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Nonstandard Representations of Unbounded Self-Adjoint Oprators

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1. Introduction

In nonstandard analysis, *standardizations* of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g. an internal measure space is converted into a measure space in the standard sense, called Loeb space ([1][2][3][4]). The standardization of an internal Hilbert space $\mathcal{H}$ is called the *nonstandard hull* of $\mathcal{H}$, written as $\hat{\mathcal{H}}$ (Henson and Moore [5]). Then the standardization of an internal operator A on $\mathcal{H}$ with finite norm is naturally defined on $\mathcal{H}$, In this paper, the standardization of A shall be called the *standard part* of A, written as $\hat{A}$. A prominent work of Moore [6] was focused on the case where $\mathcal{H}$ is hyperfinite-dimensional, and studied hyperfinite-dimensional extension of bounded operators on $\hat{\mathcal{H}}$. On the other hand, in the case where the norm of A is not finite, it is not straightforward to give an adequate definition of the standard part of A. Albeverio et al. [4] defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and A is an internal positive symmetric operator on $\mathcal{H}$.

In this paper, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal S-bonded self-adjoint operator A on $\mathcal{H}$, as well as a general consideration on $\hat{A}$ so defined, which suggests the adequacy of the definition.

2. Preliminaries

We work in a $\aleph_1$-saturated nonstandard universe [7]. Note that every nonstandard universe constructed by a bounded ultrapower is $\aleph_1$-saturated.

Let $(V, \|\cdot\|)$ be an internal normed linear space. Define the subspaces $\mu(V, \|\cdot\|)$ and $\mathrm{fin}(V, \|\cdot\|)$ of V by

$$\mu(V, \|\cdot\|) = \{\xi \in V \mid \|\xi\| \approx 0\}, \qquad \mathrm{fin}(V, \|\cdot\|) = \{\xi \in V \mid \|\xi\| < \infty\}. \tag{1}$$

We often abbreviate them as $\mu(V)$ and $\mathrm{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \mathrm{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $\|\!|\cdot|\!\|$ on $\hat{V}$ by $\|\!|\hat{\xi}|\!\| = {}^\circ\|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i\in\mathbf{N}}$, where $\xi_i \in \mathrm{fin}(V, \|\cdot\|)$, *approximately converges to* $\xi \in V$ in the norm $\|\cdot\|$ if

$$\forall\varepsilon \in \mathbf{R}^+\ \exists n \in \mathbf{N}\ \forall k \in \mathbf{N} \quad [k > n \ \Rightarrow\ \|\xi - \xi_i\| < \varepsilon]. \tag{2}$$

A sequence $\{\xi_i\}_{i\in\mathbf{N}}$ approximately converges to $\xi \in V$ if and only if $\{\hat{\xi}_i\}_{i\in\mathbf{N}}$ converges to $\hat{\xi} \in \hat{V}$. A sequence $\{\xi_i\}_{i\in\mathbf{N}}$, where $\xi_i \in \mathrm{fin}(V, \|\cdot\|)$, is *S-$\|\cdot\|$-Cauchy* if

$$\forall \varepsilon \in \mathbf{R}^+ \ \exists n \in \mathbf{N} \ \forall k, l \in \mathbf{N} \quad [k, l > n \ \Rightarrow \ \|\xi_k - \xi_l\| < \varepsilon]. \tag{3}$$

A sequence $\{\xi_i\}_{i\in\mathbf{N}}$ is S-$\|\cdot\|$-Cauchy if and only if the sequence $\{\hat{\xi}_i\}_{i\in\mathbf{N}}$ is Cauchy.

A subset $X \subset \mathrm{fin}(V, \|\cdot\|)$ is *S-$\|\cdot\|$-complete* if for any S-$\|\cdot\|$-Cauchy sequence $\{\xi_i\}_{i\in\mathbf{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to ξ in the norm $\|\cdot\|$. The subset X is S-$\|\cdot\|$-complete if and only if $\hat{X}$ is complete in $\hat{V}$, where $\hat{X} = \{\hat{\xi} | \xi \in X\}$.

The following results, called *the hull completeness theorem*, is a fundamental property of an internal normed space $(V, \|\cdot\|)$. See Hurd and Loeb [3] for detail.

Theorem 2.1. *The subspace* $\mathrm{fin}(V)$ *is S-complete in* $\|\cdot\|$.

Corollary 2.2. (The Hull Completeness Theorem) $\hat{V}$ *is a Banach space.*

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ an internal bounded linear operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, called the *standard part* of T, is defined by the relation $\hat{T}\hat{x} = \widehat{Tx}$ for any $x \in \mathrm{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [3] and Hurd and Loeb [2].

3. Several definitions of standard parts

We give several equivalent definitions of the standard part of an internal bounded self-adjoint operator which is not S-bounded.

The following lemma, which is a basic property for self-adjointness, is used to give the first definition of standard parts (see [8]).

Lemma 3.1. *Let A be a symmetric operator on a Hilbert space $\mathcal{H}$. Then, A is self-adjoint if and only if* $\mathrm{Rng}(A \pm i) = \mathcal{H}$.

Let $\mathcal{H}$ be an internal Hilbert space, and A an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}} = \mathrm{Ker}([(A+i)^{-1}]^\frown)^\perp$. Using the unitarity of $(A+i)(A-i)^{-1}$, we can easily check that $\mathrm{Ker}([(A-i)^{-1}]^\frown)^\perp = \hat{\mathcal{K}}$.

Proposition 3.2. *There exists the unique (possibly unbounded) self-adjoint operator S on $\hat{\mathcal{K}}$ satisfying*

$$(S+i)^{-1} = [(A+i)^{-1}]^\frown | \hat{\mathcal{K}}. \tag{4}$$

Proof. We see $\|(A+i)^{-1}\| < \infty$, and $[(A+i)^{-1}]^\frown$ is an bounded normal operator on $\hat{\mathcal{H}}$. The operator $T := [(A+i)^{-1}]^\frown | \hat{\mathcal{K}}$ is a bijection from $\hat{\mathcal{K}}$ to $[(A+i)^{-1}]^\frown \hat{\mathcal{K}}$. Hence the inverse T^{-1} from $[(A+i)^{-1}]^\frown \hat{\mathcal{K}}$ to $\hat{\mathcal{K}}$ is defined. Clearly the operator $S = T^{-1} - i$ satisfies the equation (4).

We will show that S is symmetric. Let $x_1, x_2 \in \mathrm{Dom}(S)$ $(= [(A+i)^{-1}]^\frown \hat{\mathcal{K}})$. Then, we can show that there exist $\xi_i \in x_i$ such that $A\xi_i \in Sx_i$ $(i = 1, 2)$ as follows. There are $y_i \in \hat{\mathcal{K}}$ and $\eta_i \in \mathcal{H}$ such that $(S+i)^{-1}y_i = [(A+i)^{-1}]\hat{\ } y_i = x_i$ and $\eta_i \in y_i$. Let $\xi_i = (A+i)^{-1}\eta_i$. Then $\xi_i \in x_i$ and $(A+i)\xi_i = \eta_i \in y_i = (S+i)x_i$. Hence $A\xi_i \in Sx_i$. Thus, $\langle x_1, Sx_2\rangle = {}^\circ\langle \xi_1, A\xi_2\rangle = {}^\circ\langle A\xi_1, \xi_2\rangle = \langle Sx_1, x_2\rangle$. Therefore, S is symmetric.

To prove the self-adjointness, it is sufficient to show $\mathrm{Rng}(S+i) = \mathrm{Rng}(S-i) = \hat{\mathcal{K}}$ by Lemma 3.1. Clearly $\mathrm{Rng}(S+i) = \mathrm{Rng}(T^{-1}) = \hat{\mathcal{K}}$. Let $x \in \mathrm{Dom}(S)$, $\xi \in x$ and $A\xi \in Sx$. Then we have

$$\left(\frac{A-i}{A+i}\right)\hat{\ } (S+i)x = \left(\frac{A-i}{A+i}(A+i)\xi\right)\hat{\ } = (S-i)x. \tag{5}$$

Thus, by the equation (4) with $\mathrm{Ker}([(A-i)^{-1}]\hat{\ })^{\perp} = \hat{\mathcal{K}}$, we have

$$(S-i)^{-1} = [(A-i)^{-1}]\hat{\ }|\hat{\mathcal{K}}. \tag{6}$$

Therefore, we can show $\mathrm{Rng}(S-i) = \hat{\mathcal{K}}$ in the similar way to the proof of $\mathrm{Rng}(S+i) = \hat{\mathcal{K}}$. The uniqueness of S is clear. *QED*

Definition 3.3. *Under the condition of Proposition 3.2, define the self-adjoint operator* $\mathrm{st}_1(A)$ *on* $\hat{\mathcal{K}}$ *by* $(\mathrm{st}_1(A)+i)^{-1} = [(A+i)^{-1}]\hat{\ }|\hat{\mathcal{K}}$.

The operator $\mathrm{st}_1(A)$ is called the *standard part* of A. We see that $\mathrm{st}_1(A) = \hat{A}$ when A is S-bounded.

Definition 3.4. *Let A be an internal bounded operator on $\mathcal{H}$, an internal Hilbert space. Define* $\mathrm{fin}(A) \subseteq \mathcal{H}$ *by*

$$\mathrm{fin}(A) = \{\xi \in \mathrm{fin}\mathcal{H} |\ A\xi \in \mathrm{fin}\mathcal{H}\}. \tag{7}$$

Definition 3.5. *Let A be an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}}$ be the closure of the subspace* $[\mathrm{fin}(A)]\hat{\ } = \{\hat{\xi} | \xi \in \mathrm{fin}(A)\}$ *of $\hat{\mathcal{H}}$. Define the self-adjoint operator* $\mathrm{st}_2(A)$ *on* $\hat{\mathcal{K}}$ *by*

$$e^{it\,\mathrm{st}_2(A)} = \widehat{e^{itA}}|\hat{\mathcal{K}}. \quad t \in \mathbf{R}. \tag{8}$$

We see that $\{\widehat{e^{itA}}|\hat{\mathcal{K}}\}_{t\in\mathbf{R}}$ is one-parameter unitary group, since $\hat{\mathcal{K}}$ is invariant under $\widehat{e^{itA}}$ for all $t \in \mathbf{R}$. We also see that it is strongly continuous as follows. Let $\xi \in \mathrm{fin}(A)$. Then, we have $\|({}^\star d/dt)e^{itA}\xi\| = \|ie^{itA}A\xi\| < \infty$, where ${}^\star d/dt$ is the internal differentiation. This implies that $\widehat{e^{itA}}\hat{\xi}$ is continuous with respect to $t \in \mathbf{R}$. Thus, $\widehat{e^{itA}}$ is strongly continuous on $\mathrm{fin}(A)^{\wedge\perp\perp}$. Hence by Stone's theorem, $\mathrm{st}_2(A)$ is uniquely defined.

If A is S-bounded, $\mathrm{st}_2(A)$ coincides with $\hat{A}$ defined in Section 2. This is seen from the following:

Proposition 3.6. *Let A be an internal S-bounded self-adjoint operator. Then,*

$$\widehat{e^{itA}} = e^{it\hat{A}}, \tag{9}$$

for all $t \in \mathbf{R}$.

Proof. For any infinitesimal $\epsilon \in {}^*\mathbf{R}_0^+$,

$$\epsilon^{-1}(e^{i\epsilon A} - I) \approx iA, \tag{10}$$

holds, because

$$\begin{aligned}\|\epsilon^{-1}(e^{i\epsilon A} - I) - iA\| &= \|\epsilon^{-1}\sum_{\nu=2}^{*\infty}(i\epsilon A)^\nu/\nu!\| \le \epsilon^{-1}\sum_{\nu=2}^{*\infty}(\epsilon\|A\|)^\nu/\nu! \\ &= \epsilon^{-1}(e^{\epsilon\|A\|} - 1) - \|A\| \approx 0.\end{aligned}$$

Thus, by the permanence principle,

$$\forall\delta \in \mathbf{R}_+, \exists\epsilon \in \mathbf{R}_+, |t| < \epsilon \Rightarrow \|t^{-1}(e^{itA} - I) - iA\| < \delta. \tag{11}$$

Hence, we have

$$\lim_{\epsilon\to 0}\|\epsilon^{-1}(\widehat{e^{i\epsilon A}} - \hat{I}) - i\hat{A}\| = 0. \tag{12}$$

Thus we have $(d/dt)\widehat{e^{itA}}|_{t=0} = i\hat{A}$, where d/dt is the usual differentiation. Because $(\widehat{e^{itA}})_{t\in\mathbf{R}}$ is one-parameter unitary group, it follows that $\widehat{e^{itA}} = e^{it\hat{A}}$. *QED*

Let $E(\cdot)$ be an internal projection-valued measure on ${}^*\mathbf{R}$, i.e., for each internal Borel set $\Omega \subseteq {}^*\mathbf{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that

(1) $E(\phi) = 0, \quad E({}^*\mathbf{R}) = I$

(2) If $\Omega = \bigcup_{n=1}^{*\infty}\Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = \text{s-lim}_{N\to{}^*\infty}\sum_{n=1}^N E(\Omega_n)$

(3) $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.

For $r \in {}^*\mathbf{R}$, let $\mathcal{H}_r = \text{Rng}(E(-r,r))$, the range of $E((-r,r))$. Let $D(E) = \bigcup_{r\in\mathbf{R}^+}\mathcal{H}_r \cap \text{fin}\mathcal{H}$. $D(E)$ is called the *standardization domain* of $E(\cdot)$. Clearly, $\widehat{D(E)}^{\perp\perp} = (\bigcup_{r\in\mathbf{R}^+}\hat{\mathcal{H}}_r)^{\perp\perp}$.

For $a \in \mathbf{R}$, define the orthogonal projection $\hat{E}_{\text{st}}(-\infty, a]$ by

$$\begin{aligned}\hat{E}_{\text{st}}(-\infty, a] &= \sup\{\hat{E}(-K, a+\epsilon]|\hat{D}(E)^{\perp\perp}|K, \epsilon \in \mathbf{R}^+\} &(13)\\ &= \text{s-lim}_{n\to\infty}\hat{E}(-n, a+\frac{1}{n}]|\hat{D}(E)^{\perp\perp}. &(14)\end{aligned}$$

Then we see

$$\begin{aligned}\text{s-lim}_{a\to-\infty}\hat{E}_{\text{st}}(-\infty, a] &= 0 &(15)\\ \text{s-lim}_{\epsilon\downarrow 0}\hat{E}_{\text{st}}(-\infty, a+\epsilon] &= \hat{E}_{\text{st}}(-\infty, a] &(16)\end{aligned}$$

$$a < b \Rightarrow \hat{E}_{\text{st}}(-\infty, a] \le \hat{E}_{\text{st}}(-\infty, b]. \tag{17}$$

Hence, $\hat{E}_{\text{st}}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbf{R}$.

Definition 3.7. *For any internal bounded self-adjoint operator A, define the self-adjoint operator $\text{st}_3(A)$ on $\hat{D}(E)^{\perp\perp}$ by*

$$\text{st}_3(A) = \int \lambda d\hat{E}_{\text{st}}(\lambda). \tag{18}$$

Proposition 3.8. *Let A be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of A. Then*

$$\hat{D}(E)^{\perp\perp} = \widehat{\mathrm{fin}(A)}^{\perp\perp}. \tag{19}$$

Proof. $\hat{D}(E)^{\perp\perp} \subseteq \mathrm{fin}(A)^{\frown\perp\perp}$ is clear. To prove $\hat{D}(E)^{\perp\perp} \supseteq \mathrm{fin}(A)^{\frown\perp\perp}$, it is sufficient to show that for any $\hat{x} \in \mathrm{fin}(A)^{\frown}$ there is a sequence $\hat{x}_n \in \hat{D}(E)$ $(n \in \mathbf{N})$ such that $\hat{x}_n \to \hat{x}$. Let $x_n = E(-n,n)x$ $(n \in {}^*\mathbf{N})$. Notice that $\|A(x - x_n)\| \geq n\|x - x_n\|$. Suppose $\|x - x_n\| > \epsilon$ for all $n \in \mathbf{N}$. By the permanence principle, there is $N \in {}^*\mathbf{N}_\infty$ such that $\|x - x_N\| > \epsilon$. Hence, $\|A(x - x_n)\| \geq N\|x - x_N\| > N\epsilon \sim \infty$. This contradicts $\|A(x - x_N)\| \leq \|Ax\| < \infty$. *QED*

Theorem 3.9. *Let A be an internal bounded self-adjoint operator. Then,*

$$\mathrm{st}_2(A) = \int \lambda d\hat{E}_{\mathrm{st}}(\lambda), \tag{20}$$

and hence $\mathrm{st}_2(A) = \mathrm{st}_3(A)$.

Proof. It is sufficient to show

$$\langle \hat{x}, \exp(it\,\mathrm{st}_2(A))\hat{x}\rangle = \int e^{it\lambda}\langle \hat{x}, d\hat{E}_{\mathrm{st}}(\lambda)\hat{x}\rangle \tag{21}$$

for all $\hat{x} \in \mathrm{fin}(A)^{\frown\perp\perp}$. Define the internal Borel measure μ by $\mu(d\lambda) = \langle x, E(d\lambda)x\rangle$. Let $L\mu$ denote the Loeb measure of μ, and $L'\mu$ the Borel measure on $\mathbf{R}$ defined by $L'\mu(\Omega) = L\mu(\mathrm{st}^{-1}[\Omega])$. We can check that $L'\mu$ is well-defined (i.e., $\mathrm{st}^{-1}[\Omega]$ is $L\mu$-measurable for any Borel set $\Omega \subseteq \mathbf{R}$). We also see that $L\mu$ is supported by $\mathrm{fin}\,{}^*\mathbf{R}$, since $L\mu({}^*\mathbf{R} \setminus \mathrm{fin}\,{}^*\mathbf{R}) \leq {}^\circ\langle x, E({}^*\mathbf{R} \setminus (-n,n))x\rangle = {}^\circ\|(1 - E(-n,n))x\|^2 \leq (1/n^2)\,{}^\circ\|Ax\|^2$ for all $n \in \mathbf{N}$. Therefore

$$\begin{aligned}
\langle \hat{x}, \exp(it\,\mathrm{st}_2(A))\hat{x}\rangle &= \langle \hat{x}, \widehat{e^{itA}x}\rangle \\
&= {}^\circ\langle x, e^{itA}x\rangle \\
&= {}^{\circ *}\int_{{}^*\mathbf{R}} e^{it\lambda} d\mu(\lambda) \\
&= \int_{{}^*\mathbf{R}} {}^\circ e^{it\lambda} dL\mu(\lambda) \\
&= \int_{\mathbf{R}} e^{it\lambda} dL'\mu(\lambda).
\end{aligned}$$

On the other hand, for $a, b \in \mathbf{R}$ with $a < b$,

$$\begin{aligned}
L'\mu(a,b) &= L\mu\Big(\bigcup_{\epsilon \in \mathbf{R}^+} (a+\epsilon, b-\epsilon)\Big) \\
&= \lim_{\epsilon \downarrow 0} {}^\circ\langle x, E(a+\epsilon, b-\epsilon)x\rangle \\
&= \lim_{\epsilon \downarrow 0} \langle \hat{x}, \hat{E}(a+\epsilon, b-\epsilon)\hat{x}\rangle \\
&= \langle \hat{x}, \operatorname{s-lim}_{\epsilon \downarrow 0} \hat{E}(a+\epsilon, b-\epsilon)\hat{x}\rangle \\
&= \langle \hat{x}, \hat{E}_{\mathrm{st}}(a,b)\hat{x}\rangle.
\end{aligned}$$

Hence, $L'\mu(\Omega) = \langle \hat{x}, \hat{E}_{\mathrm{st}}(\Omega)\hat{x}\rangle$ for any Borel set $\Omega \subseteq \mathbf{R}$. *QED*

Let $C \in \mathbf{R}$ be a positive constant, and h be an internal Borel function from ${}^*\mathbf{R}$ to ${}^*\mathbf{C}$ satisfying the following properties:

$$h(x) \approx h(y) \quad \text{iff} \quad x \approx y \quad \text{for all } x, y \text{ with } |x|, |y| < \infty,$$
$$|h(x)| < C \quad \text{for all } x \in {}^*\mathbf{R}.$$

Define the function $\hat{h} : \mathbf{R} \to \mathbf{C}$ by

$$\hat{h}(x) = {}^\circ h(x),$$

for $x \in \mathbf{R}$. We see that $\hat{h}$ is injective and continuous. Let A be an internal bounded self-adjoint operator. Notice that $h(A)$ is an S-bounded internal normal operator.

Theorem 3.10. *There exists the unique self-adjoint operator B on* $\mathrm{fin}(A)^{\wedge\perp\perp}$ *such that*

$$\hat{h}(B) = \widehat{h(A)}|\mathrm{fin}(A)^{\wedge\perp\perp}. \tag{22}$$

Moreover, B equals to $\mathrm{st}_3(A)$.

Proof. By the argument similar to the proof of Theorem 3.9, we can show

$$\begin{aligned} \langle \hat{x}, \widehat{h(A)}\hat{x} \rangle &= \int_{\mathbf{R}} \hat{h}(\lambda) dL'\mu(\lambda) \\ &= \int_{\mathbf{R}} \hat{h}(\lambda) \langle \hat{x}, d\hat{E}_{\mathrm{st}}(\lambda)\hat{x} \rangle \end{aligned}$$

for any $\hat{x} \in \mathrm{fin}(A)^{\wedge\perp\perp}$. Thus,

$$\widehat{h(A)}|\mathrm{fin}(A)^{\wedge\perp\perp} = \int_{\mathbf{R}} \hat{h}(\lambda) d\hat{E}_{\mathrm{st}}(\lambda).$$

Because $\hat{h}$ is injective, the unique self-adjoint operator B satisfying (22) is $\mathrm{st}_3(A) = \int_{\mathbf{R}} \lambda d\hat{E}_{\mathrm{st}}(\lambda)$. *QED*

Corollary 3.11. *Definition 3.3, 3.5 and 3.7 are equivalent, that is,* $\mathrm{st}_1(A) = \mathrm{st}_2(A) = \mathrm{st}_3(A)$.

Proof. Let $h(x) = 1/(x + i)$. *QED*

In section 2, $\hat{A}$ is defined only when A is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where A is an internal bounded self-adjoint operator which is not S-bounded; $\hat{A} := \mathrm{st}_1(A) = \mathrm{st}_2(A) = \mathrm{st}_3(A)$.

Definition 3.12. *Let A be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let D be an (external) subspace of* $\mathrm{fin}\mathcal{H}$. *A is* standardizable *on D if $D \subset$* $\mathrm{fin}(A)$ *and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $\hat{A}_D$ with domain $\hat{D} = \{\hat{x} | x \in D\}$, called the* standard part *of A on D, by*

$$\hat{A}_D \hat{x} = \widehat{Ax}, \quad x \in D. \tag{23}$$

Clearly, A is standardizable on D if and only if $D \subset \mathrm{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

Lemma 3.13. *An internal bounded operator A is standardizable on* $\mathrm{fin}(A^*A)$.

Proof. First, we prove $\mathrm{fin}(A^*A) \subset \mathrm{fin}(A)$ as follows. Suppose that $\xi \in \mathrm{fin}(A)$. Let $E(\cdot)$ be the internal spectral projection-valued mesure of the self-adjoint operator A^*A. Then, $\|A\xi\|^2 = \langle \xi, A^*A\xi\rangle = \langle \xi, E[0,1]A^*A\xi\rangle + \langle \xi, (I - E[0,1])A^*A\xi\rangle \leq \langle \xi, E[0,1]A^*A\xi\rangle + \langle \xi, (I - E[0,1])(A^*A)^2\xi\rangle \leq \langle \xi, E[0,1]A^*A\xi\rangle + \|A^*A\xi\|^2 < \infty$. Thus, $\xi \in \mathrm{fin}(A)$. Second, suppose $x \approx 0$ and $\|A^*Ax\| < \infty$. Then, $\|Ax\|^2 = \langle x, A^*Ax\rangle \leq \|x\|\|A^*Ax\| \approx 0$. *QED*

Corollary 3.14. *If $D \subseteq \mathrm{fin}\mathcal{H}$ is invariant under A and A^*, A is standardizable on D.*

The operator B in the above proof is called a *hyperfinite extension* of A [6].
We use the following lemma in the proof of Theorem 3.16.

Lemma 3.15. *Let A be a symmetric operator with domain $D \subset \mathcal{H}$, a Hilbert space. Let $D_1 \subset D$ be a dense linear sebset of $\mathcal{H}$ and suppose that $A|D_1$ is essentially self-adjoint. Then, A is essentially self-adjoint and $\overline{A} = \overline{A|D_1}$.*

Theorem 3.16. *Let A be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of A. Then,*

$$\hat{A} = \overline{\hat{A}_{D(E)}} = \overline{\hat{A}_{\mathrm{fin}(A^2)}} \tag{24}$$

Proof. We can show that $\hat{A}_{D(E)}$ is essentially self-adjoint e.g. by Nelson's analytic vector theorem. Hence, it has one and only one self-adjoint extension, its closure. Thus, it is sufficient to show that $\hat{A}$ is an extension of $\hat{A}_{D(E)}$. If $E(-r,r)\xi = \xi$ $(r \in \mathbf{R}^+, \xi \in \mathcal{H})$, then $\hat{E}_{\mathrm{st}}(-s,s)\hat{\xi} = \hat{\xi}$ $(s \in \mathbf{R}^+, s > r)$. Thus, $\hat{A}_D\hat{\xi} = \widehat{A\xi} = \overline{[{}^*\int_{-s}^{s} \lambda dE(\lambda)]^\frown \hat{\xi}} = \int_{-s}^{s} \lambda d\hat{E}_{\mathrm{st}}(\lambda)\hat{\xi} = \int \lambda d\hat{E}_{\mathrm{st}}(\lambda)\hat{\xi} = \mathrm{st}_3(A) = \hat{A}\hat{\xi}$. Therefore $\hat{A} = \overline{\hat{A}_{D(E)}}$. $\overline{\hat{A}_{D(E)}} = \overline{\hat{A}_{\mathrm{fin}(A^2)}}$ follows from $D(E) \subseteq \mathrm{fin}(A^2)$ and Lemma 3.15. *QED*

4. The domain of $\hat{A}$

Definition 4.1. *For an internal bounded self-adjoint operator A on $\mathcal{H}$, define $D(A)$ by*

$$D(A) = \{\xi \in \mathrm{fin}\mathcal{H} \mid \textit{for all } t \in \mathbf{R}_0^+,\ e^{-t|A|}A\xi \approx A\xi \in \mathrm{fin}\mathcal{H}\}.$$

Clearly, $D(A)$ is a subspace of $\mathcal{H}$.

Proposition 4.2. *An internal bounded self-adjoint operator A is standardizable on $D(A)$.*

Proof. Let $\xi \in D(A)$ and $\|\xi\| \approx 0$. We can easily check $\|e^{-t|A|}A\| < \infty$ for all $t > 0$, $t \not\approx 0$. Hence, ${}^\circ\|A\xi\| \leq {}^\circ\|e^{-t|A|}A\xi\| + {}^\circ\|(1-e^{-t|A|})A\xi\|$. By the S-boundedness of $e^{-t|A|}A$, the first term equals 0, and by the definition of $D(A)$, the second term equals 0. Thus we have ${}^\circ\|A\xi\| = 0$. *QED*

The following lemmas are easily shown.

Lemma 4.3. *Let $f : {}^*\mathbf{N} \to {}^*\mathbf{R}^+$ be internal and increasing. If $f(M) < \infty$ for some $M \sim \infty$, then*

$$\lim_{n\to\infty} {}^\circ f(n) < \infty.$$

Lemma 4.4. *Under the same condition to Lemma 4.3, there is $K \sim \infty$ such that for all $L \sim \infty$,*

$$f(K) \approx f(L) \quad \textit{if} \quad L \leq K.$$

Proposition 4.5. *Let $\xi \in \mathrm{fin}(\mathcal{H})$. For sufficiently large $t \approx 0$,*

$$e^{-t|A|}\xi \in D(A). \tag{25}$$

Proof. Applying Lemma 4.4 to $f(n) = \|e^{-|A|/n}A\xi\|$, we find that for sufficiently small $K \sim \infty$ and $L \sim \infty$, $e^{-|A|/K}A\xi \approx e^{-|A|/L}A\xi$. Thus, for sufficiently large $s \approx 0$ and $t \approx 0$, $e^{-s|A|}A\xi \approx e^{-t|A|}A\xi$. Hence for all $x \approx 0$, $x > 0$,

$$e^{-x|A|}Ae^{-t|A|}\xi = e^{-(x+t)|A|}A\xi \approx Ae^{-t|A|}\xi.$$

Therefore, $e^{-t|A|}\xi \in D(A)$. *QED*

Theorem 4.6. *Let $E(\cdot)$ be the spectral resolution of A and $E_K = E(-K, K)$ for $K \in {}^*\mathbf{R}^+$. For any $\xi \in \mathrm{fin}(A)$,*

$$\xi \in D(A) \quad \textit{iff} \quad A\xi \approx E_K A\xi \quad \textit{for all } K \sim \infty. \tag{26}$$

Remark. The right-hand condition is equivalent to

$$\lim_{\substack{K\to\infty \\ K\in\mathbf{R}}} {}^\circ\|(I - E_K)A\xi\| = 0. \tag{27}$$

Proof. Suppose that $\xi \in \mathrm{fin}(A)$ and $A(I - E_K)\xi \approx 0$ for all $K \sim \infty$. For any $t \approx 0$, there exists a $K \sim \infty$ such that $tK \approx 0$. Thus,

$$\begin{aligned}
\|e^{-t|A|}A\xi - A\xi\|^2 &\approx \|e^{-t|A|}E_K A\xi - E_K A\xi\|^2 \\
&= \|\int_{-K}^{K} e^{-t|\lambda|}\lambda - \lambda dE(\lambda)\xi\|^2 \\
&= \int_{-K}^{K} |(e^{-t|\lambda|} - 1)\lambda|^2 \|dE(\lambda)\xi\|^2 \\
&\leq \sup_{|\lambda|<K} |e^{-t|\lambda|} - 1|^2 \int_{-K}^{K} \lambda^2 \|dE(\lambda)\xi\|^2 \\
&= \sup_{|\lambda|<K} |e^{-t|\lambda|} - 1|^2 \|E_K A\xi\|^2 \\
&\approx 0.
\end{aligned}$$

Hence $\xi \in D(A)$.

Conversely, suppose $\xi \in D(A)$ ($\subset \mathrm{fin}(A)$). Applying Lemma 4.4 to $f(n) = \|E_n A\xi\|$, we see that for sufficiently small $K \sim \infty$ and $L \sim \infty$ ($L \leq K$),

$$\|E_L A\xi\| \approx \|E_K A\xi\|.$$

Thus, $(E_K - E_L)A\xi \approx 0$, since $\|E_L A\xi - E_K A\xi\|^2 = \|E_K A\xi\|^2 - \|E_L A\xi\|^2 \approx 0$. Let $t \in \mathbf{R}_0^+$ satisfy $tK \sim \infty$ so that

$$\begin{aligned} & \|E_K A\xi - e^{-t|A|}A\xi\| \\ = \; & \|\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi - \int_{(-{}^*\infty,-K)\cup(K,{}^*\infty)} e^{-t|\lambda|}\lambda dE(\lambda)\xi\| \\ \leq \; & \|\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi\| + e^{-tK}\|A\xi\| \\ \approx \; & \|\int_{-K}^{K} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi\|. \end{aligned}$$

Let $L \sim \infty$ satisfy $tL \approx 0$, so that the above

$$\begin{aligned} \leq \; & \|\int_{-L}^{L} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi\| + \|\int_{(-K,K)\setminus(-L,L)} \lambda(1 - e^{-t|\lambda|})dE(\lambda)\xi\| \\ \leq \; & |1 - e^{-tL}|\|A\xi\| + \|(E_K - E_L)A\xi\| \\ \approx \; & 0. \end{aligned}$$

Thus, for sufficiently small $K \sim \infty$ and for any $t \approx 0$ such that $tK \sim \infty$,

$$E_K A\xi \approx e^{-t|A|}A\xi \approx A\xi.$$

Since $\|A\xi - E_K A\xi\| \geq \|A\xi - E_{K'}A\xi\| > 0$ if $K < K'$, we have $E_{K'}A\xi \approx A\xi$ holds for any $K' \sim \infty$. *QED*

Proposition 4.7. *Let $\xi \in \mathrm{fin}(A)$. Then, $E_K\xi \in D(A)$ for sufficiently small $K \sim \infty$.*

Proof. Applying Lemma 4.4 to $f(n) = \|E_n A\xi\|$, we find that for sufficiently small $K, L \sim \infty$, $E_K A\xi \approx E_L A\xi$. Thus, if $L \sim \infty$, $L \leq K$, then $\|(1 - E_L)E_K A\xi\| = \|(E_K - E_L)A\xi\| \approx 0$. If $L > K$, clearly $(1 - E_L)E_K A\xi = 0$. Hence for all $L \sim \infty$, $E_K A\xi \approx E_L E_K A\xi$. Thus $E_K\xi \in D(A)$ by Theorem 4.6. *QED*

Corollary 4.8. $[\mathrm{fin}(A)]^\frown = [D(A)]^\frown$, *i.e., if $\xi \in \mathrm{fin}(A)$, then there is $\eta \in D(A)$ such that $\eta \approx \xi$.*

Example We have seen that the following relations hold:

$\mathrm{fin}(A^2) \subset D(A) \subset \mathrm{fin}(A) \subset \mathrm{fin}\mathcal{H}$,

$[\mathrm{fin}(A^2)]^\frown \subset [D(A)]^\frown = [\mathrm{fin}(A)]^\frown \subset \hat{\mathcal{H}}$,

$$[\mathrm{fin}(A^2)]^{\frown\perp\perp} = [D(A)]^{\frown\perp\perp} = [\mathrm{fin}(A)]^{\frown\perp\perp} \subset \hat{\mathcal{H}}.$$

An example of A such that $\mathrm{fin}(A) \setminus D(A) \neq \emptyset$ is given as follows. Let ν be an infinite hypernatural number, and $\mathcal{H} = {}^*\mathbf{C}^\nu$, ν-dimensional internal Hilbert space. Define the internal self-adjoint operator A on $\mathcal{H}$ by $A(x_1, x_2, ..., x_\nu) = (x_1, 2x_2, ..., \nu x_\nu)$. Let $\xi = (0, 0, ..., 0, \nu^{-1})$. Then we see $\xi \in \mathrm{fin}(A) \setminus D(A)$ from Theorem 4.6.

We also find $D(A) \setminus \mathrm{fin}(A^2) \neq \emptyset$; let $\eta = (1^{-2}, 2^{-2}, ..., \nu^{-2})$, then we easily see $\eta \in D(A) \setminus \mathrm{fin}(A^2)$. Moreover we find $\hat{\eta} \in [D(A)]^\frown \setminus [\mathrm{fin}(A^2)]^\frown$. In fact, if $\eta' \approx \eta$, then ${}^\circ\|A^2\eta'\| \geq \lim_{\substack{n\to\infty \\ n\in\mathbf{N}}} {}^\circ\|A^2 E_n \eta'\| = \lim_{\substack{n\to\infty \\ n\in\mathbf{N}}} {}^\circ\|A^2 E_n \eta\| = \lim_{\substack{n\to\infty \\ n\in\mathbf{N}}} \sqrt{n} = \infty$. Thus, we have $\hat{\eta} \notin [\mathrm{fin}(A^2)]^\frown$ by Theorem 4.6.

Theorem 4.9. *Let $\xi \in \mathrm{fin}(A)$, then*

$$\xi \in D(A) \quad \textit{iff} \quad \lim_{\substack{t\downarrow 0 \\ t\not\approx 0}} \left(\frac{e^{-t|A|} - 1}{t} \xi \right)^\frown = -\widehat{|A|\xi}. \tag{28}$$

Proof. Suppose that the right-hand side does not hold. In other words, suppose that

$$\exists \varepsilon \in \mathbf{R}^+ \; \forall n \in \mathbf{N} \; \exists t \in {}^*\mathbf{R},\; 0 < t < \frac{1}{n} \;\wedge\; \left\| \left(\frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \tag{29}$$

By permanence,

$$\exists \varepsilon \in \mathbf{R}^+ \; \exists N \in {}^*\mathbf{N}_\infty \; \exists t \in {}^*\mathbf{R},\; 0 < t < \frac{1}{n} \;\wedge\; \left\| \left(\frac{e^{-t|A|} - 1}{t} + |A| \right) \xi \right\| > \varepsilon. \tag{30}$$

That is, there is positive infinitesimal t such that $t^{-1}(e^{-t|A|} - 1)\xi \not\approx -|A|\xi$.

Thus, for some $\eta \in \mathrm{fin}(\mathcal{H})$,

$$\Re \left\langle \eta,\; \frac{e^{-t|A|} - 1}{t} \xi \right\rangle \not\approx \Re\langle \eta, -|A|\xi \rangle.$$

Let $f(t) = \Re\langle \eta, e^{-t|A|}\xi \rangle$. By the mean value theorem, for some $s \in {}^*\mathbf{R}$ with $0 < s < t$,

$$f'(s) = \frac{f(t) - f(0)}{t} = \Re \left\langle \eta,\; \frac{e^{-t|A|} - 1}{t} \xi \right\rangle \not\approx \Re\langle \eta, -|A|\xi \rangle.$$

Therefore, by the difinition of $D(A)$, we have $\xi \in \mathrm{fin}(A) \setminus D(A)$.

Conversely, suppose $\xi \in \mathrm{fin}(A) \setminus D(A)$. Then, there is positive infinitesimal t_0 satisfying $e^{-t_0|A|}|A|\xi \not\approx |A|\xi$. Let $\eta = (|A| - e^{t_0|A|}|A|)\xi$ ($\in \mathrm{fin}(\mathcal{H})$). Then this is equivalent to

$$\langle \eta, e^{-t_0|A|}|A|\xi \rangle \not\approx \langle \eta, |A|\xi \rangle. \tag{31}$$

Let $f(x) = \langle \eta, e^{-x|A|}\xi \rangle$ ($x \in {}^*\mathbf{R}^+$). We see that f' is increasing and $-\infty < f' < 0$, and hence f is decreasing and $0 < f < \infty$. The relation (31) is equivalent to

$$f'(t_0) \overset{>}{\not\approx} f'(0), \tag{32}$$

We have $f(x) \geq f'(t_0)(x-t_0)+f(t_0)$. Thus we have

$$0 > \frac{f(x)-f(0)}{x} \geq \frac{f'(t_0)(x-t_0)+f(t_0)-f(0)}{x}. \tag{33}$$

Let $F(x) = [f'(t_0)(x-t_0)+f(t_0)-f(0)]/x$, then for $c \in {}^*\mathbf{R}^+$,

$$F(ct_0) = f'(t_0)\left(1-\frac{1}{c}\right)+\frac{1}{c}\frac{f(t_0)-f(0)}{t_0}. \tag{34}$$

By the mean value theorem and $-\infty < f'(x) < 0$, we have $|(f(x)-f(0))/x| < \infty$. Hence $F(ct_0) \approx f'(t_0)$ for all $c \sim \infty$. Thus, by (32) and (33),

$$0 > \frac{f(ct_0)-f(0)}{ct_0} \geq F(ct_0) \underset{\not\approx}{>} f'(0), \tag{35}$$

for all $c \sim \infty$. Thus there is $\varepsilon \in \mathbf{R}^+$ such that for sufficiently large $x \approx 0$, $\frac{f(x)-f(0)}{x} - f'(0) > \varepsilon$. By the permanence principle, for sufficiently small $x \in \mathbf{R}^+$, $\frac{f(x)-f(0)}{x} - f'(0) > \varepsilon$. We can check the relations

$$\left\langle \eta, \left(\frac{e^{-x|A|}-1}{x}\right)\xi \right\rangle = \frac{f(x)-f(0)}{x}, \quad \langle \eta, |A|\xi\rangle = -f'(0), \qquad \frac{e^{-x|A|}-1}{x} > -|A|,$$

for $x > 0$. Therefore, using the increasingness of $(e^{-x\|A\|-1})/x$, x, we have

$$\lim_{\substack{x\downarrow 0\\ x\not\approx 0}} {}^\circ\left\langle \eta, \frac{e^{-x|A|}-1}{x}\xi \right\rangle \neq \langle \eta, -|A|\xi\rangle.$$

QED

Theorem 4.10. *Let A be an internal bounded self-adjoint operator. Then, $\hat{A} = \hat{A}_{D(A)}$.*

Proof. By Theorem 3.16 and Lemma 3.15, it suffices to show that $\hat{A}_{D(A)}$ is a closed extension of $\hat{A}_{\mathrm{fin}(A^2)}$. If $\xi \in \mathrm{fin}(A^2)$, for any $K \sim \infty$, $\|(1-E_K)A\xi\| \leq \frac{1}{K}\|(1-E_K)A^2\xi\| \leq \frac{1}{K}\|A^2\xi\| \approx 0$. Hence $\xi \in D(A)$, and hence $\hat{A}_{D(A)}$ is an extension of $\hat{A}_{\mathrm{fin}(A^2)}$.

To prove that $\hat{A}_{D(A)}$ is closed, it suffices to show that $\hat{D}(A) = [D(A)]^\frown$ is complete in the norm $\|\!|\cdot|\!\|_A$ defined by $\|\!|\hat{\xi}|\!\|_A = \|\hat{\xi}\| + \|\hat{A}\hat{\xi}\|$. Define the internal norm $\|\cdot\|_A$ on $\mathcal{H}$ by $\|\xi\|_A = \|\xi\| + \|A\xi\|$. We can check $\|\!|\hat{\xi}|\!\|_A = {}^\circ\|\xi\|_A$ for $\xi \in D(A)$.

By Theorem 2.1, $\mathrm{fin}(A)$ is S-$\|\cdot\|_A$-complete. Hence, if the sequence $\{\xi_i\}_{i\in\mathbf{N}} \subset D(A)$ ($\subset \mathrm{fin}(A)$) is S-$\|\cdot\|_A$-Cauchy, then there is $\xi \in \mathrm{fin}(A)$ such that $\{\xi_i\}$ approximately converges to ξ in the norm $\|\cdot\|_A$. This ξ is shown to be in $D(A)$ as follows. Regarding Theorem 4.6, and $\xi_i \in D(A)$ ($i < \infty$), this relation leads to ${}^\circ\|(I-E_K)A\xi_\nu\| = \lim_{i\to\infty} {}^\circ\|(I-E_K)A\xi_i\| = 0$, for any $K \sim \infty$. Therefore, from Theorem 4.6, we have $\xi \in D(A)$ and hence any Cauchy sequence in $\widehat{D(A)}$ converges in $\widehat{D(A)}$ in the norm $\|\!|\cdot|\!\|_A$. *QED*

Theorem 4.11. *The domain $D(A)$ is maximal. That is, if $D(A) \subset S \subset \mathrm{fin}(\mathcal{H})$ and A is standardizable on S, then $S = D(A)$.*

Proof. Suppose that $D(A) \subset S \subset \mathrm{fin}(\mathcal{H})$ and that A is standardizable on S. Let $\eta \in S$. By Corollary 4.8 and $\eta \in \mathrm{fin}(A)$, there is $\xi \in D(A)$ such that $\xi \approx \eta$. By the definition of $D(A)$ and the standardizability on S, for all positive infinitesimal t, $e^{-t|A|}A\eta \approx e^{-t|A|}A\xi \approx A\xi \approx A\eta$, since $\|e^{-t|A|}\| \leq 1$. Thus, $\eta \in D(A)$. *QED*

Proposition 4.12. *Let A be an internal positive operator on $\mathcal{H}$. Then, for any $\eta \in \mathrm{fin}(A^{\frac{1}{2}})$,*

$$\inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle = \inf_{\alpha \sim \infty} {}^\circ\langle \eta, E_\alpha A\eta\rangle. \tag{36}$$

Proof. Suppose $\eta \approx \xi$. If $\alpha < \infty$, $\langle \eta, E_\alpha A\eta\rangle \approx \langle \xi, E_\alpha A\xi\rangle \leq \langle \xi, A\xi\rangle$, that is,

$$\forall \varepsilon \in \mathbf{R}^+,\ \forall \alpha < \infty, \quad \langle \eta, E_\alpha A\eta\rangle \leq \langle \xi, A\xi\rangle + \varepsilon,$$

Thus, by the permanence principle,

$$\forall \varepsilon \in \mathbf{R}^+,\ \exists K \sim \infty,\ \forall \alpha \leq K, \quad \langle \eta, E_\alpha A\eta\rangle \leq \langle \xi, A\xi\rangle + \varepsilon.$$

By saturation,

$$\exists K \sim \infty,\ \forall \varepsilon \in \mathbf{R}^+,\ \forall \alpha \leq K, \quad \langle \eta, E\alpha A\eta\rangle \leq \langle \xi, A\xi\rangle + \varepsilon.$$

Hence we have

$$\exists K \sim \infty, \quad {}^\circ\langle \eta, E_K A\eta\rangle \leq {}^\circ\langle \xi, A\xi\rangle.$$

It follows that $\inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle \geq \inf_{\alpha \sim \infty} {}^\circ\langle \eta, E_\alpha A\eta\rangle$.

On the other hand, we see that for all $\alpha \sim \infty$, $\|\eta - E_\alpha\eta\|^2 \leq \alpha^{-1}\|A^{\frac{1}{2}}(\eta - E_\alpha\eta)\|^2 \leq \alpha^{-1}\|A^{\frac{1}{2}}\eta\|^2 \approx 0$. Hence,

$$\forall \alpha \sim \infty, \quad \inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle \leq {}^\circ\langle E_\alpha\eta, AE_\alpha\eta\rangle = {}^\circ\langle \eta, E_\alpha A\eta\rangle.$$

Thus it follows that $\inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle \leq \inf_{\alpha \sim \infty} {}^\circ\langle \eta, E_\alpha A\eta\rangle$. *QED*

Proposition 4.13. *Let A be an internal positive operator and $\eta \in \mathrm{fin}(A)$. Then,*

$$\inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle = \langle \hat{\eta}, \hat{A}\hat{\eta}\rangle. \tag{37}$$

Proof. From Proposition 4.12, we see $\inf_{\xi \approx \eta} {}^\circ\langle \xi, A\xi\rangle = \inf_{\alpha \sim \infty} {}^\circ\langle \eta, E_\alpha A\eta\rangle$. By Theorem 4.10 and Proposition 4.7, for sufficiently small $\alpha \sim \infty$, ${}^\circ\langle \eta, E_\alpha A\eta\rangle = {}^\circ\langle E_\alpha\eta, AE_\alpha\eta\rangle = \langle \widehat{E_\alpha\eta}, \hat{A}\widehat{E_\alpha\eta}\rangle = \langle \hat{\eta}, \hat{A}\hat{\eta}\rangle$. *QED*

Definition 4.14. *Let A be a internal bounded positive operator, and $D \subset \mathrm{fin}(A^{\frac{1}{2}})$. The sespuilinear form $\langle \cdot, A\cdot\rangle$ is* standardizable *on D if $\langle \xi_1, A\eta_1\rangle \approx \langle \xi_2, A\eta_2\rangle$ for all $\xi_1, \xi_2, \eta_1, \eta_2 \in D$ with $\xi_1 \approx \xi_2$ and $\eta_1 \approx \eta_2$.*

Proposition 4.15. *Let D be a subspace of* $\text{fin}(\mathcal{H})$ *and* $A \geq 0$. *Then,* $\langle \cdot, A\cdot \rangle$ *is standardizable on D if and only if $A^{\frac{1}{2}}$ is standardizable on D.*

Proof. Suppose that $A^{\frac{1}{2}}$ is standardizable on D. Then $A^{\frac{1}{2}}\xi \approx A^{\frac{1}{2}}\eta$ for any $\xi, \eta \in D$ with $\xi \approx \eta$. Thus, $\langle \xi, A\xi \rangle = \|A^{\frac{1}{2}}\xi\|^2 \approx \|A^{\frac{1}{2}}\eta\|^2 = \langle \eta, A\eta \rangle$. Conversely, suppose that $\langle \cdot, A\cdot \rangle$ is standardizable on D. Then for any $\xi, \eta \in D$ with $\xi \approx \eta$, $\|A^{\frac{1}{2}}\xi - A^{\frac{1}{2}}\eta\|^2 = \|A^{\frac{1}{2}}(\xi - \eta)\|^2 = \langle \xi - \eta, A(\xi - \eta) \rangle \approx 0$. *QED*

Corollary 4.16. *The set $D(A^{\frac{1}{2}})$ is a maximal domain of $\langle \cdot, A\cdot \rangle$, and $^\circ\langle \xi, A\eta \rangle = \langle \widehat{A^{\frac{1}{2}}}\hat{\xi}, \widehat{A^{\frac{1}{2}}}\hat{\eta} \rangle$ for any $\xi, \eta \in D(A^{\frac{1}{2}})$.*

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