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On rationality of eigenvalues of the Cartan matrix of a finite group

和田倶幸 (TOMOYUKI WADA) 東京農工大学・工

This is a joint work with M. Kiyota and M. Murai.

G : a finite group
F : an algebraically closed field of characteristic $p > 0$
B : a block of the group algebra FG with defect group D of order p^d
$\mathrm{IBr}(B)$: the set of irreducible Brauer characters in B
$l(B) := |\mathrm{IBr}(B)|$
$C_B = (c_{ij})$: the Cartan matrix of B i.e. c_{ij} is the multiplicity of an irreducible FG-module S_j in a projective cover P_i of S_i as a composition factor, where S_j and P_i belong to B
$\rho(B)$: the Perron-Frobenius eigenvalue of C_B i.e. the largest eigenvalue of C_B

It is well known that all $\mathbb{Z}$-elementary divisors of C_B are a power of p, the largest one is $p^d = |D|$ and the others are smaller than p^d, but eigenvalues need not be an integer. For example, $\rho(B_0) = (7 + \sqrt{33})/2$ for the principal 2-block B_0 of the alternating group A_5 of degree 5. So the following question is fundamental.

Question. When do eigenvalues and elementary divisors of C_B coincide ?

If eigenvalues and elementary divisors coincide, then sure $\rho(B) = |D|$. Does the converse hold ?

We shoud first consider the case $\rho(B) = |D|$. This case really occurs in the following situation.

FACTS

(A) Theorem 4.4 in [K-W]. *If G is p-solvable, then $\rho(B) \leq |D|$, and the equality holds if and only if the height of $\varphi = 0$ for all $\varphi \in IBr(B)$.*

(B) Proposition 4.7 in [K-W]. *If D is cyclic, then $\frac{|D|}{p}+1 \leq \rho(B) \leq |D|$, and $\rho(B) = |D|$ if and only if the Brauer tree of B is a star and the exceptional vertex, if it exists, is at the center if and only if* $C_B = \begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m \\ m & \cdots & m & m+1 \end{pmatrix}$, *where m is the multiplicity, $em+1 = |D|$ and $e = l(B)$.*

(C) (Erdmann) *If B is tame i.e. $p = 2$ and D is dihedral, generalized quaternion or semi dihedral, then $l(B) = 1, 2$ or 3 and in this case $\rho(B) = |D|$ if and only if the following hold.*
(i) $l(B) = 1$.
(ii) $D \simeq E_4$ (elementary abelian group of order 4) and $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$,
$\rho(B) = 4 = |D|$. Example. $G = PSL(2, q), q \equiv 3 (\text{mod } 8),\ B = B_0$.
(iii) $D \simeq Q_8$ (the quaternion group of order 8) and $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, $\rho(B) = 8 = |D|$.
Example. $G = SL(2, q), q \equiv 3 (\text{mod } 8),\ B = B_0$.

(D) Proposition 4.3 in [K-W]. *If $D \lhd G$, then $\rho(B) = |D|$.*

So we should next consider when eigenvalues and elementary divisors coincide in the examples above.

Proposition 1. *Let G be a p-solvable group and B a block of FG. Then the following are equivalent.*
(1) eigenvalues and elementary divisors of C_B coincide.
(2) $\rho(B) = |D|$.
(3) the height of $\varphi = 0$ for all $\varphi \in IBr(B)$.
(4) $f = {}^t(f_1, \cdots, f_{l(B)})$ is an eigenvector of C_B, where $f_i = \varphi_i(1)$ for $\varphi_i \in \mathrm{IBr}(B)$.

Proof. We have already proved equivalence on (2), (3) and (4) in [K-W]. So we should prove (3) $\rightarrow$ (1).

(a) (Isaacs [I], Rukolaĭne) *Let G be a p-solvable group and η_G the character afforded by the projective cover of the trivial FG-module. Then $\eta_G(x)$ is a power of p for all p-regular* *element $x \in G$.*

(b) Let G be a p-solvable group, B a block of FG of full defect. If the height of $\varphi = 0$ for all $\varphi \in \mathrm{IBr}(B)$, then eigenvalues and elementary divisors of C_B coincide.

proof of (b). Let P_G be a projective cover of the trivial FG-module and S_i, P_i be a simple B-module, its projective cover, respectively. Then, since dim S_i is not divided by p, $P_G \otimes S_i \simeq P_i$ for all $1 \leq i \leq l(B)$. Let Φ_B be an $l(B) \times l(B)$ matrix $(\varphi_i(x_j))$, where $\{x_1, \cdots, x_{l(B)}\}$ be a set of representatives of p-regular classes associated with B. Then the above statement means that $C_B\Phi_B = \Phi_B\ diag\{\eta_G(x_1), \cdots, \eta_G(x_{l(B)})\}$. This implies that $\eta_G(x_i)$ is an eigenvalue of C_B and ${}^t(\varphi_1(x_i), \cdots, \varphi_{l(B)}(x_i))$ is its eigenvector for $1 \leq i \leq l(B)$. Since $\eta_G(x_i)$ is a power of p by (a), and Φ_G is a unimodular matrix over a complete discrete valuation ring R, $\eta_G(x_1), \cdots, \eta_G(x_{l(B)})$ are also elementary divisors of C_B.

(c) We have a conclusion by Fong reduction.

Proposition 2. Let B be a block of FG with cyclic defect group D. Then the following are equivalent.
(1) eigenvalues and elementary divisors of C_B coincide.
(2) $\rho(B) = |D|$.
If one of the conditions above holds, then the set of eigenvalues of C_B is $\{|D|, 1, \cdots, 1\}$ and $\mathbf{1} = {}^t(1, \cdots, 1)$ is an eigenvector of C_B for $\rho(B)$.

Proof. If (2) holds, then $C_B = \begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m \\ m & \cdots & m & m+1 \end{pmatrix}$, where m is the multiplicity, $em + 1 = |D|$ and $e = l(B)$. It is clear that the set of eigenvalues and elementary divisors of C_B are equal to $\{|D|, 1, \cdots, 1\}$. $\mathbf{1}$ is an eigenvector for $\rho(B)$. Let $\boldsymbol{x}_i := {}^t(1, 0, \cdots, -1, \cdots, 0)$ (i.e. the column vector whose first entry is 1 and the i th entry is -1, and others are 0), then $C_B\boldsymbol{x}_i = \boldsymbol{x}_i$ for $2 \leq i \leq l(B)$.

Proposition 3. If B is a block of FG with a normal defect group D, then the set of eigenvalues and elementary divisors of C_B are equal to $\{|C_D(x_1)|, \cdots, |C_D(x_{l(B)})|\}$, where $\{x_1, \cdots, x_{l(B)}\}$ is a set of representatives of p-regular classes of G associated with B. In this case, $\boldsymbol{f} = {}^t(f_1, \cdots, f_{l(B)})$ is an eigenvector for $|D|$.

Proof. Let $M := \Phi_B\ diag\{|C_D(x_1)|, \cdots, |C_D(x_{l(B)})|\}\ \Phi_B{}^{-1}$. Then by [A-C-S] $C_B = C_{\overline{B}}M$, where $\overline{B}$ is a homomorphic image of B of an aigebra epimorphism $\tau : FG \longrightarrow F\overline{G}$, for $\overline{G} = G/D$. Now since D is a defect group of B, if $\overline{B} = \overline{B}_1 + \cdots + \overline{B}_r$, each $\overline{B}_i$ is a block of $F\overline{G}$ of defect 0. So $C_{\overline{B}}$ is the identity $l(B) \times l(B)$ matrix. This means $C_B = \Phi_B \ diag\{|C_D(x_1)|, \cdots, |C_D(x_{l(B)})|\} \ {\Phi_B}^{-1}$. Since Φ_B is a unimodular matrix over R, we have a conclusion (see Proposition 4.3 in [K-W]).

Proposition 4. *Let B be a tame block of FG. Then the following are equivalent.*
(1) eigenvalues and elementary divisors of C_B coincide.
(2) $\rho(B) = |D|$.
If one of the conditions above holds, then $\mathbf{1} = {}^t(1, \cdots, 1)$ ia an eigenvector of C_B for $|D|$.

Proof. In [E] Erdmann has classified all tame blocks of finite group algebras via Morita equivalence. There are 6 types of Cartan matrices of blocks with dihedral defect group, 5 types with generalized quaternion defect group and 10 types with semi dihedral defect group. We can check any types of Cartan matrices have no rational largest eigenvalues without two cases. For example, there is the case that D is semi dihedral, $l(B) = 3$, and

$C_B = \begin{pmatrix} 8 & 4 & 4 \\ 4 & s+2 & 2 \\ 4 & 2 & 3 \end{pmatrix}$, where $s = |D|/4$. Now $|D| \geq 16$, then $s \geq 4$. If $s = 4$, then a minimal and a maximal row sum of C_B is 9 and 16, respectively and so $9 < \rho(B) < 16$. If $\rho(B)$ is an integer, then it must be a power of $2 \leq |D|$ by Corollary 4.6 in [K-W]. This is a contradiction. Then $s \geq 8$. We have $s+2 = c_{22} < \rho(B) <$ maximal row sum $= s+8$. So if $\rho(B) \in \mathbf{Z}$, then $\rho(B) = 2s$ or $4s$. If $\rho(B) = 2s$, then $2 < s < 8$. This contradicts to $s \geq 8$. Similarly, if $\rho(B) = 4s$, then $s = 2$ and this is also a contradiction. We can also prove that $\rho(B)$ is not an integer for other cases.

There remain two types of Cartan matrices. One of them is the case that D is dihedral, $l(B) = 3$ and $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & m+1 & m \\ 1 & m & m+1 \end{pmatrix}$, where $m = |D|/4$. In this case, if $\rho(B) \in \mathbf{Z}$, then $m = 1$ and so $\rho(B) = |D| = 4$. This can actually occur, for example, when $B = B_0(PSL(2, q)), q \equiv 3 (\text{mod } 8)$. The other case is that D is generalized quaternion, $l(B) = 3$ and $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & k+2 & k \\ 2 & k & k+2 \end{pmatrix}$, where $k = |D|/4$. In this case, if $\rho(B) \in \mathbf{Z}$, then $k = 2$ and $\rho(B) = |D| = 8$. This can also actually occur, for example, when $B = B_0(SL(2, q)), q \equiv 3 (\text{mod } 8)$.

From these results the following questions seem natural.

Question 1. For any finite group G, and any block B of FG, are the following equivalent?
(1) eigenvalues and elementary divisors of C_B coincide.
(2) $\rho(B) = |D|$.

Furthermore,

Question 2. For any finite group G, and any block B of FG, does $\rho(B) \in \mathbf{Z}$ mean $\rho(B) = |D|$?

We have an affirmative answer for Question 2 if D is cyclic or B is tame as is shown later.

Further examples for $\rho(B) = |D|$.

If two algebras A, B are Morita equivalent, then the Cartan matrix of A and B coincide, and then $\rho(A) = \rho(B)$. On the other hand, if a defect group D of a block B of FG is normal in G, then $\rho(B) = |D|$. Suppose a block B of FG is Morita equivalent to its Brauer correspondent b of $FN_G(D)$. Then we have $\rho(B) = \rho(b) = |D|$. There are several such examples, in verifying Broué conjecture.

Broué conjecture. Let B be a block of FG with abelian defect group D. Then B and its Brauer correspondent b are derived equivalent.

If two algebras A, B are Morita equivalent, then they are derived equivalent. Several authors prove that Morita equivalence between B and b actually occur in the following groups.

(1) $p = 2, D \simeq E_4$ (tame), D is a Sylow p-subgroup of G.
(Erdmann) $G = PSL(2, q), q \equiv 3(\text{mod } 8) \Longrightarrow B_0(G) \sim B_0(N_G(D))$ (Morita equivalent).

(2) $p = 3, D \simeq E_9$, D is a Sylow p-subgroup of G.
(Koshitani-Kunugi) $G = PSU(3, q^2), 3||q + 1 \Longrightarrow B_0(G) \sim B_0(N_G(D))$ (Morita equivalent).
(Miyachi) $G = GL(5, q), 3||q + 1 \Longrightarrow B_0(G) \sim B_0(N_G(D))$ (Morita equivalent).

(3) Non abelian case, $p = 2, D \simeq Q_8$ (tame), D is a Sylow p-subgroup of G.
(Erdmann) $G = SL(2, q), q \equiv 3(\text{mod } 8) \Longrightarrow B_0(G) \sim B_0(N_G(D))$ (Morita equiva-

lent).

Several authors have shown that B and b are derived equivalent but not Morita equivalent. Such Cartan matrices of B and b are sure different and further $\rho(B)$ and $\rho(b)$ are different. So the following question might have an affirmative answer.

Question 3. Let B be a block of FG and b a Brauer correspondent of B i.e. b is a block of $FN_G(D)$ with $b^G = B$. Then are the following equivalent ?
(1) $\rho(B) \in \mathbf{Z}$.
(2) $B \sim b$ (Morita equivalent).

Question 3 is affirmative in the case that D is cyclic or B is tame. Suppose D is cyclic. If $\rho(B) \in \mathbf{Z}$, then $\rho(B) = |D|$ from the first inequality in case(B). Then the second inequality means that the Brauer tree of B is a star with an exceptional vertex, if it exists, is at the center. This is the same shape of Brauer tree for b. The Brauer tree of B determines every shape of projective indecomposable FG-modules and star means that every projective indecomposable FG-module is unisearial, and then any indecomposable FG-module is uniquely determined as a homomorphic image of a projective indecomposable FG-module. Thus B and b are Morita equivalent.

Suppose that B is tame. Among 21 types there are just two types of Cartan matrices of tame blocks which satisfy $\rho(B) \in \mathbf{Z}$ by [E]. Blocks belonged to any one of these two types are Morita equivalent by [E].

As assertion (2) in Question 3 is very strong, we might as well adopt (2)' in stead of (2)

(2)' there are a finite group $\tilde{H}$ and a block $\tilde{b}$ with $D(\tilde{b}) \lhd \tilde{H}, D(\tilde{b}) \simeq D(B)$ such that $B \sim \tilde{b}$ (Morita equivalent).

If G is a p-solvable group with p-length 1, then in Question 3 $\rho(B) = |D|$ means (2)'. We do not have an affirmative answer to Question 3 even if G is a p-solvable group. If D is abelian, then recently, it has been shown $B \sim b$ (Morita equivalent) in [H-L].

In known cases (A),(B),(C),(D), the Cartan matrices satisfying $\rho(B) = |D|$ have $\boldsymbol{f} = {}^t(f_1, \cdots, f_{l(B)})$ or $\mathbf{1} = {}^t(1, \cdots, 1)$ as its eigenvector. Except the case that G is p-solvable, all vectors above are dimension vectors for simple modules in b. So it seems natural to ask Question 3 in this sence. In particular, if G is p-solvable with abelian defect group, then [H-L] implies that dimension vectors $\boldsymbol{f}_B$ of B and $\boldsymbol{f}_b$ of b are proportional and so $\boldsymbol{f}_B = c\boldsymbol{f}_b$ for some $c \in \mathbf{N}$.

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Tomoyuki Wada
Department of Mathematics
Tokyo University of Agriculture and Technology
Saiwai-cho 3-5-8, Fuchu, Tokyo 183-0054, Japan
E-mail address: wada@cc.tuat.ac.jp