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A Linear-Time Algorithm to Find Independent Spanning Trees in Maximal Planar Graphs

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Abstract: Given a graph G, a designated vertex r and a natural number k, we wish to find k " independent " spanning trees of G rooted at r, that is, k spanning trees such that, for any vertex v, the k paths connecting r and v in the k trees are internally disjoint in G. In this paper we give a linear-time algorithm to find k independent spanning trees in a k-connected maximal planar graph rooted at any vertex.

Key ward: graph, algorithm, independent spanning trees

1 Introduction

Given a graph $G = (V, E)$, a designated vertex $r \in V$ and a natural number k, we wish to find k spanning trees $T_1, T_2, \cdots, T_k$ of G such that, for any vertex v, the k paths connecting r and v in $T_1, T_2, \cdots, T_k$ are internally disjoint in G, that is, any two of them have no common intermediate vertices. Such k trees are called *k independent spanning trees of G rooted at r*. Five independent spanning trees are drawn in Fig. 1 by thick lines. Independent spanning trees have applications to fault-tolerant protocols in networks [BI96, DHSS84, IR88, OIBI96].

Given a graph $G = (V, E)$ of n vertices and m edges, and a designated vertex $r \in V$, one can find two independent spanning trees of G rooted at any vertex in linear time if G is biconnected [BTV96, BTV99, IR88], and find three independent spanning trees of G rooted at any vertex in $O(mn)$ and $O(n^2)$ time if G is triconnected [BTV96, BTV99, CM88]. It is conjectured that, for any $k \geq 1$, every k-connected graph has k independent spanning trees rooted at any vertex [KS92, ZI89]. For general graphs with $k \geq 4$ the conjecture is still open, however, for planar graphs the conjecture is verified by Huck for $k = 4$ [H94] and $k = 5$ [H99] (i.e., for all planar graphs, since every planar graph has a vertex of degree at most 5 [W96, p269] means there is no 6-connected planar graph). The proof in [H99] yields an algorithm to actually find k independent spanning trees in a k-connected planar graph, but it takes time $O(n^3)$. On the other hand, for k-connected maximal planar graphs we can find k independent spanning trees in linear time for $k = 2$ [BTV96, BTV99, IR88], $k = 3$ [BTV96, BTV99, S90] and $k = 4$ [MTNN98].

In this paper we give a simple linear-time algorithm to find five independent spanning trees of a 5-connected maximal planar graph rooted at any designated vertex. Note that, since there

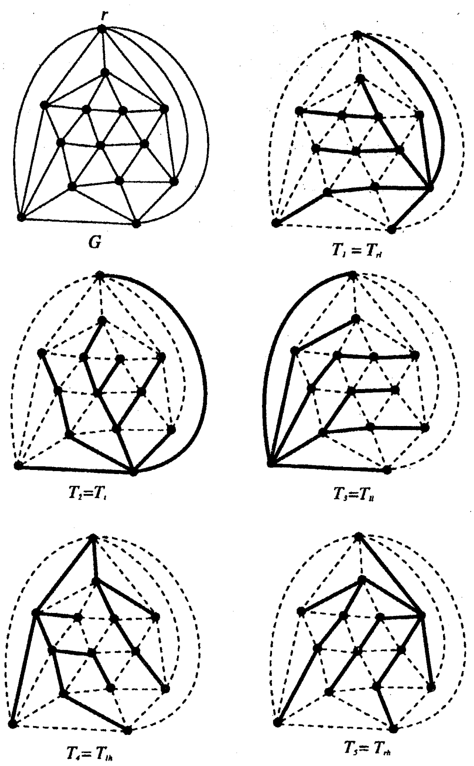

図 1: Five independent spanning trees T_1, T_2, T_3, T_4 and T_5 of a graph G rooted at r.

is no 6-connected planar graph, our result, together with previous results [BTV96, BTV99, IR88, MTNN98, S90], yields a linear-time algorithm to find k independent spanning trees in a k-connected maximal planar graph rooted at any designated vertex. Our algorithm is based on a " 5-canonical decomposition" of a 5-connected maximal planar graph, which is a generalization of an st-numbering [E79], a canonical ordering [K96], a canonical decomposition [CK93, CK97], a canonical 4-ordering [KH94] and a 4-canonical decomposition [MTNN98, NRN97].

The remainder of the paper is organized as follows. In Section 2 we introduce some definitions. In Section 3 we present our algorithm to find five independent spanning trees based on a 5-canonical decomposition. In Section 4 we give an algorithm to find a 5-canonical decomposition. Finally we put conclusion in Section 5.

2 Preliminaries

In this section we introduce some definitions.

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. Throughout the paper we denote by n the number of vertices in G, and we always assume that $n > 5$. An edge joining vertices u and v is denoted by (u, v). The *degree* of a vertex v in G, denoted by $d(v, G)$ or simply by $d(v)$, is the number of neighbors of v in G. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. A graph G is *k-connected* if $\kappa(G) \geq k$. A *path* in a graph is an ordered list of distinct vertices $v_1, v_2, \cdots, v_l$ such that $v_{i-1}v_i$ is an edge for all i, $2 \leq i \leq l$. We say that two paths having common start and end vertices are *internally disjoint* if their intermediate vertices are disjoint. We also say that a set of paths having common start and end vertices are *internally disjoint* if every pair of paths in the set are internally disjoint.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A planar graph G is *maximal* if all faces including the outer face are triangles in some planar embedding of G. Essentially each maximal planar graph has a unique planar embedding except for the choice of the outer face. A *plane graph* is a planar graph with a fixed planar embedding. The *contour* $C_o(G)$ of a biconnected plane graph G is the clockwise (simple) cycle on the outer face. We write $C_o(G) = (w_1, w_2, \cdots, w_h)$ if the vertices $w_1, w_2, \cdots, w_h$ on $C_o(G)$ appear in this order.

3 Algorithm

In this section we give our algorithm to find five independent spanning trees of a 5-connected maximal planar graph rooted at any designated vertex.

Given a 5-connected maximal planar graph $G = (V, E)$ and a designated vertex $r \in V$, we first find a planar embedding of G in which r is located on $C_o(G)$. Let $G' = G - \{r\}$ be the plane subgraph of G induced by $V - \{r\}$. In Fig. 2 (a) G is drawn by solid and dotted lines, and G' by solid lines. Since G is 5-connected, $d(r) \geq 5$. We may assume that all the neighbors $r_1, r_2, \cdots, r_{d(r)}$ of r in G appear on $C_o(G')$ clockwise in this order. Now $C_o(G') = (r_1, r_2, \cdots, r_{d(r)})$. We add to G' two new vertices r_b and r_t, join r_b with r_1,r_2 and r_3, and join r_t with $r_4, r_5, \cdots, r_{d(r)}$. Let G'' be the resulting plane graph, where vertices r_1, r_b, r_3, r_4, r_t and $r_{d(r)}$ appear on $C_o(G'')$ clockwise in this order. Fig. 2 (b) illustrates G''.

Let $\Pi = (W_1, W_2, \cdots, W_m)$ be a partition of the vertex set $V - \{r\}$ of G'. We denote by G_k, $1 \leq k \leq m$, the plane subgraph of G'' induced by $\{r_b\} \bigcup W_1 \bigcup W_2 \bigcup \cdots \bigcup W_k$. We denote by $\overline{G_k}$, $0 \leq k \leq m-1$, the plane subgraph of G'' induced by $W_{k+1} \bigcup W_{k+2} \bigcup \cdots \bigcup W_m \bigcup \{r_t\}$. We assume that if $1 < k \leq m$ and $W_k = \{u_1, u_2, \cdots, u_l\}$ then vertices $u_1, u_2, \cdots, u_l$ consecutively appear on $C_o(G_k)$ clockwise in this order. Note that for $k = 1$ we don't assume such a condition. A partition $\Pi = (W_1, W_2, \cdots, W_m)$ of $V - \{r\}$ is called a *5-canonical decomposition* of G' if the following three conditions (co1)–(co3) are satisfied.

(co1) $W_1 = \{r_1, r_2, r_3\} \bigcup \{u_2, u_3, \cdots, u_{d(r_2)-2}\}$, where vertices $u_2, u_3, \cdots, u_{d(r_2)-2}$ are the neighbors of r_2 except r_1, r_3, r_b, and $W_m = \{r_{d(r)-1}, r_{d(r)}\}$

(co2) For each k, $1 \leq k \leq m$, G_k is triconnected, and for each k, $0 \leq k \leq m-1$, $\overline{G_k}$ is biconnected (See Fig. 3.); and

(co3) For each k, $1 < k < m$, one of the following two conditions holds (See Fig. 3. The vertices in W_k are drawn in black dots):

(a) $|W_k| \geq 2$, and each vertex $u \in W_k$ satisfies $d(u, G_k) = 3$ and $d(u, \overline{G_{k-1}}) \geq 3$; and

(b) $|W_k| = 1$, and the vertex $u \in W_k$ satisfies $d(u, G_k) \geq 3$ and $d(u, \overline{G_{k-1}}) \geq 2$.

Fig. 2 (b) illustrates a 5-canonical decomposition of $G' = G - \{r\}$, where G' are drawn in solid lines and each set W_i is indicated by an oval drawn in a dotted line. A 5-canonical decomposition is a generalization of an "*st*-numbering" [E79], a "canonical ordering" [K96], a "canonical decomposition" [CK93, CK97], a "canonical 4-ordering" [KH94] and a "4-canonical decomposition" [MTNN98, NRN97].

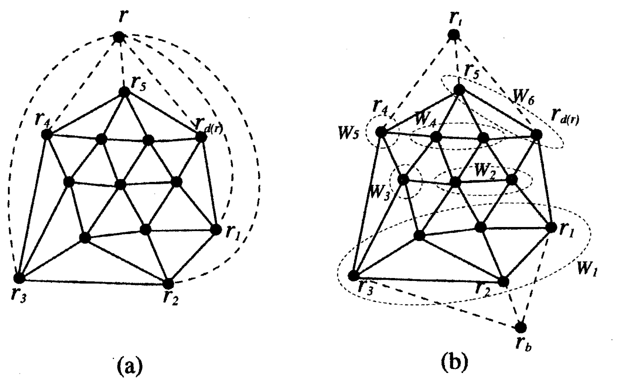

図 2: (a) Five-connected plane graph G and (b) plane graph G''.

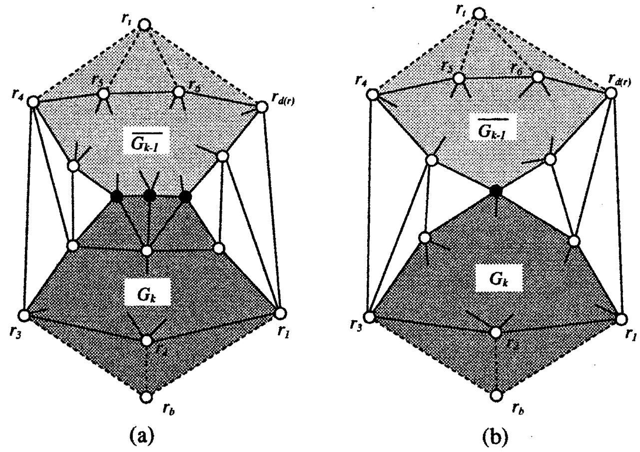

図 3: Two conditions for (co3).

We have the following lemma. We will give a proof of Lemma 3.1 in Section 4.

Lemma 3.1 Let $G = (V, E)$ be a 5-connected maximal plane graph, and let r be a designated vertex on $C_o(G)$. Then $G' = G - \{r\}$ has a 5-canonical decomposition Π. Furthermore Π can be found in linear time.

We need a few more definitions to describe our algorithm. For a vertex $v \in V - \{r\}$ we write $N(v) = \{v_1, v_2, \cdots, v_{d(v)}\}$ if $v_1, v_2, \cdots, v_{d(v)}$ are the neighbors of vertex v in G'' and appear around v clockwise in this order. To each vertex $v \in V - \{r\}$ we assign five edges incident to v in G'' as *the right leg* $rl(v)$, *the tail* $t(v)$, *the left leg* $ll(v)$, *the left hand* $lh(v)$ and *the right hand* $rh(v)$ as follows. We will show later that such an assignment immediately yields five independent spanning trees of G. Let $v \in W_k$ for some k, $1 \leq k \leq m$, then there are the following four cases to consider.

Case 1: $k = 1$. (See Fig. 4(a).)

Now $W_1 = \{r_1, r_2, r_3\} \bigcup \{u_2, u_3, \cdots, u_{d(r_2)-2}\}$. We may assume that vertices $u_2, u_3, \cdots, u_{d(r_2)-2}$ consecutively appear on $C_o(G_1)$ clockwise in this order. Let $u_1 = r_3, u_0 = r_b, u_{d(r_2)-1} = r_1$ and $u_{d(r_2)} = r_b$. For each $u_i \in W_1 - \{r_2\}$ we define $rl(u_i) = (u_i, u_{i+1})$, $t(u_i) = (u_i, r_2)$, $ll(u_i) = (u_i, u_{i-1})$, $lh(u_i) = (u_i, v_1)$, and $rh(u_i) = (u_i, v_{d(u_i)-3})$ where we assume $N(u_i) = \{u_{i-1}, v_1, v_2, \cdots, v_{d(u_i)-3}, u_{i+1}, r_2\}$. For r_2 we define $rl(r_2) = (r_2, r_1)$, $t(r_2) = (r_2, r_b)$, $ll(r_2) = (r_2, r_3)$, $lh(r_2) = (r_2, u_2)$, and $rh(r_2) = (r_2, u_{d(r_2)-2})$.

Case 2: W_k satisfies Condition (a) of (co3). (See Fig. 4(b).)

Let $W_k = \{u_1, u_2, \cdots, u_l\}$. Since $d(u_i, G_k) = 3$ for each vertex u_i and G is maximal planar, vertices $u_1, u_2, \cdots, u_l$ have exactly one common neighbor, say v, in G_k. Let u_0 be the vertex on $C_o(G_k)$ preceding u_1, and let u_{l+1} be the vertex on $C_o(G_k)$ succeeding u_l. For each $u_i \in W_k$ we define $rl(u_i) = (u_i, u_{i+1})$, $t(u_i) = (u_i, v)$, $ll(u_i) = (u_i, u_{i-1})$, $lh(u_i) = (u_i, v_1)$, and $rh(u_i) = (u_i, v_{d(u_i)-3})$ where we assume $N(u_i) = \{u_{i-1}, v_1, v_2, \cdots, v_{d(u_i)-3}, u_{i+1}, v\}$.

Case 3: W_k satisfies Condition (b) of (co3). (See Fig. 4(c).)

Let $W_k = \{u\}$, let u' be the vertex on $C_o(G_k)$ preceding u, and let u'' be the vertex on $C_o(G_k)$ succeeding u. Let $N(u) = \{u', v_1, v_2, \cdots, v_{d(u)-1}\}$, and let $u'' = v_x$ for some x, $3 \leq x \leq d(u) - 2$. Then $rl(u) = (u, u'')$, $t(u) = (u, v_{d(u)-1})$, $ll(u) = (u, u')$, $lh(u) = (u, v_1)$, and $rh(u) = (u, v_{x-1})$.

Case 4: $k = m$. (See Fig. 4(d).)

Now $W_m = \{r_{d(r)-1}, r_{d(r)}\}$. Let $u_0 = r_t$, $u_1 = r_{d(r)-1}$, $u_2 = r_{d(r)}$ and $u_3 = r_t$. For each $u_i \in W_k$ we define $rl(u_i) = (u_i, v_1)$, $t(u_i) = (u_i, v_{d(u_i)-3})$, $ll(u_i) = (u_i, v_{d(u_i)-2})$, $lh(u_i) = (u_i, u_{i-1})$, and $rh(u_i) = (u_i, u_{i+1})$ where we assume $N(u_i) = \{u_{i+1}, v_1, v_2, \cdots, v_{d(u_i)-2}, u_{i-1}\}$.

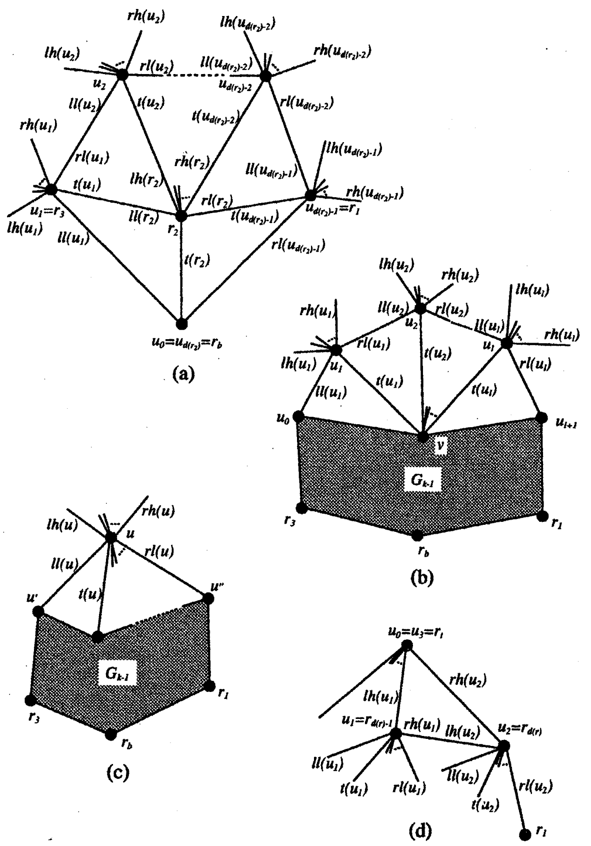

図 4: Assignment.

We are now ready to give our algorithm.

Procedure FiveTrees(G, r)
begin

1 Find a planar embedding of G such that $r \in C_o(G)$;
2 Find a 5-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$;
3 For each vertex $v \in V - \{r\}$ find $rl(v), t(v), ll(v), lh(v)$ and $rh(v)$;
4 Let T_{rl} be a graph induced by the right legs of all vertices in $V - \{r\}$;
5 Let T_t be a graph induced by the tails of all vertices in $V - \{r\}$;
6 Let T_{ll} be a graph induced by the left legs of all vertices in $V - \{r\}$;
7 Let T_{lh} be a graph induced by the left hands of all vertices in $V - \{r\}$;
8 Let T_{rh} be a graph induced by the right hands of all vertices in $V - \{r\}$;
9 Regard vertex r_b in trees T_{rl}, T_t and T_{ll} as vertex r;
10 Regard vertex r_t in trees T_{lh} and T_{rh} as vertex r;
11 **return** $T_{rl}, T_t, T_{ll}, T_{lh}$ and T_{rh} as five independent spanning trees of G.
end

We then verify the correctness of our algorithm. Assume that $G = (V, E)$ is a 5-connected maximal planar graph with a designated vertex $r \in V$, and that Algorithm FiveTrees finds a 5-canonical decomposition $\Pi = (W_1, W_2, \cdots, W_m)$ of $G - \{r\}$ and outputs $T_{rl}, T_t, T_{ll}, T_{lh}$ and T_{rh}. We first have the following lemma.

Lemma 3.2 Let $1 \le k \le m$, and let T_{rl}^k be a graph induced by the right legs of all vertices in $G_k - \{r_b\}$. Then T_{rl}^k is a spanning tree of G_k.

Proof We prove the claim by induction on k.

Clearly the claim holds for $k = 1$.

We assume that $1 \le k \le m-1$ and T_{rl}^k is a spanning tree of G_k, and we shall prove that T_{rl}^{k+1} is a spanning tree of G_{k+1}. There are the following three cases to consider.

Case 1: $k \le m-2$ and W_{k+1} satisfies Condition (a) of (co3).

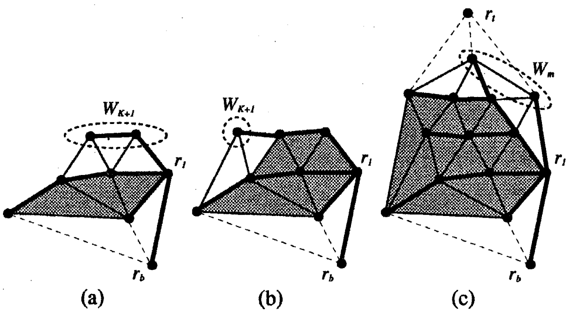

図 5: The three cases for Lemma 3.2.

Case 2: $k \le m-2$ and W_{k+1} satisfies Condition (b) of (co3).

Case 3: $k = m-1$.

For each case T_{rl}^{k+1} is a spanning tree of G_{k+1} as shown in Fig. 5; (a) for Case 1; (b) for Case 2; and (c) for Case 3. Q.E.D.

We then have the following lemma.

Lemma 3.3 $T_{rl}, T_t, T_{ll}, T_{lh}$ and T_{rh} are spanning trees of G.

Proof By Lemma 3.2, T_{rl}^m is a spanning tree of G_m, and hence T_{rl} in which r_b is regarded as r is a spanning tree of G.

Similarly T_t, T_{ll}, T_{lh} and T_{rh} are spanning trees of G. Q.E.D.

Let v be any vertex in $V - \{r\}$, and let $P_{rl}, P_t, P_{ll}, P_{lh}$ and P_{rh} be the paths connecting r and v in $T_{rl}, T_t, T_{ll}, T_{lh}$ and T_{rh}, respectively. For any vertex u in $V - \{r\}$ we write $rank(u) = k$ if $u \in W_k$; $rank(r)$ is undefined. If an edge (v, u) of G' is either a leg or a tail of vertex v, and (v, w) of G' is a hand of v, then $rank(u) \le rank(v) \le rank(w)$, and additionally if $v \ne r_2$ then $rank(u) < rank(w)$. See Fig. 4. Now we have the following lemma.

Lemma 3.4 Every pair of paths $P_1 \in \{P_{rl}, P_t, P_{ll}\}$ and $P_2 \in \{P_{lh}, P_{rh}\}$ are internally disjoint.

Proof We prove only that P_{rl} and P_{rh} are internally disjoint. Proofs for the other pairs are similar. If $v = r_1$ then $P_{rl} = (v, r)$. If $v = r_{d(r)}$ then $P_{rh} = (v, r)$. If $v = r_2$ then $P_{rl} = (v, r_1, r)$

and $P_{rh} = (v, u_{d(r_2)-2}, \cdots)$. Therefor P_{rl} and P_{rh} are internally disjoint if v is r_1, r_2 or $r_{d(r)}$. Thus we may assume that $v \neq r_1, r_2, r_{d(r)}$. Let $P_{rl} = (v, v_1, v_2, \cdots, v_l, r)$, then $v_l = r_1$. Let $P_{rh} = (v, u_1, u_2, \cdots, u_{l'}, r)$, then $u_{l'} = r_{d(r)}$. The definition of a right leg implies that $rank(v) \geq rank(v_1) \geq rank(v_2) \geq \cdots \geq rank(v_l)$, and the definition of a right hand implies that $rank(v) \leq rank(u_1) \leq rank(u_2) \leq \cdots \leq rank(u_{l'})$. Thus $rank(v_l) \leq \cdots \leq rank(v_2) \leq rank(v_1) \leq rank(v) \leq rank(u_1) \leq rank(u_2) \leq \cdots \leq rank(u_{l'})$. We furthermore have $rank(v_1) < rank(u_1)$ since $v \neq r_2$. Therefore P_{rl} and P_{rh} are internally disjoint. Q.E.D.

If $rl(v) = (v, u)$ then we say (v, u) is an *incoming right leg* of u. Similarly, if $t(v) = (v, u)$ then (v, u) is an *incoming tail* of u, and if $ll(v) = (v, u)$ then (v, u) is an *incoming left leg* of u.

We have the following lemma.

Lemma 3.5 Let $u \in V - \{r\}$, $ll(u) = (u, u')$, $rl(u) = (u, u'')$, and $N(u) = \{v_0, v_1, \cdots, v_{d(u)-1}\}$. One may assume that $u' = v_0$ and $u'' = v_z$ for some z, $3 \leq z \leq d(u) - 2$. Then all incoming right legs of u appear consecutively around u. Also all incoming tails of u appear consecutively around u, and all incoming left legs of u appear consecutively around u. Furthermore $ll(u)$, the incoming right legs, incoming tails, incoming left legs and $rl(u)$ appear clockwise around u in this order.

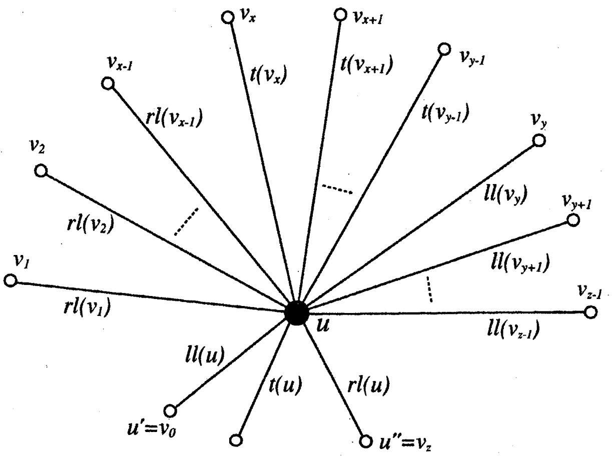

図 6: Illustration for Lemma 3.5.

Proof If $u = r_2$ then the claim is clearly holds. (In this case there is no incoming legs of u.) Thus we assume $u \neq r_2$.

If (u_i, u) is the tail of $u_i \in W_k$ then $u \in C_o(G_{k-1})$ and $u \notin C_o(G_k)$. (See Fig. 4.) Thus if $t(u_i) = (u_i, u)$ and $t(u_j) = (u_j, u)$ then $\{u_i, u_j\} \in W_k$ for some k. Therefore all incoming tails of u appear consecutively around u. (See Fig. 4.)

If $1 \leq i \leq z - 1$ and $rl(v_i) = (v_i, u)$, then $(v_{i-1}, u) \notin C_o(G_k)$, and either $t(u) = (u, v_{i-1})$, $rl(v_{i-1}) = (v_{i-1}, u)$ or $ll(u) = (u, v_{i-1})$ hold. (If $rank(v_i) = rank(u)$ then $t(u) = (u, v_{i-1})$. Otherwise assume $rank(v_i) = k$. Now edge (v_{i-1}, u) is on $C_o(G_{k-1})$. If $rank(v_{i-1}) \leq rank(u)$ then $ll(u) = (u, v_{i-1})$. If $rank(v_{i-1}) \geq rank(u)$ then $rl(v_{i-1}) = (v_{i-1}, u)$. See Fig. 4.) Thus if u has an incoming right leg e then the edge preceding e around u clockwise is either an incoming right leg of u, $t(u)$ or $ll(u)$. Since $t(u)$ and $ll(u)$ always appear consecutively around u, therefore all incoming right legs of u appear consecutively around u and $ll(u)$ precedes them. Similarly all incoming left legs of u appears consecutively around u and $rl(u)$ succeeds them. Thus the claim holds. Q.E.D.

Lemma 3.5 immediately implies the following lemma.

Lemma 3.6 A pair of paths $P_1, P_2 \in \{P_{rl}, P_t, P_{ll}\}$ may cross at a vertex u, but do not share a vertex u without crossing at u.

From the definitions of a left leg , a tail and a right leg one can immediately have the following lemma.

Lemma 3.7 Let $1 \leq k \leq m$, $u \neq r_2$ and $u \in W_k$. Then u is on $C_o(G_k)$. Let u' be the succeeding vertex of u on $C_o(G_k)$. Assume that the ordered set $N(u)$ starts with u'. Let $rl(u) = (u, v')$, $t(u) = (u, v'')$ and $ll(u) = (u, v''')$. Then v', v'', v''' appear in $N(u)$ in this order.

We then have the following lemma.

Lemma 3.8 A pair of paths $P_1, P_2 \in \{P_{rl}, P_t, P_{ll}\}$ are internally disjoint. Also P_{lh}, P_{rh} are internally disjoint.

Proof We prove only that P_{rl} and P_{ll} are internally disjoint. Proofs for the other cases are similar. Suppose for a contradiction that P_{rl} and P_{ll} share an intermediate vertex. Let w be the intermediate vertex that is shared by P_{rl} and P_{ll} and appear last on the path P_{rl} going from r to v. Now $w \neq r_2$ because r_2 has degree one in both T_{rl} and T_{ll}. Then P_{rl} and P_{ll} cross at w by Lemma 3.6. However, the claim in Lemma 3.7 holds both for $k = rank(v)$ and $u = v$ and for $k = rank(w)$ and $u = w$, and hence P_{rl} and P_{ll} do not cross at w, a contradiction. Q.E.D.

By Lemmas 3.4 and 3.8 we have the following lemma.

Lemma 3.9 $T_{rl}, T_t, T_{ll}, T_{lh}$ and T_{rh} are five independent spanning trees of G rooted at r.

Clearly the running time of Algorithm Five-Trees is $O(n)$. Thus we have the following theorem.

Theorem 3.10 Five independent spanning trees of any 5-connected maximal planar graph rooted at any designated vertex can be found in linear time.

4 Proof of Lemma 3.1

In this section we give an algorithm to find a 5-canonical decomposition. Then we show it runs in linear time. First we need some definitions.

Let $G = (V, E)$ be a 5-connected maximal plane graph, let r be a designated vertex on $C_o(G)$, and let H be a triconnected plane subgraph of G'' such that $r_b \in C_o(H)$. Let $C_o(H) = (r_b = w_1, w_2, \cdots, w_l)$.

A set of edges $(v_1, u), (v_2, u), \cdots, (v_h, u)$ in H is called a *fan with center u* if (1) $u \notin C_o(H)$, (2) the neighbors of u on $C_o(H)$ are $v_1, v_2, \cdots, v_h$, called *leaves*, and they appear in $C_o(H)$ clockwise in this order, and (3) either $h = 2$ and H does not have $edge(v_1, v_2)$, or $h \geq 3$. Assume a set of edges $(v_1, u), (v_2, u), \cdots, (v_h, u)$ is a fan F with center u Now, for $1 \leq i \leq h-1$, $v_i = w_a$ and $v_{i+1} = w_b$ hold for some a, b such that $1 \leq a < b \leq l$, and let C_i be the cycle consisting of the subpath $(w_a, w_{a+1}, \cdots, w_b)$ of $C_o(H)$ and two edges $(w_b, u), (u, w_a)$. Each plane subgraph F_i of H inside C_i (including C_i) is called a *piece* of F. F_i is called an *empty piece* if $a + 1 = b$. If F_i is an empty piece then C_i is a triangle face of H. (Since G is 5-connected, if $a + 1 = b$ then F_i has no vertex in the proper inside.) Note that by the definition if a fan has exactly two leaves then it has exactly one piece and the piece is not empty. Also note that F has exactly $h - 1$ pieces, and if $v_1 \neq r_b$ then none of pieces of F contains r_b. If none of pieces of F contains a distinct fan, then F is a *minimal* fan.

A *cut-set* is a set of vertices whose removal results in a disconnected graph. Since G is 5-connected and maximal planar, every cut-set of H consisting of three vertices has (1) exactly one vertex not in $C_o(H)$ and (2) exactly two vertices in $C_o(H)$. Thus each cut-set of H consisting of three vertices corresponds to a center of a fan and its two leaves.

We have the following lemmas.

Lemma 4.1 If a vertex $v \in C_o(H)$ is contained in none of fans of H (Note that, however, v may be contained in a piece of a fan.), then $H - \{v\}$ is triconnected, where $H - \{v\}$ is the plane subgraph of H obtained from H by deleting v and all edges incident to v.

Lemma 4.2 If all pieces of a fan $F = (v_1, u), (v_2, u), \cdots, (v_h, u)$ of H is empty (Now $d(v_1) \geq 4$, $d(v_h) \geq 4$ and, for $j = 2, 3, \cdots, h-1$, $d(v_j) = 3$.) and $u \neq r_2$, then $H - \{v_2, v_3, \cdots, v_{h-1}\}$ is triconnected, where $H - \{v_2, v_3, \cdots, v_{h-1}\}$ is a plane subgraph of H obtained from H by deleting $v_2, v_3, \cdots, v_{h-1}$ and all edges incident to them.

Now we give our algorithm to find a 5-canonical decomposition.

First, by Condition (co1) we can find W_m. Now $\overline{G_{m-1}}$ is biconnected since $\overline{G_{m-1}}$ is a triangle cycle. Since $G = (V, E)$ is 5-connected, the vertex set $V - \{r\}$ induces a 4-connected graph G'. And G_m is obtained from G' by adding a new vertex r_b adjacent three vertices of G'. Now G_m is triconnected since a graph obtained from a k-connected graph G by adding a new vertex adjacent k vertices of G is also k-connected [W96, p145]. Also G_{m-1} is triconnected, since otherwise G_{m-1} has a cut-set S with two or less vertices and then $S \bigcup W_m$ is a cut-set of G with four or less vertices, a contradiction. Thus for $k = m-1$ and m, G_k is triconnected, and for $k = m-1$, $\overline{G_k}$ is biconnected. Clearly $r_1, r_2, r_3 \notin W_m$.

Then, inductively assume that we have chosen $W_m, W_{m-1}, \cdots, W_{i+1}$ such that for each $k = i, i+1, \cdots, m$, G_k is triconnected, and for each $k = i, i+1, \cdots, m-1$, $\overline{G_k}$ is biconnected, $r_1, r_2, r_3 \notin W_m \bigcup W_{m-1} \bigcup \cdots \bigcup W_{i+1}$ and each W_k, $k = i+1, i+2, \cdots, m$, satisfies either (co1) or (co3). Now we can choose W_i as follows. We have two cases. If G_i has exactly one vertices in the proper inside of G_i then it is r_2 and we have done by setting all vertices in G_i except r_b as W_1. Otherwise we can find $W_i \subseteq V - W_m \bigcup W_{m-1} \bigcup \cdots \bigcup W_{i+1}$ such that (1) G_{i-1} is triconnected, (2) $\overline{G_{i-1}}$ is biconnected, (3) $r_1, r_2, r_3 \notin W_i$, (4) W_i satisfies (co3), as follows.

Let $F = (v_1, u), (v_2, u), \cdots, (v_h, u)$ be a minimal fan of G_i. Note that G_i always has a fan $(r_b, r_2), (r_3, r_2), \cdots, (r_1, r_2)$ with center r_2 implies G_i always has a fan.

If every piece of F is empty then F has three or more leaves, and we can set $W_i = \{v_2, v_3, \cdots, v_{h-1}\}$. Now if $h \geq 4$ then W_i satisfies (a) of (co3) and G_{i-1} is triconnected by Lemma 4.2, and $\overline{G_{i-1}}$ is biconnected since each vertex in W_i has degree exactly three in G_i means each vertex in W_i has two or more neighbors in $\overline{G_i}$. Similarly if $h = 3$ then W_i satisfies (b) of (co3), and G_{i-1} is triconnected by Lemma 4.2, and $\overline{G_{i-1}}$ is biconnected as above.

Otherwise, let F' be a non-empty piece of F. Now F' has four or more vertices on $C_o(G_i)$ since otherwise G has a cut-set with four or less vertices, a contradiction. Now there exists at least one vertex of F' on $C_o(G_i)$ such that (1) it is not a leaf of F, and (2) it has two or more neighbors in $\overline{G_i}$. (Since otherwise each vertices of F' on $C_o(G_i)$ except the two leaves w_a, w_b of F has at most one neighbor in $\overline{G_i}$, and for G is maximal planar each neighbor in $\overline{G_i}$ is a common vertices, say x, and $\{u, w_a, w_b, x\}$ forms a cut-set, a contradiction.) Thus we can find W_i satisfying (b) of (co3). Now G_{i-1} is triconnected by Lemma 4.1, and $\overline{G_{i-1}}$ is biconnected.

Thus we can find a 5-canonical decomposition. By maintaining a data-structure to keep fans and the number of neighbors in $\overline{G_i}$ for each vertex, the algorithm runs in linear time.

5 Conclusion

In this paper we give a linear-time algorithm to find k independent spanning trees of a k-connected maximal planar graph rooted at any designated vertex. It is remained as future work to find a linear-time algorithm for planar graphs, which are not always maximal planar.

参考文献

[BI96] F. Bao and Y. Igarashi, *Reliable broadcasting in product networks with Byzantine faults*, Proc. 26th Annual International Symposium on Fault-Tolelant Computing (FTCS'96) (1996) 262-271.

[BTV96] G. Di Battista, R. Tamassia and L. Vismara, *Output-sensitive reporting of disjoint paths*, Technical Report CS-96-25, Department of Computer Science, Brown University (1996).

[BTV99] G. Di Battista, R. Tamassia and L.Vismara, *Output-sensitive reporting of disjoint paths*, Algorithmica, 23 (1999) 302-340.

[CK93] M. Chrobak and G. Kant, *Convex grid drawings of 3-connected planar graphs*, Technical Report RUU-CS-93-45, Department of Computer Science, Utrecht University (1993).

[CK97] M. Chrobak and G. Kant, *Convex grid drawings of 3-connected planar graphs*, International Journal of Computational Geometry and Applications, 7 (1997) 211-223.

[CM88] J. Cheriyan and S. N. Maheshwari, *Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs*, J. Algorithms, 9 (1988) 507-537.

[DHSS84] D. Dolev, J. Y. Halpern, B. Simons and R. Strong, *A new look at fault tolerant network routing*, Proc. 16th Annual ACM Symposium on Theory of Computing (1984) 526-535.

[E79] S. Even, *Graph Algorithms*, Computer Science Press, Potomac (1979).

[H94] A. Huck, *Independent trees in graphs*, Graphs and Combinatorics, 10 (1994) 29-45.

[H99] A. Huck, *Independent trees in planar graphs*, Graphs and Combinatorics, 15 (1999) 29-77.

[IR88] A. Itai and M. Rodeh, *The multi-tree approach to reliability in distributed networks*, Information and Computation, 79 (1988) 43-59.

[K96] C. Kant, *Drawing planar graphs using the cononical ordering*, Algorithmica, 16 (1996) 4-32.

[KH94] G. Kant and X. He, *Two algorithms for finding rectangular duals of planar graphs*, Proc. 19th Workshop on Graph-Theoretic Concepts in Computer Science (WG'93), Lect. Notes in Comp. Sci., 790, Springer (1994) 396-410.

[KS92] S. Khuller and B. Schieber, *On independent spanning trees*, Information Processing Letters, 42 (1992) 321-323.

[MTNN98] K. Miura, D. Takahashi, S. Nakano and T. Nishizeki, *A Linear-Time Algorithm to Find Four Independent Spanning Trees in Four-Connected Planar Graphs*, WG'98, Lect. Notes in Comp. Sci., 1517, Springer (1998) 310-323.

[NRN97] S. Nakano, M. S. Rahman and T. Nishizeki, *A linear time algorithm for four-partitioning four-connected planar graphs*, Information Processing Letters, 62 (1997) 315-322.

[OIBI96] K. Obokata, Y. Iwasaki, F. Bao and Y. Igarashi, *Independent spanning trees of product graphs and their construction*, Proc. 22nd Workshop on Graph-Theoretic Concepts in Computer Science (WG'96), Lect. Notes in Comp. Sci., 1197 (1996) 338-351.

[S90] W. Schnyder, *Embedding planar graphs on the grid*, Proc. 1st Annual ACMSIAM Symp. on Discrete Algorithms, San Francisco (1990) 138-148

[W96] D. B. West, *Introduction to Graph Teory*, Prentice Hall (1996)

[ZI89] A. Zehavi and A. Itai, *Three tree-paths*, J. Graph Theory, 13 (1989) 175-188.