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On an univalence criterion

Dorina Răducanu, Shigeyoshi Owa and Paula Curt

Abstract

The method of subordination chains is used to establish an univalence criterion for holomorphic mappings in the open unit ball B^n in $\mathbf{C}^n$. The authors consider an univalence criterion for holomorphic functions in B^n.

1 Introduction

Let $\mathbf{C}^n$ be the space of n-complex variables $z = (z_1, ..., z_n)$ with Euclidean inner product $\langle z, w\rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ and the norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$.

Let B^n be the open unit ball in $\mathbf{C}^n$, i.e $B^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$. We denote by $\mathcal{L}(\mathbf{C}^n)$ the space of continuous linear operators from B^n into $\mathbf{C}^n$, i.e the $n \times n$ complex matrices $A = (A_{jk})$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| < 1\}, \quad A \in \mathcal{L}(\mathbf{C}^n)$$

and

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbf{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings

$$f(z) = (f_1(z), ..., f_n(z)), \ z \in B^n$$

from B^n into $\mathbf{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has local holomorphic inverse at each point in B^n or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1\le j,k\le n}$$

is nonsingular at each point $z \in B^n$.

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \le \|z\|$, for all $z \in B^n$.

Let $f, g \in H(B^n)$. Then f is said to be *subordinate* to g(written by $f \prec g$) in B^n if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$. A function $L : B^n \times [0,\infty) \to \mathbf{C}^n$ is said to be the *subordination chain* if $L(\cdot,t)$ is holomorphic and univalent in B^n, $L(0,t) = 0$ for all $t \in [0,\infty)$, and $L(z,s) \prec L(z,t)$ whenever $0\le s \le t < \infty$.

The subordination chain $L(z,t)$ is called a *normalized subordination chain* if $DL(0,t) = e^t I$ for all $t \ge 0$.

The following result concerning subordination chains is due to J. A. Pfaltzgraff [5].

Theorem 1 *Let $L(z,t) = e^t z + ...$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ such that:*

(i) $L(\cdot,t) \in H(B^n)$ for all $t \in [0,\infty)$.

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in B^n$.

Let $h(z,t)$ be a function from $B^n \times [0,\infty)$ into $\mathbf{C}^n$ which satisfies the following condition:

(iii) $h(\cdot,t) \in H(B^n), h(0,t) = 0, Dh(0,t) = I$ and $\mathrm{Re}\,\langle h(z,t), z\rangle \ge 0$ for all $t \ge 0$ and $z \in B^n$.

(iv) For each $T > 0$ and $r \in (0,1)$, there is a number $K = K(r,T)$ such that $\|h(z,t)\| \le K(r,T)$ when $\|z\| \le r$ and $t \in [0,T]$.

(v) For each $z \in B^n, h(\cdot,t)$ is a measurable function on $[0,\infty)$.

Suppose $h(z,t)$ satisfies

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\,h(z,t) \quad a.e.\ t \ge 0, \quad for\,all\; z \in B^n.$$

Further, suppose there is a sequence $\{t_m\}_{m\geq 0}, t_m > 0$ *which is increasing to* ∞ *such that*

$$\lim_{m\to\infty} e^{-t_m} L(z, t_m) = F(z)$$

locally uniformly in B^n.

Then for each $t \in [0,\infty), L(\cdot,t)$ *is univalent in* B^n.

A version of Theorem 1 for subordination chains which are not normalized is due to P. Curt [2].

Theorem 2 *Let* $L(z,t) = a_1(t) z + \ldots,\ a_1(t) \neq 0$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbf{C}^n$ *such that:*

(*i*) *For each* $t \geq 0, L(\cdot,t) \in H(B^n)$.

(*ii*) $L(z,t)$ *is a locally absolutely continuous function of* $t \in [0,\infty)$, *locally uniformly with respect to* $z \in B^n$.

(*iii*) $a_1(t) \in C^1[0,\infty)$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$.

(*iv*) $h(\cdot,t) \in H(B^n)$ *for all* $t \in [0,\infty)$.

(*v*) *For each* $z \in B^n, h(z,\cdot)$ *is a measurable function on* $[0,\infty)$.

(*vi*) $h(0,t) = 0$ *and* $\mathrm{Re}\,\langle\, h(z,t,), z\rangle\ \geq 0$ *for all* $t \geq 0$ *and* $z \in B^n$.

(*vii*) *For each* $T > 0$ *and* $r \in (0,1)$, *there exists a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$ *when* $\|z\| \leq r$ *and* $t \in [0,T]$.

Suppose that $h(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\, h(z,t) \quad a.e.\ t \geq 0, for\, all\, z \in B^n. \tag{1}$$

Further suppose that there exists a sequence $\{t_m\}_{m\geq 0}, t_m > 0$ *which is increasing to* ∞ *such that*

$$\lim_{m\to\infty}\frac{L(z,t_m)}{a_1(t_m)}=F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $t\in[0,\infty), L(\cdot,t)$ *is univalent in* B^n.

2 Main results

By using Theorem 2 we obtain an univalence criterion which generalize two univalence criteria for holomorphic mappings in B^n.

Theorem 3 *Let* f,g *be holomorphic functions in* B^n *satisfying the conditions:*

1° g *is locally univalent in* B^n.

2° $f(0)=g(0)=0$ *and* $Df(0)=Dg(0)=I$.

Let α *be a real number with* $\alpha\geq 2$. *If*

$$\left\|(Dg(z))^{-1}Df(z)-\frac{\alpha}{2}I\right\|<\frac{\alpha}{2} \tag{3}$$

and

$$\Big\|\|z\|^2\left[Dg(z)^{-1}Df(z)-I\right] + (1-\|z\|^{\alpha})(Dg(z))^{-1}D^2g(z)(z,\cdot)+\left(1-\frac{\alpha}{2}\right)I\Big\|\leq\frac{\alpha}{2} \tag{4}$$

for all $z\in B^n$, *then* f *is an univalent function in* B^n.

Remark 1

The second derivative of function $g \in H(B^n)$ is a symmetric bilinear operator $D^2g(z)(\cdot,\cdot)$ on $\mathbf{C}^n \times \mathbf{C}^n$ and $D^2g(z)(w,\cdot)$ is the linear operator obtained by restricting $D^2g(z)$ to $\{w\} \times \mathbf{C}^n$. The linear operator $D^2g(z)(z,\cdot)$ has the matrix representation

$$D^2g(z)(z,\cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 g_k(z)}{\partial z_j \partial z_m} z_m\right)_{1\le j,k\le n}.$$

Proof. We define the function $L(z,t)$ by

$$L(z,t) = f\left(e^{-t}z\right) + \left(e^{\alpha t} - 1\right) e^{-t} Dg\left(e^{-t}z\right)(z), \quad (z,t) \in B^n \times [0,\infty). \quad (5)$$

We have to show that $L(z,t)$ satisfies the conditions of Theorem 2 and hence $L(\cdot,t)$ is univalent in B^n, for all $t \in [0,\infty)$. It is easy to check that $a_1(t) = e^{(\alpha-1)t}$ and hence $a_1(t) \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $a_1 \in C^1[0,\infty)$.

We have

$$L(z,t) = a_1(t)\, z + (holomorphic\, term).$$

Thus $\lim_{t\to\infty} \frac{L(z,t)}{a_1(t)} = z$, locally uniform with respect to B^n and hence (2) holds with $F(z) = z$. Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem 2. From (5) we obtain

$$DL(z,t) = \frac{\alpha}{2} e^{(\alpha-1)t} Dg\left(e^{-t}z\right)\left[I - E(z,t)\right], \quad (6)$$

where, for all $(z,t) \in B^n \times [0,\infty), E(z,t)$ is the linear operator defined by

$$\begin{aligned} E(z,t) \;&=\; \left(1 - \frac{2}{\alpha}\right) I - \frac{2}{\alpha} e^{-\alpha t} \left[\left(Dg\left(e^{-t}z\right)\right)^{-1} Df\left(e^{-t}z\right) - I\right] \\ &\quad -\frac{2}{\alpha}\left(1 - e^{-\alpha t}\right)\left(Dg\left(e^{-t}z\right)\right)^{-1} D^2g\left(e^{-t}z\right)\left(e^{-t}z,\cdot\right). \quad (7) \end{aligned}$$

For $t = 0, E(z,0) = -\frac{2}{\alpha}\left[(Dg(z))^{-1} Df(z) - \frac{\alpha}{2} I\right]$, it follows

$$\|E(z,0)\| < 1, for\ all\ z \in B^n. \tag{8}$$

For $t > 0, E(\cdot,t) : \overline{B^n} \to \mathcal{L}(\mathbf{C}^n)$ is holomorphic. By using the weak maximum modulus theorem[4], we obtain that $\|E(z,t)\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout $\overline{B^n}$.

If $z = 0$ and $t > 0$, then we have

$$\|E(0,t)\| = \left\|\left(1 - \frac{2}{\alpha}\right) I\right\| = \left|1 - \frac{2}{\alpha}\right| < 1. \tag{9}$$

We also have

$$\|E(z.t)\| < \max_{\|w\|=1} \|E(w,t)\|.$$

If we let $u = e^{-t}w$, with $\|w\| = 1$, then $\|u\| = e^{-t}$ and from (4) we obtain

$$\|E(w,t)\| = \frac{2}{\alpha}\Big\| \|u\|^{\alpha} \cdot \left[(Dg(u))^{-1} Df(u) - I\right] + (1 - \|u\|^{\alpha})(Dg(u))^{-1} D^2 g(u)(u,\cdot) + \left(1 - \frac{\alpha}{2}\right) I\Big\| \le 1.$$

Since $\|E(z,t)\| < 1$ for all $(z,t) \in B^n \times [0,\infty)$, it follows $I - E(z,t)$ is an invertible operator.

Further the calculation shows that

$$\begin{aligned} \frac{\partial L(z,t)}{\partial t} &= \frac{\alpha}{2} e^{(\alpha-1)t} Dge^{-t} z \left[I + E(z,t)\right](z) \\ &= DL(z,t)\left[I - E(z,t)\right]^{-1}\left[I + E(z,t)\right](z). \end{aligned}$$

Hence $L(z,t)$ satisfies the differential equation (1) for all $t \geq 0$ and $z \in B^n$, where

$$h(z,t) = [I - E(z,t)]^{-1} [I + E(z,t)](z). \tag{10}$$

It remains to show that $h(z,t)$ satisfies the conditions (iv) (v) (vi) and (vii) of Theorem 2. Clearly, $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$. The inequality

$$\begin{aligned} \|h(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| &\leq \|E(z,t)\| \, \|(h(z,t) + z)\| \\ &< \|(h(z,t) + z)\| \end{aligned}$$

implies Re $\langle h(z,t), z\rangle \geq 0$, for all $z \in B^n$ and $t \geq 0$.
By using the inequality

$$\left\|[I - E(z,t)]^{-1}\right\| \leq [1 - \|E(z,t)\|]^{-1},$$

we obtain

$$\|h(z,t)\| \leq \frac{1 + \|E(z,t)\|}{1 - \|E(z,t)\|} \|z\|.$$

Since the conditions of Theorem 2 are satisfied it follows that the function $L[z,t), t \geq 0$ is univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

Remark 2

1) If $g = f$ and $\alpha = 2$, then Theorem 3 becomes the n-dimensional version of Beker's univalence criterion [5].

2) For $\alpha = 2$ we obtain an univalence criterion due to P. Curt and D. Răducanu [3].

References

[1] J. Becker, *Loewnersche Differentialgleichung und Schichtheits - Kriterionn*, Math. Ann. 202(1973), 321-335.

[2] P. Curt, *A generalization in n-dimensional complex space of Ahlfos and Becker criterion for univalence*, Studia Math .49(1994), 31-38.

[3] P. Curt and D. Răducanu, *On an univalence condition*, (to appear).

[4] E. Hille and R.S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Collaq. Publ. 31(1957).

[5] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in* $\mathbf{C}^n$, Math. Ann. 210(1974), 55-68.

Dorina Răducanu:
Department of Mathematics
Transilvania University
Braşov 2200
România

Shigeyoshi Owa:
Department of Mathematics
Kinki University
Higashi - Osaka, Osaka 577-8502
Japan

Paula Curt:
Faculty of Mathematics
Kogălniceanu 1
Cluj-Napoca 3400
România