Molecular Gas Dynamics

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Supplementary Notes and Errata

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  [2006] → [2007]
  59, (to be published) → **60**, 147–163
- [136] Liu, T.-P. and S.-H. Yu [2006a]:
  [2006a] → [2006]
- [137] Liu, T.-P. and S.-H. Yu [2006b]:
  [2006b] → [2007]
  59, (to be published) → **60**, 295–356

The corresponding corrections in the text

- p. 166, the first line in the third paragraph:
  [2004b, 2006a] → [2004b, 2006]
- p. 166, the last line in the third paragraph:
  [2006b] → [2007]
- p. 183, the second line:
  [2006] → [2007]

Errata

- p. 9, the 7th line:
  specular condition $\rightarrow$ specular reflection
- p. 27, the 3rd line of Footnote 26:
  Eq. (1.99) $\rightarrow$ a linear combination of Eqs. (1.99) and (1.101)
- p. 27, the 6th line of Footnote 26:
  except for a common constant factor $\rightarrow$ except for a common constant factor and additive functions (say, f_a in $\hat{H}$ and f_{bi} in $\hat{H}_i$ in their second order) satisfying $\mathrm{Sh}\partial f_a/\partial\hat{t} + \partial f_{bi}/\partial x_i = 0$
- p. 81, the 4th line in Footnote 7:
  u_{iGm} $\rightarrow$ $u_{iGm} - u_{jGm}n_j n_i$
  or ϕ_{eGm} $\rightarrow$ of ϕ_{eGm}
- p. 83, the first line in Footnote 14:
  u_{iGm} $\rightarrow$ $u_{iGm} - u_{jGm}n_j n_i$
- p. 504, the first line in Footnote 24:
  damin $\rightarrow$ domain
- p. 505, Eq. (A.60):
  $$\left|\frac{1}{\sin\theta_c}\frac{\partial\sin^2\theta_e}{\partial\theta_c}\right| \rightarrow \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}\sin^2\theta_e}{\mathrm{d}\theta_c}\right|$$
  $$\left|\frac{1}{\sin\theta_c}\frac{\partial b^2}{\partial\theta_c}\right| \rightarrow \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}b^2}{\mathrm{d}\theta_c}\right|$$
- p. 506, the 13th line [The line next to Eq. (M-A.63)]:
  with respect to θ_c $\rightarrow$ with respect to θ_α
- p. 639, the 3rd line in Ref. [262]:
  gs $\rightarrow$ gas

Supplementary Notes

In the present supplementary notes, the letter M is attached to the labels of sections, equations, etc. in the book "Molecular Gas Dynamics" to avoid confusion.

1 Chapter M-1

1.1 Supplement to Foonote M-26 in Chapter M-1

Footnote M-26 is supplemented with more explicit mathematical expressions for the process given there. Take the non-dimensional form of the equation for the H function, i.e., Eq. (M-1.72):

$$\mathrm{Sh}\frac{\partial \hat{H}}{\partial \hat{t}} + \frac{\partial \hat{H}_i}{\partial x_i} = \frac{1}{k}\hat{G}, \tag{1}$$

where

$$\left.\begin{aligned} &\hat{H}(x_i,\hat{t}) = \int \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \quad \hat{H}_i(x_i,\hat{t}) = \int \zeta_i \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \\ &\hat{G} = -\frac{1}{4}\int (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)\ln\left(\frac{\hat{f}'\hat{f}'_*}{\hat{f}\hat{f}_*}\right)\widehat{B}\mathrm{d}\Omega\mathbf{d}\boldsymbol{\zeta}_*\mathbf{d}\boldsymbol{\zeta} \le 0, \end{aligned}\right\} \tag{2}$$

with $\hat{c}_0 = c_0(2RT_0)^{3/2}/\rho_0$. The perturbed form of the velocity distribution function $\hat{f}$ is defined by

$$\hat{f} = E(1+\phi), \tag{3}$$

where

$$E = \frac{1}{\pi^{3/2}}\exp(-\zeta^2).$$

Let ε be a small quantity. Here, we take the case in which ϕ is of the order of ε, and examine the terms of the order of ε^2 of Eq. (1). The perturbed function ϕ is expressed as

$$\phi = \phi_1\varepsilon + \phi_2\varepsilon^2 + \cdots. \tag{4}$$

Corresponding to the expansion, the macroscopic variables, i.e., ω, u_i, P, etc., $\hat{H}$, $\hat{H}_i$, and $\hat{G}$ are also expressed as

$$h = h_1\varepsilon + h_2\varepsilon^2 + \cdots, \tag{5a}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1\varepsilon + \hat{H}_2\varepsilon^2 + \cdots, \tag{5b}$$

$$\hat{H}_i = \hat{H}_{i0} + \hat{H}_{i1}\varepsilon + \hat{H}_{i2}\varepsilon^2 + \cdots, \tag{5c}$$

$$\hat{G} = \hat{G}_0 + \hat{G}_1\varepsilon + \hat{G}_2\varepsilon^2 + \cdots, \tag{5d}$$

where h represents the perturbed macroscopic variables, ω, u_i, P, etc., and the quantities ϕ_n, h_n, $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are of the order of unity. Then, with the aid of the expanded forms of Eqs. (M-1.78a)–(M-1.78f), $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are

expressed as

$$\hat{H}_0 = -\frac{3}{2} - \ln \pi^{3/2}\hat{c}_0, \tag{6a}$$

$$\begin{aligned}\hat{H}_1 &= (1 - \ln \pi^{3/2}\hat{c}_0)\int E\phi_1 \mathbf{d}\boldsymbol{\zeta} - \int \zeta^2 E\phi_1 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)\omega_1 - \frac{3}{2}P_1,\end{aligned} \tag{6b}$$

$$\begin{aligned}\hat{H}_2 &= (1 - \ln \pi^{3/2}\hat{c}_0)\int E\phi_2 \mathbf{d}\boldsymbol{\zeta} - \int \zeta^2 E\phi_2 \mathbf{d}\boldsymbol{\zeta} + \frac{1}{2}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)\omega_2 - \left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{1}{2}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta},\end{aligned} \tag{6c}$$

$$\hat{H}_{i0} = 0, \tag{7a}$$

$$\begin{aligned}\hat{H}_{i1} &= (1 - \ln \pi^{3/2}\hat{c}_0)\int \zeta_i E\phi_1 \mathbf{d}\boldsymbol{\zeta} - \int \zeta_i \zeta^2 E\phi_1 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)u_{i1} - \left(Q_{i1} + \frac{5}{2}u_{i1}\right),\end{aligned} \tag{7b}$$

$$\begin{aligned}\hat{H}_{i2} &= (1 - \ln \pi^{3/2}\hat{c}_0)\int \zeta_i E\phi_2 \mathbf{d}\boldsymbol{\zeta} - \int \zeta_i \zeta^2 E\phi_2 \mathbf{d}\boldsymbol{\zeta} + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)(u_{i2} + \omega_1 u_{i1}) - \left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) \\ &\quad + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta},\end{aligned} \tag{7c}$$

$$\hat{G}_0 = 0, \tag{8a}$$

$$\hat{G}_1 = 0, \tag{8b}$$

$$\hat{G}_2 = -\frac{1}{4}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{*1})^2 \widehat{B}\mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \leq 0. \tag{8c}$$

With the aid of these expressions, the ε and ε^2-order expressions of Eq (1) are given as

$$\begin{aligned}\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} &= (1 - \ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i}\right) \\ &\quad - \left[\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right)\right],\end{aligned} \tag{9a}$$

$$\begin{aligned}\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} &= (1 - \ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2} + \omega_1 u_{i1})}{\partial x_i}\right) \\ &\quad - \mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) - \frac{\partial}{\partial x_i}\left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) \\ &\quad + \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}\right).\end{aligned} \tag{9b}$$

Substituting the series expansion (5a) into the conservation equation (M-1.87), we have

$$\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i} = 0, \tag{10a}$$

$$\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2} + \omega_1 u_{i1})}{\partial x_i} = 0. \tag{10b}$$

Similarly, from the conservation equation (M-1.89), we have

$$\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right) = 0, \tag{11a}$$

$$\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i2} + Q_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) = 0. \tag{11b}$$

With the aid of the expanded forms (10a)–(11b) of the conservation equations (M-1.87) and (M-1.89), Eqs. (9a) and (9b) are reduced to, for the solution of the Boltzmann equation (M-1.47) or (M-1.75a),

$$\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} = 0, \tag{12a}$$

$$\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} = \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}\right). \tag{12b}$$

Thus, the $o(\varepsilon^2)$ terms being neglected in Eq. (1), it is reduced to

$$\begin{aligned}\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ = -\frac{1}{2k}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{1*})^2 \widehat{B}\mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \le 0.\end{aligned} \tag{13}$$

This expression does not contain ϕ_2.

(Section 1.1: Version 4-00)

2 Chapter M-3

2.1 Processes of solution of the systems in Section M-3.7.2 (July 2007)

The processes of solutions of the fluid-dynamic-type equations derived in Section M-3.7.1 are straightforward and may not need explanation. For the equations in Section M-3.7.2, some explanation may be better to be given. The discussion will be made on the basis of the boundary conditions in Section M-3.7.3 for a simple boundary where the shape of the boundary is invariant and its velocity component normal to it is zero.

2.1.1 "Incompressible Navier–Stokes set"

Consider the initial and boundary-value problem of Eqs. (M-3.265)–(M-3.268), i.e.,

$$\frac{\partial P_{S1}}{\partial x_i} = 0, \tag{14}$$

$$\frac{\partial u_{iS1}}{\partial x_i} = 0, \tag{15a}$$

$$\frac{\partial u_{iS1}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_{S2}}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2}, \tag{15b}$$

$$\frac{5}{2}\frac{\partial \tau_{S1}}{\partial \tilde{t}} - \frac{\partial P_{S1}}{\partial \tilde{t}} + \frac{5}{2}u_{jS1}\frac{\partial \tau_{S1}}{\partial x_j} = \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S1}}{\partial x_j^2}, \tag{15c}$$

$$\frac{\partial u_{iS2}}{\partial x_i} = -\frac{\partial \omega_{S1}}{\partial \tilde{t}} - \frac{\partial \omega_{S1}u_{iS1}}{\partial x_i}, \tag{16a}$$

$$\begin{aligned}
&\frac{\partial u_{iS2}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS2}}{\partial x_j} + u_{jS2}\frac{\partial u_{iS1}}{\partial x_j} \\
&= -\frac{1}{2}\left(\frac{\partial P_{S3}}{\partial x_i} - \omega_{S1}\frac{\partial P_{S2}}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{iS2}}{\partial x_j} + \frac{\partial u_{jS2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{kS2}}{\partial x_k}\delta_{ij}\right) \\
&\quad - \frac{\gamma_1\omega_{S1}}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2} + \frac{\gamma_4}{2}\frac{\partial}{\partial x_j}\left[\tau_{S1}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)\right] - \frac{\gamma_3}{3}\frac{\partial}{\partial x_i}\frac{\partial^2 \tau_{S1}}{\partial x_j^2},
\end{aligned} \tag{16b}$$

$$\begin{aligned}
&\frac{3}{2}\frac{\partial P_{S2}}{\partial \tilde{t}} + \frac{3}{2}u_{jS1}\frac{\partial P_{S2}}{\partial x_j} + \frac{5}{2}\left(\frac{\partial P_{S1}u_{jS2}}{\partial x_j} - \frac{\partial \omega_{S2}}{\partial \tilde{t}} - \frac{\partial(\omega_{S2}u_{jS1} + \omega_{S1}u_{jS2})}{\partial x_j}\right) \\
&= \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S2}}{\partial x_j^2} + \frac{5\gamma_5}{4}\frac{\partial}{\partial x_j}\left(\tau_{S1}\frac{\partial \tau_{S1}}{\partial x_j}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)^2,
\end{aligned} \tag{16c}$$

where

$$P_{S1} = \omega_{S1} + \tau_{S1}, \quad P_{S2} = \omega_{S2} + \omega_{S1}\tau_{S1} + \tau_{S2}. \tag{17}$$

From Eq. (14), P_{S1} is a function of $\tilde{t}$, i.e.,

$$P_{S1} = f_1(\tilde{t}). \tag{18}$$

In an unbounded-domain problem where the pressure at infinity is specified (or the pressure is specified at some point), $P_{S1} = f_1(\tilde{t})$ is known, but in a bounded-domain problem of a simple boundary, $f_1(\tilde{t})$ is unknown at this moment and is determined later. Let u_{iS1} and τ_{S1} as well as $f_1(\tilde{t})$ be given at time $\tilde{t}$ in such a way that u_{iS1} satisfies Eq. (15a). Taking the divergence of Eq. (15b) and using Eq. (15a), we have

$$\frac{\partial^2 P_{S2}}{\partial x_i^2} = -2\frac{\partial u_{jS1}}{\partial x_i}\frac{\partial u_{iS1}}{\partial x_j}. \tag{19}$$

On a simple boundary, the derivative of P_{S2} normal to it is found to be expressed with u_{iS1} and its space derivatives by multiplying Eq. (15b) by the normal vector to the boundary.[1] In the unbounded-domain problem, where $f_1(\tilde{t})$ is known, P_{S2} is determined by Eq. (19). In the bounded-domain problem, P_{S2} is determined by Eq. (19) except for an additive function of $\tilde{t}$ [say, $f_2(\tilde{t})$]. Anyway, $\partial P_{S2}/\partial x_i$ is independent of this ambiguity. From Eq. (15b), $\partial u_{iS1}/\partial \tilde{t}$ at $\tilde{t}$ is determined, irrespective of $f_2(\tilde{t})$, in such a way that $\partial(\partial u_{iS1}/\partial x_i)/\partial \tilde{t} = 0$ for the above choice of P_{S2}. Thus, the solution u_{iS1} of Eqs. (15a) and (15b) is determined by Eq. (15b) with the supplementary condition (19) instead of Eq. (15a). From Eq. (15c), $(5/2)\partial\tau_{S1}/\partial\tilde{t} - \partial P_{S1}/\partial\tilde{t}$ or $(5/2)\partial\tau_{S1}/\partial\tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is determined, i.e.,

$$(5/2)\partial\tau_{S1}/\partial\tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t} = G(x_i, \tilde{t}), \tag{20}$$

where

$$G(x_i, \tilde{t}) = -\frac{5}{2}u_{jS1}\frac{\partial\tau_{S1}}{\partial x_j} + \frac{5\gamma_2}{4}\frac{\partial^2\tau_{S1}}{\partial x_j^2}. \tag{21}$$

Thus, τ_{S1} is determined in the unbounded-domain problem, but τ_{S1} has ambiguity owing to $f_1(\tilde{t})$ in the bounded-domain problem. The undetermined function $f_1(\tilde{t})$ is determined in the following way.

In the bounded-domain problem whose boundary is a simple boundary, the mass of the gas in the domain is invariant with respect to $\tilde{t}$. The condition at the leading order is

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \omega_{S1}\mathbf{d}\boldsymbol{x} = 0, \tag{22}$$

where V indicates the domain (or its volume in the later). With the aid of Eq. (17), we have

$$\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V - \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} = 0. \tag{23}$$

On the other hand, from Eq. (20),

$$-\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V + \frac{5}{2}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} = \int_V G(x_i, \tilde{t})\mathbf{d}\boldsymbol{x}. \tag{24}$$

From Eqs. (23) and (24), we obtain $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ and $\mathrm{d}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x}/\mathrm{d}\tilde{t}$ as

$$\left.\begin{aligned} \frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}} &= \frac{2}{3V}\int_V G(x_i, \tilde{t})\mathbf{d}\boldsymbol{x}, \\ \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} &= \frac{2}{3}\int_V G(x_i, \tilde{t})\mathbf{d}\boldsymbol{x}. \end{aligned}\right\} \tag{25}$$

That is, $f_1(\tilde{t})$ in the bounded-domain problem [and thus the solution τ_{S1} of Eq. (15c)] is determined.

[1] The time-derivative term vanishes owing to the boundary condition mentioned in the first paragraph of Section 2.1.

The analysis of the higher-order equations is similar; for example, from Eqs. (16a)–(16c), u_{iS2}, τ_{S2}, and P_{S3} are determined in the unbounded-domain problem, but $f_2(\tilde{t})$, u_{iS2}, τ_{S2}, and P_{S3}, except for an additive function of $\tilde{t}$ in P_{S3}, are determined in the bounded-domain problem.[2] Let u_{iS2}, τ_{S2}, and $f_2(\tilde{t})$ be given at $\tilde{t}$ in such a way that Eq. (16a) is satisfied.[3] Taking the divergence of Eq. (16b) and using Eq. (16a) and the results obtained above, we find that P_{S3} is governed by the Poisson equation

$$\frac{\partial^2 P_{S3}}{\partial x_i^2} = \text{Inhomogeneous term}, \tag{26}$$

where the inhomogeneous term consists of u_{iS2}, P_{S2}, and the functions determined in the preceding analysis. On a simple boundary, the derivative of P_{S3} normal to it being known,[4] P_{S3} is determined by this equation, except for an additive function of $\tilde{t}$ [say, $f_3(\tilde{t})$] in the bounded-domain problem. Then, from Eq. (16b), $\partial u_{iS2}/\partial\tilde{t}$ at $\tilde{t}$ is determined irrespective of $f_3(\tilde{t})$. From Eq. (16c), $\partial(3P_{S2} - 5\omega_{S2})/\partial\tilde{t}$ [or $\partial(5\tau_{S2} - 2P_{S2})/\partial\tilde{t}$] at $\tilde{t}$ is determined. Thus, u_{iS2} and τ_{S2} (except for the additive function $2f_2/5$ in the bounded-domain problem) [thus, ω_{S2} (except for the additive function $3f_2/5$)] are determined. In the bounded-domain problem, where the boundary is a simple boundary, the condition of invariance of the mass of the gas in the domain at the corresponding order is[5]

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_V \omega_{S2} \mathbf{d}\boldsymbol{x} = 0. \tag{27}$$

With the aid of Eq. (17), $\mathrm{d}f_2(\tilde{t})/\mathrm{d}\tilde{t}$ at $\tilde{t}$ is determined as $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is done.

To summarize, the solution $(u_{iS1}, P_{S1}, \tau_{S1}, P_{S2})$ of the initial and boundary-value problem of Eqs. (14)–(15c) is determined, with an additive arbitrary function $f_2(\tilde{t})$ in P_{S2} in a bounded-domain problem of a simple boundary, when the initial data of u_{iS1}, P_{S1}, τ_{S1}, and P_{S2} satisfy Eqs. (15a) and (19). The additive function $f_2(\tilde{t})$ does not affect the other variables. The function $f_2(\tilde{t})$ is determined in the next-order analysis. In other words, the solution $(u_{iS1}, P_{S1}, \tau_{S1})$ of Eqs. (14)–(15c) is determined consistently by Eqs. (14), (15b), and (15c) with the supplementary condition (19), instead of Eq. (15a), when the initial data of u_{iS1}, P_{S1}, and τ_{S1} satisfy Eq. (15a). Naturally, the initial P_{S2} is required to satisfy Eq. (19). This process is natural for numerical computation.

[2] Note that, with the aid of Eq. (17), the time-derivative term $\frac{3}{2}\partial P_{S2}/\partial\tilde{t} - \frac{5}{2}\partial\omega_{S2}/\partial\tilde{t}$ in Eq. (16c) is transformed into $\frac{5}{2}\partial\tau_{S2}/\partial\tilde{t} - \partial P_{S2}/\partial\tilde{t} + \frac{5}{2}\partial\omega_{S1}\tau_{S1}/\partial\tilde{t}$.

[3] The time derivative $\partial\omega_{S1}/\partial\tilde{t}$ is known from $\partial\tau_{S1}/\partial\tilde{t}$, $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$, and Eq. (17).

[4] Shift the discussion of the boundary condition for P_{S2} to the next order.

[5] The contribution of the Knudsen-layer correction to the mass in the domain is of a higher order, though it is required to ω_{S2}.

2.1.2 Ghost-effect equations (M-3.275)–(M-3.278b):

Consider the initial and boundary-value problem of Eqs. (M-3.275)–(M-3.278b), i.e.,

$$\hat{p}_{SB0} = \hat{p}_0(\tilde{t}), \tag{28}$$

$$\hat{p}_{SB1} = \hat{p}_1(\tilde{t}), \tag{29}$$

$$\frac{\partial \hat{\rho}_{SB0}}{\partial \tilde{t}} + \frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{30a}$$

$$\begin{aligned}
&\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial \tilde{t}} + \frac{\partial \hat{\rho}_{SB0}\hat{v}_{jSB1}\hat{v}_{iSB1}}{\partial x_j} \\
&\quad = -\frac{1}{2}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\Gamma_1(\hat{T}_{SB0})\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] \\
&\qquad + \frac{1}{2\hat{p}_0}\frac{\partial}{\partial x_j}\left\{\Gamma_7(\hat{T}_{SB0})\left[\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j} - \frac{1}{3}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2\delta_{ij}\right]\right\},
\end{aligned} \tag{30b}$$

$$\frac{3}{2}\frac{\partial \hat{\rho}_{SB0}\hat{T}_{SB0}}{\partial \tilde{t}} + \frac{5}{2}\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{T}_{SB0}}{\partial x_i} = \frac{5}{4}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right), \tag{30c}$$

where $\hat{p}_0$ and $\hat{p}_1$ depend only on $\tilde{t}$, and

$$\left.\begin{aligned}
&\hat{p}_{SB0} = \hat{\rho}_{SB0}\hat{T}_{SB0},\ \hat{p}_{SB1} = \hat{\rho}_{SB1}\hat{T}_{SB0} + \hat{\rho}_{SB0}\hat{T}_{SB1}, \\
&\hat{p}_{SB2} = \hat{\rho}_{SB2}\hat{T}_{SB0} + \hat{\rho}_{SB1}\hat{T}_{SB1} + \hat{\rho}_{SB0}\hat{T}_{SB2},
\end{aligned}\right\} \tag{31}$$

$$\hat{p}^*_{SB2} = \hat{p}_{SB2} + \frac{2}{3\hat{p}_0}\frac{\partial}{\partial x_k}\left(\Gamma_3(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right). \tag{32}$$

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p} = \hat{\rho}\hat{T}$) at time $\tilde{t}$ be given; thus, $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$ ($\hat{p}_{SB0}$), etc., including $\hat{p}_{SB2}$, are given. Then $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{\rho}_{SB0}\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{T}_{SB0}/\partial\tilde{t}$ at $\tilde{t}$ are given by Eqs. (30a)–(30c); thus, the future $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{T}_{SB0}$ (also $\hat{p}_{SB0}$) are determined. However, the future $\hat{p}_{SB0}$, as well as $\hat{p}_{SB0}$ at $\tilde{t}$, is required to be independent of x_i owing to Eq. (28). Taking this point into account, we discuss how the solution is determined. For convenience of the discussion, transform Eq. (30c) in the form

$$\frac{\partial \hat{p}_{SB0}}{\partial \tilde{t}} = \mathcal{P}, \tag{33}$$

where

$$\mathcal{P} = -\frac{5}{3}\hat{p}_{SB0}\frac{\partial \hat{v}_{iSB1}}{\partial x_i} + \frac{5}{6}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right).$$

First, consider the case where $\hat{p}$ (thus, $\hat{p}_{SB0}$, $\hat{p}_{SB1}$, etc.) is specified at some point, e.g., at infinity. Then, from Eq. (28), $\hat{p}_0(\tilde{t})$ is a given function of $\tilde{t}$, and

$\hat{p}_{SB0}$ is determined. The initial value of $\hat{p}_{SB0}$ is uniform, i.e., $\hat{p}_{SB0} = \hat{p}_0(0)$. On the other hand, from Eq. (33), the variation of $\partial\hat{p}_{SB0}/\partial\tilde{t}$ is also determined by the data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and their space derivatives at $\tilde{t}$. This must coincide with the corresponding data given by Eq. (28), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$. Substituting this relation into Eq. (33), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}, \tag{34}$$

which requires a relation among $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ for all $\tilde{t}$, since $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is given. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ are required to satisfy Eq. (34), and the time derivative of Eq. (34) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}. \tag{35}$$

With the aid of Eqs. (30a)–(30c) and (33), the left-hand side of Eq. (35) is expressed in the form without the time-derivative terms, i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$, as follows:

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right) = -\frac{1}{2}\hat{p}_{SB0}\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right) + \mathrm{fn}_1,$$

where fn_1 is a given function of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives. Thus, the condition (35) is reduced to an equation for $\hat{p}^*_{SB2}$, i.e.,

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right) = \mathrm{Fn}, \tag{36}$$

where

$$\mathrm{Fn} = \frac{2}{\hat{p}_0}\left(\mathrm{fn}_1 + \frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}\right).$$

The boundary condition for $\hat{p}^*_{SB2}$ in Eq. (36) on a simple boundary is derived by multiplying Eq. (30b) by the normal n_i to the boundary. In this process, the contribution of its time-derivative terms vanishes.[6] Thus, $\hat{p}^*_{SB2}$ (or $\hat{p}_{SB2}$) is determined in the present case, where $\hat{p}$ (thus, $\hat{p}_{SB2}$) is specified at some point. The solution $\hat{p}^*_{SB2}$ of Eq. (36) being substituted into Eq. (30b), Eqs. (30a)–(30c) with the first relation in Eq. (31) are reduced to the equations for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(= \hat{p}_{SB0}) = \hat{p}_0$ and that Eq. (34) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(= \hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (28), since

[6] The discussion is similar to that in Footnote 1.

Eq. (36) or (35) with the condition (34) at the initial state guarantees Eq. (34), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (28) and (30a)–(30c) with Eqs. (31) and (32) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$ consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. In the above system, $\hat{p}_{SB2}$ is determined. On the other hand, the variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by the counterpart of Eq. (33) at the order after next. The situation is similar to that at the leading order, where Eqs. (28), with a given $\hat{p}_0$, and (33) determine $\hat{p}_{SB0}$ independently. The analysis can be carried out in a similar way. Let $\hat{p}_{SB2}$ determined by Eq. (36) be indicated by $(\hat{p}_{SB2})_0$ and the equation for $\partial\hat{p}_{SB2}/\partial\tilde{t}$, or the counterpart of Eq. (33) at the order after next, be put in the form

$$\frac{\partial\hat{p}_{SB2}}{\partial\tilde{t}} = \mathcal{P}_2, \tag{37}$$

where $\mathcal{P}_2$ is a given function of $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \leq 2)$, and their space derivatives. For the consistency, $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is substituted for $\partial\hat{p}_{SB2}/\partial\tilde{t}$ in Eq. (37), i.e.,

$$\mathcal{P}_2 = \frac{\partial(\hat{p}_{SB2})_0}{\partial\tilde{t}}, \tag{38}$$

where $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is known. This requires a relation among $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \leq 2)$, and their space derivatives. This condition is equivalently replaced by the following two conditions: Equation (38) is applied only for the initial state, and the time derivative of Eq. (38), i.e.,

$$\frac{\partial\mathcal{P}_2}{\partial\tilde{t}} = \frac{\partial^2(\hat{p}_{SB2})_0}{\partial\tilde{t}^2},$$

has to be satisfied for all $\tilde{t}$. The $\partial\hat{\rho}_{SBm}/\partial\tilde{t}$, $\partial\hat{v}_{iSBm+1}/\partial\tilde{t}$, $\partial\hat{T}_{SBm}/\partial\tilde{t}$ $(m \leq 2)$ in $\partial\mathcal{P}_2/\partial\tilde{t}$ being replaced by the counterparts of Eqs. (30a)–(30c) and (33) at the corresponding order, an equation for $\hat{p}_{SB4}$ for all $\tilde{t}$ is derived.[7] The conclusion is that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced and, instead, that the condition (36) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (28), (30a)–(32) is determined consistently by Eqs. (30a)–(32) with the aid of the supplementary condition (36), instead of Eq. (28), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (28) and (34), where $\hat{p}_0(\tilde{t})$ is a known function of $\tilde{t}$ from the boundary condition.

Secondly, consider a bounded-domain problem of a simple boundary. In contrast to the first case, $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is unknown because no condition is imposed on $\hat{p}_{SB0}$ on a simple boundary. However, in a bounded-domain problem of a

[7] The conditions on the odd-order $\hat{p}_{SB2n+1}$'s are derived by the analysis starting from the condition (29) that $\hat{p}_{SB1}$ is independent of x_i.

simple boundary, the mass of the gas in the domain is invariant with respect to $\tilde{t}$, i.e., at the leading order,

$$\frac{\mathrm{d}\int_V \hat{\rho}_{SB0}\mathbf{d}\boldsymbol{x}}{\mathrm{d}\tilde{t}} = 0, \tag{39}$$

where V indicates the domain under consideration. Using the first relation of Eq. (31), i.e., $\hat{\rho}_{SB0} = \hat{p}_0/\hat{T}_{SB0}$, in Eq. (39), we have

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x} = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}\frac{\partial \hat{T}_{SB0}}{\partial \tilde{t}}\mathbf{d}\boldsymbol{x}. \tag{40}$$

Using Eq. (30c) for $\partial\hat{T}_{SB0}/\partial\tilde{t}$ in Eq. (40), we find that the variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is expressed with $\hat{p}_0$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ as follows:

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}} = P(\tilde{t}), \tag{41}$$

where

$$P(\tilde{t}) = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}\left[\frac{5}{6\hat{\rho}_{SB0}}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) - \frac{5}{3}\hat{v}_{iSB1}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right]\mathbf{d}\boldsymbol{x} \\ \times\left(\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}. \tag{42}$$

With this expression of $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, we can carry out the analysis in a similar way to that in the first case.

The variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ or $\partial\hat{p}_{SB0}/\partial\tilde{t}$ is also determined by Eq. (33). The two $\partial\hat{p}_{SB0}/\partial\tilde{t}$'s given by Eq. (41) with Eq. (42) and Eq. (33) have to be consistent. Thus, substituting Eq. (41) with Eq. (42) into $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (33), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}P(\tilde{t}), \tag{43}$$

where $P(\tilde{t})$ is given by Eq. (42). This must hold for all $\tilde{t}$ for consistency. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$ are required to satisfy Eq. (43), and the time derivative of Eq. (43) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}P(\tilde{t})}{\mathrm{d}\tilde{t}}. \tag{44}$$

Using Eqs. (30a), (30b), and (33) for the time derivatives $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (44), we find that $\hat{p}^*_{SB2}$ at $\tilde{t}$ is determined by the equation

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) + \mathcal{L}\left(\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) = \mathrm{Fn}, \tag{45}$$

where Fn is a given functional of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives, and $\mathcal{L}(\partial\hat{p}^*_{SB2}/\partial x_i)$ is a given linear functional of $\partial\hat{p}^*_{SB2}/\partial x_i$, i.e.,

$$\mathcal{L}\left(\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right) = -\frac{1}{\hat{p}_0}\int_V \frac{1}{\hat{T}_{SB0}}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\mathbf{d}\boldsymbol{x}\left(\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}.$$

On a simple boundary, the derivative of $\hat{p}^*_{SB2}$ normal to the boundary is specified. Thus, $\hat{p}^*_{SB2}$ is determined except for an additive function of $\tilde{t}$. The solution $\hat{p}^*_{SB2}$ of Eq. (45) being substituted into Eq. (30b), the result is independent of the additive function. Thus, Eqs. (30a)–(30c) with the first relation in Eq. (31) and the above $\hat{p}^*_{SB2}$ substituted are reduced to those for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$, which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(=\hat{p}_{SB0}) = \hat{p}_0$ and that Eq. (43) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(=\hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (28), since Eq. (45) or (44) with the condition (43) at the initial state guarantees Eq. (43), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (28) and (30a)–(30c) with Eqs. (31) and (45) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$, except for an additive function of $\tilde{t}$ in $\hat{p}_{SB2}$, consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. First, the undetermined additive function in $\hat{p}_{SB2}$ is determined by the condition of invariance of the mass of the gas in the domain at the order after next as $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is determined.[8] The $\partial\hat{p}_{SB2}/\partial\tilde{t}$ or $\hat{p}_{SB2}$ determined in this way is indicated by $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ or $(\hat{p}_{SB2})_0$. On the other hand, the variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by Eq. (37) or the counterpart of Eq. (33) at the order after next. The two results must coincide. The discussion from here is the same as that given from the sentence starting from Eq. (37) to the end of the paragraph. The results are that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced, and that the condition (45) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (28), (30a)–(30c) is determined consistently by Eqs. (30a)–(30c) with the aid of the supplementary condition (45), instead of Eq. (28), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (28) and (43).

2.2 Notes on basic equations in classical fluid dynamics

2.2.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

[8] The Knudsen-layer correction to $\hat{\rho}_{SB1}$, already determined (see Footnote 7), contributes to the mass at this order.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \tag{46}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = 0, \tag{47}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}v_i^2\right)\right] + \frac{\partial}{\partial X_j}\left[\rho v_j\left(e + \frac{1}{2}v_i^2\right) + v_i p_{ij} + q_j\right] = 0, \tag{48}$$

where ρ is the density, v_i is the flow velocity, e is the internal energy per unit mass, p_{ij}, which is symmetric with respect to i and j, is the stress tenor, and q_i is the heat-flow vector. The pressure p and the internal energy e are given by the equations of state as functions of T and ρ, i.e.,

$$p = p(T, \rho), \quad e = e(T, \rho). \tag{49}$$

Especially, for a perfect gas,

$$p = R\rho T, \quad e = e(T). \tag{50}$$

Equations (47) and (48) are rewritten with the aid of Eq. (46) in the form

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0, \tag{51}$$

$$\rho\frac{\partial}{\partial t}\left(e + \frac{1}{2}v_i^2\right) + \rho v_j \frac{\partial}{\partial X_j}\left(e + \frac{1}{2}v_i^2\right) + \frac{\partial}{\partial X_j}(v_i p_{ij} + q_j) = 0. \tag{52}$$

The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $\mathrm{D}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$

Multiplying Eq. (51) by v_i we obtain the equation for the variation of kinetic energy as

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v_i^2\right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \tag{53}$$

Another form of Eq. (48), where Eq. (53) is subtracted from Eq. (52), is given as

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = -p_{ij}\frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \tag{54}$$

Noting the thermodynamic relation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{55}$$

where s is the entropy per unit mass, and Eq. (46), Eq. (54) is rewritten as

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T}\left[(p_{ij} - p\delta_{ij})\frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j}\right]. \tag{56}$$

Equation (56) expresses the variation of the entropy of a fluid particle.

Equations (46)–(49) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor p_{ij} and the heat-flow vector q_i are assumed in some ways. The *Navier–Stokes set of equations* (or the *Navier–Stokes equations*) is Eqs. (46)–(49) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) - \mu_B\frac{\partial v_k}{\partial X_k}\delta_{ij}, \tag{57}$$

$$q_i = -\lambda\frac{\partial T}{\partial X_i}, \tag{58}$$

where μ, μ_B, and λ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of T and ρ. The *Euler set of equations* (or the *Euler equations*) is Eqs. (46)–(49) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \tag{59}$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (57) and (58), the entropy variation is expressed in the form[9]

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T}\left[\frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)^2 + \mu_B\left(\frac{\partial v_k}{\partial X_k}\right)^2 + \frac{\partial}{\partial X_i}\left(\lambda\frac{\partial T}{\partial X_i}\right)\right]. \tag{60}$$

For the Euler equations, for which p_{ij} and q_i are given by Eq.(59), the entropy of a fluid particle is invariant, i.e.,

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = 0. \tag{61}$$

[9]Note the following transformation:

$$\begin{aligned}
&\frac{\partial v_i}{\partial X_j}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\\
&= \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij} + \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\\
&= \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)^2 + \frac{1}{3}\frac{\partial v_l}{\partial X_l}\delta_{ij}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right).
\end{aligned}$$

The second term in the last expression is easily seen to vanish.

For an *incompressible fluid*, the first relation of Eq. (49) is replaced by[10]

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{62}$$

Thus, from Eqs. (46) and (62),

$$\frac{\partial v_i}{\partial X_i} = 0. \tag{63}$$

Equation (57) for the Navier–Stokes-stress tensor reduces to

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right). \tag{64}$$

The first term on the right-hand side of Eq. (54) reduces to

$$\begin{aligned} -p_{ij} \frac{\partial v_i}{\partial X_j} &= -\left[p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j} \\ &= \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2. \end{aligned}$$

Thus, Eq. (54) reduces to

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right). \tag{65}$$

To summarize, the Navier–Stokes equations for incompressible fluid are

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{66a}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j} \left[\mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right], \tag{66b}$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right), \tag{66c}$$

with the incompressible condition (62) being supplemented, i.e.,

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{67}$$

[10] The density is invariant along fluid-particle paths. If ρ is of uniform value ρ_0 initially, it is a constant, i.e.,

$$\rho = \rho_0.$$

In a time-independent (or steady) problem, the density is constant along streamlines.

2.2.2 Boundary condition for the Euler set

In Section M-3.5, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentun-conservation equations of the Euler set are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{68}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{69}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{70}$$

where ρ is the density, which is uniform, (u, v) is the flow velocity, and p is the pressure. Owing to Eq. (68), the stream function Ψ can be introduced as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{71}$$

Eliminating p from Eqs. (69) and (70), we have[11]

$$u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = 0, \tag{72}$$

where Ω is the vorticity, i.e.,

$$\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}. \tag{73}$$

From Eqs. (71) and (72),

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Omega}{\partial y} = 0. \tag{74}$$

[11] The following equation is formed from them:

$$\partial \mathrm{Eq.}\,(69)/\partial y - \partial \mathrm{Eq.}\,(70)/\partial x = 0.$$

This equation shows that Ω is a function of Ψ,[12] i.e.,

$$\Omega = F(\Psi). \tag{75}$$

This functional relation between Ω and Ψ is a local relation, and therefore F may be a multivalued function of Ψ. From Eqs. (73) and (75),

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi). \tag{76}$$

Consider a boundary-value problem in a simply-connected bounded domain, where Ψ is given on the boundary ($\Psi = \Psi_B$). Introduce a coordinate s ($0 \leq s < S$) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial \Psi_B / \partial s < 0$, and the fluid flows out from the domain on the boundary where $\partial \Psi_B / \partial s > 0$. When F is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose F on the part where $\partial \Psi_B / \partial s < 0$ or $\partial \Psi_B / \partial s > 0$. For example, take the case where $\partial \Psi_B / \partial s < 0$ for $0 < s < S_m$ and $\partial \Psi_B / \partial s > 0$ for $S_m < s < S$, and choose the distribution $\Omega_B(s)$ of Ω along the boundary for the part $0 < s < S_m$. By the choice of Ω_B, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi = \Psi_B(s)$ between

[12] This can be seen with the aid of theorems on implicit functions (see References M-[47, 48, 267]). The proof is outlined here. The Ω and Ψ are functions of x and y :

$$\Omega = \Omega(x, y), \quad \Psi = \Psi(x, y). \tag{$*$}$$

Solving the second equation with respect to x, we have

$$x = \hat{x}(\Psi, y). \tag{$**$}$$

With this relation into Eq. ($*$),

$$\Omega = \Omega(\hat{x}(\Psi, y), y) = \hat{\Omega}(\Psi, y), \tag{$\sharp$a}$$

$$\Psi = \Psi(\hat{x}(\Psi, y), y) = \hat{\Psi}(\Psi, y). \tag{$\sharp$b}$$

That is, Ω is expressed as a function of Ψ and y. From Eqs. ($\sharp$a) and ($\sharp$b),

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = \frac{\partial \Omega(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Omega(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Omega(x, y)}{\partial y}, \tag{$\sharp\sharp$a}$$

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = 0. \tag{$\sharp\sharp$b}$$

On the other hand,

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = \frac{\partial \Psi(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y}.$$

Thus,

$$\frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y} = 0. \tag{$\ddagger$}$$

From Eqs. (74), ($\sharp\sharp$a) and ($\ddagger$), we have

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \hat{\Omega}(\Psi).$$

Ψ and s on the part $0 < s < S_m$, i.e., $s(\Psi)$, and noting the relation (75), we find that F is given by

$$F(\Psi) = \Omega_B(s(\Psi)). \tag{77}$$

Then, the boundary-value problem is fixed. That is, Eq. (76) is fixed as[13]

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \Omega_B(s(\Psi)), \tag{78}$$

and the boundary condition is given as $\Psi = \Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (73), the solution of the above system automatically satisfies condition $\Omega = \Omega_B(s)$ along the boundary for $0 < s < S_m$. We cannot choose the distribution of Ω on the boundary for $S_m < s < S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (65) with $\mu = \lambda = 0$, i.e.,

$$u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} = 0, \quad \text{or} \quad \frac{\partial \Psi}{\partial y}\frac{\partial e}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial e}{\partial y} = 0, \tag{79}$$

where e is the internal energy. Thus, e is a function of Ψ, i.e.,

$$e = F_1(\Psi). \tag{80}$$

In the above boundary-value problem, therefore, e can be specified on the the part $(0 < s < S_m)$ of the boundary, but no condition can be specified on other part $(S_m < s < S)$ and vice versa.[14]

To summarize, we can specify three conditions for Ψ, Ω, and e on the part $\partial\Psi_B/\partial s < 0$ $(\partial\Psi_B/\partial s > 0)$ of boundary but one condition for Ψ on the other part $\partial\Psi_B/\partial s > 0$ $(\partial\Psi_B/\partial s < 0)$. The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

2.2.3 Ambiguity of pressure in the incompressible Navier–Stokes system

It may be better to note ambiguity of the solution of the initial and boundary-value problem of the incompressible Navier–Stokes equations in a bounded domain of simple boundaries.

[13] There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x, y) = \text{const}$ inside the domain is not excluded.

[14] From the second relation on e of Eq. (49) and the uniform-density condition, the condition on e can be replaced by the condition on the temperature T.

Consider the Navier–Stokes equations for an incompressible fluid, i.e.,

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{81a}$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{81b}$$

$$\rho\frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda\frac{\partial T}{\partial X_j}\right), \tag{81c}$$

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} = 0, \tag{81d}$$

where e, μ, and λ are functions of T and ρ.

Consider the initial and boundary-value problem of Eqs. (81a)–(81d) in a bounded domain D on the boundary ∂D of which v_i and T are specified as $v_i = v_{wi}$ and $T = T_w$ (v_{wi} and T_w are, respectively, the surface velocity and temperature of the boundary satisfying $\int_{\partial D} v_{wi} n_i \mathrm{d}S = 0$, n_i : the unit normal vector to the boundary) and no condition is imposed on ρ and p. Let $(v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)})$ be a solution of the initial and boundary-value problem. Let $P^{(a)}$ be an arbitrary function of t, independent of x_i, that vanishes at initial time $t = 0$, i.e., $P^{(a)} = f(t)$ with $f(0) = 0$. Put

$$(v_i, \rho, T, p) = (v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)} + P^{(a)}).$$

Then, e, μ, and λ corresponding to the new (v_i, ρ, T, p) are equal to $e^{(s)}$, $\mu^{(s)}$, and $\lambda^{(s)}$ respectively, because they are determined by ρ and T. The new (v_i, ρ, T, p) satisfy the equations (81a)–(81d) and the initial and boundary conditions.

2.2.4 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set. This statement should be made precise. The difference is briefly explained in the book "Molecular Gas Dynamics" in connection with the equations derived by the S expansion from the Boltzmann equation in Sections M-3.2.4 and M-3.7.2. Here, we explicitly show the process of analysis from the compressible Navier–Stokes set. The resulting set of equations no longer has ambiguity of pressure in contrast to the incompressible Navier–Stokes set. Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq. (M-1.74) in Section M-1.10 as

follows:

$$\mathrm{Sh}\frac{\partial\omega}{\partial\hat{t}}+\frac{\partial(1+\omega)u_i}{\partial x_i}=0, \tag{82}$$

$$\mathrm{Sh}\frac{\partial(1+\omega)u_i}{\partial\hat{t}}+\frac{\partial}{\partial x_j}\left((1+\omega)u_iu_j+\frac{1}{2}P_{ij}\right)=0, \tag{83}$$

$$\begin{aligned}&\mathrm{Sh}\frac{\partial}{\partial\hat{t}}\left[(1+\omega)\left(\frac{3}{2}(1+\tau)+u_i^2\right)\right]\\&+\frac{\partial}{\partial x_j}\left[(1+\omega)u_j\left(\frac{3}{2}(1+\tau)+u_i^2\right)+u_i(\delta_{ij}+P_{ij})+Q_j\right]=0.\end{aligned} \tag{84}$$

The nondimensional stress tensor P_{ij}, and heat-flow vector Q_i are expressed as[15]

$$P_{ij}=P\delta_{ij}-\frac{\mu_0(2RT_0)^{1/2}}{p_0L}(1+\bar{\mu})\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \tag{85a}$$

$$Q_i=-\frac{\lambda_0T_0}{Lp_0(2RT_0)^{1/2}}(1+\bar{\lambda})\frac{\partial\tau}{\partial x_i}. \tag{85b}$$

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu=\mu_0(1+\bar{\mu}),\quad \lambda=\lambda_0(1+\bar{\lambda}),$$

where μ_0 and λ_0 are, respectively, the values of the viscosity μ and the thermal conductivity λ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of τ and ω. The first relation of the equation of state [Eq. (50)] is expressed as

$$P=\omega+\tau+\omega\tau. \tag{86}$$

Take a small parameter ε, and consider the case where

$$u_i=O(\varepsilon),\quad \omega=O(\varepsilon),\quad \tau=O(\varepsilon),\quad \mathrm{Sh}=O(\varepsilon), \tag{87a}$$

$$\frac{\mu_0(2RT_0)^{1/2}}{p_0L}=\gamma_1\varepsilon,\quad \frac{\lambda_0T_0}{Lp_0(2RT_0)^{1/2}}=\frac{5}{4}\gamma_2\varepsilon, \tag{87b}$$

thus,

$$P=O(\varepsilon),\quad \bar{\mu}=O(\varepsilon),\quad \bar{\lambda}=O(\varepsilon).$$

According to the definition of u_i in Eq. (M-1.74), ε is of the order of the Mach number. In view of this and the definition of the Prandtl number $\mathrm{Pr}=5R\mu/2\lambda$ (see Section M-3.1.9), γ_1 and γ_2 are, respectively, of the orders of $1/\mathrm{Re}$ and $1/\mathrm{PrRe}$ (Re: the Reynolds number). According to Eq. (M-1.48a), the condition $\mathrm{Sh}=O(\varepsilon)$ in Eq. (87a) means that the time scale t_0 of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous

[15] For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B=0$.

diffusion. Thus, we are considering the case where the Mach number is small, the Reynolds and Prandtl numbers are of the order of unity, and the time scale of variation of the system is of the order of the time scale of viscous diffusion. We can take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality. Then,

$$\mathrm{Sh} = \varepsilon. \tag{88}$$

Corresponding to the above situation, u_i, ω, P, and τ are expanded in power series of ε, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots, \tag{89a}$$
$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots, \tag{89b}$$
$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots, \tag{89c}$$
$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots, \tag{89d}$$
$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots, \tag{89e}$$
$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots, \tag{89f}$$
$$P_{ij} = P_1\delta_{ij}\varepsilon + P_{ij2}\varepsilon^2 + \cdots, \tag{89g}$$
$$Q_i = Q_{i2}\varepsilon^2 + \cdots. \tag{89h}$$

Substituting Eqs. (89a)–(89h) with Eqs. (87b) and (88) into Eqs. (82)–(84) with Eqs. (85a) and (85b), and arranging the same-order terms of ε, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \qquad \frac{\partial P_1}{\partial x_i} = 0, \qquad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

$$\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$
$$\frac{\partial u_{i1}}{\partial \hat{t}} + \frac{\partial u_{i1}u_{j1}}{\partial x_j} + \frac{1}{2}\frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right) = 0,$$
$$\frac{3}{2}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left(\frac{5}{2}u_{j2} + \frac{5}{2}P_1 u_{j1} - \frac{5}{4}\gamma_2\frac{\partial \tau_1}{\partial x_j}\right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (82) and (84) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{90}$$

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{91a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{91b}$$

$$\frac{5}{2}\frac{\partial \tau_1}{\partial \hat{t}} - \frac{\partial P_1}{\partial \hat{t}} + \frac{5}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{91c}$$

$$\frac{\partial u_{i2}}{\partial x_i} = -\frac{\partial \omega_1}{\partial \hat{t}} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \tag{92a}$$

$$\begin{aligned}
&\frac{\partial u_{i2}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i2}}{\partial x_j} + u_{j2}\frac{\partial u_{i1}}{\partial x_j} \\
&= -\frac{1}{2}\left(\frac{\partial P_3}{\partial x_i} - \omega_1\frac{\partial P_2}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right) \\
&\quad - \frac{\gamma_1\omega_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left[\bar{\mu}_1\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)\right],
\end{aligned} \tag{92b}$$

$$\begin{aligned}
&\frac{3}{2}\frac{\partial P_2}{\partial \hat{t}} + \frac{3}{2}u_{j1}\frac{\partial P_2}{\partial x_j} + \frac{5}{2}\left(P_1\frac{\partial u_{j2}}{\partial x_j} - \frac{\partial \omega_2}{\partial \hat{t}} - \frac{\partial(\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j}\right) \\
&= \frac{5\gamma_2}{4}\frac{\partial}{\partial x_i}\left(\frac{\partial \tau_2}{\partial x_i} + \bar{\lambda}_1\frac{\partial \tau_1}{\partial x_i}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)^2,
\end{aligned} \tag{92c}$$

where

$$P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1\tau_1. \tag{93}$$

These equations are very similar to Eqs. (M-3.265)–(M-3.268) [or Eqs. (14)–(17)] obtained by the S expansion of the Boltzmann equation in Section M-3.7.2 (or Section 2.1.1). The solution is determined in the same way as the solution of the S-expansion system is done in Section 2.1.1. What should be noted is the determination of $P_1, P_2, \cdots$ in a bounded-domain problem. They are determined by the condition of invariance of the mass of the gas in the domain with the aid of higher-order equations in the same way as $P_{S1}, P_{S2}, \cdots$ in the S-expansion system (see Section 2.1.1).

In order to compare Eqs. (91a)–(91c) and (93) with the incompressible Navier–Stokes equations (81a)–(81d), we will rewrite the latter equations for the situation where the former equations are derived. The starting equations are Eqs. (82)–(85b)[16] and the nondimensional form of Eq. (62), i.e.,

$$\mathrm{Sh}\frac{\partial \omega}{\partial \hat{t}} + u_i\frac{\partial \omega}{\partial x_i} = 0, \tag{94}$$

instead of Eq. (86).[17] The analysis is carried out in a similar way and the

[16] As the internal energy e, $3RT/2$ $[= 3RT_0(1+\tau)/2]$ is chosen for consistency.
[17] From Eqs. (82) and (94), we have $\partial u_i/\partial x_i = 0$.

equations corresponding to Eqs. (91a)–(91c) are[18]

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{95a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{95b}$$

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{95c}$$

Equations (95a) and (95b) are, respectively, of the same form as Eqs. (91a) and (91b). Equation (91c) is rewritten with the aid of Eqs. (90) and (93) as

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} - \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right) = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{96}$$

The difference of Eq. (91c) or (96) from Eq. (95c) is

$$\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i},$$

which vanishes for an incompressible fluid. The work W done on unit volume of fluid by pressure, given by $-p_0(2RT_0)^{1/2}\partial(1+P)u_i/\partial x_i$, is transformed with the aid of Eq. (92a) in the following way:

$$\begin{aligned}
\frac{W}{p_0(2RT_0)^{1/2}} &= -\frac{\partial(1+P)u_i}{\partial x_i} \\
&= -\frac{\partial u_{i1}}{\partial x_i}\varepsilon - \left(P_1\frac{\partial u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i}\right)\varepsilon^2 + \cdots \\
&= -\frac{\partial u_{i2}}{\partial x_i}\varepsilon^2 + \cdots \\
&= \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right)\varepsilon^2 + \cdots .
\end{aligned}$$

The work vanishes up to the order considered here for an incompressible fluid, because $\partial u_i/\partial x_i = 0$ and $\partial P_1/\partial x_i = 0$ (see Footnote 18). That is, Eq. (91c) differs from Eq. (95c) by the amount of the work done by pressure.

To summarize, the mass and momentum-conservation equations of the set derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small are of the same form as the corresponding equations for the incompressible Navier–Stokes set, but the energy-conservation equation differs by the work done by pressure.[19] Incidentally, we have already

[18] We also obtain $\partial P_1/\partial x_i = 0$.

[19] When the density ρ is uniform initially, for which ρ is a constant for an incompressible fluid, the viscosity and thermal conductivity are constants, and heat production by viscosity is neglected, Eqs. (95a)–(95c) can be compared directly with Eqs. (91a)–(91c) and (93), without carrying expansion, and the same results are obtained.

seen that the pressure in the incompressible Navier–Stokes set has ambiguity of an additive function of time irrespective of the size of the parameters in contrast to the pressure in the former set derived from the compressible Navier–Stokes set.

3 Chapter M-9

3.1 Processes of solution of the equations with the ghost effect of infinitesimal curvature (July 2007)

The way in which Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e), including the time-dependent case with the additional time-derivative terms given by Eq. (M-9.42) or the mathematical expressions next to Eq. (M-9.59), contain the pressure terms, $(\hat{p}_{\mathfrak{S}0}, \hat{p}_{\mathfrak{S}2})$ or (P_{01}, P_{02}, P_{20}), is different from the way in which the Navier–Stokes equations (M-3.265)–(M-3.266c) do the pressure terms, (P_{S1}, P_{S2}). In Section M-9.4, we consider the time-independent solution of Eqs. (M-9.49a)–(M-9.50e) [Eqs. (M-9.56)–(M-9.57d)] that is uniform with respect to $\bar{\chi}$. Here, it may be better to explain how a solution of Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e) in a general case or a time-dependent solution that depends on χ or $\bar{\chi}$ is obtained. Incidentally, the boundary conditions for the time-dependent case are derived in the same way as in Section M-3.7.3. Naturally from the derivation of the equations, the domain of a gas is in a straight pipe or channel of infinite length whose axis is in the x or χ direction.

3.1.1 Equations (M-9.33)–(M-9.39b):

Take Eqs. (M-9.33)–(M-9.39b) with the additional time-derivative terms given by Eq. (M-9.42), i.e.,[20]

$$\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial y} = \frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial z} = 0, \tag{97}$$

$$\frac{\partial \hat{\rho}_{\mathfrak{S}0}}{\partial \hat{t}} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{z\mathfrak{S}1}}{\partial z} = 0, \tag{98}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \hat{t}} + \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)& \\ = -\frac{1}{2}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} + \frac{1}{2}\frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right),&\end{aligned} \tag{99}$$

[20] Equation (M-9.33) is replaced by its equivalent form (97).

$$
\begin{aligned}
&\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \hat{t}} + \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} - \frac{1}{c^2}\hat{v}_{x\mathfrak{S}0}^2\right) \\
&\quad = -\frac{1}{2}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) \\
&\qquad + \frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] \\
&\qquad + \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right\} \\
&\qquad + \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)\right\}, \qquad (100)
\end{aligned}
$$

$$
\begin{aligned}
&\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \hat{t}} + \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) \\
&\quad = -\frac{1}{2}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial z} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) \\
&\qquad + \frac{1}{2}\frac{\partial}{\partial y}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) \\
&\qquad + \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]\right\} \\
&\qquad + \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]\right\}, \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{5\hat{\rho}_{\mathfrak{S}0}}{2}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \hat{t}} + \frac{5}{2}\hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) \\
&\quad - \frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}} - \hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} \\
&\qquad = \frac{5}{4}\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{5}{4}\frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) + \Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right], \qquad (102)
\end{aligned}
$$

and the subsidiary relations

$$
\hat{p}_{\mathfrak{S}0}(\chi, \hat{t}) = \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0}, \qquad (103a)
$$

$$\hat{p}^{c}_{\mathfrak{S}2} = \hat{p}_{\mathfrak{S}2} + \frac{2\Gamma_1}{3}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) + \frac{\Gamma_7}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]$$
$$+ \frac{2}{3\hat{p}_{\mathfrak{S}0}}\left[\frac{\partial}{\partial y}\left(\Gamma_3 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_3 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]$$
$$- \frac{2\Gamma_9}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right], \tag{103b}$$

where Γ_1, Γ_2, Γ_3, Γ_7, Γ_8, and Γ_9 are short forms of the functions $\Gamma_1(\hat{T}_{\mathfrak{S}0})$, $\Gamma_2(\hat{T}_{\mathfrak{S}0}), \ldots, \Gamma_9(\hat{T}_{\mathfrak{S}0})$ of $\hat{T}_{\mathfrak{S}0}$ defined in Section M-A.2.9.

Consider the solution of the initial and boundary-value problem of Eqs. (97)–(103b).

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p} = \hat{\rho}\hat{T}$) at time $\hat{t}$ be given; thus, $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{T}_{\mathfrak{S}0}$ ($\hat{p}_{\mathfrak{S}0}$) etc., including $\hat{p}_{\mathfrak{S}2}$, are given. Then $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t}$ at $\hat{t}$ are given by Eqs. (98)–(103b); thus, the future $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ (also $\hat{p}_{\mathfrak{S}0}$) are determined. However, the future $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z, as well as $\hat{p}_{\mathfrak{S}0}$ at $\hat{t}$, owing to Eq. (97). Taking this into account, we will discuss how the solution is obtained by this system consistently.

First, transform Eq. (102) with the aid of Eqs. (98) and (103a) in the following form:

$$\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}} = \mathcal{P}, \tag{104}$$

where

$$\mathcal{P} = -\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) - \hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi}$$
$$+ \frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right] + \frac{2}{3}\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]. \tag{105}$$

For $\hat{p}_{\mathfrak{S}0}$ to be independent of y and z [see Eq. (97)], $\mathcal{P}$ as well as the initial data of $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z. Noting that $\hat{p}_{\mathfrak{S}0}$ is independent of y and z, and taking the average of Eq. (105) over the cross section S of the pipe or channel,[21] we have another expression $\mathfrak{P}$ of $\mathcal{P}$, explicitly uniform with

[21] (i) In a channel, where the gas extends from $z = -\infty$ to $z = \infty$, the integral $\int_S A\mathrm{d}y\mathrm{d}z$ per unit length in z, per a period in z, etc. should be considered. Otherwise, it can be infinite.

(ii) Note that $\hat{v}_{y\mathfrak{S}1}n_y + \hat{v}_{z\mathfrak{S}1}n_z = 0$ on a simple boundary where $n_i = (0, n_y, n_z)$ is the normal to the boundary.

respect to y and z, i.e.,

$$\mathfrak{P} = -\frac{5}{3}\frac{\partial\overline{\hat{v}_{x\mathfrak{S}0}}}{\partial\chi}\hat{p}_{\mathfrak{S}0} - \overline{\hat{v}_{x\mathfrak{S}0}}\frac{\partial\hat{p}_{\mathfrak{S}0}}{\partial\chi} + \frac{5}{6}\overline{\left[\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]}$$
$$+ \frac{2}{3}\Gamma_1\overline{\left[\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]}, \tag{106}$$

where

$$\overline{A} = \int_S A\mathrm{d}y\mathrm{d}z \Big/ \int_S \mathrm{d}y\mathrm{d}z.$$

The expression (106) is noted to be independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$. The two expressions (105) and (106) must give the same result, i.e.,

$$\mathcal{P} = \mathfrak{P},$$

or

$$-\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial\chi} + \frac{\partial\hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial\hat{v}_{z\mathfrak{S}1}}{\partial z}\right) - \hat{v}_{x\mathfrak{S}0}\frac{\partial\hat{p}_{\mathfrak{S}0}}{\partial\chi}$$
$$+\frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right] + \frac{2}{3}\Gamma_1\left[\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]$$
$$= \mathfrak{P}, \tag{107}$$

when Eq. (97) holds, and vice versa. The condition (107) for all $\hat{t}$ is equivalently replaced by the two conditions that the initial data of $\hat{p}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ satisfy Eqs. (97) and (107) and that the time derivative of Eq. (107) holds for all $\hat{t}$, i.e.,

$$\frac{\partial\mathcal{P}}{\partial\hat{t}} = \frac{\partial\mathfrak{P}}{\partial\hat{t}}. \tag{108}$$

Using Eqs. (98)–(101) and (104) for $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ ($\hat{\rho}_{\mathfrak{S}0}\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t} = \partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t} - \hat{T}_{\mathfrak{S}0}\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$) in $\partial\mathcal{P}/\partial\hat{t}$ derived from Eq. (105), we find that $\partial\mathcal{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ in the form

$$\frac{\partial\mathcal{P}}{\partial\hat{t}} = \frac{5}{6}\hat{p}_{\mathfrak{S}0}\left[\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}}{\partial z}\right)\right] + \mathrm{Fn}_1, \tag{109}$$

where Fn_1 is a given function of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. The expression (106) of $\mathfrak{P}$ being independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$, its time derivative $\partial\mathfrak{P}/\partial\hat{t}$ does not contain $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$ and $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$. Therefore, with the aid of Eqs. (98), (99), and (102), $\partial\mathfrak{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives, i.e.,

$$\frac{\partial\mathfrak{P}}{\partial\hat{t}} = \mathrm{Fn}_2(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{p}_{\mathfrak{S}0}, \text{ and their space derivatives}), \tag{110}$$

where Fn_2 is a given functional of its arguments. From Eqs. (108), (109), and (110), we have

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial z}\right)=\mathrm{Fn}, \tag{111}$$

where $\mathrm{Fn}=6(\mathrm{Fn}_2-\mathrm{Fn}_1)/5\hat{p}_{\mathfrak{S}0}$, and therefore, Fn is a given functional of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. This is the equation for $\hat{p}^c_{\mathfrak{S}2}$ over a cross section of the pipe or channel.

The boundary condition for $\hat{p}^c_{\mathfrak{S}2}$ on a simple boundary is obtained by multiplying Eqs. (99)–(101) by the normal $n_i=(0,n_y,n_z)$ to the boundary; In this process, the contribution of their time-derivative terms vanishes because $\hat{v}_{y\mathfrak{S}1}n_y+\hat{v}_{z\mathfrak{S}1}n_z=0$; Then, the $n_y\partial\hat{p}^c_{\mathfrak{S}2}/\partial y+n_z\partial\hat{p}^c_{\mathfrak{S}2}/\partial z$ is imposed as the boundary condition. Thus, $\hat{p}^c_{\mathfrak{S}2}$ is determined by Eq. (111) except for an additive function of $\hat{t}$ and χ. With this $\hat{p}^c_{\mathfrak{S}2}$ substituted into Eqs. (100) and (101), $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ are determined by Eqs. (98)–(103b) independently of the additive function in $\hat{p}^c_{\mathfrak{S}2}$ in such a way that $\partial(\partial\hat{p}_{\mathfrak{S}02}/\partial y)/\partial\hat{t}=\partial(\partial\hat{p}_{\mathfrak{S}0}/\partial z)/\partial\hat{t}=0$ and $\partial(\partial\mathcal{P}/\partial y)/\partial\hat{t}=\partial(\partial\mathcal{P}/\partial z)/\partial\hat{t}=0$. That is, the solution $(\hat{\rho}_{\mathfrak{S}0},\hat{v}_{x\mathfrak{S}0},\hat{v}_{y\mathfrak{S}1},\hat{v}_{z\mathfrak{S}1},\hat{T}_{\mathfrak{S}0})$ of Eqs. (97)–(103b) is determined by Eqs. (98)–(103b) with the aid of the supplementary condition (111), instead of Eq. (97), when the initial condition for $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ is given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(=\hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.

Equations (97)–(103b) are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. The equation for $\partial\hat{p}_{\mathfrak{S}2}/\partial\hat{t}$, corresponding to Eq. (104), is derived at the order after next. However, owing to the consistency of $\hat{p}_{\mathfrak{S}0}$, $\hat{p}_{\mathfrak{S}2}$ is already determined by Eq. (111) except for an additive function of χ and $\hat{t}$. The situation is similar to that at the leading order. That is, $\hat{p}_{\mathfrak{S}0}$ and $\hat{p}_{\mathfrak{S}2}$ are, respectively, determined by Eqs. (97) and (111), each with an additive function of χ and $\hat{t}$ and also by Eqs. (104) and the counterpart of Eq. (104) at the order after next. Thus, the higher-order analysis can be carried out in a similar way. The results are that an additional initial condition and an equation for $\hat{p}_{\mathfrak{S}4}$, the counter part of Eq. (111), are introduced and that the condition (111) is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ (thus also $\hat{p}_{\mathfrak{S}0}$).

To summarize, the solution $(\hat{\rho}_{\mathfrak{S}0},\hat{v}_{x\mathfrak{S}0},\hat{v}_{y\mathfrak{S}1},\hat{v}_{z\mathfrak{S}1},\hat{T}_{\mathfrak{S}0})$ of Eqs. (97)–(103b) is determined by Eqs. (98)–(103b) with the aid of the supplementary condition (111), instead of Eq. (97), when the initial data of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ are given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(=\hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.[22] The results are not affected by the higher-order analysis.

[22] If $\mathcal{P}$ is independent of y and z, $\mathcal{P}=\mathfrak{P}$ by definition.

3.1.2 Equations (M-9.49a)–(M-9.50e):

Take Eqs. (M-9.49a)–(M-9.50e) with the additional time-derivative terms given in the first mathematical expressions after Eq. (M-9.59), i.e.,

$$\frac{\partial P_{01}}{\partial \bar{\chi}} = \frac{\partial P_{01}}{\partial y} = \frac{\partial P_{01}}{\partial z} = 0, \quad P_{01} = \omega + \tau, \tag{112a}$$

$$\frac{\partial P_{02}}{\partial y} = \frac{\partial P_{02}}{\partial z} = 0, \tag{112b}$$

$$\frac{\partial u_x}{\partial \bar{\chi}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{113a}$$

$$\frac{\partial u_x}{\partial \hat{t}} + u_x \frac{\partial u_x}{\partial \bar{\chi}} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{2}\frac{\partial P_{02}}{\partial \bar{\chi}} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right), \tag{113b}$$

$$\frac{\partial u_y}{\partial \hat{t}} + u_x \frac{\partial u_y}{\partial \bar{\chi}} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} - \frac{u_x^2}{C^2} = -\frac{1}{2}\frac{\partial P_{20}}{\partial y} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right), \tag{113c}$$

$$\frac{\partial u_z}{\partial \hat{t}} + u_x \frac{\partial u_z}{\partial \bar{\chi}} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{2}\frac{\partial P_{20}}{\partial z} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right), \tag{113d}$$

$$\frac{\partial \tau}{\partial \hat{t}} - \frac{2}{5}\frac{\partial P_{01}}{\partial \hat{t}} + u_x \frac{\partial \tau}{\partial \bar{\chi}} + u_y \frac{\partial \tau}{\partial y} + u_z \frac{\partial \tau}{\partial z} = \frac{\gamma_2}{2}\left(\frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2}\right). \tag{113e}$$

The qualitative difference of this set of equations from the set (97)–(103b) is the absence of the time-derivative term in Eq. (113a) that corresponds to Eq. (98).

Consider the solution of the initial and boundary-value problem of Eqs. (112a)–(113e). Let u_x, u_y, u_z, and τ at $\hat{t}$ be given in such a way that Eq. (113a) is satisfied. Integrating Eq. (113a) over the cross section of the channel or pipe $\left[\int_S \text{Eq. (113a)} \mathrm{d}y\mathrm{d}z\right]$, we find that $\int_S u_x \mathrm{d}y\mathrm{d}z$ depends only on $\hat{t}$,[23] i.e.,

$$\int_S (\partial u_x/\partial \bar{\chi}) \mathrm{d}y\mathrm{d}z = 0, \tag{114}$$

where S indicates the cross section. Applying Eqs. (112b), (113a), and (114) to the equation $\partial \int \text{Eq. (113b)} \mathrm{d}y\mathrm{d}z/\partial \bar{\chi}$, we have $\partial^2 P_{02}/\partial \bar{\chi}^2$ as

$$\frac{\partial^2 P_{02}}{\partial \bar{\chi}^2} = \frac{\partial}{\partial \bar{\chi}} \overline{\left[-2\frac{\partial u_x^2}{\partial \bar{\chi}} + \gamma_1 \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\right]}, \tag{115}$$

where

$$\overline{A} = \int_S A \mathrm{d}y\mathrm{d}z \Big/ \int_S \mathrm{d}y\mathrm{d}z.$$

Thus, $\partial P_{02}/\partial \bar{\chi}$ and P_{02} are determined if they are specified at a point in the gas. Here, we consider this case.[24] Using Eq. (113a) in the sum of $\partial[\text{Eqs. (113b)}]/\partial \bar{\chi}$,

[23] See Footnote 21, with $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$ being replaced by u_y and u_z.

[24] (i) Imagine the case of the Poiseuille flow.
(ii) Here, P (thus, P_{01}) is specified at some point. Then, P_{01} is a given function of $\hat{t}$.

∂[Eqs. (113c)]$/\partial y$, and ∂[Eqs. (113d)]$/\partial z$, we obtain the equation for P_{20} in the form

$$\frac{\partial^2 P_{20}}{\partial y^2} + \frac{\partial^2 P_{20}}{\partial z^2} = \mathrm{Fn}(u_x, u_y, u_z, \text{ and their space derivatives}), \tag{116}$$

where Fn is a given functional of the variables in the parentheses, and the time derivatives are absent owing to Eq. (113a). Thus, the right-hand side of Eq. (116) is known. This equation is the Poisson equation for P_{20} over the cross section S. Its boundary condition is obtained in a way similar to how the condition for $\hat{p}^c_{\mathfrak{S}2}$ in Eq. (111) is derived. Thus, P_{20} over each cross section is determined except for an additive function of $\hat{t}$ and $\bar{\chi}$. This ambiguity does not influence $\partial P_{20}/\partial y$ and $\partial P_{20}/\partial z$.

With P_{02} and P_{20} prepared above into Eqs. (113b)–(113e), the time derivatives $\partial u_x/\partial\hat{t}$, $\partial u_y/\partial\hat{t}$, $\partial u_z/\partial\hat{t}$, and $\partial\tau/\partial\hat{t}$ are determined in such a way that $\partial(\partial u_x/\partial\bar{\chi} + \partial u_y/\partial y + \partial u_z/\partial z)/\partial\hat{t} = 0$ owing to the above choice of P_{20}.[25] Thus, the solution (u_x, u_y, u_z, τ) of Eqs. (112b), (113a)–(113e) is determined by Eqs. (113b)–(113e) with the aid of the supplementary conditions (115) and (116) for P_{02} and P_{20}, instead of Eqs. (112b) and (113a). This process is natural for numerical computation. The undetermined additive function of $\bar{\chi}$ and $\hat{t}$ in P_{20}, which does not affect the solution (u_x, u_y, u_z, τ), is determined by the higher-order equation derived from that for $\partial\hat{v}_{x\mathfrak{S}2}/\partial\hat{t}$ (see Section 3.1.1), in a way similar to that in which P_{02} is determined by Eq. (113b). In the higher-order equation, P_{20} plays the same role as P_{02} in Eq. (113b); Equation (116) corresponds to Eq. (112b), and P_{20} and P_{02} are determined by these equations, each with an additive function of $\bar{\chi}$ and $\hat{t}$.

4 Appendix M-A

4.1 Parity of the collision integral: Supplement to M-A 2.7

In Appendix M-A 2.7, we discussed the parity of the linearized collision integral. It may be better to explain a similar property of the collision integral defined by Eq. (M-1.9), i.e.,

$$\begin{aligned}
&\hat{J}(\hat{f}, \hat{g}) = \frac{1}{2}\int (\hat{f}'\hat{g}'_* + \hat{f}'_*\hat{g}' - \hat{f}\hat{g}_* - \hat{f}_*\hat{g})\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{\zeta}_*, \\
&\widehat{B} = \widehat{B}(|\boldsymbol{\alpha}\cdot\boldsymbol{V}|/|\boldsymbol{V}|, |\boldsymbol{V}|), \\
&\hat{f} = \hat{f}(\zeta_i),\ \hat{f}_* = \hat{f}(\zeta_{i*}),\ \hat{f}' = \hat{f}(\zeta'_i),\ \hat{f}'_* = \hat{f}(\zeta'_{i*}), \\
&\text{and a similar notation for } \hat{g},\ \hat{g}_*,\ \hat{g}',\ \text{and } \hat{g}'_*, \\
&\zeta'_i = \zeta_i + \alpha_j V_j \alpha_i,\ \zeta'_{i*} = \zeta_{i*} - \alpha_j V_j \alpha_i,\ \zeta_{i*} = V_i + \zeta_i.
\end{aligned} \tag{117}$$

[25] Note that P_{01} is known (Footnote 24).

Here, we discuss the relation of the parity of $\hat{J}(\hat{f},\hat{g})$ with respect to a component (ζ_1, ζ_2, or ζ_3) of the variable $\boldsymbol{\zeta}$ to that of $\hat{f}$ and $\hat{g}$. Put the integral (117) in the sum

$$\hat{J}(\hat{f},\hat{g}) = \frac{1}{2}\left(I\!V + I\!I\!I - I\!I - I\right), \tag{118}$$

where

$$I = \int \hat{f}_* \hat{g} \widehat{B}\, \mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V}, \tag{119a}$$

$$I\!I = \int \hat{f} \hat{g}_* \widehat{B}\, \mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V}, \tag{119b}$$

$$I\!I\!I = \int \hat{f}'_* \hat{g}' \widehat{B}\, \mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V}, \tag{119c}$$

$$I\!V = \int \hat{f}' \hat{g}'_* \widehat{B}\, \mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V}, \tag{119d}$$

and discuss each term separately.[26] In Eqs. (119a)–(119d), the variable of integration is changed from $\boldsymbol{\zeta}_*$ to $\boldsymbol{V}$ $(= \boldsymbol{\zeta}_* - \boldsymbol{\zeta})$. The following change of the variables

$$\widetilde{V}_1 = -V_1, \quad \widetilde{V}_s = V_s, \quad \widetilde{\alpha}_1 = -\alpha_1, \quad \widetilde{\alpha}_s = \alpha_s \quad (s = 2, 3) \tag{120}$$

is performed in the integrals I, $I\!I$, $I\!I\!I$, and $I\!V$. Noting that

$$\zeta_{i*} = V_i + \zeta_i, \qquad |\widetilde{V}_i| = |V_i|, \qquad \widetilde{\alpha}_i \widetilde{V}_i = \alpha_i V_i, \tag{121}$$

we can transform the integrals I, $I\!I$, $I\!I\!I$, and $I\!V$ in the following way, where the subscript s indicates $s = 2$ and 3:

$$\begin{aligned} I(\zeta_1,\zeta_s) &= \int \hat{f}(V_1+\zeta_1, V_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\widehat{B}\left(|\alpha_i V_i|/|V_i|, |V_i|\right)\mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V} \\ &= \int \hat{f}(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\, \widehat{B}\left(|\widetilde{\alpha}_i \widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\, \mathbf{d}\widetilde{\boldsymbol{V}}; \end{aligned} \tag{122a}$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in I, we have

$$I\!I(\zeta_1,\zeta_s) = \int \hat{f}(\zeta_1,\zeta_s)\hat{g}(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\, \widehat{B}\left(|\widetilde{\alpha}_i \widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\, \mathbf{d}\widetilde{\boldsymbol{V}}; \tag{122b}$$

$$\begin{aligned} I\!I\!I(\zeta_1,\zeta_s) &= \int \hat{f}(V_i+\zeta_i-\alpha_j V_j \alpha_i)\hat{g}(\zeta_i+\alpha_j V_j \alpha_i)\widehat{B}\left(|\alpha_i V_i|/|V_i|, |V_i|\right)\mathrm{d}\Omega(\boldsymbol{\alpha})\, \mathbf{d}\boldsymbol{V} \\ &= \int \hat{f}(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j \widetilde{V}_j \widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j \widetilde{V}_j \widetilde{\alpha}_s) \\ &\qquad \times \hat{g}(\zeta_1-\widetilde{\alpha}_j \widetilde{V}_j \widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j \widetilde{V}_j \widetilde{\alpha}_s)\widehat{B}\left(|\widetilde{\alpha}_i \widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\, \mathbf{d}\widetilde{\boldsymbol{V}}; \end{aligned} \tag{122c}$$

[26] The separation is made only for convenience of explanation.

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III, we have

$$\begin{aligned}\mathit{IV}(\zeta_1,\zeta_s) = \int \hat{f}(\zeta_1 - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\ \times \hat{g}(-\widetilde{V}_1 + \zeta_1 + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s + \zeta_s - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\ \times \widehat{B}\left(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}. \end{aligned} \tag{122d}$$

Now we examine the parity of the integrals I, II, III, and IV with respect to ζ_1 on the basis of Eqs. (122a)–(122d). Here, we introduce the notation: (i) the parity of $\hat{f}$ (or $\hat{g}$) is indicated by the subscript attached to it, i.e., the subscript E is attached when it is even and the subscript O when it is odd; (ii) the first subscript of I, II, III, and IV indicates the parity of $\hat{f}$ in them and the second indicates the parity of $\hat{g}$. First, when $\hat{f}$ and $\hat{g}$ are even functions of ζ_1.

$$\begin{aligned} \mathit{I}_{EE}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1 + \zeta_1, \widetilde{V}_s + \zeta_s)\hat{g}_E(\zeta_1,\zeta_s) \\ &\quad \times \widehat{B}\left(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \int \hat{f}_E(\widetilde{V}_1 - \zeta_1, \widetilde{V}_s + \zeta_s)\hat{g}_E(-\zeta_1,\zeta_s) \\ &\quad \times \widehat{B}\left(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|\right)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \mathit{I}_{EE}(-\zeta_1,\zeta_s), \end{aligned} \tag{123a}$$

where the last relation holds owing to the first relation of Eq. (122a); Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in I_{EE}, we have

$$\mathit{II}_{EE}(\zeta_1,\zeta_s) = \mathit{II}_{EE}(-\zeta_1,\zeta_s); \tag{123b}$$

$$\begin{aligned} \mathit{III}_{EE}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1 + \zeta_1 + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s + \zeta_s - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\ &\quad \times \hat{g}_E(\zeta_1 - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \int \hat{f}_E(\widetilde{V}_1 - \zeta_1 - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s + \zeta_s - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\ &\quad \times \hat{g}_E(-\zeta_1 + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \mathit{III}_{EE}(-\zeta_1,\zeta_s); \end{aligned} \tag{123c}$$

Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in III_{EE}, we have

$$\mathit{IV}_{EE}(\zeta_1,\zeta_s) = \mathit{IV}_{EE}(-\zeta_1,\zeta_s). \tag{123d}$$

When both $\hat{f}$ and $\hat{g}$ are odd with respect to ζ_1,

$$\begin{aligned} \mathit{I}_{OO}(\zeta_1,\zeta_s) &= \int \hat{f}_O(-\widetilde{V}_1 + \zeta_1, \widetilde{V}_s + \zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \int \hat{f}_O(\widetilde{V}_1 - \zeta_1, \widetilde{V}_s + \zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\ &= \mathit{I}_{OO}(-\zeta_1,\zeta_s); \end{aligned} \tag{124a}$$

Interchanging the arguments of $\hat{f}_O$ and $\hat{g}_O$ in II_{OO}, we have

$$II_{OO}(\zeta_1,\zeta_s) = II_{OO}(-\zeta_1,\zeta_s); \tag{124b}$$

$$\begin{aligned}
III_{OO}(\zeta_1,\zeta_s) &= \int \hat{f}_O(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= \int \hat{f}_O(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= III_{OO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{124c}$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III_{OO}, we have

$$IV_{OO}(\zeta_1,\zeta_s) = IV_{OO}(-\zeta_1,\zeta_s). \tag{124d}$$

When $\hat{f}$ is even and $\hat{g}$ is odd with respect to ζ_1,

$$\begin{aligned}
I_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(\widetilde{V}_1-\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -I_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{125a}$$

$$\begin{aligned}
II_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(\zeta_1,\zeta_s)\hat{g}_O(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(-\zeta_1,\zeta_s)\hat{g}_O(\widetilde{V}_1-\zeta_1, \widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -II_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{125b}$$

$$\begin{aligned}
III_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -III_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{125c}$$

$$
\begin{aligned}
\mathit{IV}_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(\zeta_1 - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(-\widetilde{V}_1 + \zeta_1 + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s + \zeta_s - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(-\zeta_1 + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s + \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(\widetilde{V}_1 - \zeta_1 - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s + \zeta_s - \widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\mathit{IV}_{EO}(-\zeta_1,\zeta_s). \qquad (125\mathrm{d})
\end{aligned}
$$

For I_{OE}, II_{OE}, III_{OE}, and IV_{OE}, interchanging the role of $\hat{f}$ and $\hat{g}$, respectively, in II_{EO}, I_{EO}, IV_{EO}, and III_{EO}, we have

$$
\begin{aligned}
I_{OE}(\zeta_1,\zeta_s) &= -I_{OE}(\zeta_1,\zeta_s), && (126\mathrm{a}) \\
\mathit{II}_{OE}(\zeta_1,\zeta_s) &= -\mathit{II}_{OE}(\zeta_1,\zeta_s), && (126\mathrm{b}) \\
\mathit{III}_{OE}(\zeta_1,\zeta_s) &= -\mathit{III}_{OE}(\zeta_1,\zeta_s), && (126\mathrm{c}) \\
\mathit{IV}_{OE}(\zeta_1,\zeta_s) &= -\mathit{IV}_{OE}(\zeta_1,\zeta_s). && (126\mathrm{d})
\end{aligned}
$$

The parity is common to I, II, III, and IV. Therefore, the parity of $\hat{J}(\hat{f},\hat{g})$ is the same as I, i.e.,

$$
\begin{aligned}
\hat{J}(\hat{f}_E,\hat{g}_E)(\zeta_1,\zeta_s) &= \hat{J}(\hat{f}_E,\hat{g}_E)(-\zeta_1,\zeta_s), && (127\mathrm{a}) \\
\hat{J}(\hat{f}_O,\hat{g}_O)(\zeta_1,\zeta_s) &= \hat{J}(\hat{f}_O,\hat{g}_O)(-\zeta_1,\zeta_s), && (127\mathrm{b}) \\
\hat{J}(\hat{f}_E,\hat{g}_O)(\zeta_1,\zeta_s) &= -\hat{J}(\hat{f}_E,\hat{g}_O)(-\zeta_1,\zeta_s), && (127\mathrm{c}) \\
\hat{J}(\hat{f}_O,\hat{g}_E)(\zeta_1,\zeta_s) &= -\hat{J}(\hat{f}_O,\hat{g}_E)(-\zeta_1,\zeta_s). && (127\mathrm{d})
\end{aligned}
$$

Obviously, the same parity holds for the other components, i.e., ζ_2, ζ_3, of $\boldsymbol{\zeta}$.

(Section 4.1: Version 4-00)