Supplement to Kinetic Theory and Fluid Dynamics

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Bibliography

Update of bibliography

- [161] Sone, Y., M. Handa, and H. Sugimoto [2002]:
    (to be published)   →   299–332
- p. 151, the eighth line from the bottom:
    Sone, Handa & Doi [unpublished] → Sone, Handa & Doi [2003]
    The following reference is to be added to Bibliography:
Sone, Y., M. Handa, and T. Doi (2003), Ghost effect and bifurcation in a gas between coaxial circular cylinder with different temperatures, *Phys. Fluids* **15**, 2903–2915.
- p. 159, the sixth line
    Sone & Doi [unpublished] → Sone & Doi [2003]
    The following reference is to be added to Bibliography:
Sone, Y. and T. Doi (2003), Bifurcation of and ghost effect on temperature field in the Bénard problem of a gas in the continuum limit, *Phys. Fluids* **15**, 1405–1423.

Errata

- p. 11, the 10th line

specular condition → specular reflection

- p. 33, the first line in Eq. (3.31c):

$$\frac{\partial S_{ijG0}}{\partial x_i} \quad \rightarrow \quad \frac{\partial S_{ijG0}}{\partial x_j}$$

- p. 33, the first line in Eq. (3.31d):

$$\frac{\partial S_{ijG1}}{\partial x_i} \quad \rightarrow \quad \frac{\partial S_{ijG1}}{\partial x_j}$$

- p. 36, Footnote 7:

Footnote 7 should be replaced by the following statement:

In some cases, the boundary data u_{wi}, τ_w, or ω_w (thus, P_w) are unknown beforehand and may depend on the Knudsen number, for example, the surface temperature of a particle set freely in a gas. To include the case, the data u_{wi}, τ_w, ω_w, and P_w also are expanded in power series of k in the following discussion. Even when they are independent of k, the ϕ_w depends on k because ϕ_w generally depends on ϕ ($\zeta_i n_i < 0$).

- p. 104, the second line in Footnote 18:

only Ω_8 and Θ_8 → (Ω_8, Θ_8), (Ω_9, Θ_9), and (Ω_{10}, Θ_{10})

- p. 146, the fifth line:

$\hat{Y}_1(\tilde{\eta}) = \hat{Y}_1(\tilde{\eta})$ → $\hat{Y}_1(\tilde{\eta}) = Y_1(\tilde{\eta})$

- p. 177, Eq. (6.53):

The following note should be made to Eq. (6.53):

Equation (6.54), which is derived from Eqs. (6.50), (6.52), and $\hat{v}_{wi} n_i = 0$, is used in the derivation of Eq. (6.53) from the solvability condition. See the second paragraph in page 130 of Y. Sone, Molecular Gas Dynamics (Birkhäuser, Boston, 2007).

To improve the indefinite expression, the third paragraph on p. 177, which starts from its 9th line from the bottom and ends at the second line on p. 178, should be replaced by the following paragraph:[1]

Now return to the discussion of $\hat{f}_{Vm}$. The first relation of the solvability condition (6.48) with $m = 1$, where

$$Ih_1 = \zeta_i n_i(\partial \hat{f}_{V0}/\partial y), \tag{6.51}$$

reduces to

$$\frac{\partial(\hat{\rho}_{V0} \hat{v}_{iV0} n_i)}{\partial y} = 0, \tag{6.52}$$

from which with the aid of Eqs. (6.50) and $\hat{v}_{wi} n_i = 0$, we have

$$\hat{v}_{iV0} n_i = 0. \tag{6.54}$$

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[1] In the new paragraph, the numbering (6.54) comes earlier than (6.53) to avoid the correction of the quotations in the rest of the book.

With the use of Eq. (6.54), the scalar product of the second relation of Eq. (6.48) and n_i reduces to

$$\partial \hat{p}_{V0}/\partial y = 0. \tag{6.53}$$

Thus, $\hat{p}_{V0}$ is a function of χ_1 and χ_2 only, i.e.,

$$\hat{p}_{V0} = \hat{p}_{V0}(\chi_1, \chi_2). \tag{6.55}$$

The remaining three relations [or the remaining two of the second relation of Eq. (6.48) and its third relation] reduce to identities with the aid of Eq. (6.54).

- p.215, the third line in Footnote 6:

    $M_n \quad \rightarrow \quad -M_n$

- p. 220, the fifth line:

    $F_s \quad \rightarrow \quad -M_n F_s$

    $F_b \quad \rightarrow \quad -M_n F_b$

Supplementary Notes

In the present supplementary notes, the letter K is attached to the labels of equations and sections, etc. in the book *Kinetic Theory and Fluid Dynamics* and the letter M is attached to those in Y. Sone, *Molecular Gas Dynamics* (Birkhäuser, Boston, 2007) to avoid confusion. The two books, *Kinetic Theory and Fluid Dynamis* and *Molecular Gas Dynamics*, themselves are, respectively, referred to as KF and MGD.

1 Chapter K-2

1.1 Bulk viscosity

The assumptions (K-2.14) and (K-2.15) for the stress tensor and heat-flow vector in classical gas dynamics are what is to be studied by kinetic theory (see KF). For a monatomic gas, consisting of identical molecules whose intermolecular potential is spherically symmetric, which is discussed in KF, the bulk viscosity is easily seen to vanish. From Eqs. (K-2.2d) and (K-2.2f),[2]

$$p_{ii} = 3p. \tag{1}$$

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[2] For molecules with internal degree of freedom (e.g., rotational and vibrational freedoms), this freedom contributes to the integrands of Eqs. (K-2.2c)–(K-2.2g). Thus, Eq. (1) does not generally hold. (More precisely, the velocity distribution function f depends also on the variables of the internal degree of freedom of a molecule. The integration with respect to these variables in Eqs. (K-2.2a)–(K-2.2g) has to be carried out. The angular momentum due to the rotation of molecules of infinitesimal size per unit mass is negligible even when the energy of rotation is not negligible.) The density ρ and the specific internal energy e can be clearly defined whether the gas is in an equilibrium state or not. The specific internal energy e/i_f per unit freedom of a molecule is taken as $RT/2$, i.e., $e = i_f RT/2$, where i_f is the degree of freedom of a molecule; thus, the relation between e and T is independent of the state of

On the other hand, the trace of the first relation of Eq. (K-2.15) is

$$p_{ii} = 3p - 3\mu_B \frac{\partial v_i}{\partial X_i}.$$

Thus, from the two relations, we have

$$\mu_B = 0. \tag{2}$$

<div align="right">(Section 1.1: Version 3-00)</div>

2   Chapter K-4

2.1   Notes on basic equations in classical fluid dynamics

2.1.1   Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = 0, \tag{4}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}v_i^2\right)\right] + \frac{\partial}{\partial X_j}\left[\rho v_j\left(e + \frac{1}{2}v_i^2\right) + v_i p_{ij} + q_j\right] = 0, \tag{5}$$

where ρ is the density, v_i is the flow velocity, e is the internal energy per unit mass, p_{ij}, which is symmetric with respect to i and j, is the stress tenor, and q_i is the heat-flow vector. The pressure p and the internal energy e are given by the equations of state as functions of T and ρ, i.e.,

$$p = p(T, \rho), \quad e = e(T, \rho). \tag{6}$$

Especially, for a perfect gas,

$$p = R\rho T, \quad e = e(T). \tag{7}$$

Equations (4) and (5) are rewritten with the aid of Eq. (3) in the form

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0, \tag{8}$$

$$\rho \frac{\partial}{\partial t}\left(e + \frac{1}{2}v_i^2\right) + \rho v_j \frac{\partial}{\partial X_j}\left(e + \frac{1}{2}v_i^2\right) + \frac{\partial}{\partial X_j}(v_i p_{ij} + q_j) = 0. \tag{9}$$

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the gas (equilibrium or nonequilibrium). The pressure is defined by the equation of state, i.e., the perfect gas relation $p = \rho RT$; thus, except for a monatomic gas without internal degree of freedom, the pressure differs generally from the isotropic part of stress tensor in a nonequilibrium state.

The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $\mathrm{D}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$

Multiplying Eq. (8) by v_i we obtain the equation for the variation of kinetic energy as

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{1}{2} v_i^2 \right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \tag{10}$$

Another form of Eq. (5), where Eq. (10) is subtracted from Eq. (9), is given as

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p_{ij} \frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \tag{11}$$

Noting the thermodynamic relation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T \frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{12}$$

where s is the entropy per unit mass, and Eq. (3), Eq. (11) is rewritten as

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T} \left[(p_{ij} - p\delta_{ij}) \frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j} \right]. \tag{13}$$

Equation (13) expresses the variation of the entropy of a fluid particle.

Equations (3)–(6) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor p_{ij} and the heat-flow vector q_i are assumed in some ways. The *Navier–Stokes set of equations* (or the *Navier–Stokes equations*) is Eqs. (3)–(6) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) - \mu_B \frac{\partial v_k}{\partial X_k} \delta_{ij}, \tag{14}$$

$$q_i = -\lambda \frac{\partial T}{\partial X_i}, \tag{15}$$

where μ, μ_B, and λ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of T and ρ. The *Euler set of equations* (or the *Euler equations*) is Eqs. (3)–(6) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \tag{16}$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (14) and (15), the

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entropy variation is expressed in the form[3]

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T} \left[\frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \mu_B \left(\frac{\partial v_k}{\partial X_k} \right)^2 + \frac{\partial}{\partial X_i} \left(\lambda \frac{\partial T}{\partial X_i} \right) \right].$$
(17)

For the Euler equations, for which p_{ij} and q_i are given by Eq.(16), the entropy of a fluid particle is invariant, i.e.,

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = 0.$$
(18)

For an *incompressible fluid*, the first relation of Eq. (6) is replaced by[4]

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \ \text{ or } \ \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0.$$
(19)

Thus, from Eqs. (3) and (19),

$$\frac{\partial v_i}{\partial X_i} = 0.$$
(20)

Equation (14) for the Navier–Stokes-stress tensor reduces to

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right).$$
(21)

The first term on the right-hand side of Eq. (11) reduces to

$$-p_{ij} \frac{\partial v_i}{\partial X_j} = - \left[p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j}$$
$$= \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2.$$

Thus, Eq. (11) reduces to

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right).$$
(22)

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[3] Note the following transformation:

$$\frac{\partial v_i}{\partial X_j} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)$$
$$= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} + \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)$$
$$= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{3} \frac{\partial v_l}{\partial X_l} \delta_{ij} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right).$$

The second term in the last expression is easily seen to vanish.

[4] The density is invariant along fluid-particle paths. If ρ is of uniform value ρ_0 initially, it is a constant, i.e.,

$$\rho = \rho_0.$$

In a time-independent (or steady) problem, the density is constant along streamlines.

To summarize, the Navier–Stokes equations for incompressible fluid are

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{23a}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j}\left[\mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)\right], \tag{23b}$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda \frac{\partial T}{\partial X_j}\right), \tag{23c}$$

with the incompressible condition (19) being supplemented, i.e.,

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{24}$$

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2.1.2 Boundary condition for the Euler set

In Chapter K-7, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation equations of the Euler set are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{25}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{26}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{27}$$

where ρ is the density, which is uniform, (u, v) is the flow velocity, and p is the pressure. Owing to Eq. (25), the stream function Ψ can be introduced as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{28}$$

Eliminating p from Eqs. (26) and (27), we have[5]

$$u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = 0, \tag{29}$$

where Ω is the vorticity, i.e.,

$$\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}. \tag{30}$$

From Eqs. (28) and (29),

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Omega}{\partial y} = 0. \tag{31}$$

This equation shows that Ω is a function of Ψ,[6] i.e.,

$$\Omega = F(\Psi). \tag{32}$$

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[5] The following equation is formed from them:

$$\partial \mathrm{Eq.}\,(26)/\partial y - \partial \mathrm{Eq.}\,(27)/\partial x = 0.$$

[6] This can be seen with the aid of theorems on implicit functions (see Bronshtein & Semendyayev [1997], Buck [1965], Takagi [1961]):

$$\Omega = \Omega(x,y), \quad \Psi = \Psi(x,y). \tag{$*$}$$

Solving the second equation with respect to x, we have

$$x = \hat{x}(\Psi, y). \tag{$**$}$$

With this relation into Eq. ($*$),

$$\Omega = \Omega(\hat{x}(\Psi,y), y) = \hat{\Omega}(\Psi, y), \tag{$\sharp$a}$$

$$\Psi = \Psi(\hat{x}(\Psi,y), y) = \hat{\Psi}(\Psi, y). \tag{$\sharp$b}$$

That is, Ω is expressed as a function of Ψ and y. From Eqs. ($\sharp$a) and ($\sharp$b),

$$\frac{\partial \hat{\Omega}(\Psi,y)}{\partial y} = \frac{\partial \Omega(\hat{x}(\Psi,y),y)}{\partial y} = \frac{\partial \Omega(x,y)}{\partial x}\frac{\partial \hat{x}(\Psi,y)}{\partial y} + \frac{\partial \Omega(x,y)}{\partial y}, \tag{$\sharp\sharp$a}$$

$$\frac{\partial \hat{\Psi}(\Psi,y)}{\partial y} = 0. \tag{$\sharp\sharp$b}$$

On the other hand,

$$\frac{\partial \hat{\Psi}(\Psi,y)}{\partial y} = \frac{\partial \Psi(\hat{x}(\Psi,y),y)}{\partial y} = \frac{\partial \Psi(x,y)}{\partial x}\frac{\partial \hat{x}(\Psi,y)}{\partial y} + \frac{\partial \Psi(x,y)}{\partial y}.$$

Thus,

$$\frac{\partial \Psi(x,y)}{\partial x}\frac{\partial \hat{x}(\Psi,y)}{\partial y} + \frac{\partial \Psi(x,y)}{\partial y} = 0. \tag{$\ddagger$}$$

From Eqs. (31), ($\sharp\sharp$a) and ($\ddagger$), we have

$$\frac{\partial \hat{\Omega}(\Psi,y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \hat{\Omega}(\Psi).$$

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This functional relation between Ω and Ψ is a local relation, and therefore F may be a multivalued function of Ψ. From Eqs. (30) and (32),

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi). \tag{33}$$

Consider a boundary-value problem in a simply-connected bounded domain, where Ψ is given on the boundary ($\Psi = \Psi_B$). Introduce a coordinate s ($0 \le s < S$) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial \Psi_B / \partial s < 0$, and the fluid flows out from the domain on the boundary where $\partial \Psi_B / \partial s > 0$. When F is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose F on the part where $\partial \Psi_B / \partial s < 0$ or $\partial \Psi_B / \partial s > 0$. For example, take the case where $\partial \Psi_B / \partial s < 0$ for $0 < s < S_m$ and $\partial \Psi_B / \partial s > 0$ for $S_m < s < S$, and choose the distribution $\Omega_B(s)$ of Ω along the boundary for the part $0 < s < S_m$. By the choice of Ω_B, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi = \Psi_B(s)$ between Ψ and s on the part $0 < s < S_m$, i.e., $s(\Psi)$, and noting the relation (32), we find that F is given by

$$F(\Psi) = \Omega_B(s(\Psi)). \tag{34}$$

Then, the boundary-value problem is fixed. That is, Eq. (33) is fixed as[7]

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \Omega_B(s(\Psi)), \tag{35}$$

and the boundary condition is given as $\Psi = \Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (30), the solution of the above system automatically satisfies condition $\Omega = \Omega_B(s)$ along the boundary for $0 < s < S_m$. We cannot choose the distribution of Ω on the boundary for $S_m < s < S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (22) with $\mu = \lambda = 0$, i.e.,

$$u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} = 0, \quad \text{or} \quad \frac{\partial \Psi}{\partial y}\frac{\partial e}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial e}{\partial y} = 0, \tag{36}$$

where e is the internal energy. Thus, e is a function of Ψ, i.e.,

$$e = F_1(\Psi). \tag{37}$$

In the above boundary-value problem, therefore, e can be specified on the the part ($0 < s < S_m$) of the boundary, but no condition can be specified on other part ($S_m < s < S$) and vice versa.[8]

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[7] There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x, y) = $ const inside the domain is not excluded.

[8] From the second relation on e of Eq. (6) and the uniform-density condition, the condition on e can be replaced by the condition on the temperature T.

To summarize, we can specify three conditions for Ψ, Ω, and e on the part $\partial \Psi_B / \partial s < 0$ ($\partial \Psi_B / \partial s > 0$) of boundary but one condition for Ψ on the other part $\partial \Psi_B / \partial s > 0$ ($\partial \Psi_B / \partial s < 0$). The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

2.1.3 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set. This statement should be made precise. The difference is briefly explained in the book "Kinetic Theory and Fluid Dynamics" in connection with the equations derived by the S expansion from the Boltzmann equation in Section K-4.3. Here, we explicitly show the process of analysis from the compressible Navier–Stokes set. In Section K-4.3, the time-independent case is discussed. Here, we discuss the problem without this restriction. The following discussion is applied to the former case simply by eliminating the time-derivative terms.

Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq. (K-2.57) in Section K-2.10 as follows:

$$\mathrm{Sh}\frac{\partial \omega}{\partial \hat{t}} + \frac{\partial (1+\omega)u_i}{\partial x_i} = 0, \tag{38}$$

$$\mathrm{Sh}\frac{\partial (1+\omega)u_i}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left((1+\omega)u_iu_j + \frac{1}{2}P_{ij}\right) = 0, \tag{39}$$

$$\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left[(1+\omega)\left(\frac{3}{2}(1+\tau) + u_i^2\right)\right]$$
$$+ \frac{\partial}{\partial x_j}\left[(1+\omega)u_j\left(\frac{3}{2}(1+\tau) + u_i^2\right) + u_i(\delta_{ij} + P_{ij}) + Q_j\right] = 0. \tag{40}$$

The nondimensional stress tensor P_{ij}, and heat-flow vector Q_i are expressed as[9]

$$P_{ij} = P\delta_{ij} - \frac{\mu_0(2RT_0)^{1/2}}{p_0 L}(1+\bar{\mu})\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \tag{41a}$$

$$Q_i = -\frac{\lambda_0 T_0}{Lp_0(2RT_0)^{1/2}}(1+\bar{\lambda})\frac{\partial \tau}{\partial x_i}. \tag{41b}$$

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu = \mu_0(1+\bar{\mu}), \quad \lambda = \lambda_0(1+\bar{\lambda}),$$

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[9] For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$ (see Section 1.1).

where μ_0 and λ_0 are, respectively, the values of the viscosity μ and the thermal conductivity λ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of τ and ω. The first relation of the equation of state [Eq. (7)] is expressed as

$$P = \omega + \tau + \omega\tau. \tag{42}$$

Take a small parameter ε, and consider the case where

$$u_i = O(\varepsilon), \quad \omega = O(\varepsilon), \quad \tau = O(\varepsilon), \quad \mathrm{Sh} = O(\varepsilon), \tag{43a}$$

$$\frac{\mu_0(2RT_0)^{1/2}}{p_0 L} = \gamma_1\varepsilon, \quad \frac{\lambda_0 T_0}{Lp_0(2RT_0)^{1/2}} = \frac{5}{4}\gamma_2\varepsilon, \tag{43b}$$

thus,

$$P = O(\varepsilon), \quad \bar{\mu} = O(\varepsilon), \quad \bar{\lambda} = O(\varepsilon).$$

According to the definition of u_i in Eq. (K-2.57), ε is of the order of the Mach number. In view of this and the definition of the Prandtl number $\mathrm{Pr} = 5R\mu/2\lambda$ (see Section K-3.9), γ_1 and γ_2 are, respectively, of the orders of $1/\mathrm{Re}$ and $1/\mathrm{PrRe}$ (Re : the Reynolds number). According to Eq. (K-2.41a), the condition $\mathrm{Sh} = O(\varepsilon)$ in Eq. (43a) means that the time scale t_0 of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous diffusion. Thus, we are considering the case where the Mach number is small, the Reynolds and Prandtl numbers are of the order of unity, and the time scale of variation of the system is of the order of the time scale of viscous diffusion. We can take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality. Then,

$$\mathrm{Sh} = \varepsilon. \tag{44}$$

Corresponding to the above situation, u_i, ω, P, and τ are expanded in power series of ε, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots, \tag{45a}$$

$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots, \tag{45b}$$

$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots, \tag{45c}$$

$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots, \tag{45d}$$

$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots, \tag{45e}$$

$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots, \tag{45f}$$

$$P_{ij} = P_1\delta_{ij}\varepsilon + P_{ij2}\varepsilon^2 + \cdots, \tag{45g}$$

$$Q_i = Q_{i2}\varepsilon^2 + \cdots. \tag{45h}$$

Substituting Eqs. (45a)–(45h) with Eqs. (43b) and (44) into Eqs. (38)–(40) with Eqs. (41a) and (41b), and arranging the same order terms of ε, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \quad \frac{\partial P_1}{\partial x_i} = 0, \quad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

$$\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + \frac{\partial u_{i1} u_{j1}}{\partial x_j} + \frac{1}{2}\frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right) = 0,$$

$$\frac{3}{2}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left(\frac{5}{2}u_{j2} + \frac{5}{2}P_1 u_{j1} - \frac{5}{4}\gamma_2\frac{\partial \tau_1}{\partial x_j}\right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (38) and (40) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{46}$$

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{47a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{47b}$$

$$\frac{5}{2}\frac{\partial \tau_1}{\partial \hat{t}} - \frac{\partial P_1}{\partial \hat{t}} + \frac{5}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{47c}$$

$$\frac{\partial u_{i2}}{\partial x_i} = -\frac{\partial \omega_1}{\partial \hat{t}} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \tag{48a}$$

$$\frac{\partial u_{i2}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i2}}{\partial x_j} + u_{j2}\frac{\partial u_{i1}}{\partial x_j}$$
$$= -\frac{1}{2}\left(\frac{\partial P_3}{\partial x_i} - \omega_1\frac{\partial P_2}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right)$$
$$- \frac{\gamma_1 \omega_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left[\bar{\mu}_1\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)\right], \tag{48b}$$

$$\frac{3}{2}\frac{\partial P_2}{\partial \hat{t}} + \frac{3}{2}u_{j1}\frac{\partial P_2}{\partial x_j} + \frac{5}{2}\left(P_1\frac{\partial u_{j2}}{\partial x_j} - \frac{\partial \omega_2}{\partial \hat{t}} - \frac{\partial(\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j}\right)$$
$$= \frac{5\gamma_2}{4}\frac{\partial}{\partial x_i}\left(\frac{\partial \tau_2}{\partial x_i} + \bar{\lambda}_1\frac{\partial \tau_1}{\partial x_i}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)^2, \tag{48c}$$

where
$$P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1 \tau_1. \tag{49}$$

These equations are very similar to Eqs. (M-3.265)–(M-3.268) [or Eqs. (K-3.87)–(K-3.90) for the time-independent case] obtained by the S expansion of the Boltzmann equation in Section M-3.7.2 (or Section K-3.2.2 for the time independent case).

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In order to compare Eqs. (47a)–(47c) and (49) with the incompressible Navier–Stokes equations (23a)–(24), we will rewrite the latter equations for the situation where the former equations are derived. The starting equations are Eqs. (38)–(41b)[10] and the nondimensional form of Eq. (19), i.e.,

$$\mathrm{Sh}\frac{\partial \omega}{\partial \hat{t}} + u_i \frac{\partial \omega}{\partial x_i} = 0, \tag{50}$$

instead of Eq. (42).[11] The analysis is carried out in a similar way and the equations corresponding to Eqs. (47a)–(47c) are[12]

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{51a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{51b}$$

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{51c}$$

Equations (51a) and (51b) are, respectively, of the same form as Eqs. (47a) and (47b). Equation (47c) is rewritten with the aid of Eqs. (46) and (49) as

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} - \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right) = \frac{5}{4}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{52}$$

The difference of Eq. (47c) or (52) from Eq. (51c) is

$$\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i},$$

which vanishes for an incompressible fluid. The work W done per unit time on unit volume of fluid by pressure, given by $-p_0(2RT_0)^{1/2}L^{-1}\partial(1+P)u_i/\partial x_i$, is transformed with the aid of Eqs. (46), (47a), and (48a) in the following way:

$$\frac{W}{p_0(2RT_0)^{1/2}L^{-1}} = -\frac{\partial(1+P)u_i}{\partial x_i}$$

$$= -\frac{\partial u_{i1}}{\partial x_i}\varepsilon - \left(P_1\frac{\partial u_{i1}}{\partial x_i} + u_{i1}\frac{\partial P_1}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i}\right)\varepsilon^2 + \cdots$$

$$= -\frac{\partial u_{i2}}{\partial x_i}\varepsilon^2 + \cdots$$

$$= \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right)\varepsilon^2 + \cdots.$$

The work vanishes up to the order considered here for an incompressible fluid, because $\partial u_i/\partial x_i = 0$ and $\partial P_1/\partial x_i = 0$ (see Footnotes 11 and 12). That is,

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[10] As the internal energy e, $3RT/2$ $[= 3RT_0(1+\tau)/2]$ is chosen for consistency.
[11] From Eqs. (38) and (50), we have $\partial u_i/\partial x_i = 0$.
[12] We also obtain $\partial P_1/\partial x_i = 0$.

Eq. (47c) differs from Eq. (51c) by the amount of the work done by pressure. Thus, naturally, the temperature τ_1 fields in the two cases are different owing to this difference.

To summarize, the mass and momentum-conservation equations (47a) and (47b) of the set derived from the compressible Navier–Stokes set [Eqs. (38)–(41b) and (42)] under the situation given by Eqs. (43a) and (43b) with small ε are of the same form as those equations (51a) and (51b) of the corresponding set derived from the incompressible Navier–Stokes set [Eqs. (38)–(41b) and (50)], but the energy-conservation equations (47c) and (51c) of the two sets differ by the work done by pressure.[13] The density ω_1 obtained from Eqs. (46)–(47c) with the first relation of Eq. (49) does not generally satisfy the incompressible condition (50) with $\omega = \omega_1$ and $u_i = u_{i1}$.[14] Both the density and temperature fields (ω_1, τ_1) are different in the two sets. The variation of the density ω_1 along a particle path is due to the first relation of Eq. (49). Even if the temerature τ_1 varies according to Eq. (51c), the density ω_1 determined by the first relation of Eq. (49) does not generally satisfy the incompressible condition.

Finally, it may be noted that under the situation (43a), the solenoidal condition for u_{i1}, i.e., Eq. (47a) or (51a), is derived only from the mass conservation equation (38) without the help of the incompressible condition (50).

3 Appendix K-A

3.1 Boundary condition for Euler equations

In Appendix K-A.10, we discussed the boundary condition for the linearized Euler equations for simple examples. Related discussion is given in Section 2.1.2 in this notes.

4 Appendix K-C

4.1 Numerical procedure for discrete ξ_i and η (Supplement to Sections K-C.2 and K-C.3)

In Appendix K-C, a numerical method of solution of conservation equations, e.g., fluid-dynamic equations, by a kinetic equation is discussed. As noted there, the

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[13] When the density ρ is uniform initially, for which ρ is a constant for an incompressible fluid, the viscosity and thermal conductivity are constants, and heat production by viscosity is neglected, Eqs. (51a)–(51c) can be compared directly with Eqs. (47a)–(47c) and (49), without carrying expansion, and the same results are obtained.

[14] It is easily seen that the velocity u_{i1} vanishes, the pressure P_1 is a constant, and the temperature τ_1 (thus, the density ω_1) varies with time in initial-value problems where the velocity is zero and the temperature is nonuniform (strictly, non-harmonic) initially, and the pressure is time-independent at infinity. Thus, the incompressible condition is not satisfied. See also the example given in Section K-4.10.3, where the velocity vanishes and the density varies with time, and further, the temperature field is quite different from the incompressible case owing to the time-dependent boundary condition on P_{S1}, corresponding to P_1 here.

range of the variables ξ_i and η may be a set of discrete points. Obviously, the integrals with respect to ξ_i and η in the discussion in Section K-C.2 are taken to be the summation over the discrete points because the velocity distribution function f is the collection of delta functions of ξ_i and η with their center at the discrete points. To solve the conservation equations by kinetic-equation approach, a finite number of discrete ξ_i's and η's are sufficient. The number depends on the size of the conservation equations. Here, we will describe the procedure of solution of conservation equations by a kinetic equation with the formulas expressed in discrete ξ_i and η.

The Navier–Stokes equations (K-C.34a)–(K-C.34c) are taken as an example. In this case, $r = 0, 1, \ldots, 4$, i or $j = 1, 2, 3$, and γ_r's are chosen as

$$\gamma_0 = 1, \quad \gamma_i = \xi_i, \quad \gamma_4 = \xi_i^2 + (\alpha - 3)\eta^2, \tag{53}$$

where $\alpha \geq 3$ is a constant.[15] From the Navier–Stokes equations (K-C.34a)–(K-C.34c), we can identify ρ_r and $\hat{H}_i^r$ as follows:

$$\rho_0 = \rho, \tag{54a}$$

$$\rho_i = \rho u_i, \tag{54b}$$

$$\rho_4 = \rho(u_j^2 + 2e), \tag{54c}$$

$$\hat{H}_i^0 = \rho u_i, \tag{55a}$$

$$\hat{H}_i^j = \rho u_i u_j + p_{ij}, \tag{55b}$$

$$\hat{H}_i^4 = \rho u_i(u_j^2 + 2e) + 2u_j p_{ij} + 2q_i, \tag{55c}$$

where

$$e = \frac{\alpha RT}{2}, \tag{56a}$$

$$p = \rho RT, \tag{56b}$$

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) - \mu_B \frac{\partial u_k}{\partial x_k} \delta_{ij}, \tag{56c}$$

$$q_i = -\lambda \frac{\partial T}{\partial x_i}. \tag{56d}$$

The velocity distribution function $f^{(m,n)}$ of Chapman–Enskog type that gives a given set $(\rho_r, \hat{H}_i^r)$ is obtained by the following simultaneous linear algebraic

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[15]The constant α is the freedom of a molecule of the gas under consideration. For a monatomic gas without internal degree of freedom, $\alpha = 3$.

equations, which correspond to Eqs. (K-C.1) and (K–C.5):[16]

$$\sum_{m,n} f^{(m,n)} = \rho_0, \tag{57a}$$

$$\sum_{m,n} \xi_i^{(m)} f^{(m,n)} = \rho_i, \tag{57b}$$

$$\sum_{m,n} [(\xi_i^{(m)})^2 + (\alpha - 3)(\eta^{(n)})^2] f^{(m,n)} = \rho_4, \tag{57c}$$

$$\sum_{m,n} \xi_i^{(m)} \xi_j^{(m)} f^{(m,n)} = \hat{H}_i^j \quad (j \le i), \tag{57d}$$

$$\sum_{m,n} \xi_i^{(m)} [(\xi_j^{(m)})^2 + (\alpha - 3)(\eta^{(n)})^2] f^{(m,n)} = \hat{H}_i^4. \tag{57e}$$

The set of points $(\xi_i^{(m)}, \eta^{(n)})$'s has to be chosen in the way that the determinant formed with the coefficients of $f^{(m,n)}$ does not vanish and that the number of the points is 14 [5 for ρ_r, 6 for $\hat{H}_i^j$ $(j \le i)$, and 3 for $\hat{H}_i^4$] in the present case.[17] If one chooses more points for one's convenience, one should impose some extra conditions to make $f^{(m,n)}$ unique.[18]

Once the way to determine the velocity distribution function $f^{(m,n)}$ of the Chapman–Enskog type is fixed, we can proceed to the next procedure described in Section K-C.3. That is, (i) for given initial data ρ_r, u_i and T, compute $\hat{H}_i^r$ by Eqs. (55b)–(56d); (ii) construct the corresponding velocity distribution function $f^{(m,n)}$ of Chapman–Enskog type by Eqs. (57a)–(57e); (iii) determine the solution of the free-molecular equation (K-C.35) at the next time step with the velocity distribution function constructed in the step (ii) as the initial condition; (iv) compute ρ_r from the resulting velocity distribution function by Eqs (57a)–(57c); (v) return to the step (i) and repeat the process. Then, we can obtain the solution of the Navier–Stokes equations (K-C.34a)–(K-C.34c).

To determine the solution of the free-molecular equation in the step (iii) of the preceding paragraph, we need the boundary condition compatible with the solution of Chapman–Enskog type. For the Navier–Stokes equations, u_i and T are generally specified on the boundary of bodies, but ρ is not specified. The boundary value of ρ is determined with ρ and u_i at the previous time step by Eq. (K-C.34a). Then, we know the boundary data ρ_r. Together with the ρ, u_i, and T inside gas obtained by the free-molecular solution starting at the preceding time step, we can determine the boundary data of $\hat{H}_i^r$ by

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[16] Note that the relation between ρ_i and $f^{(m,n)}$ and that between $\hat{H}_i^0$ and $f^{(m,n)}$ are the same. That is, these relations are commonly expressed by Eq. (57b).

[17] i) When the determinant vanishes, we cannot obtain the solution for arbitrary data on the right-hand side.

ii) In the case of $\alpha = 3$ (monatomic gas), the left-hand side of Eq. (57c) is derived from Eq. (57d) with $i = j$. The right-hand side of Eq. (57c) is obtained in the same way because the bulk viscosity vanishes ($\mu_B = 0$) for a monatomic gas. Thus, Eq. (57c) is unnecessary in the analysis.

[18] Sometimes, it is convenient to choose more $(\xi^{(m)}, \eta^{(n)})$ than required to avoid awkward distribution $f^{(m,n)}$.

Eqs. (55b)–(56d). Then, the boundary data $f^{(m,n)}$ compatible with the solution of Chapman–Enskog type are determined by Eqs. (57a)–(57e).

In the process of solution of the free-molecular equation (K-C.35) by a finite difference method, one can make use of a simplifying process introduced in the lattice-Boltzmann-equation approach. Then, the step (iii) in the above procedure is simplified. Let $\boldsymbol{X}$ be a lattice point in x_i space, and let Δt be the time step of computation. We arrange the lattice points $\sum \boldsymbol{X}'s$ and $\boldsymbol{\xi}^{(m)\prime}s$ in such a way that $\boldsymbol{X}+\boldsymbol{\xi}^{(m)}\Delta t$ is also some lattice point in x_i. Then, $f^{(m,n)}$ at $t+\Delta t$ is obtained by shifting $f^{(m,n)}$ at t among the lattice points $\boldsymbol{X}$'s. For example, take rectangular lattices for x_i, where the lattice points $\boldsymbol{X}$'s are expressed as $\boldsymbol{X} = (m_1, m_2, m_3)\Delta x$ with m_i being an integer $(0, \pm 1, 2, \cdots)$. Then, we choose $\boldsymbol{\xi}^{(m)} = (k_1, k_2, k_3)\Delta x/\Delta t$, where k_i is, for example, $k_i = 0, \pm 1$. Then,

$$f(m_i\Delta x, k_i\xi, t + \Delta t) = f((m_i - k_i)\Delta x, k_i\xi, t),$$

where $\xi = \Delta x/\Delta t$.[19] With the above choice of $\boldsymbol{\xi}^{(m)}$, $(m_i - k_i)\Delta x$ is a lattice point of x_i or $\boldsymbol{X}$. Other examples of the lattice point system are found, for example, in Qian, Succi & Orszag [1995], and Kataoka [submitted].

In Section K-C.3, we discussed the process and validity of solving the conservation equations, or the validity of the procedure of solution given in this section, by making use of the free molecule equation on the basis of the discussion in Section K-C.2. In this discussion, we implicitly assumed the stability of solution of Eq. (K-C.6) with Eq. (K-C.7). Further, it is also assumed that $|\hat{H}_i^r(\rho_r, \nabla\rho_r) - \hat{H}_i^r(\rho_r^c, \nabla\rho_r^c)| = O((\Delta t)^2)$ for $\hat{H}_i^r$ at the step (i) after step (iv) (or at time $t+\Delta t$) when $|\rho_r - \rho_r^c| = O((\Delta t)^2)$ [Eq. (K-C.40)] for the present Navier–Stokes equations.[20] More rigorously, the condition about the size used in the discussions in Section K-C.3 should be uniformly bounded in some neighborhood of the solution of Eq. (K-C.6). The conditions are mainly determined by the property of solution of the Navier–Stokes equations [generally, Eqs (K-C.12) and (K-C.13)]. Some examples of the numerical computation of the Navier–Stokes equations by the present method are carried out by Kataoka [submitted].

(Section 4.1: Version 3-00)

References

[1] Bronshtein, I. N. and K. A. Semendyayev (1997), *Handbook of Mathematics* (Springer-Verlag, Berlin).

[2] Buck, C. R. (1965), *Advanced Calculus, Second Edition* (McGraw-Hill, New York).

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[19] The appropriate lattice ratio of Δx and Δt (thus, ξ) depends on the conservation equations (i.e., the Navier–Stokes equations) under consideration.

[20] This assumption corresponds to the assumption on the space variation of ρ_r and ρ_r^c. For the Euler equations, this assumption is not required becasue the space-derivative terms are absent in $\hat{H}_i^r$.

[3] Kataoka, T. (submitted), New lattice Boltzmann scheme for the compressible Navier–Stokes equations, *Physical Review letters.*

[4] Qian, Y. H., S. Succi, and S. A. Orszag (1995), Recent advances in lattice Boltzmann computing, in: D. Stanffer, ed., *Ann. Rev. Comput. Phys. III* (World Publishing Inc., Singapore).

[5] Takagi, T. (1961), *Kaisekigairon* (*Introduction to Analysis*), *Third Edition* (Iwanami, Tokyo). (in Japanese)