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# Inertial Sets for Phase Transition Models Induced by the Variational Principles

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## 0. Introduction

We consider one-dimensional non-isothermal phase separation model with constraints in the following form, denoted by (PSC):={(0.1)-(0.6)}:

$$e := \theta + \lambda(w) \quad \text{in } Q := \Omega \times (0, +\infty), \tag{0.1}$$

$$e_t - (\alpha(\theta))_{xx} + \nu\theta = f(x) \quad \text{in } Q, \tag{0.2}$$

$$w_t - \{-\kappa w_{xx} + g(w) + \beta(w) - \alpha(\theta)\lambda'(w)\}_{xx} = 0 \quad \text{in } Q, \tag{0.3}$$

$$\pm[\alpha(\theta)]_x(\pm L, t) + n_0\alpha(\theta(\pm L, t)) = h_\pm \quad \text{for } t > 0, \tag{0.4}$$

$$w_x(\pm L, t) = w_{xxx}(\pm, t) = 0 \quad \text{for } t > 0, \tag{0.5}$$

$$\theta(x, 0) = \theta_0(x), \quad w(x, 0) = w_0(x) \quad \text{in } \Omega. \tag{0.6}$$

Here, $\Omega := (-L, L)$ with a given finite number $L > 0$; $\alpha$ and $\beta$ are non-decreasing and smooth functions; $\lambda$ and $g$ are sufficiently smooth functions; $\lambda'$ is the derivative of $\lambda$; $\nu$, $\kappa$ and $n_0$ are positive constants; $f$, $h_\pm$, $\theta_0$ and $w_0$ are given data.

Physically, this model describes the non-isothermal phase separation phenomena of the binary alloys composed by two components A and B. The original model with $\nu = 0$ was introduced by Penrose and Fife [13] and in it $\theta$ represents the absolute temperature and $w$ the conserved order parameter. Actually, we see from the kinetic equation (0.3) and the boundary conditions (0.5) of $w$ that

$$\frac{d}{dt}\int_{-L}^{L} w(t, x) = 0 \quad \text{for any } t > 0,$$

that is,

$$\int_{-L}^{L} w(x, t) = \int_{-L}^{L} w_0(x)dx =: m_0 \quad \text{for any } t \geq 0.$$

Roughly speaking, in our model the mass quantity is conserved. From this point of view, throughout this paper it is convenient to introduce a new function $v$ by the relation $v := w - m_0$ and consider this function $v$ instead of $w$. Here, you note that the fact

$$\int_{-L}^{L} v(x,t)dx = 0 \quad \text{for any } t \geq 0.$$

The typical examples of $\alpha$ and $\beta$ are

$$\alpha(\theta) := -\frac{1}{\theta} \quad \text{for any } \theta > 0$$

and

$$\beta(w) := k_0 \log \frac{1+w}{1-w} \quad \text{for any } w \in (-1,1) \quad \text{with some constant } k_0 > 0.$$

Since the domain of $\beta$ is restricted in the interval $(-1,1)$, this model is a kind of the phase separation models with constraints. For these models, there have already been some works which guarantees the global existence and uniqueness of solutions (cf. [2], [9], [14]). But, in these papers they assumed that $\lambda$ is convex and this assumption is essential.

Recently, in [12] we discussed the weak well-posedness (i.e. (global) existence, uniqueness and weakly continuous dependence upon the data of the solution) without the assumption that $\lambda$ is convex for the case $\nu \geq 0$ and in [7] we constructed the global attractor for the case $\nu > 0$.

But, it is not sufficient to discuss the asymptotic behavior as $t \to +\infty$ because we have at least two questions for the global attractor. One is to investigate the structure of the global attractor. The other is to give the estimate of the speed under which any solution is attracted to the global attractor. In order to give the answers to these questions we use the notion of inertial set (sometimes it is called the exponential attractor), which was established by Eden, Foias, Nicolaenko and Temam in [3], for the semigroup associated with our system. In consequence, we proved in this paper that the global attractor has a finite fractal dimension and the inertial set uniformly attracts all solutions starting from some compact set.

**Notation.** We fix a positive number $L$, and put $\Omega := (-L, L)$. For simplicity we use the following notation:

(1) In $H := L^2(\Omega)$, the usual inner product is denoted by $(\cdot,\cdot)_H$ and the norm by $|\cdot|_H$.

(2) $V := H^1(\Omega)$ is the Hilbert space with the inner product $(\cdot,\cdot)_V$ given by

$$(v,z)_V := (v_x, z_x)_H + n_0\{v(L)z(L) + v(-L)z(-L)\} \quad \text{for any } v,\ z \in V$$

and the norm $|\cdot|_V := (\cdot,\cdot)_V^{\frac{1}{2}}$. The dual space of $V$ is denoted by $V^*$, and the duality pair between $V^*$ and $V$ is denoted by $\langle\cdot,\cdot\rangle_{V^*,V}$. Furthermore, the duality mapping $F : V \longrightarrow V^*$ is defined by

$$\langle Fv, z\rangle_{V^*,V} = (v,z)_V \quad \text{for any } v,\ z \in V.$$

(3) $V^*$ is the Hilbert space equipped with the inner product $(\cdot,\cdot)_{V^*}$ given by

$$(v,z)_{V^*} := \langle v, F^{-1}z\rangle_{V^*,V} (= \langle z, F^{-1}v\rangle_{V^*,V}) \quad \text{for any } v,\ z \in V^*.$$

The corresponding norm $|v|_{V^*}$ is given by $|F^{-1}v|_V$.

(4) $H_0$ is the subspace of $H$ defined by

$$H_0 := \left\{ z \in H;\ \int_\Omega z(x)dx = 0 \right\}.$$

Then, $H_0$ is the Hilbert space by succeeding to the inner product of $H$, that is, the inner product $(\cdot,\cdot)_{H_0}$ in $H_0$ is given by

$$(v,z)_{H_0} := (v,z)_H \quad \text{for any } v,\ z \in H_0.$$

Moreover, we define a projection operator $\pi_0$ from $H$ onto $H_0$ by

$$\pi_0[z](x) := z(x) - \frac{1}{2L}\int_\Omega z(y)dy \quad \text{for any } x \in \Omega.$$

(5) $H^1(\Omega)$, $H^2(\Omega)$ and $H^3(\Omega)$ are the usual Sobolev spaces; especially, we distinguish $H^1(\Omega)$ from $V$ because of the difference of the inner products throughout this paper.

(6) $V_0 := H_0 \cap H^1(\Omega)$ is the Hilbert space with the norm $|\cdot|_{V_0}$ and the inner product $(\cdot,\cdot)_{V_0}$ given by

$$(v,z)_{V_0} := (v_x, z_x)_H \quad \text{for any } v,\ z \in V_0.$$

The dual space of $V_0$ is denoted by $V_0^*$, and the duality pair between $V_0^*$ and $V_0$ is denoted by $\langle\cdot,\cdot\rangle_{V_0^*,V_0}$. Furthermore, the duality mapping $F_0 : V_0 \longrightarrow V_0^*$ is defined by

$$\langle F_0 v, z\rangle_{V_0^*,V_0} = (v,z)_{V_0} \quad \text{for any } v,\ z \in V_0.$$

(7) $V_0^*$ is the Hilbert space equipped with inner product $(\cdot,\cdot)_{V_0^*}$ given by

$$(v,z)_{V_0^*} := \langle v, F_0^{-1}z\rangle_{V_0^*,V_0} (= \langle z, F_0^{-1}v\rangle_{V_0^*,V_0}) \quad \text{for any } v,\ z \in V_0^*.$$

The corresponding norm $|v|_{V_0^*}$ is also given by $|F_0^{-1}v|_{V_0}$.

(8) $\mathcal{H} := V^* \times V_0^*$, which is the Hilbert space with the inner product

$$(U,\overline{U})_{\mathcal{H}} := (e,\overline{e})_{V^*} + (v,\overline{v})_{V_0^*} \quad \text{for any } U := [e,v],\ \overline{U} := [\overline{e},\overline{v}] \in \mathcal{H}.$$

(9) $\mathcal{E} := H \times V_0$, which is the Hilbert space with the inner product

$$(U,\overline{U})_{\mathcal{E}} := (e,\overline{e})_{H} + (v,\overline{v})_{V_0} \quad \text{for any } U := [e,v],\ \overline{U} := [\overline{e},\overline{v}] \in \mathcal{E}.$$

(10) By $\Delta_N$ we mean the Laplacian $\Delta$ with homogeneous Neumann boundary condition, namely, $\Delta v = v_{xx}$ in $\Omega$ with $v_x(\pm L) = 0$; $-\Delta_N$ is the maximal monotone operator in $H_0$ with the domain $D(-\Delta_N) := \{v \in H^2(\Omega) \cap H_0;\ v_x(\pm L) = 0\}$.

## 1. Known results

In this section, let us recall some results established in [7, 12].

Throughout this paper we consider our system under the following assumptions:

(A1) $\alpha$ is a strictly increasing function of $C^2$-class from $(0,+\infty)$ onto $(-\infty,0)$ such that

$$|\alpha(r)| \geq \frac{c_0}{r} \quad \text{for any } r > 0$$

for some suitable positive constant $c_0$ and

$$\lim_{r\uparrow+\infty} \alpha(r) = 0, \quad \lim_{r\downarrow 0} \alpha(r) = -\infty.$$

(A2) $\beta$ is a non-decreasing function of $C^2$-class from $D(\beta) := (-1,1)$ onto $R$ such that

$$\lim_{r\downarrow -1} \beta(r) = -\infty, \quad \lim_{r\uparrow 1} \beta(r) = +\infty.$$

We fix a non-negative primitive $\hat{\beta}$ of $\beta$: note $(-1,1) \subset D(\hat{\beta}) \subset [-1,1]$.

(A3) $\lambda$ is a $C^3$-function on $R$ with compact support.

(A4) $g$ is a $C^2$-function on $R$ with compact support; we fix a primitive $\hat{g}$ of $g$ such that $\hat{g} \geq 0$ on $R$.

**(A5)** $m_0 \in (-1,1)$, $\nu > 0$, $\kappa > 0$ and $n_0 > 0$.

**(A6)** $f \in H$ and $h_\pm$ are negative constants.

**(A7)** $\theta_0 \in H, \quad v_0 \in V_0$.

Now, we define a solution to (PSC):={(0.1)-(0.5)} in a weak variational sense.

**Definition 1.1.** Let $0 < T < +\infty$, $m_0 \in (-1,1)$ and define $f^* \in V^*$ by

$$\langle f^*, z\rangle_{V^*,V} := (f,z)_H + h_+ z(L) + h_- z(-L) \quad \text{for any } z \in V.$$

Moreover, we define a new function $v$ by $v := w - m_0$. Then, we call a couple of functions $[e,v]$ a solution to (PSC) on $[0,T]$ if the following properties (i)-(iv) are satisfied:

**(i)** $e := \theta + \lambda(v+m_0) \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H) (\subset C_w([0,T];H))$.

**(ii)** $v \in W^{1,2}(0,T;V_0^*) \cap L^\infty(0,T;V_0) (\subset C_w([0,T];V_0))$.

**(iii)** $\alpha(\theta) \in L^2(0,T;V)$ and

$$e_t(t) + F\alpha(\theta(t)) + \nu\theta(t) = f^* \quad \text{in } V^* \text{ for a.e. } t \in (0,T).$$

**(iv)** $\beta(v+m_0) \in L^2(0,T;H)$ and

$$F_0^{-1} v_t(t) - \kappa\Delta_N v(t) + \pi_0[g(v(t)+m_0) + \beta(v(t)+m_0) - \alpha(\theta(t))\lambda'(v(t)+m_0)]$$
$$= 0 \quad \text{in } H_0 \qquad \text{for a.e. } t \in (0,T).$$

Given initial data $e_0 \in H$ and $v_0 \in V_0$, $[e,v]$ is called a solution to the Cauchy problem (PSC; $e_0, v_0$):={(0.1)-(0.6)} on $[0,T]$ if it is a solution to (PSC) on $[0,T]$ with initial data $e(0) = e_0$ and $v(0) = v_0$.

Moreover, $[e,v]$ is called a global solution to (PSC) if it is a solution to (PSC) on $[0,T]$ for any finite time $T > 0$.

Under these situations, we relate the results in [7, 12]. To do so, first of all, we introduce a functional $\Phi$ on $H \times H_0$ in the following way:

$$\Phi(e,v) := j(e - \lambda(v+m_0)) + \Psi(v) \quad \text{for any } [e,v] \in H \times H_0,$$

where

$$j(z) := \begin{cases} \displaystyle\int_\Omega \hat{\alpha}(z(x))dx, & \hat{\alpha}(z) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\Psi(v) := \begin{cases} \frac{\kappa}{2}|v|_{V_0}^2 + \int_\Omega \hat{g}(v(x)+m_0)dx + \int_\Omega \hat{\beta}(v(x)+m_0)dx, \\ \qquad\qquad \text{if } v \in V_0 \text{ with } \hat{\beta}(v+m_0) \in L^1(\Omega), \\ +\infty, \qquad \text{otherwise;} \end{cases}$$

note that $\Psi$ is non-negative on $H_0$ by (A2) and (A4). Now, we define a subset $D$ of $\mathcal{E}$ by

$$D := \{[e,v] \in \mathcal{E};\ \Phi(e,v) < +\infty\}.$$

Then, according to the results of [7, 12] we see that for each $m_0 \in (-1,1)$ and $[e_0, v_0] \in D$ the Cauchy problem (PSC;$e_0, v_0$) has one and only one global solution $[e, v]$. Furthermore for any two initial data $[e_{0i}, v_{0i}] \in D$ the global solutions $[e_i, v_i]$ to (PSC;$e_{0i}, v_{0i}$) $(i = 1,2)$ satisfy

$$\begin{aligned} |e_2(t) - e_1(t)|_{V^*}^2 + |v_2(t) - v_1(t)|_{V_0^*}^2 + C_1 \int_s^t |v_2(\tau) - v_1(\tau)|_{V_0}^2 d\tau \\ \leq \exp\left(C_2 \int_s^t (1 + |\alpha(\theta_1(\tau))|_V^2 + |\alpha(\theta_2(\tau))|_V^2) d\tau\right) \\ \times (|e_2(s) - e_1(s)|_{V^*}^2 + |v_2(s) - v_1(s)|_{V_0^*}^2) \\ \text{for any } s,\ t \text{ with } 0 \leq s \leq t < +\infty. \end{aligned} \tag{1.1}$$

for some suitable positive constants $C_i$ $(i = 1,2)$, which are independent of initial data in $D$.

Hence, we can define a dynamical system $\{S(t)\} := \{S(t);\ t \geq 0\}$ on $D$ associated with (PSC) by for each $[e_0, v_0] \in D$, $[e(t), v(t)] = S(t)[e_0, v_0]$ is a global solution to (PSC;$e_0, v_0$).

Moreover, we have already obtained the following properties (S1)-(S6) as well as the above facts:

(S1) $S(0) = I$ on $D$.

(S2) $S(t+s) = S(t)S(s)$ for any $t,\ s \geq 0$.

(S3) $D$ is positively invariant under $\{S(t)\}_{t\geq 0}$, namely, $S(t)D \subset D$ for any $t \geq 0$.

(S4) If $[e_{0n}, v_{0n}] \in D$, $[e_{0n}, v_{0n}] \longrightarrow [e, v]$ in $\mathcal{H}$ and $\{\Phi(e_{0n}, v_{0n})\}$ is bounded, then $S(\cdot)[e_{0n}, v_{0n}] \longrightarrow S(\cdot)[e_0, v_0]$ in $C([0,T];\mathcal{H})$ for every $0 < T < +\infty$. Moreover, if $e_{0n} \longrightarrow e_0$ weakly in $H$, then $S(\cdot)[e_{0n}, v_{0n}] \longrightarrow S(\cdot)[e_0, v_0]$ in $C_w([0,T];\mathcal{E}) \cap C_w([\delta,T]; H \times D(-\Delta_N))$ for every $0 < \delta < T < +\infty$.

Before stating the statement (S5) and (S6), we have to prepare a functional $J$ with some properties. For each $\eta > 0$ let us consider a functional $J_\eta$ on $D$ which is defined by

$$J_\eta(e,v) := \Phi(e,v) - \langle e, \alpha(\theta_0)\rangle_{V^*,V} + \eta|e|_H^2 + C_3(\eta) \quad \text{for any } [e,v] \in D,$$

where a pair $[\theta_0, \alpha(\theta_0)] \in H \times V$ is a unique pair satisfying

$$(\alpha(\theta_0), z)_V + \nu(\theta_0, z)_H = \langle f^*, z\rangle_{V^*,V}$$

and $C_3(\eta)$ are chosen, depending only on $\eta$, so that

$$J_\eta(e,v) \geq \frac{\eta}{2}|e - \lambda(v+m_0)|_H^2 \quad \text{for any } [e,v] \in D.$$

This is a Lyapunov-like functional for our system. Actually, the following inequality of Gronwall's type holds: there exist $\eta_1 > 0$ and $N_0 > 0$, which are independent of the initial data $[e_0, v_0] \in D$, such that

$$\frac{d}{dt}J(e(t),v(t)) + \eta_1 J(e(t),v(t)) \leq N_0 \quad \text{for a.e. } t \geq 0, \tag{1.2}$$

where $J := J_{\eta_1}$ and $[e(t), v(t)] = S(t)[e_0, v_0]$ for any $[e_0, v_0] \in D$; for the proof of (1.2) we leave to the paper [7] and it is omitted in this paper. But, we emphasized that we used the positiveness of $\nu$ to prove (1.2), namely, $\nu(> 0)$ plays an important role to obtain the above inequality.

Now, we state (S5) and (S6):

**(S5)** (Global estimate) For each finite time $T > 0$ and bounded subset $B(\subset \mathcal{E})$ with $\sup_{[e,v]\in B} J(e,v) < +\infty$ there exists a positive constant $T(B,T)$, depends upon $B$ and $T$, such that

$$|t^{\frac{1}{2}}v_t|_{L^\infty(0,T;V_0^*)} + |t^{\frac{1}{2}}v_t|_{L^2(0,T;V_0)} + |t^{\frac{1}{2}}\alpha(\theta)|_{L^\infty(0,T;V)} + |t^{\frac{1}{2}}v|_{L^\infty(0,T;H^2(\Omega))}$$
$$+|t^{\frac{1}{2}}\beta(v+m_0)|_{L^\infty(0,T;H)} \leq M(B,T)$$

for any solutions $[e(\cdot), v(\cdot)]$ with initial datum $[e_0, v_0] \in D$.

**(S6)** [7; Lemma 4.2] (Existence of an absorbing set) There exists a subset $B_0$ of $D$ satisfying the following properties (i)-(iii):

**(i)** $B_0$ is weakly compact in $\mathcal{E}$ and $\sup_{[e,v]\in B_0} J(e,v) < +\infty$.

**(ii)** $B_0$ is arcwise connected in the weak topology of $\mathcal{E}$.

(iii) For each subset $B$ of $D$ with $\sup_{[e,v]\in B} J(e,v) < +\infty$ there exists a finite time $t_B > 0$ such that

$$S(t)B \subset B_0 \quad \text{for any } t \geq t_B.$$

As a result of (S1)-(S6) we have the following theorem.

**Theorem 1.1.** (cf. [7; Theorem 3.1]; Existence of a global attractor) *Assume that* (A1)-(A6) *hold. Then the set*

$$\mathcal{A} := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B_0}^{V^*\times V_0}$$

*satisfies* (i)-(iii) *below, where* $\overline{X}^{V^*\times V_0}$ *denotes the closure of* $X$ *in* $V^* \times V_0$*:*

(i) *$\mathcal{A}$ is compact and connected in the weak topology of $H \times (H^2(\Omega) \cap H_0)$.*

(ii) *$\mathcal{A}$ is invariant under $\{S(t)\}$, namely, $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$.*

(iii) *for each subset $B$ ($\subset D$) with* $\sup_{[e,v]\in B} J(e,v) < +\infty$

$$\lim_{t\to+\infty} \mathrm{dist}_{V^*\times V_0}(S(t)B, \mathcal{A}) = 0,$$

*where for any subsets $X$, $Y$* of $V^* \times V_0$

$$\mathrm{dist}_{V^*\times V_0}(X,Y) := \sup_{x\in X}\{\inf_{y\in Y} |x-y|_{V^*\times V_0}\}.$$

Throughout this paper, we call $\mathcal{A}$ a global attractor for the dynamical system $\{S(t)\}$ on $D$ associated with (PSC).

## 2. Main Theorem

We consider our system (PSC) under the same assumptions and use the same notation as in the previous section.

Before stating our main theorem in this paper, we introduce some notions, which are important to investigate the large-time behavior of solutions to (PSC).

**Definition 2.1.** Let $X$ be compact in $\mathcal{H}$ and $\mathcal{M}$ is a subset of $X$. Then, $\mathcal{M}$ is called an inertial set in $X$ for $\{S(t)\}$, if $\mathcal{M}$ has the following properties (IS1)-(IS4):

(IS1) $\mathcal{A} \subset \mathcal{M} \subset X$.

(IS2) $\mathcal{M}$ has a finite fractal dimension.

**(IS3)** $\mathcal{M}$ is positively invariant under $\{S(t)\}$, that is,

$$S(t)\mathcal{M} \subset \mathcal{M} \quad \text{for any } t \geq 0.$$

**(IS4)** There exist positive constants $c_1$ and $c_2$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)X, \mathcal{M}) \leq c_1 e^{-c_2 t} \quad \text{for any } t \geq 0.$$

**Remark 2.1.** From (IS1) and (IS2), we see that the fractal dimension of $\mathcal{A}$ is also finite. Moreover, by using the fact that the Hausdorff dimension is less than or equal to the fractal dimension it follows that the Hausdorff dimension of $\mathcal{A}$ is finite, too.

**Definition 2.2.** Let $T$ be a Lipschitz continuous mapping on $X$ with respect to the strong topology of $\mathcal{H}$. Then, we call that $T$ has a squeezing property on $X$ with respect to the strong topology of $\mathcal{H}$, if there is an orthogonal projection $P$, with finite rank, such that

$$|TU_2 - TU_1|_{\mathcal{H}} \leq \frac{1}{8}|U_2 - U_1|_{\mathcal{H}}$$

holds for any pair of $U_1$, $U_2 \in X$ satisfying

$$|P(TU_2 - TU_1)|_{\mathcal{H}} \leq |(I - P)(TU_2 - TU_1)|_{\mathcal{H}}.$$

Our main theorem is follows.

**Theorem 2.1.** *There exist a compact subset $\mathcal{X}$ of $\mathcal{H}$ and a finite time $t^*$ such that $S^* := S(t^*)$ has a squeezing property on $\mathcal{X}$ as well as the Lipschitz continuity on $\mathcal{X}$ with respect to the strong topology of $\mathcal{H}$.*

And by applying the results of Eden, Foias, Nicolaenko and Temam (cf. [3]) we get the following corollary to Theorem 2.1.

**Corollary to Theorem 2.1.** *There exists an inertial set $\mathcal{M}$ in $\mathcal{X}$ for $\{S(t)\}$ and the fractal dimension of $\mathcal{M}$ is dominated by the number*

$$N_* \max\left\{1, \frac{\log(16\mathrm{Lip}(S^*)) + 1}{\log 2}\right\},$$

*where* $\mathrm{Lip}(S^*)$ *is a Lipschitz constant of $S^*$ and $N_*$ is the rank of the orthogonal projection $P := P^*$ appearing in the squeezing property of $S^*$.*

## 3. Proof of Theorem 2.1

In this section, we give some lemmas, which are tools to prove Theorem 2.1. But, we will not to write their proofs and they are written in [5] in detail.

As the first lemma, we give the global uniform estimates of global solutions starting from the absorbing set $B_0$ given in (S6).

**Lemma 3.1.** *For any global solution* $[e(\cdot), v(\cdot)] := [\theta(\cdot) + \lambda(v(\cdot) + m_0), v(\cdot)]$ *with initial datum* $[e_0, v_0] \in B_0$*, the following estimates hold:*

(i) *There exists a positive constant* $R_0$*, depending upon the absorbing set* $B_0$*, such that*

$$|v_t(t)|_{V_0^*} + |v(t)|_{H^2(\Omega)} + |\alpha(\theta(t))|_V + |\beta(v(t) + m_0)|_H \le R_0$$
$$\text{for any } t \ge t_{B_0} + 1$$

*and*

$$\sup_{t \ge t_{B_0}+1} |v_t|_{L^2(t,t+3;V_0)} \le R_0,$$

*where* $t_{B_0}$ *is a finite time satisfying*

$$S(t)B_0 \subset B_0 \quad \text{for any } t \ge t_{B_0}.$$

(ii) (cf. [6; Lemma 3.1]) *There exist positive and finite constants* $\theta_*$ *and* $\theta^*$ *and a finite time* $t_1 (> t_{B_0} + 1)$ *such that*

$$\theta_* \le \theta := e - \lambda(v + m_0) \le \theta^* \quad \text{on } [-L, L] \times [t_1, +\infty).$$

(iii) *There exists a positive constant* $\varepsilon_0$ *such that*

$$-1 + \varepsilon \le v + m_0 \le 1 - \varepsilon_0 \quad \text{on } [-L, L] \times [t_1, +\infty),$$

*where* $t_1$ *is the same number as in* (ii).

It is easy from the global estimate (S5) to prove (i). And the proves of (ii) and (iii) are quite similar to those of Lemma 3.1 in [6]. We will omit them in this paper.

**Remark 3.1.** From Lemma 3.1 without loss of generality we may assume that $\alpha$ is a bi-Lipschitz strictly increasing function in $C^2$-class with $\alpha'' \in L^\infty_{loc}(R)$ and $\beta$ is a non-decreasing continuous function on $[-1, 1]$ as well as continuous on $R$ in $C^2$-class with compact support, respectively, as long as we consider the solutions to (PSC) on $[t_1, +\infty)$ with the initial data in $B_0$. Moreover, we see that any solution $[e(\cdot), v(\cdot)]$ to (PSC) with initial datum $[e_0, v_0] \in B_0$ has the following regularities:

$$e \in W^{1,2}_{loc}([t_1, +\infty); H) \cap L^\infty([t_1, +\infty); V), \quad \alpha(\theta) \in L^2_{loc}([t_1, +\infty); H^2(\Omega)),$$

$$v \in L^\infty([t_1, +\infty); H^3(\Omega)).$$

From now we assume that $\alpha$ and $\beta$ satisfy the properties in Remark 3.1, respectively.

In the next lemma, we will give some global uniform estimate with respect to $\theta_t$ and $v_t$. And this lemma plays a quite important role to prove Theorem 2.1.

**Lemma 3.2.** *Let $[e(\cdot), v(\cdot)]$ be any solution to* (PSC) *with initial datum $[e_0, v_0] \in B_0$. Then, there exists a positive constant $R_3$ such that for each $s \geq t_1$ and $T > 0$*

$$\sup_{s \leq t \leq s+T} \{(t-s)|\theta_t(t)|_H^2\} + \sup_{s \leq t \leq s+T} \{(t-s)|v_t(t)|_{V_0}^2\}$$

$$+ \int_s^{s+T} (t-s)|(\alpha(\theta))_t(t)|_V^2 dt + \int_s^{s+T} (t-s)|v_{tt}(t)|_{V_0^*}^2 dt \leq R_3$$

$$for\ any\ [e_0, v_0] \in B_0.$$

**Proof.** To prove this lemma we consider the following system: for each $\mu \in (0, 1)$, $s \in [t_1, +\infty)$ and $T > 0$

$$e_t^{\mu,s} - (\alpha(\theta^{\mu,s}))_{xx} + \nu\theta^{\mu,s} = f(x) \quad \text{in } Q_{s,T} := (-L, L) \times (s, s+T), \tag{3.1}$$

$$v_t^{\mu,s} - \{\mu v_t^{\mu,s} - \kappa v_{xx}^{\mu,s} + g(v^{\mu,s} + m_0) + \beta(v^{\mu,s} + m_0)$$
$$-\alpha(\theta^{\mu,s})\lambda'(v^{\mu,s} + m_0)\}_{xx} = 0 \tag{3.2}$$
$$\text{in } Q_{s,T},$$

$$\pm(\alpha(\theta^{\mu,s}))_x(\pm L, t) + n_0\alpha(\theta^{\mu,s}(\pm L, t)) = h_\pm \quad \text{for any } t \in (s, s+T), \tag{3.3}$$

$$v_x^{\mu,s}(\pm L, t) = v_{xxx}^{\mu,s}(\pm L, t) = 0 \quad \text{for any } t \in (s, s+T), \tag{3.4}$$

$$e^{\mu,s}(s) = e(s) \quad v^{\mu,s}(s) = v(s), \tag{3.5}$$

where $[e(s), w(s)]$ is any solution to (PSC) at time $t = s$ with initial datum in $B_0$. For this system we have already known the following results (cf. [9, 12]):

(1) The above system has one and only one solution $[e^{\mu,s}(\cdot), v^{\mu,s}(\cdot)]$ on $[s, s+T]$ satisfying the following properties:

(i) $e^{\mu,s} \in W^{1,2}(s, s+T; H) \cap L^\infty(s, s+T; V)$.

(ii) $v^{\mu,s} \in L^\infty(s, s+T; H^2(\Omega))$, $v_t^{\mu,s} \in C([s, s+T]; H_0)$,

$v_t^{\mu,s} \in L^\infty(s, s+T; V_0)$, $v_{tt}^{\mu,s} \in L^2(s, s+T; H_0)$.

(iii) $\alpha(\theta^{\mu,s}) \in L^\infty(s, s+T; V)$ and

$$e_t^{\mu,s}(t) + F\alpha(\theta^{\mu,s}(t)) + \nu\theta^{\mu,s}(t) = f^* \quad \text{in } V^* \text{ for a.e. } t \in [s, s+T].$$

(iv) $\beta(v^{\mu,s} + m_0) \in L^\infty(s, s+T; H)$ and

$$(F_0^{-1} + \mu I)v_t^{\mu,s}(t) - \kappa\Delta_N v^{\mu,s}(t)$$

$$+\pi_0[g(v^{\mu,s}(t) + m_0) + \beta(v^{\mu,s}(t) + m_0) - \alpha(\theta^{\mu,s}(t))\lambda'(v^{\mu,s}(t) + m_0)] = 0$$

$$\text{in } H_0 \quad \text{for a.e. } t \in [s, s+T].$$

(2) For each $T > 0$ there exists a positive constant $R_1 := R_1(T)$ such that

$$|e^{\mu,s}|_{W^{1,2}(s,s+T;V^*)} + |v^{\mu,s}|_{W^{1,2}(s,s+T;V_0^*)} + \mu^{\frac{1}{2}}|v^{\mu,s}|_{W^{1,2}(s,s+T;H_0)}$$

$$+|j(\theta^{\mu,s})|_{L^\infty(s,s+T)} + |v^{\mu,s}|_{L^\infty(0,T;V_0)} + |\alpha(\theta^{\mu,s})|_{L^2(s,s+T;V)}$$

$$+|v^{\mu,s}|_{L^2(s,s+T;H)} + |\beta(v^{\mu,s} + m_0)|_{L^2(s,s+T;H)} \le R_1$$

for any $\mu \in (0,1]$, $s \ge t_1$ and $[e_0, v_0] \in B_0$.

(3) For each $T > 0$ there exists a positive constant $R_2 := R_2(T)$ such that

$$|\theta_t^{\mu,s}|_{L^2(s,s+T;H)} + \sup_{s\le t\le s+T} |\alpha(\theta^{\mu,s}(t))|_V + \sup_{s\le t\le s+T} |v_t^{\mu,s}(t)|_{V_0^*}$$

$$+\mu^{\frac{1}{2}} \sup_{s\le t\le s+T} |v_t^{\mu,s}(t)|_{H_0} + |v_t^{\mu,s}|_{L^2(s,s+T;V_0)} \le R_2$$

for any $\mu \in (0,1]$, $s \ge t_1$ and $[e_0, v_0] \in B_0$.

From these estimates, we note that there exist positive constants $R_i$ $(4 \le i \le 6)$ such that

$$R_4 \le \alpha'(\theta^{\mu,s}) \le R_5 \quad \text{on } [-L, L] \times [s, s+T],$$

$$-R_6 \le \theta^{\mu,s} \le R_6 \quad \text{on } [-L, L] \times [s, s+T]$$

for any $\mu \in (0,1]$, $s \ge t_1$ and $[e_0, v_0] \in B_0$. And we put

$$R_7 := \max_{|r|\le R_6} |\alpha''(r)| + \max_{|r|\le R_6} |\alpha(r)|.$$

Now, we use the above fact and calculate $(d/dt)(3.1) \times (\alpha(\theta^{\mu,s}))_t(t)$ in $H \times H$ to obtain

$$\frac{d}{dt}\int_{-L}^{L} \frac{\alpha'(\theta^{\mu,s}(x,t))|\theta_t^{\mu,s}(x,t)|^2}{2}dx + (1-\varepsilon)|(\alpha(\theta^{\mu,s}))_t(t)|_V^2$$

$$
\begin{aligned}
&(\lambda'(v^{\mu,s}(t)+m_0)v_{tt}^{\mu,s}(t),(\alpha(\theta^{\mu,s}))_t(t))_H \\
&\le \frac{c_0R_7^2}{2\varepsilon R_4^4}\left(\int_{-L}^{L}\alpha'(\theta^{\mu,s}(x,t))|\theta_t^{\mu,s}(x,t)|^2dx\right) \\
&\quad\times\left(\int_{-L}^{L}\frac{\alpha'(\theta^{\mu,s}(x,t))|\theta_t^{\mu,s}(x,t)|^2}{2}dx\right) \\
&\quad+c_1^2L(\lambda')R_5|v_t^{\mu,s}(t)|_{V_0}^2\int_{-L}^{L}|\theta_t^{\mu,s}(x,t)|^2dx \\
&\qquad\text{for a.e. } t\in[s,s+T]
\end{aligned}
\tag{3.6}
$$

for some suitable positive constants $c_1$ and $c_2$.

Secondly, we take the inner product between $(d/dt)(3.2)$ and $v_{tt}^{\mu,s}(t)$ in $H_0$ to obtain

$$
\begin{aligned}
&|v_{tt}^{\mu,s}(t)|_{V_0^*}^2+\mu|v_{tt}^{\mu,s}(t)|_{H_0}^2+\frac{d}{dt}\left\{\frac{\kappa}{2}|v_t^{\mu,s}(t)|_{V_0}^2\right\} \\
&\le 3\varepsilon'|v_{tt}^{\mu,s}(t)|_{V_0^*}^2+R_8|v_t^{\mu,s}(t)|_{V_0}^2+(\lambda'(v^{\mu,s}(t)+m_0)v_{tt}^{\mu,s}(t),(\alpha(\theta^{\mu,s}))_t(t))_H \\
&\qquad\text{for a.e. } t\in[s,s+T],
\end{aligned}
\tag{3.7}
$$

where $R_8$ is a suitable positive constant, which is independent of $\mu\in(0,1]$, $s\ge t_1$ and $[e_0,v_0]\in B_0$.

Now we choose $\varepsilon=1/2$, $\varepsilon'=1/6$ and add (3.6) to (3.7) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{-L}^{L}\frac{\alpha'(\theta^{\mu,s}(x,t))|\theta_t^{\mu,s}(x,t)|^2}{2}dx+\frac{\kappa}{2}|v_t^{\mu,s}(t)|_{V_0}^2\right\} \\
&\quad+\frac{1}{2}|(\alpha(\theta^{\mu,s}))_t(t)|_V^2+\frac{1}{2}|v_{tt}^{\mu,s}(t)|_{V_0^*}^2+\mu|v_{tt}^{\mu,s}(t)|_{H_0}^2 \\
&\le R_9(|\theta_t^{\mu,s}(t)|_H^2+1)\left(\int_{-L}^{L}\frac{\alpha'(\theta^{\mu,s}(x,t))|\theta_t^{\mu,s}(x,t)|^2}{2}dx+\frac{\kappa}{2}|v_t^{\mu,s}(t)|_{V_0}^2\right) \\
&\qquad\text{for a.e. } t\in[s,s+T]
\end{aligned}
\tag{3.8}
$$

for some suitable constant $R_9>0$.

By applying the Gronwall's lemma to the inequality $(3.8)\times(t-s)$ and using (3), we derive that there exists a positive constant $R_{10}$ such that

$$
\begin{aligned}
&\sup_{s\le t\le s+T}\{(t-s)|\theta_t^{\mu,s}(t)|_H^2\}+\sup_{s\le t\le s+T}\{(t-s)|v_t^{\mu,s}(t)|_{V_0}^2\} \\
&+\int_s^{s+T}|(\alpha(\theta^{\mu,s}))_t(t)|_V^2dt+\int_s^{s+T}(t-s)|v_{tt}^{\mu,s}(t)|_{V_0^*}^2dt
\end{aligned}
$$

$$+\mu\int_s^{s+T}(t-s)|v_{tt}^{\mu,s}(t)|_{H_0}^2dt \le R_{10}$$

$$\text{for any } \mu\in(0,1], \quad s\ge t_1 \quad \text{and} \quad [e_0,v_0]\in B_0.$$

By letting $\mu\downarrow 0$, we obtain this lemma. $\diamond$

In the next step, we will construct $\mathcal{X}$ and give the linearrized system of (PSC) on $\mathcal{X}$.

We define the subset $\mathcal{X}$ of $V^*\times V_0^*$ by

$$\mathcal{X}:=\bigcup_{t\ge t_1}S(t)B_0\subset B_0,$$

where $t_1$ is the same number in Section 3. Then, it is easy to check that $\mathcal{X}$ satisfies the following lemma.

**Lemma 3.3.** *$\mathcal{X}$ satisfies the following properties* (i)-(iv):

(i) *$\mathcal{X}$ is compact and connected in $V^*\times V_0^*$ as well as bounded in $V\times(H_0\cap H^3(\Omega))$.*

(ii) *$\mathcal{X}$ is positively invariant for $\{S(t)\}_{t\ge0}$, namely, $S(t)\mathcal{X}\subset\mathcal{X}$ for all $t\ge0$.*

(iii) *$\mathcal{X}$ is an absorbing set for $\{S(t)\}_{t\ge0}$.*

(iv) *For any $t\ge0$, $S(t)$ is Lipschitz on $\mathcal{X}$ with respect to the norm of $\mathcal{H}$.*

Now, let $[e_{0i},v_{0i}]\in\mathcal{X}$ $(i=1,2)$ be any two elements and put

$$[e_i(t),v_i(t)]:=S(t)[e_{0i},v_{0i}],\quad \theta_i:=e_i-\lambda(v_i+m_0),\quad i=1,2,$$

$$e:=e_2-e_1,\quad v:=v_2-v_1,\quad \theta:=\theta_2-\theta_1.$$

Then it is easy to see that the difference equations of $[e,v]$ is described by

$$e_t(t)+F(\alpha(\theta_2(t))-\alpha(\theta_1(t)))+\nu\theta(t)=0 \quad \text{in } V^* \quad \text{for a.e. } t\ge0, \tag{4.1}$$

$$F_0^{-1}v_t(t)-\kappa\Delta_N v(t)+\pi_0[p_2(t)-p_1(t)]=0 \quad \text{in } H_0 \quad \text{for a.e. } t\ge0, \tag{4.2}$$

$$e(0)=e_0:=e_{02}-e_{01},\quad v(0)=v_0:=v_{02}-v_{01}, \tag{4.3}$$

where

$$p_i:=g(v_i+m_0)+\beta(v_i+m_0)-\alpha(\theta_i)\lambda'(v_i+m_0)\quad (i=1,2).$$

Next, in order to rewrite the above difference equation into the linearlized equation we introduce the functions $\sigma_i$ $(1 \le i \le 7)$ from $R^m$ into $R$ defined by

$$\begin{aligned}
\sigma_1(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0))dr, \\
\sigma_2(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\
&\qquad \times \lambda'(v_1 + r(v_2 - v_1) + m_0)dr, \\
\sigma_3(v_1, v_2) &= \int_0^1 \lambda'(v_1 + r(v_2 - v_1) + m_0)dr, \\
\sigma_4(v_1, v_2) &= \int_0^1 g'(v_1 + r(v_2 - v_1) + m_0)dr, \\
\sigma_5(v_1, v_2) &= \int_0^1 \beta'(v_1 + r(v_2 - v_1) + m_0)dr, \\
\sigma_6(e_1, e_2, v_1, v_2) &= \int_0^1 \alpha'(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\
&\qquad \times (\lambda'(v_1 + r(v_2 - v_1) + m_0))^2 dr \\
&\quad - \int_0^1 \alpha(e_1 + r(e_2 - e_1) - \lambda(v_1 + r(v_2 - v_1) + m_0)) \\
&\qquad \times \lambda''(v_1 + r(v_2 - v_1) + m_0)dr
\end{aligned}$$

and

$$\sigma_7 := \sigma_4 + \sigma_5 + \sigma_6,$$

where $m = 4$ if $i = 1,\ 2,\ 6,\ 7$ and $m = 2$ if $m = 3,\ 4,\ 5$.
Then, it is easily seen that (4.1) and (4.2) can be rewritten in the following form;

$$e_t(t) + F(\sigma_1(t)e(t) - \sigma_2(t)v(t)) + \nu e(t) - \nu\sigma_3(t)v(t) = 0 \quad \text{in } V^* \tag{4.4}$$

$$\text{for a.e. } t \ge 0,$$

$$F_0^{-1}v_t(t) - \kappa\Delta_N v(t) + \pi_0[\sigma_7(t)v(t) - \sigma_2(t)e(t)] = 0 \quad \text{in } H_0 \tag{4.5}$$

$$\text{for a.e. } t \ge 0,$$

where $\sigma_i(t) := \sigma_i(e_1(t), e_2(t), v_1(t), v_2(t))$ $(1 \le i \le 7)$.

At first, we note that the following lemma hold.

**Lemma 3.4.** *There exist positive constants $M_1$ and $M_2$ such that*

$$\sum_{i=1}^{7} |\sigma_i(x,t)| \le M_1 \quad \textit{and} \quad \sigma_1(x,t) \ge M_2, \quad \forall (x,t) \in [-L,L] \times [0,+\infty),$$

*where* $[e_i(\cdot), v_i(\cdot)]$ $(i = 1,\ 2)$ *are solutions to (PSC) with initial data* $[e_{0i}, v_{0i}] \in \mathcal{X}$.

Next from Remark 3.1 for each $t \ge 0$ we define an operator $B(t)$ with domain $\mathcal{Y} := D(B(t)) = V \times (D(-\Delta_N) \cap H^3(\Omega))$ and range in $\mathcal{H}$ by

$$\begin{aligned}(B(t)W, \overline{W})_{\mathcal{H}} \quad &:= (F(\sigma_1(t)e - \sigma_2(t)v), \bar{e})_{V^*} \\ &\quad + (F_0[-\kappa\Delta_N v + \pi_0[\sigma_7(t)v - \sigma_2(t)e]], \bar{v})_{H_0}\end{aligned}$$

$$\text{for any } W := [e,v] \in \mathcal{Y} \quad \text{and} \quad \overline{W} := [\bar{e}, \bar{v}] \in \mathcal{H}.$$

Here, we note from Remark 3.1 the fact that $\mathcal{X} \subset \mathcal{Y}$. Moreover, by means of $B(t)$, the system (4.5) and (4.6) is equivalent to the following evolution equation:

$$U_t(t) + B(t)U(t) + G(t)U(t) = 0 \quad \text{in } \mathcal{H} \quad \text{for a.e. } t \ge 0, \tag{4.6}$$

where $U(t) := [e(t), v(t)]$ and $G$ is an operator in $\mathcal{H}$ defined by

$$G(t)U := [\nu e - \nu\sigma_3(t)v, 0] \quad \text{for any } U := [e,v] \in \mathcal{H}. \tag{4.7}$$

As to the operators $B(t)$ and $G(t)$ we easily get the following lemmas. Furthermore, the constants $M_i$ $(3 \le i \le 8)$ in this lemma are independent of any solutions $\{e_i, v_i\}$ $(i = 1, 2)$ starting from $\mathcal{X}$.

**Lemma 3.5.** *The following properties* (i)-(vi) *are fulfilled:*

(i) *There exists a positive constant $M_3$ such that*

$$|(B(t)U, U)_{\mathcal{H}}| \le M_3 |U|_{\mathcal{E}}^2 \quad \textit{for any } U \in \mathcal{Y} \textit{ and } t \ge 0.$$

(ii) *There exists a positive constant $M_4$ and $M_5$ such that*

$$|U|_{\mathcal{E}}^2 \le M_4 (B(t)U, U)_{\mathcal{H}} + M_5 |v|_{V_0^*} \quad \textit{for any } U \in \mathcal{Y} \textit{ and } t \ge 0.$$

(iii) *There exists a positive constant $M_6$ such that*

$$|(G(t)U, U)_{\mathcal{H}}| \le M_6 |U|_{\mathcal{H}}^2 \quad \textit{for any } U \in \mathcal{H} \textit{ and } t \ge 0.$$

(iv) *There exists a positive constant $M_7$ such that*

$$|(B(t)U, G(t)U)_{\mathcal{H}}| \le M_7 |U|_{\mathcal{E}}^2 \quad \textit{for any } U \in \mathcal{Y} \textit{ and } t \ge 0.$$

(v) *For each $t \geq 0$, we define an operator $B_t(t)$ from $H \times H_0$ into itself by*

$$B_t(t)W := [(\sigma_1)_t(t)e - (\sigma_2)_t(t)v, \pi_0[(\sigma_7)_t(t)v - (\sigma_2)_t(t)e]]$$

$$\text{for any } W := [e, v] \in H \times H_0.$$

*Then, there exists a positive constant $M_8$ such that*

$$|(B_t(t)W, W)_{H\times H_0}| \leq M_8 \left\{ \sum_{i=1}^{2} |(\alpha(\theta_i))_t(t)|_V + \sum_{i=1}^{2} |(v_i)_t(t)|_{V_0} \right\}$$

$$\times (|e|_H^2 + |v|_{H_0}^2)$$

$$\text{for any } W := [e, v] \in H \times H_0 \quad \text{and} \quad \text{a.e. } t \geq 0,$$

*where for each $i = 1, 2$ $[e_i(\cdot), v_i(\cdot)] := [\theta_i(\cdot) + \lambda(v_i(\cdot) + m_0), v_i(\cdot)]$ are solutions to* (PSC) *with initial data $[e_{0i}, v_{0i}] \in \mathcal{X}$.*

(vi) *Let $Z \in W^{1,2}_{loc}(R_+; \mathcal{H})$ such that $Z(t) \in \mathcal{Y}$ for a.e. $t \geq 0$. Then,*

$$\frac{d}{dt}(B(t)Z(t), Z(t))_{\mathcal{H}} = (B_t(t)Z(t), Z(t))_{H\times H_0} + 2(B(t)Z(t), Z_t(t))_{\mathcal{H}}$$

$$\text{for a.e. } t \geq 0.$$

By using the above lemmas, we can actually prove Theorem 2.1, i.e., we can check the existence of a finite time $t^*$ and the squeezing property of $S^* := S(t^*)$.

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