Selection 3

COMPUTER SIMULATION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AND NON-PERIODIC OSCILLATIONS

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Abstract

There occur periodic and non-periodic oscillations in nonlinear oscillatory systems with periodic external force. These phenomena are examined by analyzing nonlinear differential equations describing the systems, i.e., the mathematical models of the phenomena. However, if every solution of the equations lacks the stability which would be associated with realizability in a physical system, how can the behavior of the corresponding oscillatory system be explained?

This paper discusses the relation between non-periodic oscillations in the computer-simulated systems and the exact solutions of second-order nonlinear ordinary differential equations of class D, that is, of dissipative systems for large displacements.

1. INTRODUCTION

In nonlinear oscillatory systems with periodic external force there sometimes occur steady non-periodic oscillations in addition to periodic oscillations whose fundamental frequency is the same as, or equal to a rational multiple of, the external frequency. These oscillatory phenomena are examined by analyzing differential equations, the mathematical models for the phenomena, by considering appropriate aspects of the phenomena. In particular, the evolution of the state as time progresses is studied by the behavior of the solutions of the differential equations, i.e., the movement of a representative point in the phase space. However, considering that, among the solutions of the differential

equation, only stable solutions can represent a realizable long-term oscillatory state in the actual physical system, how can the phenomenon be explained, and what kind of oscillations will be observed, in the system where every solution of the differential equation exhibits instability? When no analytical solutions of nonlinear differential equations can be expected, the problems presented here should be examined not only to clarify the phenomenon but also to assess the validity of numerical methods for the solution of the differential equations, because computer simulation techniques are spreading far and wide.

In this paper, non-periodic steady oscillations are observed in computer-simulated solutions of the second-order nonlinear ordinary differential equation

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = e(t) \tag{1}$$

where $e(t)$ is a periodic function of the period L. Then the phenomena are described in terms of the appropriate mathematical concepts of recurrence.

2. OBSERVATION OF NON-PERIODIC OSCILLATIONS IN COMPUTER-SIMULATED SYSTEMS

As the differential equation (1) is periodic in t with the period L, the behavior of the system can effectively be analyzed by applying the transformation theory.* Let us denote by T the transformation which transforms the phase plane at $t = 0$ into itself at $t = L$ following the solution curves of the first-order system derived from Eq. (1) by the usual substitution $y = dx/dt$. Then, according to the definition of the transformation or the mapping T, the behavior of the solution passing through the point P at $t = 0$ in the phase plane is expressed by a complete sequence $\cdots, P_{-2}, P_{-1}, P, P_1, P_2, \cdots$ ($P_n = T^n P$, $n = 0, \pm 1, \pm 2, \cdots$) generated by the point P.

In this section, let us perform the computer simulation for the equations describing a forced oscillatory system and a forced self-oscillatory system as special cases of Eq. (1), and observe the movements of images P_n under the transformation T, and the global aspect of the solutions in the phase plane.

2.1 Non-periodic Oscillations in a Forced Oscillatory System

*For the concepts of the transformation theory used in this paper, see the Appendix and Refs. [2-10].

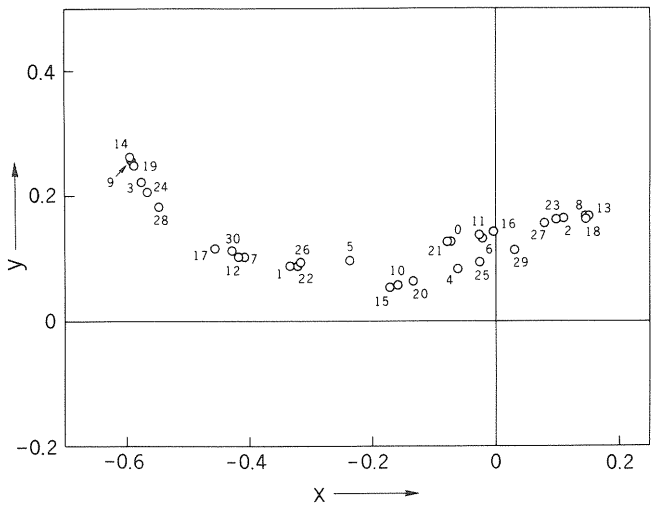

Fig. 1 Example of a point sequence representing steady oscillation which occurs in the system described by Eqs. (3), the parameters being $k = 0.2$, $B = 0.3$ and $B_0 = 0.08$.

Let us consider the Duffing equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos t + B_0 \tag{2}$$

which describes a forced oscillatory system. This equation represents a mathematical model for a series-resonant circuit containing a saturable inductor under the impression of d.c. and sinusoidal voltage [1].

As is well known, the oscillation described by Eq. (2) may include resonant, non-resonant, higher-harmonic and subharmonic oscillations. However, whether Eq. (2) has non-periodic steady solutions has not been clarified, but the computer simulation result reveals the existence of a steady oscillation which must be considered non-periodic. In the following, let us give a few examples of such oscillations.

Figure 1 shows an example of the behavior of images P_n moving in the xy plane under the transformation T in a system described by the first-order simultaneous equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -ky - x^3 + B\cos t + B_0 \tag{3}$$

with $k = 0.2, B = 0.3$ and $B_0 = 0.08$. Numerals attached to points in this

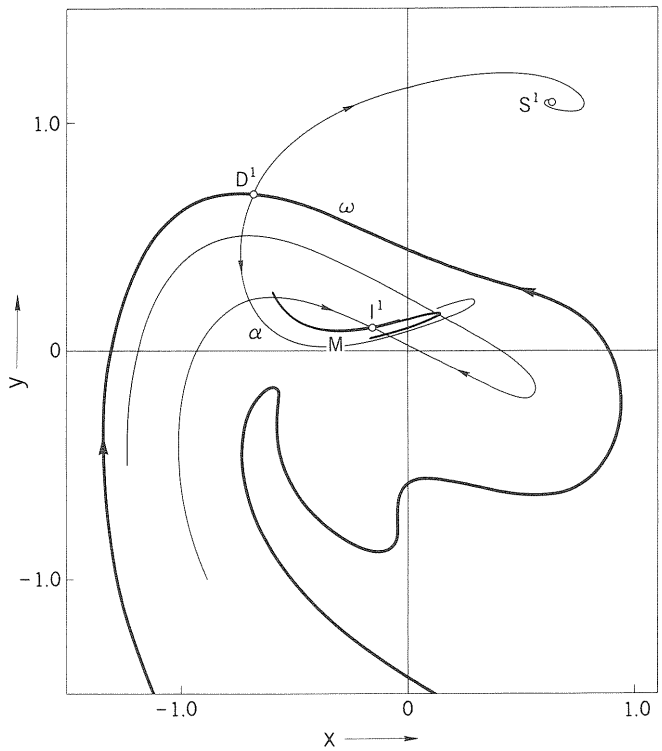

Fig. 2 Fixed points and invariant curves of the mapping for Eqs. (3), the parameters being $k = 0.2$, $B = 0.3$ and $B_0 = 0.08$.

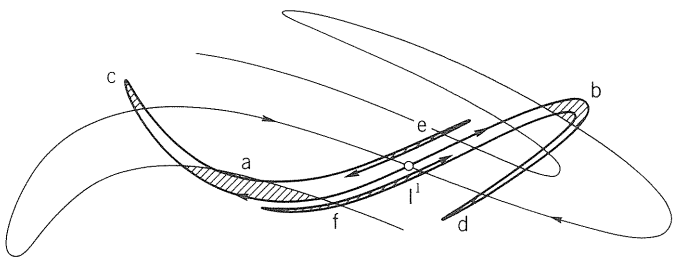

Fig. 3 Partial details of α- and ω-branches of the inversely unstable fixed point I^1.

figure indicate the order of the successive images under the transformation T. They are counted after the transient state has decayed. In every simulation, the pattern of this non-periodic oscillation in the phase plane (the shape of the point set representing the steady state) is reproducible, but the movement of

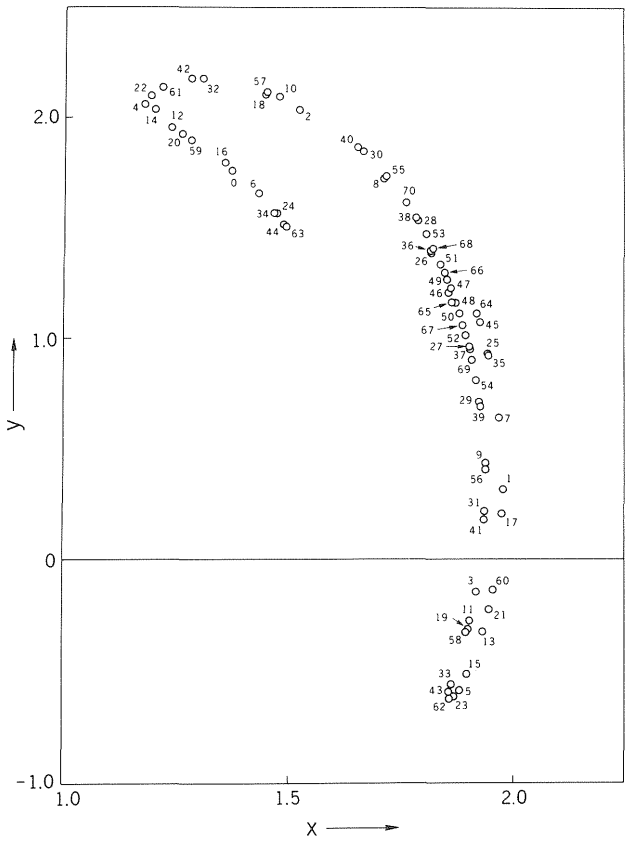

Fig. 4 Example of a point sequence representing steady oscillation which occurs in the system described by Eqs. (3), the parameters being $k = 0.2$, $B = 1.2$ and $B_0 = 0.85$.

the images cannot be reproduced over long intervals of time. This tendency is particularly remarkable in simulation by analog computer. In digital simulation, the movement of the images differs considerably depending on the integration step size, and the schemes for numerical integration. Figure 2 shows the global aspect of the fixed points and the invariant curves of the mapping in the xy plane for this case. In the figure, points S^1, D^1 and I^1 are completely stable, directly unstable and inversely unstable fixed points, respectively. The arrow on the invariant curve indicates the direction of the movement of the images under the transformation T. The ω-branches (the heavy solid line) of the directly unstable fixed point divide the phase plane into two regions. When the initial point is given inside the region containing S^1, the system settles down to the periodic oscillation represented by the completely stable fixed point S^1. The

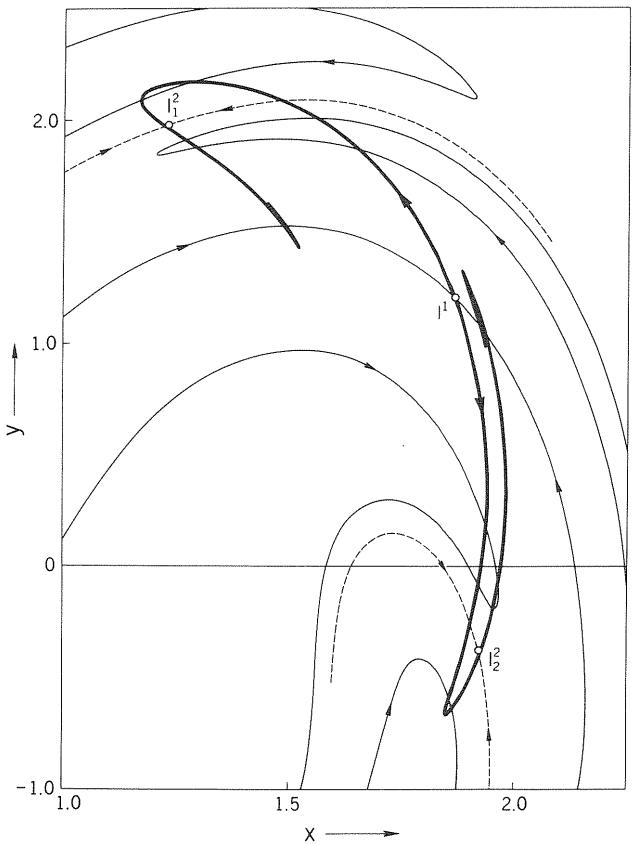

Fig. 5 Fixed point, periodic points and invariant curves of the mapping for Eqs. (3), the parameters being $k = 0.2$, $B = 1.2$ and $B_0 = 0.85$.

steady oscillation when the initial point is given inside the region containing I^1 is non-periodic as shown in Fig. 1. The image P_n representing the steady state continues to move in the neighborhood of the α-branches (medium heavy solid line) of the inversely unstable fixed point I^1.

Figure 3 shows the aspect of α-branches (heavy solid line) and ω-branches (thin solid line) of the point I^1 schematically. Both branches intersect with each other and form an infinite number of doubly asymptotic points. The α-branches are asymptotic to themselves and are confined inside the bounded region, while the ω-branches extend to infinity. The shaded region a in the figure is mapped onto the regions b, c, d, $\cdots$ successively under the transformation T. From these results, the configuration of α- and ω-branches of I^1 and the behavior of the points on the branches under the transformation T can be understood.

Next, let us consider an example in which the external force takes a slightly larger amplitude. Figure 4 shows the simulation result of a point sequence representing the steady state of the images under the transformation T, defined by Eqs. (3) with $k = 0.2$, $B = 1.2$ and $B_0 = 0.85$. As in Fig. 1, the image traces out a pattern resembling a segment of curve. Figure 5 shows the fixed point, periodic points and invariant curves of the mapping for the corresponding parameter values. The heavy solid lines represent the α-branches, the thin solid lines the ω-branches of the inversely unstable fixed point I^1. As to the inversely unstable 2-periodic points I_1^2 and I_2^2, only the ω-branches are shown by the thin dashed lines. The images under the transformation T starting from any point in the phase plane continue to move, after a sufficiently long lapse of time, in the neighborhood of the α-branches of the inversely unstable fixed point I^1.

2.2 Non-periodic Oscillations in a Forced Self-Oscillatory System

Let us consider the equation of the form (1)

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x^3 = B\cos\nu t \tag{4}$$

which describes a self-oscillatory circuit containing a negative-resistance element with the injection of a sinusoidal signal [2].

When a periodic force is applied to a self-oscillatory system, the phenomenon of synchronization occurs in a certain band of the external frequency; the system exhibits harmonic, higher-harmonic or subharmonic oscillation having period the same as, an integral multiple or submultiple of, the driving frequency. If the amplitude and frequency of the external force do not permit such synchronization, then the oscillation in the system becomes non-periodic. The non-periodic oscillation is one of two types, almost periodic oscillation and others. The almost periodic oscillation is liable to occur when the amplitude of the external force is relatively small, and can be described by a completely stable invariant closed curve in the phase plane. In this section, let us observe the other type of non-periodic oscillations in computer-simulated systems.

Figure 6 shows an example of a point sequence representing steady oscillation in the system described by

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \mu(1 - x^2)y - x^3 + B\cos\nu t \tag{5}$$

with $\mu = 0.2$, $B = 17$ and $\nu = 4$. These points are plotted after the transient state has decayed. Even if many more images are plotted than shown in the figure, the movement of images is not periodic. The set of points resembles

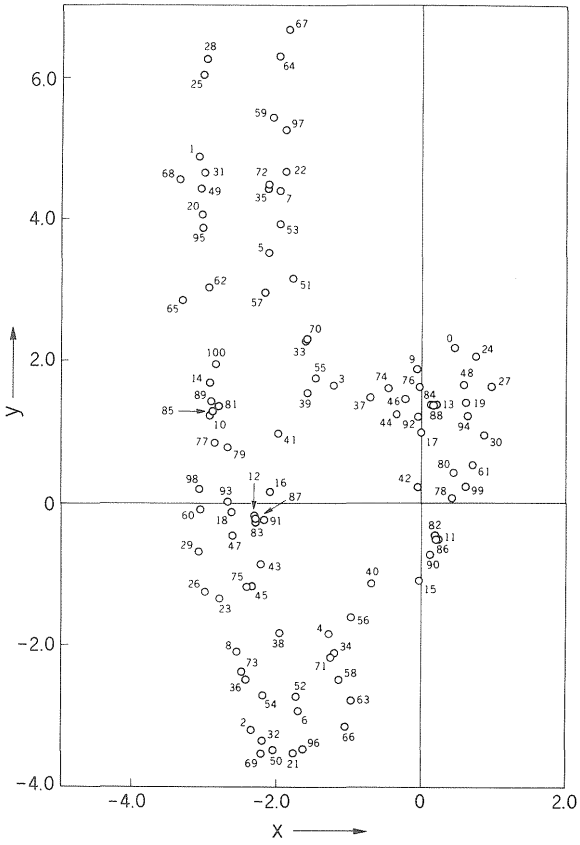

Fig. 6 Example of a point sequence representing steady oscillation which occurs in the system described by Eqs. (5), the parameters being $\mu = 0.2$, $B = 17$ and $\nu = 4$.

a bumpy ring pattern in the xy plane. The movement of images over long intervals of time is not reproducible, as with the previous examples. Figure 7 shows the global aspect of the fixed point, periodic points and invariant curves of the mapping T of this case. In this figure, the symbol ${}^iD_j^n$ indicates the i-th directly unstable n-periodic point, and the suffix j ($j = 1, 2, \cdots, n$) represents the order of the successive movements of the images in the n-periodic group under the transformation T. Similar symbols are applied to the completely stable (S), completely unstable (U) and inversely unstable (I) periodic points and fixed points. As seen in this figure, there are four directly unstable 2-periodic points ${}^1D_j^2$ and ${}^2D_j^2$ and four inversely unstable 2-periodic points ${}^1I_j^2$ and ${}^2I_j^2$ ($j = 1, 2$). The α-branches (heavy line) and ω-branches (thin line) of these points are distinguished by solid lines and dashed lines, respectively.

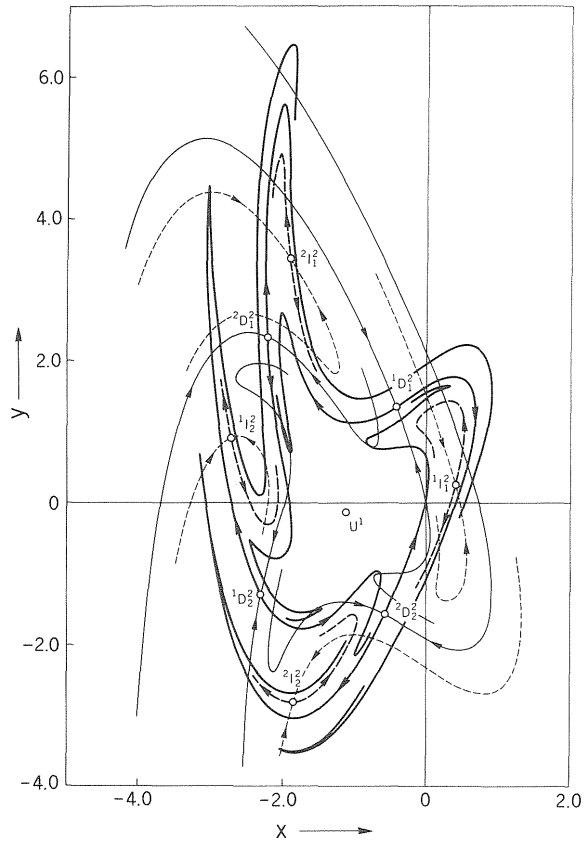

Fig. 7 Fixed point, periodic points and invariant curves of the mapping for Eqs. (5), the parameters being $\mu = 0.2$, $B = 17$ and $\nu = 4$.

Also, these α-branches are asymptotic to each other, forming a ring-like domain enclosing the completely unstable fixed point U^1. The image P_n representing the steady state in Fig. 6 continues to move in the neighborhood of these α-branches.

Finally, let us examine a more complicated example. Figure 8 shows a point sequence representing the steady oscillation of the images P_n in the system described by Eqs. (5) with $\mu = 0.2$, $B = 1.8$ and $\nu = 0.6$. Figure 9 shows the global phase-plane portrait for this case. As seen in these figures, there exist no completely stable or completely unstable points in this system.

The numerical examples given above were obtained by use of analog and digital computers. In the course of computer experiments, special attention was paid to the following points.

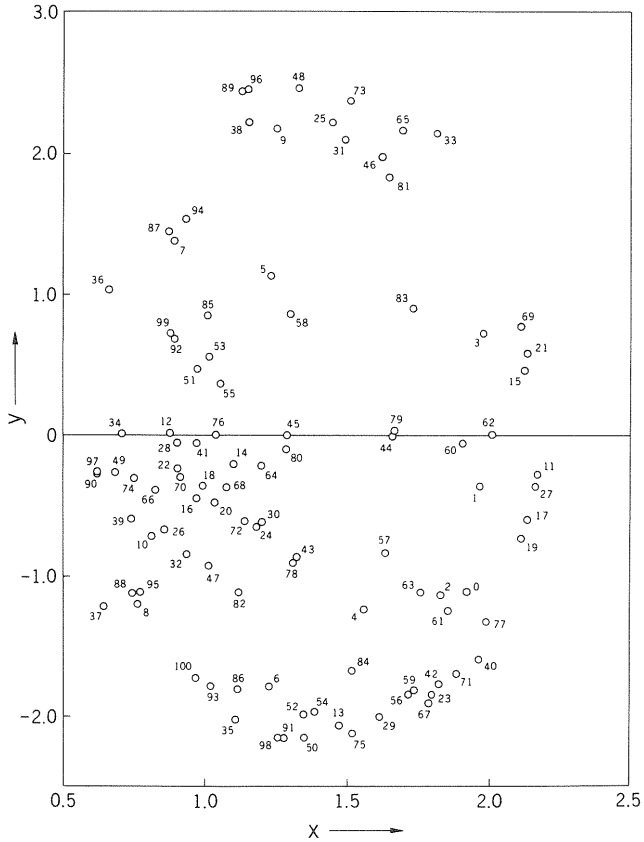

Fig. 8 Example of a point sequence representing steady oscillation which occurs in the system described by Eqs. (5), the parameters being $\mu = 0.2$, $B = 1.8$ and $\nu = 0.6$.

1. In the analog simulation, the amplitude of the external sinusoidal force must be kept constant.
2. In the digital simulation, considering that the Eq. (1) is a periodic system with the period L, the numerical integration has been carried by choosing the step size h to be an integral submultiple $h = L/N$ of the period L for the interval $0 \leq t \leq Nh$, and by iterating this procedure.

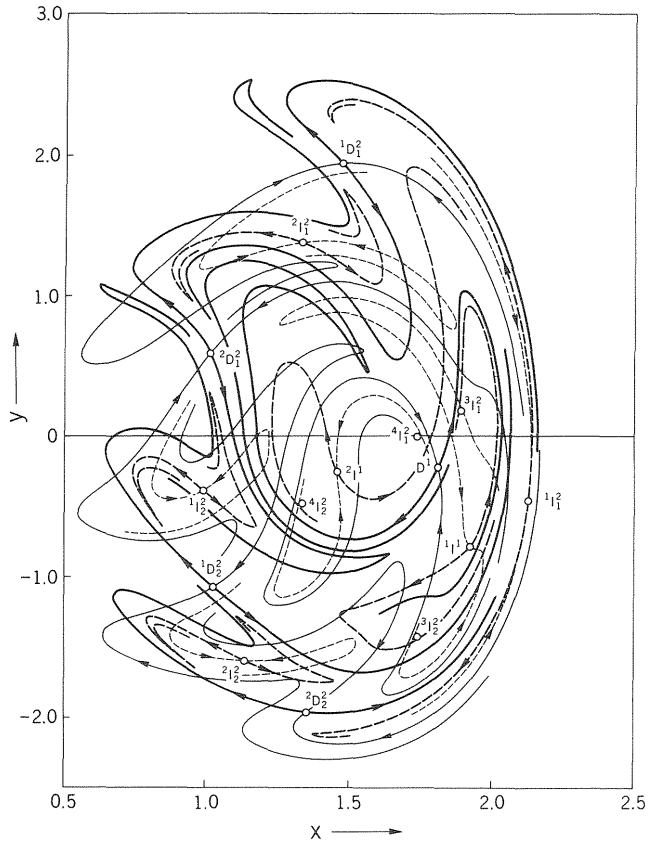

Fig. 9 Fixed points, periodic points and invariant curves of the mapping for Eqs. (5), the parameters being $\mu = 0.2$, $B = 1.8$ and $\nu = 0.6$.

3. DISCUSSION

In the preceding section non-periodic steady oscillations have been observed in computer simulated solutions of the forced oscillatory system (3) and of the forced self-oscillatory system (5).† These differential equations are of class D, i.e., dissipative systems for large displacements, and consequently have maximum finite invariant sets.‡ This set is a positively asymptotically stable connected closed set and its configuration represents the behavior of the solutions of

†The systems treated in the preceding section are supposed to be structurally stable. Hence, the systems considered in this section are assumed to possess this structural stability.

‡The area of the maximum finite invariant set is zero for the transformation T defined by Eqs. (3) with $k > 0$.

the differential equation for $t \to \infty$.[§] However, all points of the maximum finite invariant set do not always represent steady solutions and the set may contain points representing transient states. Here, let us momentarily close our eyes to the stability required for the expression of realizable physical states, but let us define steady solutions in the mathematical sense as solutions corresponding to complete sequences on the xy plane which are stable according to Poisson; then all steady solutions can be expressed by the set of pseudo-recurrent points. The set of these pseudo-recurrent points belongs to the set of central points, which is obtained by subtracting the wandering points of all orders from the maximum finite invariant set. The resulting set includes all minimal sets representing recurrent motions, harmonic sets representing almost periodic motions and periodic sets representing periodic motions.

Here, let us consider the first example mentioned in Sec. 2.1. The maximum finite invariant set in Fig. 2 is the union of three fixed points S^1, D^1 and I^1 and the closures of α-branches of points D^1 and I^1. The set of non-wandering points within this set can be taken as the two fixed points S^1 and D^1 and the closure M of the α-branches of the point I^1.[¶] These are mutually separated invariant closed sets. Among them, the fixed point S^1 and the invariant closed set M are positively asymptotically stable. When a positively asymptotically stable invariant closed set such as the fixed point S^1 consists of a single periodic set, the periodic solution passing through the point S^1 is asymptotically stable according to Lyapunov. This stability permits the correction of errors and disturbances in the computer simulation to maintain the periodic oscillation in the system, and establishes that the periodic solution through S^1 is a physically realizable steady motion. Since the invariant closed set M is positively asymptotically stable, the image under the transformation T of any neighboring point of M is asymptotic to M and will not deviate from M. However, M contains numerous minimal sets, quasi-minimal sets and higher order wandering points, all of which appear to be unstable, and so it seems that any given solution passing through these points can not possess asymptotic stability individually. Hence, the phenomenon can only be explained by assuming that the totality of solutions contained in M represents the steady behavior of the computer-

[§]Let M be a bounded invariant closed set in the phase plane. A set M is called positively stable if, for any neighborhood U of the set M, there can be found a neighborhood V of the set M such that if $P \in V$, then P_n $(= T^n P,\ n > 0) \in U$. A set M is called positively asymptotically stable if M is positively stable and if there can be found a neighborhood V of the set M such that if $P \in V$, then Ω_P $(= \omega$-limit set of $P) \subset M$. This stability is an extension of the concept of orbital (asymptotic) stability of the solutions of differential equations.

[¶]Whether all points in M are non-wandering points must be examined. Here, after consideration, M is assumed not to contain wandering points. If M contains wandering points, the set excluding them should be taken as M.

simulated system. That is, the representative point exhibiting the oscillatory state in the system continues to move randomly in the neighborhood of the numerous solutions (e.g. minimal sets) contained in M under the influence of errors, disturbances and minute variations of the system and so on. Therefore, although any particular solution within the invariant closed set M in the phase plane exhibits instability which is incompatible with the description of a physically realizable long-term steady behavior, still the set M as a whole represents the total configuration of the phenomenon in the phase plane. Considering this nature, let us call the oscillatory phenomenon exhibited by the solutions contained in the invariant closed set M a randomly transitional oscillation.

In the example given in Fig. 5, all points in the maximum finite invariant set are non-wandering and this set is a positively asymptotically stable invariant closed set representing the randomly transitional oscillation.

Next, let us consider the examples given in Sec. 2.2. The maximum finite invariant set in Fig. 7 is the connected closed region surrounded by the closure of the α-branches of the directly and inversely unstable 2-periodic points D^2 and I^2. In this case, the positively asymptotically stable invariant closed set consisting of non-wandering points is the invariant set obtained by excluding the completely unstable fixed point U^1 and its neighboring wandering points from the maximum finite invariant set. This set can be considered as the closure of the α-branches of the 2-periodic points D^2 and I^2. As seen in Figs. 6 and 7, the configuration of this set has the appearance of a bumpy ring domain enclosing the point U^1.

The last example given in Figs. 8 and 9 is so complicated that more detailed results than shown in the figures can not be expected from the simulation study.

As pointed out in this section, the steady oscillations in physical systems can be represented by positively asymptotically stable invariant closed sets of non-wandering points. However, when the action of this system which restores the displacements of the images deviated from the invariant closed set due to disturbances is weaker than the effect of these stochastic quantities, it becomes difficult to represent the actual phenomenon by this set. Such steady oscillations are frequently observed in the physical systems with small dissipation.

Theoretically, it is interesting to decompose the structure of a positively asymptotically stable invariant closed set of non-wandering points into the minimal sets and quasi-minimal sets contained in this invariant closed set, and to determine their numbers, properties and mutual relations, and also the asymptotic behavior of images under T in the neighborhood of these sets. However, simulation studies by their nature can give only partial and incomplete results in this direction.

4. CONCLUSION

In this paper, non-periodic steady oscillations have been observed in computer simulated systems described by second-order nonlinear ordinary differential equations. The equations under study are of class D, and should be structurally stable. It has been shown that, in spite of the deterministic character of the equations, there occur random oscillations in the computer simulated systems affected by stochastic quantities such as errors, external disturbances and so on. These non-periodic oscillations can be explained by the random movement of an actual representative point among solutions (e.g. minimal sets) contained in a positively asymptotically stable invariant closed set of non-wandering points. Each individual minimal set within the positively asymptotically stable invariant set is unstable, and no single minimal set can describe the steady oscillations which are observed. Further, it is proposed that this type of oscillation should be called randomly transitional oscillations.

It will be easily seen from the preceding discussion that, however accurate are the computers used, this type of random oscillation occurs inevitably so long as the computers are not ideal. This randomly transitional oscillation is typically observed in computer simulated systems of not only second-order nonlinear ordinary differential equations, but also multi-variable nonlinear equations. Further, the oscillations are supposed to be actual steady states in a wide range of nonlinear oscillatory systems such as electronic circuits containing nonlinear elements. Hence, from the point of view of the theory of nonlinear oscillations, even when apparently random oscillations are observed, deterministic nonlinear equations can still capture the essential nature of the phenomenon. Therefore, it can be considered that deterministic equations are appropriate mathematical models of such phenomenon.

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APPENDICES

I. Review of Transformation Theory

The behavior of solutions of the differential equation (1) can be studied effectively by the topological methods originated by H. Poincaré and developed by G. D. Birkhoff and others. In this Appendix, the basic concepts of the transformation theory of differential equations are reviewed.

The first-order simultaneous equations rewritten from the differential equation (1) define a one-to-one, continuous and orientation-preserving transformation T of the phase plane into itself. The infinite sequence of points $\cdots$, P_{-2}, P_{-1}, P, P_1, P_2, $\cdots$ consisting of images P_n generated by applying the transformation T^n ($n = 0, \pm 1, \pm 2, \cdots$) on an arbitrary point P in the phase plane is called a complete sequence of P and a point sequence P, P_1, P_2, $\cdots$ is called a positive half-sequence of P. An accumulation point of the positive half-sequence of P is called an ω-limit point of P and the totality of these points is called an ω-limit set Ω_P of P. Similarly, for a negative half-sequence P, P_{-1}, P_{-2}, $\cdots$, α-limit point and α-limit set A_P of P can be defined. The union of a complete sequence and its α- and ω-limit sets is called a complete group.

When the image $TE = \{P;\ T^{-1}P \in E\}$ obtained by applying the transformation T to a point set E coincides with E itself, the set E is said to be invariant with respect to T. The above-mentioned α- and ω-limit sets and the complete group are examples of invariant closed sets.[‖]

A point P is called positively stable according to Poisson if it is an ω-limit point of P, and negatively stable according to Poisson if it is an α-limit point of P. A point both positively and negatively stable according to Poisson is called stable according to Poisson. As seen in this definition, if P is positively stable according to Poisson, then every point of the complete sequence of P is also positively stable according to Poisson. A similar statement can be made for points negatively stable and stable according to Poisson.

Next, let us explain the concept of the set of central points introduced by G. D. Birkhoff. If an arbitrary connected region σ in the phase plane is not intersected by any of its images $\cdots, \sigma_{-2}, \sigma_{-1}, \sigma_1, \sigma_2, \cdots$ under the transformation T^n $(n = \pm 1, \pm 2, \cdots)$, σ is called a wandering region and its points wandering points. A point in the phase plane which is contained in no wandering region is called a non-wandering point. Among the non-wandering points are α- and ω-limit points. The totality of non-wandering points in the phase plane constitutes an invariant closed set M^1 with respect to T. Let us assume that M^1 is non-empty and is not identical with the phase plane itself[**] and let the complement of M^1 be denoted by W^1, then the points of W^1 are known to be asymptotic to M^1 on indefinite iteration of T or T^{-1}. Now let us take the set M^1 as fundamental instead of the entire plane. A connected region σ which contains points of M^1 is called wandering with respect to M^1 if the set $\sigma \cap M^1$ of points common to σ and M^1 is intersected by none of its images under the transformation T^n $(n = \pm 1, \pm 2, \cdots)$. The points of M^1 which are contained in such a region are called wandering with respect to M^1, and their totality is denoted by W^2. The set $M^2 = M^1 - W^2$ is a non-empty invariant closed set consisting of the points which are non-wandering with respect to M^1, and the points of W^2 tend toward M^2 asymptotically on indefinite iteration of T or T^{-1}. In case $M^1 = M^2$, M^1 is called non-wandering with respect to itself. In case M^2 is not identical with M^1, i.e. a proper subset of M^1, the process may be carried one step further yielding the set M^3 of points which are non-wandering with respect to M^2. Thus the process is continued reaching the set which is non-wandering with respect to itself. In case, however, that

[‖]As for the maximum finite invariant sets, fixed points, periodic points, invariant closed curves, doubly asymptotic points, etc., refer to [2, 8]

[**]Since the transformation under consideration is defined by the differential equation of class D, the set M^1 of non-wandering points is included in the maximum finite invariant set and hence non-empty.

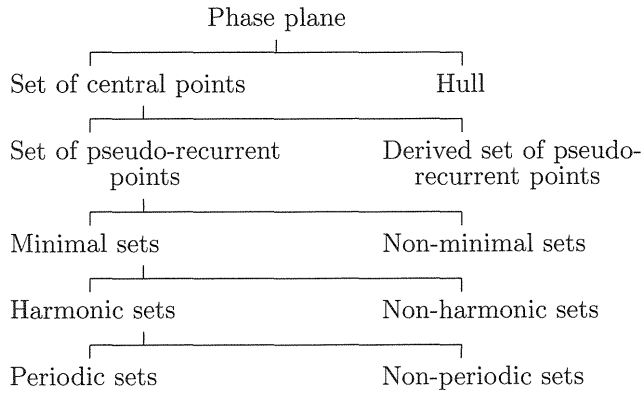

Fig. A Decomposition of a phase plane.

no such set appears after a finite number of steps, an infinite sequence M^1, M^2, $\cdots$ appears which satisfies $M^1 \supset M^2 \supset M^3 \supset \cdots$. In such a case, let us denote their intersection $\cap_{n=1}^{\infty} M^n$ by M^ω, then M^ω is a non-empty invariant closed set, to which the same process may be applied successively, yielding $M^{\omega+1}$, then $M^{\omega+2}, \cdots$. The sequence of sets thus obtained $M^1 \supset M^2 \supset \cdots \supset M^\omega \supset M^{\omega+1} \supset M^{\omega+2} \supset \cdots \supset M^{\omega 2} \supset M^{\omega 2+1} \supset \cdots$ is a well-ordered sequence and each set is non-empty and is a proper subset of all those preceding it. It is known that the sequence terminates with a definite ordinal r of Cantor's second ordinal class, i.e., $M^r = M^{r+1} = \cdots \neq \phi$ (ϕ: empty set). The points of M^r are called central points, and a complete sequence of central points is called a central motion. The set of points which is non-wandering with respect to itself, such as M^r, is said to possess the property of regional recurrence. Points of the sets W^2, $W^3, \cdots$ are called wandering points of higher orders, and a set of points outside the central points is called a hull.

In conclusion, let us decompose a set of central points. A point P is called pseudo-recurrent if it is both an α- and ω-limit point of its own complete sequence. As it has been proven that the set of central points is identical with the union of the set of pseudo-recurrent points and their derived set, a central point is either a pseudo-recurrent point or an accumulation point of pseudo-recurrent points. A pseudo-recurrent point is stable according to Poisson, and the complete group of a pseudo-recurrent point is called a quasi-minimal set. Specifically, a quasi-minimal set is called minimal if it is non-empty, closed and invariant, and has no proper subset possessing these three properties. A point of a minimal set is called recurrent, and the complete sequence of a recurrent point is called a recurrent motion. Further, if a complete sequence in a minimal set is almost periodic, then such a minimal set is called harmonic and points of

a harmonic set are called almost periodic points. Finally, as the most limited case, if a complete sequence in a harmonic set is periodic, such a harmonic set is called periodic, and points of a periodic set are called periodic or fixed points. The above-mentioned items are summarized in Fig. A.

II. Supplements to the Numerical Examples

Details of the fixed and periodic points appearing in the numerical examples described in Secs. 2.1 and 2.2 are listed in Tables 1 and 2. The characteristic numbers in the Tables represent eigenvalues of the transformation matrix obtained by linearizing T or T^2 in the neighborhood of the fixed points. The numerical integration was performed on the FACOM 230-60 computer by using the RKG method with an integration step size of one-sixtieth of the period L. The authors wish to express their sincere thanks to the staff of the computer center at Kyoto University.

Table 1 Fixed points and periodic points of Eq. (3) and their characteristic numbers

Parameter	Point	Fixed point, periodic point		Characteristic number	
		x_0	y_0	m_1	m_2
$k = 0.2$	S^1	0.639	1.085	$0.0994 \pm 0.524i$	
$B = 0.3$	D^1	−0.677	0.687	2.42	0.118
$B_0 = 0.08$	I^1	−0.154	0.103	−0.147	−1.94
$k = 0.2$	I^1	1.866	1.204	−0.198	−1.43
$B = 1.2$	I_1^2	1.230	1.979	−0.0291	−2.79
$B_0 = 0.85$	I_2^2	1.923	−0.380	−0.0291	−2.79

Table 2 Fixed points and periodic points of Eq. (5) and their characteristic numbers

Parameter	Point	Fixed point, periodic point		Characteristic number	
		x_0	y_0	m_1	m_2
	U^1	-1.132	-0.142	$-0.615 \pm 0.861i$	
	${}^1D_1^2$	-0.421	1.345	7.35	0.0898
	${}^1D_2^2$	-2.305	-1.294	7.35	0.0898
$\mu = 0.2$	${}^2D_1^2$	-2.218	2.320	7.35	0.0898
$B = 17$	${}^2D_2^2$	-0.578	-1.568	7.35	0.0898
$\nu = 4$	${}^1I_1^2$	0.410	0.239	-0.108	-3.74
	${}^1I_2^2$	-2.718	0.905	-0.108	-3.74
	${}^2I_1^2$	-1.910	3.454	-0.108	-3.74
	${}^2I_2^2$	-1.856	-2.809	-0.108	-3.74
	D^1	1.813	-0.214	22.3	0.0715
	${}^1I^1$	1.934	-0.787	-0.775	-1.35
	${}^2I^1$	1.465	-0.249	-0.775	-1.35
	${}^1D_1^2$	1.478	1.943	10.4	0.0841
	${}^1D_2^2$	1.028	-1.076	10.4	0.0841
	${}^2D_1^2$	1.016	0.589	10.4	0.0841
$\mu = 0.2$	${}^2D_2^2$	1.357	-1.962	10.4	0.0841
$B = 1.8$	${}^1I_1^2$	2.136	-0.462	-0.285	-2.52
$\nu = 0.6$	${}^1I_2^2$	0.999	-0.397	-0.285	-2.52
	${}^2I_1^2$	1.347	1.372	-0.285	-2.52
	${}^2I_2^2$	1.145	-1.603	-0.285	-2.52
	${}^3I_1^2$	1.896	0.182	-0.0513	-29.4
	${}^3I_2^2$	1.741	-1.442	-0.0513	-29.4
	${}^4I_1^2$	1.741	-0.010	-0.0513	-29.4
	${}^4I_2^2$	1.339	-0.481	-0.0513	-29.4