}{dx} + a \cdot \quad (a \in \boldsymbol{C}) \tag{2.5}$$

play an important role. Note that, for any function u with $u, \theta u \in C(I)$, we have

$$\theta_a J_\Phi u = J_\Phi \theta_a u \quad (a \in \boldsymbol{C}) \tag{2.6}$$

The basic properties of θ_a are stated as follows:

Lemma 2.3. *Let n be any nonnegative integer.*

(i) *If* Re $a > 0$ *then θ_a: $\mathcal{C}^{n+1,\alpha}(I) \to \mathcal{C}^{n,\alpha}(I)$ is bijective, and the inverse of θ_a is given by $j_a := J_{t^{a-1}}$.*

(ii) *If* Re $a < 0$ *then θ_a: $\mathcal{C}^{n+1,\alpha}(I) \to \mathcal{C}^{n,\alpha}(I)$ is surjective, and is not injective*: Ker $\theta_a = \boldsymbol{C}x^{-a}$.

Proof. (i) It is clear that θ_a is a bounded operator from $\mathcal{C}^{n+1,\alpha}(I)$ to $\mathcal{C}^{n,\alpha}(I)$. Let $\theta_a u = 0$ for $u \in \mathcal{C}^{n+1,\alpha}(I)$. This equation can be solved as $u = Cx^{-a}$ with a constant C. By Re $a > 0$, $u \in \mathcal{C}^{n+1,\alpha}(I)$ implies $C = 0$. This shows that θ_a is injective. Moreover, for each $v \in \mathcal{C}^{n,\alpha}(I)$, the function $(j_a v)(x) = x^{-a}\int_0^x t^{a-1}v(t)dt$ satisfies $\theta_a j_a v = v$. It only remains to show that $j_a v \in \mathcal{C}^{n+1,\alpha}(I)$. By (2.6) we have $\theta^n j_a v = j_a \theta^n v$, which yields

$$\theta^{n+1} j_a v = \theta^n v - a j_a \theta^n v. \tag{2.7}$$

Let $w := j_a\theta^n v$. Then, by the definition of j_a, $w, \theta w \in B(I)$. So it follows from the equality

$$y^\alpha w(y) - x^\alpha w(x) = \int_x^y t^{\alpha-1}(\theta w(t) + \alpha w(t))dt.$$

that $w \in \mathcal{C}^{0,\alpha}(I)$. By (2.7) and the assumption $v \in \mathcal{C}^{n,\alpha}(I)$ we have $\theta^{n+1}j_a v \in \mathcal{C}^{0,\alpha}(I)$. This shows that $j_a v \in \mathcal{C}^{n+1,\alpha}(I)$.

(ii) In the case Re $a < 0$, the solution $u = Cx^{-a}$ of $\theta_a u = 0$ belongs to $\mathcal{C}^{n+1,\alpha}(I)$. Hence Ker $\theta_a = \boldsymbol{C}x^{-a}$. Moreover, for each $v \in \mathcal{C}^{n,\alpha}(I)$, the function $u(x) := -x^{-a}\int_x^{x_0} t^{a-1}v(t)dt$, where we take $x_0 \in I$ so that $\pm x_0 > 0$ according to $\pm x > 0$, satisfies $\theta_a u = v$. Clearly $u \in B(I)$. In view of $\theta_a u = v$ we obtain $\theta_a\theta^n u = \theta^n v$. This equation can be solved as $\theta^n u(x) = -x^{-a}(\int_x^{x_0} t^{a-1}\theta^n v(t)dt + C)$. From this it follows that $\theta^n u, \theta^{n+1}u \in C(I)$. Hence, by

$\theta^{n+1}u = \theta^n v - a\theta^n u$ and a similar argument to that in the proof of (i), we have $u \in \mathcal{C}^{n+1,\alpha}(I)$. Thus θ_a is surjective. □

By means of $\theta_1 = \theta + 1$, (2.6), Lemma 2.3-(i), we get the following:

Lemma 2.4. *Let $0 < \alpha < 1 - \delta < 1$, and assume that*

$$\Phi(t) \in C^1(0,1),\ |\Phi(t)|,\ |t(1-t)\Phi'(t)| \le Mt^{\epsilon-1}(1-t)^{\delta-1} \tag{2.8}$$

with some constants $M, \epsilon > 0$. Then, for any nonnegative integer n, J_Φ is a bounded linear operator from $\mathcal{C}^{n,\alpha}(I)$ to $\mathcal{C}^{n,\alpha+\delta}(I)$.

Proof. By means of [2, Lemma 5.1], J_Φ is a bounded operator from $\mathcal{C}^{0,\alpha}(I)$ to $\mathcal{C}^{0,\alpha+\delta}(I)$. For $u \in \mathcal{C}^{1,\alpha}(I)$ we obtain

$$\begin{aligned}\theta_1^{-1}J_\Phi\theta_1 u(x) = J_1 J_\Phi \theta_1 u(x) &= \int_0^1 dt \int_0^1 \Phi(s)(\theta_1 u)(xst)ds \\ &= \int_0^1 \Phi(s)ds \int_0^1 (\theta_1 u)(xst)dt = \int_0^1 \Phi(s)u(xs)ds = J_\Phi u(x).\end{aligned}$$

Hence J_Φ is a bounded linear operator from $\mathcal{C}^{1,\alpha}(I)$ to $\mathcal{C}^{1,\alpha+\delta}(I)$. By using the same argument repeatedly, we complete the proof. □

Remark. From Lemmas 2.3-(i) and 2.4, we have the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{n+1,\alpha}(I) \arrow[r, "J_\Phi"] \arrow[d, "\theta_1"', "\cong"] & \mathcal{C}^{n+1,\alpha+\delta}(I) \arrow[d, "\cong"', "\theta_1"] \\ \mathcal{C}^{n,\alpha}(I) \arrow[r, "J_\Phi"'] & \mathcal{C}^{n,\alpha+\delta}(I) \end{tikzcd}$$

where the vertical maps are isomorphisms, thanks to Lemma 2.3-(i).

The following formula will be used later:

Lemma 2.5. *If Φ, $\theta\Phi \in L^1[0,1]$, then $(z+1)D_\Phi(z) = \Phi(1) - D_{\theta\Phi}(z)$ $(\operatorname{Re} z \ge 0)$. In particular,*

$$\Phi(1) = 0 \implies D_{\theta\Phi}(z) = -(z+1)D_\Phi(z) \quad (\operatorname{Re} z \ge 0). \tag{2.9}$$

Proof. Integrating by parts we get

$$(z+1)D_\Phi(z) = \left[t^{z+1}\Phi(t)\right]_0^1 - \int_0^1 \theta\Phi(t)t^z dt.$$

Noting that $\Phi, \theta\Phi \in L^1[0,1]$ implies $\Phi(t) \in C(0,1]$ and $\lim_{t\to 0} t\Phi(t) = 0$ we complete the proof. □

We can now establish Theorem 2.2 for the case where $D_\Phi(z)$ has no zeros in $\operatorname{Re} z > 0$:

Proposition 2.6. *Let $\Phi(t) \in C^{n+2}(0,1)$ satisfy $(\mathrm{I})_n$-$(\mathrm{II})_n$ in Theorem 2.2. If $D_\Phi(z) \ne 0$ in $\operatorname{Re} z \ge 0$ then, for each nonnegative integer k, the operator $J_\Phi : \mathcal{C}^{k,\alpha}(I) \to \mathcal{C}^{k+n,\alpha+\delta}(I)$ is an isomorphism.*

Proof. By (2.9) we have for $\operatorname{Re} z \geq 0$, $D_{\theta^n\Phi}(z) = (-1)^n(z+1)^n D_\Phi(z)$. Hence, by assumption, $D_{\theta^n\Phi}(z) \neq 0$ for $\operatorname{Re} z \geq 0$. This enables us to apply [2, Theorem B] to $\theta^n\Phi$. Thus we conclude that $J_{\theta^n\Phi}$ is an isomorphism of $\mathcal{C}^{0,\alpha}(I)$ onto $\mathcal{C}^{0,\alpha+\delta}(I)$. On the other hand, using $(\mathrm{II})_n$, we have $\theta_1 J_\Phi u = -J_{\theta\Phi}u$ for $u \in C^1(I \setminus \{0\})$. Repeating this procedure we get $\theta_1^n J_\Phi u = (-1)^n J_{\theta^n\Phi} u$ for $u \in C(I)$. Since $\theta_1^n \colon \mathcal{C}^{n,\alpha+\delta}(I) \to \mathcal{C}^{0,\alpha+\delta}(I)$ is an isomorphism, this shows that $J_\Phi \colon \mathcal{C}^{0,\alpha}(I) \to \mathcal{C}^{n,\alpha+\delta}(I)$ becomes an isomorphism. For $k \geq 1$, the assertion follows from the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^{k,\alpha}(I) \arrow[r, "J_\Phi"] \arrow[d, "\theta_1^k"', "\cong"] & \mathcal{C}^{k+n,\alpha+\delta}(I) \arrow[d, "\cong"', "\theta_1^k"] \\
\mathcal{C}^{0,\alpha}(I) \arrow[r, "J_\Phi"'] & \mathcal{C}^{n,\alpha+\delta}(I)
\end{tikzcd}$$

The proof is complete. □

In order to treat the case where $D_\Phi(z)$ has zeros in $\operatorname{Re} z > 0$, we employ the function

$$\Phi_a(t) := j_{a+1}\Phi(t) = J_{t^a}\Phi(t) = \int_0^1 s^a \Phi(st)\,ds = \frac{1}{t^{a+1}}\int_0^t s^a\Phi(s)\,ds. \tag{2.10}$$

We pick out some properties of $\Phi_a(t)$:

Lemma 2.7. *Let* $\operatorname{Re} a > 0$.

(i) *If* $\Phi \in L^1[0,1]$ *then* $\Phi_a \in L^1[0,1]$. *Moreover*

$$(z-a)D_{\Phi_a}(z) = -(D_\Phi(z) - D_\Phi(a)) \quad (\operatorname{Re} z \geq 0). \tag{2.11}$$

In particular,

$$D_\Phi(a) = 0 \implies D_{\Phi_a}(z) = -(z-a)^{-1}D_\Phi(z) \quad (\operatorname{Re} z \geq 0). \tag{2.12}$$

(ii) *Assume that* $\Phi, \theta\Phi \in L^1[0,1]$. *Then*

$$\theta\Phi_a = \Phi - (a+1)\Phi_a. \tag{2.13}$$

Moreover

$$\theta\Phi_a = (\theta\Phi)_a. \tag{2.14}$$

Proof. (i) By (2.10), we have

$$\int_0^1 |\Phi_a(t)|dt \leq \int_0^1 \frac{dt}{|t^{a+1}|}\int_0^t |s^a||\Phi(s)|ds = \int_0^1 |s|^{Re\, a}|\Phi(s)|ds \int_s^1 \frac{dt}{t^{Re\, a+1}} < \infty.$$

This shows that $\Phi_a \in L^1[0,1]$. Similarly we get (2.11).

(ii) A simple computation yields (2.13). Integrating by parts we have

$$(\theta\Phi)_a(t) = \frac{1}{t^{a+1}}\int_0^t s^{a+1}\Phi'(s)ds = \Phi(t) - \frac{a+1}{t^{a+1}}\int_0^t s^a\Phi(s)ds = \Phi(t) - (a+1)\Phi_a(t).$$

Combining this with (2.13) we obtain (2.14). □

Lemma 2.7 leads to:

Lemma 2.8. *Assume that Φ satisfies the conditions* $(\mathrm{I})_n$-$(\mathrm{II})_n$ *in Theorem* 2.2, *and let* $a_1, \dots, a_N$ *be zeros of* $D_\Phi(z)$ *in* $\mathrm{Re}\, z > 0$. *Then* $\Phi_{a_1 a_2 \cdots a_N}$ *satisfies the conditions* $(\mathrm{I})_{n+N}$-$(\mathrm{II})_{n+N}$ *in Theorem* 2.2.

Proof. We shall prove the lemma by induction with respect to N. By means of (2.13) and (2.14), one can obtain

$$\theta^{n+1}\Phi_{a_1} = \theta^n \Phi - (a_1+1)(\theta^n\Phi)_{a_1} = At^\epsilon(1-t^\beta)^{\delta-1} + R_1,$$

where $R_1(t) := R(t) - (a_1+1)(\theta^n\Phi)_{a_1}$. As is easily checked, $R_1(t)$ satisfies (2.4) if we replace ρ by $\min(\rho, \delta)$, and ν by $\min(\nu, \epsilon)$. This shows that Φ_{a_1} satisfies $(\mathrm{I})_{n+1}$. By the assumption $D_\Phi(a_1) = 0$ we have $\Phi_{a_1}(1) = 0$. Hence, in view of (2.13), Φ_{a_1} satisfies $(\mathrm{II})_{n+1}$. Thus we have proved the lemma for $N = 1$.

We now suppose that $\Phi_{a_1\cdots a_m}$ satisfies $(\mathrm{I})_{n+m}$-$(\mathrm{II})_{n+m}$. Then $\theta^m\Phi_{a_1\cdots a_m}$ satisfies $(\mathrm{I})_n$-$(\mathrm{II})_n$. Moreover, by (2.9) and (2.12), one can get $D_{\theta^m\Phi_{a_1\cdots a_m}}(z) = (z+1)^m(z-a_1)^{-1}\cdots(z-a_m)^{-1}D_\Phi(z)$. Accordingly $D_{\theta^m\Phi_{a_1\cdots a_m}}(a_{m+1}) = 0$. This enables us to apply the lemma for $N = 1$ to $\theta^m\Phi_{a_1\cdots a_m}$. Thus it follows that $(\theta^m\Phi_{a_1\cdots a_m})_{a_{m+1}}$ satisfies $(\mathrm{I})_{n+1}$-$(\mathrm{II})_{n+1}$. Using (2.14) repeatedly we have $(\theta^m\Phi_{a_1\cdots a_m})_{a_{m+1}} = \theta^m\Phi_{a_1\cdots a_{m+1}}$, and so, $\theta^m\Phi_{a_1\cdots a_{m+1}}$ satisfies $(\mathrm{I})_{n+1}$-$(\mathrm{II})_{n+1}$. This shows that $\Phi_{a_1\cdots a_{m+1}}$ satisfies $(\mathrm{I})_{n+m+1}$-$(\mathrm{II})_{n+m+1}$. □

Combining tools θ_{-a} and $\Phi_a(t)$ we have:

Lemma 2.9. *Let* $0 < \alpha < 1 - \delta < 1$, *and assume that* Φ *satisfies* (2.8). *Let* a *be a zero of* $D_\Phi(z)$ *in* $\mathrm{Re}\, z > 0$. *Then, for each nonnegative integer* k, J_{Φ_a} *is a bounded linear operator from* $\mathcal{C}^{k,\alpha}(I)$ *to* $\mathcal{C}^{k+1,\alpha+\delta}(I)$ *and satisfies* $J_{\Phi_a}\theta_{-a} = -J_\Phi$. *Namely we have the commutative diagram:*

$$\begin{tikzcd}
\mathcal{C}^{k+1,\alpha}(I) \arrow[r, "J_\Phi"] \arrow[d, "-\theta_{-a}"'] & \mathcal{C}^{k+1,\alpha+\delta}(I) \\
\mathcal{C}^{k,\alpha}(I) \arrow[ur, "J_{\Phi_a}"'] &
\end{tikzcd}$$

Proof. By an interchange of the order of integration we have for $f \in \mathcal{C}^{k,\alpha}(I)$,

$$\begin{aligned} J_{\Phi_a}f(x) &= \int_0^1 \frac{1}{t^{a+1}} f(xt)dt \int_0^t s^a\Phi(s)ds = \int_0^1 s^a\Phi(s)ds\int_s^1 \frac{1}{t^{a+1}}f(xt)dt \\ &= x^a\int_0^1 s^a\Phi(s)ds\int_{xs}^x \frac{1}{\tau^{a+1}}f(\tau)d\tau. \end{aligned} \tag{2.15}$$

By differentiating this and using the assumption $D_\Phi(a) = 0$, it follows that

$$\theta J_{\Phi_a}f = -J_\Phi f + aJ_{\Phi_a}f.$$

As is readily seen, Φ_a satisfies (2.8). Hence, by Lemma 2.4, $J_\Phi f, J_{\Phi_a}f \in \mathcal{C}^{k,\alpha+\delta}(I)$. This proves that $J_{\Phi_a}f \in \mathcal{C}^{k+1,\alpha+\delta}(I)$. Integrating by parts we get for $u \in \mathcal{C}^{k+1,\alpha}(I)$,

$$\int_{xs}^x \frac{1}{\tau^{a+1}}(\theta_{-a}u)(\tau)d\tau = \int_{xs}^x \frac{u'(\tau)}{\tau^a}d\tau - a\int_{xs}^x \frac{u(\tau)}{\tau^{a+1}}d\tau = \frac{u(x)}{x^a} - \frac{u(xs)}{(xs)^a}.$$

Substituting this to (2.15) we conclude that $J_{\Phi_a}\theta_{-a}u = D_\Phi(a)u - J_\Phi u = -J_\Phi u$ for $u \in \mathcal{C}^{k+1,\alpha}(I)$. The proof is complete. □

We are now in a position to give the

Proof of Theorem 2.2. By the assumption $(\mathrm{I})_n$ we obtain

$$D_{\theta^n\Phi}(z) = A\int_0^1 t^{\epsilon+z-1}(1-t^\beta)^{\delta-1}dt + \int_0^1 R(t)t^z dt = \frac{A}{\beta}B\left(\frac{\epsilon+z}{\beta},\delta\right) + \int_0^1 R(t)t^z dt,$$

where $B(p,q)$ is the beta function. Since $B\left((\epsilon+z)/\beta,\delta\right)$ and $\int_0^1 R(t)t^z dt$ are holomorphic in $\mathrm{Re}\, z > -\epsilon$ and in $\mathrm{Re}\, z > -\nu$ respectively, the function $D_{\theta^n\Phi}(z)$ is holomorphic in the region $\mathrm{Re}\, z > -\min(\epsilon,\nu)$. Moreover it follows from Stirling's formula that $B\left((\epsilon+z)/\beta,\delta\right)$ has the order $\mathrm{O}(|z|^{-\delta})$ as $|z|\to\infty$, uniformly in the region mensioned above. Furthermore, by the equality $\int_0^1 R(t)t^z dt = (z+1)^{-1}\left(R(1) - \int_0^1 R'(t)t^{z+1}dt\right)$, $\int_0^1 R(t)t^z dt$ has the order $\mathrm{o}(|z|^{-\delta})$ as $|z|\to\infty$, uniformly in the region mensioned above. Hence, thanks to $A\neq 0$, the function $D_{\theta^n\Phi}(z)$ has at most a finite number of zeros $a_1,\dots,a_N$ counted by multiplicity in $\mathrm{Re}\, z>0$. But, by (2.9), we have $D_{\theta^n\Phi}(z) = (-1)^n(z+1)^n D_\Phi(z)$. Therefore $D_\Phi(z)$ has the same zeros as of $D_{\theta^n\Phi}(z)$. Since, in the case $N=0$, Theorem 2.2 is no other than Proposition 2.6, we let $N\geq 1$.

By Lemma 2.8, $\Phi_{a_1\cdots a_N}$ satisfies the conditions $(\mathrm{I})_{n+N}$-$(\mathrm{II})_{n+N}$. Moreover, in view of (2.12), we have $D_{\Phi_{a_1\cdots a_N}}(z) = (-1)^N(z-a_1)^{-1}\cdots(z-a_N)^{-1}D_\Phi(z)$, which has no zeros in $\mathrm{Re}\, z\geq 0$. Therefore, by Proposition 2.6, $J_{\Phi_{a_1\cdots a_N}}$ ia an isomorphism of $\mathcal{C}^{i,\alpha}(I)$ onto $\mathcal{C}^{i+n+N,\alpha+\delta}(I)$ for each nonnegative integer i. But, using Lemma 2.9 repeatedly, we obtain for any $u\in\mathcal{C}^{i+N,\alpha}(I)$,

$$\begin{aligned} J_\Phi u &= -J_{\Phi_{a_1}}\theta_{-a_1}u = (-1)^2 J_{\Phi_{a_1a_2}}\theta_{-a_2}\theta_{-a_1}u \\ &= \cdots = (-1)^N J_{\Phi_{a_1\cdots a_N}}\theta_{-a_N}\cdots\theta_{-a_1}u. \end{aligned} \tag{2.16}$$

This situation can be indicated by the following diagram:

$$\begin{tikzcd} \mathcal{C}^{i+N,\alpha}(I) \arrow[rr, "J_\Phi"] \arrow[d, "(-1)^N\theta_{-a_N}\cdots\theta_{-a_1}"'] & & \mathcal{C}^{i+n+N,\alpha+\delta}(I) \\ \mathcal{C}^{i,\alpha}(I) \arrow[urr, "J_{\Phi_{a_1\cdots a_N}}"'] \arrow[r, phantom, "\cong"] & & \end{tikzcd} \tag{2.17}$$

Since $(-1)^N\theta_{-a_N}\cdots\theta_{-a_1}$ is surjective, which follows from Lemma 2.3-(ii), we conclude that $J_\Phi\colon \mathcal{C}^{i+N,\alpha}(I)\to\mathcal{C}^{i+n+N,\alpha+\delta}(I)$ is surjective. Moreover the kernel of J_Φ coincides with that of the differential operator $\theta_{-a_N}\cdots\theta_{-a_1}$. Let $b_1,\dots,b_m$ be the distinct numbers in $\{a_1,\dots,a_N\}$ and suppose b_p has multiplicity N_p $(p=1,2,\dots,m)$. Then, as is well known, solution set for $\theta_{-a_N}\cdots\theta_{-a_1}u=0$ is spanned by N functions

$$|x|^{b_p}(\log|x|)^\ell \quad (\ell=0,1,\dots,N_p-1;\ p=1,2,\dots,m).$$

Since these functions belong to $\mathcal{C}^{i+N,\alpha}(I)$ we conclude that $\dim\mathrm{Ker}\, J_\Phi = N$ in the case $ab=0$ and $2N$ in the case $ab<0$. Thus we have proved Theorem 2.2 for $k\geq N$. Due to the commutative diagram in Proposition 2.6, we complete the proof.

3. Inverse Bifurcation Problem

In this section we discuss the inverse bifurcation problem mentioned in §1. We first recall the following scheme:

Lemma 3.1. *Let X, Y, Z be Banach spaces, let F be a C^p-map $(p \geq 1)$ of a neighborhood of 0 in $X \times Y$ into Z, and assume* (1.2), (1.3), (1.4). *Then there are a neighborhood $U_1 \times V$ of $(0,0)$ in $X_1 \times Y$, a neighborhood U of 0 in X, and a C^p-diffeomorphism G of $U_1 \times V$ onto U such that $F(G(x_1, y), y) = 0$ for each $y \in V$, $x_1 \in U_1$.*

Proof. Let X_2 be a complementary closed subspace of X_1 and set $\tilde{F}((x_1, y), x_2) := F(x_1 + x_2, y)$. Then $\tilde{F}$ is a C^p-map of a neighborhood of 0 in $(X_1 \times Y) \times X_2$ to Z. By (1.4), $\tilde{F}_{x_2}((0,0),0)$ is an isomorphism on X_2 onto Z. Hence, by applying the implicit function theorem to $\tilde{F}$, there exist a neighborhood $U_1 \times V$ of $(0,0)$ in $X_1 \times Y$, a neighborhood U of 0 in X, and a C^p-map φ of $U_1 \times V$ onto U such that $\tilde{F}((x_1, y), \varphi(x_1, y)) = 0$, and moreover the derivative of $\varphi(x_1, y)$ is computed as, for each $\dot{x}_1 \in X_1, \dot{y} \in Y$,

$$
\begin{aligned}
\varphi_{(x_1,y)}(0,0)(\dot{x}_1, \dot{y}) &= -(F_x(0,0)|_{X_2})^{-1} \tilde{F}_{(x_1,y)}((0,0),0)(\dot{x}_1, \dot{y}) \\
&= -(F_x(0,0)|_{X_2})^{-1} \{F_x(0,0)\dot{x}_1 + F_y(0,0)\dot{y}\} \\
&= -(F_x(0,0)|_{X_2})^{-1} F_y(0,0)\dot{y} \ \in X_2,
\end{aligned}
\tag{3.1}
$$

due to $X_1 = \mathrm{Ker} F_x(0,0)$. We now define $G(x_1, y) := x_1 + \varphi(x_1, y)$. Then $G(x_1, y)$ is a C^p-map of $U_1 \times V$ onto U and satisfies $F(G(x_1, y), y) = 0$. From (3.1) we have $G_{(x_1,y)}(\dot{x}_1, \dot{y}) = \dot{x}_1 - (F_x(0,0)|_{X_2})^{-1} F_y(0,0)\dot{y}$. In view of (1.3) this operator is an isomorphism of $X_1 \times Y$ onto X. The proof is complete. □

For a function spaces setting, let $0 < \alpha < 1/2$, let $I := [a, b]$ be a bounded interval with $a < 0 < b$, and introduce the function spaces

$$
X_\alpha := \left\{ g(h) \in C^1(I) \,|\, g(0) = g'(0) = 0, \ \sup_{h,k \in I, h \neq k} \frac{|g'(k) - g'(h)|}{|k - h|^\alpha} < \infty \right\},
\tag{3.2}
$$

(3.3)

$$
Y_\alpha := \left\{ \mu(h) \in C(I) \,|\, \mu(0) = 0, h\mu'(h) \in C(I), \ \sup_{h,k \in I, h \neq k} \frac{\left||k|^{3/2}\mu'(k) - |h|^{3/2}\mu'(h)\right|}{|k - h|^{\alpha + 1/2}} < \infty \right\}.
$$

Throughout this section, we assume that the first eigenfunction $v_1(x)$ (normalized by $v_1(0) = 1$) of (1.7) satisfies: (A1) $v_1''(0) < 0$, (A2) $v_1'(x) < 0$ $(0 < x \leq 1)$. Under these assumptions, if g and $\mu := \lambda - \lambda_1$ are sufficiently small then the solution v of (1.6) satisfies $|v(x)| \leq |v(0)|$ $(0 \leq x \leq 1)$. This observation, together with [1, Lemma 3.1], yields:

Lemma 3.2. *Let X_α, Y_α be Banach spaces defined in* (3.2), (3.3) *with $0 < \alpha < 1/2$ and assume* (A1), (A2). *Then, F defined by* (1.9) *is a C^1-map of a neighbourhood of* $(0,0)$ *in $X_\alpha \times Y_\alpha$ to Y_α. The Fréchet derivative $F_\mu(0,0)$ is written in the form* (1.11), *and hence is an isomorphism of Y_α onto Y_α. Moreover $F_g(0,0)$ is written as* (1.14) *with the function $\Psi(t)$ defined in* (1.13).

We shall apply Theorem 2.2 to J_Φ with Φ defined by

$$\Phi(t) := \Psi(t)t^{\alpha+1} = \frac{t^{\alpha+2}}{v_1'(1)v_1'(v_1^{-1}(t))}, \tag{3.4}$$

relying on the following

Lemma 3.3. *Let $\tilde{Y}$ be the function space defined by*

$$\tilde{Y}_\alpha := \left\{\phi(h) \in C^1(I) \mid \phi(0) = \phi'(0) = 0, \sup_{h,k\in I, h\neq k} \frac{||k|^{1/2}\phi'(k) - |h|^{1/2}\phi'(h)|}{|k-h|^{\alpha+1/2}} < \infty\right\}.$$

Then we have the commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^{1,\alpha}(I) \arrow[r, "J_\Phi"] \arrow[d, "h|h|^\alpha\cdot"', "\cong"] & \mathcal{C}^{1,\alpha+1/2}(I) \arrow[d, "\cong"', "h|h|^\alpha\cdot"] \\
X_\alpha \arrow[r, "J_\Psi"] \arrow[dr, "F_g(0,0)"'] & \tilde{Y}_\alpha \arrow[d, "\cong"', "h^{-1}\cdot"] \\
 & Y_\alpha
\end{tikzcd}$$

Proof. We define

$$E_{\alpha,\delta} := \left\{f(h) \in C^1(I) \mid f(0) = f'(0) = 0, \sup_{h,k\in I, h\neq k} \frac{||k|^\delta f'(k) - |h|^\delta f'(h)|}{|k-h|^{\alpha+\delta}} < \infty\right\}.$$

An elementary estimation shows that $|h|^{\alpha+1}\cdot : \mathcal{C}^{1,\alpha+\delta}(I) \to E_{\alpha,\delta}$ is an isomorphism, provided that $0 < \alpha \le \alpha + \delta \le 1$. Noting that $X_\alpha = E_{\alpha,0}, \tilde{Y}_\alpha = E_{\alpha,1/2}$, we see that the vertical maps are isomorphisms. Other assertions are also easily checked. $\square$

We now introduce the function

$$\zeta_q(z) := \int_0^1 v_1(x)^z dx \quad \left(= -\int_0^1 \frac{t^z}{v_1'(v_1^{-1}(t))} dt\right).$$

Let us assume $q(x) \in C^2[0,1]$ in addition to (A1), (A2). Then an elementary calculation shows that $-\frac{1}{v_1'(v_1^{-1}(t))}$ is expressed as

$$-\frac{1}{v_1'(v_1^{-1}(t))} = A(1-t^2)^{-1/2} + Q(t),$$

where $A := (-v_1''(0))^{-1/2}$ and Q is a function in $C[0,1] \cap C^2[0,1)$ satisfying $|Q'(t)| \le M(1-t)^{-1/2}$, $|Q''(t)| \le M(1-t)^{-3/2}$ with some positive constant M. Hence it follows from the same argument as in Theorem 2.2 that $\zeta_q(z)$ is holomorphic in $\operatorname{Re} z > -1$ with at most finitely many zeros on $\operatorname{Re} z \ge \eta$ for each $\eta > -1$. Since $\zeta_q(\bar{z}) = \overline{\zeta_q(z)}$ for $\operatorname{Re} z > -1$ and

$\zeta_q(z) > 0$ for any real $z > -1$, the zeros of $\zeta_q(z)$ in $\operatorname{Re} z > -1$ consist of nonreal conjugate complex numbers. Noting

$$\zeta_q(z + \alpha + 2) = -v_1'(1) D_\Phi(z), \tag{3.5}$$

we arrive at:

Theorem 3.4. *Suppose that q is a real function of class $C^2[0,1]$ satisfying* (A1), (A2), *and $\alpha \in (0, 1/2)$ is any number such that*

$$\zeta_q(z) \neq 0 \quad (\operatorname{Re} z = \alpha + 2). \tag{3.6}$$

Let $\Gamma(g)$ be the first bifurcating branch of g defined in §1. Then there exist a neighborhood U of 0 in X_α and a neighborhood V of 0 in Y_α such that:

(i) *For each $g \in U$ there is a unique $\mu \in V$ such that the section in $a \leq h \leq b$ of the first bifurcating branch $\Gamma(g)$ coincides with $\{(\lambda_1 + \mu(h), h) \mid a \leq h \leq b\}$. The assignment $\pi \colon g \to \mu$ is a C^1-map from U onto V.*

(ii) *If $\zeta_q(z)$ has no zeros in the region $\operatorname{Re} z > \alpha + 2$ then π is a C^1-diffeomorphism of U onto V.*

(iii) *If $\zeta_q(z)$ has zeros $\sigma_1 \pm i\tau_1, \dots . \sigma_m \pm i\tau_m$ with multiplicities $N_1, \dots, N_m$ in the region $\operatorname{Re} z > \alpha + 2$. then. by letting X_1 be $4(N_1 + \cdots + N_m)$- dimensional subspace of X_α spanned by*

$$\begin{gathered} \chi_{[a,0]} h|h|^{\sigma_p - 2} \sin(\tau_p \log|h|)(\log|h|)^\ell, \ \chi_{[a,0]} h|h|^{\sigma_p - 2} \cos(\tau_p \log|h|)(\log|h|)^\ell \\ \chi_{[0,b]} h|h|^{\sigma_p - 2} \sin(\tau_p \log|h|)(\log|h|)^\ell, \ \chi_{[0,b]} h|h|^{\sigma_p - 2} \cos(\tau_p \log|h|)(\log|h|)^\ell \\ (\ell = 0, .., N_p - 1; \ p = 1, 2, .., m), \end{gathered} \tag{3.7}$$

there is a neighborhood U_1 of 0 in X_1 and a C^1-diffeomorphism G of $U_1 \times V$ onto U such that, for $(g, \mu) \in U \times V$, $\pi(g) = \mu$ is equivalent to $g = G(x_1, \mu)$ with $x_1 \in U_1$.

Proof. It is easily checked that Φ defined by (3.4) satisfies $(\mathrm{I})_0$ in Theorem 2.2 with $\epsilon = \nu = \alpha + 3$, $\beta = 2$, $\delta = \rho = 1/2$. Relation (3.5) and assumption (3.6) imply that Φ is non-resonant. Hence, by Theorem 2.2 with $k = 1$, $J_\Phi : \mathcal{C}^{1,\alpha}(I) \to \mathcal{C}^{1,\alpha+1/2}(I)$ is surjective. In view of (3.5), the zeros b_p of $D_\Phi(z)$ in $\operatorname{Re} z > 0$ are written as $b_p = \sigma_p \pm i\tau_p - \alpha - 2$ and therefore $\operatorname{Ker} J_\Phi$ is a $4(N_1 + \cdots + N_p)$-dimensional subspace of $\mathcal{C}^{1,\alpha}(I)$ spanned by the functions $\chi_{[a,0]}|h|^{\sigma_p \pm i\tau_p - \alpha - 2}(\log|h|)^\ell$, $\chi_{[0,b]}|h|^{\sigma_p \pm i\tau_p - \alpha - 2}(\log|h|)^\ell$. This, together with Lemma 3.3, shows that $F_g(0,0) : X_\alpha \to Y_\alpha$ is surjective and the kernel $\operatorname{Ker} F_g(0,0)$ is a $4(N_1 + \cdots + N_p)$-dimensional subspace of X_α spanned by the functions in (3.7). We now apply Lemma 3.1 with $X = X_\alpha$, $Y = Z = Y_\alpha$ to complete the proof. □

Remark. From the property of $\zeta_q(z)$ mentioned above, the assumption (3.6) is satisfied except for at most finitely many $\alpha \in (0, 1/2)$.

We wish to point out (without the proof) that any nonnegative multiple of 4 in Theorem 3.4 can be actually realized by a concrete family of Sturm-Liouville operators.

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