# STUDIES ON THE SUB-INTERVAL OPTIMIZATION TECHNIQUE TO CONTROL SYSTEMS DESIGN

Toshiro ONO

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ERRATA

| page & line | error | correction |
|---|---|---|
| p.ii, ↓ l.7 | is carried | are carried |
| p.ix, ↓ l.8 | $\phi^+$ or $\phi^-$ | $\phi^+$ and $\phi^-$ respectively |
| p.xii, ↑ l.8 | adaptive performance functional | adaptive weighting factors |
| p.1 ↓ l.14 | equation from | equation form |
| p.4, ↓ l.15 | lengthly | lengthily |
| p.4 ↑ l.4 | expensive | be expensive |
| p.5, ↓ l.5 | literature.(49)(44)(30)(6) | literature.(49)(44)(30)(16) |
| p.5, ↓ l.7 | one-line | on-line |
| p.5, ↓ l.10 | functional.(6)(44) | functional.(16)(44) |
| p.5, ↑ l.7 | is suitably | are suitably |
| p.7, ↓ l.17 | J.D. Peason | J.D. Pearson |
| p.23, ↑ l.7 | will be, on the other hand, occurred at | will, on the other hand, occur at |
| p.23, ↑ l.2 | will be occurred at | will occur at |
| p.25, ↓ l.11 | have to be | has to be |
| p.29, ↓ l.3 | to provide | to providing |
| p.29, ↓ l.3 | for an infinite | for a finite |
| p.41, ↑ l.3 | Eqs.(2.2-5) and (2.2-6) | Eqs.(2.2-5) and (2.2-7) |
| p.42, ↓ l.5 | the initial condition | the terminal condition |
| p.42, ↑ l.4 | the initial condition | the terminal condition |
| p.52, ↑ l.12 | a switching line | a switching function |
| p.67, ↓ l.3 | (iii) compute the | (3) compute the |
| p.67, ↓ l.5 | the step (ii) | the step (2) |
| p.67, ↓ l.6 | the step (iv) | the step (4) |
| p.81, ↓ l.10 | the step (iii) | the step (3) |
| p.84, ↑ l.1 | Eq.(3.7-6) | Eq.(4.7-6) |
| p.94, ↑ l.1 | is also shown in | is also shown in Fig. 4.9. |
| p.95, ↑ l.5 | that the mathematical | the mathematical |
| p.111, ↓ l.9 | Eqs.(5.3-18) and (5.3-17) | Eqs.(5.3-8) and (5.3-17) |
| p.113, ↑ l.3 | since the feedback | the feedback |
| p.117, ↓ l.9 | affected by a system parameter but by a performance weight $\lambda$, | affected by a performance weight $\lambda$ but by a syetem parameter, |
| p.117, ↓ l.12 | $a$ and $\mu$ | $a$ and $\lambda$ |

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| p.121, ↓ l.10 | It should emphasized | It should be emphasized |
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| p.128, ↓ l.12 | and $c$ denotes | and $-c$ denotes |
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| p.132, ↓ l.3 | system parameters, i.e., | system parameters, |
| p.132, ↓ l.10 | the minute solid | the broken solid |
| p.133, ↓ l.6 | Eq.(7.3-4) | Eq.(6.3-4) |
| p.135, ↓ l.5 | is not so | is less |
| p.135, ↓ l.9 | very much the same | the very same |
| p.136, ↓ l.5 | of a saturable | of a saturating |
| p.137, ↑ l.3 | the desired solution Eq.(6.3-20) is | the desired solution is |
| p.138, ↑ l.3 | is also illustrated | are also illustrated |
| p.143, ↓ l.15 | of same level to | of same level as |
| p.148, ↑ l.6 | reversed time by | reversed time is expressed by |
| p.149, ↑ l.8 | By substituting Eq.$(A-5)_2$ into the function $z(\tau)$ | By using Eqs.$(A-4)_2$, $(A-5)_1$ and $(A-5)_2$, |
| p.149, ↑ l.5 | considering facts | considering the facts |
| p.152, ↓ l.11 | the interative | the iterative |
| p.152, ↑ l.7 | Eq.(3.2-4) | Eq.(4.2-4) |
| ↑ l.2 | is stated | are stated |
| p.159, ↓ l.10 | The present technique can be found to provide | It can be found that the present technique provides |
| p.162, ↑ l.11 | An Optimization Theory for Feedback Control System Design; Inform. Contr., Vol. 3, No.1, March, 1960, pp.32-59 | Optimization Theory and the Design of Feedback Control Systems; McGraw-Hill Book Co., New York, N.Y. (1963) |
| p.163, ↑ l.10 | of Fac. Eng., | of the Fac. Eng., |

# STUDIES ON THE SUB-INTERVAL OPTIMIZATION TECHNIQUE TO CONTROL SYSTEMS DESIGN

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# PREFACE

During the past ten years, considerable interest has arisen in the subject of modern control theory. This body of theory is a direct growth of the desire to place the theory of automatic control on a firmer mathematical background. Much of the work done in this area is derived from the theory and the design of optimum control, and the problems are usually formulated and solved in the time domain using the concepts of state matrix theory. Numerous contributions have been made in this pursuit, unfortunately with considerable difficulties in practical solutions of such a mathematical problem as a two point boundary value problem, or an initial value problem which describes the necessary conditions for optimality. The solution of these problems is in general very complicated. Except for very simple cases, it cannot be done without the aid of a modern high-speed digital computer. As a result, although many of the techniques are very useful and are often necessary in applications, the language and concepts of optimal control are often beyond the comprehension of practicing control engineers.

In this paper, the author proposes from an engineer's point of view various methods of designing pseudo-optimal control systems without the direct solution of these rather complicated mathematical problems mentioned above and much attentions are emphasized upon the relationship between concepts and the evolution of ideas rather than upon mathematical rigor. This paper is divided into three main parts.

In Part I, the fundamental concept of sub-interval optimization and its applications to the synthesis of bounded control systems with deterministic inputs are presented.

In Chapter 1, the fundamental concept of a pseudo-optimization which is named as a sub-interval optimization is described to obtain the quasi-optimum structure of control system with bounded control.

The purpose of Chapter 2 is to show that the present concept of sub-interval optimization applied to a final-value control problem provides us a physically meaningful optimum solution.

The description in Chapter 3 is concerned with the quasi-optimum control of a second order vibratory system with bounded control. A few extensions of the sub-interval optimization technique is carried out to derive the quasi-optimum switching line. An experimental study on the derived switching line by a digital computer is also presented in this chapter.

Part II deals with the near-optimum approaches to the synthesis of control systems under random environments.

An analytical method for synthesis of a near-optimum control system with random inputs is described in Chapter 4 by introducing the Taylor-Cauchy transformation. Discussions on the proposed numerical procedure from mathematical viewpoints are also developed in detail with digital simulation studies in this chapter.

The presentation in Chapter 5 is concerned with stochastic synthesis of an optimum final-value control system with control energy constraint under random environments. Detailed descriptions on control characteristics of the final-value control system are also carried out.

Part III concerns with an on-line computer optimization approach to non-linear control systems.

The description in Chapter 6 is devoted to establish the fundamental concept of an on-line computer utilization. This serves an extension of the concept of sub-interval optimization cited in Chapter 1 to the more complicated design problem of non-linear control systems. Two illustrative examples provide the detailed aspects of the present concept. Further discussions are also carried out, emphasizing the exploration of the concept to the design problems of adaptive, or self-optimalizing systems.

In every stage of this work, the author has benefited from many invaluable suggestions and stimulating discussions with many individuals. First of all, he wishes to express his sincere appreciation to Dr. Yoshikazu Sawaragi, Professor of Kyoto University, for his helpful suggestions, criticism, and guidance. To Dr. Yoshifumi Sunahara, Assistant Professor of Kyoto University, the author is very grateful for his help in reviewing this manuscript and for making numerous constructive criticisms.

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CONTENTS

## List of Principal Symbols

PART I

| | |
|---|---|
| $t$; | time variable |
| $\tau$; | reversed time variable |
| $\rho$; | dummy time variable |
| $s$; | complex-frequency variable |
| "min"; | symbol representing the minimization of what appears to its right |
| "^"; | symbol representing the normalization of what appears to its below |
| $e_1(t)$, $e_2(t)$; | state variables of the error signal in a control system |
| $x_1(t)$, $x_2(t)$; | state variables of a controlled system |
| $x_{1d}$, $x_{2d}$; | desired values of the state variables $x_1(t)$, $x_2(t)$ at the final instant of control operation |
| $u(t)$; | control signal to a controlled system |
| $\overline{u}(t)$; | optimum control signal |
| $v(t)$; | desired signal to a control system |
| $J_2$; | integral-error-squared value for a finite control interval |
| $J_f$; | performance functional of final-value controls |

$M_2$; integral-error-squared value for an infinite control interval

$\phi(e_1, e_2; \tau)$; minimum performance functional when a system starts from the state $(e_1, e_2)$ at the time $\tau$

$\phi^+, \phi^-$; solutions of linear partial differential equations

$k_i(\tau)$; coefficients in quadratic form of the minimum performance functional

$k_i^+(\tau), k_i^-(\tau)$; coefficients in quadratic form of solution $\phi^+$ or $\phi^-$

$H$; Hamiltonian function

$p_i(t)$; auxiliary variables in the Hamiltonian function

$z$; switching function

$z^+_{RES}, z^-_{RES}$; switching functions of restricted-optimal control for positive and negative trajectories respectively

$T$; time duration of the whole control operation

$T_{sub-i}$; time duration of the $i$-th sub-interval

$\bar{t}_i$; critical switching time in the $i$-th sub-interval

$T_i$; rest of control interval after the $i$-th switching operation abided by the sub-interval optimization technique

$t_s, \tau$; switching instant of restricted-optimal control

$T_s$; critical value of the time duration

$a, b, k, w_n$; parameters in a controlled system

$c_1, c_2, \ell_1, \ell_2, d_1, \beta$; constants

$L$; value of the magnitude constraint on a control variable

$\mu$; weighting factor in a performance functional

PART II

$t$; time variable

$\tau$; reversed time variable

$T$; final instant of control operation

$\rho$; dummy time variable

| | |
|---|---|
| $\zeta$; | complex variable |
| $s$; | complex-frequency variable |
| $\mathcal{E}(\cdot)$; | symbol representing the ensemble average under the condition of observing the present state variables of a system |
| $\mathcal{T}_c[\cdot]$ ; | symbol representing the direct Taylor-Cauchy transform |
| $\mathcal{T}_c^{-1}[\cdot]$; | symbol representing the inverse Taylor-Cauchy transform |
| "min"; | symbol representing the minimization of what appears to its right |
| ${}^tA$ ; | transposed form of a matrix $A$ |
| $e(t)$, $e_i(t)$; | state vector of the error signal in a control system and its component |
| $x(t)$, $x_i(t)$; | state vector and its component of a controlled system |
| $x_{id}$ ; | desired value of the state variable $x_i$ at the final instant of control operation |
| $u(t)$, $u_i(t)$; | control signal vector to a controlled system and its component |
| $\bar{u}(t)$, $\bar{u}_i(t)$; | optimum control vector and its component |
| $v(t)$, $v_i(t)$; | desired signal vector and its component |
| $\Xi(t)$, $\xi_i(t)$; | purely random signal vector and its component |
| $\alpha(t)$, $\beta(t)$, $\xi(t)$, $\eta(t)$; | purely random signals |
| $A(t)$; | state matrix |
| $D(t)$; | driving matrix |
| $Q(t)$; | matrix expressing the cost of state |
| $R(t)$; | matrix expressing the cost of control |
| $I$, $R_f$, $R_{ft}$ ; | performance functionals |
| $\lambda_i$, $\lambda$, $\mu$ ; | weighting factors in performance functionals |
| $\phi(e;\tau)$; | minimum performance functional when a system starts from the state $e$ at the time $\tau$ |
| $k_0(\tau)$, $k_i(\tau)$, $k_{ij}(\tau)$; | coefficients in quadratic form of the minimum performance functional |
| $k_i^\infty$ , $k_{ij}^\infty$ ; | stable solutions of the parameters in the minimum performance functional |

$k_m(\tau)$, $k_p(\tau)$, $k_d(\tau)$; time-dependent control coefficients

$a(t)$, $b(t)$; random parameters in a controlled system

$\bar{\alpha}$, $A^2$; mean and variance of a random parameter $a(t)$

$\bar{\beta}$, $B^2$; mean and variance of a random parameter $b(t)$

$m$, $\sigma^2$; mean and variance of a purely random signal $\xi(t)$

$m(t)$, $m_i(t)$; mean vector of a purely random signal $\Xi(t)$ and its component

$\Sigma(t)$, $\sigma_{ij}^2(t)$; variance-covariance matrix of a purely random signal $\Xi(t)$ and its component

$a$, $d$, $r$; constants representing system parameters

$\underline{K}_1(\zeta)$, $K_i(\zeta)$; vector-valued complex function and its component corresponding to $K_1(\tau)$ and $k_i(\tau)$ in a real variable respectively

$K_2(\zeta)$, $K_{ij}(\zeta)$; complex matrix and its component corresponding to $K_2(\tau)$ and $k_{ij}(\tau)$ in a real variable respectively

$k_{i,n}$, $k_{ij,n}$; Taylor-Cauchy transforms of $K_i(\zeta)$, and $K_{ij}(\zeta)$

$K_p(\zeta)$; arbitrary component of complex matrix $\underline{K}_2(\zeta)$ or vector $K_1(\zeta)$

$k_{p,n}$; Taylor-Cauchy transform of $K_p(\zeta)$

$\delta_n$; Kronecker's delta

$\underline{A}(\zeta)$, $Q(\zeta)$, $W(\zeta)$; complex matrices

$\underline{A}_t(\zeta_m)$, $Q_t(\zeta_m)$, $\underline{W}_t(\zeta_m)$; $m$ times translated forms of matrices $\underline{A}(\zeta)$, $\underline{Q}(\zeta)$ and $\underline{W}(\zeta)$ respectively

$\underline{m}(\zeta)$, $m_i(\zeta)$; vector-valued complex function and its component

$\underline{m}_t(\zeta_m)$; $m$ times translated form of $\underline{m}(\zeta)$

$N^{(m)}$; positive number

$\zeta_0^{(m)}$; constant

$\delta$, $\xi$, $\eta$, $K$; constants

PART III

$j$; integer expressing dummy time

$k$; integer expressing time

$\triangle$ ; sampling period

$e(k)$, ${}^{t}e(k)$; state vector of the error signal in a control system at the $k$-th sampling instant and its transposed form

$e(t)$; error signal in a control system

$u(k)$, ${}^{t}u(k)$; control vector to a control system at the $k$-th sampling instant and its transposed form

$\bar{u}(k)_i$ ; optimum control variable

$\bar{u}(t)$; optimum control signal

$v(t)$; desired signal to a control system

$y(k)$; state vector of the related systems at the $k$-th sampling instant

$I(t)_i$ ; performance index

$J_i$ ; performance functional

$J_{pi}$ ; state adaptive performance functional

$\tilde{J}_{pi}$ ; generalized state adaptive performance functional

$p\,[e(k)]$, $Q\,[e(k)]$ ; state adaptive performance functionals

$p\,[e(k), y(k)\,;k]$ , $Q\,[e(k)\,, y(k);\,k]$ ; generalized state adaptive performance functional

$p\,(k+i)$, $Q\,(k+i)$; time dependent weighting factors

$i$; integer expressing the duration of a sub-interval

$n$; highest order in the dynamics of a controlled system

$a$, $b$, $L$; parameters in a controlled system

$M$; value of the magnitude constraint on a control variable

$\beta$; constant

## Introduction

The mathematical description of the design problems of optimum control is shown in various text books in detail.(35)(47)(48) The author reviews it briefly through a short discussion about some of the problems which are related to the present work, and the fundamental concept of the methods of attack is outlined.

### 1. Review of Optimum Design Problems

Most of design problems of optimum control are developed in the time domain using the concept of state and matrix theory.
In general, the basic approach to the problem is as follows :

(1) Define a cost function or performance index (some precise mathematical measure of "goodness") of the control system to be designed.

(2) Determine the dynamical characteristics of the controlled system or plant in differential or difference equation from relating the state variables and control variables.

(3) Specify certain equality or inequality constraints in the state variables or control variables, or both.

The objective is as follows: Given (2), optimize (1) subject to (3). To make the idea more precise, let us consider the case that the dynamical characteristics of the controlled system as shown in Fig. 1 may be well-described by

$$\dot{x} = \frac{dx}{dt} = f(x, u, t) , \tag{1}$$

where $x \equiv x(t)$ and $u \equiv u(t)$ are the system state vector and the control vector, respectively, and they are defined by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \text{and} \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix} . \tag{2}$$

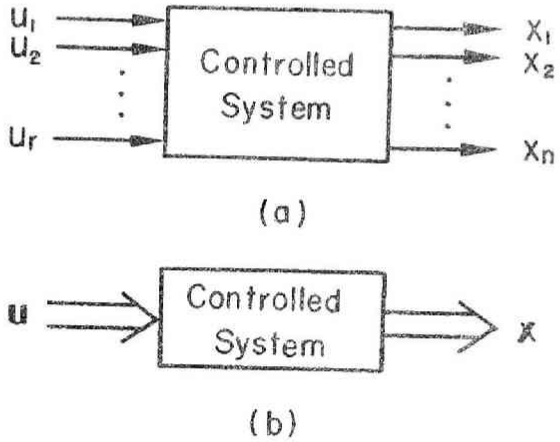


Fig. 1 Block diagrams of a controlled system

The control vector consists of $r$ control variables. At each moment, both the control variables $u_i$ and the state variables $x_i$ must satisfy the inequality constraints

$$g(u) \leq 0 \quad \text{and} \quad h(x) \leq 0 \tag{3}$$

which reflect the restrictions imposed upon the control system. For instance, when the control variables are bounded in magnitude, the first inequality in Eq. (3) becomes

$$a_i \geq u_i \geq -b_i \quad (i = 1, 2, \cdots\cdots, r) . \tag{4}$$

The control vector which satisfies the constraint conditions is referred to as an admissible control vector.

The form of a cost function or performance index yields the many types of optimum control problems. Among them, three basic modes of optimum control are of fundamental importance. They are the minimum-time control problem, the final-value control problem, and the minimum-integral control problem.

The minimum-time control problem may be stated as the determination of an admissible control vector $u$ so that the system is taken from a specified initial state $x_0$ to a desired final state $x_f$ in the shortest possible time.

The final-value control problem may be stated as the determination of an admissible control vector $u$ such that, in a given time interval $T$, the system is taken from an initial state $x_0$ into a state in which one or a combination of the state variables (for example, $x_1$) becomes as small as possible or as large as possible, and the remaining state variables have fixed values within physical limits. In other words, for example, it is stated as the determination of the admissible control vector which minimizes or maximizes the functional

$$J_f \equiv x_1(T) = F_1 \left(x_0 , x , u , t\right)\Big|_{t=T}, \tag{5}$$

or

$$J_f \equiv \sum_{i=1}^{k} \lambda_i x_i(T) = F_2 \left(x_0 , x , u , t\right)\Big|_{t=T}. \tag{6}$$

In Eq. (6), $k \leq n$ , and $x_i(T)$ denotes the $i$-th state variable evaluated at the end of control operation. A final-value controller is then designed to achieve a desired response at one instant only, the response at the earlier instant being arbitrary within physical limits.

The minimum-integral control problem involves the optimization of a system with respect to an integral. This optimum-control problem may be stated as the determination of an admissible control vector $u$ in such a manner that the integral

$$I = \int_{t_0}^{t_f} F\left(x , u , \rho\right) d\rho \tag{7}$$

reduces to a minimum during the time of movement $t_f - t_0$.

In a general formulation, the design problem of optimum control stated above is usually viewed as a variational problem. There are many possible variational methods for minimizing or maximizing a functional. The commonly used methods in control system design are:

(1) The calculus of variations.[4]

(2) The Maximum Principle of Pontryagin.[42]

(3) Dynamic Programming.[2]

The application of these variational methods to design problems gives us two types of difficult mathematical problems which describe the necessary conditions for optimality. That is to say, the method of the calculus of variations or the Maximum Principle provides a two point boundary value problem.[42] The method of Dynamic Programming, on the other hand, formulates an initial value problem[3] with respect to a partial differential equation. The solution of these problems are often very complicated and except for very simple cases it normally requires lengthly preliminary computations which are unwieldy for present-day computers.

## 2. Fundamental Concept of a Pseudo-Optimization

As the author has already stated in the previous paragraph, the solution of these difficult and rather unfeasible mathematical problems is inevitable to obtain an optimum solution which minimizes or maximizes the performance index. Even if such an optimum solution could be found, the resulting control $u$ would very likely be difficult to realize if it is impossible to instrument. Furthermore, an approximate solution to the exact optimization problem may have serious convergence difficulties and expensive to instrument. Hence we may abandon the hope of finding either the exact or the approximate solution to optimum control problems. The scope of these problems has increased to the point where the solutions may not be economically feasible unless some valid formulation of a sub-optimal or pseudo-optimal problem in such a way that the solution will be reasonably simple and feasible to instrument. Several approximation techniques for sub-optimal solution are already described in the literature.[49][44][30][6] This paper developes a new concept which is directly based on approximating the optimization problem by introducing the concept of one-line control schema,[30] instead of basing the control law either on a simplified or approximate set of model relations describing the controlled process,[41][49] or on approximating the minimum performance functional.[6][44] The latter approaches are often undesirable because the resulting approximate problem may still be a very difficult optimization problem, especially under the situation of bounded control variables. In contrast, however, the concept of a suitable approximation to the original optimization problem through on-line control features yields a pseudo-optimal control policy very quickly. That is to say, in the formulation of the optimization problem, a time interval for performing optimization (optimization interval) is of considerable importance. This must be an appreciable portion of the time interval over which the control is performed (operation interval), since one is usually interested in the performance over the entire operation interval, for instance, $t_f - t_0$ in Eq. (7). However, if a few restrictions on a duration of the optimization interval is suitably assumed, the solution of a restricted problem may be simplified and carried out analytically. Thus, to ease the required calculation for the original optimization problem, a shorter fictitious sub-interval into the future from any given time can be taken as a tentative optimization interval. The duration of a fictitious sub-interval depends upon the state of the system. The optimization over the whole operation (control) interval is carried out through an on-line scheme which recomputes, utilizing the best information available at present, the simplified optimization problem at discrete intervals of time. Such a philosophy leading to pseudo-optimal policies will be explored in the following chapters.

# PART I

# CONCEPT OF SUB-INTERVAL OPTIMIZATION AND ITS APPLICATIONS TO THE SYNTHESIS OF BOUNDED CONTROL SYSTEMS WITH DETERMINISTIC INPUTS

# CHAPTER 1 SYNTHESIS OF QUASI-OPTIMUM CONTROL SYSTEMS WITH BOUNDED CONTROL

## 1.1 Introductory Remarks

The majority of theoretical studies[18][25] on the problem of designing the optimum control with bounded control has been concerned with problems of the minimum-time control which minimizes the time required for the system from the initial state to the desired state.

In recent years, by using such concepts of modern optimization techniques as R. Bellman's Dynamic Programming[2] and L. S. Pontryagin's Maximum Principle,[42] much attention has been focused on solving problems of the optimum control with more general performance criteria. A. T. Fuller[22][23] has calculated the optimum switching line minimizing the integral-error-squared value for the control system whose transfer function of the controlled element with the magnitude constraint on its input is $k/s^2$. L. N. Fitsner[17] has also considered the same problem and derived an approximate solution. Subsequently, J. D. Peason,[40] P. J. Brennan and A. P. Roberts[9] have respectively showed that one of the exact solutions established by A. T. Fuller was in close agreement with the digital and analog computations of a two point boundary value problem obtained by L. S. Pontryagin's Maximum Principle. W. H. Wonham[50] has verified the optimum system of A. T. Fuller's by solving the Bellman-Hamilton-Jacobi partial differential equation. A. T. Fuller[24] has also discussed the fact that his optimum solution satisfies the Maximum Principle. Although the analytical method established by A. T. Fuller provides the solution in the closed form, this is not applicable to other types of controlled systems owing to their specialized features.

In this chapter, from a slightly different point of view, an analytical method of designing the quasi-optimum control system with bounded control is described by introducing the concept of sub-interval optimization based upon the restricted-optimal control law which gives us the best possible strategy in a certain restricted control situation. The present attention is directed to determining the optimum switching function of restricted-optimal control by using R. Bellman's Dynamic Programming.[2] A new graphical technique for the determination of quasi-optimum switching lines is also presented by applying the optimum switching function of restricted-optimal control.

## 1.2 Formulation of the Problem and Definition of the Restricted-Optimal Control

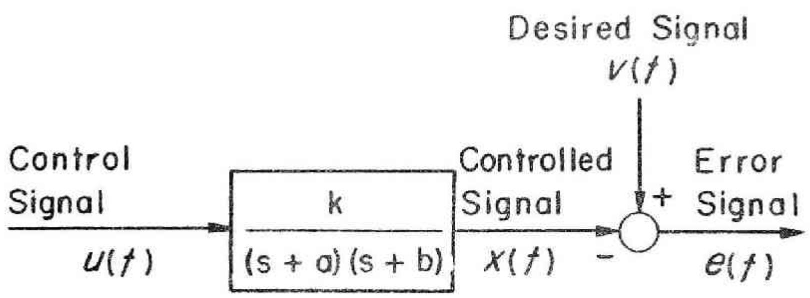


Fig. 1.1 Block diagram of the system to be considered

We consider a system as shown in Fig. 1.1, in which the dynamical characteristic of the controlled system is represented by a transfer function of the form $k/(s+a)(s+b)$ between the control signal $u(t)$ and the controlled signal $x(t)$. The desired input $v(t)$ is assumed to be a step signal with the magnitude $d_1$. The control signal is subjected to a magnitude constraint as

$$|u(t)| \leq L, \tag{1.2-1}$$

where $L$ is a pre-assigned positive constant.

The principal problem considered here is to design the controller so that the integral-error-squared value for a finite control interval, namely,

$$J_2 = \int_0^T e(\rho)^2 \, d\rho \tag{1.2-2}$$

is minimized, for any initial state of the system at the time $t=0$. In Eq. (1.2-2), $T$ is also a pre-assigned constant which expresses the final instant of control operation.

Another problem to minimize the integral-error-squared value for an infinite control interval,

$$M_2 = \int_0^\infty e(\rho)^2 \, d\rho \tag{1.2-3}$$

is simultaneously treated as the special case of the former problem shown by Eq. (1.2-2).

Let the state variables of the controlled system be expressed by $x_1(t)$ and $x_2(t)$. The system dynamics can be expressed as

$$\left.\begin{aligned} \dot{x}_1(t) &= -b\, x_1(t) + x_2(t), \qquad & x_1(0) &= c_1 \\ \dot{x}_2(t) &= -a\, x_2(t) + k\, u(t), \qquad & x_2(0) &= c_2 \end{aligned}\right\}, \tag{1.2-4}$$

where $x_1(t)=x(t)$ and "." expresses the symbol representing the differentiation with respect to the time variable $t$. By using the corresponding state variables of the error signal, $e_1(t)$ and $e_2(t)$, and considering the fact that the desired input is a step signal, Eq. (1.2-4) becomes

$$\left.\begin{aligned} \dot{e}_1(t) &= -b\, e_1(t) + e_2(t)\ , \quad e_1(0) = l_1 \equiv d_1 - c_1 \\ \dot{e}_2(t) &= -a\, e_2(t) + a b d_1 - k u(t),\ e_2(0) = l_2 \equiv b d_1 - c_2 \end{aligned}\right\}. \qquad (1.2\text{-}5)$$

Let us define the new variables as*

$$\left.\begin{aligned} \hat{t} &= a\, t \\ \hat{e}_1 &= a^2 e_1 / kL \\ \hat{e}_2 &= a e_2 / kL \\ \hat{u} &= u / L \end{aligned}\right\}\ , \qquad (1.2\text{-}6)$$

Eqs. (1.2-1), (1.2-2) and (1.2-5) become

$$|\,\hat{u}(\hat{t})\,| \le 1\ , \qquad (1.2\text{-}7)$$

$$J_2 = (kL)^2 / a^5 \cdot \int_0^{\hat{T}} \hat{e}_1(\rho)^2\, d\rho\ , \qquad (1.2\text{-}8)$$

$$\left.\begin{aligned} \dot{\hat{e}}_1(\hat{t}) &= -\beta\, \hat{e}_1(\hat{t}) + \hat{e}_2(\hat{t}),\quad \hat{e}_1(0) = \hat{l}_1 \\ \dot{\hat{e}}_2(\hat{t}) &= -\hat{e}_2(\hat{t}) + \beta d_1 - u(t),\quad \hat{e}_2(0) = \hat{l}_2 \end{aligned}\right\}\ , \qquad (1.2\text{-}9)$$

where $\beta \equiv b/a$, $\hat{T} \equiv aT$, $\hat{d}_1 \equiv a^2 d_1 / kL$, $\hat{l}_1 \equiv a^2 l_1 / kL$ and $\hat{l}_2 \equiv a\, l_2 / kL$,

From Eq. (2.2-8), we obtain a relation;

$$\min_{\substack{u \\ |u| \le L}} J_2 = (kL)^2 / a^5 \min_{\substack{\hat{u} \\ |\hat{u}| \le 1}} \hat{J}_2\ , \qquad (1.2\text{-}10)$$

where

$$\hat{J}_2 = \int_0^{\hat{T}} \hat{e}_1(\rho)^2\, d\rho\ . \qquad (1.2\text{-}11)$$

The problem is therefore reduced to find the normalized control signal $\hat{u}(\hat{t})$ which minimizes the functional $\hat{J}_2$ given by Eq. (1.2-11) taking the constraints shown by Eqs. (1.2-7) and (1.2-9) into account.

---

* This transformation ceases to be valid in the case where $a = b = 0$. The treatment for this limiting case will be presented in Appendix-A.

For the simplicity of present description, we shall omit the chapeau "^" henceforth, unless it is necessary.

By applying the concept of Dynamic Programming,(3) the problem is reduced to solve the following partial differential equation;

$$\frac{\partial \phi}{\partial \tau}=\min_{\substack{u \\ |u|\leq 1}} \{ e_1^2 - (\beta e_1 - e_2)\frac{\partial \phi}{\partial e_1} - (e_2 - \beta d_1 + u)\frac{\partial \phi}{\partial e_2} \}, \qquad (1.2\text{-}12)$$

with the initial condition;

$$\phi=0 \qquad (\tau=0), \qquad (1.2\text{-}13)$$

where $\phi=\phi(e_1, e_2; \tau)$ and this is defined by

$$\phi(e_1, e_2; \tau)=\min_{\substack{u \\ |u|\leq 1}} \int_0^{\tau} e_1(\rho)^2 d\rho, \qquad (1.2\text{-}14)$$

and $\tau$ expresses an auxiliary time variable which is introduced for the convenience of present discussion and is called the reversed time variable, i.e., $\tau\equiv T-t$.

From the right hand side of Eq. (1.2-12), the optimum control signal $u(t)$ which makes Eq. (1.2-11) minimum can be obtained as

$$\left.\begin{array}{l} \bar{u}(t)=\mathrm{sgn}[z(\tau)] \\ z(\tau)\equiv \partial\phi/\partial e_2 \end{array}\right\}, \qquad (1.2\text{-}15)$$

where

$$\left.\begin{array}{lll} \mathrm{sgn}[z] = 1 & & (z>0) \\ \quad = -1 & & (z<0) \end{array}\right\}. \qquad (1.2\text{-}16)$$

From Eq. (1.2-15), the configuration of the optimum control system with bounded control becomes a relay control system as shown in Fig. 1.2, where the on-off state of the relay element depends upon the sign of the function $z(\tau)$ in Eq. (1.2-15).*

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* In the later discussion, the terminology "switching function" is used to express $z(\tau)$

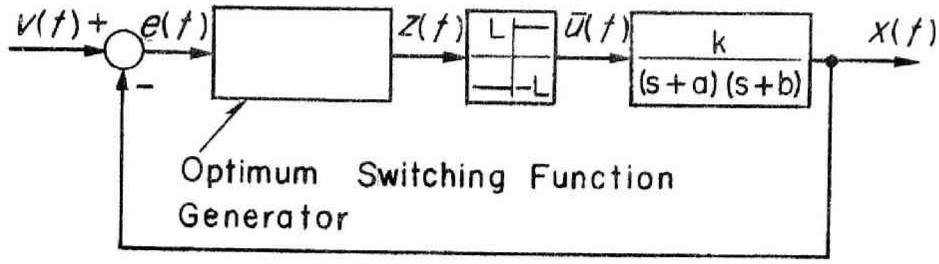


Fig. 1.2 Optimum configuration of the control system with bounded control

In order to obtain the switching function, we substitute Eq. (1.2-15) for $u(t)$ into Eq. (1.2-12), the following non-linear partial differential equation is derived;

$$\frac{\partial \phi}{\partial \tau} = e_1^2 - (\beta e_1 - e_2)\frac{\partial \phi}{\partial e_1} - (e_2 - \beta d_1 + \mathrm{sgn}\,[z])\frac{\partial \phi}{\partial e_2}\,. \qquad (1.2\text{-}17)$$

As the solution of Eq. (1.2-17), if we assume that the following quadratic form;(34)

$$\begin{aligned}\phi\,(e_1, e_2\,;\,\tau) = {} & k_0(\tau) + k_1(\tau)\,e_1 + k_2(\tau)\,e_2 \\ & + k_3(\tau)\,e_1 e_2 + k_4(\tau)\,e_1^2 + k_5(\tau)\,e_2^2\,,\end{aligned} \qquad (1.2\text{-}18)$$

then Eq. (1.2-17) with the initial condition Eq. (1.2-13) becomes the following set of non-linear simultaneous differential equations;

$$\left.\begin{aligned} k_0'(\tau) &= (\beta d_1 - \mathrm{sgn}\,[z(\tau)])\,k_2(\tau) \\ k_1'(\tau) &= -\beta k_1(\tau) + (\beta d_1 - \mathrm{sgn}\,[z(\tau)])\,k_3(\tau) \\ k_2'(\tau) &= k_1(\tau) - k_2(\tau) + 2\,(\beta d_1 - \mathrm{sgn}\,[z(\tau)])k_5(\tau) \\ k_3'(\tau) &= 2k_4(\tau) - (\beta+1)\,k_3(\tau) \\ k_4'(\tau) &= 1 - 2\beta k_4(\tau) \\ k_5'(\tau) &= k_3(\tau) - 2k_5(\tau) \end{aligned}\right\}, \qquad (1.2\text{-}19)$$

with the initial conditions;

$$k_i(0)=0 \quad (i=0,1,2,\cdots\cdots,5), \tag{1.2-20}$$

where "$\prime$" expresses the differentiation with respect to the reversed time variable $\tau$ and the switching function $z(\tau)$ is expressed by

$$z(\tau)\equiv k_2(\tau)+k_3(\tau)e_1+2k_5(\tau)e_2 . \tag{1.2-21}$$

On the other hand, changing the variable $t$ by the reversed time variable $\tau$ and substituting $\bar{u}(\tau)$ for $u(t)$ into Eq. (1.2-9), we have

$$\left.\begin{aligned} e_1'(\tau) &= \beta e_1(\tau) - e_2(\tau), & e_1(T)&=l_1 \\ e_2'(\tau) &= e_3(\tau)-\beta d_1+\mathrm{sgn}\,[z(\tau)], & e_2(T)&=l_2 \end{aligned}\right\} . \tag{1.2-22}$$

The design problem for a finite control interval is therefore reduced to a two point boundary value problem which consists of Eqs. (1.2-19) and (1.2-22). Because of the difficulty in solving a two point boundary value problem the following assumption is made.

Assumption: The final instant, $T$, of the control operation is suitably chosen beforehand so that the optimal control might be performed by no more than one switching operation within this control interval, $[0,T]$, which depends on both the initial conditions and the system parameters.

Furthermore, the terminology, "restricted-optimal control", is defined as follows:

Definition: The terminology, "restricted-optimal control", is introduced to define the restricted control situation satisfying the assumption mentioned above.*

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* The importance of this specialized control situation will be discussed in section 1.4.

## 1.3 Determination of Switching Functions realizing the Restricted-Optimal Control

We express the time instant of the switching operation with respect to the reversed time by $\tau=\tau_s$. This time instant is equivalent to $t_s$ in the real time, $t$.

Assuming that the sign of relay output at $\tau=0$ is negative, Eq. (1.2-17) can be replaced by the combination of the following two linear partial differential equations;

$$\frac{\partial \phi^+}{\partial \tau}=e_1^2-(\beta e_1-e_2)\frac{\partial \phi^+}{\partial e_1}-(e_2-\beta d_1-1)\frac{\partial \phi^+}{\partial e_2} \quad \text{for } \frac{\partial \phi}{\partial e_2}<0, \tag{1.3-1$_1$}$$

$$\frac{\partial \phi^-}{\partial \tau}=e_1^2-(\beta e_1-e_2)\frac{\partial \phi^-}{\partial e_1}-(e_2-\beta d_1+1)\frac{\partial \phi^-}{\partial e_2} \quad \text{for } \frac{\partial \phi}{\partial e_2}>0. \tag{1.3-1$_2$}$$

By using the relations;

$$\left.\begin{aligned} &\phi^+\big|_{\tau=0}=0 \\ &\phi^-\big|_{\tau=\tau_s}=\phi^+\big|_{\tau=\tau_s} \\ &\frac{\partial \phi^-}{\partial e_2}\Big|_{\tau=\tau_s}=\frac{\partial \phi^+}{\partial e_2}\Big|_{\tau=\tau_s} \end{aligned}\right\}, \tag{1.3-2}$$

the solution of Eq. (1.2-17) is obtained as

$$\left.\begin{aligned} \phi&=\phi^+ \quad \text{for } \tau_s\geq\tau\geq 0 \\ &=\phi^- \quad \text{for } T\geq\tau\geq\tau_s \end{aligned}\right\}. \tag{1.3-3}$$

The two-point boundary value problem stated in the previous section can therefore be converted into the initial value problem as follows:

$$\left.\begin{array}{ll} k_0'^{+}(\tau)=(\beta d_1+1)\,k_2^{+}(\tau), & k_0^{+}(0)=0 \\ k_1'^{+}(\tau)=-\beta k_1^{+}(\tau)+(\beta d_1+1)\,k_3^{+}(\tau), & k_1^{+}(0)=0 \\ k_2'^{+}(\tau)=k_1^{+}(\tau)-k_2^{+}(\tau)+2\,(\beta d_1+1)\,k_5^{+}(\tau), & k_2^{+}(0)=0 \\ k_3'^{+}(\tau)=2\,k_4^{+}(\tau)-(\beta+1)\,k_3^{+}(\tau), & k_3^{+}(0)=0 \\ k_4'^{+}(\tau)=1-2\beta k_4^{+}(\tau), & k_4^{+}(0)=0 \\ k_5'^{+}(\tau)=k_3^{+}(\tau)-2k_5^{+}(\tau), & k_5^{+}(0)=0 \end{array}\right\}, \qquad (1.3\text{-}4)_1$$

and

$$\left.\begin{array}{ll} k_0'^{-}(\tau)=(\beta d_1-1)\,k_2^{-}(\tau), & k_0^{-}(\tau_s)=k_0^{+}(\tau_s) \\ k_1'^{-}(\tau)=-\beta k_1^{-}(\tau)+(\beta d_1-1)\,k_3^{-}(\tau), & k_1^{-}(\tau_s)=k_1^{+}(\tau_s) \\ k_2'^{-}(\tau)=k_1^{-}(\tau)-k_2^{-}(\tau)+2\,(\beta d_1-1)\,k_5^{-}(\tau), & k_2^{-}(\tau_s)=k_2^{+}(\tau_s) \\ k_3'^{-}(\tau)=2k_4^{-}(\tau)-(\beta+1)\,k_3^{-}(\tau)\,. & k_3^{-}(\tau_s)=k_3^{+}(\tau_s) \\ k_4'^{-}(\tau)=1-2\beta k_4^{-}(\tau), & k_4^{-}(\tau_s)=k_4^{+}(\tau_s) \\ k_5'^{-}(\tau)=k_3^{-}(\tau)-2k_5^{-}(\tau), & k_5^{-}(\tau_s)=k_5^{+}(\tau_s) \end{array}\right\}, \qquad (1.3\text{-}4)_2$$

where $k_i^{-}(\tau)$ and $k_i^{+}(\tau)$ $(i=0,1,\ldots\ldots,5)$ express respectively the coefficients in the quadratic forms of solutions;

$$\phi^{+}(e_1,e_2\,;\,\tau)=k_0^{+}(\tau)+k_1^{+}(\tau)\,e_1+k_2^{+}(\tau)\,e_2 +k_3^{+}(\tau)\,e_1e_2+k_4^{+}(\tau)\,e_1^2+k_5^{+}(\tau)\,e_2^2\,, \qquad (1.3\text{-}5)_1$$

and

$$\phi^{-}(e_1,e_2\,;\,\tau)=k_0^{-}(\tau)+k_1^{-}(\tau)\,e_1+k_2^{-}(\tau)\,e_2 +k_3^{-}(\tau)\,e_1e_2+k_4^{-}(\tau)\,e_1^2+k_5^{-}(\tau)\,e_2^2\,. \qquad (1.3\text{-}5)_2$$

By a similar procedure presented in Appendix-A, from Eqs. $(1.3\text{-}4)_1$ and $(1.3\text{-}4)_2$, the optimum switching functions of restricted-optimal control for positive trajectories of the system response become as follows* :

(1) In the case where $a=b=0$, the switching function is

$$z^{+}_{\mathrm{RES}}\equiv\frac{1}{4}\tau^2+\frac{e_1}{kL}+\frac{2}{3}\tau\,\frac{e_2}{kL}\,. \qquad (1.3\text{-}6)$$

* The optimum switching functions for negative trajectories can easily be obtained by considering the symmetric property of the switching line with respect to the origin of the phase coordinate, except for the case where $a\neq 0$ and $b\neq 0$.

(2) In the case where $a \neq 0$ and $b = 0$, the switching function is

$$z^{+}_{\text{RES}} \equiv [a\tau + \exp(-a\tau) - 1]^2 + 2[a\tau + \exp(-a\tau) - 1]\,\frac{a^2 e_1}{kL} + [2\{a\tau + \exp(-a\tau) - 1\} - \{1 - \exp(-a\tau)\}^2]\,\frac{a e_2}{kL}. \tag{1.3-7}$$

(3) In the case where $a \neq 0$ and $b \neq 0$, the switching function is

$$z^{+}_{\text{RES}} \equiv \frac{a^{+}}{\beta^2 (1-\beta)^2}\,[1 - \exp(-b\tau) - \beta\{1 - \exp(-a\tau)\}]^2$$

$$+ \left[\frac{1}{\beta(1+\beta)} - \frac{1}{\beta(1-\beta)}\exp(-2b\tau) + \frac{2}{(1-\beta)(1+\beta)}\exp\{-(b+a)\tau\}\right]\frac{a^2 e_1}{kL}$$

$$+ \left[\frac{1}{\beta(1+\beta)} - \frac{1}{(1-\beta)^2}\exp(-2a\tau) + \frac{4}{(1+\beta)(1-\beta)^2}\exp\{-(b+a)\tau\}\right.$$

$$\left. - \frac{1}{\beta(1-\beta)^2}\exp(-2b\tau)\right]\frac{a e_2}{kL}, \tag{1.3-8}$$

where $a^{+} \equiv abd_1/kL + 1$ and $\beta \equiv b/a$.

## 1.4 Sub-Interval Optimization Technique

### 1.4-1 Basic Concept of a Sub-Interval Optimization Technique

As the author has already stated, since there is no items to be mentioned here in the case where the restricted-optimal control can be realized during the time interval $[0, T]$, then the technique introduced here is to provide a pseudo-optimization strategy for the control interval which does not satisfy the assumption realizing the restricted-optimal control.

The fundamental idea may be stated along the illustration of Fig. 1.3 as follows:

(1) By taking into account of both initial conditions and system parameters, determine the fictitious final instant $T_{\text{sub-1}}$ of sub-interval which satisfies the assumption realizing the restricted-optimal control.

(2) Solve the fictitious restricted-optimal control problem where

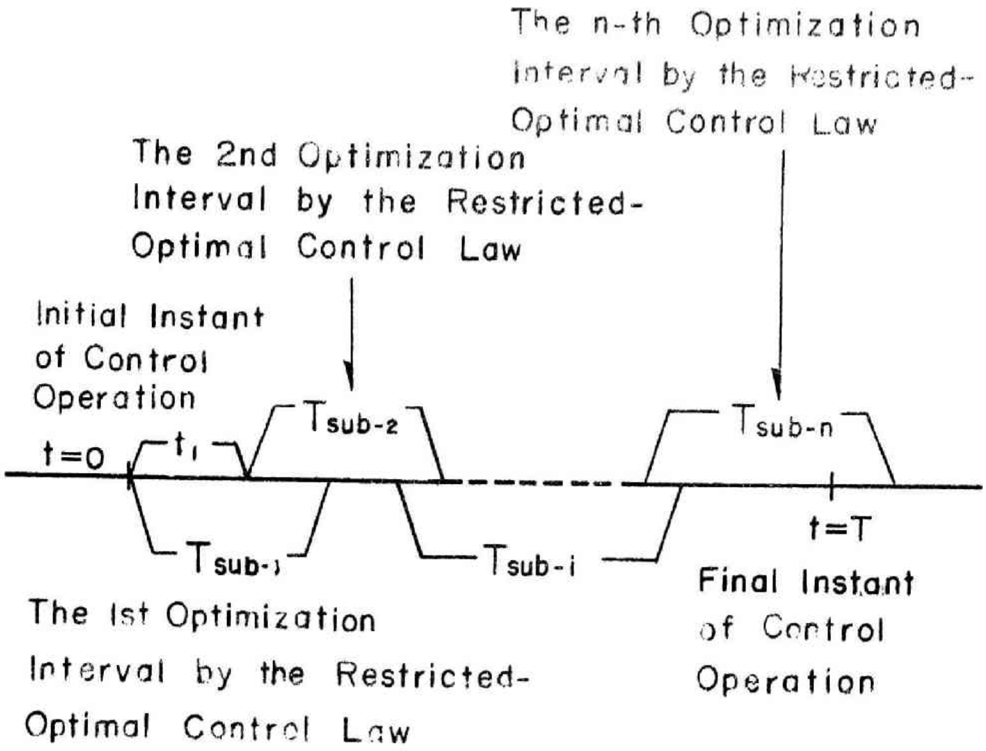


Fig. 1.3 Translation of the sub-interval with respect to time

$T_{\text{sub-1}}$ is tentatively considered as the final instant and decide the control law for the optimization of the first fictitious sub-interval.

(3) Compute the system states at the terminus of the first fictitious interval $T_{\text{sub-1}}$ and examine whether the rest interval satisfies the assumption with respect to the computed system states or not.

(4) If the terminus of the first fictitious interval $T_{\text{sub-1}}$ does not satisfy the assumption, then compute the value of $e_1$-coordinate corresponding to $e_2 = 0$ after the first change over. Determine the second fictitious final instant $T_{\text{sub-2}}$ by considering the values of $e_1$, calculated above and $e_2 = 0$ as the initial conditions for the given optimization problem and decide the control law for the second fictitious interval.

(5) Continue the iterative procedure listed above until the n-th extension involves the final instant of time $T$.

Fig. 1.4 shows the logic flow chart of the pseudo-optimization procedure. Since the necessary condition realizing the restricted-optimal control is expressed as an inequality which is described by system initial conditions, system parameters and the final instant of control operation, then the necessary condition is assumed to be expressed by the relation, $f(l_1, l_2; \beta; T) \leq 0$, in Fig. 1.4. However, it is in general difficult to derive the relation $T \leq g(l_1, l_2; \beta)$ from $f(l_1, l_2; \beta; T) \leq 0$ because it is a transcendetal inequality. So the following graphical method is applied to determine the final instant of a fictitious optimization interval without using the function $f(l_1, l_2; \beta; T) = 0$.

### 1.4-2 Geometrical Interpretation of the Necessary Condition realizing the Restricted-Optimal Control

As a preliminary consideration, let us calculate numerically the intersections of the system response trajectories with the switching line expressed as $z^{\pm}_{\mathrm{RES}} = 0$ in the phase plane.* For simplicity, we consider the limiting case where the system parameters, $a$ and $b$ in Eq. (1.2-5), become zero. From Eq. (1.3-6), the normalized switching line of restricted-optimal control for positive trajectories in this case is

$$\frac{1}{4}\hat{\tau}^2 + \frac{2}{3}\hat{\tau}\hat{e}_2 + \hat{e}_1 = 0 , \tag{1.4-1}$$

where

$$\left.\begin{aligned} \hat{\tau} &= \tau \\ \hat{e}_1 &= e_1 / kL \\ \hat{e}_2 &= e_2 / kL \end{aligned}\right\} , \tag{1.4-2}$$

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* Strictly speaking, the two dimensional state space must be used for this term.[27]

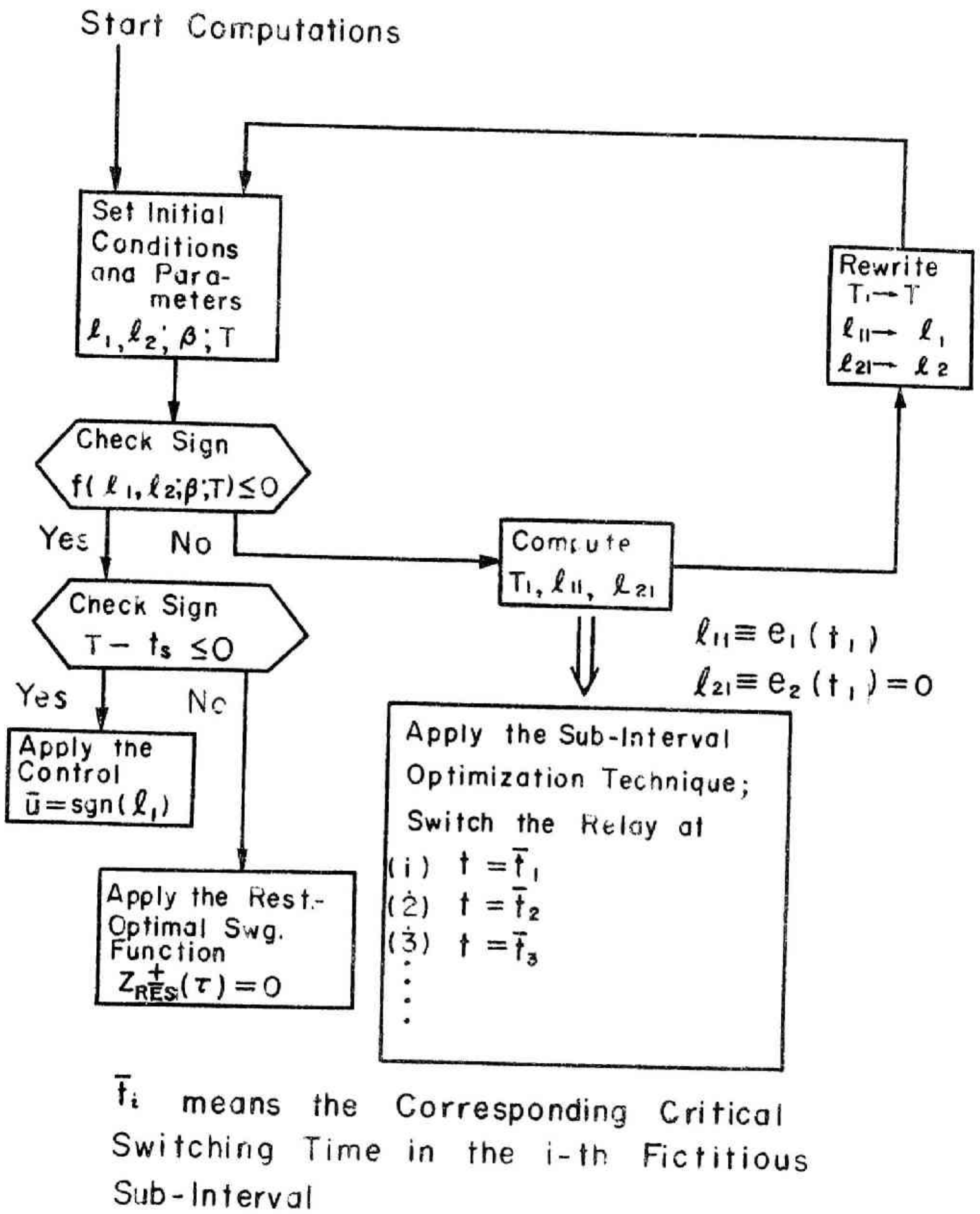


Fig. 1.4 Logic flow chart of a pseudo-optimization procedure

On the other hand, by setting $a=b=0$ in Eq. (1.2-5) and integrating the second relation of Eq. (1.2-5) with respect to the variable $t$, the following equation which expresses the normalized $e_2$-coordinate

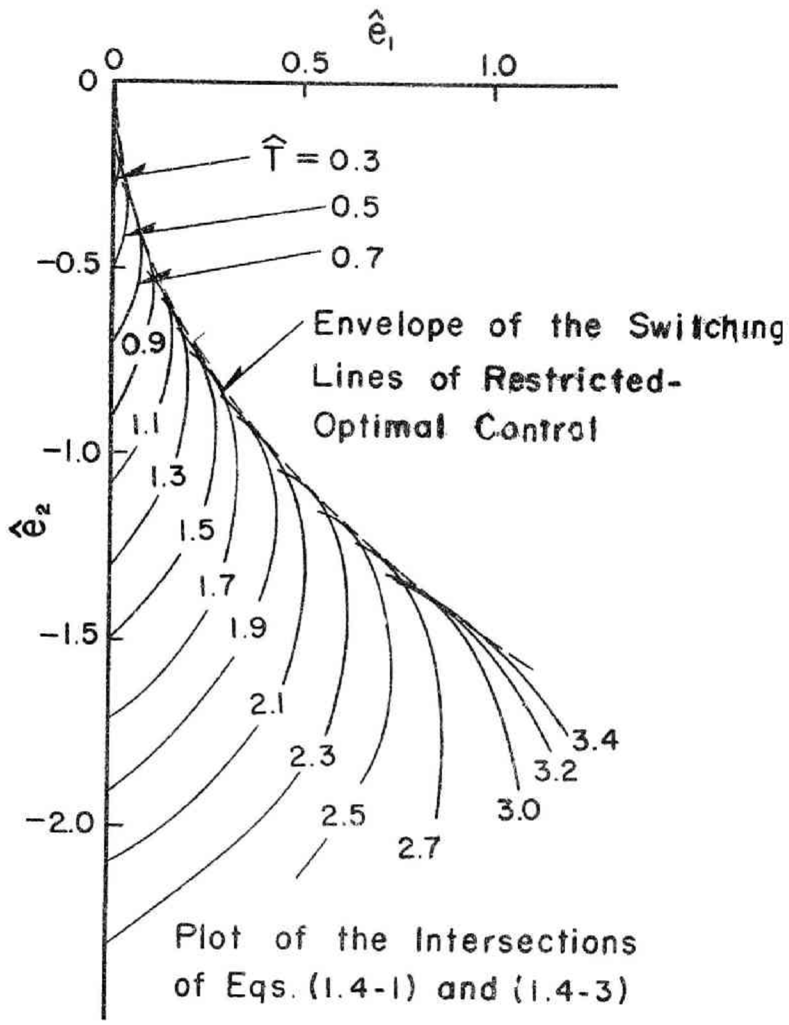


Fig. 1.5 Plot of the switching lines of restricted-optimal control for $a=b=0$

of the positive trajectory of the system response can be obtained as

$$\hat{e}_2 = \hat{l}_2 - (\hat{T} - \hat{\tau}), \qquad (1.4\text{-}3)$$

where $\hat{l}_2 = l_2 / kL$ and $\hat{T} = T$ respectively.

By using Eqs. (1.4-1) and (1.4-3) and considering the value of $\hat{T}$ as a parameter, we calculate the intersections stated above and the results are plotted as the solid curves in Fig. 1.5, in which we can set the initial condition $\hat{l}_2$ to be zero without any loss of generality. In Fig. 1.5, as well in the following figures showing the switching lines of restricted-optimal control, it must be noticed that only the switching lines for positive trajectories are illustrated. The switching lines for negative trajectories are easily plotted from the symmetric property of a switching line.

In order to explain the role of restricted-optimal switching in detail, we consider the particular value of $\hat{T}=1.1$ and shows schematically the figure of switching line as shown in Fig. 1.6 by extracting the curve corresponding to $\hat{T}=1.1$ from Fig. 1.5. In Figs. 1.5 and 1.6, the dotted line expresses the envelope of the plot of the intersections, which indicates the right boundary region of the restricted-optimal switching. In other words, this means that the plots of the point of restricted-optimal switching do not locate in the hatched area of Fig. 1.6. The left boundary region is obtained as the $\hat{e}_2$-axis from the consideration of taking the limit of the variable $\tau$ to zero in Eq. (1.4-1). Furthermore, from the definition of the restricted-optimal control, it is evident that the effective(optimal) switching line of restricted-optimal control becomes the part of the curve running downward from the point of tangent to the envelope $Q$ if we fixed the final instant of control operation $\hat{T}$ to be 1.1. Since we can draw a positive trajectory so as to run across at the point $Q$, we express the point of intersection between this positive trajectory and the $\hat{e}_1$-axis by $Q'$. Since we can also calculate the $\hat{e}_1$-coordinate of the intersection $Q'$, if we represent this by the symbol $\bar{l}_{10}$, then the relation $\hat{l}_1 \geq \bar{l}_{10}$ has the equivalent physical meaning of the necessary condition realizing the restricted-optimal control in the case where $\hat{T}=1.1$, in Eq. (1.2-2) and $\hat{l}_2=0$, $a=b=0$ in Eq.(1.2-5). Since the envelope means a set of the critical point for the fixed $\hat{T}$ like $Q$, it is concluded that the envelope gives the boundary switching line of the restricted-optimal control. It is revealed from the

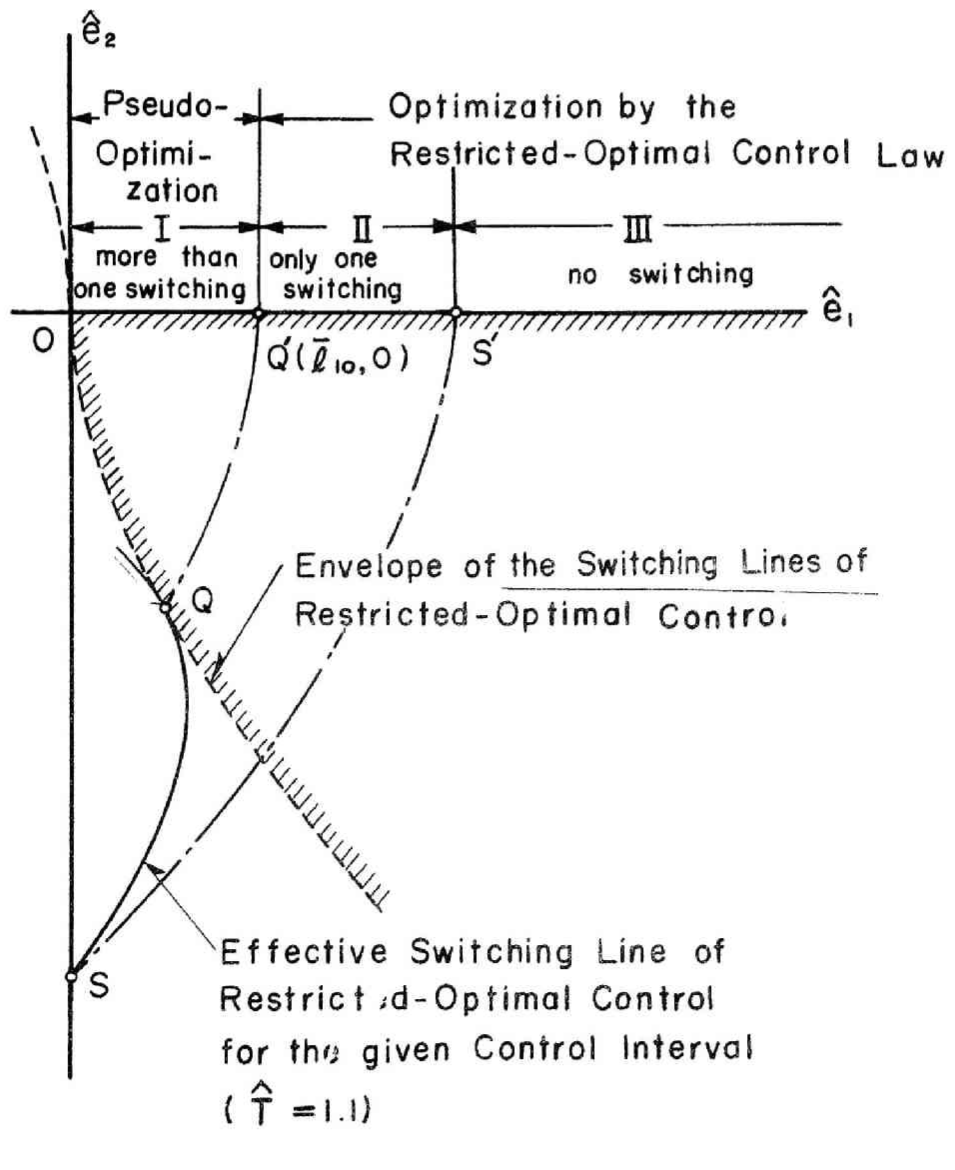


Fig. 1.6 Illustration of the geometrical meaning of the switching lines of restricted-optimal control

discussions presented above that, by obtaining the envelope of the restricted-optimal switching line in the phase plane through the graphical method, we can get the necessary condition realizing the restricted-optimal control in a geometrical sense.

1.4-3 Geometrical Interpretation of the Sub-Interval Optimization Technique

As the author has already mentioned, the basic concept of the sub-interval optimization technique is introduced to give a pseudo-optimization rule for the control interval which does not satisfy the assumption realizing the restricted-optimal control. This means that the concept provides a pseudo-switching rule for a smaller initial condition $\hat{l}_1$ than $\bar{l}_{10}$ in case of a fixed control interval.

In this paragraph, let us consider the geometrical meaning of sub-interval optimization technique.

Fig. 1.7 shows schematically the basic concept of sub-interval optimization technique from the graphical point of view. In Fig. 1.7, let us consider the optimization problem with respect to the initial state $F(\hat{l}_1, \hat{l}_2)$ from which the response trajectory will across the $\hat{e}_1$-axis at the point $R'$ nearer to the origin than the point $Q'$. In this case the response trajectory meets the boundary line for the realization of restricted-optimal control at the point $R$. The intersection of the response trajectory with the restricted-optimal switching line derived for the given control interval will be, on the other hand, occurred at the point $R''$. Then we must introduce the sub-interval optimization technique for this problem. The procedure stated in 1.4-1 can be interpreted as follows: Firstly we consider the fictitious sub-interval $[0, T_{\text{sub-1}}]$ of which the critical point of restricted-optimal switching will be occurred at $R$. Then the response trajectory switched at $R$ will run toward the point $R_1'$.

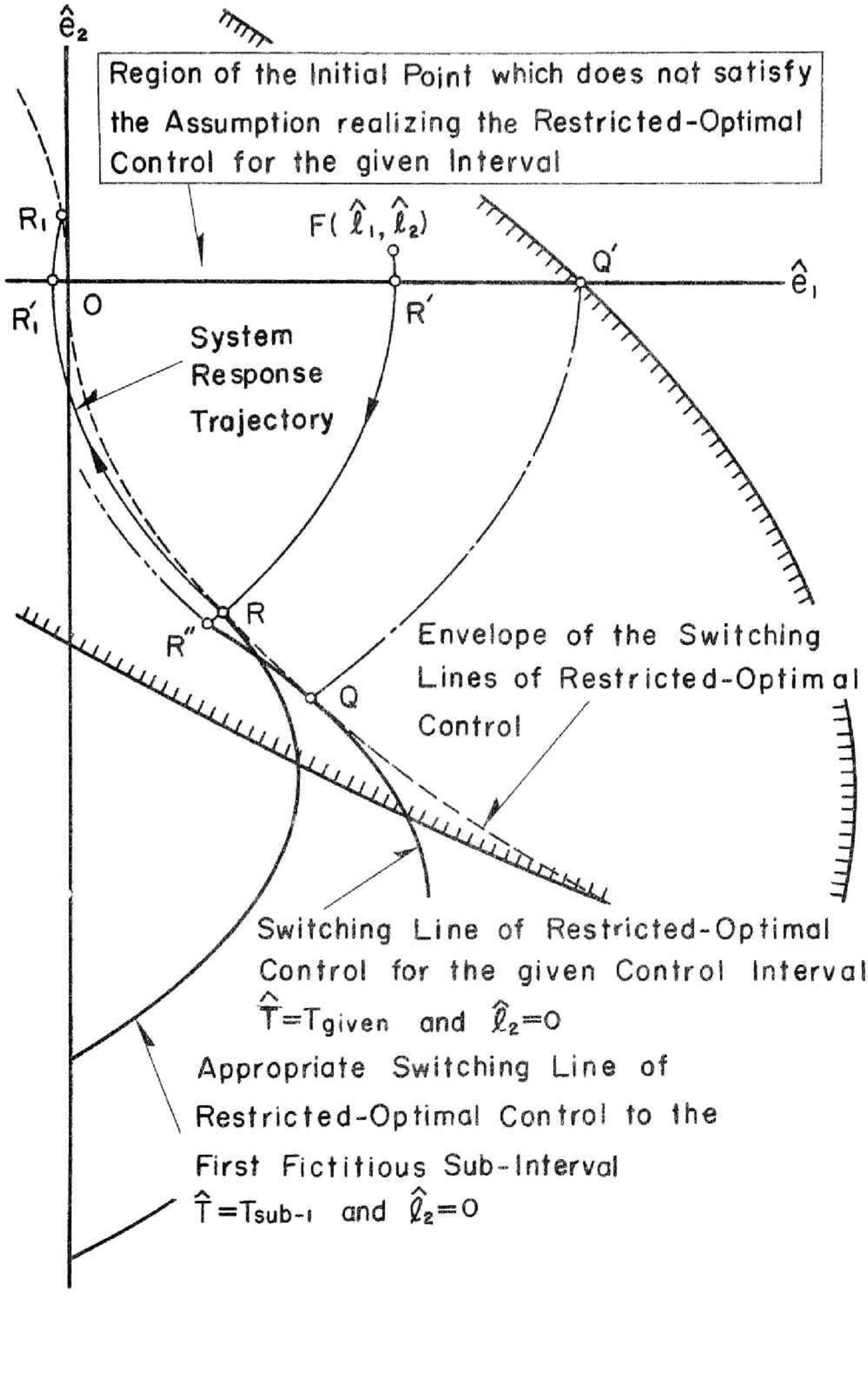


Fig. 1.7 Illustration of the geometrical meaning of sub-interval optimization technique

Since the time duration $t_1$ which the response changes the states from these of the point $F$ to these of $R_1'$ can easily be obtained, then we can calculate both the rest of control interval $T_1 = T - t_1$ and the $\hat{e}_1$-coordinate of the point $R_1'$. Secondly, we must examine if the rest of control interval satisfies the assumption realizing the restricted-optimal control regarding to the newly computed system states, i.e., the $\hat{e}_1$-coordinate of $R_1'$. By considering the results of examination mentioned above, we may carry out either the determination of switching time by using the restricted-optimal switching rule, or the further computation deriving the second fictitious sub-interval. The procedure presented above have to be continued until the final step.

## 1.5 Quasi-Optimum Stationary Switching Lines

From discussions stated in the previous paragraph, it turns out that the envelope of switching lines of the restricted-optimal control provides a pseudo-optimal switching rule for the infinite-control-interval problem which minimizes the functional $M_2$ given by Eq. (1.2-3).

Definition: A switching line which realizes a pseudo-minimization of the functional $M_2$ is referred to as a quasi-optimum stationary switching line.

This section is devoted to show the plausibility of using the envelope of switching lines of restricted-optimal control as a quasi-optimum stationary switching line.

### 1.5-1 In the case where $a = b = 0$.

It is in general difficult to derive the analytical form of the envelope stated above. Fortunately, however, it can be done in this limiting case.

Letting $\hat{l}_2 = 0$ in Eq. (1.4-3) and substituting Eq.(1.4-3) into Eq. (1.4-1), Eq.(1.4-1) becomes

$$\frac{1}{4}(\hat{e}_2+\hat{T})^2+\frac{2}{3}(\hat{e}_2+\hat{T})\hat{e}_2+\hat{e}_1=0. \qquad (1.5\text{-}1)$$

Differentiation of Eq. (1.5-1) with respect to the parameter $\hat{T}$ gives us

$$\frac{1}{2}(\hat{e}_2+\hat{T})+\frac{2}{3}\hat{e}_2=0. \qquad (1.5\text{-}2)$$

From Eqs. (1.5-1) and (1.5-2), we obtain the quasi-optimum stationary switching line for positive trajectories of the system response as

$$\hat{e}_1-\frac{4}{9}\hat{e}_2{}^2=0. \qquad (1.5\text{-}3)$$

By considering the symmetric property of the switching line, the quasi-optimum stationary switching line can be derived as

$$\hat{e}_1+0.4444\ \hat{e}_2|\hat{e}_2|=0. \qquad (1.5\text{-}4)$$

On the other hand, since the optimum solution derived by A. T. Fuller[(23)] is

$$\hat{e}_1+0.4446\ \hat{e}_2|\hat{e}_2|=0, \qquad (1.5\text{-}5)$$

then the plausibility of our proposal is analytically verified.

1.5-2 In the case where $a\neq 0$ and $b=0$

From Eq. (1.3-7), we can derive the normalized switching line as

$$\{\hat{\tau}+\exp(-\hat{\tau})-1\}^2+2\{\hat{\tau}+\exp(-\hat{\tau})-1\}\hat{e}_1$$
$$+[2\{\hat{\tau}+\exp(-\hat{\tau})-1\}-\{1-\exp(-\hat{\tau})\}^2]\hat{e}_2=0, \qquad (1.5\text{-}6)$$

where $\hat{\tau}$, $\hat{e}_1$ and $\hat{e}_2$ are the normalized variables respectively expressed by

$$\left.\begin{aligned}\hat{\tau}&=a\tau\\ \hat{e}_1&=a^2e_1/kL\\ \hat{e}_2&=ae_2/kL\end{aligned}\right\}. \qquad (1.5\text{-}7)$$

By integrating the second relation in Eq. (1.2-22), the following relation holds;

$$\hat{e}_2 = \hat{l}_2 \exp\{-(\hat{T}-\hat{\tau})\} - [1-\exp\{-(\hat{T}-\hat{\tau})\}] . \qquad (1.5\text{-}8)$$

By letting $\hat{l}_2 = 0$ in Eq. (1.5-8), and performing a similar procedure described previously, we can obtain numerical plots as shown in Figs. 1.8 and 1.9. In Fig. 1.8, the quasi-optimum stationary switching line is plotted by the dotted line. It is also interesting to compare

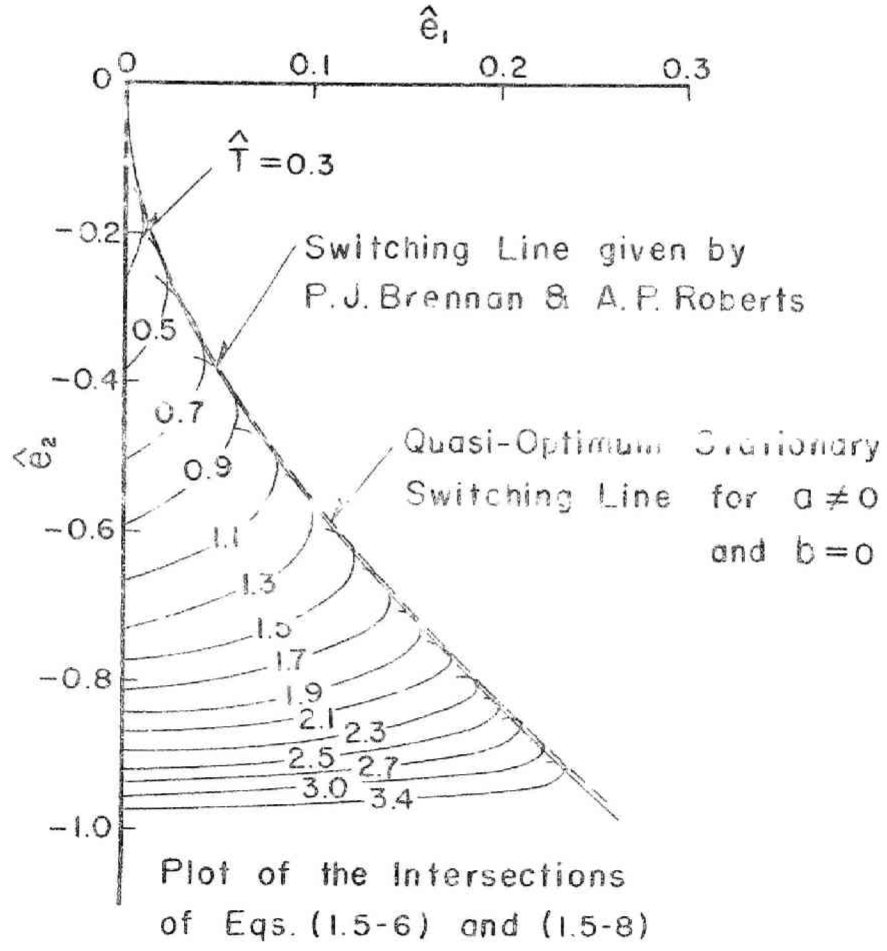


Fig. 1.8 Plot of the switching lines of restricted-optimal control for $a \neq 0$ and $b = 0$

our result with the switching line obtained by P. J. Brennan and A. P. Roberts which is also shown by the boldfaced solid line. Fig. 1.9 shows the quasi-optimum stationary switching lines with the value of $a$ as a parameter.

### 1.5-3 In the case where $a \neq 0$ and $b \neq 0$

In this case, from Eq. (1.3-8), the normalized switching line becomes

$$
\begin{aligned}
&\frac{1}{\beta^2(1-\beta)^2}\left(1+\frac{abd_1}{kL}\right)\left[1-\exp(-\beta\hat{\tau})-\beta\{1-\exp(-\hat{\tau})\}\right]^2 \\
&+\left[\frac{1}{\beta(1+\beta)}-\frac{1}{\beta(1-\beta)}\exp(-2\beta\hat{\tau})+\frac{2}{(1-\beta)(1+\beta)}\exp\{-(\beta+1)\hat{\tau}\}\right]\hat{e}_1 \\
&+\left[\frac{1}{\beta(1+\beta)}-\frac{1}{(1-\beta)^2}\exp(-2\hat{\tau})\right. \\
&\quad\left.+\frac{4}{(1+\beta)(1-\beta)^2}\exp\{-(\beta+1)\hat{\tau}\}-\frac{1}{\beta(1-\beta)^2}\exp(-2\beta\hat{\tau})\right]\hat{e}_2=0, \qquad (1.5\text{-}9)
\end{aligned}
$$

where $\hat{\tau}$, $\hat{e}_1$ and $\hat{e}_2$ are the normalized variables given by Eq. (1.5-7). By a similar way the results are obtained as shown in Figs. 1.10 and 1.11, where we set $d_1 = 0$, *i.e.*, $\alpha^+ = 1$, for simplicity. In Fig. 1.10, the quasi-optimum stationary switching line is plotted by the dotted line. Fig. 1.11 shows the quasi-optimum stationary switching lines with the value of $\beta$ as a parameter.

In the cases where system parameters are not equal to zero simultaneously, it is difficult to derive analytically the equation of the envelope. In order to perform it, we shall make the use of a curve-fitting procedure.[10] However, our discussions are confined to show the plausibility of the substitution of the quasi-optimum stationary switching line for the optimum stationary switching line.

## 1.6 Quasi-Optimum Non-stationary Switching Line

In this section, we consider the determination of the quasi-optimum switching line for the finite control interval where the switching functions, Eqs. (1.3-6), (1.3-7) and (1.3-8), of restricted-optimal control cease to be optimal.

Our attention is directed to utilizing the optimum switching line of restricted-optimal control with the quasi-optimum stationary switching line, and to provide a pseudo-optimal switching rule for an infinite-control-interval problem which minimizes the functional $J_2$ given by Eq. (1.2-2).

Definition; A switching line which realizes a pseudo-minimization of the functional $J_2$ is referred to as a quasi-optimum non-stationary switching line.

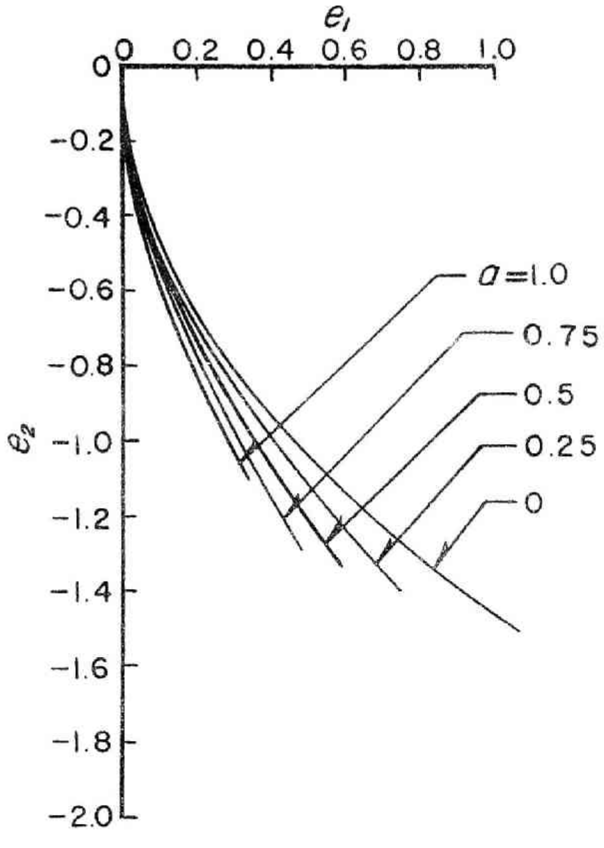


Fig. 1.9 Behavior of a quasi-optimum stationary switching line affected by the changes of a system parameter $a$ ($b=0$)

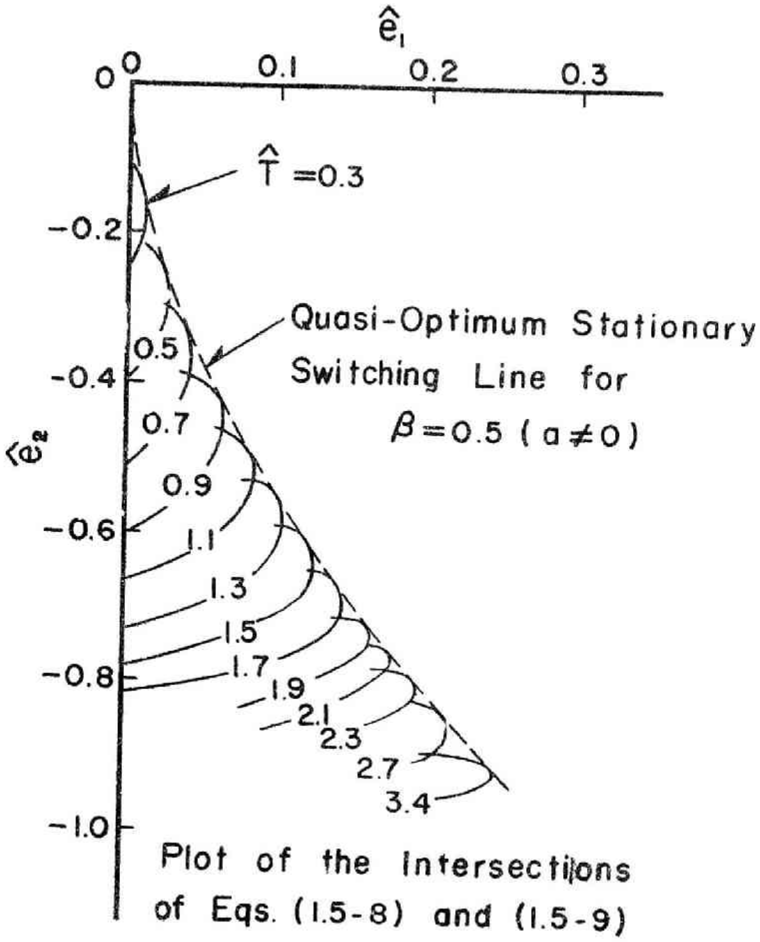


Fig. 1.10 Plot of the switching lines of restricted-optimal control for $a \neq 0$ and $b \neq 0$ ($d_1 = 0$)

In order to examine the possibility of using a quasi-optimum stationary switching line as a quasi-optimum switching line in a longer control interval than that of restricted-optimal control, we evaluate the difference of the control performances between the system with the quasi-optimum stationary switching line and that of the optimum one with the restricted-optimal switching line. The comparison is numerically carried out under the most undesirable situation where two switching lines yield the most different feature. Figs. 1.12, 1.13 and 1.14 show the graphical comparison of these control performances in the case where $a = b = 0$. In Figs. 1.12, 1.13 and 1.14,

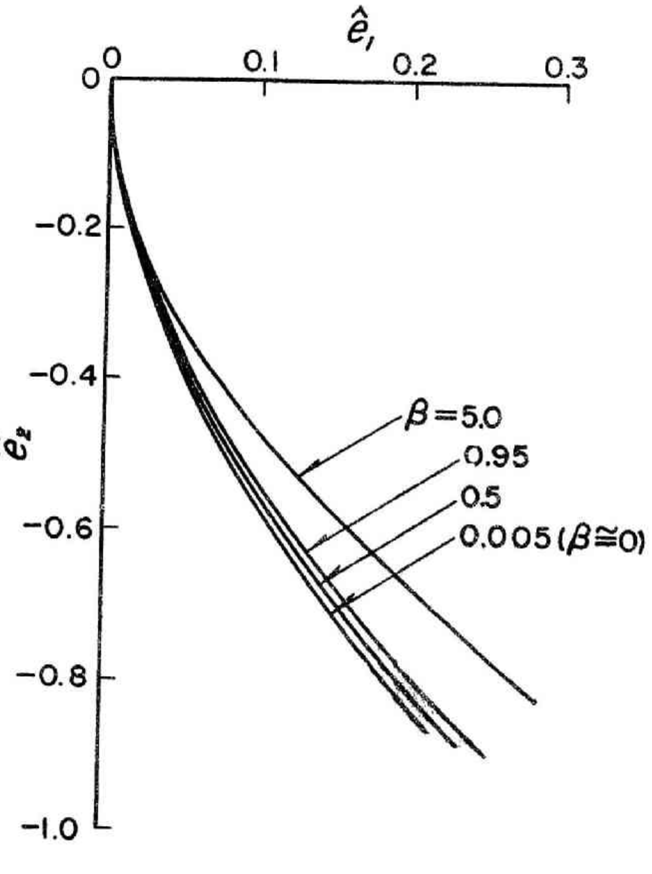


Fig. 1.11 Behavior of a quasi-optimum stationary switching line affected by the changes of system parameters $\beta \equiv b/a$ ($d_1 = 0$)

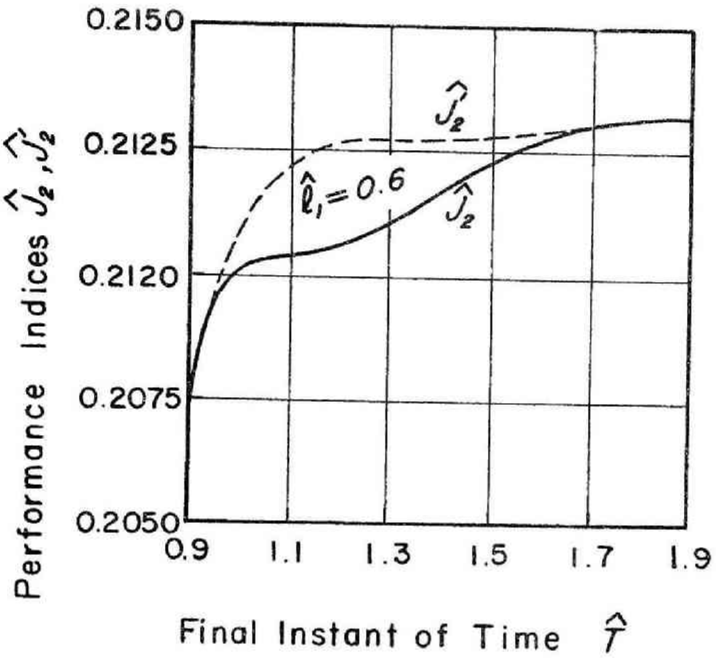


Fig. 1.12 Graphical comparison of performance indices ($\hat{l}_1 = 0.6$, $\hat{l}_2 = 0$)

$\hat{J}_2$ denotes the performance index of the optimum system with the restricted-optimal switching line. On the other hand, $\hat{J}_2'$ expresses the performance index of the system with the quasi-optimum stationary switching line. It is evident from Figs. 1.12, 1.13 and 1.14 that, since the maximum discrepancy between $\hat{J}_2$ and $\hat{J}_2'$ is only limited to $1 \sim 1.1$ percent of the value of $\hat{J}_2'$, and furthermore since it occurs in the most undesirable usage of a quasi-optimum stationary switching line, then from the viewpoint of practical engineering the instantaneous difference between $\hat{J}_2$ and $\hat{J}_2'$ with respect to time in the range of a longer control interval than that of restricted-optimal control may thus be neglected. This means that the quasi-optimum stationary

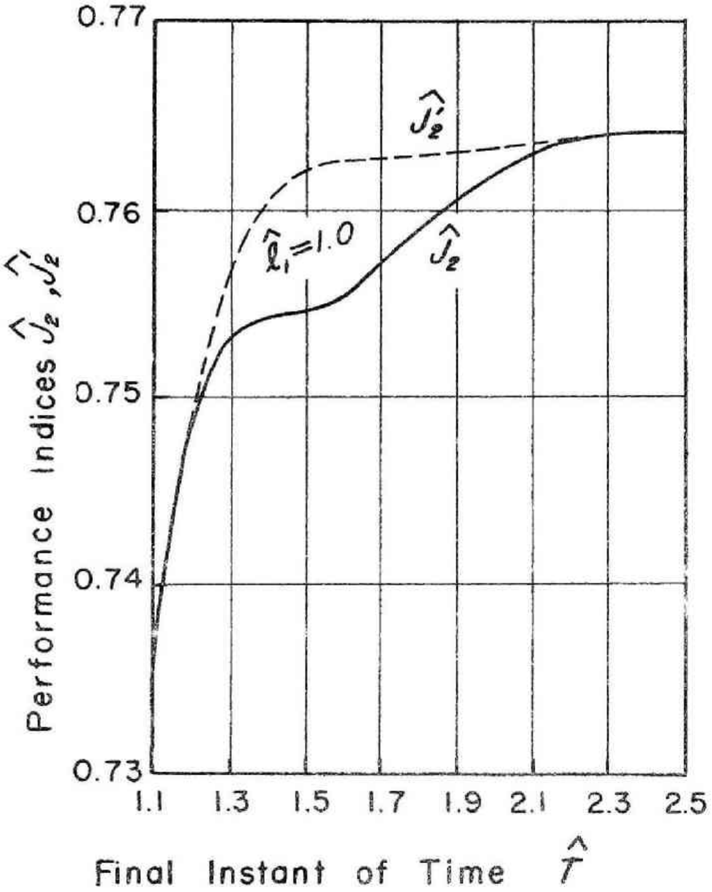


Fig. 1.13 Graphical comparison of performance indices

$(\hat{l}_1 = 1.0 \,,\ \hat{l}_2 = 0)$

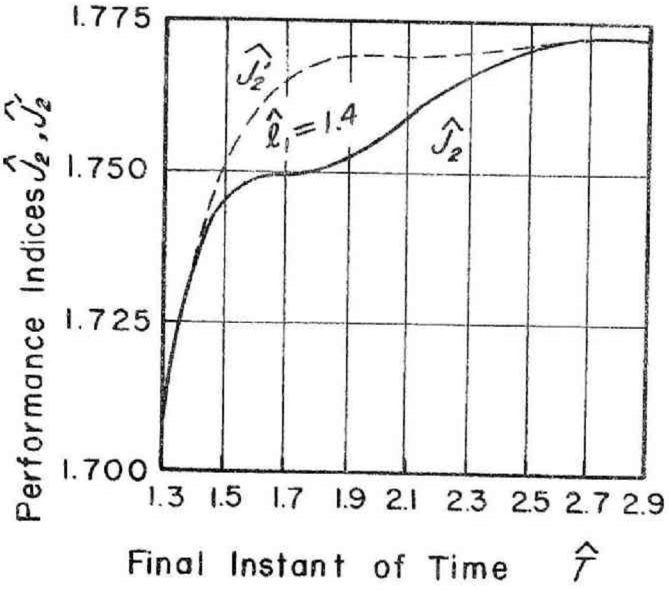


Fig. 1.14 Graphical comparison of performance indices
($\hat{l}_1 = 1.4$, $\hat{l}_2 = 0$)

switching line may be used for a pseudo-optimum non-stationary switching line in the time range stated above. The quasi-optimum non-stationary switching line for an arbitrary finite control interval can, therefore, be obtained by connecting the restricted-optimal switching line with the quasi-optimum stationary one. The graphical procedure is illustrated by an example as shown in Fig. 1.15. Fig. 1.16 shows an example of the response of the system subjected to a step input for the finite control interval, $\hat{T} = 1.3$ and $\hat{l}_2 = 0$. In Fig. 1.16, the points indicated by $C_{11}$, $C_{12}$ and $C_{13}$ show initial points of trajectories and the points represented by $E_{11}$, $E_{12}$ and $E_{13}$ indicate their termini respectively.

The discussions developed for the case where $a = b = 0$, are also

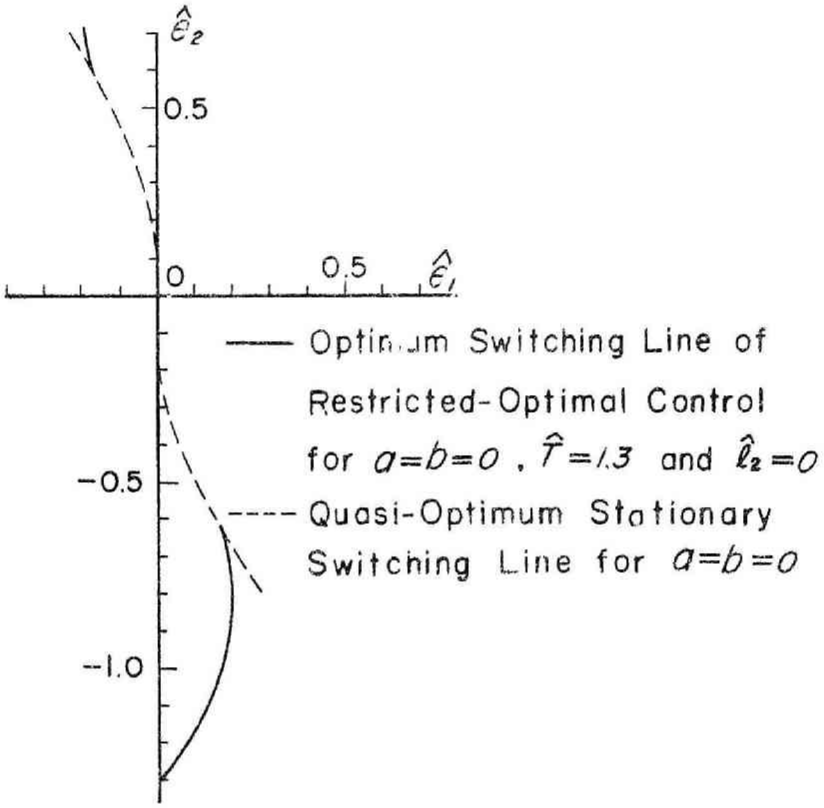


Fig. 1.15 Graphical determination of a quasi-optimum non-stationary switching line ($\hat{T}=1.3$ and $\hat{\ell}_2=0$)

extendable to the cases where $a$ and $b$ are not equal to zero.

## 1.7 Configuration of the Quasi-Optimum Control System with Bounded Control

From the discussions given in the previous sections, the block diagrams of the quasi-optimum control system for a finite control interval and the one for an infinite control interval can respective-

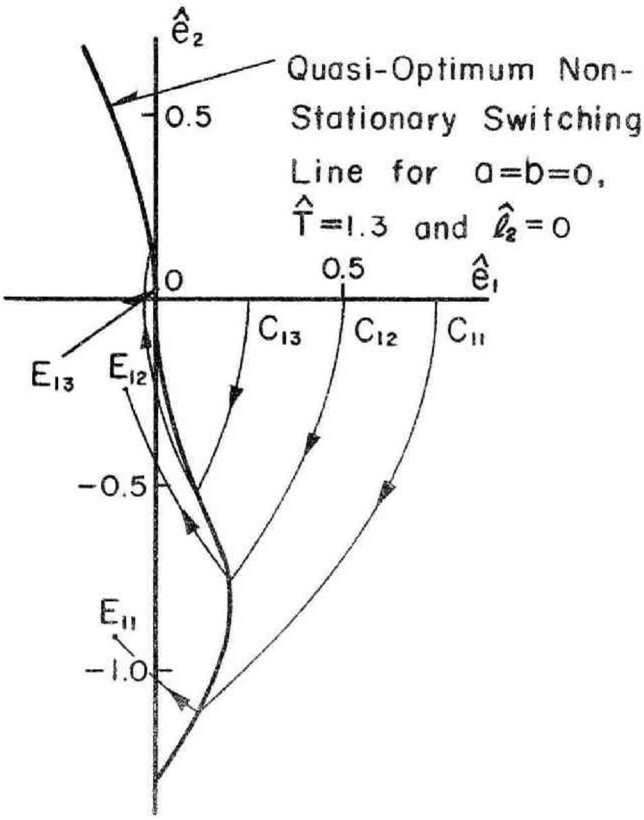


Fig. 1.16 Illustrative example of the response trajectories of the system with a quasi-optimum non-stationary switching line ($\hat{T}=1.3$ and $\hat{l}_2=0$)

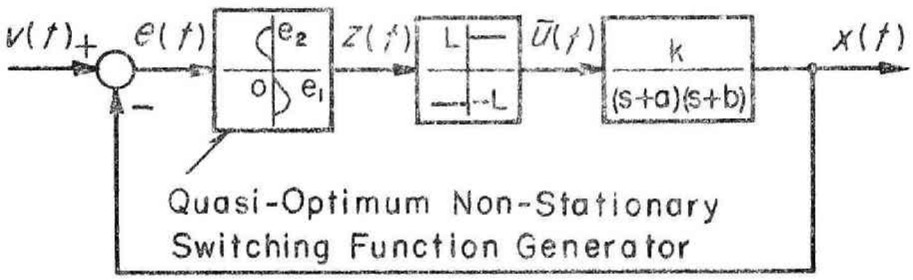


Fig. 1.17 Block diagram of the quasi-optimum control system for a finite control interval

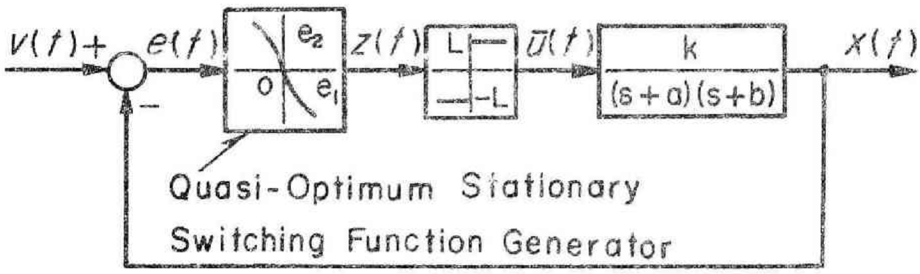


Fig. 1.18 Block diagram of the quasi-optimum control system for an infinite control interval

ly be shown in Figs. 1.17 and 1.18.

The quasi-optimum switching function generators in both Fig. 1.17 and 1.18 mean the physical realization of the switching lines obtained by the procedure stated above. Although it is important to consider the practical realization of switching lines given by the author, this will not be discussed in this chapter.

## 1.8 Concluding Remarks

In this chapter, an analytical technique for the synthesis of quasi-optimum control systems with bounded control is developed.

The principal line of attack is directed to convert the two point boundary value problem formulated by using the method of Dynamic Programming into an initial value problem providing the necessary condition for optimality in a certain restricted control situation, which is referred to as a restricted-optimal control. A new concept of sub-interval optimization based upon the restricted-optimal control is introduced to provide a pseudo-optimum control strategy. By applying the concept, the quasi-optimum stationary and non-stationary switching lines are obtained by a graphical procedure.

The quasi-optimum switching lines are to be constructed as the non-linear function of the state variables of the error signal. It will be expected that the method proposed here can be extended to any system with two state variables even if when performance index is the integral of the modulus of the error signal raised to a positive power not equal to zero.

# CHAPTER 2 SYNTHESIS OF AN OPTIMUM FINAL-VALUE CONTROL SYSTEM WITH BOUNDED CONTROL

## 2.1 Introductory Remarks

The author has already proposed in the previous chapter an approximate method of designing the optimum control system with bounded control which minimizes the integral-error-squared criterion. In this chapter, the previous method is extended to the case of a final-value control problem. A linear second-order dynamical system with bounded control is considered as a controlled plant.

Our present attention is directed to showing that the concept of sub-interval optimization technique provides a physically meaningful optimum solution without solving a two point boundary value problem.[3]

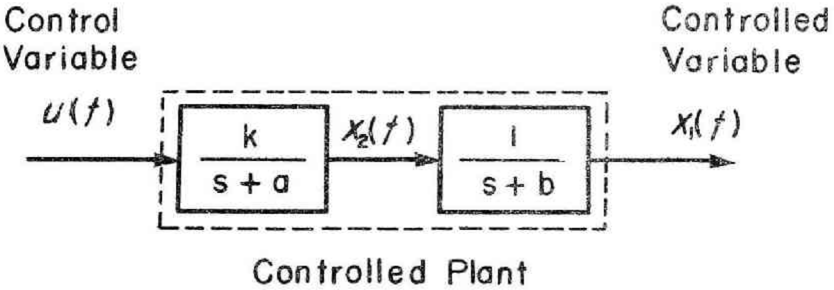


Fig. 2.1 Block diagram of the system to be considered

## 2.2 Statement of the Problem

We consider a linear second order controlled plant as shown in Fig. 2.1, of which the dynamical characteristic is represented by a transfer function of the form $k/(s+a)(s+b)$ between the control variable $u(t)$ and the controlled variable $x(t)$. The control variable is subjected to a constraint on its magnitude as

$$|u(t)| \leq L, \qquad (2.2\text{-}1)$$

where $L$ is a pre-assigned positive constant. By using the state variables of the controlled plant, the plant dynamics can be expressed as

$$\left.\begin{aligned} \dot{x}_1(t) &= -b\,x_1(t) + x_2(t), & x_1(0) &= c_1 \\ \dot{x}_2(t) &= -a x_2(t) + k u(t), & x_2(0) &= c_2 \end{aligned}\right\}, \qquad (2.2\text{-}2)$$

where the symbol "." expresses the differentiation with respect to a time variable. In Eq. (2.2-2), $c_1$ and $c_2$ denote the initial states of the controlled variables respectively.

The problem considered here is to design the controller which minimizes the performance functional

$$J_f = [x_{1d} - x_1(T)]^2 + \mu\,[x_{2d} - x_2(T)]^2 \qquad (2.2\text{-}3)$$

for any initial states of the controlled plant at the time $t=0$. In Eq. (2.2-3), $x_{1d}$ and $x_{2d}$ respectively express the desired values corresponding to the state variables $x_1(t)$ and $x_2(t)$ of the controlled plant at $t=T$, where $T$ is also a pre-assigned constant which expresses the final instant of control operation. For the convenience of our discussion, we assume $x_{1d} = x_{2d} = 0$ in Eq.(2.2-3) without any loss of generality.

The design problem is therefore to find the control variable $u(t)$ which minimizes Eq. (2.2-3) under the constraints given by Eqs. (2.2-1) and (2.2-2).

Let us define the new variables as*

$$\left.\begin{aligned} \hat{t} &= a t \\ \hat{x}_1 &= a^2 x_1 / kL \\ \hat{x}_2 &= a x_2 / kL \\ \hat{u} &= u/L \end{aligned}\right\} , \tag{2.2-4}$$

Eqs. (2.2-1), (2.2-2) and (2.2-3) become

$$|\hat{u}(\hat{t})| \leq 1 \tag{2.2-5}$$

$$J_f = (kL)^2/a^4 \cdot \{ \hat{x}_1(\hat{T})^2 + \hat{\mu}\,\hat{x}_2(\hat{T})^2 \} , \tag{2.2-6}$$

and

$$\left.\begin{aligned} \dot{\hat{x}}_1(\hat{t}) &= -\beta\hat{x}_1(\hat{t}) + \hat{x}_2(\hat{t}) , \quad \hat{x}_1(0) = \hat{c}_1 \equiv a^2 c_1 / kL \\ \dot{\hat{x}}_2(\hat{t}) &= -\hat{x}_2(\hat{t}) + \hat{u}(\hat{t}) , \quad \hat{x}_2(0) = \hat{c}_2 \equiv a c_2 / kL \end{aligned}\right\} , \tag{2.2-7}$$

where $\hat{\mu} \equiv a^2 \mu$ , $\beta \equiv b/a$ and $\hat{T} \equiv aT$.

From Eq. (2.2-6), we obtain a relation;

$$\min_{\substack{u \\ |u| \leq L}} J_f = (kL)^2/a^4 \cdot \min_{\substack{\hat{u} \\ |\hat{u}| \leq 1}} \hat{J}_f \tag{2.2-8}$$

where

$$\hat{J}_f \equiv \hat{x}_1(\hat{T})^2 + \hat{\mu}\,\hat{x}_2(\hat{T})^2 . \tag{2.2-9}$$

The problem is therefore reduced to the one which minimizes the functional $\hat{J}_f$ , given by Eq. (2.2-9), taking the constraints shown by Eqs. (2.2-5) and (2-2-6) into account. For simplicity of the present description we shall omit the chapeau "^" henceforth, unless it is necessary.

---

* This transformation ceases to be valid in the case where the plant parameters $a$ and $b$ become zero simultaneously. It is necessary to treat the case where $a=b=0$ separately.

## 2.3 Configuration of the Optimum Final-Value Control System

By applying the well-known concept of Dynamic Programming, the problem is reduced to solve the following partial differential equation;

$$-\frac{\partial\phi}{\partial t}=\min_{\substack{u\\|u|\leq 1}}\left\{(-\beta x_1+x_2)\frac{\partial\phi}{\partial x_1}+(-x_2+u)\frac{\partial\phi}{\partial x_2}\right\}, \tag{2.3-1}$$

with the initial condition;

$$\phi=x_1(T)^2+\mu x_2(T)^2 \quad , \tag{2.3-2}$$

where $\phi\equiv\phi(x_1, x_2, \tau)$ is defined by

$$\phi(x_1\ ,\ x_2\ ;\ \tau)=\min_{\substack{u\\|u|\leq 1}}\ \{\ x_1(T)^2+\mu\, x_2(T)^2\ \}\ , \tag{2.3-3}$$

and $\tau\equiv T-t$ denotes an auxiliary time variable which is introduced for the convenience of the later description and is called the reversed time variable.

From Eq. (2.3-1) the optimum control variable $\bar{u}(t)$ can be obtained as

$$\left.\begin{aligned}\bar{u}(t)&=-\operatorname{sgn}\,[\,z(\tau)\,]\\ z(\tau)&\equiv\partial\phi/\partial x_2\end{aligned}\right\}, \tag{2.3-4}^*$$

where the function $\phi$ is the solution of the non-linear partial differential equation;

$$-\frac{\partial\phi}{\partial t}=-(\beta x_1-x_2)\frac{\partial\phi}{\partial x_1}-x_2\frac{\partial\phi}{\partial x_2}-\frac{\partial\phi}{\partial x_2}\ \operatorname{sgn}\left[\frac{\partial\phi}{\partial x_2}\right] \tag{2.3-5}$$

with the initial condition given by Eq. (2.3-2).

Thus the optimum configuration of control system subjected to the constraint on its control variable becomes a bang-bang control system with an optimum switching function as shown in Fig. 2.2.

---

* $\operatorname{sgn}[z]=1\ (z>0),\ -1\ (z<0)$

In Eq. (2.3-5), if we assume the solution $\phi$ is of a quadratic form;

$$\phi = k_0(\tau) + k_1(\tau)x_1 + k_2(\tau)x_2 + k_3(\tau)x_1x_2 + k_4(\tau)x_1^2 + k_5(\tau)x_2^2 \ , \qquad (2.3\text{-}6)$$

then the following set of non-linear simultaneous differential equations can be derived as

$$\left.\begin{aligned}
k_0'(\tau) &= -k_2(\tau)\ \mathrm{sgn}\ [z(\tau)] \\
k_1'(\tau) &= -\beta k_1(\tau) - k_3(\tau)\ \mathrm{sgn}\ [z(\tau)] \\
k_2'(\tau) &= k_1(\tau) - k_2(\tau) - 2k_5(\tau)\ \mathrm{sgn}\ [z(\tau)] \\
k_3'(\tau) &= 2k_4(\tau) - (\beta+1)k_3(\tau) \\
k_4'(\tau) &= -2\beta k_4(\tau) \\
k_5'(\tau) &= k_3(\tau) - 2k_5(\tau)
\end{aligned}\right\} , \qquad (2.3\text{-}7)_1$$

with the initial conditions;

$$\left.\begin{aligned}
&k_i(0) = 0 \quad (i = 0,\ 1,\ 2,\ 3) \\
&k_4(0) = 1\ ,\ k_5(0) = \mu
\end{aligned}\right\} , \qquad (2.3\text{-}7)_2$$

where "$'$" expresses the differentiation with respect to the reversed time variable $\tau$ and the switching function $z(\tau)$ is expressed by

$$z(\tau) \equiv k_2(\tau) + k_3(\tau)x_1 + 2k_5(\tau)x_2 \ . \qquad (2.3\text{-}8)$$

On the other hand, replacing the time variable $t$ by the reversed

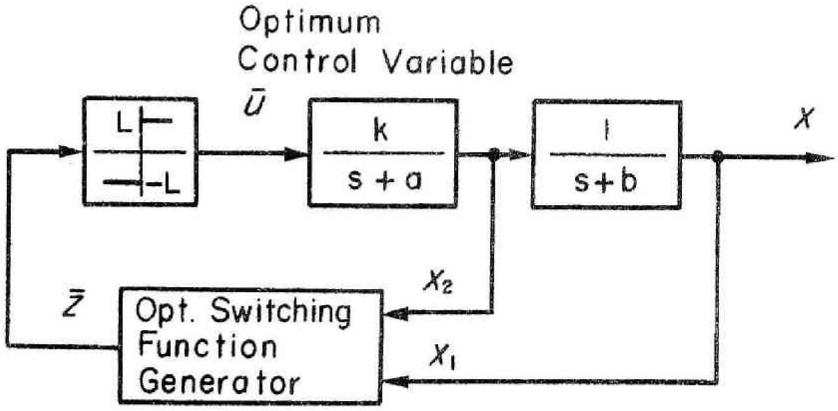


Fig. 2.2 Configuration of the optimum control system

time variable $\tau$ and substituting $\bar{u}(t)$ for $u(t)$ into Eq. (2.2-7), we have

$$\left.\begin{aligned} x_1'(\tau) &= \beta x_1(\tau) - x_2(\tau) \\ x_2'(\tau) &= x_2(\tau) + \operatorname{sgn}\left[k_2(\tau) + k_3(\tau)x_1 + 2k_5(\tau)x_2\right] \\ x_1'(T) &= c_1\ , \quad x_2(T) = c_2 \end{aligned}\right\} . \qquad (2.3\text{-}9)$$

The determination of switching function is therefore reduced to solve the two point boundary value problem which consists of Eqs. (2.3-7) and (2.3-9).

## 2.4 Switching Functions realizing the Restricted-Optimal Control

Assuming that the final instant of control operation $\hat{T}$ satisfies the assumption realizing the restricted-optimal control and performing a similar derivation presented in Appendix-A, the coefficients necessary for the switching function of restricted-optimal control can be obtained from the following set of equations;

$$\left.\begin{aligned} k_1'^{+}(\tau) &= -\beta k_1^{+}(\tau) - k_3(\tau)\ , & k_1^{+}(0) &= 0 \\ k_2'^{+}(\tau) &= k_1^{+}(\tau) - k_2^{+}(\tau) - 2k_5(\tau)\ , & k_2^{+}(0) &= 0 \end{aligned}\right\} \qquad (2.4\text{-}1)_1$$

for the positive relay output at $t=0$, and for the negative relay output at $t=0$,

$$\left.\begin{aligned} k_1'^{-}(\tau) &= -\beta k_1^{-}(\tau) + k_3(\tau)\ , & k_1^{-}(0) &= 0 \\ k_2'^{-}(\tau) &= k_1^{-}(\tau) - k_2^{-}(\tau) + 2k_5(\tau)\ , & k_2^{-}(0) &= 0 \end{aligned}\right\} , \qquad (2.4\text{-}1)_2$$

where the functions $k_3(\tau)$ and $k_5(\tau)$ are respectively derived from the last three equations in Eq. $(2.3\text{-}7)_1$ as

$$k_3(\tau) = \frac{2}{1-\beta}\{\exp(-2\beta\tau) - \exp[-(\beta+1)\tau]\}\ , \qquad (2.4\text{-}2)$$

and

$$k_5(\tau) = \frac{1}{(1-\beta)^2}\{\exp(-\tau) - \exp(-\beta\tau)\}^2 + \mu\exp(-2\tau)\ . \qquad (2.4\text{-}3)$$

From Eqs. $(2.4\text{-}1)_1$, $(2.4\text{-}1)_2$, (2.4-2) and (2.4-3), the switching function of the restricted-optimal control can be expressed as

$$z^{\pm}_{RES}(\tau) \equiv k_2^{\pm}(\tau) + k_3(\tau)x_1 + 2k_5(\tau)x_2 \quad , \tag{2.4-4}$$

where

$$\begin{aligned} k_2^{\pm}(\tau) = &\mp \frac{2}{\beta(1-\beta)^2}\{\exp(-\beta\tau) - \exp(-\tau)\} \\ &\times [\{1-\exp(-\beta\tau)\} - \beta\{1-\exp(-\beta\tau)\}] \\ &\mp 2\mu\exp(-\tau)\{1-\exp(-\tau)\} \end{aligned} \tag{2.4-5}$$

In Eq. (2.4-4), $z^{+}_{RES}(\tau)$ expresses the switching function for positive trajectories corresponding to the positive sign of the relay output at $t=0$, and $z^{-}_{RES}(\tau)$ denotes the one for negative trajectories.

Since Eq. (2.4-4) with Eqs. (2.4-2), (2.4-3) and (2.4-5) expresses the switching function with respect to the normalized variable defined by Eq. (2.2-4), the switching function with respect to the original variables is expressed as

$$\begin{aligned} z^{\pm}_{RES} \equiv &\mp \frac{1}{\beta(1-\beta)^2}\{\exp(-b\tau) - \exp(-a\tau)\} \times \\ &[1-\exp(-b\tau) - \beta\{1-\exp(-a\tau)\}] \mp 2\mu\exp(-\tau)\{1-\exp(-\tau)\} \\ &+ \frac{1}{1-\beta}\{\exp(-2b\tau) - \exp[-(b+a)\tau]\}\frac{a^2x_1}{kL} \\ &+ [\frac{1}{(1-\beta)^2}\{\exp(-a\tau) - \exp(-b\tau)\}^2 + a^2\mu\exp(-2a\tau)]\frac{ax_2}{kL} \ , \end{aligned} \tag{2.4-6}$$

where $\beta \equiv b/a$.

From Eq. (2.4-4), we can also derive the switching functions of restricted-optimal control for the other cases where $a$ or $b$, or both are equal to zero as follows*:

(1) In the case where $a \neq 0$ and $b = 0$, the switching function is

---

* The detailed procedure of derivation is presented in Appendix B.

$$z^{\pm}_{\mathrm{RES}} \equiv \mp\{1-\exp(-a\tau)\}[\{a\tau+\exp(-a\tau)-1\}-a^2\mu\exp(-a\tau)]$$
$$+\{1-\exp(-a\tau)\}\frac{a^2x_1}{kL}$$
$$+[\{1-\exp(-a\tau)\}^2+a^2\mu\exp(-2a\tau)]\frac{ax_2}{kL} \qquad (2.4\text{-}7)$$

(2) In the case where $a=b=0$ , the switching function is

$$z^{\pm}_{\mathrm{RES}} \equiv \mp(\tau^3+2\mu\tau)+2\tau\frac{x_1}{kL}+2(\tau^2+\mu)\frac{x_2}{kL}\ . \qquad (2.4\text{-}8)$$

## 2.5 Application of the Sub-Interval Optimization Technique to a Final-Value Control Problem

It is evident from the exposition of sub-interval optimization technique in 1.4 that the pseudo-optimization rule is based upon the restricted-optimal control. Then, the behaviors of the restricted-optimal switching line must be investigated before the introduction of pseudo-optimization procedure.

### 2.5-1 Restricted-Optimal Switching Line

As a preliminary discussion, let us calculate numerically the intersections of the system response trajectories with the switching lines expressed as $z^{\pm}_{\mathrm{RES}}=0$ in the phase plane. For simplicity, we consider the case of a double-integrator plant where system parameters $a$ and $b$ in Eq. (2.2-2), become zero. From Eq. (2.4-8), the switching line of restricted-optimal control for negative trajectories in this case is expressed as

$$\tau^3+2\tau\hat{x}_1+2\tau^2\hat{x}_2+2\mu(\tau+\hat{x}_2)=0\ . \qquad (2.5\text{-}1)$$

where both $\hat{x}_1$ and $\hat{x}_2$ are the normalized state variables respectively defined by

$$\hat{x}_1=x_1/kL \quad \text{and} \quad \hat{x}_2=x_2/kL\ . \qquad (2.5\text{-}2)$$

On the other hand, by substituting the condition $a=b=0$ into Eq. (2.2-2) and integrating with respect to the reversed time $\tau$, the system response corresponding to the negative relay output at $t=0$ are obtained as

$$\left.\begin{aligned} \hat{x}_1(\tau) &= \hat{c}_1 + \hat{c}_2(T-\tau) - (T-\tau)^2/2 \\ \hat{x}_2(\tau) &= \hat{c}_2 - (T-\tau) \end{aligned}\right\} , \qquad (2.5\text{-}3)$$

where $\hat{c}_1 \equiv c_1/kL$ and $\hat{c}_2 \equiv c_2/kL$.

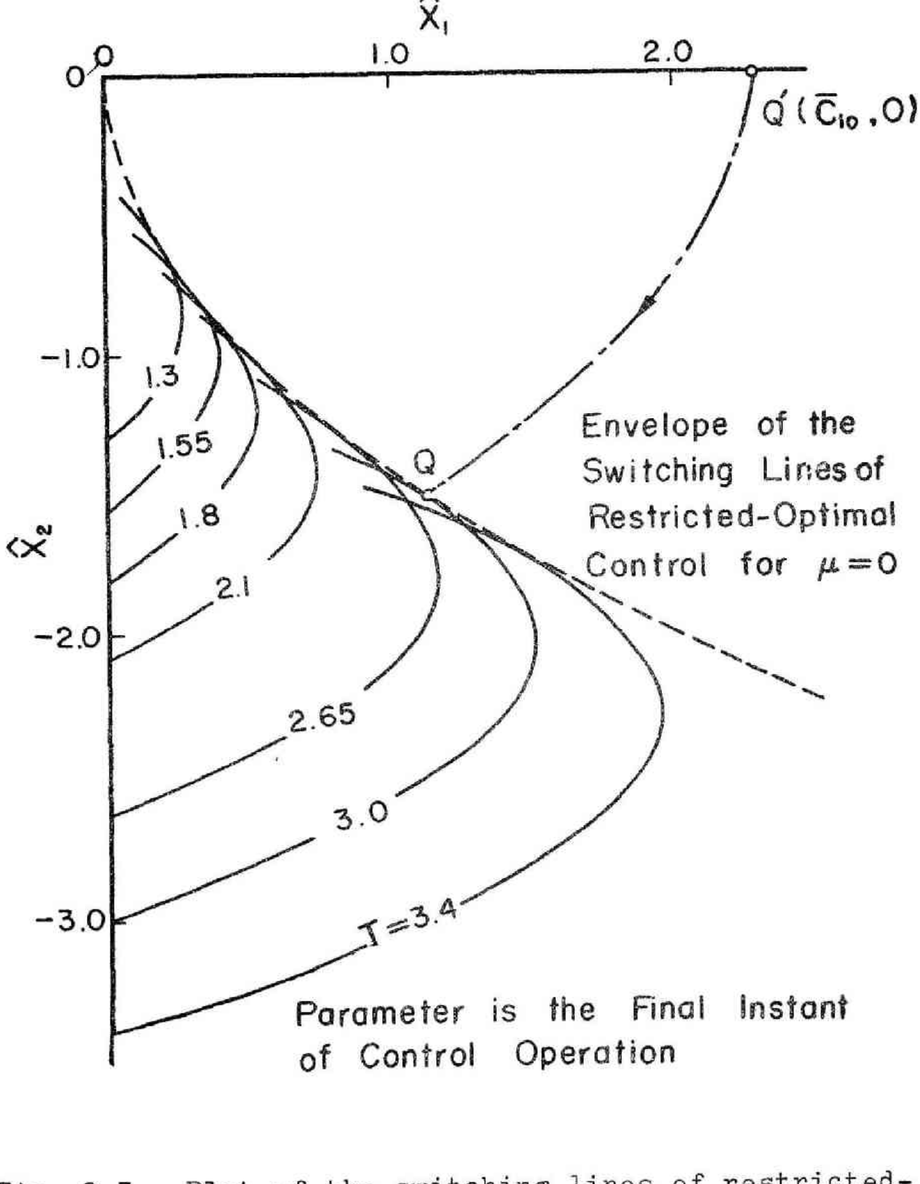


Fig. 2.3 Plot of the switching lines of restricted-optimal control in the case where $\mu=0$

By using Eqs. (2.5-1) and (2.5-3) and considering the value of $T$ or $\mu$ as a parameter, we calculate the intersections stated above and the results are plotted as the solid curves in Figs. 2.3, 2.4, and 2.5 in which we can set the initial condition $\hat{c}_1$ to be zero without loss of generality. Fig. 2.3 shows the loci of intersections with respect to the initial condition $\hat{c}_1$ and the control interval $T$, where $\mu = 0$. Fig. 2.4 illustrates, on the other hand, an example of the loci of intersections where $\mu \neq 0$, and the behavior of the loci of intersections is plotted in Fig. 2.5 considering the value of $\mu$ as a parameter, where $T=3.0$.

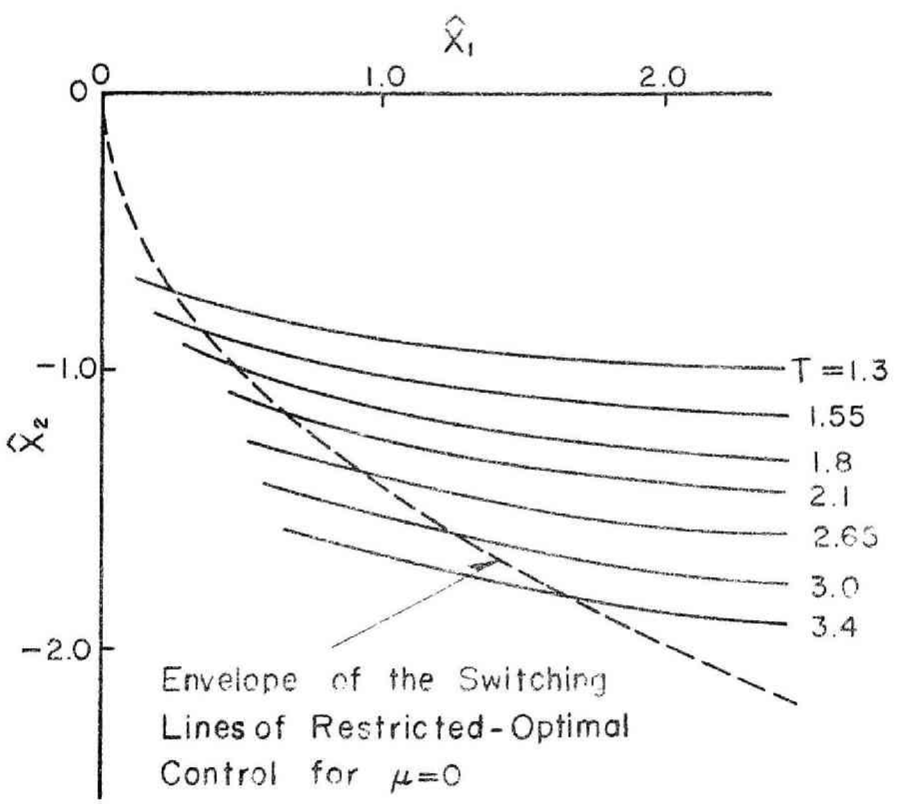


Fig. 2.4 Plot of the switching lines of restricted-optimal control in the case where $\mu = 1.0$

2.5-2 Envelope of the Restricted-Optimal Switching Lines for $\mu = 0$

In the problem presented in Chapter 1, the switching lines of restricted-optimal control have always the envelope which plays an important role in the sub-interval optimization technique. In this problem, on the contrary, the existence of an envelope **depends** upon the value of $\mu$. In other words, there exists an envelope in the case where $\mu = 0$. In the case where $\mu \neq 0$, no such an envelope exists. However, there is no problem in application of the sub-interval optimization technique. For, it turns out from Fig. 2.5 the envelope of switching lines for $\mu = 0$ gives a set of critical point

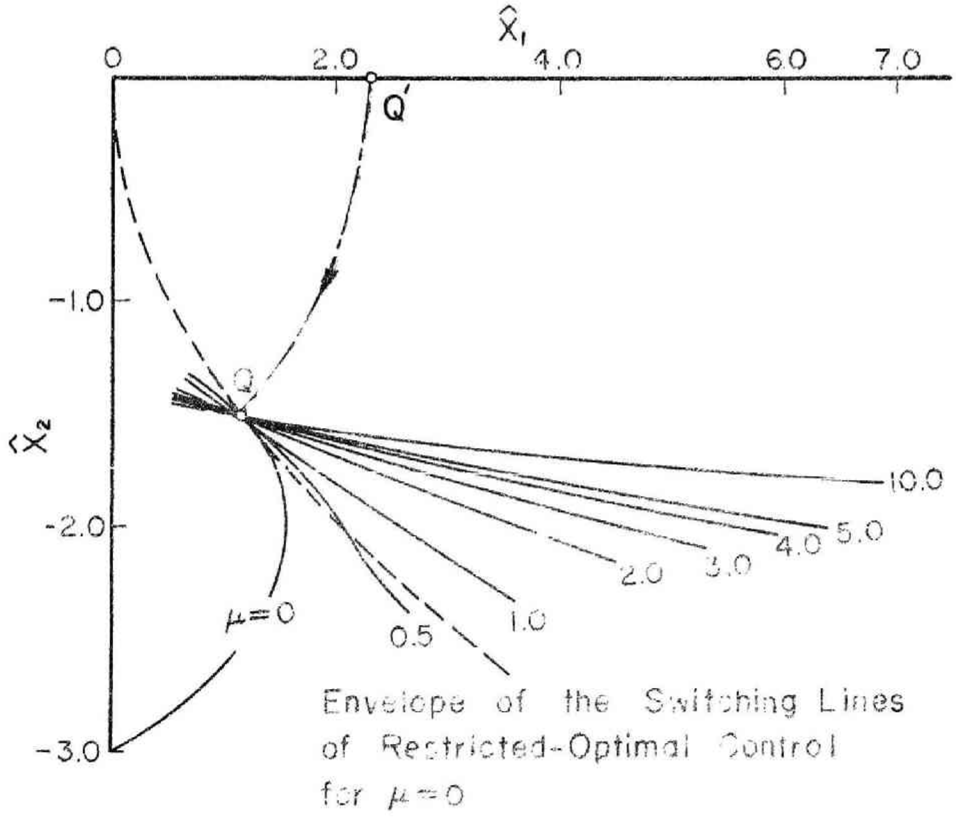


Fig. 2.5 Plot of the switching lines of restricted-optimal control in the case where $T = 3.0$

for the fixed $T$, like $Q$ in Fig. 1.6 or 1.7, even if in the case where $\mu \neq 0$. Mathematical verification of the above presentation is as follows: In order to perform the verification, it is enough to show that the $\mu$-free relation satisfies the equation expressing the envelope of switching lines for $\mu=0$.

By eliminating the parameter $\mu$ from Eq. (2.5-1), $\mu$-free relation is expressed as

$$\left.\begin{array}{l} \tau^3+2\tau\,\hat{x}_1+2\tau^2\,\hat{x}_2=0 \\ \tau+\hat{x}_2=0 \end{array}\right\}, \tag{2.5-4}$$

where $\tau$ is a parameter.

Elimination of the parameter $\tau$ from Eq. (2.5-4) gives us the relation

$$2\,\hat{x}_1-\hat{x}_2^2=0\ . \tag{2.5-5}$$

On the other hand, setting $\mu=0$ in Eq. (2.5-1), the elimination of the variable $\tau$ from Eq. (2.5-1) provides us an equation expressing the envelope of switching lines for $\mu=0$ as

$$2\,\hat{x}_1-\hat{x}_2^2=0\ . \tag{2.5-6}$$

Since Eq. (2.5-6) coincides with Eq. (2.5-5), then the verification is over.

### 2.5-3 Physical Meaning of the Sub-Interval Optimization Technique

According to the presentation developed in 1.4-3, the envelope of switching lines realizing the restricted-optimal control gives a pseudo-optimal switching rule for the control interval which does not satisfy the assumption realizing the restricted-optimal control.

On the other hand, it turns out in Eq. (2.5-6) that the equation expressing the envelope coincides with the optimum switching line for a minimum-time control problem.[18] In other words, this fact means that the point corresponding to $R_1'$ in Fig. 1.7 coincides with the origin. In general, as concerns a relay control system with time-minimum switching function, it can be considered that the origin becomes a stable equilibrium point. Then the response trajectory once reached the origin should be stayed there during the rest of control interval. In that control situation, there exists the region of initial points, any point of which can be transferred to the origin within the time duration of $T$. This region is expressed as the intersection of two regions defined by the following inequalities;

$$\hat{c}_1 \leq -\frac{1}{4}(\hat{c}_2 + T)^2 + \frac{1}{2}T^2 \ , \qquad (2.5\text{-}7)_1$$

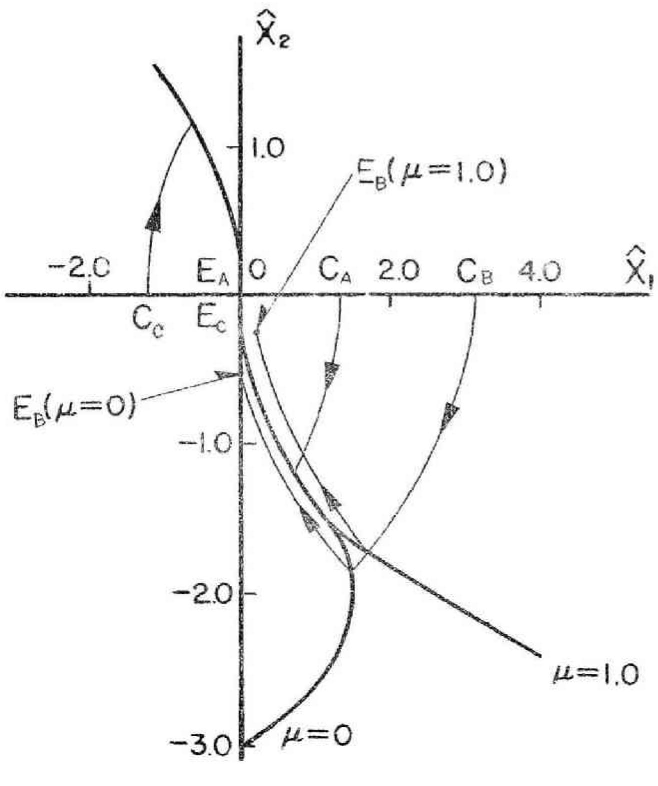


Fig. 2.6 Illustration of response trajectories of the optimum control system

and

$$\hat{c}_1 \geq \frac{1}{4}(\hat{c}_2 - T)^2 - \frac{1}{2}T^2 \quad . \tag{2.5-7$_2$}$$

Then, the switching rule obtained by the sub-interval optimization technique is as follows: If the initial states of the system satisfy Eqs. $(2.5\text{-}7)_1$ and $(2.5\text{-}7)_2$ simultaneously, we use the minimum-time switching function, Eq. (2.5-6). If not, then we use the switching function given by Eq. (2.4-8). An example of response trajectories of the system with such a switching logic cited above is illustrated in Fig. 2.6, where both cases of $\mu=0$ and $\mu\neq 0$ are simultaneously shown. In Fig. 2.6, points expressed by $C_A$, $C_B$ and $C_C$ are initial points and points indicated by $E_A$, $E_B$ and $E_C$ are corresponding final points of trajectories. Since the switching lines for $\mu\neq 0$ as shown in Figs. 2.4, 2.5 and 2.6 are meaningful only for negative trajectories, then there never occurs a sliding or chattering mode[18] along the switching line of restricted-optimal control. An example of realization of such a switching line is illustrated in Fig. 2.7. As shown in Fig. 2.7 the optimum switching function generator consists of three parts, i.e, the minimum-time switching function generator, the restricted-optimal switching function generator and the switching rule selector. The former switching function generator is realization of Eq. (2.5-6) and the latter is instrumentation of Eq. (2.4-8). The switching rule selector corresponds to a channel selector based upon Eqs. $(2.5\text{-}7)_1$ and $(2.5\text{-}7)_2$.

It is emphasized that the sub-interval optimization technique in this case does not provide a pseudo-optimal switching rule but an physically meaningful optimum switching rule. It is also examined that these properties of the sub-interval optimization technique applied to the final-value control problem are independent of the mathematical form of controlled systems of second order with real poles.

## 2.6 Concluding Remarks

In this chapter, a method of designing the optimum final-value control system with bounded control is described. The controlled plant of linear second order dynamics with bounded control is treated directly, without assigning any penalties on the performance criterion. A physically meaningful optimum solution is obtained by applying the concept of sub-interval optimization technique.

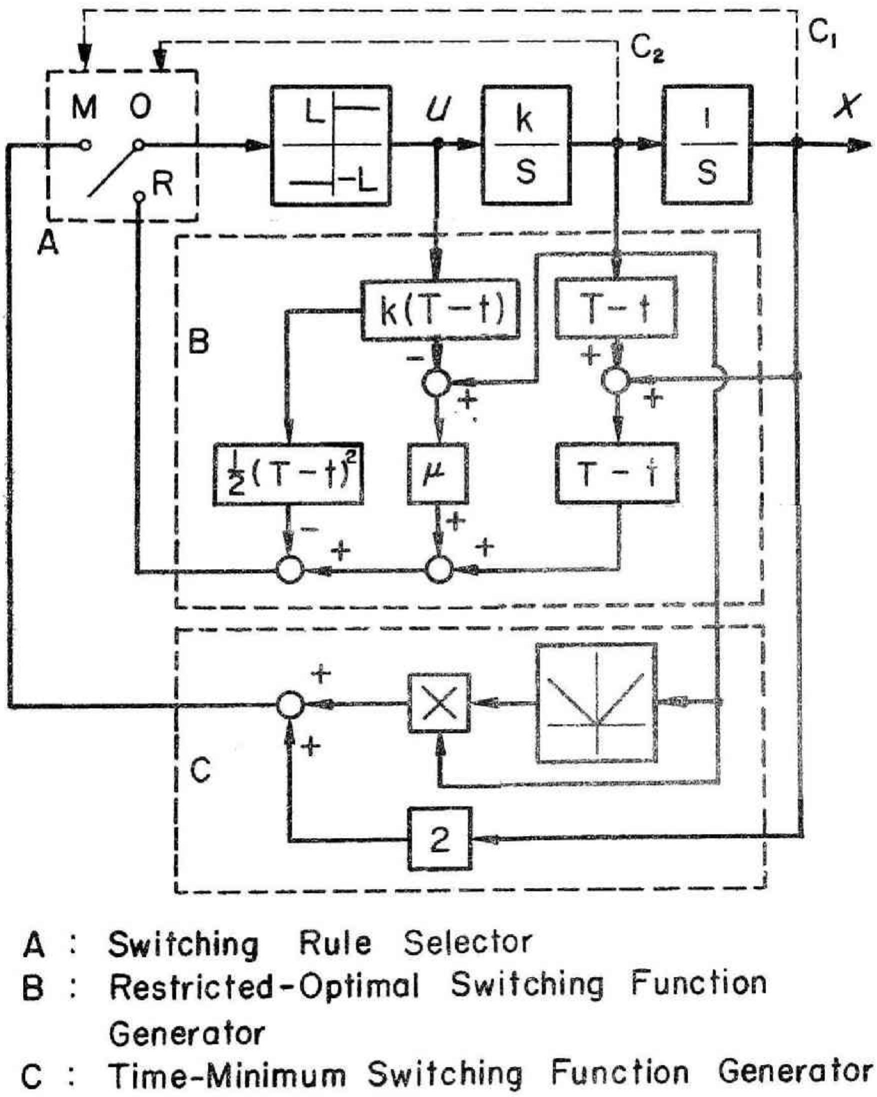


Fig. 2.7 Configuration of the optimum final-value control system with bounded control

## CHAPTER 3 CONSIDERATIONS ON THE QUASI-OPTIMUM CONTROL OF A SECOND ORDER VIBRATORY SYSTEM WITH BOUNDED CONTROL

### 3.1 Introductory Remarks

In the previous two sections, the author shows several examples in which the proposed concept of pseudo-optimization is effectively applied to derive a physically meaningful quasi-optimum solution for the original control problem. However, the presentation has been confined to the control of second order systems except for a vibratory one. The purpose of this chapter is to obtain the corresponding quasi-optimum switching line which minimizes the functional $J_2$ or $M_2$ in Chapter 1 for a second order vibratory system. An experimental study by a digital computer is also carried out to examine the derived quasi-optimum switching line.

### 3.2 Statement of the Problem

Let us consider the controlled system as shown in Fig. 3.1, of which dynamical characteristic is described by using the state variables of the error signal as

$$\left.\begin{aligned} \dot{e}_1(t) &= e_2(t) , & e_1(0) &= l_1 \\ \dot{e}_2(t) &= -w_n^2 e_1(t) - k u(t) , & e_2(0) &= 0 \end{aligned}\right\} , \qquad (3.2\text{-}1)$$

where $e_1 = e$ , $e_2 = \dot{e}$ , and "." denotes the differentiation with respect to $t$. The control signal $u(t)$ is bounded as

$$| u(t) | \leq L , \qquad (3.2\text{-}2)$$

where $L$ is a pre-assigned positive constant.

The problem considered here is to design the controller which

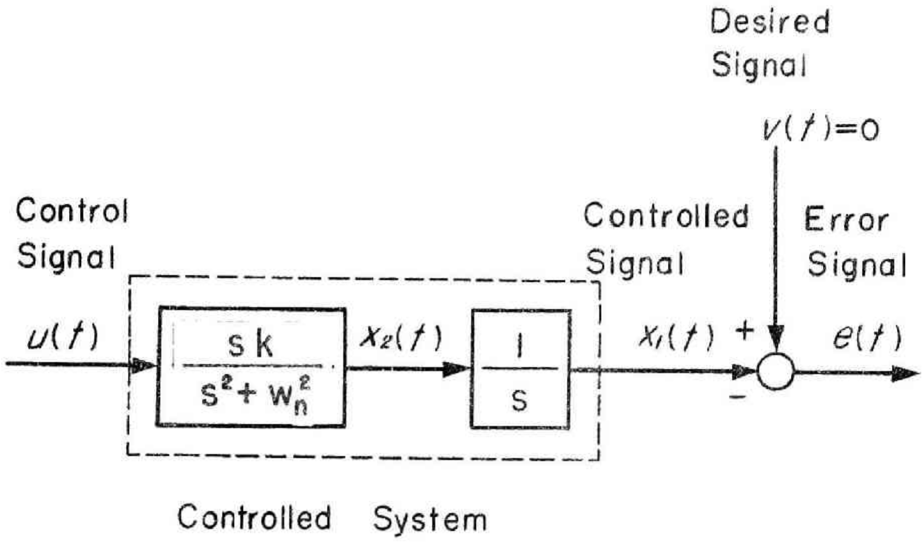


Fig. 3.1 Block diagram of the system to be considered

minimizes the performance functional $J_2$ or $M_2$ in Chapter 1, for any initial states of the controlled system at $t=0$, i.e., $e_1(0)=l_1$ .

By performing a similar consideration as presented in Chapter 1, the original problem is reduced to solve a partial differential equation and to determine the optimum switching function for the relay type control element. That is to say, the necessary condition for optimality in regard to the minimization of $J_2$ is expressed by the following partial differential equation;

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial \tau} = \min_{\substack{u \\ |u|\leq L}} \left\{ e_1^2 + e_2 \frac{\partial \phi}{\partial e_1} - w_n^2 e_1 \frac{\partial \phi}{\partial e_2} - ku \frac{\partial \phi}{\partial e_2} \right\} \\ &\phi = 0 \quad (\tau = 0) \end{aligned}\right\}, \tag{3.2-3}$$

where

$$\phi\ (e_1, e_2\ ;\ \tau) = \min_{\substack{u \\ |u| \leq L}} \int_0^{\tau} e_1(\rho)^2\, d\rho\ , \tag{3.2-4}$$

and $\tau = T - t$ .

From Eq. (3.2-3), since the optimum control signal $\bar{u}$ becomes as

$$\left.\begin{aligned} &\bar{u} = L\ \mathrm{sgn}\ [z] \\ &z \equiv \partial\phi / \partial e_2 \end{aligned}\right\}\ , \tag{3.2-5}$$

in order to determine the optimum switching function $z$,then, we must solve the partial differential equation as follows:

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial\tau} = e_1{}^2 + e_2\frac{\partial\phi}{\partial e_1} - w_n^2 e_1 \frac{\partial\phi}{\partial e_2} - kL\left|\frac{\partial\phi}{\partial e_2}\right| \\ &\phi = 0 \qquad (\tau = 0) \end{aligned}\right\}, \tag{3.2-6}$$

## 3.3 Switching Function realizing the Restricted-Optimal Control

Let us consider the solution of Eq.(3.2-6) from the view point of the restricted-optimal control. Assuming that the final instant of control operation $T$ satisfies the assumption realizing the restricted-optimal control, by following the procedure of derivation presented in Chapter 1, we can calculate formally the switching functions for positive and negative trajectories as follows:

$$z^{\pm}_{\mathrm{RES}} \equiv k_2^{\pm}(\tau) + k_3(\tau)\, e_1 + 2k_5(\tau)\, e_2\ , \tag{3.3-1}$$

where

$$k_2^{\pm}(\tau) = \pm\frac{kL}{w_n^4}\ (1 - \cos w_n\tau)^2\ , \tag{3.3-2$_1$}$$

$$k_3(\tau) = \frac{1}{2w_n^2}\ (1 - \cos 2w_n\tau) \tag{3.3-2$_2$}$$

and

$$k_5(\tau) = \frac{1}{4w_n^3}\ (2w_n\tau - \sin 2w_n\tau)\ . \tag{3.3-2$_3$}$$

Hence $z_{RES}^{+}$ denotes the switching function for positive trajectories and $z_{RES}^{-}$ does for negative trajectories respectively.

## 3.4 Envelope of the Switching Lines of Restricted-Optimal Control

Let us calculate numerically the intersection of positive trajectories with the corresponding switching line of restricted-optimal control.

The equation expressing positive trajectories is described as

$$\left.\begin{aligned} e_1(\tau) &= -kL + (l_1 + kL)\cos w_n(T-\tau) \\ e_2(\tau) &= -(l_1 + kL)\, w_n \sin w_n(T-\tau) \end{aligned}\right\} . \tag{3.4-1}$$

By using Eqs. (3.3-1) and (3.4-1), we get the loci of the intersection

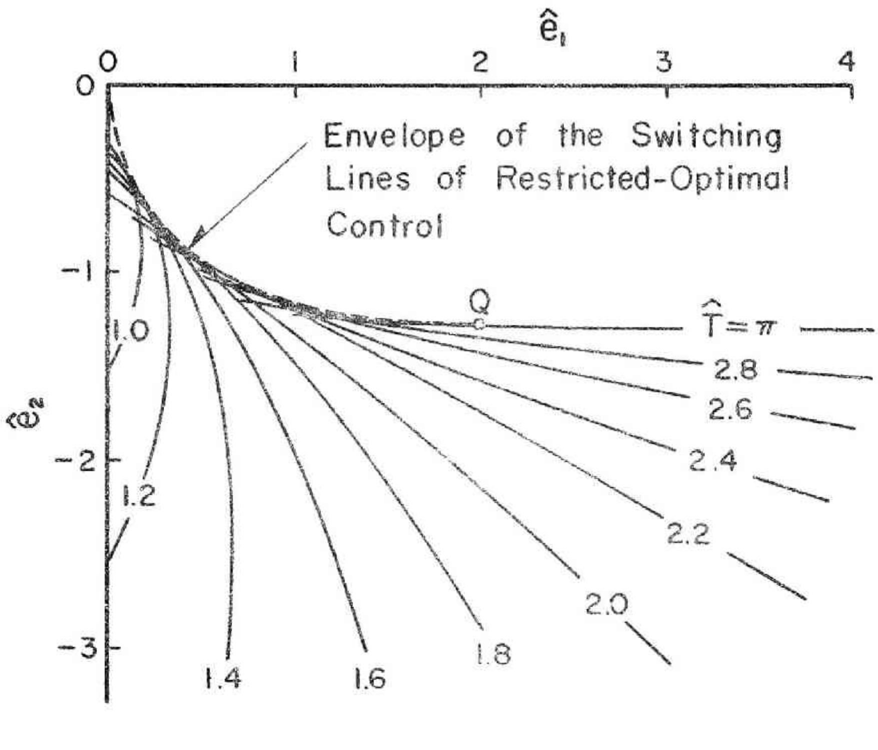


Fig. 3.2 (a) Plot of the switching lines of restricted-optimal control

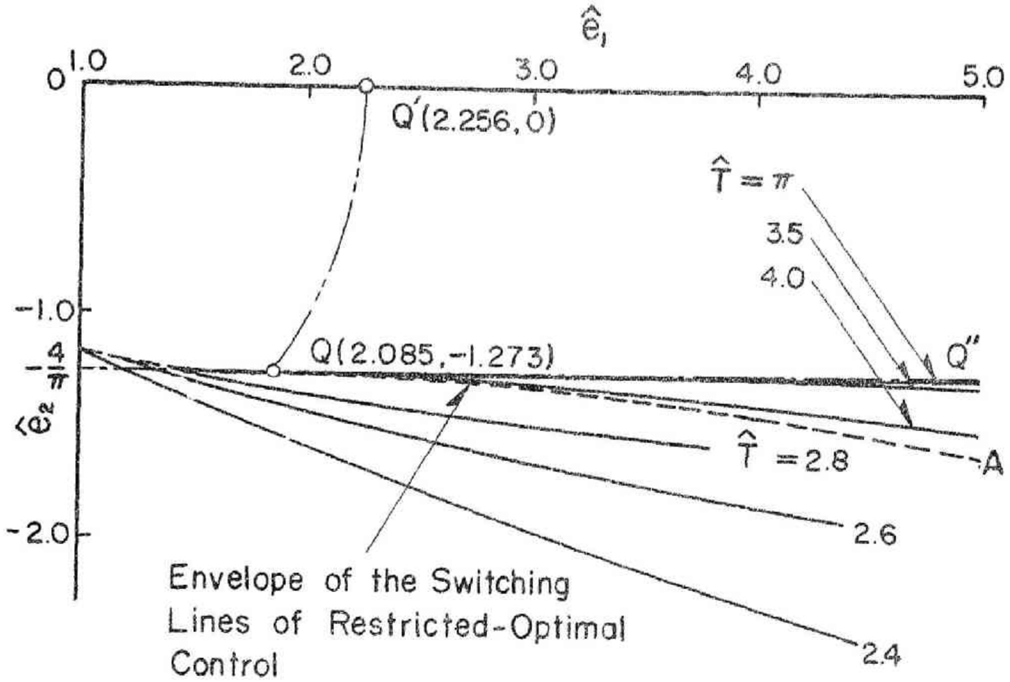


Fig. 3.2(b) Plot of the switching lines of restricted-optimal control

as shown in Figs. 3.2 (a) and 3.2 (b) where $\hat{e}_1 \equiv w_n^2 e_1 / kL$, $\hat{e}_2 \equiv w_n e_2 / kL$, and $\hat{T} \equiv w_n T$ is taken as a parameter. In both figures the broken line is the envelope of switching lines of restricted-optimal control. Fig. 3.3 illustrates the behavior of envelope affected by increasing the final instant $\hat{T}$. In Figs. 3.2 (a), (b) and 3.3, the point $Q$ expresses a stationary point on $\widehat{OA}$, at which the relation between the envelope and switching lines is essentially changed. That is, in Fig. 3.2 (b), while the part of envelope $\widehat{OQ}$ expresses the right boundary line of restricted-optimal switching as well as the envelopes

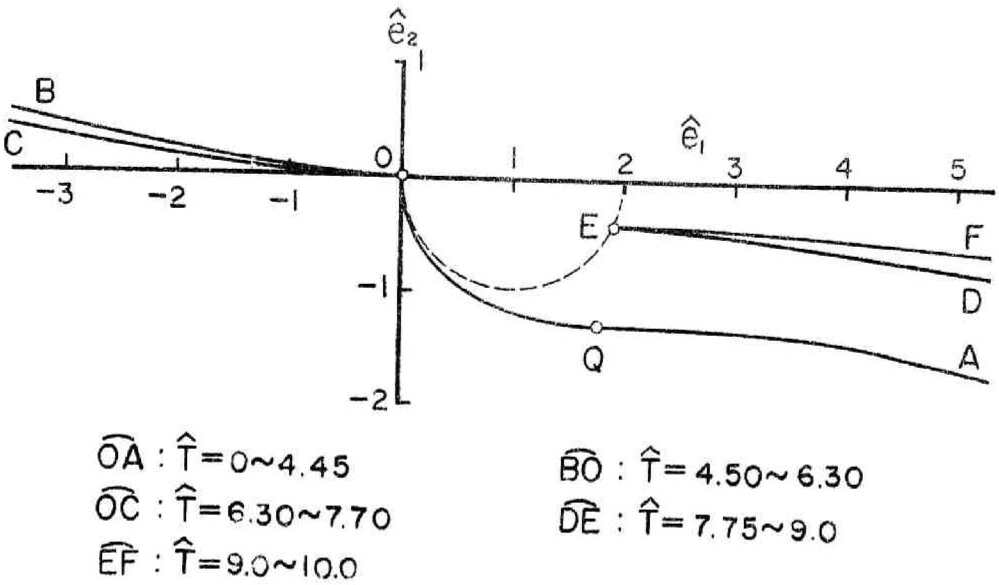


Fig. 3.3 Plot of the envelope of switching lines of restricted-optimal control

in Chapter 1, the part of envelope $\widehat{QA}$ denotes the apparent left boundary line of restricted-optimal switching for $\hat{T} = 3.5$ and $4.0$.

It is evident from Figs. 3.2 (b) and 3.3 that the envelope of the switching lines of restricted-optimal control shows a strange feature from the envelopes illustrated in Figs. 1.5, 1.8 and 1.10. In other words, for the controlled systems discussed in Chapter 1 there exists a continuous envelope uniquely and it gives us the corresponding geometrical meaning to the necessary condition realizing the restricted-optimal control without regarding the value of control duration $\hat{T}$. On the contrary, in this case the envelope has several discontinuities as shown in Fig. 3.3. It seems to be natural to consider that the occurrence of the strange phenomenon concerning the envelope is related to the fact that the controlled system has no real poles. In such a controlled system, if we pre-assign a larger value than a certain critical one to the final instant $\hat{T}$, the pre-assigned $\hat{T}$ does not satisfy the assumption realizing the restricted-optimal control whatever large values may be tentatively assumed for the initial state $\hat{l}_1 \equiv w_n^2 l_1 / kL$. Therefore, the physical interpretation of the envelope in this case should be considerably changed from what we have discussed in 1.4-2. Then it is necessary to find the critical value of $\hat{T}$, or the corresponding phase point.

In order to derive a critical point, let us consider the time-dependent characteristics of the switching function given by Eq. (3.3-1). Since Eq. (3.3-1) is a time-variant linear switching line, then it is enough to examine the time-dependent characteristics of its gradient and intersection to the $\hat{e}_1$-axis.

For simplicity, let us consider the switching function for positive trajectories. The corresponding switching line is rewritten as

$$\hat{e}_2 = -\frac{2(1-\cos\hat{\tau})^2}{2\hat{\tau}-\sin 2\hat{\tau}} - \frac{1-\cos 2\hat{\tau}}{2\hat{\tau}-\sin 2\hat{\tau}}\hat{e}_1 \quad , \tag{3.4-2}$$

where $\hat{\tau} \equiv w_n \tau$.

By using Eq. (3.4-2), we can calculate the time-dependency of both gradient and intersection to the $\hat{e}_1$-axis, and the results are plotted in Fig. 3.4. In Fig. 3.4 the solid line shows the time-dependency of gradient and the broken line illustrates that of the intersection.

Fig. 3.4 emphasizes that the switching line becomes parallel to the $\hat{e}_1$-axis at the time $\hat{\tau} = n\pi$ $(n=1,2,\ldots)$. On the other hand, since it is evident from Eq. (3.4-1) that the phase trajectory is a circle and it revolutes in time $2\pi$, then the only meaningful time interval

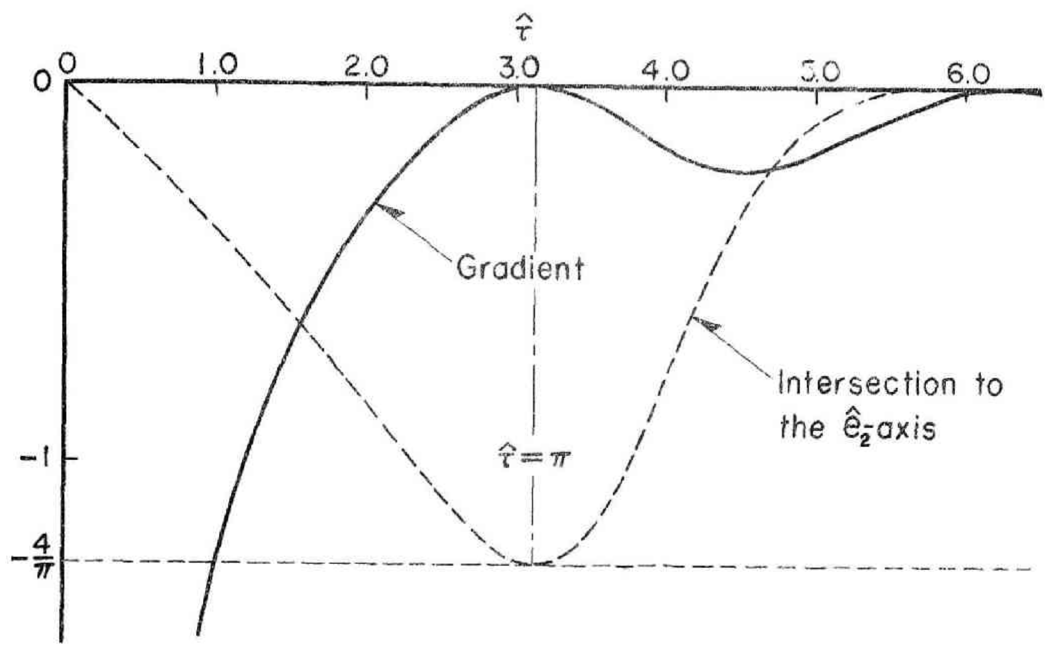


Fig. 3.4 Time dependency of the gradient and intersection of the switching line

for the restricted-optimal switching is confined to $0 \leq \hat{\tau} \leq 2\pi$ . By considering the behavior of the switching line, it is concluded that there should exist a stationary point along the line $\hat{e}_2 = -\frac{4}{\pi}$ . This is the very point expressed as $Q$ in Figs. 3.2 (a), (b) and 3.3. Since the $\hat{e}_1$ -coordinate of $Q$ is obtained numerically as $\hat{e}_1 = 2.085$, then we can calculate the corresponding final instant $\hat{T}_S$ by using Eq. (3.4-1). That is, since the normalized form of the second relation in Eq. (3.4-1) becomes

$$\hat{e}_2 = -(\hat{l}_1 + 1) \sin (\hat{T} - \hat{\tau}) \quad , \tag{3.4-3}$$

then we get the relation as

$$\hat{T} = \hat{\tau} - \sin^{-1}\left\{ \frac{\hat{e}_2}{\hat{l}_1 + 1} \right\} . \tag{3.4-4}$$

On the other hand, by eliminating a time variable $\hat{\tau}$ from Eq. (3.4-1) we have the equation of phase trajectories;

$$(\hat{e}_1 + 1)^2 + (\hat{e}_2)^2 = (\hat{l}_1 + 1)^2 . \tag{3.4-5}$$

Since by eliminating $\hat{l}_1$, from Eqs. (3.4-4) and (3.4-5), we derive the relation

$$\hat{T} = \hat{\tau} - \sin^{-1}\left\{ \frac{\hat{e}_2}{\sqrt{(\hat{e}_1 + 1)^2 + (\hat{e}_2)^2}} \right\} , \tag{3.4-6}$$

then by setting

$$\left.\begin{aligned} \hat{e}_2 &= -4/\pi \\ \hat{e}_1 &= 2.085 \\ \hat{\tau} &= \pi \end{aligned}\right\} \tag{3.4-7}$$

in Eq. (3.4-6), we obtain the corresponding final instant

$$\hat{T}_S = 3.543 . \tag{3.4-8}$$

On the other hand, the corresponding $\hat{e}_1$-coordinate, $\bar{l}_{10}$, of initial point $Q'$ can be calculated from Eqs. (3.4-5) and (3.4-7) as

$$\bar{l}_{10} = 2.256 . \tag{3.4-9}$$

It is evident from Fig. 3.2 (a) that as concerns the part of envelope $\widehat{OQ}$ the circumstance is quite similar to the derived envelopes in Chapter 1. Then it is of no doubt to use the part of envelope $\widehat{OQ}$ as the geometrical expression of the necessary condition realizing the restricted-optimal control for the range of initial state $\hat{l}_1 \leq \bar{l}_{10}$. On the contrary, no rigourous informations on the switching rule for a larger control interval $\hat{T} > \hat{T}_S$ will be expected from the switching function of restricted-optimal control. However, it comes from Fig. 3.2 (b) that the line $\overline{QQ''}$(a tangent at $Q$) seems to play the role in an apparent boundary for formally calculated switching lines of restricted-optimal control with respect to $\hat{l}_1 > \bar{l}_{10}$. Then, let us make an adventurous supposition on the quasi-optimum stationary switching line for $\hat{l}_1 > \bar{l}_{10}$.

Supposition: The compound curve which is synthesized by using the curves $\widehat{OQ}$ and $\overline{QQ''}$ may be used for the quasi-optimum stationary switching line for positive trajectories.

## 3.5 Experimental Considerations on the Quasi-Optimum Stationary Switching Line

This section is devoted to show the plausibility of the supposition cited above. The method of attack is application of L.S. Pontryagin's Maximum Principle[42] and solution of the derived two point boundary value problem is carried out by a digital computer.

### 3.5-1 Formulation of a Two Point Boundary Value Problem

According to the Maximum Principle, the Hamiltonian function in this case has the form

$$H = p_1 e_2 + p_2(-w_n^2 e_1 + ku) + p_3 e_1^2 . \tag{3.5-1}$$

Furthermore, we obtain the system of equations

$$\left.\begin{aligned} \dot{p}_1 &= w_n^2 p_2 - 2p_3 e_1 , & p_1(T) &= 0 \\ \dot{p}_2 &= -p_1 , & p_2(T) &= 0 \\ \dot{p}_3 &= 0 , & p_3(T) &= -1 \end{aligned}\right\} . \tag{3.5-2}$$

By taking the maximization of Eq. (3.5-1) with respect to $u$, we get the optimum control variable $\bar{u}$ as

$$\bar{u} = -L \operatorname{sgn}[p_2] . \tag{3.5-3}$$

Then, substitution of Eq. (3.5-3) into Eq. (3.2-1) gives us the equation of optimum trajectories

$$\left.\begin{aligned} \dot{e}_1 &= e_2 , & e_1(0) &= l_1 \\ \dot{e}_2 &= -w_n^2 e_1 + kL \operatorname{sgn}[p_2] , & e_2(0) &= 0 \end{aligned}\right\} . \tag{3.5-4}$$

It is evident from Eq. (3.5-3) that solution of a two point boundary value problem which consists of Eqs. (3.5-2) and (3.5-4) is inevitable to obtain the explicit form of optimum control variable.

3.5-2 Solution of the Two Point Boundary Value Problem by a Digital Computer

First let us normalize the problem. By using the relations

$$\left.\begin{aligned} \hat{t} &= w_n t \\ \hat{e}_1 &= w_n^2\, e_1 / kL \\ \hat{e}_2 &= w_n\, e_2 / kL \end{aligned}\right\} \qquad (3.5\text{-}5)$$

in Eqs. (3.5-2) and (3.5-4), we get the following system of normalized equations

$$\left.\begin{aligned} \dot{\hat{e}}_1 &= \hat{e}_2 , & \hat{e}_1(0) &= \hat{l}_1 \\ \dot{\hat{e}}_2 &= -\hat{e}_1 + \mathrm{sgn}\,[\hat{P}_2] , & \hat{e}_2(0) &= 0 \\ \dot{\hat{P}}_1 &= \hat{P}_2 + 2\hat{e}_1 , & \hat{P}_1(\hat{T}) &= 0 \\ \dot{\hat{P}}_2 &= -\hat{P}_1 , & \hat{P}_2(\hat{T}) &= 0 \end{aligned}\right\} , \qquad (3.5\text{-}6)$$

where $\hat{P}_1 \equiv w_n^3 P_1 / kL$ , and $\hat{P}_2 \equiv w_n^4 P_2 / kL$ . For the convenience of computation, we replace a time variable $\hat{t}$ by the reversed time variable $\hat{\tau} = \hat{T} - \hat{t}$. Then Eq. (3.5-6) becomes

$$\left.\begin{aligned} \hat{e}_1' &= -\hat{e}_2 , & \hat{e}_1(\hat{T}) &= \hat{l}_1 \\ \hat{e}_2' &= \hat{e}_1 - \mathrm{sgn}\,[\hat{P}_2] , & \hat{e}_2(\hat{T}) &= 0 \\ \hat{P}_1' &= -\hat{P}_2 - 2\hat{e}_1 , & \hat{P}_1(0) &= 0 \\ \hat{P}_2' &= \hat{P}_1 , & \hat{P}_2(0) &= 0 \end{aligned}\right\} . \qquad (3.5\text{-}7)$$

Then, the next step is to solve Eq. (3.5-7).

It is a difficult problem even though we use a computer. However, since the author's desire is to check the quasi-optimum stationary switching line, then we set suitably small arbitrary values to the initial conditions of $\hat{e}_1$ and $\hat{e}_2$, and we only solve a set of simultaneous equations without assigning the value of $T$ and $\hat{l}_1$. For this purpose an analog computer seems to be suitable(9) because the solution is given by a chart of trajectories. From the viewpoint of accuracy, however, the author uses a digital computer. Fig. 3.5 shows the result of the computation, where two trajectories are plotted by solid

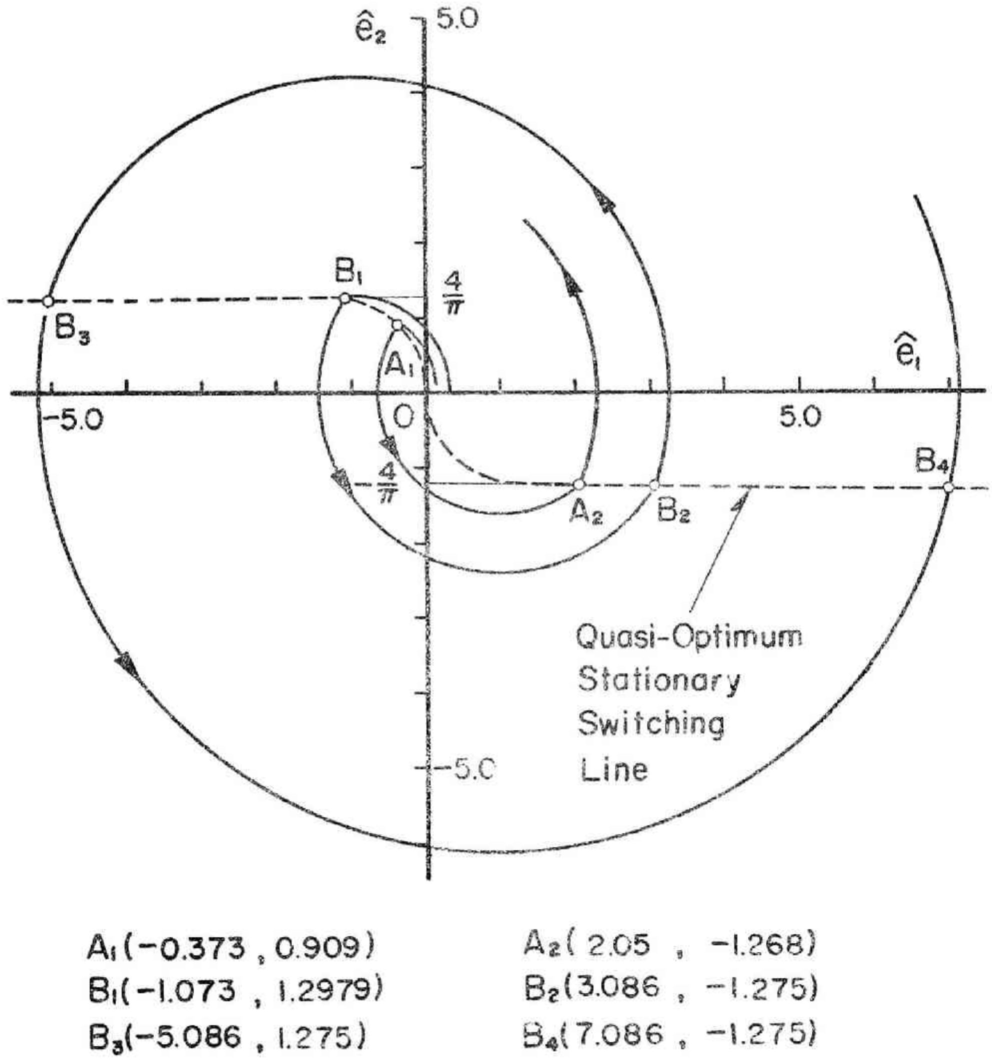


Fig. 3.5 Plot of optimum trajectories

curves, and the broken curve is the quasi-optimum stationary switching line. It is reasonable to consider that the points expressed by $B_2$, $B_3$ and $B_4$ are corresponding to switching points which provide us a stationary switching rule. Furthermore, since the $\hat{e}_2$-coordinate of these points have the same value of magnitude 1.275, then it is concluded that the switching line for a large initial condition of $\hat{l}_1$ becomes parallel to the $\hat{e}_1$-axis. On the other hand, since the quasi-optimum switching rule for a large initial coordinate $\hat{l}_1 > \bar{l}_{10}$ supposed by the author is $\hat{e}_2 = \pm\frac{4}{\pi} = \pm 1.273$, then it turns out from Fig. 3.5 that the supposition cited above may be concluded to be plausible.

## 3.6 Further Discussions

### 3.6-1 On the Sub-Interval Optimization Technique

As the author has presented in 3.4, the switching lines and their envelope of restricted-optimal control are only meaningful in the case where $\hat{T} \leq \hat{T}_s$. Furthermore, this switching rule is free from the initial value of $\hat{l}_1$. In other words, the sub-interval optimization technique based upon the restricted-optimal switching rule is only meaningful for the case where $\hat{T} \leq \hat{T}_s$. On the contrary, for the case where $\hat{T} > \hat{T}_s$ some modifications should be necessary to the sub-interval optimization technique described in 1.4. The modified part of procedure is concerned with the determination of fictitious sub-interval $T_{\text{sub-}i}$, and is expressed as follows:

(1) Check whether the control interval $T$ is less than $3.54/w_n$ or not. If not, jump to the step (4).

(2) Switch a relay at the time $\bar{t}_i$ $(i = 1, 2 \ldots\ldots)$

$$\bar{t}_i = \frac{1}{w_n} \sin^{-1}\left\{\frac{4kL}{\pi\,(w_n^2 l_{1,\,i-1} + kL)}\right\}, \qquad (3.6\text{-}1)$$

where $l_{1,i-1}$ denotes an $e_1$-coordinate of the intersection of a trajectory to the $e_1$-axis, i.e., $l_{1,i-1} = e_1(t')$, $e_2(t') = 0$.

(iii) Compute the traveling time $t_1$ which is defined in 1.4-3, and check whether the rest of control interval, $T_1 = T - t_1$, is larger than $T_s$ or not. If so, repeat from the step (ii).

(iv) Apply the procedure presented in 1.4.

The physical interpretation of the above procedure is that the length of sub-interval is supposed to be equal to $\hat{\tau}_s / w_n$.

3.6-2 On the Quasi-Optimum Non-Stationary Switching Line

Since the definition and the synthesizing method of the quasi-optimum non-stationary switching line are presented in 1.5, then no further discussions seem to be necessary on the quasi-optimum non-stationary switching line which minimizes the functional $J_2$ in Chapter 1, i.e., Eq. (1.2-2).

## 3.7 Concluding Remarks

In this chapter, the problem of determining the quasi-optimum stationary and non-stationary switching lines is treated for a second order vibratory controlled system. The method of attack is application of sub-interval optimization technique presented in 1.4. A supposition is made to derive a quasi-optimum stationary switching line, which is used to determine the length of a sub-interval (fictitious optimization interval) in the case where both the initial value of error $l_1$ and the control interval $T$ are larger than the corresponding critical values. The derived quasi-optimum stationary switching line is examined by an experimental study, and is found to be reasonable.

# PART II SYNTHESIS OF NEAR-OPTIMUM CONTROL SYSTEMS UNDER RANDOM ENVIRONMENTS

# CHAPTER 4 ANALYTICAL METHOD FOR THE SYNTHESIS OF A NEAR-OPTIMUM CONTROL SYSTEM WITH RANDOM INPUTS

## 4.1 Introductory Remarks

During the past decade, design problems of optimum control systems have received widespread attention owing to the increasing demand for control systems of high level control performance. Design techniques of optimum control systems have also been developed by various research workers, based upon such mathematical approaches as R. Bellman's Dynamic Programming,[3] L.S. Pontryagin's Maximum Principle[42] and others.[21]

Recent investigations by C. W. Merriam, 111,[34] J. D. Kramer, Jr.,[31] J. J. Florentin,[20] S. Katz[28] and J. D. Pearson[41] have presented the design problem of optimum control systems from the viewpoint of Dynamic Programming. All of these works have provided the equations of optimum control for machine computations to determine the configuration of optimum control systems. That is, using concepts of Dynamic Programming, the synthetical problem on the optimum control system with random inputs can be reduced to solve the partial differential equations whose solution yields the value of performance index. Especially, in the case of problems with linear plant dynamics, generalized quadratic performance indices, random signals, all with time-varying parameters, the partial differential equations which must be treated are reduced to a set of non-linear ordinary differential equations whose form is well suited to machine computation.

In this chapter, from a practical viewpoint, an analytical method for the synthesis of near-optimum control systems with random inputs is developed without the direct usage of a digital computer. Our principal line of attack is made by using the method of Taylor-Cauchy transform by A. A. Wolf, et al,[32] which is effective for the solution of non-linear ordinary differential equation of higher order. A numerical procedure which provides an approximate solution of a set of non-linear ordinary differential equations cited above is developed. Use is made of conceptual extension of the concept of sub-interval optimization.

## 4.2 Mathematical Formulation of the Problem

As shown in Fig. 4.1, we consider that the controlled system or the plant which is to be controlled is adequately represented by a set of known differential equations;

$$\left.\begin{aligned} \dot{x} &= A(t)\,x + D(t)\,u \\ x(0) &= x_0 \equiv \text{col.}\,[x_1^0, x_2^0, \cdots\cdots, x_n^0] \end{aligned}\right\} , \tag{4.2-1}$$

where $x = x(t)$ (an $n$-column vector, i.e., $x \equiv$ col. $[x_1, x_2, \cdots\cdots, x_n]$ and $u = u(t)$ (an $r$-column vector, i.e., $u \equiv$ col. $[u_1, u_2, \cdots\cdots, u_r]$ represent the state of the controlled signal and the control signal to the system respectively. In Eq. (4.2-1), $A(t)$ (an $n \times n$ time-dependent matrix) and $D(t)$ (an $n \times r$ time-dependent matrix) are the state matrix and the driving matrix of the system respectively.

It is also assumed that there is a random desired signal $v = v(t)$ (an $n$-vector valued function of time, i.e., $v \equiv$ col. $[v_1, v_2, \cdots\cdots, v_n]$) with known statistical properties. The performance index of the system is given by a performance functional;

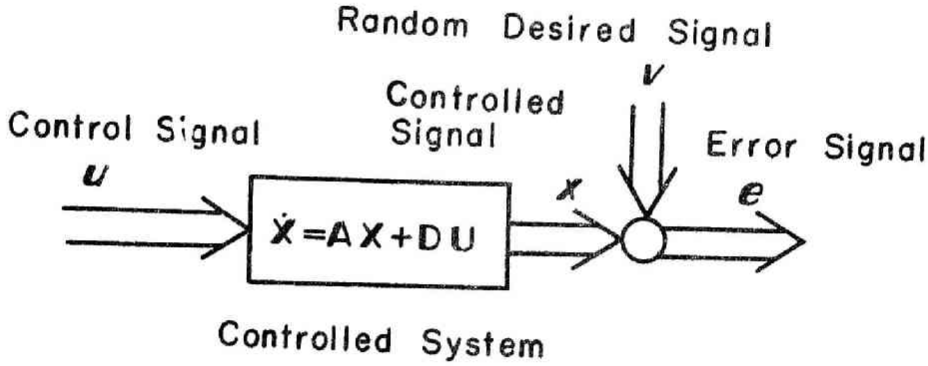


Fig. 4.1  Block diagram of the system to be considered

$$I\,(e(0),0\,;u\,(t')=\varepsilon\{\int_0^T[{}^t e(\rho)Q\,(\rho)\,e(\rho)+{}^t u(\rho)R\,(\rho)\,u(\rho)]\,d\rho\}\,,\qquad (0\leq t'\leq T) \tag{4.2-2}$$ *

where $e=e(t)$ is the system error (an $n$-column vector, i.e., $e\equiv$col. $[e_1,\ e_2,\ \cdots\cdots,\ e_n]$ ) defined by

$$e(t)=v(t)-x(t)\ . \tag{4.2-3}$$

In Eq. (4.2-2), $Q(t)$ and $R(t)$ express $n\times n$ and $r\times r$ symmetric time-dependent matrices respectively. Furthermore, $\varepsilon$ denotes the ensemble average of what appears to its right under the condition of observing the present state variables of the system.

In order to present the statistical properties of the random desired signal, it is convenient to rewrite the system equation, Eq. (4.2-1), in terms of the error signal $e(t)$ and the desired signal $v(t)$. That is, by substituting Eq. (4.2-3) for $x(t)$ into Eq. (4.2-1), the following equation can be derived;

* ${}^tA$ denotes the transpose of a matrix $A$ .

$$\left.\begin{array}{l}\dot{e}=A(t)\,e-D(t)u+\Xi\\ e(0)=e_0\equiv\mathrm{col.}\,[e_1^0,\ e_2^0\cdots\cdots,\ e_n^0\,]\end{array}\right\}\ ,\qquad(4.2\text{-}4)$$

where $\Xi\equiv\Xi(t)$ (an $n$-column vector, i.e., $\Xi\equiv\mathrm{col.}\,[\xi_1,\ \xi_2,\ldots\ldots,\xi_n]$) is the apparent random disturbance to the system described by Eq. (4.2-4) and it is defined by

$$\Xi=\dot{v}-A(t)\ v\ .\qquad(4.2\text{-}5)$$

We assume that the apparent random disturbance, $\Xi$ , is a multi-dimensional Gaussian white noise process with mean value $m=m(t)$ (an $n$-column vector, i.e., $m\equiv\mathrm{col.}\ [m_1,\ m_2,\ldots\ldots m_n]$ ) and variance-covariance $\Sigma=\Sigma(t)$ (an $n\times n$ positive definite matrix, i.e., $\Sigma=\|\sigma_{ij}^2\|$ ). The desired signal is thus assumed to be equivalent to the output signal of the system, which is expressed by Eq. (4.2-5), subjected to the Gaussian white noise process $\Xi$ mentioned above. The optimum control problem is to determine the control signal $u$ which minimizes the performance functional, Eq. (4.2-2). We observe that the optimum control signal $u$ depends upon such important factors as (a) the initial state of the system output $x(0)=x_0$ (or the initial state of the system error $e(0)=e_0$), (b) the statistical properties of the desired signal $v(t)$ and (c) the description of a performance functional.

## 4.3 Equations of Optimum Control

With the help of Appendix C, the following partial differential equation of the optimum control can be obtained;

$$\frac{\partial\phi}{\partial\tau}=\min_{u}\{{}^t eQe+{}^t uRu+{}^t[Ae-Du+m]\frac{\partial\phi}{\partial e}$$
$$+\frac{1}{2}\,{}^t[\frac{\partial}{\partial e}]\ \Sigma\ [\frac{\partial}{\partial e}]\ \phi\ \}\ ,\qquad(4.3\text{-}1)$$

where $\tau$ denotes an auxiliary time variable defined by $\tau\equiv T-t$ and referred to as a reversed time. As the optimum control signal $\bar{u}$ which minimizes Eq. (4.3-1), from Eq. (4.3-1), we have

$$u=\frac{1}{2}R^{-1}\,{}^tD\frac{\partial\phi}{\partial e}\qquad(4.3\text{-}2)$$

If $R$ is singular, then $R^{-1}$ is to be taken in the sense of the generalized inverse of matrix,[39] which roughly means that a suitable set of linear equations can be described by this expression. By substituting Eq. (4.3-2) for $u$ in Eq. (4.3-1), it follows that

$$\frac{\partial\phi}{\partial\tau}=-\frac{1}{4}\,{}^t\left[\frac{\partial\phi}{\partial e}\right]DR^{-1}\,{}^tD\left[\frac{\partial\phi}{\partial e}\right]+{}^te\,Qe$$
$$+{}^t\left[Ae+m\right]\frac{\partial\phi}{\partial e}+\frac{1}{2}\,{}^t\left[\frac{\partial}{\partial e}\right]\Sigma\left[\frac{\partial}{\partial e}\right]\phi\ . \qquad (4.3\text{-}3)$$

No special conditions on state of the system have to be met at the final instant, $\tau=0$ (i.e., $t=T$), and the boundary conditions are

$$\phi\,(e\,;\,0)=0 \quad \text{for any } e\ . \qquad (4.3\text{-}4)$$

We assume that

$$\phi\,(e\,;\,\tau)=k_0(\tau)+{}^tK_1(\tau)e+{}^teK_2(\tau)\,e\ , \qquad (4.3\text{-}5)$$

where $k_0(\tau)$ is a scalar, ${}^tK_1(\tau)$ is a vector (an $n$-row vector, i.e., ${}^tK_1\equiv$ row $[k_1,\ k_2,\ \cdots\cdots,\ k_n]$), and $K_2(\tau)$ is a matrix (an $n\times n$ positive definite symmetric matrix, i.e., $K_2=\|k_{ij}\|$, $k_{ij}=k_{ji}$ ), and all $k$-coefficients depend only on the reversed time .

Substituting Eq. (4.3-5) for $\phi(e\,;\tau)$ into Eq. (4.3-3), and making extensive use of the properties of symmetric matrices, the following non-linear differential equations can be obtained for the coefficients in expanded series Eq. (4.3-5);

$$\left.\begin{aligned}\dot{k}_0&=-\frac{1}{4}\,{}^tK_1DR^{-1}\,{}^tDK_1+\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}^{2}\,k_{ij}\\ {}^t\dot{K}_1&=-{}^tK_1DR^{-1}\,{}^tDK_2+{}^tK_1A+2\,{}^tmK_2\\ \dot{K}_2&=Q-K_2DR^{-1}\,{}^tDK_2+{}^tAK_2+K_2A\end{aligned}\right\}\ , \qquad (4.3\text{-}6)$$

with boundary conditions $k_0(0)={}^tK_1(0)=K_2(0)=0$. Therefore, using the coefficients satisfying Eq. (4.3-6), we can rewrite Eq. (4.3-2) as follows;

$$\bar{u}=\frac{1}{2}R^{-1}\,{}^tD\,(K_1+2K_2\,e)\ . \qquad (4.3\text{-}7)$$

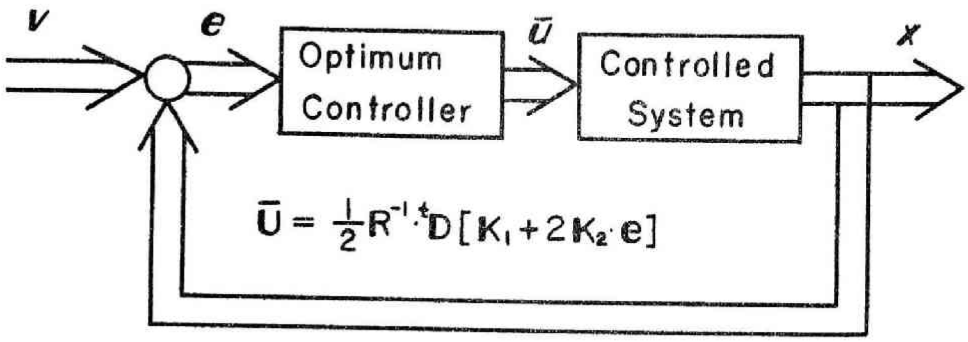


Fig. 4.2 Configuration of the optimum control system

The structure of the optimum control system with a Gaussian random signal can be described by a typical feedback system with time-varying control elements as shown in Fig. 4.2.

The design problem of optimum control systems can thus be reduced to solve a set of simultaneous non-linear differential equations;

$$\left.\begin{aligned} {}^t\dot{K}_1 &= -K_1 DR^{-1}\,{}^tDK_2 + {}^tK_1 A + 2\,{}^tmK_2\,, & {}^tK_1(0) &= 0 \\ \dot{K}_2 &= Q - K_2 DR^{-1}\,{}^tDK_2 + {}^tAK_2 + K_2 A\,, & K_2(0) &= 0 \end{aligned}\right\}. \qquad (4.3\text{-}8)$$

The following discussions are, therefore, focused on providing an analytical technique for the solution of Eq. (4.3-8). Use is made of an extension of the method of Taylor-Cauchy transform introduced by A.A. Wolf, et al.(32)

## 4.4 Taylor-Cauchy Transformation

The direct form, $h_{n,\nu}$ , of Taylor-Cauchy transformation of a function $H^{(\nu)}(\zeta)$, where $\nu$ corresponds the order of the highest derivatives of the system equation, is defined by

$$\mathcal{T}_c\,[H^{(\nu)}(\zeta)] = h_{n,\nu} = \frac{1}{2\pi j}\int_c \frac{H^{(\nu)}(\zeta)}{\zeta^{n+1}}\,d\zeta\ , \qquad (4.4\text{-}1)$$

and its inverse form is

$$\mathcal{T}_c^{-1}[h_{n,\nu}] = H^{(\nu)}(\zeta) = \sum_{n=0}^{\infty} h_{n,\nu}\,\zeta^n\ , \qquad (4.4\text{-}2)$$

where $n$ is a running discrete positive index taking values $0, 1, 2, \ldots$. In Eqs. (4.4-1) and (4.4-2), $H^{(\nu)}(\zeta)$ and $c$ respectively express the $\nu$-th derivative of a function $H(\zeta)$ with respect to a complex variable $\zeta$ and a closed contour enclosing the singularities of $H^{(\nu)}(\zeta)$ in the $\zeta$-plane.

With the help of Appendix D, the author summarizes several useful formulae for the present discussion in the following:

(1) A formula of the direct transformation for the product, $F_p(\zeta)\cdot H_q(\zeta)$ of two complex functions is presented by

$$\mathcal{T}_c\,[F_p(\zeta)H_q(\zeta)] = f_{p,n}\,\overline{h}_q + F_{n,u}^{p,q}\ , \qquad (4.4\text{-}3)$$

where

$$\left.\begin{aligned} F_{n,u}^{p,q} &= 0 && \text{for} \quad n=0 \\ &= \sum_{u=0}^{n-1} \frac{f_{p,u}\,h_{q,n-u-1}}{n-u} && \text{for} \quad n \geq 1 \end{aligned}\right\}, \qquad (4.4\text{-}4)_1$$

$$\overline{h}_q \equiv H_q(0)\ , \qquad (4.4\text{-}4)_2$$

$$f_{p,n} = \mathcal{T}_c\,[F_p(\zeta)] = \frac{1}{2\pi j}\int_c \frac{F_p(\zeta)}{\zeta^{n+1}}\,d\zeta\ , \qquad (4.4\text{-}4)_3$$

and

$$h_{q,n} = \mathcal{T}_c\,[\dot{H}_q(\zeta)] = \frac{1}{2\pi j}\int_c \frac{\dot{H}_q(\zeta)}{\zeta^{n+1}}\,d\zeta\ . \qquad (4.4\text{-}4)_4$$

(2) A formula of the direct transformation for the triple product, $H_p(\zeta)F_q(\zeta)H_r(\zeta)$, of three complex functions is described by

$$\mathcal{T}_C\left[H_p(\zeta)F_q(\zeta)H_r(\zeta)\right]$$

$$=\bar{h}_p\bar{h}_r f_{q,n}+\bar{h}_p F^{q,r}_{n,u}+\bar{h}_r F^{q,p}_{n,u}+D^{q,p,r}_{n,w} \quad . \tag{4.4-5}$$

where

$$\left.\begin{aligned} D^{q,p,r}_{n,w} &= 0 && \text{for} \quad n=0,1 \\ &= \sum_{w=0}^{n-2} f_{q,w}\, C^{p,r}_{n-w,u} && \text{for} \quad n\geq 2 \end{aligned}\right\}, \tag{4.4-6}_1$$

$$\left.\begin{aligned} C^{p,r}_{n,u} &= 0 && \text{for} \quad n=0,1 \\ &= \sum_{u=0}^{n-2} \frac{h_{p,u}\, h_{r,n-u-2}}{(u+1)\ (n-u-1)} && \text{for} \quad n\geq 2 \end{aligned}\right\}, \tag{4.4-6}_2$$

$$\bar{h}_p \equiv H_p(0) \qquad \text{and} \qquad \bar{h}_r \equiv H_r(0). \tag{4.4-6}_3$$

and $h_{p,n}$, $h_{q,n}$ and $f_{r,n}$ express the direct transformations of $\dot{H}_p(\zeta)$, $\dot{H}_q(\zeta)$ and $F_r(\zeta)$ respectively.

## 4.5 The Method of solving the Equations of Optimum Control

We consider the equations of optimum control described by

$$\left.\begin{aligned} {}^t\dot{K}_1 &= -{}^tK_1 W K_2 + {}^tK_1 A + 2\,{}^t m K_2 \ , & {}^tK_1(0) &= 0 \\ \dot{K}_2 &= Q - K_2 W K_2 + {}^tA K_2 + K_2 A \ , & K_2(0) &= 0 \end{aligned}\right\}, \tag{4.5-1}$$

where $W$ denotes an $n\times n$ time dependent matrix equivalent to $DR^{-1}\,{}^tD$ in Eq. (4.3-8).

In order to apply the Taylor-Cauchy transformation to Eq. (4.5-1), we extend the real time functions $K_1(\tau)$, $K_2(\tau)$, $A(\tau)$, $m(\tau)$, $W(\tau)$ and $Q(\tau)$ to complex time functions $\underline{K}_1(\zeta)$, $\underline{K}_2(\zeta)$, $\underline{A}(\zeta)$, $\underline{m}(\zeta)$, $\underline{W}(\zeta)$ and $\underline{Q}(\zeta)$, respectively, by using the principle of analytical continuation.(1) Thus, Eq. (4.5-1) becomes

$$\left.\begin{aligned} {}^t\dot{\underline{K}}_1(\zeta) &= -{}^t\underline{K}_1(\zeta)\underline{W}(\zeta)\underline{K}_2(\zeta) + {}^t\underline{K}_1(\zeta)\underline{A}(\zeta) + 2\,{}^t\underline{m}(\zeta)\underline{K}_2(\zeta)\ , & {}^t\underline{K}_1(0) &= 0 \\ \dot{\underline{K}}_2(\zeta) &= \underline{Q}(\zeta) - \underline{K}_2(\zeta)\underline{W}(\zeta)\underline{K}_2(\zeta) + {}^t\underline{A}(\zeta)\underline{K}_2(\zeta) + \underline{K}_2(\zeta)\underline{A}(\zeta)\ , & \underline{K}_2(0) &= 0 \end{aligned}\right\}. \tag{4.5-2}$$

where both ${}^t\dot{\underline{K}}_1(\zeta)$ and $\dot{\underline{K}}_2(\zeta)$ express the derivatives of complex functions, $\underline{K}_1(\zeta)$ and $\underline{K}_2(\zeta)$ respectively. In Eq. (4.5-2), both ${}^t\underline{K}_1(\zeta)$ and ${}^t\underline{m}(\zeta)$ are complex functions of vector form. On the other hand, $\underline{K}_2(\zeta)$, $\underline{W}(\zeta)$, $\underline{A}(\zeta)$, and $\underline{Q}(\zeta)$ are matrices defined by

$$ {}^t\underline{K}_1(\zeta) = \mathrm{row}\,[K_1(\zeta), K_2(\zeta), \cdots\cdots, K_n(\zeta)] \,, \qquad (4.5\text{-}3)_1 $$

$$ {}^t\underline{m}(\zeta) = \mathrm{row}\,[M_1(\zeta), M_2(\zeta), \cdots\cdots, M_n(\zeta)] \,, \qquad (4.5\text{-}3)_2 $$

$$ \underline{K}_2(\zeta) = \| K_{ij}(\zeta) \| \,, \quad (i, j = 1, 2, \cdots\cdots, n) \,, \qquad (3.5\text{-}3)_3 $$

$$ \underline{W}(\zeta) = \| W_{ij}(\zeta) \| \,, \quad (i, j = 1, 2, \cdots\cdots, n) \,, \qquad (4.5\text{-}3)_4 $$

$$ \underline{A}(\zeta) = \| A_{ij}(\zeta) \| \,, \quad (i, j = 1, 2, \cdots\cdots, n) \,, \qquad (4.5\text{-}3)_5 $$

and

$$ \underline{Q}(\zeta) = \| Q_{ij}(\zeta) \| \,, \quad (i, j = 1, 2, \cdots\cdots, n) \,. \qquad (4.5\text{-}3)_6 $$

By performing the direct transformation on Eq. (4.5-2), we have

$$ \left.\begin{aligned} \mathcal{T}_c[{}^t\dot{\underline{K}}_1(\zeta)] &= -\mathcal{T}_c[{}^t\underline{K}_1(\zeta)\,\underline{W}(\zeta)\,\underline{K}_2(\zeta)] \\ &\quad + \mathcal{T}_c[{}^t\underline{K}_1(\zeta)\underline{A}(\zeta)] + 2\mathcal{T}_c[{}^t\underline{m}(\zeta)\,\underline{K}_2(\zeta)] \\ \mathcal{T}_c[\dot{\underline{K}}_2(\zeta)] &= \mathcal{T}_c[\underline{Q}(\zeta)] - \mathcal{T}_c[\underline{K}_2(\zeta)\underline{W}(\zeta)\,\underline{K}_2(\zeta)] \\ &\quad + \mathcal{T}_c[{}^t\underline{A}(\zeta)\underline{K}_2(\zeta)] + \mathcal{T}_c[\underline{K}_2(\zeta)\,\underline{A}(\zeta)] \end{aligned}\right\} , \qquad (4.5\text{-}4) $$

where

$$ \begin{aligned} \mathcal{T}_c[{}^t\dot{\underline{K}}_1(\zeta)] &= \mathrm{row}\,(\mathcal{T}_c[\dot{K}_1(\zeta)], \mathcal{T}_c[\dot{K}_2(\zeta)], \cdots\cdots \\ &\qquad \cdots\cdots, \mathcal{T}_c[\dot{K}_n(\zeta)]\,) \\ &= \mathrm{row}\,(k_{1,n}, k_{2,n}, \cdots\cdots, k_{n,n}) \,, \end{aligned} \qquad (4.5\text{-}5)_1 $$

$$ \begin{aligned} \mathcal{T}_c[{}^t\underline{K}_1(\zeta)\underline{A}(\zeta)] = \mathrm{row}\,\Big(&\sum_{r=1}^{n}\mathcal{T}_c[K_r(\zeta)A_{r1}(\zeta)] \,, \\ &\sum_{r=1}^{n}\mathcal{T}_c[K_r(\zeta)A_{r2}(\zeta)] \,, \cdots\cdots \\ &\cdots\cdots, \sum_{r=1}^{n}\mathcal{T}_c[K_r(\zeta)A_{rn}(\zeta)]\,\Big) \,, \end{aligned} \qquad (4.5\text{-}5)_2 $$

$$\mathcal{T}_c\left[{}^t\underline{m}(\zeta)\underline{K}_2(\zeta)\right] = \text{row}\ \Big(\sum_{r=1}^{n}\mathcal{T}_c\left[M_r(\zeta)K_{r1}(\zeta)\right],$$
$$\sum_{r=1}^{n}\mathcal{T}_c\left[M_r(\zeta)K_{r2}(\zeta)\right], \cdots\cdots$$
$$\cdots\cdots, \sum_{r=1}^{n}\mathcal{T}_c\left[M_r(\zeta)K_{rn}(\zeta)\right]\Big), \qquad (4.5\text{-}5)_3$$

$$\mathcal{T}_c\left[{}^t\underline{K}_1(\zeta)\underline{W}(\zeta)\underline{K}_2(\zeta)\right] = \text{row}\ \Big(\sum_{p=1}^{n}\sum_{q=1}^{n}\mathcal{T}_c\left[K_p(\zeta)W_{pq}(\zeta)K_{q1}(\zeta)\right],$$
$$\sum_{p=1}^{n}\sum_{q=1}^{n}\mathcal{T}_c\left[K_p(\zeta)W_{pq}(\zeta)K_{q2}(\zeta)\right], \cdots\cdots \qquad (4.5\text{-}5)_4$$
$$\cdots\cdots, \sum_{p=1}^{n}\sum_{q=1}^{n}\mathcal{T}_c\left[K_p(\zeta)W_{pq}(\zeta)K_{qn}(\zeta)\right]\Big),$$

$$\mathcal{T}_c\left[\dot{\underline{K}}_2(\zeta)\right] = \left\|\mathcal{T}_c\left[\dot{K}_{ij}(\zeta)\right]\right\| = \left\|k_{ij,m}\right\|$$
$$(i, j=1, 2, \cdots\cdots, n), \qquad (4.5\text{-}5)_5$$

$$\mathcal{T}_c\left[\underline{Q}(\zeta)\right] = \left\|\mathcal{T}_c\left[Q_{ij}(\zeta)\right]\right\|$$
$$(i, j=1, 2, \cdots\cdots, n), \qquad (4.5\text{-}5)_6$$

$$\mathcal{T}_c\left[\underline{K}_2(\zeta)\underline{A}(\zeta)\right] = \left\|\sum_{r=1}^{n}\mathcal{T}_c\left[K_{ir}(\zeta)A_{rj}(\zeta)\right]\right\|$$
$$(i, j=1, 2, \cdots\cdots, n), \qquad (4.5\text{-}5)_7$$

$$\mathcal{T}_c\left[{}^t\underline{A}(\zeta)\underline{K}_2(\zeta)\right] = \left\|\sum_{r=1}^{n}\mathcal{T}_c\left[A_{ri}(\zeta)K_{rj}(\zeta)\right]\right\|$$
$$(i, j=1, 2, \cdots, n), \qquad (4.5\text{-}5)_8$$

and

$$\mathcal{T}_c\left[{}^t\underline{K}_2(\zeta)\underline{W}(\zeta)\underline{K}_2(\zeta)\right] = \left\|\sum_{p=1}^{n}\sum_{q=1}^{n}\mathcal{T}_c\left[K_{pi}(\zeta)W_{pq}(\zeta)K_{qj}(\zeta)\right]\right\|$$
$$(i, j=1, 2, \cdots\cdots, n). \qquad (4.5\text{-}5)_9$$

By applying the direct transformation, Eqs. (4.4-3) and (4.4-5), to Eqs. $(4.5\text{-}5)_2$, $(4.5\text{-}5)_3$, $(4.5\text{-}5)_4$, $(4.5\text{-}5)_7$, $(4.5\text{-}5)_8$, and $(4.5\text{-}5)_9$, a set of recurrence relations with respect to coefficients $k_{i,m}$ and $k_{ij,m}$ ($i, j=1, 2, \ldots, n$) with $m_{i,m}$, $a_{ij,m}$, $q_{ij,m}$ and $w_{ij,m}$ can be derived, where

$$m_{i,m} = \mathcal{T}_c\left[M_i(\zeta)\right] = \frac{1}{2\pi j}\int_c \frac{M_i(\zeta)}{\zeta^{m+1}}\,d\zeta \qquad (i=1, 2, \cdots\cdots, n), \qquad (4.5\text{-}6)_1$$

$$a_{ij,m}=\mathcal{T}_c\ [A_{ij}(\zeta)]=\frac{1}{2\pi j}\int_c \frac{A_{ij}(\zeta)}{\zeta^{m+1}}\ d\zeta \qquad (i,j=1,2,\cdots\cdots,n)\ , \qquad (4.5\text{-}6)_2$$

$$q_{ij,m}=\mathcal{T}_c\ [Q_{ij}(\zeta)]=\frac{1}{2\pi j}\int_c \frac{Q_{ij}(\zeta)}{\zeta^{m+1}}\ d\zeta \qquad (i,j=1,2,\cdots\cdots n)\ , \qquad (4.5\text{-}6)_3$$

and

$$w_{ij,m}=\mathcal{T}_c\ [W_{ij}(\zeta)]=\frac{1}{2\pi j}\int_c \frac{W_{ij}(\zeta)}{\zeta^{m+1}}\ d\zeta \qquad (i,j=1,2,\cdots\cdots,n)\ . \qquad (4.5\text{-}6)_4$$

By applying the inverse transformation to $k_{i,m}$ and $k_{ij,m}$ $(i,j=1,2,\cdots\cdots,n)$, ${}^t\underline{K}_1(\zeta)$ and $\underline{K}_2(\zeta)$ can be obtained as

$$ {}^t\underline{K}_1(\zeta)=\text{row}\ \Big(\sum_{m=0}^{\infty}\frac{k_{1,m}}{m+1}\ \zeta^{m+1},\ \sum_{m=0}^{\infty}\frac{k_{2,m}}{m+1}\ \zeta^{m+1},\ \cdots\cdots\ \cdots\cdots\cdots,\ \sum_{m=0}^{\infty}\frac{k_{n,m}}{m+1}\ \zeta^{m+1}\Big)\ , \qquad (4.5\text{-}7)_1$$

and

$$\underline{K}_2(\zeta)=\Big\|\sum_{m=0}^{\infty}\frac{k_{ij,m}}{m+1}\ \zeta^{m+1}\Big\|\ , \qquad (i,j=1,2,\cdots\cdots,n)\ . \qquad (4.5\text{-}7)_2$$

The final results are expressed by

$$ {}^t\underline{K}_1(\tau)=\text{row}\ \Big(\sum_{m=0}^{\infty}\frac{k_{1,m}}{m+1}\ \tau^{m+1},\ \sum_{m=0}^{\infty}\frac{k_{2,m}}{m+1}\ \tau^{m+1},\ \cdots\cdots\cdots\ \cdots\cdots\cdots,\ \sum_{m=0}^{\infty}\frac{k_{n,m}}{m+1}\ \tau^{m+1}\Big)\ , \qquad (4.5\text{-}8)_1$$

and

$$\underline{K}_2(\tau)=\Big\|\sum_{m=0}^{\infty}\frac{k_{ij,m}}{m+1}\ \tau^{m+1}\Big\| \qquad (i,j=1,2,\cdots\cdots,n)\ . \qquad (4.5\text{-}8)_2$$

By substituting Eqs. $(4.5\text{-}8)_1$ and $(4.5\text{-}8)_2$ for $\underline{K}_1$ and $\underline{K}_2$ in Eq. (4.3-7), the complete description of the controller operation can mathematically be performed.

## 4.6 A Method of Approximate Calculations

Since Eqs. $(4.5\text{-}7)_1$ and $(4.5\text{-}7)_2$ are described by infinite series expansions, it is very tedious to calculate them directly. In order to simplify these calculations, a numerical procedure is presented in this section by using the concept of sub-interval optimization.

Assuming that $K_p(\zeta)$ expresses an arbitrary component of both functions ${}^t\underline{K}_1(\zeta)$ and $\underline{K}_2(\zeta)$, the function $K_p(\zeta)$ becomes

$$K_p(\zeta)=\sum_{n=0}^{N}\frac{k_{p,n}}{n+1}\zeta^{n+1}+\sum_{n=N+1}^{\infty}\frac{k_{p,n}}{n+1}\zeta^{n+1}. \tag{4.6-1}$$

In general, if we choose a certain number $N$, then there exists a some constant $\zeta_0'$ such that

$$K_p(\zeta)\cong\sum_{n=0}^{N}\frac{k_{p,n}}{n+1}\zeta^{n+1}, \qquad (0\leq\zeta\leq\zeta_0'). \tag{4.6-2}$$

Since Eq. (4.6-2) can be considered as the first step approximation of Eq. (4.6-1) with respect to $\zeta$, we introduce a new symbol,

$$\left.\begin{aligned} K_p(\zeta)_1&=\sum_{n=0}^{N}\frac{k_{p,n}}{n+1}\zeta^{n+1} && \text{for} \quad 0\leq\zeta\leq\zeta_0' \\ &=0 && \text{for} \quad \zeta>\zeta_0' \end{aligned}\right\}. \tag{4.6-3}$$

Under the mathematical concepts of analytical continuation,(1) we can thus rewrite Eq. (4.6-1) as follows:

$$K_p(\zeta)\cong K_p(\zeta)_1 \qquad \text{for} \quad 0\leq\zeta\leq\zeta_0' \tag{4.6-4$_1$}$$

$$=K_p(\zeta_0')_1+\sum_{n=0}^{\infty}\frac{k'_{p,n}}{n+1}(\zeta-\zeta_0')^{n+1} \quad \text{for} \quad \zeta>\zeta_0', \tag{4.6-4$_2$}$$

where $k'_{p,n}$ expresses the direct transformation of $\dot{K}_p(\zeta_1)$ which is the derivative of a translated function $K_p(\zeta_1)$, where $\zeta_1=\zeta-\zeta_0'$, and $K_p(\zeta_1)$ represents an arbitrary component of ${}^t\underline{K}_1(\zeta_1)$ and $\underline{K}_2(\zeta_1)$ satisfying the following equations;

$$\left.\begin{aligned} {}^t\dot{\underline{K}}_1(\zeta_1)&=-\underline{K}_1(\zeta_1)\underline{W}_t(\zeta_1)\underline{K}_2(\zeta_1)+{}^t\underline{K}_1(\zeta_1)\underline{A}_t(\zeta_1)+2\,{}^t\underline{m}_t(\zeta_1)\underline{K}_2(\zeta_1) \\ \dot{\underline{K}}_2(\zeta_1)&=\underline{Q}_t(\zeta_1)-\underline{K}_2(\zeta_1)\underline{W}_t(\zeta_1)\underline{K}_2(\zeta_1)+{}^t\underline{A}_t(\zeta_1)\underline{K}_2(\zeta_1)+\underline{K}_2(\zeta_1)\underline{A}_t(\zeta_1) \end{aligned}\right\}, \tag{4.6-5}$$

with initial conditions;

$$\left.\begin{aligned} {}^t\underline{K}_1(0)&={}^t\underline{K}_1(\zeta_0')_1\equiv \text{row}\,[K_1(\zeta_0')_1, K_2(\zeta_0')_1, \cdots\cdots, K_n(\zeta_0')_1] \\ \underline{K}_2(0)&=\underline{K}_2(\zeta_0')_1\equiv\|K_{ij}(\zeta_0')_1\|, \quad (i,\ j=1,2,\cdots\cdots,n) \end{aligned}\right\}, \tag{4.6-6}$$

where $\underline{W}_t(\zeta_1)$, $\underline{A}_t(\zeta_1)$, ${}^t\underline{m}_t(\zeta_1)$ and $\underline{Q}_t(\zeta_1)$ express $\underline{W}(\zeta_1+\zeta_0')$, $\underline{A}(\zeta_1+\zeta_0')$, $\underline{m}(\zeta_1+\zeta_0')$ and $\underline{Q}(\zeta_1+\zeta_0')$ respectively. Since Eq. (4.6-4)$_2$ can be written as

$$K_p(\zeta)\cong K_p(\zeta_0')_1+\sum_{n=0}^{N'}\frac{k'_{p,n}}{n+1}(\zeta-\zeta_0')^{n+1}+\sum_{n=N'+1}^{\infty}\frac{k'_{p,n}}{n+1}(\zeta-\zeta_0')^{n+1} \qquad \text{for} \quad \zeta>\zeta_0' \tag{4.6-7}$$

then we can choose a some constant $\zeta_0''$ for a certain number $N'$, and the following relation can be derived

$$K_p(\zeta) \cong K_p(\zeta_0')_1 + \sum_{n=0}^{N'} \frac{k'_{p,n}}{n+1} (\zeta - \zeta_0')^{n+1} \qquad \text{for } \zeta_0' \leq \zeta \leq \zeta_0'' . \qquad (4.6\text{-}8)$$

From Eq. (4.6-8), we can define the second step approximate solution $K_p(\zeta)_2$ , for $K_p(\zeta)$ as follows:

$$\left.\begin{aligned} K_p(\zeta)_2 &= K_p(\zeta_0')_1 + \sum_{n=0}^{N'} \frac{k'_{p,n}}{n+1} (\zeta - \zeta_0')^{n+1} && \text{for } \zeta_0' \leq \zeta \leq \zeta_0'' \\ &= 0 && \text{for } \zeta > \zeta_0'' \end{aligned}\right\} \qquad (4.6\text{-}9)$$

By repeating a similar procedure, we derive the following recurrence formula for the $m$-th step approximate solution, $K_p(\zeta)_m$ for $K_p(\zeta)$, i.e.,

$$\left.\begin{aligned} K_p(\zeta)_m &= K_p(\zeta_0^{(m-1)})_{m-1} + \sum_{n=0}^{N^{(m-1)}} \frac{k_{p,n}^{(m-1)}}{n+1} (\zeta - \zeta_0^{(m-1)})^{n+1} \\ & \qquad\qquad \text{for } \zeta_0^{(m-1)} \leq \zeta \leq \zeta_0^{(m)} \\ &= 0 \qquad\qquad \text{for } \zeta > \zeta_0^{(m)} \end{aligned}\right\}, \qquad (4.6\text{-}10)$$

where $K_p(\zeta)_0 \equiv K_p(\zeta)$, $\zeta_0^{(0)} \equiv 0$, $k_{p,n}^{(0)} \equiv k_{p,n}$, and $k_{p,n}^{(m-1)}$ expresses the direct transformation of the derivative of a translated function, $K_p(\zeta_{m-1})$, where $\zeta_{m-1} = \zeta - \zeta_0^{(m-1)}$ , and $K_p(\zeta_{m-1})$ is an arbitrary component of both ${}^t\underline{K}_1(\zeta_{m-1})$ and $\underline{K}_2(\zeta_{m-1})$ satisfying the following equations;

$$\left.\begin{aligned} {}^t\dot{\underline{K}}_1(\zeta_{m-1}) &= -{}^t\underline{K}_1(\zeta_{m-1})\underline{W}_t(\zeta_{m-1})\underline{K}_2(\zeta_{m-1}) \\ &\quad + {}^t\underline{K}_1(\zeta_{m-1})\underline{A}_t(\zeta_{m-1}) + 2\,{}^t\underline{m}_t(\zeta_{m-1})\underline{K}_2(\zeta_{m-1}) \\ \dot{\underline{K}}_2(\zeta_{m-1}) &= \underline{Q}_t(\zeta_{m-1}) - \underline{K}_2(\zeta_{m-1})\underline{W}_t(\zeta_{m-1})\underline{K}_2(\zeta_{m-1}) \\ &\quad + {}^t\underline{A}_t(\zeta_{m-1})\underline{K}_2(\zeta_{m-1}) + \underline{K}_2(\zeta_{m-1})\underline{A}_t(\zeta_{m-1}) \end{aligned}\right\}, \qquad (4.6\text{-}11)$$

with initial conditions;

$$\left.\begin{aligned} {}^t\underline{K}_1(0) &= {}^t\underline{K}_1(\zeta_0^{(m-1)})_{m-1} \equiv \text{row}\,(K_1(\zeta_0^{(m-1)})_{m-1}, K_2(\zeta_0^{(m-1)})_{m-1}, \\ &\qquad \cdots\cdots, K_n(\zeta_0^{(m-1)})_{m-1}) \\ \underline{K}_2(0) &= \underline{K}_2(\zeta_0^{(m-1)})_{m-1} \equiv \| K_{ij}(\zeta_0^{(m-1)})_{m-1} \| \\ &\qquad (i, j = 1, 2, \cdots\cdots, n) \end{aligned}\right\}. \qquad (4.6\text{-}12)$$

where $\underline{W}_t(\zeta_{m-1})$ , $\underline{A}_t(\zeta_{m-1})$, $\underline{m}_t(\zeta_{m-1})$ and $\underline{Q}_t(\zeta_{m-1})$ express $\underline{W}(\zeta_{m-1}+\zeta_0^{(m-1)})$, $\underline{A}(\zeta_{m-1}+\zeta_0^{(m-1)})$, $\underline{m}(\zeta_{m-1}+\zeta_0^{(m-1)})$ and $\underline{Q}(\zeta_{m-1}+\zeta_0^{(m-1)})$ respectively.

The successive steps are summarised as follows:

(1) Perform the direct transformation of Eq. (4.5-2) and derive a set of recurrence relations with respect to $k_{i,n}$ and $k_{ij,n}$.

(2) Calculate a sequence of coefficients $k_{i,n}$ and $k_{ij,n}$ for $n=1, 2, \ldots\ldots, N$.

(3) Plot the truncated series of Eqs. $(4.5\text{-}7)_1$ and $(4.5\text{-}7)_2$, and estimate the value of $\zeta_0'$ from a graphical point of view.

(4) From the step (iii), determine the tentative first step approximate solution.

(5) By replacing the variable $\zeta$ by $\zeta_1=\zeta-\zeta_0'$, obtain the translated differential equation Eq. (4.6-5) with Eq. (4.6-6) as the initial condition.

(6) Perform the direct transformation of the translated differential equation considering initial conditions, and derive a set of recurrence relations with respect to $k_{i,n}'$ and $k_{ij,n}'$

(7) Calculate a sequence of new coefficients $k_{i,n}'$ and $k_{ij,n}'$ for $n=1, 2, \ldots\ldots, N'$, and plot the new truncated series expansion of Eq. (4.6-7).

(8) Check whether the connection of the first step approximate solution to the second one at the estimated time $\zeta=\zeta_0'$ is smooth or not. If not, then the estimated value $\zeta_0'$ ceases to be reasonable and another smaller value must be selected as an estimate of $\zeta_0'$, and the procedure is repeated again from the step ( 4 ). If so, on the other hand, the determination of the first step approximate solution is completed.

(9) Determine the value of $\zeta_0^{(m)}$ and obtain the $m$-th step approximate solution, applying the graphical checking procedure stated above,

and repeat the procedure until the $m$-th step approximate solution comes near around the stable solution* of Eq.(4.5-2), or it covers the whole control interval $[0, T]$.

## 4.7 Illustrative Examples

Two simple examples are provided in this section to throw light upon the details of numerical procedure in 4.6.

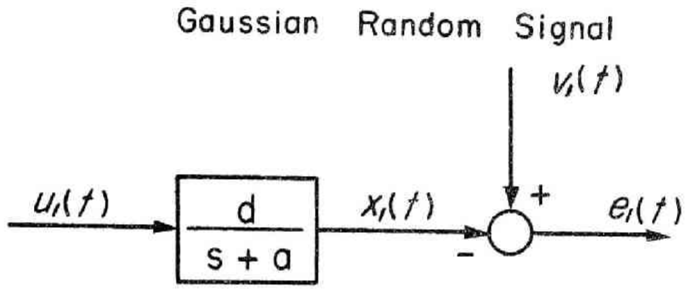


Fig. 4.3 Block diagram of the controlled system in example 1

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* If the original problem is stationary, i.e., $A, D, Q$ and $R$ are constants, then the control signal becomes a stationary function of the state as the final instant $T$ takes a large value.(26) The control coefficients ${}^{t}K_1$ and $K_2$ for the stationary state can thus be found by taking the time derivatives as zero in Eq. (4.5-1) or (4.5-2), and solving the quadratic algebraic equations with $k_i^{\infty}$ and $k_{ij}^{\infty}$ which constitute the elements of stable solutions for the stationary state.

4.7-1 Optimization of a Plant of First Order

We consider a system as shown in Fig. 4.3. The controlled system is described by

$$\dot{x}_1 = -ax_1 + du_1 , \quad x_1(0) = x_1^0 , \tag{4.7-1}$$

where $x_1 = x_1(t)$ and $u_1 = u_1(t)$ express the controlled signal and the control signal to the system respectively. As the author has already stated in 4.2, the error signal $e_1 = e_1(t)$ is related to the control signal as

$$\dot{e}_1 = -ae_1 - du_1 + \xi_1 , \tag{4.7-2}$$

where $\xi_1$ is

$$\xi_1 = \dot{v}_1 + av_1 \tag{4.7-3}$$

and $\xi_1$ can be considered as a Gaussian white noise with mean value $m_1$ and variance $\sigma_{11}^2$. The following performance index is considered here, i.e.,

$$I\,(e_1(t),\, t\,;\, u_1(t')) = \varepsilon\{\int_t^T [e_1(\rho)^2 + ru_1(\rho)^2]\, d\rho\} , \tag{4.7-4}$$

$$(t \leq t' \leq T) .$$

In Eq. (4.3-8), by letting

$$\left.\begin{array}{llll} A = -a , & D = d\ ; & m = m_1 , & \Sigma = \sigma_{11}^2 \\ Q = 1 , & R = r & & \end{array}\right\} \tag{4.7-5}$$

we have the following equations corresponding to Eq. (4.3-8) as

$$\left.\begin{array}{ll} \dot{k}_1(\tau) = -ak_1(\tau) + 2m_1 k_{11}(\tau) - \dfrac{d^2}{r}\, k_1(\tau)\, k_{11}(\tau) , & k_1(0) = 0 \\ \dot{k}_{11}(\tau) = 1 - 2ak_{11}(\tau) - \dfrac{d^2}{r}\, k_{11}(\tau)^2 , & k_{11}(0) = 0 \end{array}\right\} , \tag{4.7-6}$$

where both $k_1(\tau)$ and $k_{11}(\tau)$ express time-dependent coefficients in Eq. (4.3-5).

From Eq. (4.7-6), it follows that

$$\left.\begin{array}{ll} \dot{K}_1(\zeta) = -aK_1(\zeta) + 2m_1 K_{11}(\zeta) - \dfrac{d^2}{r}\, K_1(\zeta)\, K_{11}(\zeta) , & K_1(0) = 0 \\ \dot{K}_{11}(\zeta) = 1 - 2aK_{11}(\zeta) - \dfrac{d^2}{r}\, K_{11}(\zeta)^2 , & K_{11}(0) = 0 \end{array}\right\} . \tag{4.7-7}$$

By applying the direct transformation formulae, defined by Eqs. (4.4-3) and (4.4-5), to Eq. (4.7-7), we obtain the following equations corresponding to Eq. (4.5-4);

$$k_{\mathrm{I},n} = -\frac{a}{n} k_{\mathrm{I},n-1} + \frac{2m_1}{n} k_{\mathrm{II},n-1} - \frac{d^2}{r} \mathbb{C}^{\mathrm{I},\mathrm{II}}_{n,u} \qquad (4.7\text{-}8)_1$$

$$k_{\mathrm{II},n} = \delta_n - \frac{2a}{n} k_{\mathrm{II},n-1} - \frac{d^2}{r} \mathbb{C}^{\mathrm{II},\mathrm{II}}_{n,u} \qquad (4.7\text{-}8)_2$$

where $\delta_n$ expresses Kronecker's delta function, defined by

$$\left.\begin{aligned} \delta_n &= 1 \quad \text{for} \quad n=0 \\ &= 0 \quad \text{for} \quad n \neq 0 \end{aligned}\right\}, \qquad (4.7\text{-}9)$$

and $\mathbb{C}^{\mathrm{I},\mathrm{II}}_{n,u}$ and $\mathbb{C}^{\mathrm{II},\mathrm{II}}_{n,u}$, defined by Eq. (D-22), are

$$\left.\begin{aligned} \mathbb{C}^{\mathrm{I},\mathrm{II}}_{n,u} &= 0 \quad \text{for} \quad n = 0, 1 \\ &= \sum_{u=0}^{n-2} \frac{k_{\mathrm{I},u} k_{\mathrm{II},n-u-2}}{(u+1)(n-u-1)} \quad \text{for} \quad n \geq 2 \end{aligned}\right\} \qquad (4.7\text{-}10)$$

and

$$\left.\begin{aligned} \mathbb{C}^{\mathrm{II},\mathrm{II}}_{n,u} &= 0 \quad \text{for} \quad n = 0, 1 \\ &= \sum_{u=0}^{n-2} \frac{k_{\mathrm{II},u} k_{\mathrm{II},n-u-2}}{(u+1)(n-u-1)} \quad \text{for} \quad n \geq 2 \end{aligned}\right\}. \qquad (4.7\text{-}11)$$

The coefficients, $k_{\mathrm{I},n}$ and $k_{\mathrm{II},n}$ can be obtained by calculating the recurrence formulae given by Eqs. $(4.7\text{-}8)_1$ and $(4.7\text{-}8)_2$.

On the other hand, since, the respective inverse transformation of coefficients, $k_{\mathrm{I},n}$ and $k_{\mathrm{II},n}$ is given by

$$K_{\mathrm{I}}(\zeta) = \sum_{n=0}^{\infty} \frac{k_{\mathrm{I},n}}{n+1} \zeta^{n+1} \qquad (4.7\text{-}12)_1$$

and

$$K_{\mathrm{II}}(\zeta) = \sum_{n=0}^{\infty} \frac{k_{\mathrm{II},n}}{n+1} \zeta^{n+1} \quad , \qquad (4.7\text{-}12)_2$$

then the solution of Eq. (3.7-6) can be obtained as follows:

$$k_{1}(\tau)=\sum_{n=0}^{\infty} \frac{k_{1,n}}{n+1} \tau^{n+1} , \qquad (4.7\text{-}13)_1$$

$$k_{11}(\tau)=\sum_{n=0}^{\infty} \frac{k_{11,n}}{n+1} \tau^{n+1} . \qquad (4.7\text{-}13)_2$$

By using these results, the optimum control signal $\bar{u}_1(t)$ can finally be given by

$$\bar{u}_1(t)=\frac{d}{2r} \{ k_1 (T-t)+2k_{11} (T-t)\, e_1(t) \} , \qquad (4.7\text{-}14)$$

where $T$ is the final instant of control operation. By applying the numerical procedure stated in the previous section, we derive the approximate solutions of the first and the $m$-th steps. The first step approximate solutions, $K_1(\zeta)_1$ and $K_{11}(\zeta)_1$, are obtained by the direct application of Eq. (4.6-3) to Eqs. $(4.7\text{-}12)_1$ and $(4.7\text{-}12)_2$ as

$$\left. \begin{aligned} K_1(\zeta)_1 &= \sum_{n=0}^{N} \frac{k_{1,n}}{n+1} \zeta^{n+1} && \text{for} \quad 0 \le \zeta \le \zeta_0' \\ &= 0 && \text{for} \quad \zeta > \zeta_0' \end{aligned} \right\}, \qquad (4.7\text{-}15)_1$$

and

$$\left. \begin{aligned} K_{11}(\zeta)_1 &= \sum_{n=0}^{N} \frac{k_{11,n}}{n+1} \zeta^{n+1} && \text{for} \quad 0 \le \zeta \le \zeta_0' \\ &= 0 && \text{for} \quad \zeta > \zeta_0' \end{aligned} \right\}. \qquad (4.7\text{-}15)_2$$

By using Eq. (4.6-10), the m-th step approximate solutions, $K_1(\zeta)_m$ and $K_{11}(\zeta)_m$, can be described as

$$\left. \begin{aligned} K_1(\zeta)_m &= K_1(\zeta_0^{(m-1)})_{m-1} + \sum_{n=0}^{N^{(m-1)}} \frac{k_{1,n}^{(m-1)}}{n+1} (\zeta-\zeta_0^{(m-1)})^{n+1} \\ &\qquad \text{for} \quad \zeta_0^{(m-1)} \le \zeta \le \zeta_0^{(m)} \\ &= 0 \\ &\qquad \text{for} \quad \zeta > \zeta_0^{(m)} \end{aligned} \right\} \qquad (4.7\text{-}16)_1$$

and

$$\left. \begin{aligned} K_{11}(\zeta)_m &= K_{11}(\zeta_0^{(m-1)})_{m-1} + \sum_{n=0}^{N^{(m-1)}} \frac{k_{11,n}^{(m-1)}}{n+1} (\zeta-\zeta_0^{(m-1)})^{n+1} \\ &\qquad \text{for} \quad \zeta_0^{(m-1)} \le \zeta \le \zeta_0^{(m)} \\ &= 0 \qquad\qquad \text{for} \quad \zeta > \zeta_0^{(m)} \end{aligned} \right\}, \qquad (4.7\text{-}16)_2$$

where $k_{1,n}^{(m-1)}$ and $k_{11,n}^{(m-1)}$ express the coefficients satisfying a set of the recurrence relations as follows:

$$k_{1,n}^{(m-1)} = (-a\bar{k}_{1,m-1} + 2m_1\bar{k}_{11,m-1} - \frac{d^2}{r}\bar{k}_{1,m-1}\bar{k}_{11,m-1})\,\delta_n$$
$$- (a + \frac{d^2}{r}\bar{k}_{11,m-1})\frac{1}{n}k_{1,n-1}^{(m-1)}$$
$$+ (2m_1 - \frac{d^2}{r}\bar{k}_{1,m-1})\frac{1}{n}k_{11,n-1}^{(m-1)}$$
$$- \frac{d^2}{r}[C_{n,u}^{1,11}]^{(m-1)}, \qquad (4.7\text{-}17)_1$$

$$k_{11,n}^{(m-1)} = (1 - 2a\bar{k}_{11,m-1} - \frac{d^2}{r}\bar{k}_{11,m-1}^2)\,\delta_n$$
$$-2(a + \frac{d^2}{r}\bar{k}_{11,m-1})\frac{1}{n}k_{11,n-1}^{(m-1)} - \frac{d^2}{r}[C_{n,u}^{11,11}]^{(m-1)}, \qquad (4.7\text{-}17)_2$$

where

$$\bar{k}_{1,m-1} \equiv K_1(\zeta_0^{(m-1)})_{m-1}, \qquad (4.7\text{-}18)_1$$

$$\bar{k}_{11,m-1} \equiv K_{11}(\zeta_0^{(m-1)})_{m-1}, \qquad (4.7\text{-}18)_2$$

$$[C_{n,u}^{1,11}]^{(m-1)} = \left.\begin{cases} 0 & \text{for } n = 0, 1 \\ \displaystyle\sum_{u=0}^{n-2} \frac{k_{1,u}^{(m-1)} k_{11,n-u-2}^{(m-1)}}{(u+1)(n-u-1)} & \text{for } n \geq 2 \end{cases}\right\}, \qquad (4.7\text{-}19)_1$$

and

$$[C_{n,u}^{11,11}]^{(m-1)} = \left.\begin{cases} 0 & \text{for } n = 0, 1 \\ \displaystyle\sum_{u=0}^{n-2} \frac{k_{11,u}^{(m-1)} k_{11,n-u-2}^{(m-1)}}{(u+1)(n-u-1)} & \text{for } n \geq 2 \end{cases}\right\} \qquad (4.7\text{-}19)_2$$

As an illustrative example for the numerical procedure, we consider the case where the parameters in Eqs. $(4.7\text{-}8)_1$ and $(4.7\text{-}8)_2$ are given by

$$a = d = r = 1 \quad \text{and} \quad m_1 = 0\,. \qquad (4.7\text{-}20)$$

By choosing $N = 6$ and calculating coefficients $k_{1,n}$ and $k_{11,n}$ $(n = 1, 2, \cdots\cdots, 6)$ by using Eqs. $(4.7\text{-}8)_1$ and $(4.7\text{-}8)_2$, the first truncated series becomes

$$K_{I}(\zeta) = 0 \tag{4.7-21}_1$$

$$K_{II}(\zeta) = \zeta - \zeta^2 + 0.333333\,\zeta^3 + 0.333333\,\zeta^4 - 0.466666\,\zeta^5 + 0.155555\,\zeta^6 + 0.190476\,\zeta^7. \tag{4.7-21}_2$$

The numerical plot of the truncated series expression of $K_I(\zeta)$ and $K_{II}(\zeta)$ for the first step approximation, is shown by the curve (a) in Fig. 4.4. In Fig. 4.4, although the precise determination of location of the stationary point on the curve (a) is almost impossible from the mathematical viewpoint because of the complexity of computation, we can easily find the location only by an graphical consideration. In this case, the stationary point is seemed to be laid between $\zeta = 0.6$ and $\zeta = 0.7$. It is quite reasonable to choose the value of a constant $\zeta_0'$ less than that of the stationary point. The value of $\zeta_0'$ is,

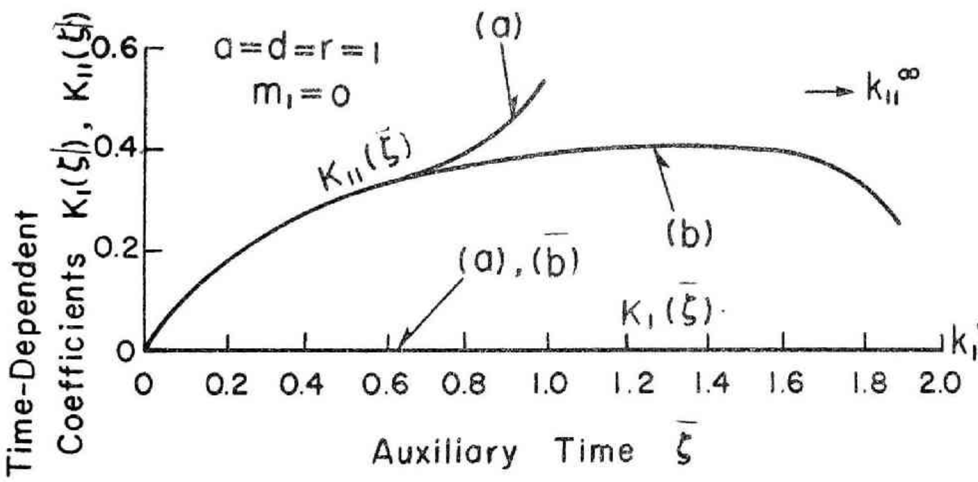



Fig. 4.4 Numerical plot of Eqs. (4.7-21) and (4.7-23)

therefore, taken as $\zeta_0' = 0.5$ . Hence, we can write the first step approximate solutions for $k_{\mathrm{I}}(\tau)$ and $k_{\mathrm{II}}(\tau)$ as

$$k_{\mathrm{I}}(\tau)_1 = 0 \qquad (0 \leq \tau \leq 0.5) \tag{4.7-22$_1$}$$

$$\begin{aligned} k_{\mathrm{II}}(\tau)_1 = {} & \tau - \tau^2 + 0.333333\,\tau^3 + 0.333333\,\tau^4 \\ & - 0.466666\,\tau^5 + 0.155555\,\tau^6 + 0.190476\,\tau^7 \\ & (0 \leq \tau \leq 0.5) \ . \end{aligned} \tag{4.7-22$_2$}$$

Calculation of Eqs. $(4.7\text{-}17)_1$ and $(4.7\text{-}17)_2$ for the case of $m = 2$ and $\zeta_0' = 0.5$ gives us the truncated series expression of $K_{\mathrm{I}}(\zeta)$ and $K_{\mathrm{II}}(\zeta)$ for the second step approximation as

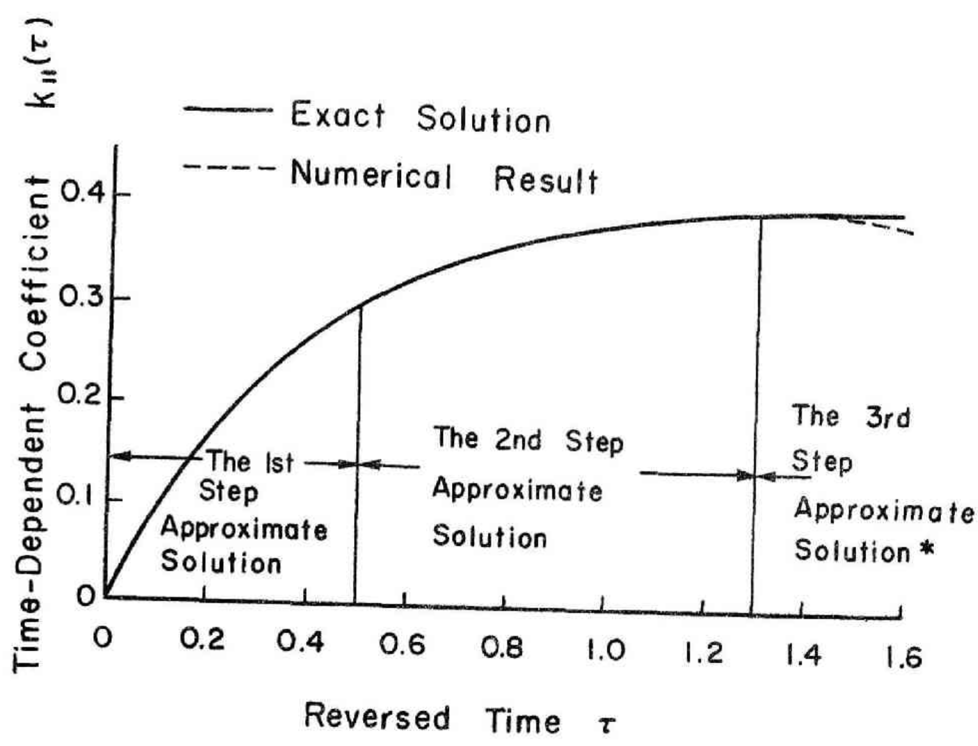
$$K_{\mathrm{I}}(\zeta) = 0 \qquad (\zeta > 0.5) \tag{4.7-23$_1$}$$

$$\begin{aligned} K_{\mathrm{II}}(\zeta) = {} & 0.301439 + 0.306275(\zeta - 0.5) - 0.398598(\zeta - 0.5)^2 \\ & + 0.314565(\zeta - 0.5)^3 - 0.143653(\zeta - 0.5)^4 \\ & + 0.004469(\zeta - 0.5)^5 + 0.054521(\zeta - 0.5)^6 \\ & - 0.051159(\zeta - 0.5)^7 + 0.024213(\zeta - 0.5)^8 \\ & - 0.001296(\zeta - 0.5)^9 - 0.008525(\zeta - 0.5)^{10} \ , \\ & (\zeta > 0.5) . \end{aligned} \tag{4.7-23$_2$}$$

Considering the plot of Eq. $(4.7\text{-}23)_2$ as shown by the curve (b) in Fig. 4.4, choice of the value (of a constant) $\zeta_0''$ can intuitively be carried out as $\zeta_0'' = 1.3$ . Then the second approximate solutions for $k_{\mathrm{I}}(\tau)$ and $k_{\mathrm{II}}(\tau)$ are expressed by

$$k_{\mathrm{I}}(\tau)_2 = 0 \qquad (0.5 \leq \tau \leq 1.3) \tag{4.7-24$_1$}$$

$$\begin{aligned} k_{\mathrm{II}}(\tau)_2 = {} & 0.30149 + 0.306275(\tau - 0.5) - 0.398598(\tau - 0.5)^2 \\ & + 0.314565(\tau - 0.5)^3 - 0.143653(\tau - 0.5)^4 \\ & + 0.004469(\tau - 0.5)^5 + 0.054521(\tau - 0.5)^6 \\ & - 0.051159(\tau - 0.5)^7 + 0.024213(\tau - 0.5)^8 \\ & - 0.001296(\tau - 0.5)^9 - 0.008525(\tau - 0.5)^{10} \ , \\ & (0.5 \leq \tau \leq 1.3) \end{aligned} \tag{4.7-24$_2$}$$




*(to be continued)

Fig. 4.5 Comparison of the numerical result with the exact solution in example 1

By repeating a similar procedure, successive calculations can numerically be carried out.

On the other hand, since the analytical solution of the second differential equation shown by Eq. (4.7-7) can precisely be obtained, then it is very interesting to compare the exact solution with the approximate one. In Fig. 4.5, the solid curve gives the exact solution. Comparison of the numerical result with the exact solution is given in Table 4.1 with the good agreement. Fig. 4.6 shows the block diagram of the optimum control system.

Table 4.1 Numerical Comparison

| time $\tau$ \ method | Exact Solution | Present Method |
|---|---|---|
| 0.0 | 0.00000 | 0.00000 |
| 0.1 | 0.09006 | 0.09029 |
| 0.2 | 0.16306 | 0.16341 |
| 0.3 | 0.22069 | 0.22109 |
| 0.4 | 0.26289 | 0.26333 |
| 0.5 | 0.30096 | 0.30144 |
| 0.6 | 0.32801 | 0.32838 |
| 0.7 | 0.34875 | 0.34904 |
| 0.8 | 0.36460 | 0.36481 |
| 0.9 | 0.37666 | 0.37684 |
| 1.0 | 0.38612 | 0.38624 |
| 1.1 | 0.39277 | 0.39283 |
| 1.2 | 0.39729 | 0.39698 |
| 1.3 | 0.40199 | 0.40157 |

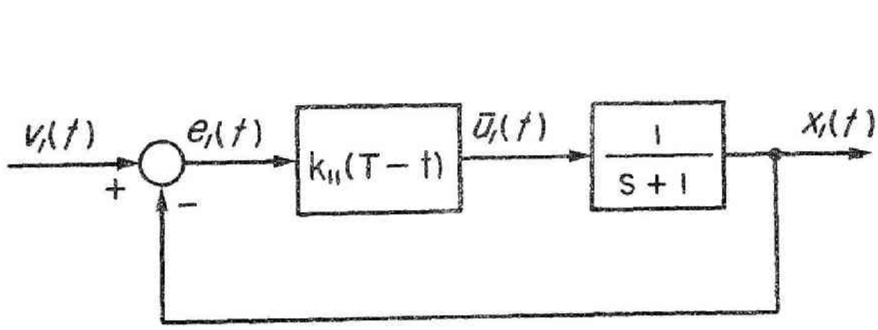



Fig. 4.6 Configuration of the optimum control system

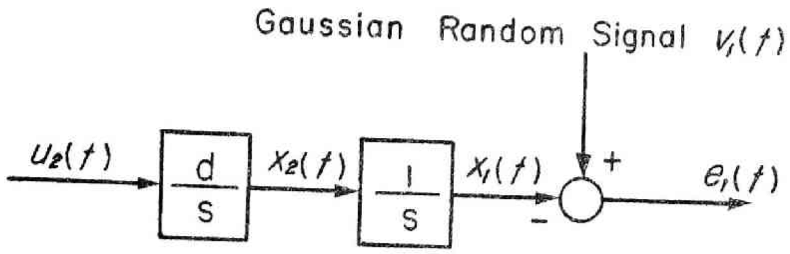



Fig. 4.7 Block diagram of the controlled system in example 2

4.7-2 Optimization of a Double-Integrator Plant

The system dynamics as shown in Fig. 4.7 is described by

$$\left.\begin{array}{ll} \dot{x}_1 = x_2 , & x_1(0) = x_1^0 \\ \dot{x}_2 = du_2 , & x_2(0) = x_2^0 \end{array}\right\} . \qquad (4.7\text{-}25)$$

The error signal $e_1 = e_1(t)$ is related with both the control signal $u_2 = u_2(t)$ and the desired random signal $v_1 = v_1(t)$ as

$$\left.\begin{array}{l} \dot{e}_1 = e_2 \\ \dot{e}_2 = -du_2 + \xi_2 \end{array}\right\} , \qquad (4.7\text{-}26)$$

where $\xi_2$ is

$$\xi_2 = \ddot{v}_1 \qquad (4.7\text{-}27)$$

and $\xi_2 = \xi_2(t)$ can be considered as a Gaussian white noise process with time-dependent mean value $m_2(t)$ and time-dependent variance $\sigma_{22}^2(t)$. As the performance functional to be minimized in this example, the following form is considered;

$$I\,(e_1(t), e_2(t),\ t\ ;\ u_2(t')) = \varepsilon\{ \int_t^T [e_1(\rho)^2 + r\, u_2(\rho)^2]\, d\rho \}\ , \qquad (4.7\text{-}28)$$
$$(t \leq t' \leq T)\ .$$

Letting

$$\left.\begin{array}{llll} A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix} , & D = \begin{Vmatrix} 0 & 0 \\ 0 & d \end{Vmatrix} \\ Q = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} , & R = \begin{Vmatrix} 0 & 0 \\ 0 & r \end{Vmatrix} \\ m = \begin{bmatrix} 0 \\ m_2(t) \end{bmatrix} , & \Sigma = \begin{Vmatrix} 0 & 0 \\ 0 & \sigma_{22}^2(t) \end{Vmatrix} \end{array}\right\} , \qquad (4.7\text{-}29)$$

Eq. (4.3-8) becomes

$$\left.\begin{array}{ll} \dot{k}_1(\tau) = 2m_2(\tau)\, k_{12}(\tau) - \frac{d^2}{r}\, k_2(\tau)\, k_{12}(\tau)\ , & k_1(0) = 0 \\ \dot{k}_2(\tau) = k_1(\tau) + 2m_2(\tau)\, k_{22}(\tau) - \frac{d^2}{r} k_2(\tau)\, k_{22}(\tau)\ , & k_2(0) = 0 \\ \dot{k}_{12}(\tau) = k_{11}(\tau) - \frac{d^2}{r} k_{12}(\tau)\, k_{22}(\tau)\ , & k_{12}(0) = 0 \\ \dot{k}_{11}(\tau) = 1 - \frac{d^2}{r}\, k_{12}(\tau)^2\ , & k_{11}(0) = 0 \\ \dot{k}_{22}(\tau) = 2\,k_{12}(\tau) - \frac{d^2}{r}\, k_{22}(\tau)^2\ , & k_{22}(0) = 0 \end{array}\right\} , \qquad (4.7\text{-}30)$$

and the corresponding optimum control signal $\bar{u}_2(t)$ is represented by

$$\bar{u}_2(t) = \frac{d}{2r}\{ k_2(T-t) + 2k_{21}\ (T-t)\, e_1(t) + 2k_{22}\ (T-t)\, e_2(t) \}\ , \qquad (4.7\text{-}31)$$

where $k_{21}(T-t)$ is equivalent to $k_{12}(T-t)$.

Application of the direct transformation to Eq. (4.7-30) gives us a set of recurrence relation with respect to coefficients, $k_{1,n}$, $k_{2,n}$ $k_{12,n}$ , $k_{11,n}$ and $k_{22,n}$, as

$$\left.\begin{array}{l} k_{1,n} = 2M_{n;u}^{2;12} - \frac{d^2}{r}\ C_{n,u}^{2,12} \\ k_{2,n} = \frac{1}{n}\, k_{1,n-1} + 2M_{n,u}^{2,22} - \frac{d^2}{r}\ C_{n,u}^{2,22} \\ k_{12,n} = \frac{1}{n} k_{11,n-1} - \frac{d^2}{r}\, C_{n,u}^{2,22} \\ k_{11,n} = \delta_n - \frac{d^2}{r}\, C_{n,u}^{12,12} \\ k_{22,n} = \frac{2}{n} k_{12,n-1} - \frac{d^2}{r}\, C_{n,u}^{22,22} \end{array}\right\} , \qquad (4.7\text{-}32)$$

where $M^{2,p}_{n,u}$ and $C^{p,3}_{n,u}$ denote

$$\left.\begin{aligned} M^{2,p}_{n,u} &= 0 && \text{for} \quad n=0 \\ &= \sum_{u=0}^{n-1} \frac{m_{2,u}k_{p,\,n-u-1}}{n-u} && \text{for} \quad n \geq 1 \end{aligned}\right\} \quad (4.7\text{-}33)$$

and

$$\left.\begin{aligned} C^{p,q}_{n,u} &= 0 && \text{for} \quad n=0,1 \\ &= \sum_{u=0}^{n-2} \frac{k_{p,u}\,k_{q,\,n-u-2}}{(u+1)(n-u-1)} && \text{for} \quad n \geq 2 \end{aligned}\right\}, (4.7\text{-}34)$$

respectively. Performing the numerical procedure with respect to a particular set of parameters, i.e., $d=r=1$ and $m_2(t)=0$, the approximate solution of the first and second steps can be derived by

$$\left.\begin{aligned} k_1(\tau)_1 &= 0 \\ k_2(\tau)_1 &= 0 \\ k_{12}(\tau)_1 &= 0.50000\tau^2 - 0.03611\tau^6 + 0.00291\tau^8 \\ k_{11}(\tau)_1 &= \tau - 0.05000\tau^5 + 0.00040\tau^9 \\ k_{22}(\tau)_1 &= 0.33333\tau^3 - 0.02619\tau^7 + 0.00212\tau^{11} \\ & \qquad\qquad\qquad (0 \leq \tau \leq 1.0) \end{aligned}\right\}, (4.7\text{-}35)$$

and

$$\begin{aligned} k_1(\tau)_2 &= 0 \\ k_2(\tau)_2 &= 0 \\ k_{12}(\tau)_2 &= 0.4625 + 0.80662(\tau-1.0) + 0.07638(\tau-1.0)^2 \\ &\quad - 0.43998(\tau-1.0)^3 - 0.14415(\tau-1.0)^4 \\ &\quad + 0.13995(\tau-1.0)^5 + 0.12841(\tau-1.0)^6 \\ &\quad - 0.01246(\tau-1.0)^7 \\ k_{11}(\tau)_2 &= 0.9500 + 0.78609(\tau-1.0) - 0.37306(\tau-1.0)^2 \\ &\quad - 0.07375(\tau-1.0)^3 - 0.43580(\tau-1.0)^4 \\ &\quad - 0.03407(\tau-1.0)^5 + 0.12008(\tau-1.0)^6 \\ &\quad + 0.10187(\tau-1.0)^7 \end{aligned}$$

$$
\left.\begin{aligned}
k_{22}(\tau)_2 = {} & 0.3100 + 0.82890(\tau - 1.0) + 0.54967(\tau - 1.0)^2 \\
& - 0.07375(\tau - 1.0)^3 - 0.43580(\tau - 1.0)^4 \\
& - 0.03407(\tau - 1.0)^5 + 0.12008(\tau - 1.0)^6 \\
& + 0.01087(\tau - 1.0)^7
\end{aligned}\right\} ,
$$

$$(1.0 \leq \tau \leq 1.8). \tag{4.7-36}$$

Numerical plots of Eqs. (4.7-35) and (4.7-36) are shown in Fig. 4.8. Table 4.2 shows the comparison of the numerical result with the computed result by the Runge-Kutta's method of fourth order. The table illustrates us the effectiveness of the proposed numerical procedure.

The block diagram of the optimum control system is also shown in

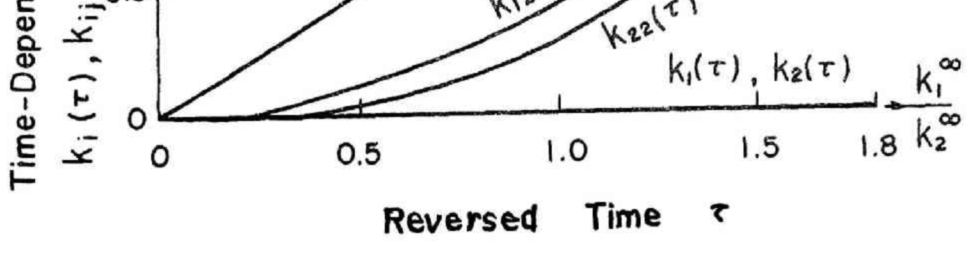



Fig. 4.8. Numerical result of example 2

## 4.8 Further Discussions on the Method of Approximate Calculation

In general, the procedure described in 4.6 provides us two difficult problems. The first is on the determination of a certain number $N^{(m+1)}$, which determines the truncation of infinite series. The second is on the estimation of a constant $\zeta_0^{(m)}$ with respect to a certain number $N^{(m+1)}$. Generally speaking, since it is almost impossible to deal with these problems by a purely mathematical way, then a graphical checking procedure has been proposed.

The author shows, in this section, that the mathematical presentation of the present method introduced to connect the $m$-th step approximate solution to the $m+1$-th step one.

Let us express an arbitrary component of Eq. (4.6-11) as

Table 4.2 Numerical Comparison

| time $\tau$ | $k_{12}(\tau)$ Present Method | $k_{12}(\tau)$ Runge-Kutta's Method | $k_{11}(\tau)$ Present Method | $k_{11}(\tau)$ Runge-Kutta's Method | $k_{22}(\tau)$ Present Method | $k_{22}(\tau)$ Runge-Kutta's Method |
|---|---|---|---|---|---|---|
| 0.1 | 0.0099 | 0.0099 | 0.0999 | 0.0999 | 0.0003 | 0.0003 |
| 0.2 | 0.0399 | 0.0399 | 0.1999 | 0.1999 | 0.0026 | 0.0026 |
| 0.3 | 0.0899 | 0.0899 | 0.2998 | 0.2998 | 0.0089 | 0.0089 |
| 0.4 | 0.1597 | 0.1597 | 0.3994 | 0.3994 | 0.0212 | 0.0212 |
| 0.5 | 0.2488 | 0.2488 | 0.4984 | 0.4984 | 0.0414 | 0.0414 |
| 0.6 | 0.3566 | 0.3566 | 0.5961 | 0.5961 | 0.0712 | 0.0712 |
| 0.7 | 0.4816 | 0.4816 | 0.6917 | 0.6917 | 0.1122 | 0.1122 |
| 0.8 | 0.6216 | 0.6216 | 0.7841 | 0.7841 | 0.1653 | 0.1653 |
| 0.9 | 0.7736 | 0.7735 | 0.8720 | 0.8719 | 0.2311 | 0.2311 |
| 1.0 | 0.9336 | 0.9331 | 0.9540 | 0.9537 | 0.3092 | 0.3090 |
| 1.1 | 1.0869 | 1.0955 | 1.0246 | 1.0279 | 0.3980 | 0.3980 |
| 1.2 | 1.2464 | 1.2552 | 1.0905 | 1.0933 | 0.4947 | 0.4956 |
| 1.3 | 1.3980 | 1.4066 | 1.1467 | 1.1489 | 0.5970 | 0.5988 |
| 1.4 | 1.5368 | 1.5448 | 1.1927 | 1.1943 | 0.7010 | 0.7039 |
| 1.5 | 1.6592 | 1.6658 | 1.2284 | 1.2297 | 0.8025 | 0.8074 |
| 1.6 | 1.7640 | 1.7671 | 1.2557 | 1.2559 | 0.8971 | 0.9057 |

$$\dot{K}_p(\zeta_{m-1})=f(K_p(\zeta_{m-1}),\zeta_{m-1}), \quad K_p(0)=\bar{k}_{p,m-1}\equiv K_p(\zeta_0^{(m-1)})_{m-1} \tag{4.8-1}$$

where $f$ is a continuously differentiable function defined for $(K_p, \zeta_{m-1})$ in a domain $-\infty\leq K_p\leq\infty$ and $0\leq \zeta_{m-1} \leq T$, and satisfies a Lipschitz condition in the defined domain.(13) The solution of Eq. (4.8-1) can be expressed by a Taylor series as

$$K_p(\zeta_{m-1})=\sum_{n=0}^{N^{(m-1)}+1}\frac{K_p^{(n)}(0)}{n!}(\zeta_{m-1})^n+\frac{(\zeta_{m-1})^{N^{(m-1)}+2}}{(N^{(m-1)}+2)!}K_p^{(N^{(m-1)}+2)}(\xi), \tag{4.8-2}$$

$$(0\leq\xi\leq\zeta_{m-1}).$$

The first term in the right hand side of Eq. (4.8-2) is written as

$$\begin{aligned}\sum_{n=0}^{N^{(m-1)}+1}\frac{K_p^{(n)}(0)}{n!}(\zeta_{m-1})^n&=K_p(0)+\sum_{n=0}^{N^{(m-1)}+1}\frac{K_p^{(n)}(0)}{n!}(\zeta_{m-1})^n\\&=\bar{k}_{p,m-1}+\sum_{n=0}^{N^{(m-1)}}\frac{K_p^{(n+1)}(0)}{(n+1)\,n!}(\zeta_{m-1})^{n+1}.\end{aligned} \tag{4.8-3}$$

Since the relation between the direct transform, $K_{p,n}^{(m-1)}$, of the solution $K_p(\zeta_{m-1})$ and the coefficient in a Taylor expansion is

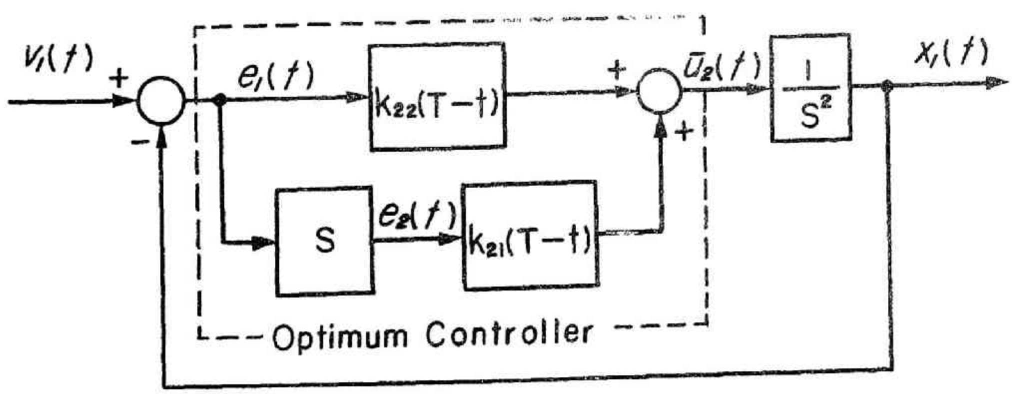



Fig. 4.9 Configuration of the optimum control system

$$k_{p,n}^{(m-1)}=\frac{K_p^{(n+1)}(0)}{n!} \tag{4.8-4}$$

then, Eq. (4.8-3) becomes

$$\sum_{n=0}^{N^{(m-1)}+1}\frac{K_p^{(n)}(0)}{n!}(\zeta_{m-1})^n=K_p(\zeta_{m-1})_m . \tag{4.8-5}$$

By using Eqs. (4.8-2) and (4.8-5), the difference between the $m$-th step approximate solution, $K_p(\zeta_{m-1})_m$, and the exact solution of $m$ times translated equation, $K_p(\zeta_{m-1})$, therefore, is expressed as

$$\begin{aligned}|K_p(\zeta_{m-1})_m-K_p(\zeta_{m-1})| &\leq \frac{(\zeta_{m-1})^{N^{(m-1)}+2}}{(N^{(m-1)}+2)!}\,|K_p^{(N^{(m-1)}+2)}(\xi)\,| \\ &\leq \frac{(\zeta_{m-1})^{N^{(m-1)}+2}}{(N^{(m-1)}+2)!}\ \max\{\,|K_p^{(N^{(m-1)}+2)}(\xi)\,|\,\}, \\ &\qquad (0\leq\xi\leq\zeta_{m-1}) .\end{aligned} \tag{4.8-6}$$

On the other hand, if we introduce the $m+1$-th step approximate solution at $\zeta_{m-1}=\zeta_0^{(m)}$, the translated equation and its initial condition respectively are expressed as

$$\dot{K}_p(\zeta_m)=f_t(K_p(\zeta_m),\zeta_m) \tag{4.8-7}$$

and

$$K_p(0)=K_p(\zeta_0^{(m)})_m=K_p(\zeta_0^{(m)})-\delta , \tag{4.8-8}$$

where $\zeta_m=\zeta-\zeta_0^{(m)}$.

By a similar procedure, we obtain the relation concerning the $m+1$ -th step approximate solution $K_p(\zeta_m)_{m+1}$ as

$$K_p(\zeta_m)=K_p(\zeta_m)_{m+1}+\frac{\zeta_m^{(N^{(m)}+2)}}{(N^{(m)}+2)!}\,K_p^{(N^{(m)}+2)}(\eta)$$
$$(0\leq\eta\leq\zeta_m), \tag{4.8-9}$$

where

$$\begin{aligned}K_p(\zeta_m)_{m+1}&=\sum_{n=0}^{N^{(m)}}\frac{k_{p,n}^{(m)}}{n+1}(\zeta_m)^{n+1}+K_p(\zeta_0^{(m)})_m \\ &=\sum_{n=0}^{N^{(m)}+1}\frac{K_p^{(n)}(0)}{n!}(\zeta_m)^n\end{aligned} \tag{4.8-10}$$

Substituting Eq.(4.8-5) into Eq.(4.8-2), differentiation of Eq. (4.8-2) with respect to $\zeta_{m-1}$ gives us

$$\dot{K}_p(\zeta_{m-1})_m = \dot{K}_p(\zeta_{m-1}) - \frac{(\zeta_{m-1})^{N^{(m-1)}+1}}{(N^{(m-1)}+1)!} K_p^{(N^{(m-1)}+2)}(\xi) \quad . \tag{4.8-11}$$

By differentiating the both side of Eq. (4.8-9) with respect to $\zeta_m$ we obtain

$$\dot{K}_p(\zeta_m)_{m+1} = \dot{K}_p(\zeta_m) - \frac{(\zeta_m)^{N^{(m)}+1}}{(N^{(m)}+1)!} K_p^{(N^{(m)}+2)}(\eta) \quad . \tag{4.8-12}$$

From Eq. (4.8-11) and (4.8-12), the following relation is derived

$$\begin{aligned} |\dot{K}_p(\zeta_{m-1})_m - \dot{K}_p(\zeta_m)_{m+1}| \leq |\dot{K}_p(\zeta_{m-1}) - \dot{K}_p(\zeta_m)| \\ + \frac{(\zeta_{m-1})^{N^{(m-1)}+1}}{(N^{(m-1)}+1)!} |K_p^{(N^{(m-1)}+2)}(\xi)| \\ + \frac{(\zeta_m)^{N^{(m)}+1}}{(N^{(m)}+1)!} |K_p^{(N^{(m)}+2)}(\eta)| \quad . \end{aligned} \tag{4.8-13}$$

In Eq. (4.8-13), taking the limit of tending $\zeta_m \to +0$ and $\zeta_{m-1} \to \zeta_0^{(m)} - 0$ simultaneously, we derive the relation as follows:

$$\begin{aligned} \lim_{\substack{\zeta_m \to +0 \\ \zeta_{m-1} \to \zeta_0^{(m)} - 0}} |\dot{K}_p(\zeta_{m-1})_m - \dot{K}_p(\zeta_m)_{m+1}| \\ \leq \lim_{\substack{\zeta_m \to +0 \\ \zeta_{m-1} \to \zeta_0^{(m)} - 0}} |\dot{K}_p(\zeta_{m-1}) - \dot{K}_p(\zeta_m)| \\ + \frac{(\zeta_0^{(m)})^{N^{(m-1)}+1}}{(N^{(m-1)}+1)!} |K_p^{(N^{(m-1)}+2)}(\xi)| \quad . \end{aligned} \tag{4.8-14}$$

With the help of Eqs. (4.8-1) and (4.8-7), the first term in the right hand side of Eq. (4.8-14) becomes

$$\lim_{\substack{\zeta_m \to +0 \\ \zeta_{m-1} \to \zeta_0^{(m)} - 0}} |\dot{K}_p(\zeta_{m-1}) - \dot{K}_p(\zeta_m)| = |f(K_p(\zeta_0^{(m)}), \zeta_0^{(m)}) - f_t(K_p(0), 0)| \quad . \tag{4.8-15}$$

By considering the definition of a translated equation, we get

$$\begin{aligned} f_t(K_p(0), 0) &= f(K_p(\zeta_0^{(m)})_m, \zeta_0^{(m)}) \\ &= f(K_p(\zeta_0^{(m)}) - \delta, \zeta_0^{(m)}) \quad . \end{aligned} \tag{4.8-16}$$

Then, substituting Eq. (4.8-16) into Eq. (4.8-15) and using the Lipschitz condition, we have a relation

$$\lim_{\substack{\zeta_m \to +0 \\ \zeta_{m-1} \to \zeta_0^{(m)} - 0}} |\dot{K}_p(\zeta_{m-1}) - \dot{K}_p(\zeta_m)| \leq K|\delta|, \tag{4.8-17}$$

where $K$ is a Lipschitz constant. The substitution of Eq. (4.8-17) into Eq. (4.8-14) gives us a relation as follows:

$$\lim_{\substack{\zeta_m \to +0 \\ \zeta_{m-1} \to \zeta_0^{(m)} - 0}} |\dot{K}_p(\zeta_{m-1})_m - \dot{K}_p(\zeta_m)_{m+1}| \leq K|\delta| + \frac{(\zeta_0^{(m)})^{N^{(m-1)}+1}}{(N^{(m-1)}+1)!} \max\{|K_p^{(N^{(m-1)}+2)}(\xi)|\},$$
$$(0 \leq \xi \leq \zeta_{m-1}). \tag{4.8-18}$$

Since the right hand side of Eq. (4.8-18) is non-negative and furthermore there is a relation as

$$|\delta| \leq K|\delta| + \frac{(\zeta_0^{(m)})^{N^{(m-1)}+1}}{(N^{(m-1)}+1)!} \max\{|K_p^{(N^{(m-1)}+2)}(\xi)|\}, \tag{4.8-19}$$

where

$$|\delta| \leq \frac{(\zeta_0^{(m)})^{N^{(m-1)}+2}}{(N^{(m-1)}+2)!} \max\{|K_p^{(N^{(m-1)}+2)}(\xi)|\}, \tag{4.8-20}$$

then, it turns out that the connecting criterion proposed in 4.6 is equivalent to improve the accuracy of the $m$-th step approximate solution $K_p(\zeta_{m-1})_m$. This is the mathematical interpretation of a graphical checking procedure presented in 4.6. Fig. 4.10 illustrates the details of procedure determining the value of $\zeta_0'$ in the earlier example, 4.7-1.

It must also be added to notice that the present method of approximate calculation is conceptually equivalent to the concept of sub-interval optimization technique described in 1.4-1. In this case, however, the pseudo-optimization in the $m$-th sub-interval $[\zeta_0^{(m-1)}, \zeta_0^{(m)}]$ is carried out by using the $m$-th step approximate solution.

## 4.9 Concluding Remarks

In this chapter, an analytical method for synthesizing an optimal control system with Gaussian random inputs is presented. By extending the method of Taylor-Cauchy transform, a near-optimal approach is described to solve a set of non-linear differential equations which yields the optimal control characteristics of the designed system. Several illustrative examples are also presented in detail to emphasize the effectiveness and validity of the proposed method.

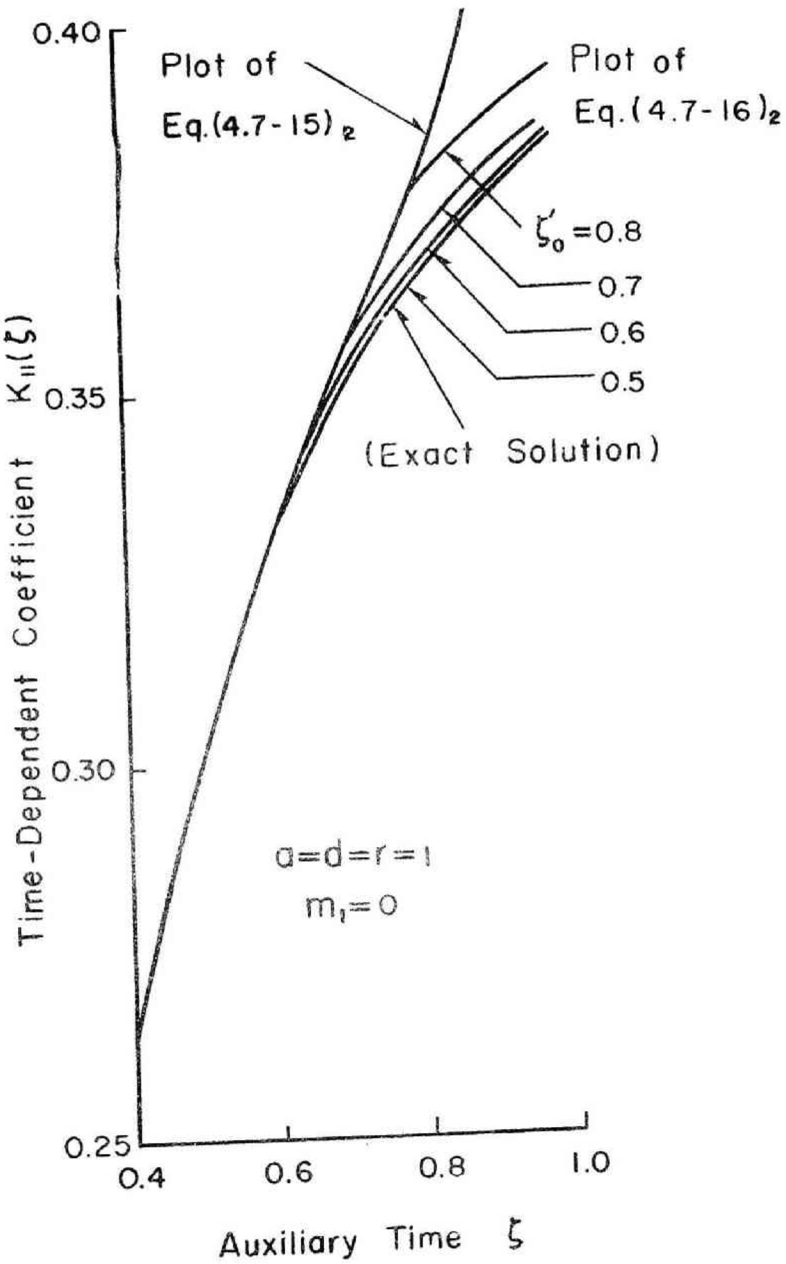



Fig. 4.10 Numerical result of $K_{11}(\zeta)$ affected by the determination of a constant $\zeta_0'$

## CHAPTER 5 CONTROL CHARACTERISTICS OF AN OPTIMUM FINAL-VALUE CONTROL SYSTEM UNDER RANDOM ENVIRONMENTS

### 5.1 Introductory Remarks

Recently there has been a growing interest in a class of control systems which are characterized by the requirement that given variables need to be controlled accurately only at a pre-assigned instant of time. Such control systems have been called the final-value or the terminal control systems. In fact, many problems in automatic controls may be reduced to this class of problems. Both the landing control problem of an aircraft and the start-up problem of a chemical reaction plant are typical examples.

Basic studies on these final-value control systems have been carried out by R.C. Booton, Jr.,[6][7][8] C.W. Steeg & M.V. Mathews,[33] A. Rosenbloom[45] and others.[37][38] Extensive studies on the design of final-value control systems are presented by L.S. Kirillova[29] or S. Katz.[28] Although many papers concerning final-value control systems have been reported, since the most of them are based upon the sophisticated configuration by R.C. Booton, Jr.,[6] then there remain many fundamental problems on the optimum structure, or optimum control characteristics of final-value control systems, especially on the optimum structure under random environments.

The author's principal object in this chapter is to explore, in detail, the characteristics of control action which a final-value control system subjected to random environments must have. The design technique presented here is the application of an optimization theory based on the concept of Dynamic Programming. Present considerations are, therefore, limited to the following two very concrete and particular stochastic control situations; (1) the time-invariant linear second order controlled syetm with an additive random disturbance, and (2) the randomly time-variant second order linear controlled system are considered under the performance index defined in the sense of final-value control with control energy constraint.

## 5.2 Mathematical Statement of Stochastic Final-Value Control Problems

In this section, we describe the mathematical background of the stochastic optimum control theory by somewhat the similar way of S. Katz.[28] Stochastic optimization problems have recently been developed based on the theory of Markov process,[15] and more rigorous discussions are found in W.M. Wonham's[51] or other papers.[20]

We assume that the dynamical behavior of a controlled system is adequately represented by a set of known ordinary differential equations which may be put into the form;

$$\left.\begin{aligned}\dot{x}_i &= x_{i+1} \qquad (i=1,2,3\cdots\cdots,n-1)\\ \dot{x}_n &= f(x_1,x_2,\cdots\cdots,x_n;u)+g(x_1,x_2,\cdots\cdots,x_n;u)\xi\end{aligned}\right\}, \quad (0\le t\le T), \tag{5.2-1}$$

where $x=(x_1, x_2, \cdots\cdots, x_n)$ is a state vector of the controlled system, $x(0)=c=(c_1, c_2, \cdots\cdots\cdots, c_n)$ is an arbitrary given initial state vector. In Eq. (5.2-1), $T$ is an arbitrary pre-assigned instant of time named the final instant of control operation or simply the final time.

For the final-value criterion functional which is to be minimized, we choose

$$R_f=\varepsilon\{\sum_{i=1}^{n-1}\lambda_i[x_{id}-x_i(T)]^2+\mu\int_0^T u(\rho)^2 d\rho\}, \tag{5.2-2)*}$$

* Strictly speaking, the functional $R_f$ is referred to as a final-value criterion with control energy constraint.

where $x_{id}$ is the desired value of a state variable $x_i$ at the final time $T$ and both $\lambda_i$ and $\mu$ are respectively assumed to be constants which are called performance weights. In Eq. (5.2-2), the symbol $\varepsilon$ expresses the ensemble average of what appears to its right under the condition of observing the state variables of the system at present instant. Our problem is to obtain the optimum control variable $\bar{u}$ which minimizes the functional (5.2-2) under the controlled system dynamic equations (5.2-1).

We consider the arbitrary time $t = t$ $(0 \leq t \leq T)$. The performance criterion functional to be minimized at the time interval left before the final time $T$, namely, at the time interval $[t, T]$ is

$$\begin{aligned} R_{f,t} &= \varepsilon\{\sum_{i=1}^{n-1} \lambda_i x_i(T)^2 + \mu \int_t^T u(\rho)^2 d\rho\} \\ &= \varepsilon\{\int_t^T [\sum_{i=1}^{n-1} 2\lambda_i x_i(\rho) x_{i+1}(\rho) + \mu u(\rho)^2] d\rho\} + \sum_{i=1}^{n-1} \lambda_i x_i(t)^2, \end{aligned} \tag{5.2-3}$$

where we assume, for convenience, without any loss of generality that the desired value of the state variable $x_i$ at $T$, namely, $x_{id}$, is zero.

Instead of minimizing Eq. (5.2-3), it is sufficient for us to consider the minimization of the following functional $I$,

$$I = \varepsilon\{\int_t^T (\sum_{i=1}^{n-1} 2\lambda_i x_i x_{i+1} + \mu u^2) d\rho\} \tag{5.2-4}$$

because both the functionals $R_{f,t}$ and $I$ attain their minima simultaneously.

To make the optimization problem more concrete, we resort to the well-known imbedding procedure of Dynamic Programming.(3) Letting

$$\phi(x; t) = \min_u \varepsilon\{\int_t^T (\sum_{i=1}^{n-1} 2\lambda_i x_i x_{i+1} + \mu u^2) d\rho\}, \tag{5.2-5}$$

and then applying the procedure presented in Appendix C, the required relation becomes

$$\frac{\partial\phi}{\partial\tau}=\min_{u}\{\sum_{i=1}^{n-1}2\lambda_i x_i x_{i+1}+\mu u^2$$

$$+\sum_{i=1}^{n-1}x_{i+1}\frac{\partial\phi}{\partial x_i}+(f+gm)\frac{\partial\phi}{\partial x_n}+\frac{\sigma^2}{2}g^2\frac{\partial^2\phi}{\partial x_n^2}\}\,. \tag{5.2-6}$$

From Eq. (5.2-5), the boundary condition for $\phi$ is apparently;

$$\phi(x;\tau)=0 \qquad (\tau=0)\,. \tag{5.2-7}$$

In Eq. (5.2-6), the expression min means the least value of what appears to its right which can be attained by variation of $u$.

## 5.3 Particular Final-Value Control Problems

In order to explore the characteristics of final-value control systems, two particular control problems are considered here, the one is concerned with the controlled system subjected to a random disturbance and the other is concerned with the controlled system containing randomly time-varying parameters.

### 5.3-1 Controlled System Subjected to a Random Disturbance

For the controlled system, we consider a second order linear system with a random disturbance $\xi(t)$ as shown in Fig. 5.1.

The dynamical characteristics of a controlled system may be described by

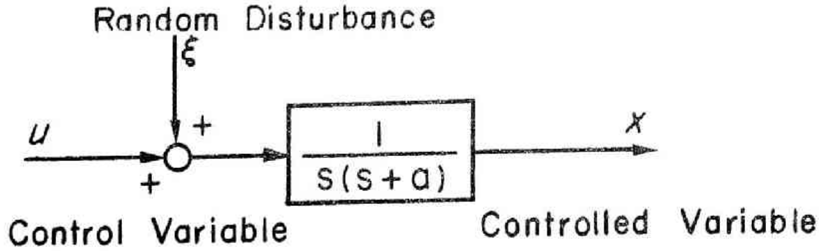



Fig. 5.1 Controlled system with a random disturbance

$$\begin{aligned} \dot{x}_1 &= x_2 \ , & x_1(0) &= c_1 \\ \dot{x}_2 &= -a x_2 + u + \xi \ , & x_2(0) &= c_2 \end{aligned} \Bigg\} \ , \tag{5.3-1}$$

where

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x} \tag{5.3-2}$$

and $a$ is a given constant. In Eqs. (5.3-1) the symbol "." is used to represent the first derivative with respect to time variable t.

As the performance criterion functional to be minimized in the sense of the final-value control, we consider

$$R_f = \varepsilon\{ \lambda x_1(T)^2 + \mu \int_0^T u(\rho)^2 \, d\rho \} \ , \tag{5.3-3}$$

where $\lambda$ and $\mu$ are respectively positive constants, with the relation;

$$\lambda + \mu = 1 \ . \tag{5.3-4}$$

We assume that the disturbance $\xi(t)$ is a stationary white Gaussian process with the mean $m$ and the variance $\sigma^2$.

Then, according to the discussion in the previous section, we can obtain the functional equation;

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \tau} &= \min_u \{ 2\lambda x_1 x_2 + \mu u^2 + x_2 \frac{\partial \phi}{\partial x_1} \\ &\qquad + (u + m - a x_2) \frac{\partial \phi}{\partial x_2} + \frac{\sigma^2}{2} \frac{\partial^2 \phi}{\partial x_2^2} \} \\ \phi &= 0 \qquad (\tau = 0) \end{aligned}\right\} , \tag{5.3-5}$$

We can thus obtain the optimum control variable $\bar{u}$ by differentiating the right hand side of Eq. (5.3-5) and putting it equal to zero, i.e.,

$$\bar{u} = -\frac{1}{2\mu} \frac{\partial \phi}{\partial x_2} \ . \tag{5.3-6}$$

Substituting Eq. (5.3-6) for the value of $\bar{u}$ as the minimum in Eq. (5.3-5), we have

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \tau} &= 2\lambda x_1 x_2 + x_2 \frac{\partial \phi}{\partial x_1} + (m - a x_2) \frac{\partial \phi}{\partial x_2} \\ &\qquad - \frac{1}{4\mu} \left(\frac{\partial \phi}{\partial x_2}\right)^2 + \frac{\sigma^2}{2} \frac{\partial^2 \phi}{\partial x_2^2} \\ \phi &= 0 \qquad (\tau = 0) \end{aligned}\right\} , \tag{5.3-7}$$

The problem is, therefore, reduced to solve the partial differential equation given by Eq. (5.3-7) and to obtain the optimal control variable $\bar{u}$ by Eq. (5.3-6).

For the purpose of solving Eq. (5.3-7), it is an usual way to assume its solution in a polynomial of $x_1$ and $x_2$ as*

$$\begin{aligned}\phi(x_1, x_2; \tau) = k_0(\tau) + k_1(\tau)\,x_1 + k_2(\tau)\,x_2 \\ + k_3(\tau)\,x_1 x_2 + k_4(\tau)\,x_1^2 + k_5(\tau)\,x_2^2\,,\end{aligned} \tag{5.3-8}$$

where $k_0(\tau)$, $k_1(\tau)$, ..... and $k_5(\tau)$ are the functions of $\tau$ only. In order to determine each coefficient $k_0(\tau)$, $k_1(\tau)$, ..... and $k_5(\tau)$, substituting Eq. (5.3-8) into Eq. (5.3-7) and equating the same power of $x_1$, $x_2$ and so on, we have a set of non-linear simultaneous ordinary differential equations;

$$\left.\begin{aligned}
k_0' &= m k_2 - k_2^2/4\mu + \sigma^2 k_5\,, & k_0(0) &= 0\\
k_1' &= m k_3 - k_2 k_3/2\mu\,, & k_1(0) &= 0\\
k_2' &= k_1 + 2m k_5 - a k_2 - k_2 k_5/\mu\,, & k_2(0) &= 0\\
k_3' &= 2\lambda + 2k_4 - a k_3 - k_3 k_5/\mu\,, & k_3(0) &= 0\\
k_4' &= -k_3^2/4\mu\,, & k_4(0) &= 0\\
k_5' &= k_3 - 2a k_5 - k_5^2/\mu\,, & k_5(0) &= 0
\end{aligned}\right\}, \tag{5.3-9}$$

where the symbol "$'$" represents the first derivative with respect to the reversed time variable $\tau$.

Although it seems very difficult to obtain the solution of the non-linear simultaneous Eq. (5.3-9), one way of solving them has been proposed in the previous chapter, using Taylor-Cauchy Transformation. The Runge-Kutta's method of fourth order[5] is, however, applied here, with the help of an electronic digital computer.

From Eqs. (5.3-6) and (5.3-8), as the optimal control variable we have

$$\bar{u} = -\frac{1}{2\mu}\frac{\partial\phi}{\partial x_2} = -\{\,k_m(\tau) + k_p(\tau)\,x_1 + k_d(\tau)\,x_2\,\}\,, \tag{5.3-10}$$

---

* The existence theorem of the solution of a type of Eq. (5.3-7) is developed by W.H. Fleming.[19]

where for simplicity, we put

$$k_m(\tau) \equiv \frac{k_2(\tau)}{2\mu}, \; k_p(\tau) \equiv \frac{k_3(\tau)}{2\mu} \quad \text{and} \quad k_d(\tau) \equiv \frac{k_5(\tau)}{\mu} \; . \tag{5.3-11}$$

It must thus be noted that the three coefficients, $k_2(\tau)$, $k_3(\tau)$ and $k_5(\tau)$ are related with the control variable. The optimum control variable which minimizes the performance functional (5.3-3) is previously given in Eq. (5.3-10). It can thus be found that the final-value control system is realized as a feedback control system with time-variant bias $k_m(\tau)$, and time-variant feedback gains $k_p(\tau)$ and $k_d(\tau)$.

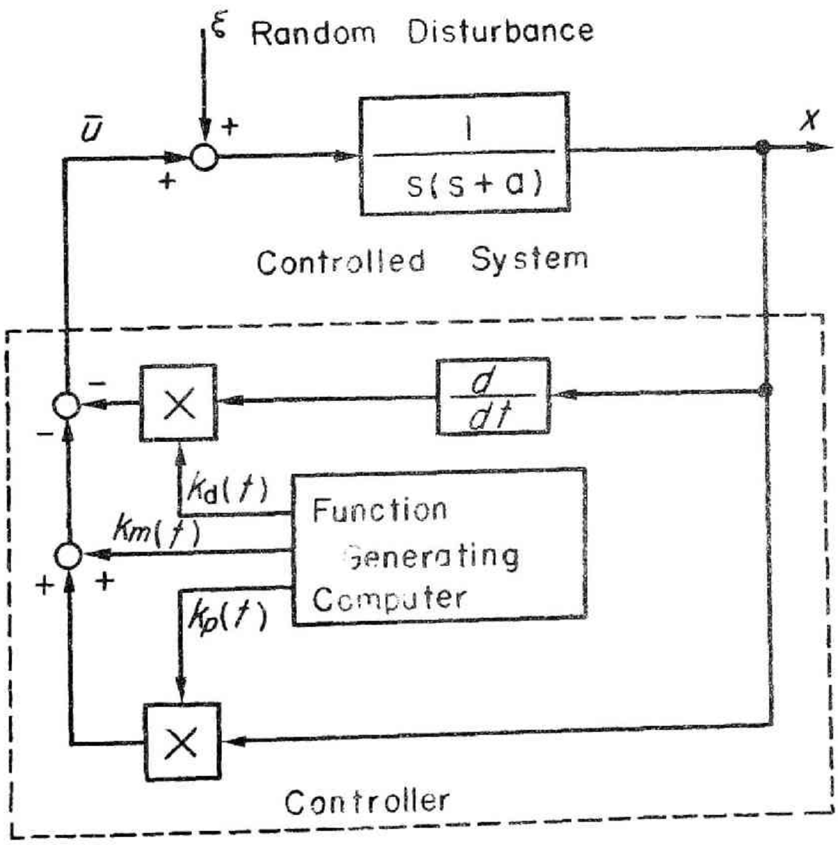



Fig. 5.2 Configuration of the final-value control system

The configuration of the final-value control system considered here is schematically shown in Fig. 5.2.

Consequently, the characteristics of the final-value control system can be investigated by examining the control coefficients $k_m(\tau)$, $k_p(\tau)$ and $k_d(\tau)$.

Control coefficients $k_p(\tau)$ and $k_d(\tau)$ are respectively plotted in Fig. 5.3, where $\sigma^2=1.0$, $m=0$, $a=0$, $\lambda=0.5$ and $\mu=0.5$. $k_m(\tau)$ is always equal to zero in the case where $m=0$, of which property will be discussed in the later section, then $k_m(\tau)$ is omitted in Fig. 5.3.

In Fig. 5.3, as well in the following Figs., it must be noticed that the horizontal axis represents the reversed time $\tau$ and that the real time $t$ moves along $\tau$-axis from right to left, so $\tau=T$ and $\tau=0$ mean the initial time $t=0$ and the final time $t=T$ respectively.

Comparing the curve of $k_p(\tau)$ with that of $k_d(\tau)$ in Fig. 5.3, it can be found that the proportional feedback gain $k_p(\tau)$ is larger in

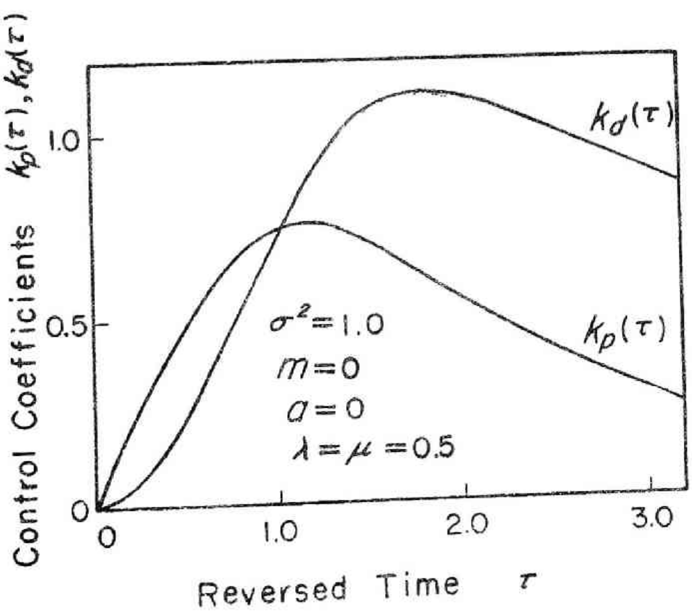



Fig. 5.3 Time dependent coefficients $k_p(\tau)$ and $k_d(\tau)$ vs. reversed time $\tau$

small $\tau$ and takes its maximum value at smaller $\tau$ than that of the derivative feedback gain $k_d(\tau)$, and that $k_p(\tau)$ decreases to zero more rapidly in large $\tau$ than $k_d(\tau)$. Considering this fact, we may conclude that, when $\tau$ is large, the final-value control system shows the operation as a derivative control system and as the value of $\tau$ decreases, the characteristics of the control action become those of a proportional control system. Further discussions will be carried out in 5.4.

5.3-2 Controlled System with Randomly Time-Varying Parameters

We consider here a system having randomy time-varying parameters for the controlled system as shown in Fig. 5.4, of which dynamic characteristics are described by

$$\left.\begin{aligned} \dot{x}_1 &= x_2\,, & x_1(0) &= c_1 \\ \dot{x}_2 &= -a(t)x_2 + b(t)u\,, & x_2(0) &= c_2 \end{aligned}\right\}\,, \tag{5.3-12}$$

where $x_1 = x$ and $x_2 = \dot{x}$ respectively. The random parameters $a(t)$ and $b(t)$ are independent with each other and are respectively assumed as

$$\left.\begin{aligned} a(t) &= \bar{\alpha} + \alpha(t) \\ b(t) &= \bar{\beta} + \beta(t) \end{aligned}\right\}\,. \tag{5.3-13}$$

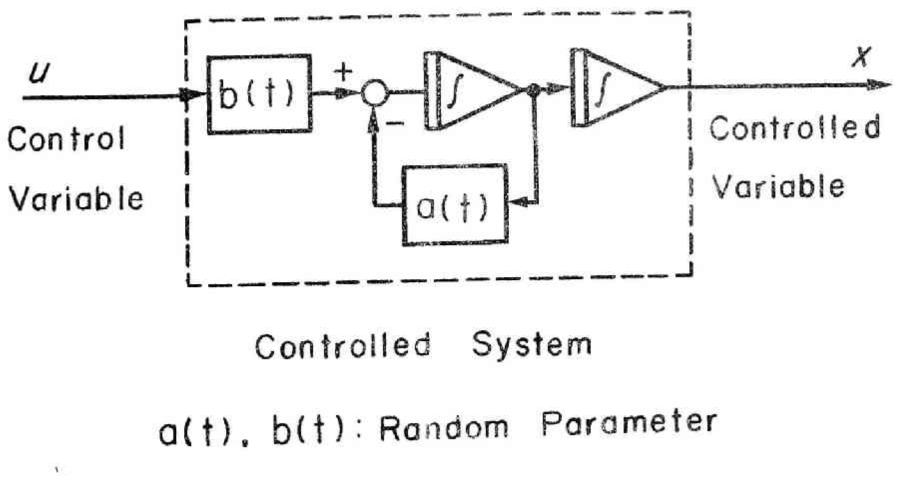



Fig. 5.4 Randomly time-variant controlled system

In Eq. (5.3-13), $\bar{a}$ and $\bar{\beta}$ are the known mean values of $a(t)$ and $b(t)$ respectively. Both $\alpha(t)$ and $\beta(t)$ express random portions of $a(t)$ and $b(t)$, which are respectively considered as stationary white Gaussian processes with zero mean and respective variances $A^2$ and $B^2$. Thus Eqs. (5.3-12) are expressed in the form;

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\bar{\alpha} x_2 + \bar{\beta} u + \sqrt{A^2 x_2^2 + B^2 u^2}\ \eta(t) \end{aligned}\right\} , \qquad (5.3\text{-}14)$$

where $\eta(t)$ is a stationary white Gaussian process with zero mean and unit variance.

We consider again Eq. (5.3-3) as the criterion functional, and according to the discussions in 5.2, we have the functional equation;

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \tau} &= \min_u \{ 2\lambda x_1 x_2 + \mu u^2 + x_2 \frac{\partial \phi}{\partial x_1} \\ &\quad - (\bar{\alpha} x_2 - \bar{\beta} u) \frac{\partial \phi}{\partial x_2} + \frac{A^2 x_2^2 + B^2 u^2}{2} \frac{\partial^2 \phi}{\partial x_2^2} \\ \phi &= 0 \qquad (\tau = 0) \end{aligned}\right\} , \qquad (5.3\text{-}15)$$

The optimum control variable $\bar{u}$ can therefore be determined by

$$\bar{\beta} \frac{\partial \phi}{\partial x_2} + 2\mu \bar{u} + B^2 \frac{\partial^2 \phi}{\partial x_2^2} \bar{u} = 0 , \qquad (5.3\text{-}16)$$

that is to say

$$\bar{u} = -\bar{\beta} \frac{\partial \phi}{\partial x_2} \Big/ \left( 2\mu + B^2 \frac{\partial^2 \phi}{\partial x_2^2} \right) . \qquad (5.3\text{-}17)$$

Then Eq. (5.3-15) becomes

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \tau} &= 2\lambda x_1 x_2 \\ &\quad + \{ \mu / (2\mu + B^2 \frac{\partial^2 \phi}{\partial x_2^2}) - 1 \} \bar{\beta}^2 \left(\frac{\partial \phi}{\partial x_2}\right)^2 / (2\mu + B^2 \frac{\partial^2 \phi}{\partial x_2^2}) \\ &\quad + x_2 \left(\frac{\partial \phi}{\partial x_1} - \bar{\alpha} \frac{\partial \phi}{\partial x_2}\right) + \frac{A^2}{2} x_2^2 \frac{\partial^2 \phi}{\partial x_2^2} \\ &\quad + \frac{B^2}{2} \bar{\beta}^2 \left(\frac{\partial \phi}{\partial x_2}\right)^2 \frac{\partial^2 \phi}{\partial x_2^2} / \left(2\mu + B^2 \frac{\partial^2 \phi}{\partial x_2}\right)^2 \\ \phi &= 0 \qquad (\tau = 0) \end{aligned}\right\} , \qquad (5.3\text{-}18)$$

By assuming the solution of Eq. (5.3-18) as the same polynomial in Eq. (5.3-8), we have

$$\left.\begin{aligned} k_0' &= -\bar{\beta}^2 k_2^2 / 4\,(\mu+B^2 k_5)\,, & k_0(0)&=0 \\ k_1' &= -\bar{\beta}^2 k_2 k_3 / 2\,(\mu+B^2 k_5)\,, & k_1(0)&=0 \\ k_2' &= k_1 - \bar{\alpha} k_2 - \bar{\beta}^2 k_2 k_5 / (\mu+B^2 k_5)\,, & k_2(0)&=0 \\ k_3' &= 2\lambda+2k_4 - \bar{\alpha} k_3 - \bar{\beta}^2 k_3 k_5 / (\mu+B^2 k_5)\,, & k_3(0)&=0 \\ k_4' &= -\bar{\beta}^2 k_3^2 / 4\,(\mu+B^2 k_5)\,, & k_4(0)&=0 \\ k_5' &= k_3 + (A^2-2\bar{\alpha})\,k_5 - \bar{\beta}^2 k_5^2 / (\mu+B^2 k_5)\,, & k_5(0)&=0 \end{aligned}\right\} . \tag{5.3-19}$$

From Eqs. (5.3-18), and (5.3-17), we have the optimal control action as

$$\bar{u} = -\{\, k_m(\tau) + k_p(\tau)\, x_1 + k_d(\tau)\, x_2 \,\}\;, \tag{5.3-20}$$

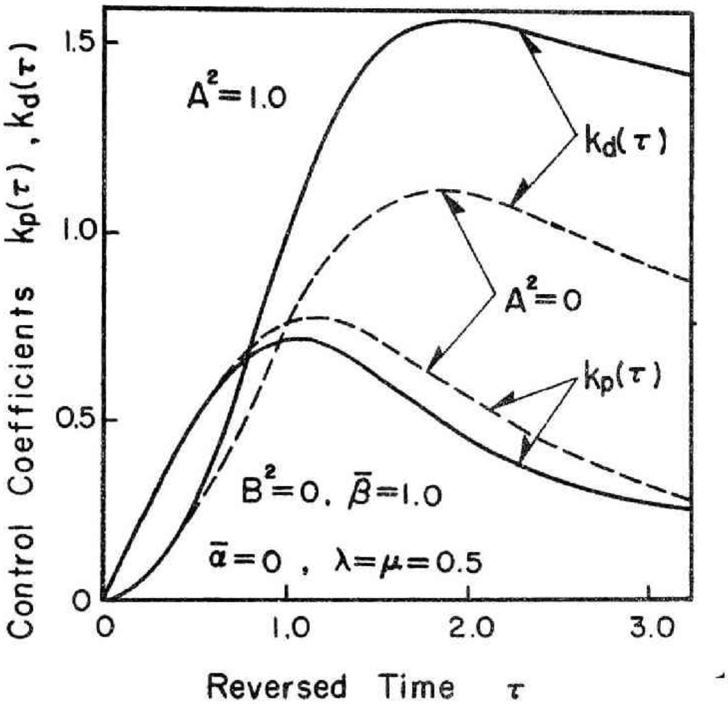
where

$$\left.\begin{aligned} k_m(\tau) &\equiv \bar{\beta}\, k_2(\tau) / 2\,(\mu+B^2 k_5) \\ k_p(\tau) &\equiv \bar{\beta}\, k_3(\tau) / 2\,(\mu+B^2 k_5) \\ k_d(\tau) &\equiv \bar{\beta}\, k_5(\tau) / (\mu+B^2 k_5) \end{aligned}\right\} . \tag{5.3-21}$$

Then it is found that the optimal final-value control system to be designed here is also realized as a feedback control system with time-variant control coefficients $k_m(\tau)$, $k_p(\tau)$ and $k_d(\tau)$. Its configuration is the same as that shown in Fig. 5.2. First, the case where $a(t)$ is a random process with the value of the variance $A^2=1.0$ and $b(t)$ is a deterministic process i.e., $B^2=0$, is numerically computed with other numerical data as $\bar{\alpha}=0$, $\bar{\beta}=1.0$ and $\lambda=\mu=0.5$.

The numerical plots of the proportional feedback gain $k_p(\tau)$ and derivative feedback gain $k_d(\tau)$ versus reversed time $\tau$ are shown in Fig. 5.5(a).

Second, the case where $a(t)$ is deterministic (i.e., $A^2=0$) and $b(t)$ is a random process (i.e., $B^2=1.0$) is computed. The results are shown in Fig. 5.5(b) with other numerical data as $\bar{\alpha}=0$, $\bar{\beta}=1.0$ and




(a) Influences of $A^2$

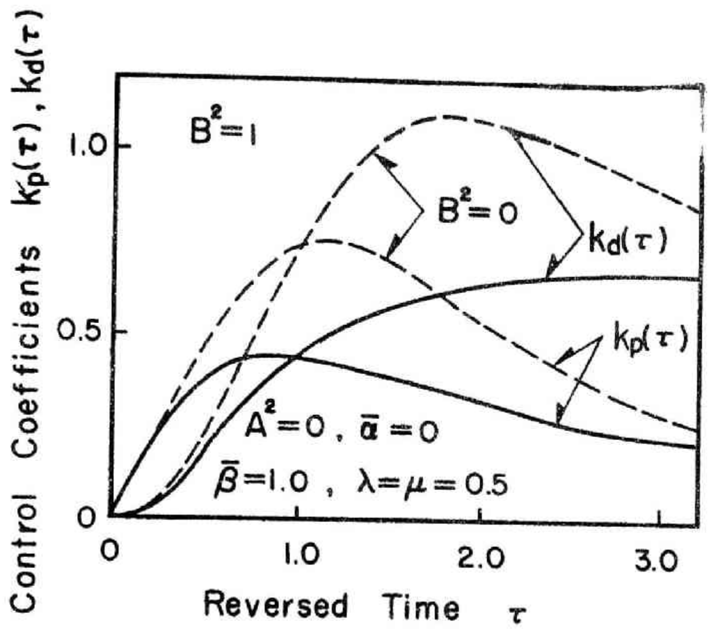
Fig. 5.5 Time dependent coefficients $k_p(\tau)$ and $k_d(\tau)$ vs. reversed time $\tau$

$\lambda=\mu=0.5$. In both Figs., the broken lines show the corresponding deterministic cases, i.e., $A^2=B^2=0$.

From Eqs. (5.3-19), it is found $k_2(\tau)$ is always equal to zero. It is obvious that

$$k_m(\tau)=0 \qquad (\tau\geq 0)\ , \tag{5.3-22}$$

that is to say, the time variant bias $k_m(\tau)$ is not necessary in any cases.




(b) Influences of $B^2$

Fig. 5.5 Time dependent coefficients $k_p(\tau)$ and $k_d(\tau)$ vs. reversed time $\tau$

## 5.4 Further Considerations

In this section, we discuss about the effects of changing the controlled system parameters and the performance weights on the control characteristics.

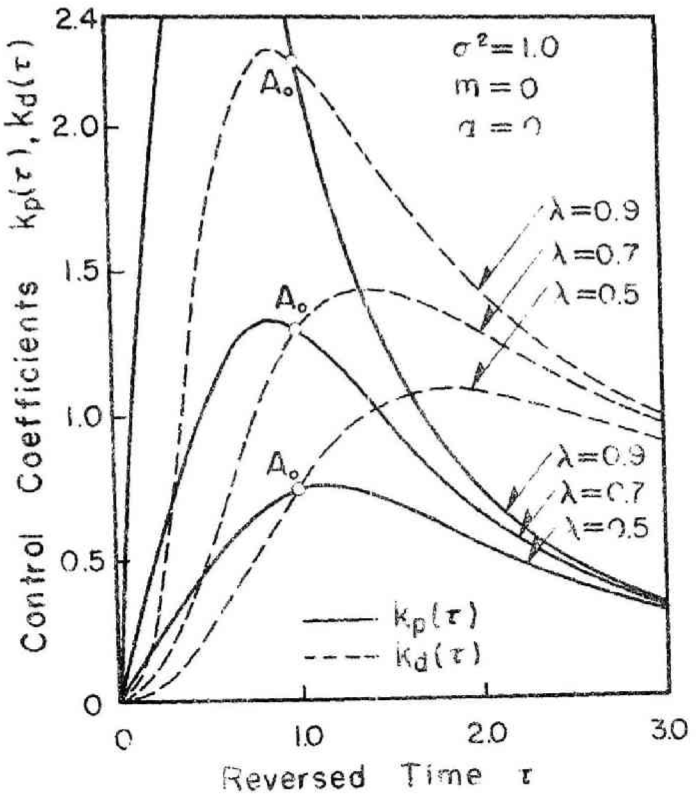
First, we consider the case where the controlled system is subjected to a random disturbance given in the previous 5.3-1.

The effects of the selection of values of $\lambda$ and $\mu$ on both the control coefficients $k_p(\tau)$ and $k_d(\tau)$ are respectively shown in Figs. 5.6 (a) and 5.6 (b) It is evident from Figs. 5.6 (a) and (b) that, as the value of $\lambda$ trends to 1, since the feedback gains $k_p(\tau)$ and $k_d(\tau)$ become to take too much large value. Then it is physically impossible to instrument the controller for a final-value control in the strict

Fig. 5.6

Influences of performance weight $\lambda$ on the time dependent coefficients $k_p(\tau)$ and $k_d(\tau)$

(a) In the case where $a=0$




(b) In the case where $a=1.0$

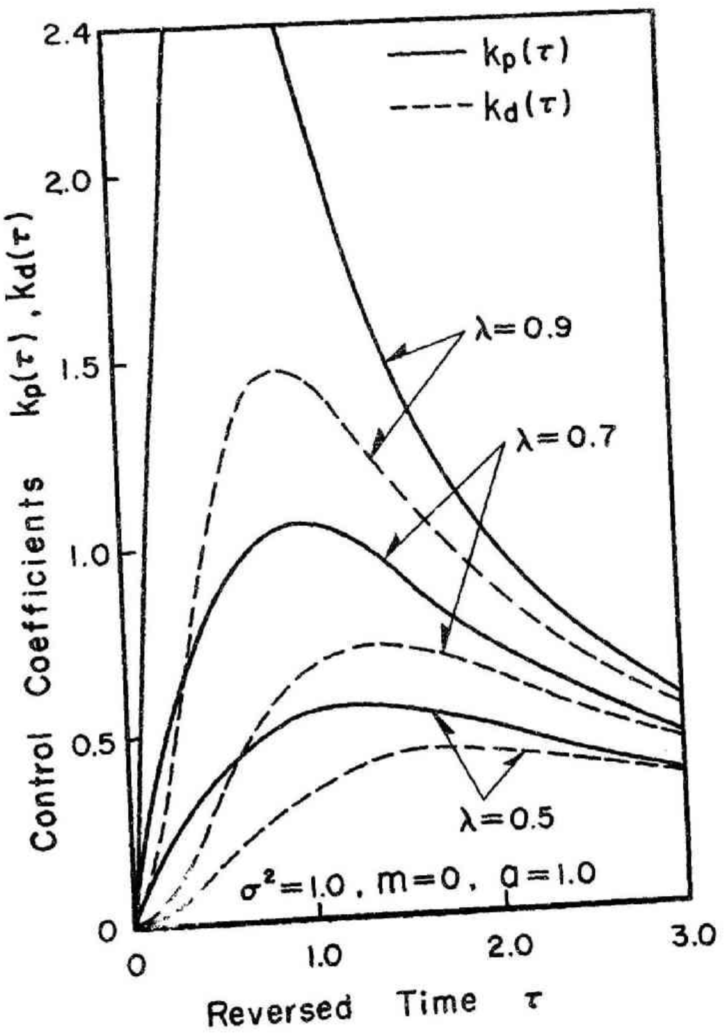
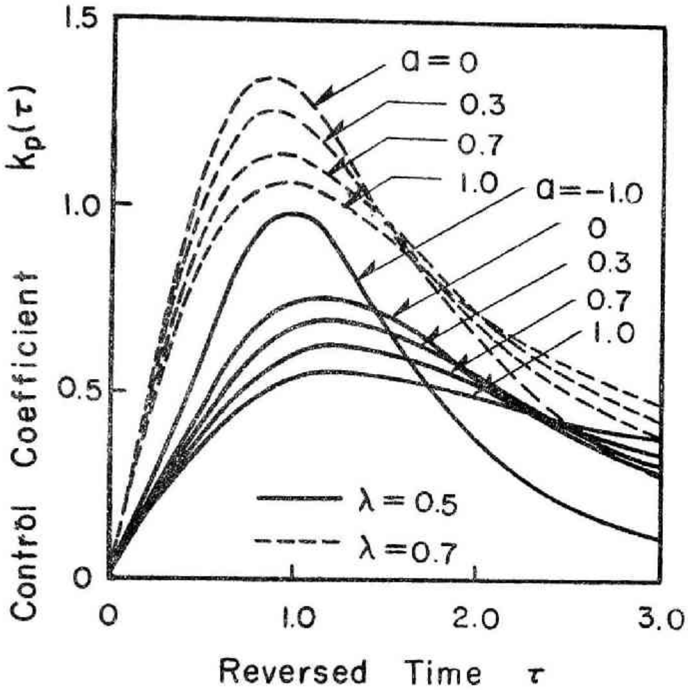
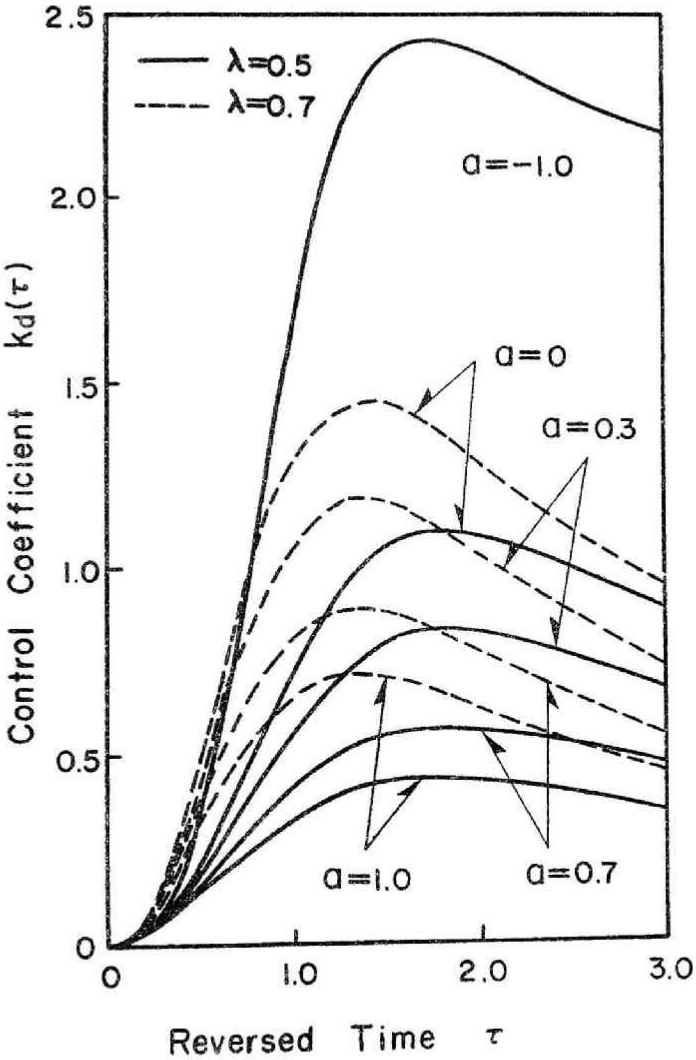


Fig. 5.7

Time dependent coefficients $k_p(\tau)$ and $k_d(\tau)$ vs. reversed time $\tau$




(a) Influences of $a$ on the coefficient $k_p(\tau)$




(b) Influences of $a$ on the coefficient $k_d(\tau)$

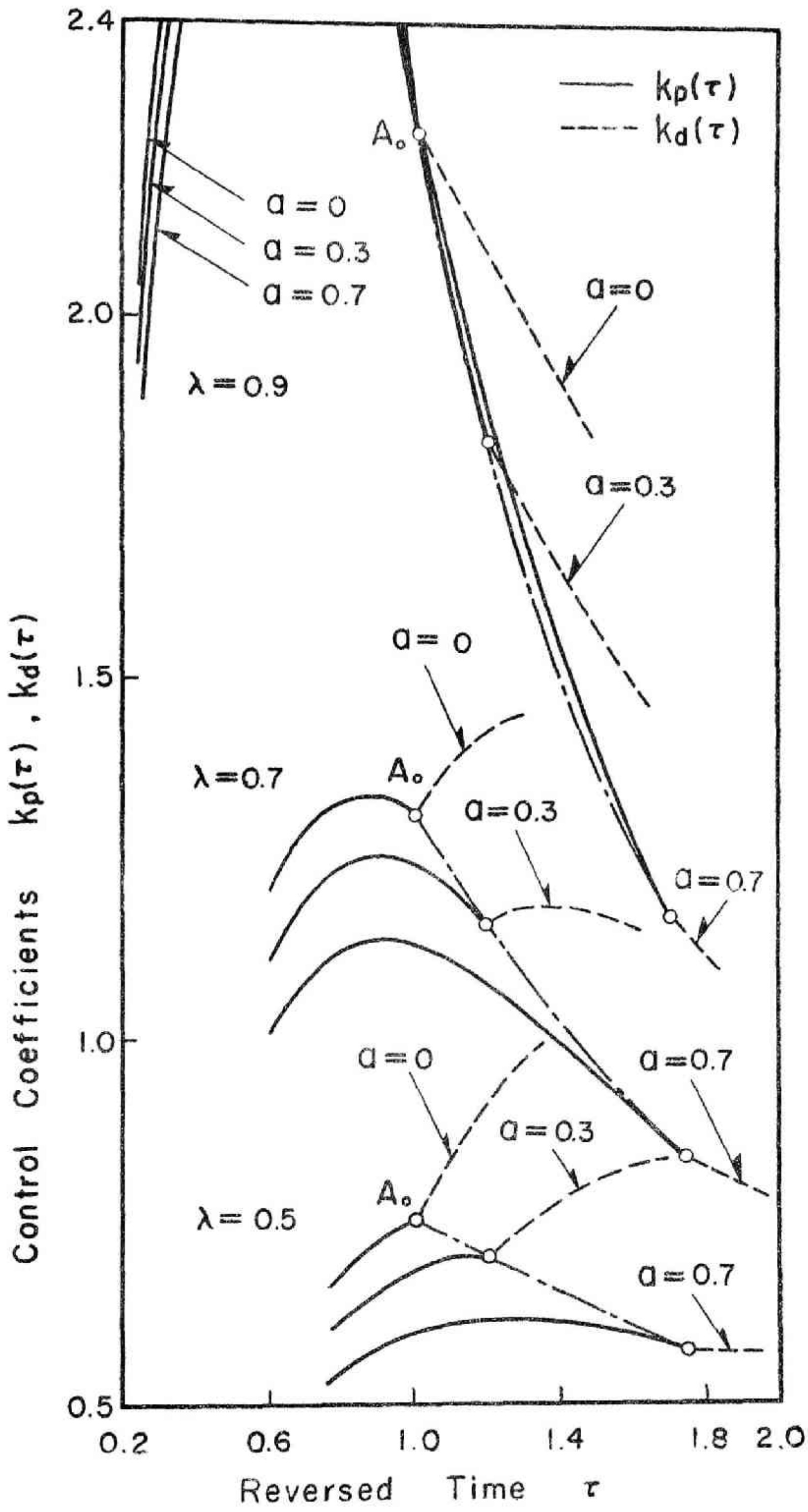



Fig. 5.8 Influences of $\lambda$ and $a$ on the point of changing control characteristics

sense, i.e., $\mu=0$ in Eq. (5.3-3). Therefore, if we are confronted to design a final-value control system in the strict sense, it is necessary to add a magnitude constraint on the elements of controller and to treat the design problem as a bounded control problem. The effects of the values of $a$ are shown in Figs. 5.7 (a) and 5.7 (b). These results reveal us, from the physical point of view, that the system may be less sensitive as the value of $a$ increases.

Since it is pressumed from Fig. 5.6 (a) that the point of changing control characteristics is not affected by a system parameter but by a performance weight $\mu$ , then it is very interesting to plot the point which is the intersection of $k_p(\tau)$ and $k_d(\tau)$. Fig. 5.8 shows the plot of such points where $a$ and $\mu$ are shown as parameters.

We consider about the effects of the presence of the mean value $m$ of the disturbance $\xi(t)$ on the control coefficients $k_m(\tau)$, $k_p(\tau)$, and $k_d(\tau)$. It is clear, from Eqs. (5.3-9), that only $k_0(\tau)$, $k_1(\tau)$ and $k_2(\tau)$ are related to $m$ . Both $k_p(\tau)$ and $k_d(\tau)$ are, therefore, independent of the value of $m$ . Only $k_m(\tau)$ is related to $m$ , and this is shown in Fig. 5.9. If $m$ is equal to zero, $k_2(\tau)$ must always be

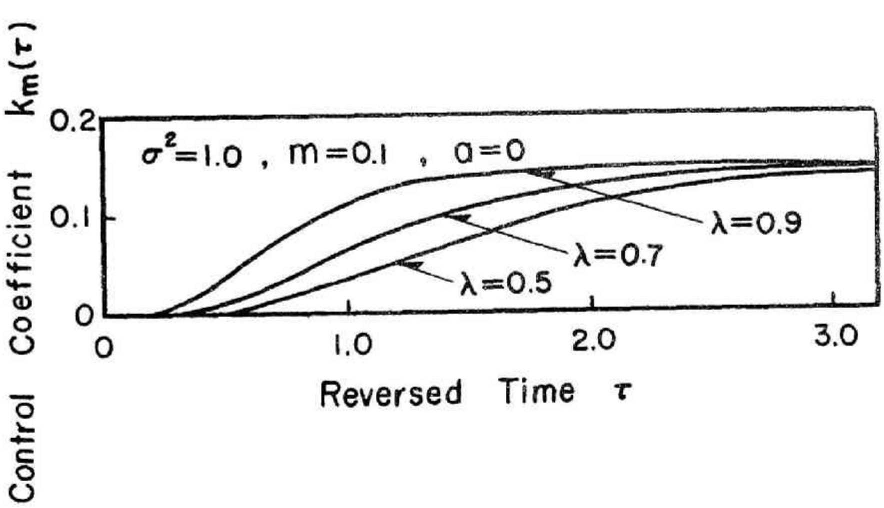



Fig. 5.9 Time dependent coefficient $k_m(\tau)$ vs. reversed time $\tau$

equal to zero from Eqs. (5.3-9). Then $k_m(\tau)$ must be zero from Eqs. (5.3-11). If $\tau$ is sufficiently large, it is expected from Eq. (5.3-9) that

$$\lim_{\tau\to\infty} k_m(\tau)=m\,,\quad \lim_{\tau\to\infty} k_p(\tau)=0 \text{ and } \lim_{\tau\to\infty} k_d(\tau)=0\,. \tag{5.4-1}$$

Therefore, $k_m(\tau)$, roughly speaking, plays a role to cancel the mean value portion of the random disturbance $\xi(t)$. Moreover Eq. (5.4-1) may express that the final-value control system behaves as a free system,[27] since feedback loops are broken, if $\tau$ is sufficiently large.

Finally, we must pay our attention to the variance $\sigma^2$ of the random disturbance $\xi(t)$. Since, from Eq. (5.3-9), we can easily find that $k_0(\tau)$ can only be affected by the values of the variance $\sigma^2$ and that all the coefficients $k_2(\tau)$, $k_3(\tau)$ and $k_5(\tau)$ are independent of $\sigma^2$, then the value of the variance $\sigma^2$ does not make any effects on the control coefficients $k_m(\tau)$, $k_p(\tau)$ and $k_d(\tau)$.

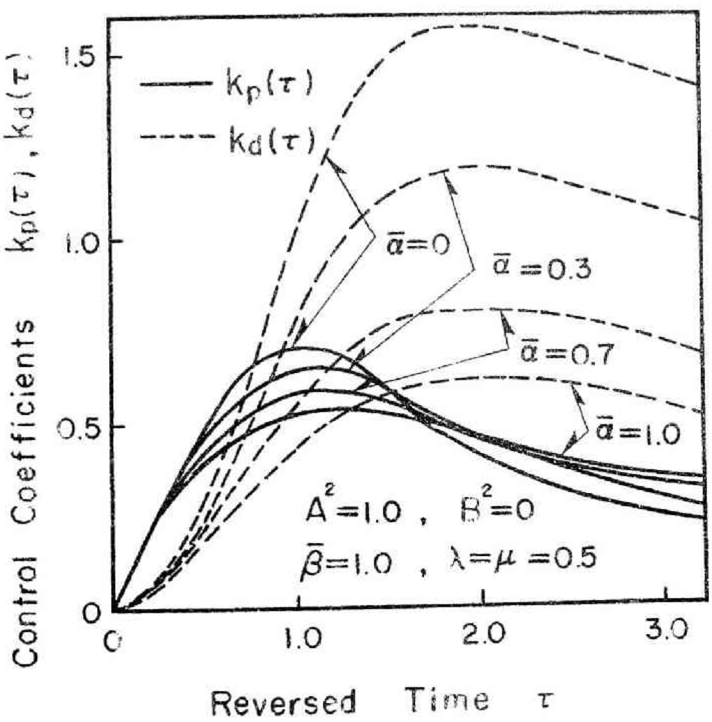



Fig. 5.10 Influences of $\bar{\alpha}$ on the control coefficients $k_p(\tau)$ and $k_d(\tau)$

Second, we consider the case where the controlled system has randomly time-varying parameters discussed in 5.3-2. The effects of changes of $\bar{\alpha}$ are shown in Fig. 5.10. From Fig. 5.10, we can find that the effects of the values of $\bar{\alpha}$ are almost similar to those discussed in the previous case where $\alpha(t)$ is not random. The effects of the value of $\lambda$ have also been examined, but the results are the same, so we will never mention them. In particular, the fact should be emphasized that if the parameters of a controlled system have randomly time-varying characteristics, then both the value of the variances $A^2$ and $B^2$ are simultaneously related to control coefficients $k_p(\tau)$ and $k_d(\tau)$. When both $A^2$ and $B^2$ are not zero, control coefficients are shown in Fig. 5.11 with the values of a set of parameters $A^2 = 1.0$, $B^2 = 1.0$ $\bar{\beta} = 1.0$, $\bar{\alpha} = 0$ and $\lambda = \mu = 0.5$.

Lastly let us consider the effect of random parameters $\alpha(t)$ and $\beta(t)$ on the control characteristics referring the results as shown in

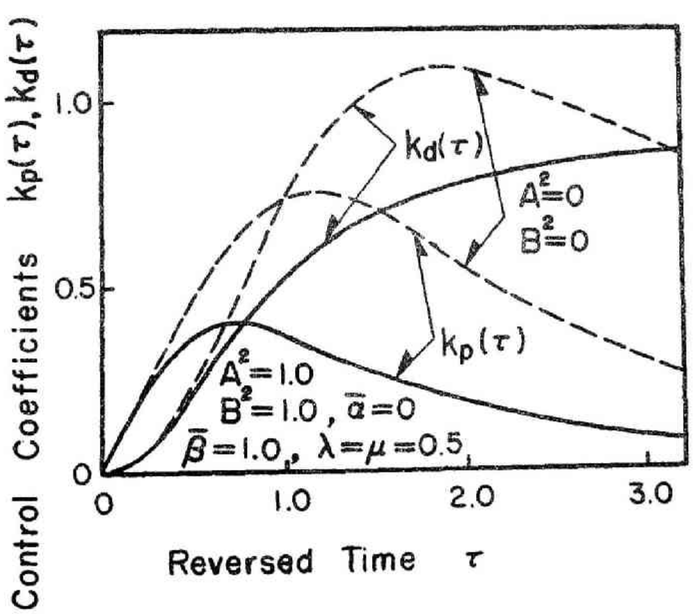



Fig. 5.11 Influences of $A^2$ and $B^2$ on the coefficients $k_p(\tau)$ and $k_d(\tau)$

Figs. 5.8 and 5.10. Fig. 5.10 shows the comparison of control coefficients subjected to random parameters. It is evident from Figs. 5.8 and 5.12 that the effect of $\alpha(t)$ is equivalent to that of decreasing the value of a system parameter $a$, that is to say, it makes the control system more sensitive. This effect is easily understood by comparing the location of the point of changing control characteristics, with that of the point where $\alpha(t)=0$ in Fig. 5.12. The effect of $\beta(t)$ seems, on the other hand, to be equivalent to that of decreasing $\lambda$. This is easily recognized by comparing the curves numbered 1 with these marked 3 in Fig. 5.12. The effect by both random parameters $\alpha(t)$ and $\beta(t)$ is that the controller becomes of less final-value control

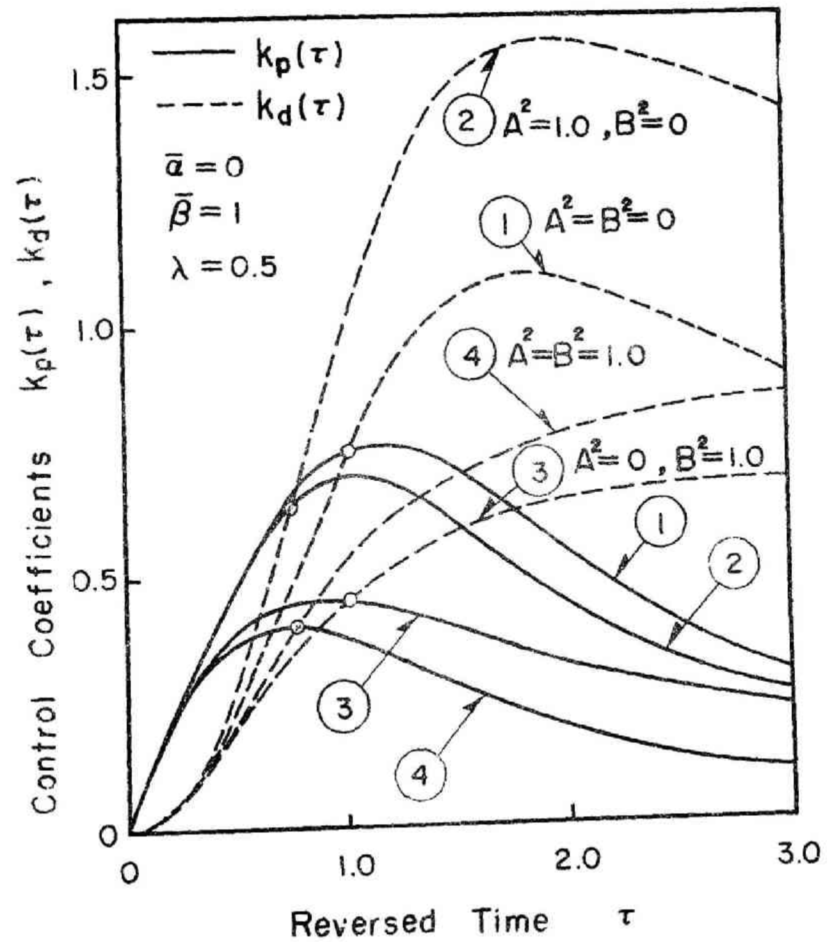



Fig. 5.12 Comparison of the effects of random parameters $\alpha(t)$ and $\beta(t)$ on the control characteristics

but of more derivative control, which is to counteract the increased sensitiveness caused by the random parameter $\alpha(t)$.

## 5.5 Concluding Remarks

In this chapter, the author has discussed about the optimum design technique of the final-value control systems and about what properties they must have. Both the final-value control problem with a linear time-invariant second order controlled system which undertakes a random disturbance, and the one with a linear randomly time-variant second order controlled system are studied in detail.

It should emphasized that the characteristics of the optimum final-value controller must become time-variant to a high degree. Roughly speaking, the final-value control system operates as a free system, that is to say, as a system with no feedback loops, at the primitive stage of the pre-assigned control duration. The derivative action becomes to play a stronger role with the lapse of time. And at the final stage, the proportional action becomes stronger.

It is also emphasized that if we are confronted to design a final-value control system in the strict sense, it is necessary to add a magnitude constraint on the elements of controller and to treat the design problem as a bounded control problem.

It is also made clear that the situations are considerably different between the controlled system subjected to a random disturbance and the one with randomly time-varying parameters.

# PART III ON-LINE COMPUTER OPTIMIZATION APPROACH TO NON-LINEAR CONTROL SYSTEMS

# CHAPTER 6 FUNDAMENTAL CONCEPT OF AN ON-LINE COMPUTER OPTIMIZATION APPROACH TO NON-LINEAR CONTROL SYSTEMS AND ITS APPLICATIONS

## 6.1 Introductory Remarks

In recent years there has been a growing interest in the idea of controlling a complex process by using high-speed modern computing facilities. Several attempts have been carrying out both in and out of this country to control such complex processes as chemical reaction plants or power supplying plants. Most of the attempts are focused on the investigations to the dynamic optimization of a complex process subjected to the change of its environments, which effect the performance of the process to be controlled. And the method of attack is application of modern control or optimization theories. As the author has already stated, the modern theories are developed upon the basis of complete mathematical description of the design problem. However, from the view point of controlling a real plant, there remain many preliminary problems in applying the theory of optimum control because all the design specifications are not given in a precise mathematical forms. That is to say, some of them may be given graphically or numerically as the data of experimental studies. Then it is necessary to determine a mathematical presentation which is almost equivalent to the experimental data. This procedure is in general considered as an inverse problem[43] which yields another type of difficult mathematical problems to be studied such as problems on the existence or uniqueness of a solution. Furthermore, since this procedure possesses the feature of "cut and try method" in some sense, then the process to be controlled becomes the more complicated, the more tedious the procedure becomes and it should be carried out through a digital computer. Therefore, the more complicated the construction of a controlled plant becomes, the more difficult the mathematical description of the optimization problem under many associated physical constraints results. Furthermore, the rigorous mathematical solution of the problem is hopeless at the present even though we utilize gigantic high-speed digital computers.

In this chapter, from a quite different point of view, the fundamental idea of on-line computer optimization approach to non-linear control systems is presented by extending the concept of sub-interval optimization stated in Chapter 1. From the view point of computer utilization, an attempt will be done to explore a practical design procedure for designing non-linear control systems subjected to the physical limitations which are not completely described in mathematical formulations. Two examples will be presented to show the idea in detail.

## 6.2 Fundamental Concept of an On-Line Computer Optimization co-operated with State Adaptive Performance Criterion

### 6.2-1 Description of the State Adaptive Performance Criterion

Among the design specifications the most important seems to be a performance specification and its mathematical presentation, a performance index or a performance functional. The determination of performance functional, although somewhat a matter of experience and ingenuity, is suggested by the performance specifications of the design problem. The allowable latitude in this determination is very much a function of dynamic difficulty of the design problem. In particular, more difficult design problems require more accurately determined performance functional if the performance specifications are to be met. The most widely used approximation is to choose a quadratic form of functional because it is feasible to treat. Especially in the case of problem controlling linear dynamical plants, the quadratic form of performance functional gives us the configuration of optimum linear feedback control systems as presented in Chapter 4 or 5. In spite of the many application of optimum linear feedback controls,[35] the non-linearities associated with many important design problems can not be avoided. Furthermore, many applications in real plant control problems require the use of non-quadratic forms of performance functional because of many physical limitations related with non-linearities.

Generally speaking in the optimization problems, however, the more complicated forms take the performance functionals, the more hopeless becomes their rigourous solution. Then it seems to be of importance to study the form of performance functional which is feasible to solve economically, and furthermore is flexible to satisfy the various types of performance specifications.

Let us trace back to the starting point of "control" and consider what the basic structure of performance functional is. Fundamentally speaking, the very object of "control" is to reduce the error itself or the corresponding "loss" or "cost" caused by the error at the next instant with irrespective of its quantity. However, for doing this there should be an infinite control power and this idea is revealed to be unfeasible. Then the object of "control" is soften to minimize the amount of "loss" or "cost" associated with control processes as soon as possible under physical limitations. There are, on the other hand, two types of "loss" or "cost" in control processes. The one is calculated with respect to the error itself or the effects caused by it, which is referred to as an error-cost. The other is calculated with regard to the expenditure of control action, which is referred to as a control-cost. Since these two cost are essentially exclusive each other, then if we only minimize the error-cost the corresponding control-cost shall be much increased. Recalling the object of "control" again, the control processes may be classified roughly into the following three phases;

Phase-I minimum-error control : If there is a large quantity of error or effects caused by it, a control action should be exhausted to make the error small as quickly as possible without regarding to the expenditure of control-cost. This phase of control is named as a "minimum-error control" phase because it is quite similar to that of minimum-time control or final-value control.

Phase-II minimum-energy control : In the case of a small quantity of error within a kind of engineering tolerance, the corresponding control should be determined by taking a heavy attention on the expenditure of control-cost. Then this phase of control is referred to as a "minimum-energy control" phase.

Phase-III minimum-error control with energy constraint : In the case where there is a considerable quantity of error, the control action should be carried out by minimizing the compound cost which is defined as the appropriately weighted sum of error-cost and control-cost.*
The title of this phase of control is a natural result.

Let us pay our attention to the phase III and consider the details of this phase. That is, in this phase of control situation, the control action is very much affected by the selection of a weighting factor. Since it is natural to consider that the selection of a weighting factor should be related to the quantity of error, then the following concept of "state adaptive cost function" or "state adaptive performance functional" is available to unify the three control phases stated above.

Definition: The cost function or the performance functional whose form is enable to change itself depending upon the present and past informations with respect to the related systems (state variables of the related systems, the rest of control energy, the

---

* (compound cost)=(error-cost)+(weighting factor)×(control-cost)

rest of operation time and so on) is referred to as a "state adaptive cost function" or a "state adaptive performance functional".

### 6.2-2 Fundamental Concept of Sub-Interval Optimization co-operated with State Adaptive Performance Functional

For on-line computer utilization, a suitable method is desired in order to perform the optimization simply and rapidly. This means that it is desirable to choose the performance functional whose structure is simple. Let us suppose that presentation is restricted to the discrete case, and consider a sub-interval optimization technique which recomputes, utilizing the best information available at present, the optimization problem at discrete time intervals of time $\Delta$. In this manner, feedback control can be obtained by measuring state variable of the actual controlled system. For this purpose, we introduce the following type of state adaptive performance functional;

$$J_{p_i}(k) = {}^t e(k+i)\, P[e(k)]\, e(k+i) + \sum_{j=0}^{i-1} {}^t u(k+j)\, Q[e(k)]\, u(k+j)\, \Delta, \qquad (6.2\text{-}1)$$

where $e(k)$ and $u(k)$ are shorthand of $e(k\Delta)$ and $u(k\Delta)$ respectively. They respectively express an error vector $e$ and a control vector $u$ at the $k$-th sampling instant, i.e., $t = k\Delta$. $i$ denotes the integer expressing the duration of sub-interval ( $k\Delta$, $k+i\,\Delta$ ), and $\Delta$ is sampling period. In Eq. (6.2-1), $P[e(k)]$ and $Q[e(k)]$ are positive semi-definite $n \times n$ and $r \times r$ matrices respectively, which express weighting factors depending on the present state of the system, and they are named as "state adaptive weighting factors". The first and second terms in the right hand side of Eq. (6.2-1) mean the error-cost and the control-cost respectively. The noteworthy point of Eq. (6.2-1) is that it is different from

$$J_i(k) = {}^t e(k+i)\, P(k+i)\, e(k+i) + \sum_{j=0}^{i-1} {}^t u(k+j)\, Q(k+j)\, u(k+j)\, \Delta. \qquad (6.2\text{-}2)$$

In other words, from Eq. (6.2-2) we can derive a time dependent or time scheduled control policy, while Eq. (6.2-1) provides us a state adaptive control policy. That is, since weighting factors $P[e(k)]$ and $Q[e(k)]$ in Eq. (6.2-1) are functions of the present state variables, then the computed optimum control vector from Eq. (6.2-1) becomes much more heavily depend upon the present state than the computed optimum control vector from Eq. (6.2-2). In other words, it results from the fact that the rule in Eq. (6.2-2) for finding the point of compromise between the error-cost and the control-cost is pre-assigned without regarding to the actual states of the system. It is also interesting that if we restrict a controlled plant to be linear the characteristics of optimum controller derived from Eq. (6.2-1) becomes non-linear while Eq. (6.2-2) provides us a linear optimum controller. The non-linear characteristica of controller derived from Eq. (6.2-1) is rely on the structure of weighting factors $P[e(k)]$ and $Q[e(k)]$. It is also evident from Eq. (6.2-1) that if an integer $i$ takes a large value, the corresponding computing time for optimization becomes long. From the view point of on-line computer utilization, however, it is desirable to shorten the computation time. Then let us pre-assign that $i=n$ or $n+1$ where $n$ expresses the highest order of the controlled plant dynamics.

Although there is a room for discussions on the determination of these structures of state adaptive weighting factors $P[e(k)]$ and $Q[e(k)]$, the state adaptive performance functional, Eq. (6.2-1), gives us several interesting points associated with the design of non-linear control systems. The following sections will throw light upon the details of the present proposal.

## 6.3 Fundamental Considerations on the Application of the Concept to the Desigm of Non-Linear Control Systems

In this section, two simple examples are presented to show the details of the idea and the characteristics of the system designed by using the concept stated in the previous section, and to point out its applicability to the design of non-linear control systems.

### 6.3-1 Example 1: Design of a First Order Non-Linear Control System

Statement of the Problem and Determination of Control Policies

Let us consider the problem of controlling a first order non-linear controlled system whose dynamics is given by

$$\dot{x}(t) = a\,x(t) + b\,x(t)^3 + u(t) \quad , \quad x(0) = -c \quad , \tag{6.3-1}$$

where $x(t)$ and $u(t)$ are the controlled signal and the control signal respectively, hence, $a$ and $b$ are constants expressing system parameters, and $c$ denotes an initial state of the controlled system. From Eq. (6.3-1), we can get the discrete presentation with respect to the error signal $e(t) = v(t) - x(t)$ as follows:

$$e\,(k+1) = e(k) + \left[a\,e(k) + b\,e\,(k)^3\right]\Delta - u\,(k)\Delta \;, \quad e\,(0) = c \quad , \tag{6.3-2}$$

where $e(k)$ and $u(k)$ are shorthand of $e(k\Delta)$ and $u(k\Delta)$ , which respectively denote the error signal and the control signal at the $k$-th sampling instant, i.e., $t = k\Delta$ . Hence $\Delta$ expresses a sampling period. Furthermore, in deriving Eq. (6.3-2) we set $v(t) = 0$ for simplicity. From Eq. (6.2-1), the corresponding state adaptive performance functional in this case is described as follows:

$$J_{pi}(k) = P[e(k)]\,e(k+i)^2 + \sum_{j=0}^{i-1} Q[e(k)]\,u(k+j)^2\,\Delta \;, \quad (i = 1, 2) \tag{6.3-3}$$

where $P[e(k)]$ and $Q[e(k)]$ are non-negative function of $e(k)$ .

Then the problem is to derive the control policy which minimizes Eq. (6.3-3) under the constraint of Eq. (6.3-2) By performing the minimization procedure* to Eqs. (6.3-2) and (6.3-3), the corresponding control policies are easily calculated as

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* The detailed procedure of derivation is presented in Appendix E

$$\overline{u}(k)_1 = R\{ (1+a\triangle)e(k)+b\triangle\, e(k)^3 \} \tag{6.3-4}$$

and

$$\overline{u}(k)_2 = -\{ z - (1+a\triangle)e(k)+b\triangle\, e(k)^3 \}/\triangle , \tag{6.3-5}$$

where $z$ is the physically meaningful solution of

$$\{ (1+a\triangle)e(k)+b\triangle\, e(k)^3 \} - \{ 1+(1+a\triangle)^2 R\triangle \} z$$
$$-4Rb(1+a\triangle)\triangle^2 z^3 - 3Rb^2\triangle^3 z^5 = 0 , \tag{6.3-6}$$

hence, $R$ in Eqs. (6.3-4) and (6.3-6) is defined by

$$R \equiv P[e(k)]/(P[e(k)]\triangle + Q[e(k)]) . \tag{6.3-7}$$

Therefore, the corresponding optimum control based upon the concept of sub-interval optimization described in 6.2-2 is carried out by realizing the control policies expressed by Eq. (6.3-4) or Eq. (6.3-5). It is evident from Eq. (6.3-4) that the control policy $\overline{u}(k)_1$ can be realized by taking the configuration of a non-linear sampled-data control system as shown in Fig. 6.1. On the contrary, the control policy $\overline{u}(k)_2$ given by Eq. (6.3-5) with Eq. (6.3-6) should be performed by an on-line computing device solving the 5th order algebraic equation, Eq. (6.3-6). Fig. 6.2 shows the simplified logic flow chart for realization of the control policy given by Eq. (6.3-5).

<u>Relations between the Control Characteristics and the Selection of State Adaptive Weighting Factors</u>

In Fig. 6.1, there are three non-linear elements except for the elements of multiplication and division. The one (NL-1) is introduced to simulate the non-linear characteristics of controlled plant. The others (NL-2 and NL-3) are introduced to play an important role in changing the phase of control. Since, by assuming the various types of non-linear functions of $e(k)$ to these weighting factors, we get the various types of controllers, then the next problem before us is to determine the appropriate function so as to satisfy the performance specifications. There is no analytical method of determination if some of the performance specifications are left to the judgement of

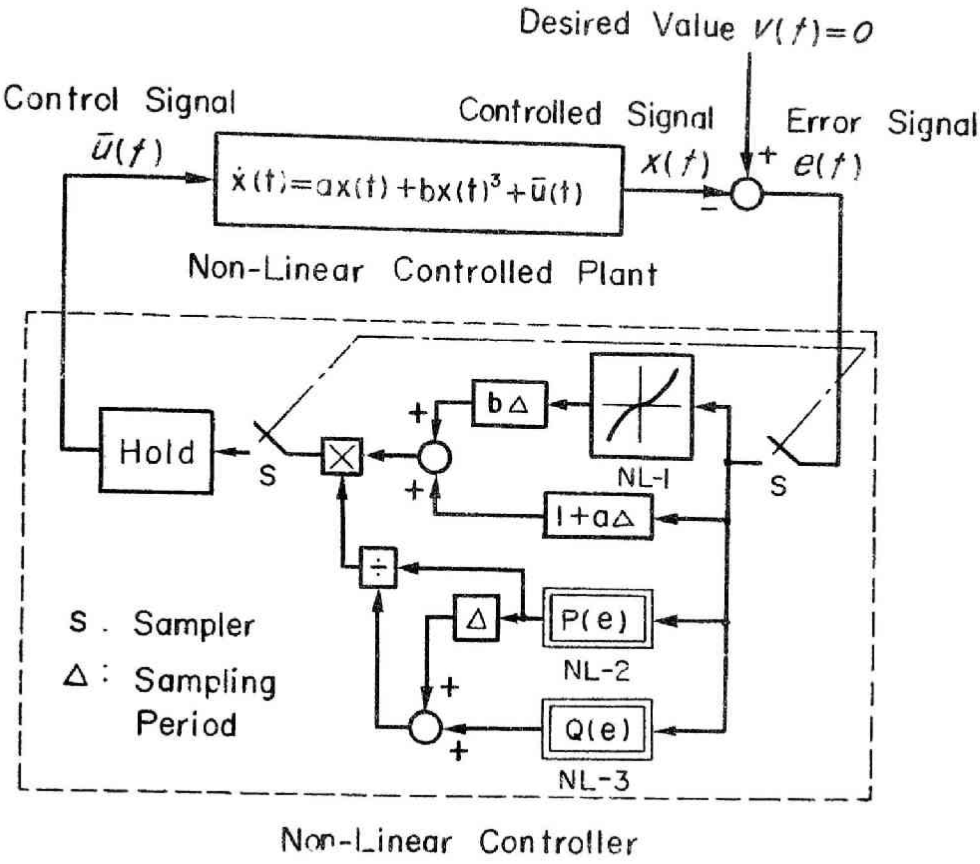
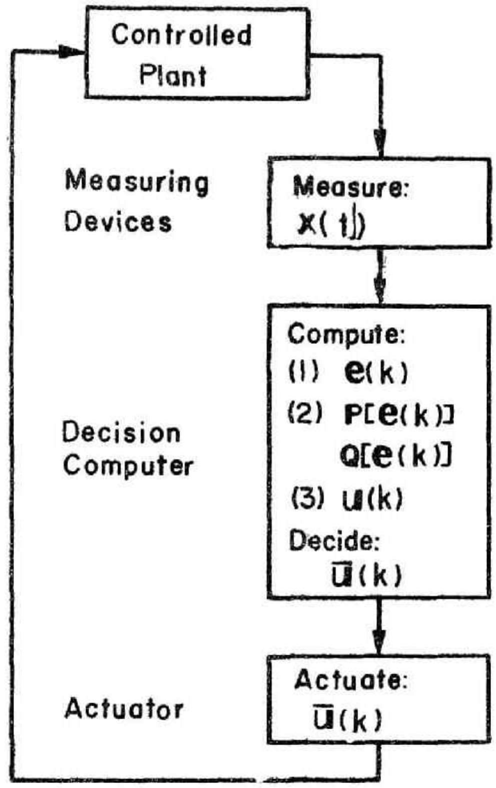



Fig. 6.1 Configuration of the non-linear sampled-data control system

well-trained designers. The only way of doing this is simulation studies, in which we can determine the appropriate form of weighting factors by the method of cut and try. Although many discussions are expected on the determination of weighting factors. Let us postpone these discussions later, and we shall consider the control characteristics of the designed system. Let us first consider the difference between two policies given by Eqs. (6.3-4) and (6.3-5). Since the control characteristics of the system is, in general, function of $P[e(k)]$ , $Q[e(k)]$ and $\Delta$ , then it is necessary to pre-assign them

Fig. 6.2

Simplified logic flow diagram of the controller



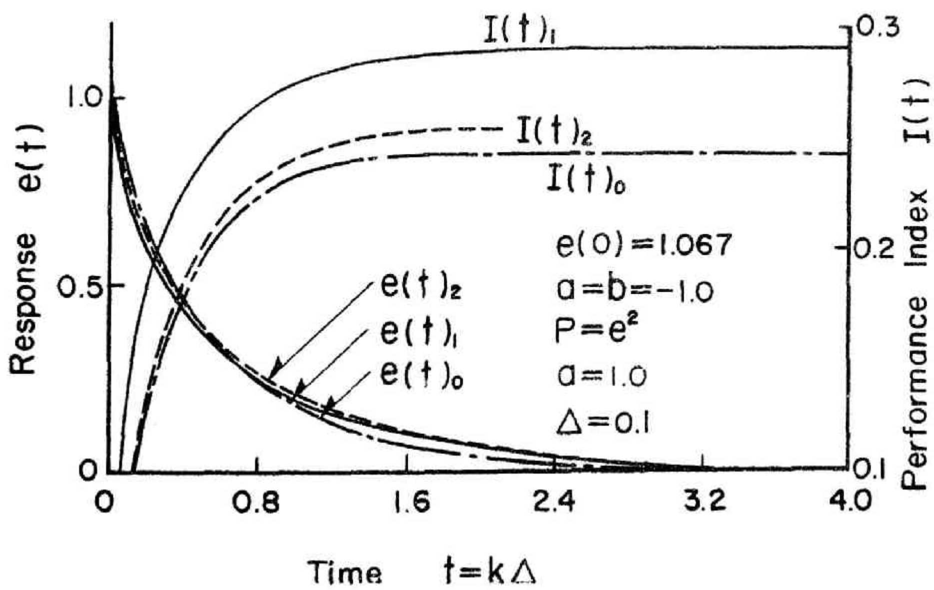




Fig. 6.3 Comparison of control characteristics

for comparing these two policies. Fig. 6.3 shows the graphical comparison where it is pre-assigned that $P=e(k)^2$ , $Q=1$ and $\Delta=0.1$ with system parameters, i.e., $a=b=-1$ , and $c=1.067$ . In Fig. 6.3, the line marked by $e(t)_i$ expresses the corresponding system response controlled by $\bar{u}(t)_i$ and the line marked by $I(t)_i$ shows the corresponding index of performance abided by $\bar{u}(t)_i$, Hence, in order to compare the control performance, the following quadratic form of index is introduced for simplicity;

$$I(k\Delta)_i=\sum_{j=0}^{k-1}\{e(j+1)^2+\bar{u}(j)_i^2\}\Delta . \qquad (6.3\text{-}7)$$

The minute solid lines show the response and the index of performance of the system which is designed by a near-optimum method[46] to minimize Eq. (6.3-7) with $k=40$.

On the other hand, for on-line computer utilization, one of the most important aspects of the method is the rapidity of computing time necessary for realization of control policy. Then it is interesting to compare the computing time for realizing the control policies $\bar{u}(k)_i$ ( $i=0, 1, 2$ ). The comparison is carried out by using a digital computer, NEAC-2101. The results are tabulated in Table 6.1.
Table 6.1 emphasizes us the rapidity of decision-making abided by the

| | Policies for Decision-making | | |
|---|---|---|---|
| | $u(k)_0$ | $u(k)_1$ | $u(k)_2$ |
| Average Time for One Decision-making | 0.5 min. | 1~2 sec. | 3~4 min. |

Table 6.1 Comparison of the computing time

policy, Eq. (6.3-4).

By considering the fact that an appropriate selection of weighting factors enables us to improve the control chracteristics of policy, Eq. (6.3-4), and that its rapidity of decision-making is very much desirable, it is expected that the further considerations on the control policy given by Eq. (7.3-4) provides us useful properties of the present concept. Fig. 6.4 illustrates the behavior of system responses affected by changing the initial condition. Fig. 6.4 points out the state adaptive characteristics of controller. That is, the figure shows the controller realizes the various phases of control action corresponding to the present state of the system. From the above mentioned state adaptive characteristics, it is evident that this type of controller has a considerable adaptability to a suddenly applied impulsive load disturbance at the output.

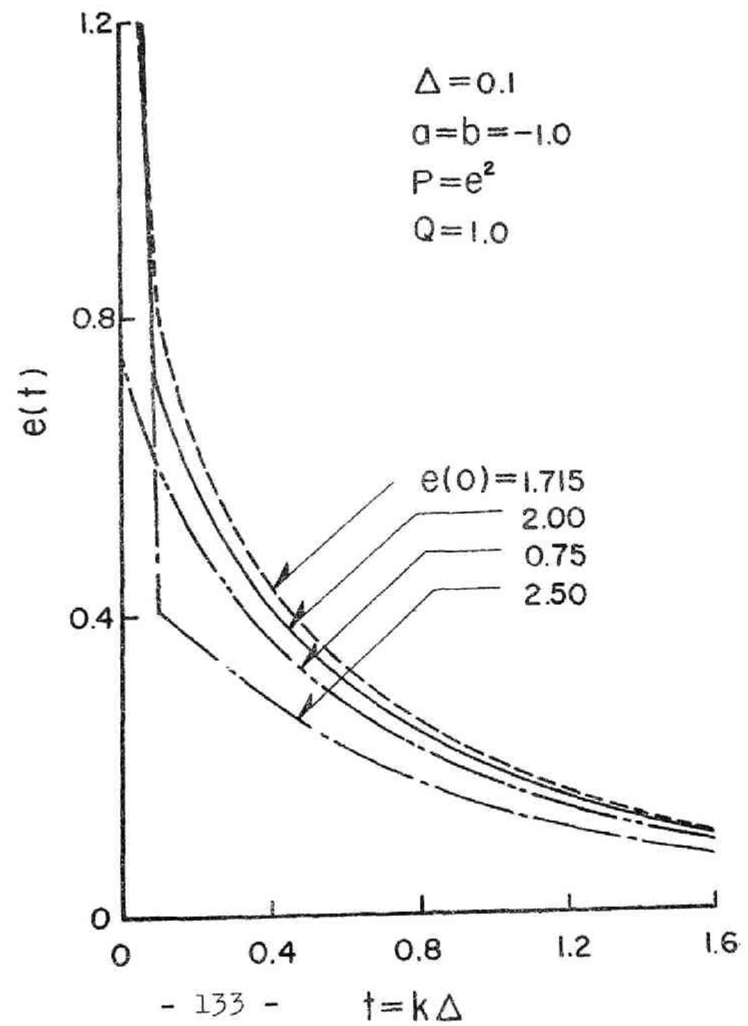



Fig. 6.4
Behavior of the system response affected by changing the initial condition

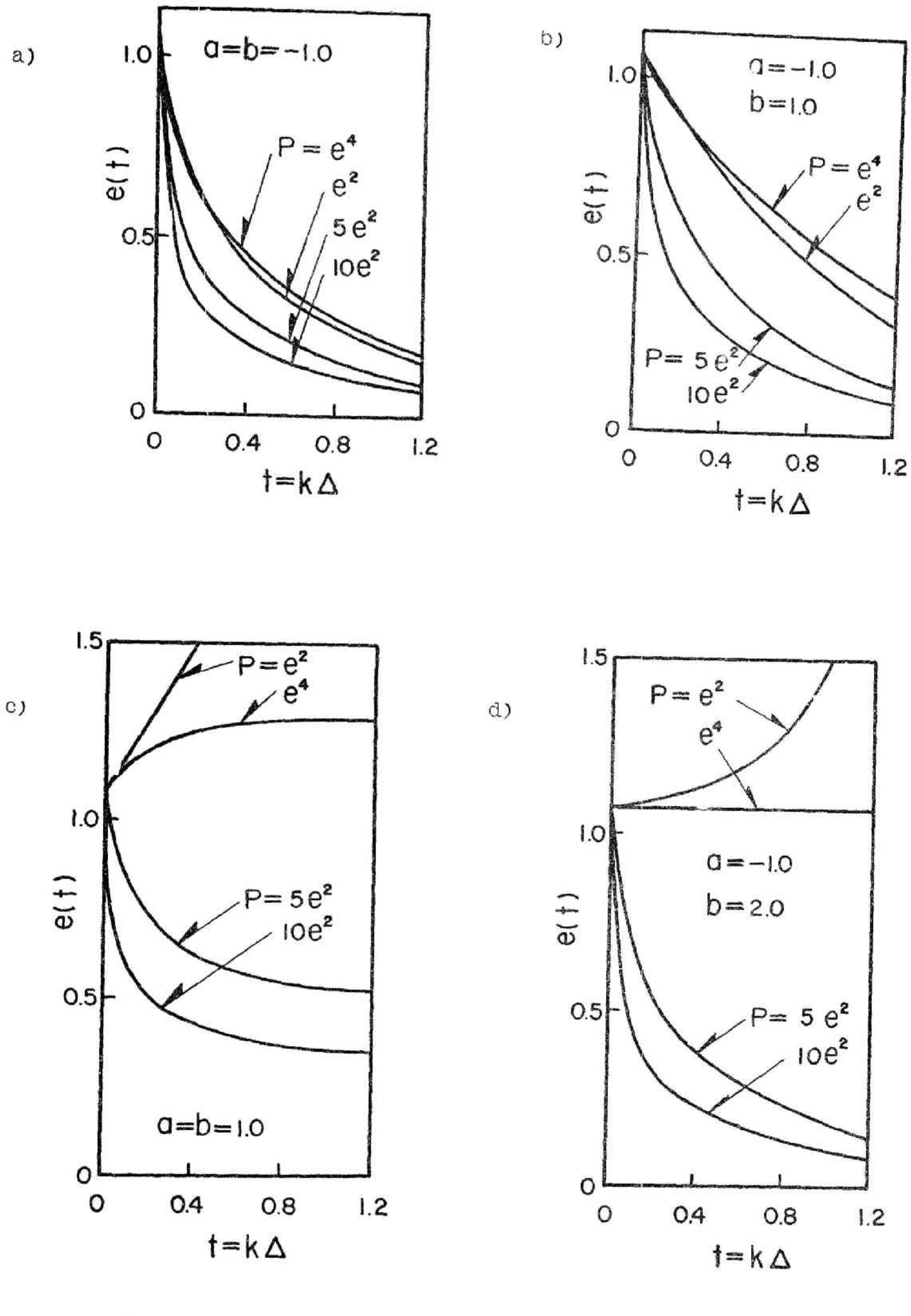



Fig. 6.5 Behavior of the system response affected by changing the form of a state adaptive weighting factor

Lastly, let us make a few discussions on the relation between system parameters and the form of state adaptive weighting factors. Fig. 6.5 illustrates the several examples of the relation where we set $Q=1$ for simplicity. It turns out from Fig. 6.5 that the effect of changing the form of $P$ on the stable system where $a=-1$, and $b=-1$ is not so remarkable than that on the unstable systems where $a=-1.0$ and $b=2.0$ or $a=b=1.0$ as respectively shown in Fig. 6.5 (c) and (d). Essentially speaking, since the characteristics of controller based upon Eq. (6.3-5) are qualitatively very much the same as these of controller cited above, then we stop the discussion on this example.

6.3-2 Example 2: Design of Non-Linear Control System with Bounded Control

In this example, from the view point of computer utilization several discussions are presented on the design problem of controller which controls the second order controlled system with both velocity and control saturation.

Statement of the Problem

Fig. 6.6 shows the system to be considered, of which the discrete

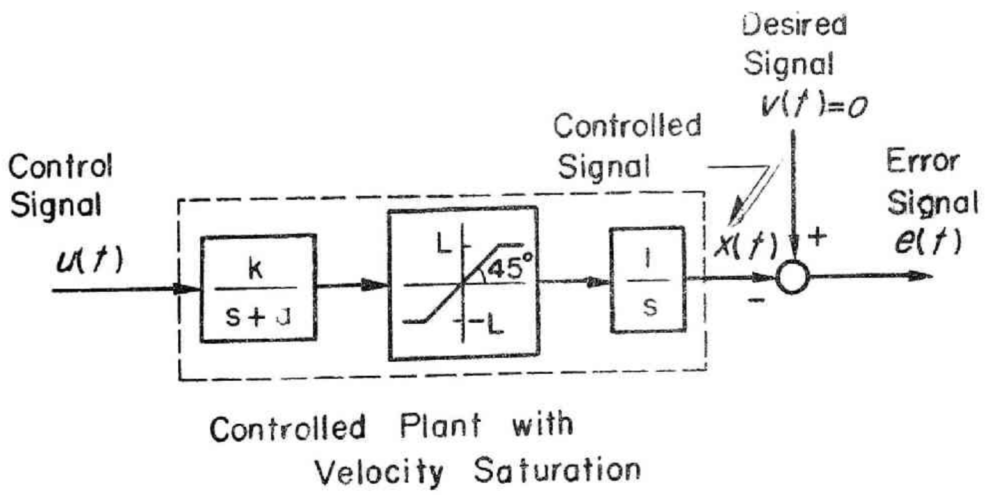



Fig. 6.6 Block diagram of the non-linear controlled system to be considered

presentation with respect to state variables of the error signal is given by

$$\left.\begin{aligned} e_1(k+1) &= e_1(k) + f(e_2(k))\Delta, \quad & e_1(0) &= c_1 \\ e_2(k+1) &= G(\Delta)\, e_2(k) - D(\Delta)\, u(k), \quad & e_2(0) &= c_2 \end{aligned}\right\}, \tag{6.3-8}$$

where $f(z)$ denotes the mathematical form of a saturable element defined by

$$\left.\begin{aligned} f(z) &= L & (z > L) \\ &= z & (-L \leq z \leq L) \\ &= -L & (z < -L) \end{aligned}\right\}. \tag{6.3-9}$$

In Eq. (6.3-8), $G(\Delta)$ and $D(\Delta)$ are given by

$$\left.\begin{aligned} G(\Delta) &= \exp(-a\Delta) \\ D(\Delta) &= b/a \cdot \{1 - \exp(-a\Delta)\} \end{aligned}\right\}, \tag{6.3-10}$$

and furthermore, the control variable $u(k)$ is subjected to the magnitude constraint

$$|u(k)| \leq M, \tag{6.3-11}$$

where $M$ is a pre-assigned constant. Since the corresponding state adaptive performance functional in this case becomes from Eq. (6.2-1) as

$$\begin{aligned} J_{pi}(k) &= {}^t e(k+i) P[e(k)]\, e(k+i) \\ &\quad + \sum_{j=0}^{i-1} Q(e_1(k), e_2(k))\, u(k+j)^2 \Delta, \\ &\qquad (i = 2, 3) \end{aligned} \tag{6.3-12}$$

where

$$P[e(k)] = \| P_{rs}(e_1(k), e_2(k)) \| \quad (r, s = 1, 2), \tag{6.3-13}$$

then the problem is to derive the control policy which minimizes Eq. (6.3-12) under the constraints of Eqs. (6.3-8) and (6.3-11).

Determination of the Control Policy

For simplicity, let us confine our considerations to the case where $i = 2$, $p_{11} = p(e_1(k))$, $P_{12} = P_{21} = P_{22} = 0$, and $Q = \beta$ (constant). Eq. (6.3-12) can thus be rewritten as

$$J_{p2} = p(e_1(k))\, e_1(k+2)^2 + Q\Delta\{u(k)^2 + u(k+1)^2\}. \tag{6.3-14}$$

By applying the optimization principle based upon Dynamic Programming, we get the following which is necessary condition for optimality;

$$Qu(k)-p(e_1(k))D(\triangle)f'(z)\{\triangle f(e_2(k+1))+e_1(k+1)\}=0, \tag{6.3-15}$$

where

$$\left.\begin{array}{ll} f'(z)=1 & (-L\leq z\leq L) \\ \quad\;\; =0 & (-L>z \ \text{ or } \ z>L) \end{array}\right\}, \tag{6.3-16}$$

and

$$z\equiv G(\triangle)e_2(k)-D(\triangle)u(k) \quad . \tag{6.3-17}$$

Since it is impossible to get the straightforward solution of $u(k)$ from Eqs. (6.3-15),(6.3-16) and (6.3-17), then we shall consider the sophisticated way of solution. If we suppose that $f'=1$, Eq. (6.3-15) gives us

$$u(k)=w(k)\equiv\frac{p(e_1(k))D(\triangle)}{p(e_1(k))D(\triangle)^2\triangle+Q}\{e_1(k)+f(e_2(k))\triangle+G(\triangle)\triangle e_2(k)\}. \tag{6.3-18}$$

On the other hand, since there is the magnitude constraint on $u(k)$, then the corresponding solution becomes

$$\left.\begin{array}{ll} u(k)=M & (w(k)>M) \\ \quad\;\; =w(k) & (-M\leq w(k)\leq M) \\ \quad\;\; =-M & (w(k)<-M) \end{array}\right\}. \tag{6.3-19}$$

However, it is necessary to check the supposition cited above. The checking can be done as follows: If the solution Eq. (6.3-19) satisfies the relation given by

$$G(\triangle)e_2(k)-L\leq D(\triangle)\,u(k)\leq G(\triangle)e_2(k)+L, \tag{6.3-20}$$

which is derived from Eqs. (6.3-16) and (6.3-17), then Eq. (6.3-19) becomes the desired solution.

On the other hand, if the solution Eq. (6.3-19) does not satisfy Eq. (6.3-20), the desired solution Eq. (6.3-20) is determined from Eqs. (6.3-15) and (6.3-20) as follows:

By letting $f'=0$ in Eq. (6.3-15), we get the solution $u(k)=0$. However, this solution yields the contradiction that it satisfies Eq. (6.3-20). Then it turns out that the desired solution should be expressed by the boundary values of Eq. (6.3-20).

$$\left.\begin{aligned}\bar{u}(k) &= \{G(\triangle)\,e_2(k)+L\}/D(\triangle), \quad (D(\triangle)\,u(k) > G(\triangle)\,e_2(k)+L) \\ &= \{G(\triangle)\,e_2(k)-L\}/D(\triangle), \quad (D(\triangle)\,u(k) < G(\triangle)\,e_2(k)-L)\end{aligned}\right\} . \qquad (6.3\text{-}21)$$

Then, we get the desired control policy as follows:

$$\left.\begin{aligned}\bar{u}(k) &= M & &(w(k) > M) \\ &= w(k) & &(-M \le w(k) \le M) \\ &= -M & &(w(k) < -M)\end{aligned}\right\} , \qquad (6.3\text{-}22)_1$$

$$(G(\triangle)\,e_2(k)) - L \le D(\triangle)\,\bar{u}(k) \le G(\triangle)\,e_2(k) + L)$$

$$\bar{u}(k) = \{G(\triangle)\,e_2(k)+L\}/D(\triangle), \quad (D(\triangle)\,\bar{u}(k) > G(\triangle)\,e_2(k)+L) \qquad (6.3\text{-}22)_2$$

$$\bar{u}(k) = \{G(\triangle)\,e_2(k)-L\}/D(\triangle), \quad (D(\triangle)\,\bar{u}(k) < G(\triangle)\,e_2(k)-L) \qquad (6.3\text{-}22)_3$$

The logic flow chart for realization of the control policy is shown in Fig. 6.7.

Discussion on Control Characteristics

In order to illustrate the characteristics of controller, we simulate the control policy Eq. (6.3-22), by a digital computer. The results are shown in Figs. 6.8, 6.9, 6.10 and 6.11. That is, Fig. 6.8 shows an example of the relation between the initial conditions and the system responses, in which we set $p = e_1(k)^2$, $Q = 1.0$, $L = 1.0$, and $M = \infty$. It is noted that the controller changes its phase relying on state variables of the system. Fig. 6.9 illustrates the trajectories of the system with bounded control where $M = 2.0$. The effects from changing the value of $\beta$ on the response is also illustrated in Fig. 6.9. Fig. 6.10 shows the behaviors of trajectories affected by changing the value of $\beta$, in which $M = 1.0$ or $2.0$. The solid curves

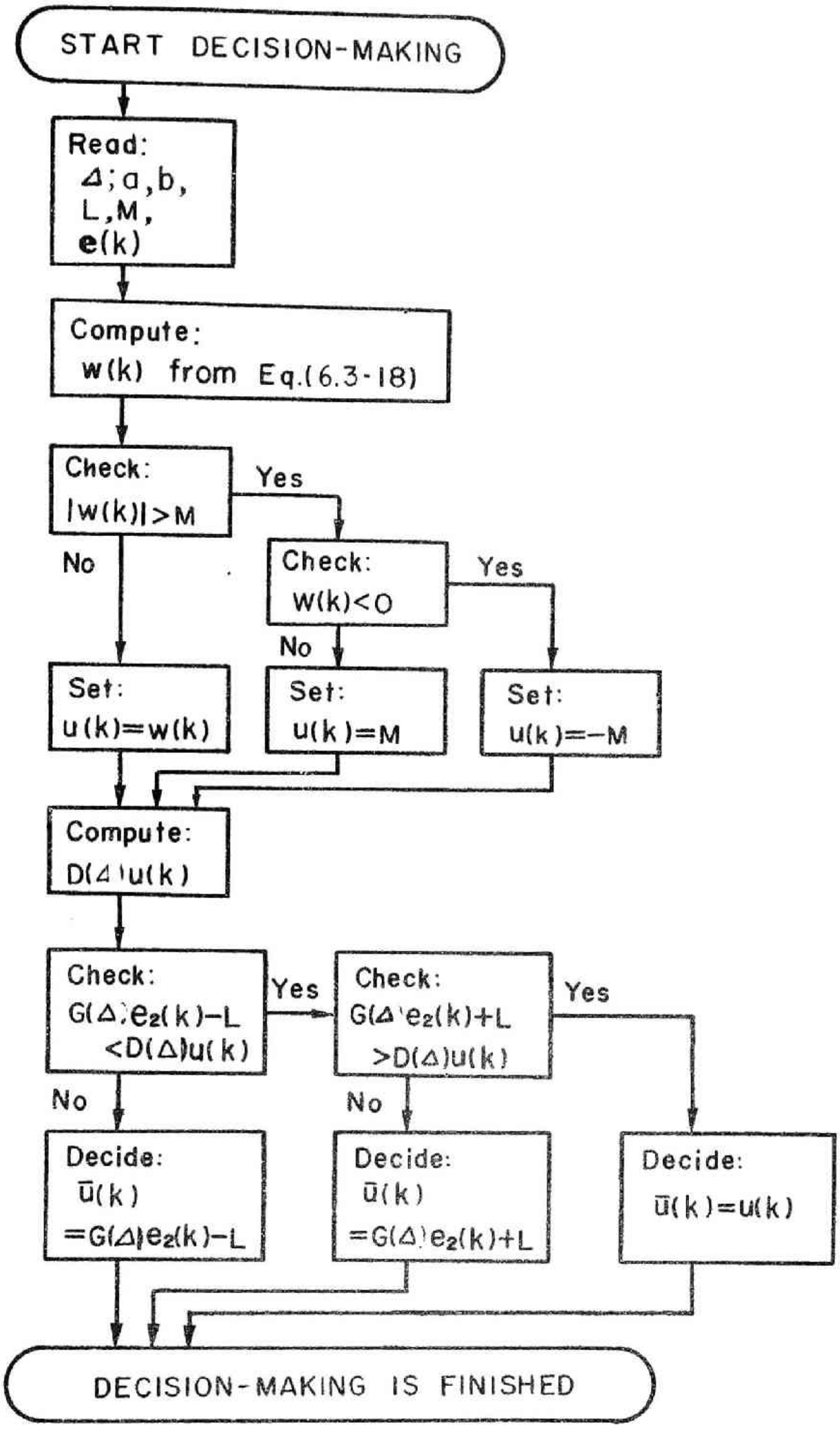



Fig. 6.7 Logic flow chart for realization of the control policy

are corresponding to the responses of the system whose controlled element is assumed to be linear. The broken lines are the responses of the system with bounded control; $|u(k)| \leq 2.0$. An Example showing the

Fig. 6.8

Illustrative example of the relation between the initial condition and the response trajectory ($M = \infty$)

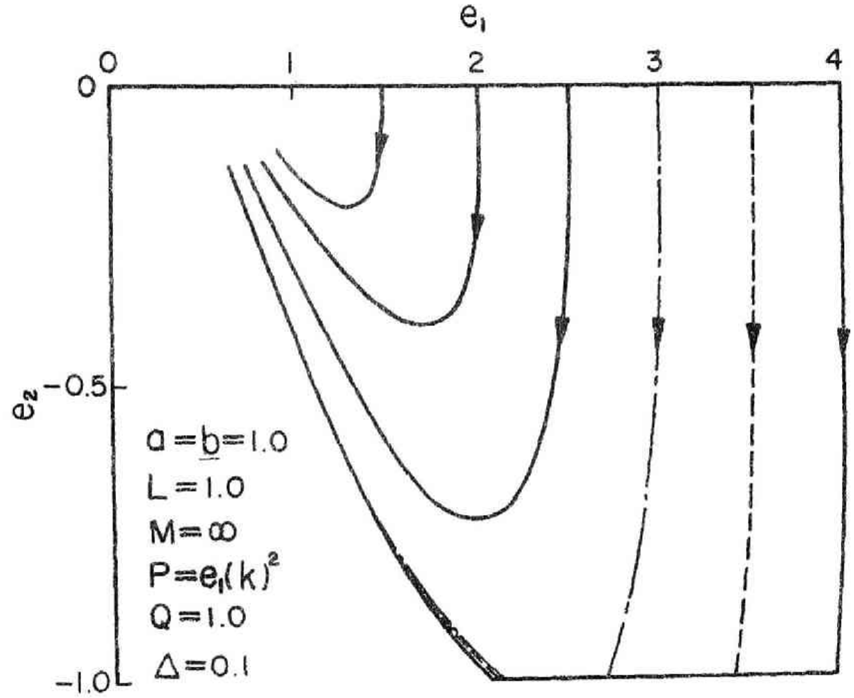
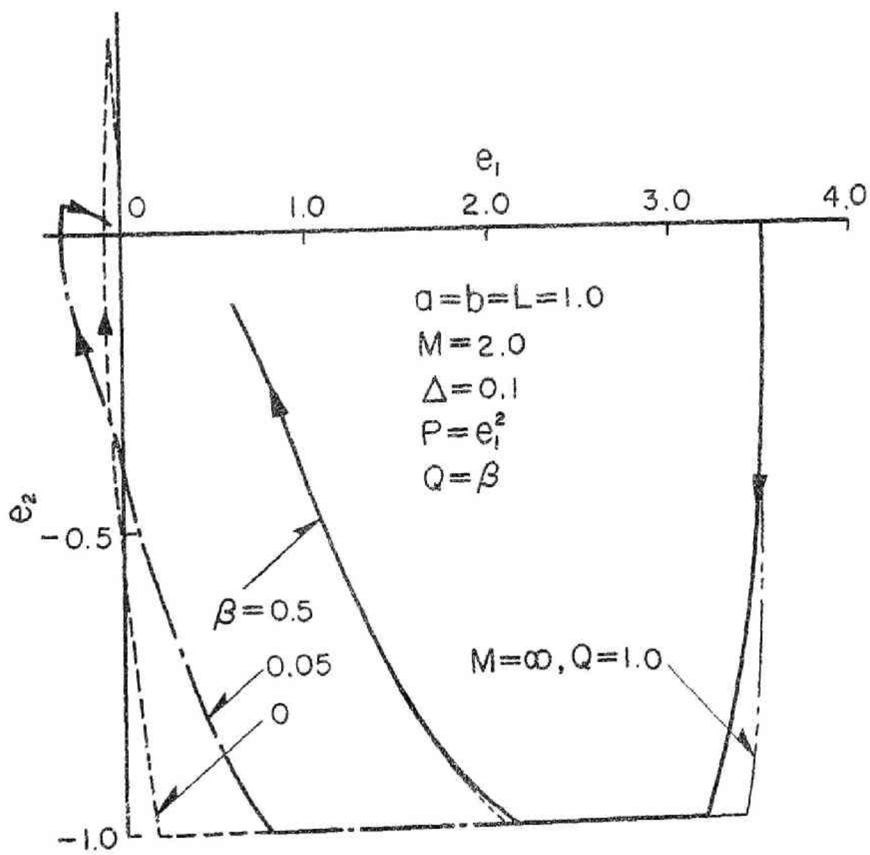



Fig. 6.9

Behavior of the trajectory affected by changing the value of both $\beta$ and $M$



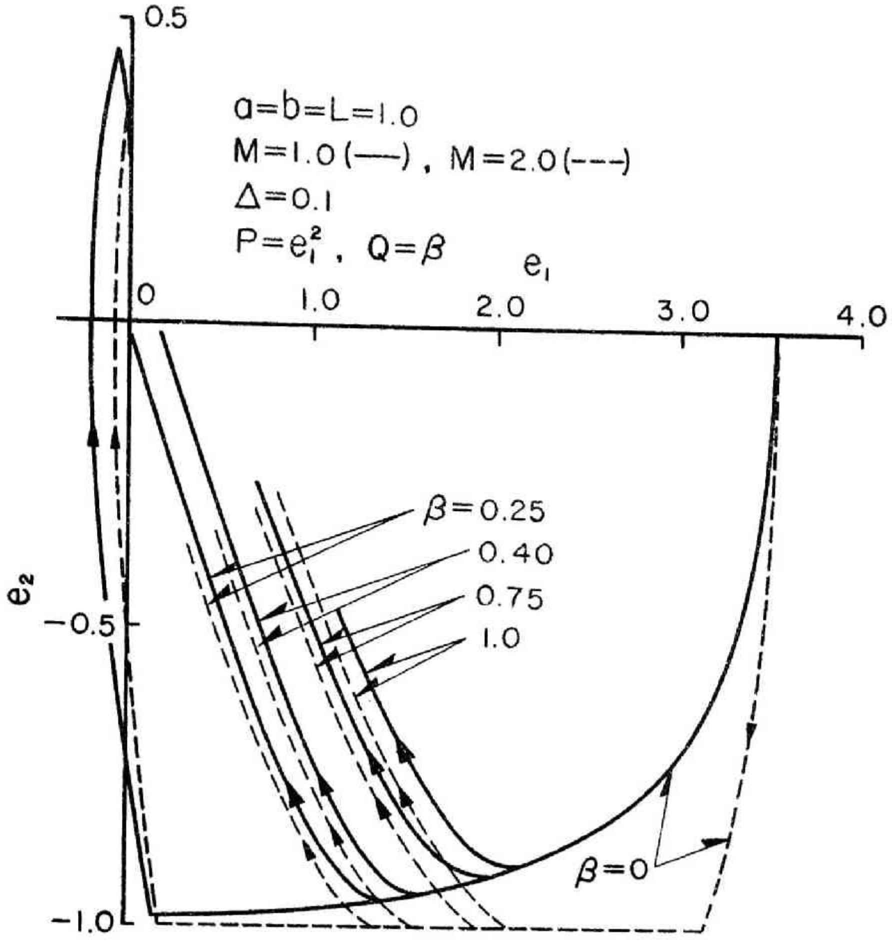



Fig. 6.10 Behavior of the trajectory affected by changing the value of both $\beta$ and $M$

effect from changing state adaptive weighting factors $P$ and $Q$ on the control characteristics is illustrated in Fig. 6.11, where $P = e_1(k)^2$ or $e_1(k)^2+1$ and $Q=\beta$. It is pointed out by simulation studies that by assuming the proper form of weighting factors, the controller based upon Eq. (6.3-22) provides us various types of non-linear control actions. Although it is interesting to consider the relation between the form of weighting factors and the total cost during the operation time interval, it seems to be beyond the scope of this example. The very point which is emphasized in two examples cited above is that the concept of sub-interval optimization technique co-operated with state adaptive performance criterion provides us the possibility of designing non-linear control systems through a direct digital simulatation study.

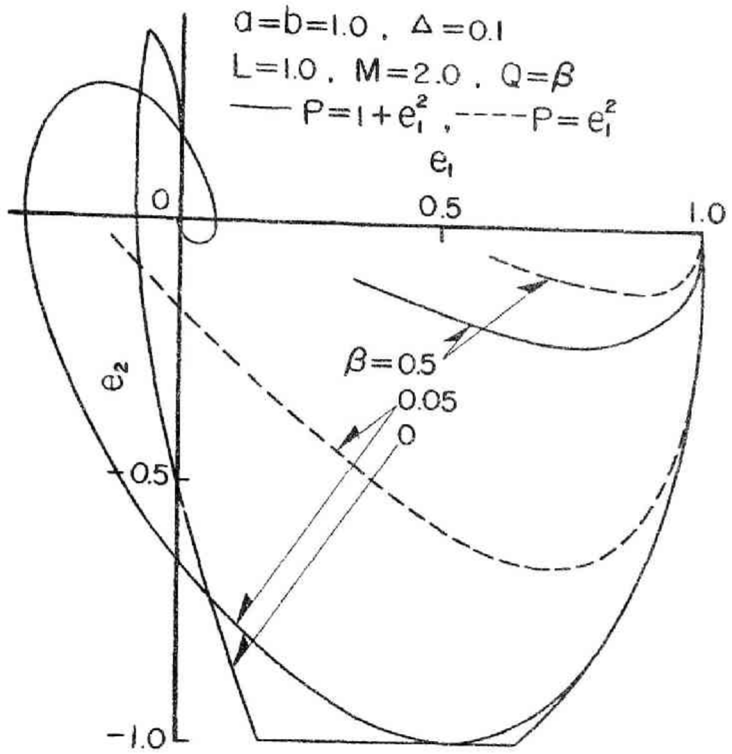



Fig. 6.11 Behavior of the trajectory affected by changing the form of state adaptive weighting factors

## 6.4 Further Discussions

In the previous section, two simple examples are presented to investigate several typical features of the controller which is designed by using the present concept. In this section, the author arranges interesting aspects of the present approach and discusses further problems associated with the present concept. The interesting points of the present approach may be summarized as follows:

(1) The computing time in optimization (decision-making) is short because the performance functional of simple structure is defined in Eq. (6.2-1).

(2) Since an integer $i$ expressing the duration of sub-interval is pre-assigned as $i = n$ or $n+1$, then a gigantic scale digital computer is not necessary to decide the control policy even though a controlled system becomes high order.

(3) Design specifications concerning nonlinearities in system dynamics and magnitude constraints on control variables are directly used, without assigning any penalty in performance functional, to decide the control policy.

On the other hand, all the discussions cited above are developed under the assumption that the proper forms of state adaptive weighting factors have been pre-assigned. For practical application of the method, however, the most important point is the determination of their functional forms so as to satisfy the given performance specifications. As the author has already pointed out in 6.1, this is a difficult problem of same level to the choice of performance functional which we encounter in applying the modern control theories to practical design problems. Then no direct mathematical approach seems to be possible to this problem.

From the view point of on-line computer utilization, however, the circumstance is very much altered. That is, we can determine an appropriate functional form through the cut and try method using a digital computer. The fundamental procedure is as follows:

(i) Assume a physically meaningful functional form of state adaptive weighting factor, decide the control policy, and calculate the corresponding response.

(ii) If the performance specifications are given as the necessary conditions with respect to the system response trajectory, then plot the computed response and adjust the functional form by the method of cut and try until the computed response becomes to satisfy the performance specifications. On the contrary, if the control performance is measured by the total cost expended during the operation interval, then calculate the corresponding total cost with respect to the computed system response, and find the optimum functional form so as to minimize the computed total cost.

Generally speaking, the procedure for finding the optimum functional form, through the method of cut and try consumes much of time. However, if we assume the nominal functional form as a finite sum of such

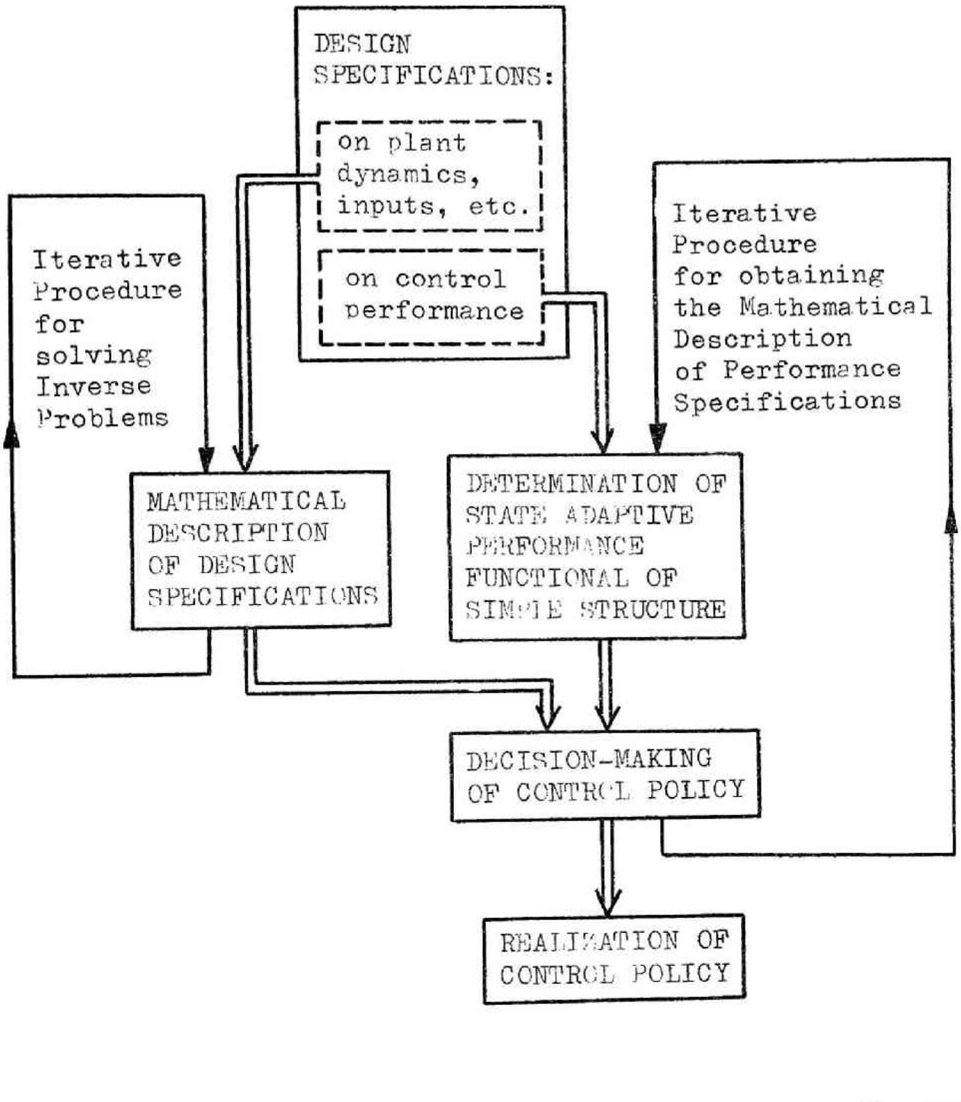



Fig. 6.12 Flow diagram of the design procedure proposed by the author

orthogonal polynomials as Hermite, Legendre, and Tschebyscheff polynominals,[14] it is expected that the determination of a suitable function expressing a state adaptive weighting factor seems to be not so tedious task because the computing time for decision-making is very much short. Flow diagram for determination of functional forms is illustrated in Fig. 6.12. For comparison, the author shows in Fig. 6.13 the flow diagram based upon modern optimization theories. The remarkable differences between two flow diagrams are as follows: The design specifications are separate into these related with the

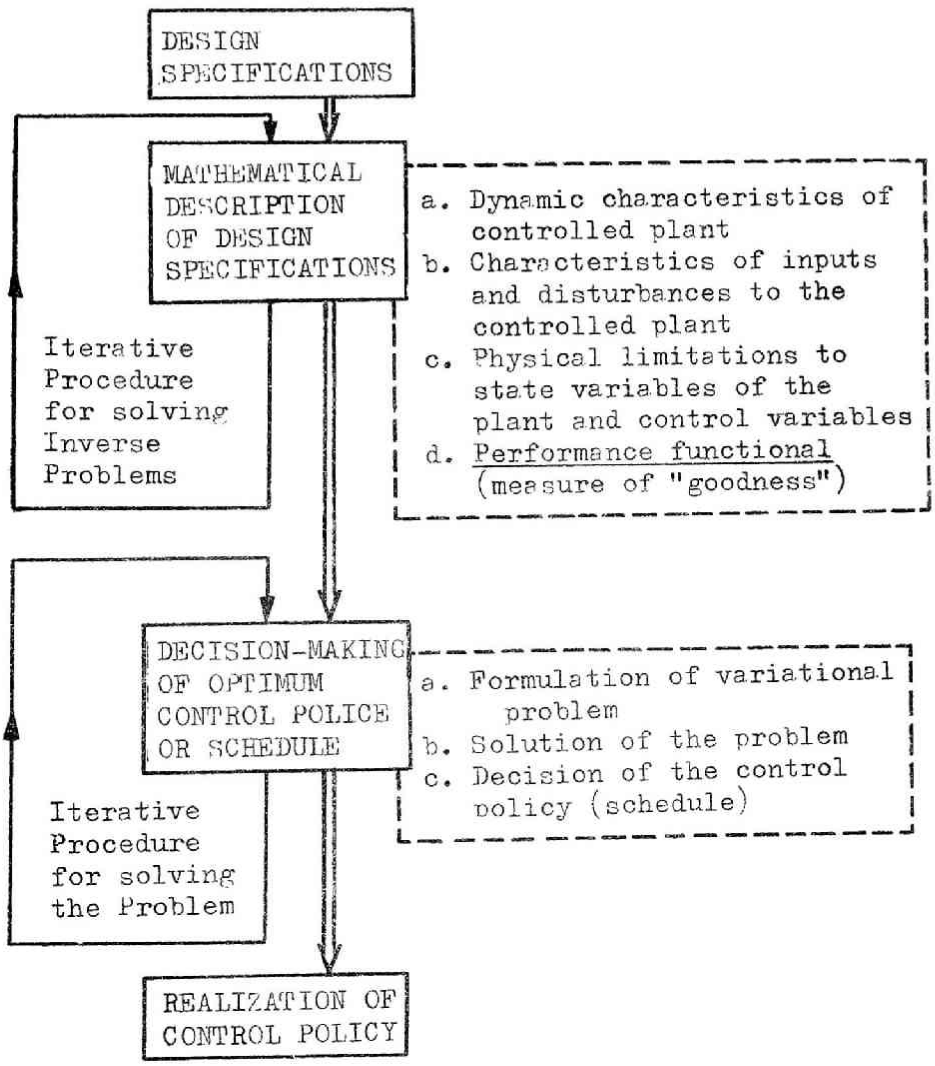



Fig. 6.13 Flow diagram of the design procedure based upon modern optimization theories

description of the physical systems, and these related with the measure of control performance. Therefore, identification procedures are separately performed in Fig. 6.12. This configuration of flow diagram is a straight forward result from the concept stated in 6.2.

The merit of this configuration will be closed up in the case where the performance specifications should be changed in accordance with the external information. In other words, such circumstances are frequently encountered in controls of real plants in industries, to which the concept of "hierarchy control"(12) or "multi-level control"(36) is introduced.

In hierarchy control, the following form of state adaptive performance functional should be introduced instead of Eq. (6.2-1);

$$J_{pi}(k)=F({}^{t}e(k+i)\,p\,[e(k),\ y(k)\ ;k]\,e(k+i)\ ;k) + \sum_{j=0}^{i-1} {}^{t}u(k+j)\,Q\,[e(k),y(k)\,;\,k]\,u\,(k+j)\triangle\ , \qquad (6.4\text{-}1)$$

where $y(k)$ denotes the present informations from the related systems, and $F$ is a scalar function of both the error-cost ${}^{t}e(k+i)pe(k+i)$, and time $k$. The remarkable points in Eq. (6.4-1) are that the state adaptive weighting factors $P$ and $Q$ become functions of $e(k)$, $y(k)$ and $k$.

Furthermore, if we should add to the decision-making computer the learning algorithm which shortens the computing time for determination of state adaptive weighting factor by considering the experience of past trials, then the very useful algorithm of computer optimization for realization of "hierarchy control" in industries could be obtained.

Although the physical interpretation of Eq. (6.4-1) should be necessary, since the further description is beyond the scope of the present investigation, then it is left for the subject of further development.

## 6.5 Concluding Remarks

In this chapter, from the view point of practical control engineers the fundamental concept of an on-line computer optimization approach to non-linear control systems is described. Two examples are presented to illustrate several interesting points of present concept and its applicability to the design problem of non-linear control systems. It turns out that according to the present concept the computing time for decision-making of control policies is very much shortened, and that by selecting a suitable form of weighting factors, we can obtain the flexible controller which realizes various types of control phases. It is also emphasized that the present concept is expected to provide a practical direct design procedure in the case where all the design specifications are not described in precisely mathematical forms.

## Appendix A : Derivation of the Switching Function of Restricted-Optimal Control for a Double-Integrator Plant

In this appendix, we shall derive the switching function of restricted-optimal control for a double-integrator plant where both system parameters $a$ and $b$ in Eq. (1.2-4) are equal to zero. Derivations for the other cases can be performed by a similar procedure.

Substituting $a=b=0$ in Eq. (1.2-4) and applying a similar method based on Dynamic Programming as stated in 1.2 to the minimization procedure, we get the following set of non-linear simultaneous differential equations corresponding to Eq. (1.2-19) with initial conditions;

$$\left.\begin{array}{ll} k_0'(\tau)=-k_2(\tau)\,kL\ \mathrm{sgn}\,[z(\tau)], & k_0(0)=0 \\ k_1'(\tau)=-k_3(\tau)\,kL\ \mathrm{sgn}\,[z(\tau)], & k_1(0)=0 \\ k_2'(\tau)=k_1(\tau)-2k_5(\tau)\,kL\ \mathrm{sgn}\,[z(\tau)], & k_2(0)=0 \\ k_3'(\tau)=2k_4(\tau), & k_3(0)=0 \\ k_4'(\tau)=1, & k_4(0)=0 \\ k_5'(\tau)=k_3(\tau), & k_5(0)=0 \end{array}\right\}. \qquad \text{(A-1)}$$

We assume that the final instant of control operation $T$ satisfies the assumption realizing the restricted-optimal control and that the instant of switching in the reversed time by $\tau=\tau_S$, which is equivalent to $t=t_S$ in the real time. Furthermore, assuming that the sign of relay output at $\tau=0$ is negative, we get the following pair of equations which is necessary for the derivation of the switching function;

$$\left.\begin{array}{ll} k_1'^{+}(\tau)=kL\,k_3(\tau), & k_1^{+}(0)=0 \\ k_2'^{+}(\tau)=k_1^{+}(\tau)+2kL\,k_5(\tau), & k_2^{+}(0)=0 \end{array}\right\}, \qquad \text{(A-2)}_1$$

and

$$\left.\begin{array}{ll} k_1'^-(\tau) = -kL\,k_3(\tau)\ , & k_1^-(\tau_s) = k_1^+(\tau_s) \\ k_2'^-(\tau) = k_1^-(\tau) - 2kL\,k_5(\tau), & k_2^-(\tau_s) = k_2^+(\tau_s) \end{array}\right\} \quad , \tag{A-2$_2$}$$

where

$$k_3(\tau) = \tau^2 \qquad \text{and} \qquad k_5(\tau) = \tau^3/3\ . \tag{A-3}$$

In Eq. $(A\text{-}2)_2$, the initial conditions are derived by assuming the continuity of the solution surface for the partial differential equation on the switching boundary. From Eq. $(A\text{-}2)_1$ and $(A\text{-}2)_2$, the coefficients $k_2^+(\tau)$ and $k_2^-(\tau)$ are obtained as

$$k_2^+(\tau) = kL/4 \cdot \tau^4 \tag{A-4$_1$}$$

and

$$\begin{aligned} k_2^-(\tau) &= -kL\,(\tau^4/4 - 2\tau_s^3\tau/3 + \tau_s^4/6) \\ &= k_2^+(\tau) - kL\,(\tau^4/2 - 2\tau_s^3\tau/3 + \tau_s^4/6)\ . \end{aligned} \tag{A-4$_2$}$$

By considering the assumption on the switching time mentioned above, we can derive the following relations from Eq. (1.2-21) as;

$$z^+(\tau) \equiv k_2^+(\tau) + k_3(\tau)e_1 + 2k_5(\tau)e_2 < 0\ , \qquad (0 \leq \tau < \tau_s) \tag{A-5$_1$}$$

and

$$z^-(\tau) \equiv k_2^-(\tau) + k_3(\tau)e_1 + 2k_5(\tau)e_2 > 0, \qquad (\tau_s < \tau \leq T) \tag{A-5$_2$}$$

By substituting Eq. $(A\text{-}5)_2$ into the function $z(\tau)$ we get the relation as

$$z^-(\tau) = z^+(\tau) - kL\,(\tau^4/2 - 2\tau_s^3\tau/3 + \tau_s^4/6)\ . \tag{A-6}$$

By considering facts that the second term in the right hand side of Eq. (A-6) is a non-positive function with respect to $\tau \geq \tau_S$ and that the continuity of the solution surface on the switching boundary is assumed, we can derive the following necessary condition which must be satisfied at the time of switching, i.e., $\tau = \tau_S$;

$$z^{+}(\tau)\,\Big|_{\tau=\tau_s-0} \equiv k_2^{+}(\tau)\,\Big|_{\tau=\tau_s-0} + k_3(\tau_s)e_1 + 2k_5(\tau_s)e_2 = 0\,. \tag{A-7}$$

By substituting Eqs. (A-3) and $(A-4)_1$ into Eq. (A-7), we have the switching function of restricted-optimal control for the positive relay output at $t=0$ as follows:

$$z^{+}_{\mathrm{RES}} \equiv \tau^2/4 + 2\tau/3 \cdot e_2/kL + e_1/kL\,. \tag{A-8}$$

On the other hand, the switching function for the negative relay output at $t=0$ can be derived by a similar way as

$$z^{-}_{\mathrm{RES}} \equiv -\tau^2/4 + 2\tau/3 \cdot e_2/kL + e_1/kL\,. \tag{A-9}$$

Eqs. (A-8) and (A-9) are the final results of this appendix.

## Appendix B : Derivation of Eqs. (2.4-7) and (2.4-8) from Eq. (2.4-4)

In this appendix, the author shows that the switching functions of restricted-optimal control, Eqs. (2.4-7) and (2.4-8), can be derived from the one, Eq. (2.4-4), as the limit of tending $a$ or $b$, or both to zero.

First we consider the derivation of Eq. (2.4-7) from Eq. (2.4-4). In Eq. (2.4-4), by taking the limit of tending $\beta$ to zero, we get the relation expressing the switching lines as

$$\lim_{\beta\to 0} z^{\pm}_{\mathrm{RES}}(\tau) \equiv \lim_{\beta\to 0} k_2^{\pm}(\tau) + \lim_{\beta\to 0} k_3(\tau)x_1 + \lim_{\beta\to 0} 2k_5(\tau)x_2 = 0\,. \tag{B-1}$$

On the other hand, the limiting forms of Eqs. (2.4-2), (2.4-3) and (2.4-5) when $\beta$ tends to zero, are respectively described as

$$\lim_{\beta\to 0} k_3(\tau) = 2\{1-\exp(-\tau)\}, \tag{B-2}$$

$$\lim_{\beta\to 0} k_5(\tau) = \{1-\exp(-\tau)\}^2 + \mu\exp(-2\tau)\,, \tag{B-3}$$

$$\lim_{\beta\to 0} k_2^{\pm}(\tau) = \mp 2\{1-\exp(-\tau)\}\,[\{\tau+\exp(-\tau)-1\}+\mu\exp(-\tau)]\,. \tag{B-4}$$

By using Eqs. (B-2), (B-3) and (B-4), Eq. (B-1) is rewritten as

$$\{1-\exp(-\tau)\}\,(\mp[\{\tau+\exp(-\tau)-1\}+\mu\exp(-\tau)]+x_1$$
$$+[\{1-\exp(-\tau)\}+\mu\exp(-2\tau)/\{1-\exp(-\tau)\}]\,x_2\,)=0 \qquad \text{(B-5)}$$

In Eq. (B-5), since the term $\{1-\exp(-\tau)\}$ is independent of the system states $x_1$ and $x_2$, and it never becomes zero except the case where $\tau=0$, then the switching function becomes

$$\mp[\{\tau+\exp(-\tau)-1\}+\mu\exp(-\tau)]+x_1$$
$$+[\{1-\exp(-\tau)\}+\mu\exp(-2\tau)/\{1-\exp(-\tau)\}]x_2, \qquad \text{(B-6)}$$

where $\tau$, $x_1$ and $x_2$ are normalized variables. By changing the normalized variables into the original variables in Eq. (B-6) and taking the relation $\hat{\mu}\equiv a^2\mu$ into account, we obtain the switching function Eq. (2.4-7), for the case where $a\neq 0$ and $b=0$. Second, let us derive Eq. (2.4-8) from Eq. (2.4-7).

Since Eq. (2.4-7) is rewritten as

$$a^2\,(\mp[a\tau+\exp(-a\tau)-1]/a^2+\mu\exp(-a\tau)]+x_1/kL$$
$$+[\{1-\exp(-a\tau)\}/a+a\mu\exp(-a\tau)/\{1-\exp(-a\tau)\}]\,x_2/kL\,), \qquad \text{(B-7)}$$

the limiting form of the switching function with respect to $a$ is expressed as

$$\lim_{a\to 0}\,(\mp\{[a\tau+\exp(-a\tau)-1]/a^2+\mu\exp(-a\tau)\}+x_1/kL$$
$$+[\{1-\exp(-a\tau)\}/a+a\mu\exp(-a\tau)/\{1-\exp(-a\tau)\}]\,x_2/kL\,). \qquad \text{(B-8)}$$

By performing the calculation of Eq. (B-8), the switching function, Eq. (2.4-8), of restricted-optimal control for the double-integrator plant is obtained.

## Appendix C : Derivation of Eq. (4.3-1) from the Viewpoint of Dynamic Programming

We consider an arbitrary time $t = t\ (0 \leq t \leq T)$. From Eq. (4.2-2), the performance functional to be minimized during the time interval left before the final time $T$, namely during the rest of control interval $[\,t,\ T\,]$ is

$$I(e(t),\, t\,;\, u\,(t')) = \varepsilon \left\{ \int_t^T [\,{}^t e(\rho) Q(\rho)\, e(\rho) + {}^t u(\rho)\, R(\rho)\, u(\rho)\,]\, d\rho \right\}, \qquad (t \leq t' \leq T)\,. \tag{C-1}$$

The minimum performance functional with respect to Eq. (C-1) is, therefore, defined as

$$\phi(e\,;\,t) = \min_{u} \varepsilon \left\{ \int_t^T [\,{}^t e Q e + {}^t u R u\,]\, d\rho \right\}. \tag{C-2}$$

Taking up the interative structure and computational scheme of R. Bellman's Dynamic Programming,[3] Eq. (C-2) yields

$$\begin{aligned}\phi\,(e\,;\,t) &= \min_{u} \Big( \varepsilon \Big\{ \int_t^{t+\Delta t} [\,{}^t eQe + {}^t uRu\,]\, d\rho \Big\} \\ &\qquad + \varepsilon \Big\{ \int_{t+\Delta t}^{T} [\,{}^t eQe + {}^t uRu\,]\, d\rho \Big\} \Big) \\ &= \min_{u} \big( [\,{}^t eQe + {}^t uRu\,]\, \Delta t + \varepsilon_{\Delta e} \{ \phi\,(e+\Delta e\,;\, t+\Delta t) \} \big)\,,\end{aligned} \tag{C-3}$$

where $\Delta t$ is a small time interval and the expression $\varepsilon_{\Delta e}$ means the ensemble average of what appears to its right with respect to the Gaussian variate $\Delta e$ .

From Eq. (3.2-4), on the other hand, the Gaussian variate $\Delta e$ for a small time interval $\Delta t$ can be described by

$$\Delta e(t) = A\,(t)\; e\,(t)\; \Delta t - D(t)\; u(t)\; \Delta t + \Xi(t)\; \Delta t\,. \tag{C-4}$$

In Eq. (C-4), $\Xi$ is a Gaussian white noise process with mean $m$ and variance-covariance $\Sigma$ . Then, the statistical properties of the Gaussian variate $\Delta e\,(t)$ is stated as follows: Its mean value and variance-covariance are $(Ae - Du + m)\Delta t$ and $\Sigma \Delta t + o(\,\Delta t\,)$ respectively, and for non-overlapping time intervals, $(t, t+\Delta t)$, $(t', t'+\Delta t')$, etc., the corresponding variates, $\Delta e(t)$, $\Delta e(t')$, etc., are statistically independent.

Therefore, using Taylor expansion formula, we can rewrite the term $\phi(e+\Delta e\ ; t+\Delta t)$ as follows:

$$\begin{aligned}\phi(e+\Delta e\,; t+\Delta t) &= \phi(e, t) + {}^t\Delta e\frac{\partial\phi}{\partial e} + \Delta t\frac{\partial\phi}{\partial t} \\ &+ \Delta t\,{}^t\Delta e\frac{\partial^2\phi}{\partial t\,\partial e} + \frac{1}{2!}\,{}^t\left[\frac{\partial}{\partial e}\right]\Delta e\,{}^t\Delta e\left[\frac{\partial}{\partial e}\right]\phi \\ &+ o(\Delta t)\end{aligned} \qquad \text{(C-5)}$$

Taking the average of the both side of Eq. (C-5) with respect to the Gaussian variate $\Delta e$, the second term, $\varepsilon_{\Delta e}\{\phi(e+\Delta e, t+\Delta t)\}$ in Eq. (C-3) is calculated as

$$\begin{aligned}\varepsilon_{\Delta e}\{\phi(e+\Delta e\,; t+\Delta t)\} &= \phi(e, t) + \Delta t\,{}^t(\mathcal{A}e - \mathcal{D}u + m)\frac{\partial\phi}{\partial e} \\ &+ \Delta t\frac{\partial\phi}{\partial t} + \frac{1}{2}\,{}^t\left[\frac{\partial}{\partial e}\right]\Sigma\Delta t\left[\frac{\partial}{\partial e}\right]\phi \\ &+ o(\Delta t).\end{aligned} \qquad \text{(C-6)}$$

Then, substituting Eq. (C-6) into Eq. (C-3) and allowing the small time interval $\Delta t$ to tend to zero. We can derive a partial differential equation as follows:

$$\begin{aligned}-\frac{\partial\phi}{\partial t} = \min_{u}\Big\{ &{}^t e Q e + {}^t u R u \\ &+ {}^t[\mathcal{A}e - \mathcal{D}u + m]\frac{\partial\phi}{\partial e} + \frac{1}{2}\,{}^t\left[\frac{\partial}{\partial e}\right]\Sigma\left[\frac{\partial}{\partial e}\right]\phi\Big\},\end{aligned} \qquad \text{(C-7)}$$

or

$$\begin{aligned}\frac{\partial\phi'}{\partial\tau} = \min_{u}\Big\{ &{}^t e Q' e + {}^t u R' u \\ &+ {}^t[\mathcal{A}'e - \mathcal{D}'u + m']\frac{\partial\phi'}{\partial e} + \frac{1}{2}\,{}^t\left[\frac{\partial}{\partial e}\right]\Sigma'\left[\frac{\partial}{\partial e}\right]\phi'\Big\},\end{aligned} \qquad \text{(C-8)}$$

where $\tau \equiv T - t$ is an auxiliary time variable, and $\phi \equiv \phi(e\,;\ \tau)'$, $Q' \equiv Q(\tau)'$, $R' \equiv R(\tau)'$, $\mathcal{A}' \equiv \mathcal{A}(\tau)'$, $\mathcal{D}' \equiv D(\tau)'$, $m' \equiv m(\tau)'$ and $\Sigma' \equiv \Sigma(\tau)'$ are respectively equivalent to $\phi(e\,; T-\tau)$, $Q(T-\tau)$, $R(T-\tau)$, $\mathcal{A}(T-\tau)$, $D(T-\tau)$, $m(T-\tau)$ and $\Sigma(T-\tau)$.

For the convenience of the present description, however, the symbols without prime, where distinction is not necessary, i.e., $\phi(\epsilon;\tau)$ $Q(\tau)$, $R(\tau)$, $A(\tau)$, $D(\tau)$, $m(\tau)$ and $\Sigma(\tau)$ are used for $\phi'$, $Q'$, $R'$, $A'$, $D'$, $m'$ and $\Sigma'$ respectively.

## Appendix D : Derivation of Eqs. (4.4-3) and (4.4-5)

First, we consider two arbitrary complex functions, $F_p(\zeta)$ and $\dot{H}_q(\zeta)$, which are defined by

$$F_p(\zeta) = \sum_{n=0}^{\infty} f_{p,n}\, \zeta^n \tag{D-1}$$

and

$$\dot{H}_q(\zeta) = \sum_{n=0}^{\infty} h_{q,n} \zeta^n , \tag{D-2}$$

where $\dot{H}_q(\zeta)$ expresses the derivative of the function $H_q(\zeta)$ with respect to a complex variable $\zeta$. (For the convenience of the present description, the symbol, $h_{q,n}$ is substituted for $h_{q,n,1}$ ). If $H_q(\zeta)$ is an analytic function in a region of the $\zeta$-plane, and if both zero and $\zeta$ are the terminal points of any path laying within this analytic region, then we can write

$$H_q(\zeta) = \int_0^{\zeta} \dot{H}_q(\zeta')\,d\zeta' + \bar{h}_q , \tag{D-3}$$

where

$$\bar{h}_q \equiv H_q(0) \tag{D-4}$$

By substituting Eq. (D-2) into Eq. (D-3) and interchanging the order of summation and integration, we can derive

$$H_q(\zeta) = \sum_{n=0}^{\infty} \frac{h_{q,n}}{n+1}\, \zeta^{n+1} + \bar{h}_q \tag{D-5}$$

Multiplication of Eq. (D-1) by Eq. (D-5) gives

$$F_p(\zeta) H_q(\zeta) = \bar{h}_q \sum_{n=0}^{\infty} f_{p,n} \zeta^n + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{h_{q,m}}{m+1}\, f_{p,n} \zeta^{m+n+1} \tag{D-6}$$

By manipulating the second term in the right hand side of Eq. (D-6), we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{h_{q,m}}{m+1}f_{p,n}\zeta^{m+n+1}=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{f_{p,u}\,h_{q,n-u-1}}{n-u}\zeta^{n}. \tag{D-7}$$

Eq. (D-6) can thus be rewritten by

$$F_p(\zeta)H_q(\zeta)=\sum_{n=0}^{\infty}\{\bar{h}_q f_{p,n}+F_{n,u}^{p,q}\}\zeta^{n}, \tag{D-8}$$

where

$$\left.\begin{aligned} F_{n,u}^{p,q}&=0 && \text{for} \quad n=0 \\ &=\sum_{u=0}^{n-1}\frac{f_{p,u}h_{q,n-u-1}}{n-u} && \text{for} \quad n\geq 1 \end{aligned}\right\}. \tag{D-9}$$

From Eq. (D-8), we can readily derive the direct transform, Eq. (4.4-3), as

$$\mathcal{T}c\,[F_p(\zeta)H_q(\zeta)]=\bar{h}_q f_{p,n}+F_{n,u}^{p,q}. \tag{D-10}$$

Next, let us consider three arbitrary functions, $\dot{H}_p(\zeta)$, $F_q(\zeta)$ and $\dot{H}_r(\zeta)$, of a complex variable $\zeta$ which are defined by

$$\dot{H}_p(\zeta)=\sum_{n=0}^{\infty}h_{p,n}\zeta^{n}, \tag{D-11}$$

$$F_q(\zeta)=\sum_{n=0}^{\infty}f_{q,n}\zeta^{n} \tag{D-12}$$

and

$$\dot{H}_r(\zeta)=\sum_{n=0}^{\infty}h_{r,n}\zeta^{n} \tag{D-13}$$

respectively. From Eqs. (D-11) and (D-13), we get

$$H_p(\zeta)=\sum_{n=0}^{\infty}\frac{h_{p,n}}{n+1}\zeta^{n+1}+\bar{h}_p \tag{D-14}$$

and

$$H_r(\zeta)=\sum_{n=0}^{\infty}\frac{h_{r,n}}{n+1}\zeta^{n+1}+\bar{h}_r \tag{D-15}$$

where $\bar{h}_p\equiv H_p(0)$, $\bar{h}_r\equiv H_r(0)$, respectively. By using Eqs. (D-14), (D-12) and (D-15), the triple product of complex functions, $H_p(\zeta)F_q(\zeta)H_r(\zeta)$, can be expressed by

$H_p(\zeta)F_q(\zeta)H_r(\zeta)$

$$= \bar{h}_p\bar{h}_r \sum_{l=0}^{\infty} f_{q,l}\,\zeta^l + \bar{h}_r \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{h_{p,m}}{m+1} f_{q,l}\,\zeta^{l+m+1}$$

$$+ \bar{h}_p \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{r,n}}{n+1} f_{q,l}\,\zeta^{l+n+1}$$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{p,m}\,h_{r,n}}{(m+1)\,(n+1)} f_{q,l}\,\zeta^{l+m+n+2} \qquad \text{(D-16)}$$

By applying Eqs. (D-7) and (D-9), the second and the third term in the right hand side of Eq. (D-16) can respectively be expressed as

$$\bar{h}_r \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{h_{p,m}}{m+1} f_{q,l}\,\zeta^{l+m+1} = \bar{h}_r \sum_{l=0}^{\infty} F_{l,u}^{q,p}\,\zeta^l \qquad \text{(D-17)}$$

and

$$\bar{h}_p \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{r,n}}{n+1} f_{q,l}\,\zeta^{l+n+1} = \bar{h}_p \sum_{l=0}^{\infty} F_{l,u}^{q,r}\,\zeta^l \,. \qquad \text{(D-18)}$$

Then, our present attention is directed to the calculation of the last term in Eq. (D-16). This term can, however, be described as

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{p,m}\,h_{r,n}}{(n+1)(n+1)} f_{q,l}\,\zeta^{l+m+n+2}$$

$$= \sum_{l=0}^{\infty}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{p,m}\,h_{r,n}}{(m+1)\,(n+1)}\,\zeta^{m+n+2}\Big\} f_{q,l}\,\zeta^l \,. \qquad \text{(D-19)}$$

Since

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{p,m}h_{r,n}}{(m+1)\,(n+1)}\,\zeta^{m+n+2} = \sum_{m=0}^{\infty}\sum_{n=2}^{m-2} \frac{h_{p,n}\,h_{r,m-n-2}}{(n+1)\,(m-n-1)}\,\zeta^m \qquad \text{(D-20)}$$

then, Eq. (D-19), becomes

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{h_{p,m}h_{r,n}}{(m+1)(n+1)} f_{q,l}\,\zeta^{l+m+n+2} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} C_{m,u}^{p,r} f_{q,l}\,\zeta^{l+m}, \qquad \text{(D-21)}$$

where a new symbol, $C_{m,u}^{p,r}$ , introduced here is

$$\left.\begin{aligned} C_{m,u}^{p,r} &= 0 && \text{for} \quad m=0,1 \\ &= \sum_{u=0}^{m-2} \frac{h_{p,u}\,h_{r,m-u-2}}{(u+1)\,(m-u-1)} && \text{for} \quad m \geq 2 \end{aligned}\right\}. \qquad \text{(D-22)}$$

Manipulation of the right hand side in Eq. (D-21) gives us the following useful relation for the present discussion;

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{h_{p,m}h_{r,n}}{(m+1)(n+1)}f_{q,l}\,\zeta^{l+m+n+2}=\sum_{l=2}^{\infty}\sum_{w=0}^{l-2}f_{q,w}\mathbb{C}_{l-w,u}^{p,r}\,\zeta^{l}\,. \tag{D-23}$$

By using a new symbol, $\mathbb{D}_{l,w}^{q,p,r}$ , defined by

$$\left.\begin{aligned}\mathbb{D}_{l,w}^{q,p,r}&=0 &&\text{for}\quad l=0,1\\ &=\sum_{w=0}^{l-2}f_{q,w}\mathbb{C}_{l-w,u}^{p,r} &&\text{for}\quad l\geq 2\end{aligned}\right\}, \tag{D-24}$$

Eq. (D-23) yields

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{h_{p,m}h_{r,n}}{(m+1)(n+1)}f_{q,l}\,\zeta^{l+m+n+2}=\sum_{l=0}^{\infty}\mathbb{D}_{l,w}^{q,p,r}\,\zeta^{l}\,. \tag{D-25}$$

Substitution of Eqs. (D-17), (D-18) and (D-25) into Eq. (D-16) gives us the triple product of complex functions i.e.,

$$\begin{aligned}H_p(\zeta)F_q(\zeta)H_r(\zeta)=\sum_{l=0}^{\infty}\{&\bar{h}_p\bar{h}_r f_{q,l}+\bar{h}_p\mathbb{F}_{l,u}^{q,r}\\ &+\bar{h}_r\mathbb{F}_{l,u}^{q,p}+\mathbb{D}_{l,w}^{q,p,r}\}\,\zeta^{l}\,.\end{aligned} \tag{D-26}$$

The direct transform can, therefore, be given by

$$\mathcal{T}_c\,[H_p(\zeta)F_q(\zeta)H_r(\zeta)]=\bar{h}_p\bar{h}_r f_{q,n}+\bar{h}_p\mathbb{F}_{n,u}^{q,r}+\bar{h}_r\mathbb{F}_{n,u}^{q,p}+\mathbb{D}_{n,w}^{q,p,r}\quad . \tag{D-27}$$

This is the final result of this appendix.

## Appendix E : Derivation of Eqs. (6.3-4), (6.3-5) and (6.3-6)

By setting $i=1$ in Eq. (6.3-3), we get

$$J_{p_1}(k)=p[e(k)]\,e(k+1)^2+Q\,[e(k)]\,\Delta u(k)^2\quad . \tag{E-1}$$

Substitution of Eq. (6.3-2) into Eq. (E-1) gives us

$$\begin{aligned}J_{p_1}(k)=p\,[e(k)]\,\{e(k)+[a\,e(k)+b\,e(k)^3]\,\Delta-u(k)\,\Delta\}^2\\ +Q\,[e(k)]\,\Delta\,u(k)^2\,.\end{aligned} \tag{E-2}$$

From Eq. (E-2), the desired control variable $\bar{u}(k)_1$ is obtained as

$$\bar{u}(k)_1 = \frac{p[e(k)]}{p[e(k)]\Delta + Q[e(k)]}\{e(k) + [ae(k) + be(k)^3]\Delta\}. \tag{E-3}$$

This is the first result of this appendix.

By applying the principle of optimality,(3) the minimized form of Eq. (6.3-3) with respect to both $u(k)$ and $u(k+1)$ can be expressed as

$$\begin{aligned}
&\min_{u(k),\,u(k+1)} \{p[e(k)]e(k+2)^2 + \sum_{j=0}^{1} Q[e(k)]u(k+j)^2\Delta\} \\
&= \min_{u(k)} (Q[e(k)]u(k)^2\Delta \\
&\quad + \min_{u(k+1)} \{p[e(k)]e(k+2)^2 + Q[e(k)]u(k+1)^2\Delta\}) .
\end{aligned} \tag{E-4}$$

On the other hand, by applying Eq. (E-3) the second term in the bracket is calculated as

$$\begin{aligned}
&\min_{u(k+1)} \{p[e(k)]e(k+2)^2 + Q[e(k)]u(k+1)^2\Delta\} \\
&= \frac{p[e(k)]Q[e(k)]}{p[e(k)]\Delta + Q[e(k)]}\{(1+a\Delta)e(k+1) + b\Delta e(k+1)^3\}^2 .
\end{aligned} \tag{E-5}$$

Therefore, substituting Eq. (E-5) into Eq. (E-4), differentiation of Eq. (E-4) with respect to $u(k)$ gives us the necessary condition which the desired control variable $\bar{u}(k)_2$ should satisfy as follows:

$$\begin{aligned}
&\bar{u}(k)_2 - R\{(1+a\Delta)e(k+1) + b\Delta e(k+1)^3\}\times \\
&\qquad \{(1+a\Delta) + 3b\Delta e(k+1)^2\} = 0 ,
\end{aligned} \tag{E-6}$$

where

$$R \equiv \frac{p[e(k)]}{p[e(k)]\Delta + Q[e(k)]} . \tag{E-7}$$

Since $u(k)$ is rewritten as

$$u(k)\Delta = (1+a\Delta)e(k) + b\Delta e(k)^3 - e(k+1) , \tag{E-8}$$

then Eq. (6.3-5) is the straight forward result from Eq. (E-8). Setting $e(k+1) = z$ in Eqs. (E-6) and (E-8), substitution of Eq. (E-8) into Eq. (E-6) provides us Eq. (6.3-6).

## Summarized Conclusions

Remarkable points emphasized in this paper are summarized as follows:

(1) A basic concept of sub-interval optimization is presented to establish the approximate methods of designing optimum control systems. A sub-interval optimization technique is proposed by introducing the concept of on-line control scheme based upon the restricted-optimal control. By using the sub-interval optimization technique a graphical method of determining quasi-optimum switching lines is established for quasi-optimum controls of second-order linear systems with bounded control.

(2) Problems on an optimum final-value control of second order linear systems with bounded control are studied. The present technique can be found to provide a physically meaningful optimum solution to the original problem.

(3) An analytical method of designing the optimum control system with random inputs is described. An approximate method of solving the set of non-linear differential equations which are associated with the design problems of a linear optimum controller is established by using the concept of sub-interval optimization.

(4) A case study on the stochastic synthesis of optimum final-value control systems with control energy constraint are carried out to investigate the control characteristics of the optimum controller under random environments. The time-variant characteristics of the optimum final-value controller is clearly shown with many numerical results.

(5) From the view point of computer utilization, a basic idea is presented to explore a powerful approach to designing non-linear control systems with many physical limitations. A remarkable point of the proposed method of computer approach is that it provides a practical direct design procedure of non-linear control systems with many physical limitations which are not completely described in mathematical formulations.

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