STUDIES ON ROBUST CONTROL OF NONLINEAR SYSTEMS INCLUDING ROBOT MANIPULATORS

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Preface

This thesis is submitted for the Doctor degree in Mechanical Engineering at Kyoto University. The research was carried out in the period from April 1988 to August 1994. Dr. Tsuneo Yoshikawa, Professor of Kyoto University, was my supervisor.

When I started to study this research, my interest was in the field of the robust trajectory control of robot manipulators. The control problem of this field is addressed in this paper from the various viewpoints: adaptive control, control with acceleration feedback or joint torque feedback, and hierarchical or digital control. Since January 1991, my attention has been attracted to robust control problems of more general nonlinear systems including robot manipulators. Especially, in most of the latter period of the research, the nonlinear H_∞ control problem has been addressed, which will provide a chance to develop robust control of nonlinear systems. A study on robust control of nonlinear systems is a new challenging field in control engineering. I hope that the results of this paper will give a basis to form the robust control theory of nonlinear systems, especially the nonlinear H_∞ control theory, and will also be useful for future research of this field.

I would like to express my gratitude to Professor Tsuneo Yoshikawa for his patient guidance and constant encouragement throughout this work. His good advice and valuable comment have enabled me to complete this thesis.

I also wish to express my deep indebtedness to Dr. Toshiharu Sugie, Associate Professor of Kyoto University, for introducing me to the robust control problem of nonlinear systems. His stimulating discussions and critical reading of the manuscripts have provided me with many valuable suggestions.

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Abstract

This paper is concerned with robust control of nonlinear systems including robot manipulators. The following results have been obtained in this research.

· **Robust trajectory control of robot manipulators:** A practical robust control design method is proposed, fully exploiting the property of manipulators, i.e., the feature that the dynamics is composed of the linear combination between unknown physical parameters and measurable parameters such as joint displacement or the feature that the acceleration sensor information or joint torque sensor information is available. In addition, a digital robust controller of robot manipulators is proposed, where the effect of the discretization of a robust controller on the control error is taken into consideration.

· **Development of foundations of robust control of nonlinear systems:** First, a characterization of the bounded real condition of nonlinear systems is given using the Hamilton-Jacobi equation with a stabilizing solution and the Hamilton-Jacobi strict inequality. The former has an important role to analyze the internal stability of nonlinear systems, while the latter has an advantage that it can simply be applied to the $\boldsymbol{H}_\infty$ control problem. The characterization by these two approaches completes the strict bounded real condition of nonlinear systems to form a basis to develop the nonlinear $\boldsymbol{H}_\infty$ control theory. Second, based on the above obtained results, some sufficient (and necessary) conditions for the solvability of the nonlinear $\boldsymbol{H}_\infty$ control problems via state feedback or output feedback are given. In addition, the obtained $\boldsymbol{H}_\infty$ state feedback control is applied to the robust stabilization problem of nonlinear system with unstructured uncertainty. Finally, a global robust stabilizability condition for nonlinear cascaded systems is analyzed, using a different approach from the nonlinear H_∞ control theory.

Contents

Chapter 1

INTRODUCTION

1.1 History of robust control

In general, it is difficult to find a model of the plant to be controlled rigorously. There mostly exists some discrepancy between the real plant and the model which is analytically or experimentally given. If we take no account of such a modeling error to design a controller based on the model, then the obtained controller frequently may make a closed loop system unstable or give an unsatisfactory control performance. Thus since the late 1970s, many researchers have developed control system design methods to make a plant stable or keep a specified control performance of the closed loop systems in spite of uncertainties such as the modeling error. Such a control theory is called the robust control theory. Although the robust control theory has many approaches at present, two approaches are focused on here, that is, an approach based on the Lyapunov Stability Theorem and an approach based on the H_∞ control theory.

Robust control based on Lyapunov Stability Theorem

Studies on the stabilization of linear systems with deterministic uncertainty began in the late 1970s, based on the Lyapunov Stability Theorem [38, 70]. After that, the Lyapunov-based robust control approach of linear systems has been extended to nonlinear setting. There are many results especially when there exist uncertainties satisfying the so-called matching condition (which is a condition for an input of a plant to act on uncertainties directly) [37, 28, 2, 32, 29]. In addition, combined with the singular perturbation theory [104, 72, 26, 71, 122, 64] or differential geometric approach [109, 14, 34], the control theory based on the Lyapunov Stability Theorem has been developed as one field of the robust control theory of nonlinear systems. This approach has also been applied to tracking control problems of robot manipulators which have strong nonlinearity [103, 39, 119, 93, 59, 1].

In the case of linear systems, on the other hand, the Lyapunov-based robust control approach has been developed as the quadratic stabilization scheme. The quadratic stabilization problem is reduced to the solvability of the Riccati equation, and the method is useful even for the uncertainty which does not satisfy the matching condition [11, 100, 98, 95, 96, 99, 97, 151, 150, 102, 62, 94].

H_∞ control theory

In the field of linear system control theory, the robust control has also been considered from the viewpoint of the frequency domain. Especially since the early 1980s, the H_∞ control theory, which was proposed by Zames, has been developed [148, 33, 36, 31]. Recently the H_∞ control theory is well acknowledged as one of the most powerful design schemes for the robust control system. It pays attention to a maximum gain of a transfer function, which is called H_∞ norm or L_2 gain, to reduce a robust control problem to a H_∞ control problem. A state space solution to the general H_∞ control problem was given by Glover and Doyle in 1988. Using the H_∞ control theory, we can solve a robust stabilization problem for the unstructured uncertainty such as the modeling error which is given by the discrepancy of the gain. In addition, the relation between the H_∞ control scheme and the quadratic stabilization scheme was clarified, and it was found in [62] that, roughly speaking, the former includes the latter. Recently, several researchers have begun to extend the H_∞ control theory of linear systems to nonlinear setting. We call it the nonlinear H_∞ control theory.

Under the above background on the robust control, this paper treats two topics of the robust control: (I) Studies on the robust control of robot manipulators based on the Lyapunov Stability Theorem, and (II) Studies on the robust control of nonlinear systems (the nonlinear H_∞ control and global robust stabilization in the absence of the matching condition). In this first chapter, the previous works on both topics are surveyed first, and then the goal and the composition of this thesis are described.

1.2 Background of robust control of robot manipulators

Trajectory control of a robot manipulator is one of fundamental problems in the field of robotics. Since a robot manipulator in general has some kind of nonlinearity, it is much important to consider the nonlinearity in the trajectory control problem. Although there exist many previous studies on the trajectory control, one of well known results was given by Luh et al. [77] in 1980, which is called the resolved acceleration control method or the dynamic control. A controller designed by this scheme is composed of linearization and servo compensation. In other words, the linearizing compensation makes the dynamics of the manipulator linear and then the resulting linearized system is compensated by a linear servo controller. Indeed, the idea of this control scheme is natural from the viewpoint of the control of nonlinear system, but it requires an exact model of the real plant to be controlled. In the case of a robot manipulator, it is usual that real values of some physical parameters such as the mass and the inertia of the arm or the friction coefficient of the joint are exactly unknown, although we can get estimated values of these parameters by some identification method. If the estimated values are used for the linearization of the manipulator, then the manipulator cannot be often controlled theoretically well, and the obtained controller may be unsatisfactory from the viewpoint of control performance, because the dynamics of the manipulator cannot be linearized completely. In addition, there may exist some disturbances such as measurement noise in the manipulator system, which often lead to unsatisfactory control performance.

Thus the robust control scheme based on the Lyapunov Stability Theorem, which has been developed for general nonlinear systems with deterministic uncertainty, began to be applied to the trajectory control of manipulators. There are mainly two approaches in this field: sliding mode control and robust control based on the Lyapunov Stability Theorem (in the local sense). The sliding mode control scheme has been developed by Itkis [58] and Utkin [126] et al., and has been applied to the trajectory control of the manipulator by Young [147] and some researchers [114, 112, 41, 143, 68, 20]. On the other hand, concerning the robust control approach based on the Lyapunov Stability Theorem (in the local sense), Ryan et al. [103] applied the Lyapunov-based robust control scheme proposed by Corless and Leitmann [28] to the trajectory control of manipulators for the first time. After that, using this approach, the robust trajectory control of manipulators has been studied by many researchers [2, 39, 119, 93, 59, 1]. Especially, Slotine [112], and Osuka and Sugie [93, 119] clarified how to design control parameters so as to achieve the specified tracking precision. In addition, the idea that the feedback gain in the above robust control is automatically adjusted according to the bound of the control error has been developed for general nonlinear systems including robot manipulators, which is called an adaptive robust control scheme [27, 21, 81, 24, 73].

Although these methods commonly adopt a high feedback gain in order to compensate for the uncertainty, the unnecessarily high gain in feedback may make the system unstable by the effect of the unmodelled dynamics, and cause unexpected phenomena such as chattering in digital control systems. Therefore from the practical viewpoint, it is crucial to make the feedback gain as small as possible without sacrificing the tracking accuracy. In the conventional robust control methods of robot manipulators, the estimation of the bound of the uncertainties is too conservative, and as a result, the feedback gain calculated by the conservative estimation on the uncertainty tends to be much larger than necessary to achieve the specified tracking precision. On the other hand, the conventional adaptive robust control methods have an advantage that the feedback gain is automatically determined without any a priori information on the uncertainty. However, the explicit quantitative relation between the tracking error bound and the design parameters is not clear at all there. In this light, it is difficult to tell whether the feedback gain is unnecessarily high or not. Also the design procedure of both the conventional robust and adaptive robust controllers of robot manipulators is much complicated. These problems in the conventional robust control of robot manipulators come from the fact that the robust control schemes for general nonlinear systems is straightforwardly applied to that of robot manipulators and the special structure of the manipulator dynamics is not fully exploited there. General formulation may result in the conservative estimation of the uncertainty, makes the controller more complicated, and makes it more difficult to clarify the relation between the design parameters and the bound of the control error.

In addition, most of robust controllers of nonlinear systems are nonlinear, so the robust nonlinear controller is usually discretized when it is implemented. Thus we need to take account of the effect of the discretization of the robust controller on the control error, but there has been no research on this topic.

It is concluded that the robust control schemes of robot manipulators in the existing literatures are not sufficient from a practical point of view. We believe that it is much significant to establish a practical robust control design scheme of robot manipulators, fully exploiting properties of the plant itself.

1.3 Background of robust control of nonlinear systems

1.3.1 Nonlinear H_∞ control

As stated in Section 1.1, various techniques on the $\boldsymbol{H}_\infty$ control theory of linear systems have been developed in the last decade, see e.g. [36, 31], and several researchers have recently attempted to extend the $\boldsymbol{H}_\infty$ control to the case of nonlinear systems. Ball and Helton [7, 8, 9], from a viewpoint of operator theory, discussed $\boldsymbol{H}_\infty$ control theory of nonlinear systems, for the first time, and connected it with the differential game theory [13, 124, 74, 12]. Van der Schaft [130] analyzed the relation between the $\boldsymbol{L}_2$ gains of nonlinear systems and their linearization, and gave a sufficient condition for the existence of smooth $\boldsymbol{H}_\infty$ state feedback control. In addition, van der Schaft [131, 127, 128] paid attention to the dissipative system theory [139, 138, 137, 85, 84, 46, 44, 45, 134], and discussed the relation between the $\boldsymbol{L}_2$ gain and the Hamilton-Jacobi equation, and applied to the state feedback case. However, there was no discussion on a stabilizing solution of the Hamilton-Jacobi equation, except for the discussion based on the linearization. Isidori and Astolfi [55, 57, 54] have derived a sufficient condition for the existence of $\boldsymbol{H}_\infty$ output feedback control as well as state feedback in the case where the Hamiltonian system does not necessarily have a hyperbolic equilibrium. Their success is based on the differential game theory and the La Salle's Invariance Principle. Using the latter, they proved the internal stability of the closed loop system. However, their sufficient condition is more restrictive than that of the linear case at the point that it requires positive definiteness of the solution of the Hamilton-Jacobi-Isaacs equation, while a positive semi-definite solution is enough in the linear case. There is also some discussion on a necessary condition for $\boldsymbol{H}_\infty$ control.

Very recently, Ball et al. [10] and van der Schaft [129] discussed a necessary condition for the existence of $\boldsymbol{H}_\infty$ output feedback control, from the dissipative system theory, and the structure of nonlinear $\boldsymbol{H}_\infty$ controllers, but the derived condition is not necessary and sufficient. There is also no analysis on the stabilizing solution of the Hamilton-Jacobi-Isaacs equation, which appears in the nonlinear H_∞ control theory, while the stabilizing solution of the Riccati equation plays an important role in the linear $\boldsymbol{H}_\infty$ theory. In addition, van der Schaft [129] gave some results about the strict $\boldsymbol{H}_\infty$ control problem, where the strict inequality condition for the $\boldsymbol{L}_2$ gain of the systems is taken into consideration. However, his results cannot be extended to an asymptotic stability case, because it is based on the linearization.

Although the former results shown above are very interesting and important, it is not satisfactory in the following sense: they do not give the answer to the following fundamental questions. (1) Can we treat the strict $\boldsymbol{H}_\infty$ problem of nonlinear systems in the case of asymptotic stability? (2) When does there exist a stabilizing solution of the Hamilton-Jacobi equation? (3) Do we really need a positive definite solution of the Hamilton-Jacobi-Isaacs equations rather than a positive semi-definite solution? (4) How do the $\boldsymbol{H}_\infty$ control (or $\boldsymbol{L}_2$ gain) results depend on the type of the stability (such as asymptotic stability or exponential stability)? (5) Can we extend the approach based on the Riccati strict inequality [149, 107] to nonlinear setting? So there is a big gap between the linear $\boldsymbol{H}_\infty$ control theory and its nonlinear version obtained so far, and one can hardly say that the essence of the H_∞ control of nonlinear systems was captured. This is mainly because the conventional methods strongly depend on the linearization or the linear H_∞ control techniques. Therefore we need a different approach, which does not depend on the linearization or the linear H_∞ control techniques, to capture the essential feature of the strict H_∞ control theory of nonlinear systems.

In addition, the robust stabilization problem is one of fundamental robust control problems to be considered. However, there was no research on the robust stabilization of nonlinear systems in terms of the nonlinear H_∞ control theory so far (although most recently van der Schaft [132] and Isidori [53] treated this topic). In order to develop the robust control theory of nonlinear systems in terms of the nonlinear H_∞ control, we need to analyze fundamental problems such as robust stabilization conditions for unstructured uncertainty.

1.3.2 Global robust stabilization of nonlinear cascaded systems

Since the 1980s, nonlinear system analysis has been studied based on the differential geometric theory [56], and some fundamental and important results have been derived. Especially, in the early 1980s, the problem on the state space linearization via coordinate transformation and nonlinear state feedback was completely solved by Su [118] or Hunt et al. [51]. After that, the idea of "zero dynamics", which corresponds to the zero of a transfer function of a linear system have been developed in the mid 1980s [16], and the idea of "normal form" of nonlinear systems have been established in 1991 [15].

Based on these studies, in the late 1980s, the stabilization problem of nonlinear systems that have the normal form structure has attracted considerable attention, and some sufficient conditions for local or global stabilization of these systems have been derived [15, 79, 17, 18, 67, 123, 105]. Since the normal form systems have the structure similar to a class of nonlinear cascaded systems, the stabilization problem for a class of nonlinear cascaded systems has also been attacked by many researchers [111, 91, 110, 19]. Needless to say, the important next step is to discuss the stabilization of the nonlinear cascaded system in the presence of uncertainty. This will be the first step to the robust stabilization of general nonlinear systems. However, the previous works described above assume that the systems to be controlled are completely known, and this assumption is crucial to prove the stability. Therefore, it is difficult to apply their methods to the case where the uncertainty exists. Although there are a few researches [109, 108, 14, 34] about robust control of uncertain systems with normal form, these are concerned with a robust output tracking control problem, not a global stabilization one.

As stated in section 1.1, on the other hand, the robust stabilization techniques based on the Lyapunov Stability Theorem have been developed for nonlinear systems in the presence of uncertainty. Most of them treat the case where the uncertainty satisfies the so-called matching condition [37, 28, 2], although some of them studied mismatched uncertainty cases such as the cone-bounded case [89, 140, 23, 25, 106] or the singular perturbation case [104, 72, 26, 71, 122, 64], which is of local nature essentially. However, in order to discuss the robust stabilization of nonlinear cascaded systems, the matching condition is too restrictive. In addition, it is difficult to apply the latter methods to the global stabilization problem.

In summary, it is still an open problem how a nonlinear system with mismatched uncertainty is globally stabilized. We believe that it is much significant to find an essence of the solution to such a problem from the viewpoint how the input and the term of the uncertainty should be cascaded in the state space description of nonlinear systems in order to globally stabilize it.

1.4 The goal and the organization of this thesis

There are two main goals in this thesis. **The first goal is to establish a robust trajectory control design method of robot manipulators:** A practical and systematic robust control design method is proposed, fully exploiting the property of manipulators, i.e., the feature that the dynamics is composed of the linear combination between unknown physical parameters and measurable parameters such as joint displacement or the feature that the acceleration sensor information or joint torque sensor information is available. In addition, a digital robust controller of robot manipulators is proposed, where the effect of the discretization of a robust controller on the control error is considered. **The second goal is to develop foundations of robust control of nonlinear systems**: First, a characterization of the bounded real condition of nonlinear systems via the Hamilton-Jacobi equation with a stabilizing solution or the Hamilton-Jacobi strict inequality is presented. Second, based on the above condition, the nonlinear $\boldsymbol{H}_\infty$ control theory is discussed. In addition, the obtained $\boldsymbol{H}_\infty$ state feedback control is applied to robust stabilization problems of nonlinear system with unstructured uncertainty. Finally, a global robust stabilizability condition for nonlinear cascaded systems is analyzed, using a different approach from the nonlinear H_∞ control theory.

The organization of this thesis is as follows. Chapters 2 to 6 are concerned with a robust trajectory control of robot manipulators, which are for the first goal. Chapters 7 to 9 are concerned with a robust control of general nonlinear systems, which are for the second goal.

Chapter 2 is concerned with a robust trajectory control of robot manipulators, where the joint displacement and the velocity are available. First, a new robust control scheme of robot manipulators is proposed, which overcomes some drawbacks of conventional robust control methods. The proposed controller has a simple structure by exploiting the special structure of the manipulator dynamics, and achieves the specified tracking precision. Next, based on the formulation of the above robust control, a new adaptive robust control scheme for manipulators is proposed, where the feedback gain is automatically adjusted based on the bound of the control error and no a priori information on uncertainty is required. Thus the feedback gain of the proposed method is almost necessary and minimum for the specified precision. To verify the advantages of the adaptive robust control method, experimental results are shown for the trajectory control of a 2 link direct-drive arm.

In chapter 3, merits of acceleration information in the robust control of robot manipulators are clarified in the case that the signal is available. First, the essential feature of the conventional methods is made clear, and it is shown what their problems are. Second, a robust control scheme using acceleration information is proposed for robot manipulators, which overcomes the above problems. Finally, the advantage of acceleration information for the proposed scheme is discussed.

Chapter 4 is concerned with the robust control of robot manipulators in the case where joint torque sensor information is available. First, a dynamic equation of the manipulator with joint torque sensors is derived, which explicitly expresses a nonlinear multivariable structure. This dynamic equation makes it possible to construct the control systems of the manipulators with joint torque sensors based on the same method as in the conventional case without the sensors. Second, based on this dynamic equation, a robust control scheme is proposed, which achieves the specified tracking precision in the presence of the modeling error including the modeling error of actuator systems. The proposed method fully exploits joint torque sensor information to compensate for the uncertainty of link and load parameters. Furthermore, an illustrative simulation result is given to show the effectiveness of the proposed control method.

In chapter 5, a digital robust control method of robot manipulators is proposed. The effect of the discretization of a robust controller on the control error is discussed theoretically. The design procedure for a digital robust control system, which is obtained by the above analysis, gives an allowable feedback gain to guarantee the specified tracking precision. Moreover, a simple idea is proposed to make the feedback gain small so as to decrease the chattering, and the effectiveness of this idea is shown by illustrative simulation results.

In chapter 6, a hierarchical robust control scheme of robot manipulators is proposed, which has a hierarchical structure with an upper level loop and a lower level loop. In the upper level loop, an input for linearizing compensation, a desired trajectory and a switching gain are computed at a low sampling frequency. In the lower level loop, a switching input is generated at a high sampling frequency. This scheme will make the computation for robust compensation very fast, so we can expect that the effect of the discretization of a robust controller on the control error is smaller. The control performance of this hierarchical system is analyzed under the consideration of the sampling period of an upper level loop and the modeling error.

Chapter 7 is concerned with the strict $\boldsymbol{H}_\infty$ control theory of nonlinear systems. First, a necessary and sufficient condition for nonlinear systems to be internally stable and to have the L_2 gain less than a specified number γ, which is called the strict bounded real condition, is given via the Hamilton-Jacobi equation with a stabilizing solution and the Hamilton-Jacobi strict inequality. The former has an important role to analyze the internal stability of nonlinear systems, while the latter has an advantage that it can simply be applied to the strict H_∞ control problem. The characterization by these two approaches completes the strict bounded real condition of nonlinear systems to form a basis to develop the strict H_∞ control theory. Second, based on the above results, some sufficient (and necessary) conditions are given for the solvability of the strict H_∞ state or output feedback control problem, which exactly correspond to the case of linear systems.

Chapter 8 is concerned with robust stabilization of nonlinear systems in terms of the nonlinear H_∞ state feedback theory. First, a robust stability condition is given for a closed loop system which is composed of a nonlinear nominal system and an unstructured uncertainty. Second, based on the obtained robust stability condition, a sufficient condition for robust stabilization by state feedback is given in terms of the solvability of some H_∞ state feedback control problem.

In chapter 9, a sufficient condition is given for a class of nonlinear cascaded systems to be globally stabilizable via state feedback in the presence of uncertainty which does not necessarily satisfy the so-called matching condition. The obtained result is an extension of the former stabilization results which treated systems without uncertainty, in the sense that the uncertainty is taken into account. In addition, considering a specified class of the systems, a more practical condition for global robust stabilization is derived.

We will use the following notations in the paper. R^n denotes an n-dimensional real Euclidean space whose norm is given by $\|\cdot\|$. $A \in R^{n\times m}$ is a $n \times m$ matrix, and $\lambda_m(A)$ and $\lambda_M(A)$ express minimum and maximum singular values of a matrix, respectively. I expresses a unit matrix.

Chapter 2

ROBUST CONTROL AND ADAPTIVE ROBUST CONTROL OF ROBOT MANIPULATORS

2.1 Introduction

In the past several years, various robust trajectory control schemes have been developed for robot manipulators with unknown parameters or disturbances, such as sliding mode control [147, 114, 112, 41, 143, 20], robust control [103, 2, 39, 119, 93, 59, 1], or adaptive robust control [27, 21, 81, 22, 24, 73]. One of the fundamental problems in this field is to control the robot manipulator so as to track the desired trajectory within a specified tracking precision in the presence of uncertainty. Especially, Slotine [112] and, Osuka and Sugie [93, 119] clarified how to design control parameters so as to achieve the specified tracking precision.

In most of the previous existing literatures, however, the design procedures are much complicated. In addition, the estimation of the bound of the uncertainties is too conservative, so the feedback gain calculated by the conservative estimation on the uncertainty is much larger than necessary to achieve the specified tracking precision. As a result, the obtained controller is impossible to be implemented in practice, since too high feedback gain excites unmodelled dynamics. It is desired that the feedback gain is necessarily and sufficiently high to achieve the specified tracking precision. In addition, the conventional adaptive robust control schemes clarify no relation between the design parameters and the specified precision.

In this chapter, a new robust control scheme for robot manipulators is proposed first, which overcomes the above shortcomings of the former robust control methods in the sense that it has the following properties: by fully exploiting the property of the structure of the dynamics, (i) the controller structure is as simple as the conventional dynamic control method [77] except that it has only a time-varying feedback gain and (ii) the estimation of the bound of the uncertainty is less conservative. Next, based on the above robust control scheme, a new adaptive robust control scheme is proposed, which, in addition to the above advantages, (iii) clarifies the explicit relation between the design parameters and the tracking precision, (iv) achieves the specified tracking precision without any a priori information on the robot uncertainty, and (v) the feedback gain of the proposed adaptive robust control is much smaller than the robust control case.

Furthermore, the validity of the proposed adaptive robust control method is experimentally verified using a 2 link Direct-Drive robot arm.

2.2 Problem statement

Consider a manipulator with n degrees of freedom whose dynamics is described by the following equation :

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{2.1}$$

where $\boldsymbol{\theta} \triangleq [\theta_1, \theta_2, \ldots, \theta_n]^T$ is the n-dimensional vector of joint displacements, $\boldsymbol{\phi}$ is the physical parameter vector with an appropriate dimension, $\boldsymbol{u}$ is the n-dimensional joint torque input vector, $\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})$ is the $n \times n$ manipulator inertia matrix, and $\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ is the n-dimensional vector that represents the nonlinear terms such as the centrifugal, Coriolis, frictional, and gravitational forces.

This system usually has the following features.

[**Feature 2.1**] *$M(\theta)$ is a positive definite matrix for any θ.* ∎

[**Feature 2.2**] *The left-hand side of (2.1) can be expressed as*

$$M(\phi,\theta)\ddot{\theta} + h(\phi,\theta,\dot{\theta}) = E(\phi)y(\theta,\dot{\theta},\ddot{\theta}) \tag{2.2}$$

where $E(\phi)$ is an appropriate dimensional matrix consisting of physical parameters, and $y(\theta,\dot{\theta},\ddot{\theta})$ is an appropriate dimensional vector whose elements are known functions of θ, $\dot{\theta}$, and $\ddot{\theta}$ (see section 2.5). ∎

In this chapter, the following assumptions are made.

[**Assumption 2.1**] *θ and $\dot{\theta}$ are measurable.* ∎

[**Assumption 2.2**] *The values of the physical parameter vector ϕ may be unknown, but it is known that ϕ exists in a certain bounded region Ω.* ∎

[**Assumption 2.3**] *A vector $\hat{\phi}$, a bounded estimate of ϕ, is given such that there exist bounded positive constants α and β which satisfy the following conditions for any non-zero vector $x \in R^n$, any non-zero and appropriate dimensional vector y, any $\theta \in R^n$, and any $\phi \in \Omega$:*

$$\alpha \parallel x \parallel^2 < x^T \hat{I} x, \qquad \hat{I} \triangleq M^{-1}(\phi,\theta)M(\hat{\phi},\theta) \tag{2.3}$$

$$\beta \parallel y \parallel \geq \parallel M^{-1}(\phi,\theta)\{E(\hat{\phi}) - E(\phi)\}y \parallel \tag{2.4}$$

∎

The existence of the positive constant α in (2.3) is dependent on the estimate $\hat{\phi}$. However, it is difficult to show rigorously when such an α exists. According to our experience on numerical analysis of various 2 DOF manipulators, there exists an α even when the difference between the real values and the estimate values of the physical parameters is 50 % of the real values. While, Khosla and Kanade [65, 66] have shown the effectiveness of a dynamic control law with the parameters estimated by their identification method. This means that the estimated value of the physical parameters are not quite different from the real values. So using the conventional identification methods such as [65, 80, 6], we believe that it would be possible to estimate the physical parameters within such accuracy that there exists a $\alpha > 0$ in (2.3). Therefore we do not think that the assumption of (2.3) is so restrictive in a practical sense. On the other hand, it is guaranteed that there exists a positive constant β, because of the fact that M is a positive definite matrix (namely M^{-1} is bounded) and that E is a constant matrix. Note that α and β are obtained by calculating the smallest and largest singular values of $\widehat{\boldsymbol{I}}$ and $\boldsymbol{M}^{-1}\{\boldsymbol{E}(\boldsymbol{\phi}) - \boldsymbol{E}(\widehat{\boldsymbol{\phi}})\}$, respectively, if the region $\boldsymbol{\Omega}$ is known.

For simplicity, the following notation are used: $\widehat{\boldsymbol{M}} \triangleq \boldsymbol{M}(\widehat{\boldsymbol{\phi}}, \boldsymbol{\theta})$, $\widehat{\boldsymbol{h}} \triangleq \boldsymbol{h}(\widehat{\boldsymbol{\phi}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, and $\widehat{\boldsymbol{E}} \triangleq \boldsymbol{E}(\widehat{\boldsymbol{\phi}})$.

Now, the following problem is considered.

[Problem 2.1] *For the robot manipulator given by (2.1) that satisfies Assumptions 2.1 to 2.3, the desired trajectory $\boldsymbol{\theta}_d(t)$ is given whose derivatives $\dot{\boldsymbol{\theta}}_d$ and $\ddot{\boldsymbol{\theta}}_d$ exist and are bounded. Also ε_P and ε_V, the tracking precision, are given. Then, find a control law such that*

$$\| \boldsymbol{e}(t) \| < \varepsilon_P, \qquad \| \dot{\boldsymbol{e}}(t) \| < \varepsilon_V \quad \forall t \geq T \tag{2.5}$$

holds for some finite time $T \geq t_0$, where $\boldsymbol{e}(t) \triangleq \boldsymbol{\theta}(t) - \boldsymbol{\theta}_d(t)$ and t_0 is an initial time. ∎

For simplicity, one may assume that $\boldsymbol{e}(t_0) = \mathbf{o}$ and $\dot{\boldsymbol{e}}(t_0) = \mathbf{o}$.

2.3 Robust control

In this section, a robust control method is proposed which determines the feedback gain using the information on $\boldsymbol{\Omega}$ so as to achieve the relation (2.5). The argument is much simpler than the former works [103, 2, 39, 119, 93, 59, 1], and gives a good insight for the robust control of robot manipulators.

In order to solve the problem in section 2.2, the following control algorithm with a constant gain λ and a time-varying gain $k(t)$ is considered.

$$\boldsymbol{u} = \widehat{\boldsymbol{M}}\{\ddot{\boldsymbol{\theta}}_d - (\lambda + k(t))\dot{\boldsymbol{e}} - \lambda k(t)\boldsymbol{e}\} + \widehat{\boldsymbol{h}} \tag{2.6}$$

Note that this algorithm is the same as in the conventional dynamic control method [77] except that the conventional algorithm consists of the fixed PD gain in (2.6).

First, the error equation which is important for the control system design is derived. Substituting (2.6) into (2.1), one obtains the following error system:

$$\ddot{\boldsymbol{e}} + (\lambda \boldsymbol{I} + k\widehat{\boldsymbol{I}})\dot{\boldsymbol{e}} + \lambda k \widehat{\boldsymbol{I}} \boldsymbol{e} = \boldsymbol{\eta} \tag{2.7}$$

where

$$\boldsymbol{\eta} \triangleq \boldsymbol{M}^{-1}(\widehat{\boldsymbol{E}} - \boldsymbol{E})\boldsymbol{y}_d$$

$$\boldsymbol{y}_d \triangleq \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}_d - \lambda \dot{\boldsymbol{e}})$$

If we have no modeling error, $\boldsymbol{\eta} = \mathbf{o}$ and $\widehat{\boldsymbol{I}} = \boldsymbol{I}$ in the error equation (2.7). So $\boldsymbol{\eta}$ can be regarded as a disturbance which results from the modeling error and $\widehat{\boldsymbol{I}}$ as one part of the feedback gain which contains uncertainty. The effect of these uncertainties, $\widehat{\boldsymbol{I}}$ and $\boldsymbol{\eta}$, on the control error $\boldsymbol{e}$ is evaluated by (2.3) and (2.4), respectively: the infimum value of the uncertain part, $\widehat{\boldsymbol{I}}$, of the feedback gain is estimated as α in (2.3), and the supremum value of the disturbance $\boldsymbol{\eta}$ by (2.4) as follows.

$$\beta \parallel \boldsymbol{y}_d \parallel \geq \parallel \boldsymbol{\eta} \parallel \tag{2.8}$$

Note that $\boldsymbol{y}_d$ is measurable, because $\ddot{\boldsymbol{\theta}}_d$ is known. Therefore, one will determine the feedback gains λ and k so as to satisfy the relation (2.5) based on the error equation (2.7) and the information on the uncertainties, α and β.

By letting

$$\gamma \triangleq \frac{\beta}{\alpha} \tag{2.9}$$

we obtain the following result.

[Theorem 2.1] *Consider the manipulator (2.1) that satisfies Assumptions 2.1 to 2.3. The desired trajectory $\boldsymbol{\theta}_d$ and the specified tracking error precision, ε_P and ε_V, are given. Moreover suppose γ is obtained from a priori knowledge of the region $\boldsymbol{\Omega}$. If the control law (2.6) whose feedback gains are given as*

$$\lambda = \frac{\varepsilon_V}{2\varepsilon_P} \tag{2.10}$$

$$k = \frac{\gamma \parallel \boldsymbol{y}_d \parallel}{\lambda \varepsilon_P} \tag{2.11}$$

is applied to the manipulator, then

$$\parallel \boldsymbol{e}(t) \parallel < \varepsilon_P, \qquad \parallel \dot{\boldsymbol{e}}(t) \parallel < \varepsilon_V \tag{2.12}$$

holds for any $t \geq t_0$. ∎

Proof: By defining a new variable $\boldsymbol{s}$ as $\boldsymbol{s} \triangleq \dot{\boldsymbol{e}} + \lambda \boldsymbol{e}$, one can reduce the error equation (2.7) to two first order differential equations as follows.

$$\dot{\boldsymbol{s}} + k\widehat{\boldsymbol{I}}\boldsymbol{s} = \boldsymbol{\eta} \tag{2.13}$$

$$\dot{\boldsymbol{e}} + \lambda \boldsymbol{e} = \boldsymbol{s} \tag{2.14}$$

First, it will be shown that the following relation is satisfied in (2.13) if the feedback gain k is determined by (2.11).

$$\parallel \boldsymbol{s} \parallel < \lambda \varepsilon_P \qquad \forall t \geq t_0 \tag{2.15}$$

To this end, a Lyapunov candidate is considered:

$$V_1 = \frac{1}{2} s^T s \tag{2.16}$$

Differentiating $V_1(t)$ along (2.13), one can see that the following relation holds, provided that $\| s \| \geq \lambda \varepsilon_P$, by using the information on the modeling error of (2.3) and (2.8), and the feedback gain k in (2.11).

$$\begin{aligned} \dot{V}_1 &= s^T(\eta - k\hat{I}s) \\ &< \beta \| y_d \| \| s \| - k\alpha \| s \|^2 \\ &\leq \| s \| (\beta \| y_d \| - k\alpha\lambda\varepsilon_P) = 0 \end{aligned} \tag{2.17}$$

Therefore, one can get (2.15) with the initial condition $s(t_0) = \mathbf{o}$.

Next in order to show the first part of (2.12), the extended error s in (2.14) is regarded as the disturbance with the condition (2.15), and the norm of e is evaluated in the same way as the evaluation of $\| s \|$. Consider the following Lyapunov candidate.

$$V_2 = \frac{1}{2} e^T e \tag{2.18}$$

Differentiating $V_2(t)$ along (2.14) and using (2.15), one obtains the following relation provided that $\| e \| \geq \varepsilon_P$.

$$\begin{aligned} \dot{V}_2 &= e^T(s - \lambda e) \\ &\leq \| e \| \| s \| - \lambda \| e \|^2 \\ &\leq \| e \| (\| s \| - \lambda\varepsilon_P) < 0 \end{aligned} \tag{2.19}$$

Therefore, one gets the first part of (2.12) with the initial condition $e(t_0) = \mathbf{o}$.

Lastly, the second part of (2.12) is proven. The relation $\| \dot{e} \| \leq \| s \| + \lambda \| e \|$ is obtained from (2.14). Therefore the second part of (2.12) is shown by using (2.10), (2.15), and the former part of (2.12). This completes the proof. ∎

The proposed robust control method has three features compared with the conventional robust control methods: (i) The controller structure is very simple because it is based on the conventional dynamic control method. (ii) The argument is much simpler (see the proof of Theorem 2.1). The employment of the error equation (2.7) (namely, (2.13) and (2.14)) enables this kind of simplification. (iii) The proposed method has a less conservative evaluation in determining the feedback gain, because the measurable signals are fully exploited by making use of Feature 2.2 of the manipulator.

2.4 Adaptive robust control

In the proposed robust control law of the previous section, γ, which expresses a bound of the uncertainty, is calculated in advance based on the knowledge of the region $\boldsymbol{\Omega}$. However γ depends on the information on $\boldsymbol{\Omega}$, and often tends to be unnecessarily high for the specified tracking precision. Therefore in this section, an adaptive robust control method is proposed which adjusts the feedback gain adaptively, that is, it estimates the parameter γ, in order to achieve the specified tracking precision without any a priori information on $\boldsymbol{\Omega}$. The estimate of γ is denoted by $\widehat{\gamma}$.

Here, the same control law is considered as the previous robust control law (2.6). Thus, the error equation is again (2.7), namely (2.13) and (2.14). It is shown that (2.5) is guaranteed by suitably adjusting the feedback gains $k(t)$ in (2.13) and λ in (2.14). As a preparation, the following lemma is given.

[Lemma 2.1] *Consider the manipulator (2.1) that satisfies Assumptions 2.1 to 2.3. Assume that positive constants ε and λ are given, and $\rho > 0$, the gain of the adaptation law, is given. Also let $\| \boldsymbol{y}_{d+} \| \triangleq \| \boldsymbol{y}_d \| + \delta$, where δ is an arbitrary small positive number. If the gain k is given by*

$$k = \frac{\widehat{\gamma} \| \boldsymbol{y}_{d+} \|}{\lambda\varepsilon} \tag{2.20}$$

$$\dot{\widehat{\gamma}} = \begin{cases} \rho \| \boldsymbol{y}_{d+} \| \| \boldsymbol{s} \| & \text{if} \quad \| \boldsymbol{s} \| \geq \lambda\varepsilon \\ 0 & \text{if} \quad \| \boldsymbol{s} \| < \lambda\varepsilon \end{cases}, \quad \widehat{\gamma}(t_0) \geq 0 \tag{2.21}$$

and the input $\boldsymbol{u}(t)$ is given by (2.6), then

$$\dot{V}_3 < 0 \qquad \text{for} \quad \| \boldsymbol{s} \| \geq \lambda\varepsilon \tag{2.22}$$

holds along the trajectory of the system (2.1), where $V_3(t)$ is a Lyapunov candidate:

$$V_3(t) = \frac{1}{2\alpha}\boldsymbol{s}^T\boldsymbol{s} + \frac{1}{2\rho}(\gamma - \widehat{\gamma})^2 \tag{2.23}$$

∎

Proof: Differentiating $V_3(t)$ along (2.13), one obtains the following relation provided that $\| \boldsymbol{s} \| \geq \lambda\varepsilon$, by using the information on the modeling error of (2.3) and (2.8), the feedback gain k given by (2.20), and the parameter adaptation law (2.21).

$$\begin{aligned}\dot{V}_3 &= \frac{1}{\alpha}s^T(\boldsymbol{\eta} - k\widehat{\boldsymbol{I}}\boldsymbol{s}) - \frac{1}{\rho}(\gamma - \widehat{\gamma})\dot{\widehat{\gamma}} \\ &< \gamma \parallel \boldsymbol{y}_{d+} \parallel\parallel \boldsymbol{s} \parallel - k \parallel \boldsymbol{s} \parallel^2 - (\gamma - \widehat{\gamma}) \parallel \boldsymbol{y}_{d+} \parallel\parallel \boldsymbol{s} \parallel \\ &\leq \parallel \boldsymbol{s} \parallel (\widehat{\gamma} \parallel \boldsymbol{y}_{d+} \parallel - k\lambda\varepsilon) = 0 \end{aligned} \tag{2.24}$$

Thus one gets $\dot{V}_3 < 0$ provided that $\parallel \boldsymbol{s} \parallel \geq \lambda\varepsilon$. This completes the proof. ∎

Remark 2.1 *If $\parallel \boldsymbol{y}_d \parallel$ is non-zero provided that $\parallel \boldsymbol{s} \parallel \geq \lambda\varepsilon$, one can replace $\parallel \boldsymbol{y}_{d+} \parallel$ by $\parallel \boldsymbol{y}_d \parallel$.* ∎

Based on the above Lemma, the following result is obtained.

[Theorem 2.2] *Consider the manipulator given by (2.1) that satisfies Assumptions 2.1 to 2.3. Suppose that $\boldsymbol{\theta}_d$ is given, and ε_P and ε_V, the specified tracking precision, are also given. Moreover the adaptation gain ρ and the initial value $\widehat{\gamma}$ are given. If $\varepsilon(< \varepsilon_P)$ is given, and the control law (2.6) whose feedback gains, λ and k, are given as (2.10), (2.20), and (2.21), is applied to the manipulator, then there exists a finite time $T(\geq t_0)$ which satisfies*

$$\parallel \boldsymbol{e}(t) \parallel < \varepsilon_P, \qquad \parallel \dot{\boldsymbol{e}}(t) \parallel < \varepsilon_V \qquad \forall t \geq T \tag{2.25}$$

∎

Proof: First, we show the boundedness of all the signals in the system. Noting that $\dot{\widehat{\gamma}} = 0$ holds when $\parallel \boldsymbol{s} \parallel < \lambda\varepsilon$, it is easily verified by using Lemma 2.1 that $V_3(t)$ of (2.23) is bounded for any t. Therefore one obtains that $\widehat{\gamma}$ and $\boldsymbol{s}$ are bounded. Let the maximum value of $\parallel \boldsymbol{s} \parallel$ be s_{max}. Then one gets

$$\parallel \boldsymbol{e}(t) \parallel < s_{max}/\lambda \qquad \forall t \geq t_0 \tag{2.26}$$

in the same way as (2.18) and (2.19) of the robust control. So $\boldsymbol{\theta}$ is bounded. Moreover from the boundedness of $\boldsymbol{e}$ and $\boldsymbol{s}$, it is obtained that $\dot{\boldsymbol{\theta}}$ is bounded. Therefore it is proved that $\boldsymbol{u}$ is bounded and so is $\ddot{\boldsymbol{\theta}}$.

Next, we prove that there exists a finite time $t_N(\geq t_0)$ such that

$$\parallel \boldsymbol{s}(t) \parallel < \lambda\varepsilon_P \qquad \forall t \geq t_N \tag{2.27}$$

We consider the time $q_{ij}(i = 0, 1; j = 1, 2, \cdots)$ such that

$$\parallel \boldsymbol{s}(q_{ij}) \parallel = \lambda\varepsilon \tag{2.28}$$

and

$$\parallel \boldsymbol{s}(t) \parallel \geq \lambda\varepsilon \qquad \forall t \in [q_{0j}, q_{1j}] \tag{2.29}$$

Since the proof of (2.27) is trivial in the case that j is finite, only the case that q_{ij} $(i = 1, 2)$ is an infinite sequence in j is considered. Then since $\boldsymbol{s}(t)$ is continuous in t and $\dot{\boldsymbol{s}}(t)$ is bounded, there exists a positive constant ε_1 which satisfies the following condition for any j:

$$| \| \boldsymbol{s}(t) \| - \| \boldsymbol{s}(q_{0j}) \| | < \varepsilon_1 \mid q_{1j} - q_{0j} \mid \qquad \forall t \in [q_{0j}, q_{1j}] \quad (2.30)$$

On the other hand, let $\Delta q_j \triangleq q_{1j} - q_{0j} (> 0)$, then Δq_j converges to zero as $j \to \infty$. This is because, if Δq_j is not a convergent sequence to zero, which means that $\sum_{j=1}^{\infty} \Delta q_j$ is infinite, then it is inconsistent with the boundedness of $\widehat{\gamma}$ obtained by (2.21). Thus there exists some N which satisfies the following condition provided that $\varepsilon_P > \varepsilon$:

$$| \Delta q_j | < \frac{\lambda(\varepsilon_P - \varepsilon)}{\varepsilon_1} \qquad \forall j \geq N \quad (2.31)$$

Therefore from (2.28), (2.30), and (2.31), one obtains that there exists a finite time $t_N (\triangleq q_{0N})$ which satisfies the condition of (2.27).

Lastly by using (2.14) and (2.27), the relation (2.25) is proved in a similar way to the second part of the proof of Theorem 2.1. This completes the proof. ∎

This theorem shows that the tracking error precision is explicitly specified based on the quantitative relation between the control error and the design parameters. Note that the above point was not clear in the conventional adaptive robust control method [27, 21, 81, 24, 73]. Furthermore, the structure of the proposed controller is much simpler than the conventional ones. For example, one needs only one parameter adaptation law (2.21), and its formulation is quite clear. The employment of Feature 2.2 enabled this kind of simplification.

Compared with the robust control method in section 2.3, the proposed adaptive robust control method has the advantages that the feedback gain is automatically determined by using the parameter adaptation law (2.21), and that $\widehat{\gamma}$ is independent of a priori information on $\boldsymbol{\Omega}$. The parameter adaptation law has the following physical interpretation. If the norm of $\boldsymbol{s}$ is greater than or equal to the specified value (i.e., $\lambda\varepsilon$), then the feedback gain k is considered to be too small, so $\widehat{\gamma}$ is increased (i.e., $\dot{\widehat{\gamma}} > 0$). Otherwise, the gain k is considered to be sufficiently large, so $\widehat{\gamma}$ is not renewed (i.e., $\dot{\widehat{\gamma}} = 0$). Thus one can expect that the adaptation mechanism produces the necessary and minimum feedback gain which provides the specified tracking accuracy. However one should note that the adaptive robust control method achieves the specified tracking accuracy only in the steady-state, while the robust control method achieves it all the time.

Finally, the proposed method is compared to the adaptive control methods (e.g. [50, 113, 92]). The proposed method adjusts the feedback gain so as to achieve the specified tracking precision, while the adaptive control methods try to do the on-line estimation of the physical parameters. Therefore these two methods are different from each other in essence. The proposed method is easy to cope with the unexpected disturbances, since the robustness of the proposed method relies on the feedback structure.

2.5 Experiment

In this section, the validity of the proposed adaptive robust control method is demonstrated by some experiments. In these experiments, the validity from the following viewpoint is considered.

<V1> Will the proposed adaptive robust control achieve the specified tracking precision after some finite time without any a priori information on the uncertainty ?

<V2> Will the feedback gain of the adaptive robust control method become smaller than that of the robust control method for the same specified tracking precision ?

For the experiment, a 2 link Direct-Drive (for simplicity, DD) arm built by Shin Meiwa Industry Co., LTD. is used (see Fig.2.1). The length of link 1 and 2 is 250(mm) and 300(mm) respectively. The joint angle and angular velocity are detected by an optical encoder at each joint, and are sent through a PI/O board to the host computer (NEC PC-9801DA with an 80387 numerical processor). The signal of each joint driving input is supplied through a D/A converter to a driver amplifier. Assembler and C language are used. Sampling period is 1.36 ms.

Now in order to design the controller, the model of this DD arm is given as follows. Let m_i, I_i, l_i, and l_{gi} denote the mass of link i, the moment of inertia of link i about the center of mass, the length of link i, and the distance between joint i and the center of mass of link

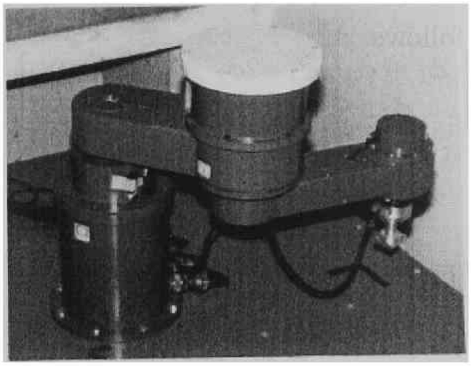

Figure 2.1: 2 link DD arm

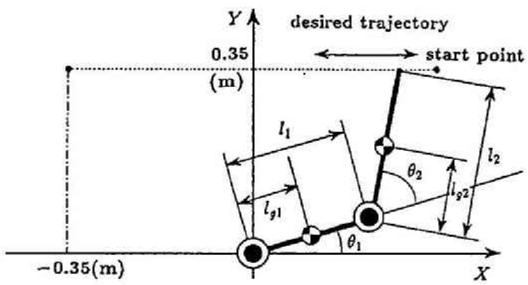

Figure 2.2: Model of 2 link arm

i ($i = 1, 2$), respectively. The physical parameters ϕ_1, ϕ_2, and ϕ_3 are defined as $\phi_1 = m_1 l_{g1}^2 + I_1 + m_2 l_1^2$, $\phi_2 = m_2 l_1 l_{g2}$, and $\phi_3 = m_2 l_{g2}^2 + I_2$. Then the dynamic equation of the manipulator shown in Fig.2.2 is described by

$$\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{2.32}$$

$$\boldsymbol{\theta} = [\theta_1, \theta_2]^T, \quad \boldsymbol{u} = [u_1, u_2]^T, \quad \boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^T$$

$$\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta}) = \begin{bmatrix} \phi_1 + \phi_3 + 2\phi_2 C_2 & \phi_3 + \phi_2 C_2 \\ \phi_3 + \phi_2 C_2 & \phi_3 \end{bmatrix}$$

$$\boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\phi_3 S_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \phi_3 S_2 \dot{\theta}_1^2 \end{bmatrix}$$

where $S_j \triangleq \sin\theta_j$, $C_j \triangleq \cos\theta_j$ $(j = 1, 2)$. Therefore $\boldsymbol{E}$ and $\boldsymbol{y}$ of (2.2) can be selected as follows.

$$E(\phi) = \begin{bmatrix} \phi_1 + \phi_3 & \phi_3 & 2\phi_2 & \phi_2 & \phi_2 & 0 \\ \phi_3 & \phi_3 & \phi_2 & 0 & 0 & \phi_2 \end{bmatrix}$$

$$y(\theta, \dot{\theta}, \ddot{\theta}) = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ C_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_2 \\ -S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ S_2\dot{\theta}_1^2 \end{bmatrix}$$

Although we do not have any a priori information on the range $\boldsymbol{\Omega}$, we assume that the estimates of ϕ_i $(i = 1, 2, 3)$, $\widehat{\phi}_i$, are given as $\widehat{\phi}_1 = 1.8$, $\widehat{\phi}_2 = 0.48$, and $\widehat{\phi}_3 = 0.49$.

The desired trajectory of the end effector is given by

$$\begin{aligned} x_d &= 0.35\cos(\pi t) \quad (\mathrm{m}) \\ y_d &= 0.35 \quad (\mathrm{m}) \quad \text{for} \quad 0 \le t \le 6.0(s) \end{aligned} \tag{2.33}$$

which has the period of 2 seconds (see Fig.2.2). The desired trajectory of each joint $(\theta_{d1}, \theta_{d2})$ is calculated from (x_d, y_d) in advance. The initial errors of position and velocity are $\boldsymbol{e}(0) = \mathbf{o}$ and $\dot{\boldsymbol{e}}(0) = \mathbf{o}$, respectively.

In this situation, the following two experiments are made.

Experiment I This experiment is concerned with [V1]. The tracking precision is specified as $\varepsilon_P = 0.03$ and $\varepsilon_V = 0.6$ in the proposed adaptive robust control. Then one obtains $\lambda = 10.0$ from (2.10) and $\varepsilon = 0.0299$ from $\varepsilon < \varepsilon_P$. Also set the initial estimate in (2.21) as $\widehat{\gamma}(0) = 0.0$, the adaptation gain in (2.21) as $\rho = 0.35$, and δ in (2.20) and (2.21) as $\delta = 0.01$.

Fig.2.3 shows the result. Fig.2.3(a) and (b) show $\| \boldsymbol{s} \|$, the norm of the extended error, and $\widehat{\gamma}$, the parameter estimate. From these figures, one can see the following: the parameter estimate $\widehat{\gamma}$ is adjusted when the norm of $\boldsymbol{s}$ is greater than or equal to $\lambda\varepsilon$, and after the second period (2 seconds), the norm of $\boldsymbol{s}$ is smaller than $\lambda\varepsilon$ and $\widehat{\gamma}$ is not renewed. Fig.2.3(c) and (d) show the feedback gain k and the control error norm $\| \boldsymbol{e} \|$, respectively. After the second period (2 seconds), an appropriate feedback gain is obtained to achieve the specified tracking error precision. A similar result concerning the velocity error $\dot{\boldsymbol{e}}$ has also been obtained. Moreover Fig.2.3(e) shows the joint driving input, and so we can see that the input is very smooth. Therefore the validity

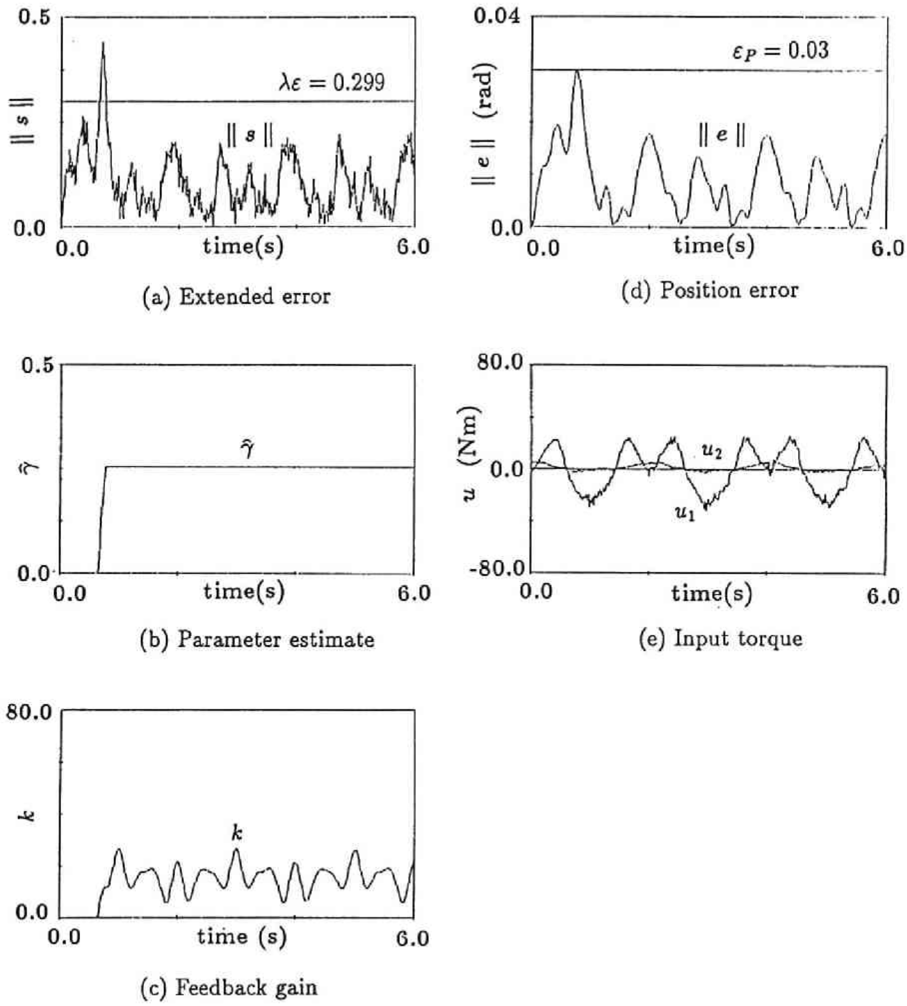

Figure 2.3: Experimental results of the adaptive robust control: (a) Extended error; (b) Parameter estimate; (c) Feedback gain; (d) Position error; (e) Input torque.

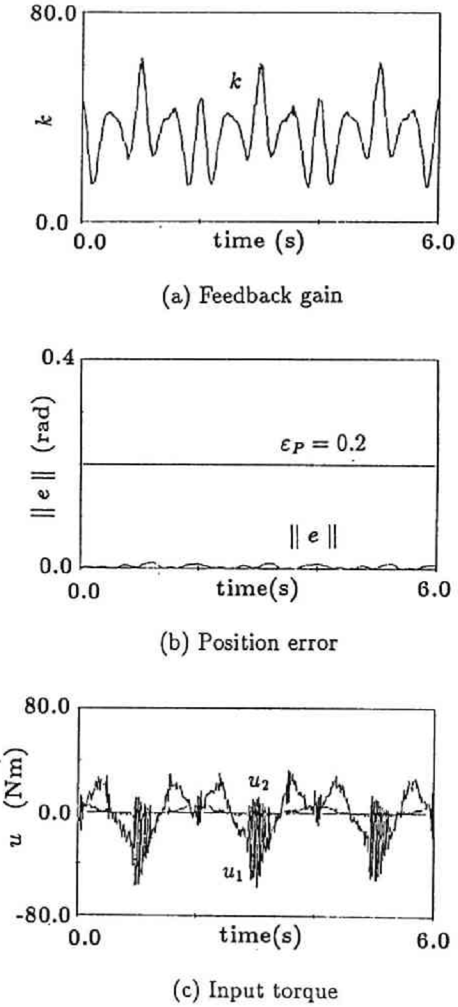

Figure 2.4: Experimental results of the robust control: (a) Feedback gain; (b) Position error; (c) Input torque.

about [V1] is verified from these results.

Experiment II In this experiment, the robust control in section 2.3 is applied to this arm and [V2] is discussed. It is assumed that the estimation error of $\phi_i (i = 1, 2, 3)$ is within $\pm 20\%$, and by using this information, γ in (2.11) is determined as $\gamma = 3.39$. First the same tracking precision as in Experiment I is set, that is, $\varepsilon_P = 0.03$ and $\varepsilon_V = 0.6$. However in this case, remarkable chattering occurred in the joint driving input, and the real trajectory could not track the desired trajectory at all. This is because the feedback gain is too large. So the specified tracking precision is set as $\varepsilon_P = 0.2$ and $\varepsilon_V = 4.0$. Then $\lambda = 10.0$. Experimental results for this case is shown in Fig.2.4. Even in this case, we can see that the chattering is very large in Fig.2.4(c), and that the real control error is about one tenth of the specified precision in Fig.2.4(b). This is the result of too high gain feedback for the specified precision as shown Fig.2.4(a). Certainly, the feedback gain depends on the information on the uncertainty, but for example, even in the case that the real parameters exist within $\pm 10\%$ of the estimated values, γ is 4 times larger than in the adaptive robust control of Experiment I. In this way the evaluation of the bound of uncertainty, γ, tends to be conservative in the robust control method. On the other hand, Fig.2.3(d) shows that the bound of the norm of control error is about 60% of the specified tracking precision in the adaptive robust control method. This means that the feedback gain is almost necessary and minimum for the specified precision. Furthermore, the adaptation gain ρ is reset as $\rho = 0.3$ in order to achieve the same specified precision by a smaller feedback gain. In this case, the feedback gain is 90 % of the case of $\rho = 0.35$, and the bound of the norm of control error is about 70% of the specified tracking precision. From these results, we confirm the usefulness of the proposed adaptive robust control law with respect to [V2].

The same results about [V1] and [V2] for several different values of ε_P and ε_V have also been obtained. These results show the validity of the proposed adaptive robust control.

Remark 2.2 *In this experimental system, there exists about ± 20 % torque distortion because of the fact that the motor driver is not adjusted well. In spite of this wrong adjustment, the proposed method has* *achieved the specified error precision. Probably this is because of the fact that the torque uncertainty is acted on the adaptive robust controller as something like the perturbation of the physical parameters.* ∎

2.6 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A new robust control scheme of robot manipulators with uncertainty has been proposed, which is almost as simple as that of the dynamic control method, and has a less conservative evaluation in determining the feedback gain, because it makes use of the effective expression (i.e., <P2>) of the dynamics of the robot manipulator.

(ii) Based on the above robust control, a new adaptive robust control scheme of robot manipulators with uncertainty has been proposed, in addition to the merits in (i), where the tracking precision is explicitly specified and, as a result, it is possible to evaluate if the feedback gain is small enough for the specified tracking precision.

(iii) By the experiment of the trajectory control of a 2 link DD arm, it has been verified that the feedback gain of the adaptive robust control method is much smaller than that of the robust control method, and is almost necessary and minimum for the specified tracking precision.

Chapter 3

ROBUST CONTROL OF ROBOT MANIPULATORS BASED ON ACCELERATION INFORMATION

3.1 Introduction

In the previous chapter, a robust tracking control method of robot manipulators is treated, where only information on position and velocity is available. In the case of the manipulator, it is relatively ease to get acceleration information by acceleration sensors, as well as the position and velocity information. So what merits will be added in the robust control, if acceleration information is available in addition to the position and velocity information? The purpose of this chapter is to clarify the merits of acceleration information in the robust control of robot manipulators.

Acceleration feedback control methods of robot manipulators have been studied since the early 1980s (e.g., [43]). In the late 1980s, a scheme which is called a disturbance observer [87, 90, 125] or a scheme which is called a time delay control scheme [83, 146, 49, 48] have been developed independently. However, both schemes are based on the same idea that the interference force in each arm or the uncertainty in the physical parameters is compensated by feeding back the difference between the input and the acceleration, and the control problem for the multi-variable system is reduced to that for a single-input single-output system on each joint. However, there is little discussion on the reason why the uncertainty is compensated by such an acceleration feedback system, which is the essence of this control system, or what problems there are in this method. In addition, in the analysis of the effect of the uncertainty on the closed loop system, the disturbance observer treats the manipulator which originally belongs to a class of nonlinear systems as a linear system with a step disturbance, and the time-delay control scheme assumes that the state at some time is equivalent to the state at the time before one step. Both methods lacks the rigorous discussion. It is needed to analyze the effect of the uncertainty on the control error rigorously.

In the first part of this chapter, the essence of the conventional acceleration feedback control system is made clear, and it is shown what their problems are. Second, a new robust control scheme based on the acceleration information is proposed, which overcomes the above problems in the conventional methods. Especially, the proposed method shows one idea on how to express the uncertainty which exists in the manipulator and how to compensate for it by fully exploiting acceleration information. Finally, the advantages of acceleration information for the proposed scheme are clarified, and simulation results show the effectiveness of the proposed scheme.

3.2 Discussion on conventional acceleration feedback systems

In this section, the essence of conventional acceleration feedback systems is discussed, and some disadvantages are clarified.

Consider a manipulator with n degrees of freedom whose dynamics is described by the following equation :

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{3.1}$$

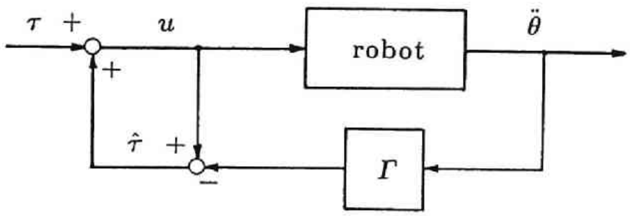

Figure 3.1: Conventional control system

where $\boldsymbol{\theta} \triangleq [\theta_1, \theta_2, \ldots, \theta_n]^T$ is the n-dimensional vector of joint displacements, $\boldsymbol{u}$ is the n-dimensional joint torque input vector, $\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta})$ is the $n \times n$ manipulator inertia matrix, and $\boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is the n-dimensional vector that represents the nonlinear terms such as centrifugal, Coriolis, frictional, and gravitational forces.

Then in the conventional approach, (3.1) is rewritten as follows.

$$\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} = \boldsymbol{u} + [(\boldsymbol{\Gamma} - \boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta}))\ddot{\boldsymbol{\theta}} - \boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})] \tag{3.2}$$

where $\boldsymbol{\Gamma} \triangleq \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, and $\gamma_i (i = 1, 2, \cdots, n)$ is a positive number. Here assume that the second term of the right-hand side in (3.2) is a disturbance to the system $\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} = \boldsymbol{u}$. Then if the angler acceleration $\ddot{\boldsymbol{\theta}}$ is available by some sensor, we can know the value of the disturbance, i.e., the second term of the right-hand side in (3.2) by calculating the term $\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} - \boldsymbol{u}$. Thus the following controller is considered.

$$\begin{cases} \boldsymbol{u} = \hat{\boldsymbol{\tau}} + \boldsymbol{\tau} \\ \hat{\boldsymbol{\tau}} = \boldsymbol{u} - \boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} \end{cases} \tag{3.3}$$

where $\hat{\boldsymbol{\tau}}$ is an input to compensate for the disturbance, and $\boldsymbol{\tau}$ is a new input. This control system is shown in Fig.3.1. The controller given by (3.3) leads to the following closed loop system.

$$\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} = \boldsymbol{\tau} \tag{3.4}$$

Therefore, a control problem for multi-input multi-output nonlinear system given by (3.1) is reduced to a control problem for a decoupled system, i.e., each joint system. In addition, in the conventional approach using acceleration information, we do not have to know the real

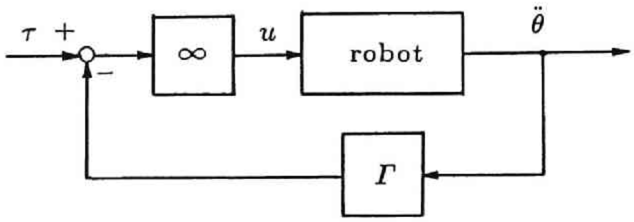

Figure 3.2: Equivalent control system

value of the physical parameters, and the nonlinearity of the manipulator is compensated for by acceleration feedback. This may be one feature of the conventional approach.

Why can the conventional approach using acceleration information compensate for the nonlinearity of the manipulator without the real values of the physical parameters and reduce a multivariable system to a decoupled system? Then Fig.3.1 is rewritten to Fig.3.2 , where we can find the essence of the conventional acceleration feedback system: the conventional system contains a infinite feedback gain at all frequency, which compensates for the nonlinearity of the manipulator. In addition, by feeding back the acceleration of each joint to each corresponding joint, namely, by making a matrix $\boldsymbol{\Gamma}$ diagonal, the system in question is decoupled.

In such a system, there are the following problems. First, we cannot define the closed loop system which includes infinite gain mathematically, although we can write the system in a block diagram. In addition, we cannot implement the system which includes the infinite gain, because the system has more or less time lag. [1] Second, it is difficult to estimate the effect of the uncertainty which cannot be compensated for when the $\hat{\tau}$ is fed back through some filter in order to eliminate the

[1] Hsia [49] and Mizutani [82] discuss the infinite gain of the closed loop system. However we discuss it here, from a different point of view in the sense that we do not share the advantages of the infinite gain of the conventional approach, while they do.

disadvantage due to the infinite gain. In fact, the disturbance observer [87, 90, 125] uses some filter such as a low-pass filter. However, the uncertainty which cannot be compensated for owing to the use of the filter is regarded as a step disturbance, and its effect on the system is discussed in the field of the linear system theory, because it is not easy to estimate its effect rigorously. In addition, in the case of the time-delay control method, which corresponds to the case that (3.3) is replaced by

$$\begin{cases} \boldsymbol{u}(t) = \hat{\boldsymbol{\tau}}(t) + \boldsymbol{\tau}(t) \\ \hat{\boldsymbol{\tau}}(t) = \boldsymbol{u}(t - t_L) - \boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}}(t - t_L) \end{cases} \tag{3.5}$$

where t_L is the time lag, the effect of the uncertainty which cannot be compensated for is not considered since supposing that $\boldsymbol{u}(t-t_L) = \boldsymbol{u}(t)$ and $\ddot{\boldsymbol{\theta}}(t - t_L) = \ddot{\boldsymbol{\theta}}(t)$ [146].

Third, the time-delay control method does not theoretically guarantee the internal stability of the closed loop system, that is, the boundedness of all the signals of the closed loop system, because the closed loop system becomes a nonlinear system including the time lag, and it is much difficult to analyze the stability [48].

The above disadvantages result from feedback of the input signal as shown in Fig.3.1. So in the next section, a new approach using acceleration information will be given to eliminate all the above disadvantages.

3.3 Robust control based on acceleration information

The discussion in section 3.2 gives the following remarks when the uncertainty such as nonlinearity is compensated for by acceleration feedback.

(a) There is no infinite gain in the closed loop system.

(b) The effect of the uncertainty which cannot be compensated for on the control error is rigorously estimated, namely, for a given desired trajectory $\boldsymbol{\theta}_d(t) \in \boldsymbol{R}^n$ with twice derivatives which are bounded, and a given design parameter $\boldsymbol{\varepsilon} \triangleq [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]^T$, the control error $\boldsymbol{e} \triangleq \boldsymbol{\theta}(t) - \boldsymbol{\theta}_d(t)$ satisfies

$$| e_i(t) | \leq \varepsilon_i \ \ \forall t \geq 0, \quad i = 1, \cdots, n$$

when $\boldsymbol{e}(0) = \mathbf{o}$ and $\dot{\boldsymbol{e}}(0) = \mathbf{o}$.

(c) It is guaranteed that all the signal of the closed loop system is bounded.

Now a control method satisfying the above specifications is given. In the same way as section 2.2, we can express the left-hand side of (3.1) as

$$\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{E}(\boldsymbol{\phi})\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.6}$$

where $\boldsymbol{E}(\boldsymbol{\phi})$ is an appropriate dimensional matrix consisting of physical parameters, and $\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ is an appropriate dimensional vector whose elements are known functions of $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ (see section 3.5).

Some assumptions are also made.

[Assumption 3.1] *$\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ are measurable.* ∎

[Assumption 3.2] *The values of the physical parameter vector $\boldsymbol{\phi}$ may be unknown, but it is known that $\boldsymbol{\phi}$ exists in a certain bounded region $\boldsymbol{\Omega}$.* ∎

[Assumption 3.3] *$\widehat{\boldsymbol{\phi}}$, a bounded estimate of $\boldsymbol{\phi}$, is given.* ∎

For simplicity, we use the notations $\widehat{\boldsymbol{M}}$, $\widehat{\boldsymbol{h}}$, and $\widehat{\boldsymbol{E}}$ in place of $\boldsymbol{M}(\widehat{\boldsymbol{\phi}}, \boldsymbol{\theta})$, $\boldsymbol{h}(\widehat{\boldsymbol{\phi}}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, and $\boldsymbol{E}(\widehat{\boldsymbol{\phi}})$.

One can get, from (3.6) and $\widetilde{\boldsymbol{E}} \triangleq \widehat{\boldsymbol{E}} - \boldsymbol{E}$,

$$\widehat{\boldsymbol{M}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \widehat{\boldsymbol{h}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{u} + \widetilde{\boldsymbol{E}}\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.7}$$

Then by regarding the second term of the right-hand side of (3.7) as a disturbance to the system $\widehat{\boldsymbol{M}}\ddot{\boldsymbol{\theta}} + \widehat{\boldsymbol{h}} = \boldsymbol{u}$, we consider the following controller.

$$\boldsymbol{u} = \boldsymbol{u}_{D1} + \boldsymbol{u}_{D2} + \boldsymbol{u}_R \tag{3.8}$$

$$\boldsymbol{u}_{D1} = \widehat{\boldsymbol{M}}\ddot{\boldsymbol{\theta}} + \widehat{\boldsymbol{h}} \tag{3.9}$$

$$\boldsymbol{u}_{D2} = -\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} \tag{3.10}$$

where $\boldsymbol{u}_{D1}$ is an input which compensates for the nonlinearity of the manipulator as possible, by using the estimated value of the physical parameters, and $\boldsymbol{u}_{D2}$ is an input which gives a desired inertia of the manipulator. A matrix $\boldsymbol{\Gamma}$ is nonsingular, but it is not necessarily diagonal. $\boldsymbol{u}_R$ is an input to compensate for the uncertainty which cannot be compensated for by $\boldsymbol{u}_{D1}$, namely, the modeling error. Then substituting (3.8) $\sim$ (3.10) into (3.7), we get

$$\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} = \boldsymbol{u}_R + \widetilde{\boldsymbol{E}}\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.11}$$

which corresponds to (3.4). Therefore, the control problem for (3.7) is reduced to that for the case of $\boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} = \boldsymbol{u}_R$ where the term $\widetilde{\boldsymbol{E}}\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ is regarded as a disturbance term. In addition, the term $\widetilde{\boldsymbol{E}}\boldsymbol{y}$ has the following property. There is a non-negative function $g_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ $(i = 1, 2, \cdots, n)$ such that, for all $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$,

$$g_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \geq \mid \boldsymbol{a}_i \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \mid \tag{3.12}$$

where $\boldsymbol{a}_i$ is the ith row vector of a matrix $\boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{E}}$. An example of g_i in the case of a 2 d.o.f manipulator is shown in section 3.5. In the next discussion, a function g_i satisfying (3.12) is assumed to be given.

Now we consider the following input as a $\boldsymbol{u}_R$.

$$\boldsymbol{u}_R = \boldsymbol{\Gamma}\{\ddot{\boldsymbol{\theta}}_d - (\boldsymbol{\Lambda} + \boldsymbol{K})\dot{\boldsymbol{e}} - \boldsymbol{K}\boldsymbol{\Lambda}\boldsymbol{e}\} \tag{3.13}$$

where $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, and $\lambda_i > 0 (i = 1, 2, \cdots, n)$ is a positive constant. $\boldsymbol{\Lambda}$ is a design parameter that specifies the control precision. $\boldsymbol{K}$ is a time-varying gain matrix, and $\boldsymbol{K} = \text{diag}\{k_1, k_2, \cdots, k_n\}$. Then substituting (3.13) into (3.11), we get the error equation:

$$\ddot{e} + (\boldsymbol{\Lambda} + \boldsymbol{K})\dot{e} + \boldsymbol{K}\boldsymbol{\Lambda}\boldsymbol{e} = \boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{E}}\boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.14}$$

or equivalently

$$\ddot{e}_i + (\lambda_i + k_i)\dot{e}_i + k_i\lambda_i e_i = \boldsymbol{a}_i \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \quad i = 1, \cdots, n \tag{3.15}$$

Based on this equation, the following theorem is obtained, where the relation between the design parameters λ_i and k_i and the control error e_i is clarified.

[Theorem 3.1] *Consider the manipulator given by (3.1) that satisfies Assumptions 3.1 to 3.3. Suppose a desired trajectory $\boldsymbol{\theta}_d$ with twice derivatives which are bounded is given, and a design parameter vector $\boldsymbol{\varepsilon}$ and a matrix $\boldsymbol{\Lambda}$ are given. Assume also that a time-varying gain k_i is given by*

$$k_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) = \frac{1}{\lambda_i \varepsilon_i} g_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.16}$$

$(i = 1, 2, \cdots, n)$. Then if the control law given by (3.8), (3.9), (3.10), and (3.13) is applied to the manipulator subject to $e(0) = \dot{e}(0) = 0$, then

$$\mid e_i(t) \mid \leq \varepsilon_i, \quad \mid \dot{e}_i(t) \mid \leq 2\lambda_i \varepsilon_i \tag{3.17}$$

holds for all $t \geq 0$, and $i = 1, 2, \cdots, n$, when the angular acceleration $\ddot{\boldsymbol{\theta}}$ is not infinite for any finite time. ∎

Proof: Define

$$s_i \triangleq \dot{e}_i + \lambda_i e_i \tag{3.18}$$

Then, from (3.15) we get

$$\dot{s}_i + k_i s_i = \boldsymbol{a}_i \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.19}$$

which is the first order differential equation. Thus the bound of s_i is estimated. Consider as a non-negative function

$$V_i = \frac{1}{2} s_i^2 \tag{3.20}$$

Then differentiating (3.20) along (3.19), one gets

$$\dot{V}_i = -s_i k_i s_i + s_i \boldsymbol{a}_i \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.21}$$

Eqns.(3.16) and (3.21) imply

$$\dot{V}_i = -\frac{g_i}{\lambda_i \varepsilon_i} \mid s_i \mid^2 + s_i \boldsymbol{a}_i \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) \tag{3.22}$$

Then if the acceleration $\ddot{\boldsymbol{\theta}}$ is not infinite for any finite time, one obtains, from (3.12) and (3.22),

$$\dot{V}_i < -\frac{g_i}{\lambda_i \varepsilon_i} \mid s_i \mid (\mid s_i \mid - \varepsilon_i \lambda_i) \qquad \forall t \geq 0 \tag{3.23}$$

which means

$$\mid s_i \mid \geq \lambda_i \varepsilon_i \quad \Rightarrow \quad \dot{V} < 0 \tag{3.24}$$

Since it is assumed that $e_i(0) = 0$ and $\dot{e}_i(0) = 0$, (3.24) implies

$$\mid s_i \mid < \lambda_i \varepsilon_i \qquad \forall t \geq 0 \tag{3.25}$$

Next, the bound of e_i is estimated. It can be verified that

$$\mid e_i \mid < \varepsilon_i \qquad \forall t \geq 0 \tag{3.26}$$

in the same way as the case of s_i. Also one can show the case of $\dot{e}_i$, by noting

$$\mid \dot{e}_i \mid < \mid s_i \mid + \lambda_i \mid e_i \mid \qquad \forall t \geq 0 \tag{3.27}$$

This completes the proof. ∎

It has been shown that the specification (b) holds under the assumption that the acceleration is not infinite in Theorem 3.1. Now based on the result of Theorem 3.1, a design procedure of the control system satisfying the specifications (b) and (c) is given.

It can be easily verified that, for a $\boldsymbol{g} = [g_1, g_2, \cdots, g_n]^T$ satisfying (3.12), there exist a matrix $\bar{\boldsymbol{K}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d)$ and a vector $\boldsymbol{f}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d)$ such that the equation

$$\boldsymbol{\Gamma K}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})(\dot{e} + \boldsymbol{\Lambda} e) = \bar{\boldsymbol{K}} \ddot{\boldsymbol{\theta}} + \boldsymbol{f} \tag{3.28}$$

holds, if one choose g_i as an appropriate function. An example on (3.28) is shown in section 3.5. Then the design procedure is as follows.

Step 1 Give a desired trajectory $\boldsymbol{\theta}_d$,$\dot{\boldsymbol{\theta}}_d$,$\ddot{\boldsymbol{\theta}}_d$ and a specified tracking precision $\boldsymbol{\varepsilon}$ and $\boldsymbol{\Lambda}$.

Step 2 Choose a function $\boldsymbol{g}$ satisfying (3.12) and find a matrix $\bar{\boldsymbol{K}}$ satisfying (3.28).

Step 3 Consider a set $\boldsymbol{\Psi}_\varepsilon$ which consists of all the element $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ to be characterized by $(\boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d)$ and (3.17) (See the definition of $\boldsymbol{\Psi}_\varepsilon$ in (3.A3) of Appendix). Then find a matrix $\boldsymbol{\Gamma}$ such that, for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\varepsilon$,

$$\det(\boldsymbol{M} - \widehat{\boldsymbol{M}} + \boldsymbol{\Gamma} + \bar{\boldsymbol{K}}) \neq 0 \tag{3.29}$$

holds.

Step 4 Give a control law which composed of (3.8), (3.9), (3.10), and (3.13).

It can be proven that a controller designed according to the above procedure guarantees the boundedness of the acceleration for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\varepsilon$, and satisfies the specifications (b) and (c) by Theorem 3.1. The proof is shown in Appendix.

In section 3.2, it is clarified that, in the case of conventional acceleration feedback systems, the nonlinearity of a manipulator is compensated for by high gain feedback, which is due to positive feedback of an input. On the other hand, the controller proposed here includes a nonlinear compensation, i.e., $\boldsymbol{u}_{D1}$, which compensates for a known nonlinearity and a feedback with respect to position and velocity, i.e. $\boldsymbol{u}_R$, which compensates for a unknown nonlinearity, as shown in Fig.3.3. Hence the proposed controller has no infinite gain.

Therefore, the controller designed according to the proposed design procedure satisfies three specifications (a), (b) and (c).

Remark 3.1 *The matrix $\boldsymbol{\Gamma}$ which corresponds to a desired inertia matrix needs to be a nonsingular one satisfying (3.29), which does not need to be positive definite.* ∎

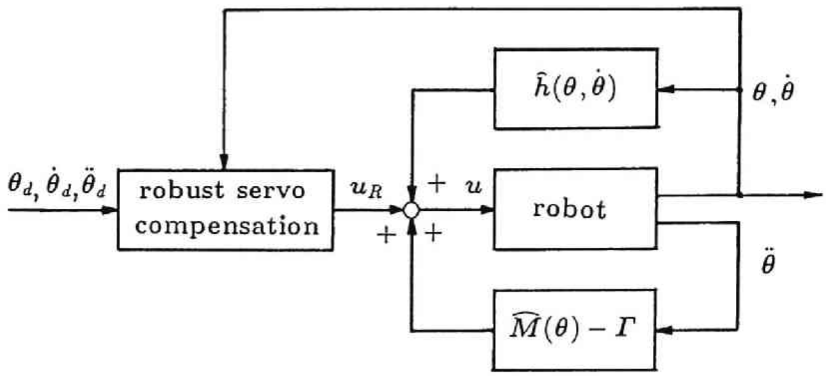

Figure 3.3: Proposed control system

3.4 Discussion

3.4.1 Advantages of acceleration information

Some merits of acceleration information are clarified here, comparing a control method using acceleration information with that without it. The former corresponds to the proposed control method, and the latter corresponds to a robust control method that is treated in chapter 2.

In the robust control method treated in chapter 2, which uses no acceleration information, the following control law is applied to a robot manipulator (3.1).

$$u = \widehat{M}\{\ddot{\theta}_d - (\lambda + k)\dot{e} - k\lambda e\} + \widehat{h} \tag{3.30}$$

where λ is a positive constant, and $k = k(\theta, \dot{\theta}, \ddot{\theta}_d - \lambda\dot{e})$. Then we get, as an error equation,

$$\begin{aligned}&\ddot{e} + (\lambda + kM^{-1}\widehat{M})\dot{e} + k\lambda M^{-1}\widehat{M}e \\ &= M^{-1}\widetilde{E}y(\theta, \dot{\theta}, \ddot{\theta}_d - \Lambda\dot{e})\end{aligned} \tag{3.31}$$

Based on this error equation, the appropriate feedback gain k is given so as to control the effect of the uncertainty. The acceleration information is not required to calculate a gain k because the term in the right-hand side of (3.31), which is concerned with the uncertainty, is independent on the acceleration $\ddot{\theta}$. However, since the uncertainty is appeared in the term with the gain k in the right-hand side of (3.31), it is difficult to estimate the effect of the uncertainty for each joint, and it is estimated for all joints altogether in terms of a singular value. This leads to a conservative estimation and a too high feedback gain. In addition, although the robust control method treated in chapter 2 also achieves the specified tracking precision, the control error is estimated in terms of the Euclidean norm, and so the control error for each joint angler cannot be specified independently. It is also required that $\boldsymbol{M}^{-1}\widehat{\boldsymbol{M}}$ is a co-positive definite matrix.

On the other hand, in the case of the robust control method proposed here, which uses the acceleration information, the effect of the uncertainty is estimated for each joint as you can easily see in (3.14), and so the control error is specified for each joint angler. In addition, we can give any desired inertia $\boldsymbol{\Gamma}$, although it is required that $\boldsymbol{\Gamma}$ and $\boldsymbol{M} - \widehat{\boldsymbol{M}} + \boldsymbol{\Gamma} + \bar{\boldsymbol{K}}$ are nonsingular.

In summary, the advantages of the use of the acceleration information are : (i) the effect of the uncertainty on the control error is estimated for each joint, and so the estimate is easier and is less conservative than the case without acceleration information, (ii) the control precision can be specified for each joint angler, and (iii) a larger class of the desired inertia $\boldsymbol{\Gamma}$ is specified.

3.4.2 Relation to the conventional control methods

It is stated that the proposed robust control law given by (3.8), (3.9), (3.10), and (3.13) includes the conventional control laws as a special case.

When $\widehat{\boldsymbol{M}} = \boldsymbol{M}$ and $\widehat{\boldsymbol{h}} = \boldsymbol{h}$ in (3.9), the control law is

$$\boldsymbol{u} = \boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{h} - \boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}} + \boldsymbol{u}_R \tag{3.32}$$

Then the control law given by (3.32) corresponds to the disturbance observer [87, 90, 125] or the time delay control [83, 146, 49, 48], regarding the term $\boldsymbol{M}\ddot{\boldsymbol{\theta}} + \boldsymbol{h} - \boldsymbol{\Gamma}\ddot{\boldsymbol{\theta}}$ in (3.32) as an observer of the disturbance. Note that, however, the proposed robust control law (3.32) does not include the infinite gain in the closed loop system.

When $\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}$ in (3.10), the control law is

$$\boldsymbol{u} = \widehat{\boldsymbol{M}}\{\ddot{\boldsymbol{\theta}}_d - (\boldsymbol{\Lambda} + \boldsymbol{K})\dot{\boldsymbol{e}} - \boldsymbol{K}\boldsymbol{\Lambda}\boldsymbol{e}\} + \widehat{\boldsymbol{h}} \tag{3.33}$$

Then the control law given by (3.33) corresponds to that in the dynamic control method [77] or in the robust control method obtained in chapter

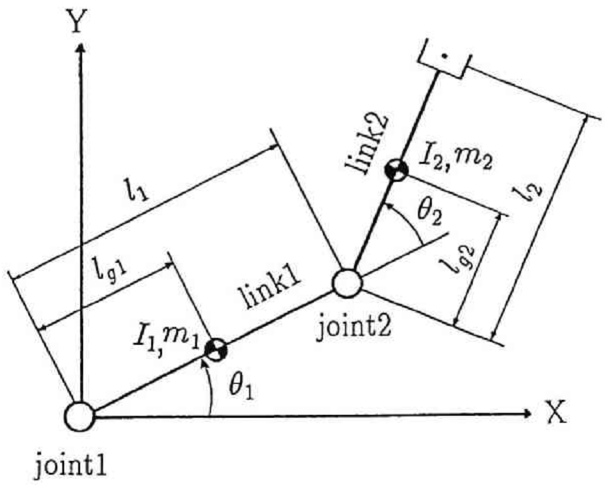

Figure 3.4: Model of 2 d.o.f. manipulator

2, which use no acceleration information. Eq. (3.33) still requires the acceleration information, but if we can estimate the maximum value of the acceleration in (3.A4) according to the design procedure in section 3.3 and the acceleration $\ddot{\boldsymbol{\theta}}$ in k is replaced by the maximum value, then the controller achieves a specified tracking precision for each joint angler and requires no acceleration information.

3.5 Simulation

This section shows a simulation result of the trajectory control of a 2 d.o.f. manipulator shown in Fig.3.4 to verify the effectiveness of the proposed control method.

Let the mass of the jth link be m_j, the moment of the inertia about the center of the mass of the jth link be I_j, the length of the jth link be l_j, the distance between the jth joint and the center of the mass of the jth link be $l_{gj}(j = 1, 2)$. Also let

$$\phi_1 = m_1 l_{g1}^2 + I_1 + m_2 l_1^2$$
$$\phi_2 = m_2 l_1 l_{g2}$$
$$\phi_3 = m_2 l_{g2}^2 + I_2$$

Table 3.1: Unknown parameters of manipulator

Unknown parameter	Inf. value	Sup. value	Real value	Nominal value
m_1 (kg)	6.0	8.0	6.0	7.0
m_2 (kg)	6.0	8.0	6.0	7.0
I_1 (kgm^2)	0.2	0.4	0.2	0.3
I_2 (kgm^2)	0.2	0.4	0.2	0.3

Then we get the dynamic equation of the manipulator in Fig.3.4:

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{3.34}$$

$$\boldsymbol{\theta} = [\theta_1, \theta_2]^T, \quad \boldsymbol{u} = [u_1, u_2]^T$$

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta}) = \begin{bmatrix} \phi_1 + \phi_3 + 2\phi_2 C_2 & \phi_3 + \phi_2 C_2 \\ \phi_3 + \phi_2 C_2 & \phi_3 \end{bmatrix}$$

$$\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\phi_2 S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \phi_2 S_2 \dot{\theta}_1^2 \end{bmatrix}$$

where $S_j \triangleq \sin\theta_j$ and $C_j \triangleq \cos\theta_j$ $(j = 1, 2)$, and the gravity term is omitted for simplicity. Set $\boldsymbol{E}$ and $\boldsymbol{y}$ as

$$\boldsymbol{E} = \begin{bmatrix} \phi_1 + \phi_3 & \phi_3 & 2\phi_2 & \phi_2 & \phi_2 & 0 \\ \phi_3 & \phi_3 & \phi_2 & 0 & 0 & \phi_2 \end{bmatrix}$$

$$\boldsymbol{y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}}) = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ C_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_2 \\ -S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ S_2\dot{\theta}_1^2 \end{bmatrix}$$

which are verified to satisfy (3.6). The real and estimated values of the physical parameters are shown in Table 3.1. Also $l_1 = l_2 = 0.5$(m) and $l_{g1} = l_{g2} = 0.25$(m) are used.

Now a robust controller is designed according to the design procedure proposed in section 3.3.

Step 1 : For the desired trajectory given by

$$\begin{aligned} \theta_{d1}(t) &= -\cos(\pi t/3) \\ \theta_{d2}(t) &= -\sin(\pi t/3) - 2.0 \qquad \text{for} \qquad 0 \le t \le 6.0 \end{aligned} \tag{3.35}$$

Set 0.01(rad) as control precision with respect to the position, and 0.02(rad/s) as control precision with respect to the velocity for each joint. Then from (3.17), we get $\varepsilon_i = 0.01$ and $\lambda_i = 1.0 (i = 1, 2)$.

Step 2 : Let $\widehat{\phi}_i$ be the estimate of ϕ_i, and

$$\alpha_i = \max \mid \phi_i - \widehat{\phi}_i \mid \qquad i = 1, 2, 3$$

Also let α_4 be an appropriate positive constant. Then if one chooses a function $\boldsymbol{g}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$ as

$$\boldsymbol{g} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} \alpha_1 + \alpha_3 & \alpha_3 & 2\alpha_2 & \alpha_2 & \alpha_2 & 0 \\ \alpha_3 & \alpha_3 & \alpha_2 & 0 & 0 & \alpha_2 \end{bmatrix} \times \begin{bmatrix} \mid \ddot{\theta}_1 \mid \\ \mid \ddot{\theta}_2 \mid \\ \mid C_2 \mid\mid \ddot{\theta}_1 \mid \\ \mid C_2 \mid\mid \ddot{\theta}_2 \mid \\ \mid S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \mid \\ \mid S_2\dot{\theta}_1^2 \mid \end{bmatrix} + \begin{bmatrix} \alpha_4 \\ \alpha_4 \end{bmatrix} \tag{3.36}$$

where $\boldsymbol{\Gamma} = \text{diag}\{\gamma_1, \gamma_2\}(\gamma_i > 0)$, one can verify that $\boldsymbol{g}$ given by (3.36) satisfies (3.12). According to this, one gets $\alpha_1 = 0.4125$, $\alpha_2 = 0.125$, and $\alpha_3 = 0.1625$. Also set $\alpha_4 = 0.01$. Then $\bar{\boldsymbol{K}}$ of (3.28) is given by

$$\bar{\boldsymbol{K}} = \begin{bmatrix} \frac{s_1}{\varepsilon_1\lambda_1} & 0 \\ 0 & \frac{s_2}{\varepsilon_2\lambda_2} \end{bmatrix} \begin{bmatrix} \alpha_1 + \alpha_3 + 2\alpha_2 \mid C_2 \mid & \alpha_3 + \alpha_2 \mid C_2 \mid \\ \alpha_3 + \alpha_2 \mid C_2 \mid & \alpha_3 \end{bmatrix} \times \begin{bmatrix} \text{sgn}(\ddot{\theta}_1) & 0 \\ 0 & \text{sgn}(\ddot{\theta}_2) \end{bmatrix}$$

where sgn(·) is a sign function.

Step 3 : Set $\gamma_1 = 5.0$ and $\gamma_2 = 5.0$ so as to satisfy (3.29) for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\varepsilon$

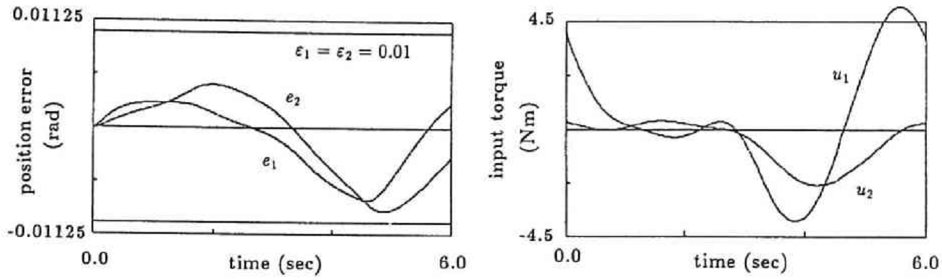

Figure 3.5: Simulation results

Step 4 : The controller is given by (3.8), (3.9), (3.10), and (3.13).

The sampling period is 1(msec).

Then simulation results are shown in Fig.3.5, which shows that the control error for each joint angle is within the specified tracking precision, and the control inputs are bounded. It is also verified that all the signals are bounded in this simulation. The simulation results show the validity of the proposed control method.

3.6 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) It has been pointed out that the conventional acceleration feedback system compensates for the uncertainty by high gain feedback essentially, and the use of acceleration feedback gain matrix which is diagonal reduces a multivariable control problem to a decoupled control problem.

(ii) The disadvantages of the conventional acceleration feedback method have been clarified; (a) there exists a infinite feedback gain in the closed loop system, (b) there is no analytic discussion on the effect of the uncertainty which cannot be compensated for on the control error, and (c) there is no discussion on the boundedness of all the signals of the closed loop systems.

(iii) A robust tracking control methods using acceleration information for a robot manipulator with uncertainties has been proposed, where the acceleration information is fully exploited and the disadvantages of the conventional control methods are overcome.

(iv) Comparing with the robust control methods without acceleration information, the advantages of the proposed method with acceleration information have been shown; (a) the effect of the uncertainty on the control error is estimated for each joint, so the estimate is less conservative, (b) the control precision can be specified for each joint angler independently, and (c) we can give a larger class of a desired inertia.

Appendix

Proof: Define the following set for a given desired trajectory $\boldsymbol{\theta}_d(t)$ and $\dot{\boldsymbol{\theta}}_d(t)$, a given vector $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_n]^T \in \boldsymbol{R}^n$, and a given matrix $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\} \in \boldsymbol{R}^{n\times n}$.

$$\begin{aligned} &\boldsymbol{\Psi}_0(\boldsymbol{\theta}_d(t), \dot{\boldsymbol{\theta}}_d(t), \boldsymbol{\xi}, \boldsymbol{\Lambda}) \\ \triangleq\ &\{(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) \mid\ \mid \theta_i - \theta_{di} \mid \le \xi_i,\ \mid \dot{\theta}_i - \dot{\theta}_{di} \mid \le 2\lambda_i\xi_i, \\ &i = 1, \cdots, n\} \end{aligned} \tag{3.A1}$$

In addition let us define the following sets given by

$$\boldsymbol{\Psi}_\omega \triangleq \boldsymbol{\Psi}_0(\boldsymbol{\theta}_d(t), \dot{\boldsymbol{\theta}}_d(t), \boldsymbol{\omega}, \boldsymbol{\Lambda}) \tag{3.A2}$$

$$\boldsymbol{\Psi}_\varepsilon \triangleq \boldsymbol{\Psi}_0(\boldsymbol{\theta}_d(t), \dot{\boldsymbol{\theta}}_d(t), \boldsymbol{\varepsilon}, \boldsymbol{\Lambda}) \tag{3.A3}$$

where $\boldsymbol{\omega} \in \boldsymbol{R}^n$ is a constant vector that satisfies $\boldsymbol{\Psi}_\varepsilon \subseteq \boldsymbol{\Psi}_\omega$, namely, $\varepsilon_i \le \omega_i$

From (3.1), (3.8), (3.9), (3.10), (3.13), (3.16), and (3.28), we get

$$\begin{aligned} &(\boldsymbol{M} - \widehat{\boldsymbol{M}} + \boldsymbol{\Gamma} - \bar{\boldsymbol{K}})\ddot{\boldsymbol{\theta}} \\ &= \boldsymbol{\Gamma}(\ddot{\boldsymbol{\theta}}_d - \boldsymbol{\Lambda}\dot{\boldsymbol{e}}) + \widehat{\boldsymbol{h}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \boldsymbol{f}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d) \end{aligned} \tag{3.A4}$$

Then it is verified that

$$\mid \ddot{\boldsymbol{\theta}} \mid < \infty, \qquad \forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\omega \tag{3.A5}$$

if the following relation holds for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\omega$.

$$\det(\boldsymbol{M} - \widehat{\boldsymbol{M}} + \boldsymbol{\Gamma} + \bar{\boldsymbol{K}}) \neq 0 \tag{3.A6}$$

Now suppose that there exists a matrix $\boldsymbol{\Gamma}$ satisfying (3.A6) for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\omega$. Let

$$\boldsymbol{\Psi}_s \triangleq \{\boldsymbol{s}(t) \mid \mid s_i \mid \leq \lambda_i \omega_i, i = 1, \cdots, n\} \tag{3.A7}$$

Then it is verified that the acceleration $\ddot{\boldsymbol{\theta}}$ is bounded for all $\boldsymbol{s} \in \boldsymbol{\Psi}_s$, noting that $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\omega$ for all $\boldsymbol{s} \in \boldsymbol{\Psi}_s$ because of the assumption on the initial state error. Hence (3.24) in the proof of Theorem 3.1 holds for all $\boldsymbol{s} \in \boldsymbol{\Psi}_s$. It can also be shown that (3.25) holds because a nonnegative function V_i of (3.20) has a maximum value when $\mid s_i \mid < \lambda_i \varepsilon_i$, and (3.17) holds. Therefore if we use a matrix Γ which satisfies (3.A6) for $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\omega$, (3.17) holds. It can also be verified that, when a specified control precision $\boldsymbol{\varepsilon}$, $\boldsymbol{\Lambda}$ are given in advance, we have only to use a matrix Γ which satisfies (3.A6) for all $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Psi}_\varepsilon$.

In summary, if we design a control law according to the design procedure given by section 3.3, then (3.17) holds, the boundedness of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ is always guaranteed, and so is the boundedness of $\ddot{\boldsymbol{\theta}}$. That is to say, the specifications (a) to (c) are achieved.

Finally, the existence of $\boldsymbol{\Gamma}$ in (3.A6) is discussed. Assume that $\boldsymbol{\Gamma}$ is given by $\boldsymbol{\Gamma} = \text{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\} (\gamma_i > 0)$, and g_i satisfying (3.12) and (3.28) is given by $g_i = \gamma_i \hat{g}_i$. Let $\boldsymbol{N} \triangleq \boldsymbol{M} - \widehat{\boldsymbol{M}} + \bar{\boldsymbol{K}}$. If

$$\det(\boldsymbol{I} + \boldsymbol{\Gamma}^{-1}\boldsymbol{N}) \neq 0 \tag{3.A8}$$

then (3.29) holds because $\det(\boldsymbol{\Gamma}) \neq 0$. Note that $\bar{\boldsymbol{K}}$ does not contain $\gamma_i (i = 1, 2, \cdots, n)$. Hence if $\det(\boldsymbol{\Gamma}) \neq 0$ holds, the fact that the eigenvalues of $\boldsymbol{\Gamma}^{-1}\boldsymbol{N}$ are -1 is equivalent to (3.A6). Thus, for example, if there exists a $\boldsymbol{\Gamma}$ such that, for all $\boldsymbol{N}$,

$$\sigma_{min}(\boldsymbol{\Gamma}) > \sigma_{max}(\boldsymbol{N}) \tag{3.A9}$$

then (3.A6) holds, where $\sigma_{min}(\cdot)$ and $\sigma_{max}(\cdot)$ are the minimum singular value and the maximum singular value, respectively.

Chapter 4

ROBUST CONTROL OF ROBOT MANIPULATORS BASED ON JOINT TORQUE INFORMATION

4.1 Introduction

In the previous chapter, a robust tracking control method of robot manipulators is treated where acceralation informations are available. In the case of a manipulator, it is not so difficult to get joint torque information using a torque sensor built in each joint. This chapter focuses on the robust control of robot manipulators in the case that the joint torque information is available.

In the past decade, various methods utilizing the information of a joint torque sensor in each joint have been developed to improve the torque control performance of actuators of a robot arm [141, 76, 5, 101, 135]. Some experiments have shown that the utilization of joint torque information is effective to compensate for the nonlinearity of the actuator such as friction.

Recently, using a different viewpoint from the above methods, robust tracking control methods using joint torque information have been proposed by Kosuge [68, 69] and Hashimoto [42]. In these methods, uncertainties of links and an end-effector are regarded as a part of load on each joint axis, and the joint torque information, which is equivalent to the load on the joint axes, is fed back to the joint driving torque to cancel out the dynamics, including uncertainty. Hence the resultant control system becomes robust. To design a controller based on this idea, a model of manipulators in the case where torque sensors are available is proposed by Kosuge [68], whose equation gives the inverse dynamics in a recursive form. Although the dynamic equation is effective in calculating the inverse dynamics, it is not convenient for control system design in the case where the coupling terms of the links and the modeling error of the actuator system are not negligible. This is because the dynamic equation has no explicit expression of the total structure which is inherently a nonlinear multi-input/multi-output system. So the dynamic equation that explicitly expresses the total structure is desired for the design of the control system.

In this chapter, a robust control method of robot manipulators based on joint torque information is proposed. First, a dynamic equation of a robot manipulator with torque sensors is derived, where a nonlinear multivariable structure is explicitly described. Some features on the structure are clarified. For instance, the coefficient matrix of the joint angular acceleration is nonsingular and lower triangular, and the total dynamics are given in a form such that the link dynamics is implicitly contained in the torque sensor signal. This dynamic equation makes it possible to design the control system of a robot manipulator with torque sensors based on the similar method used in the conventional case without torque sensors, for example, the dynamic control method [77].

Second, it is shown that the proposed dynamic equation is effective for the design of the robust control system against the uncertainty of the actuator system, which has never been considered in previous methods using torque information. The proposed controller achieves the specified tracking precision in the presence of the modeling error, where torque information is fully exploited to compensate for the uncertainty of the links and the load at the end-effector. In chapter 2, the robust tracking control method is treated in the case where the joint torque sensor is not available, which achieves the specified tracking performance against parameter uncertainties. Compared with the robust method in chapter 2, the proposed method needs less knowledge about the uncertainties. For example, no a priori information on the bounds of the uncertainties of the links and the load at the end-effector is required, because these uncertainties are compensated for by torque information. Furthermore, it is shown that the proposed method requires less computational time to calculate the control input than the conventional methods. Finally, some simulation results are given to confirm the effectiveness of the use of joint torque information.

4.2 Model of a manipulator with joint torque sensors

In this section, we derive a dynamic equation of the manipulator with joint torque sensors, and state some features of this equation. Furthermore, we consider the derived dynamic equation from the viewpoint of the control system design.

4.2.1 Derivation of dynamic equation

We consider a serial link manipulator with n rotary joints that has a joint torque sensor in each joint. As is shown in Fig.4.1 , the dynamics of each link are divided into two dynamic systems namely, the motor system and the link system, by regarding each joint torque sensor as the border, like Kosuge [68]: the motor system includes a rotor and a speed reducer, and the link system is composed of a link. Then we define coordinate frames as follows. The origin of coordinate frame Σ_i of the ith link is set on the ith joint axis. The Z axis of Σ_i is selected in such a way that it aligns with the ith joint axis, and the unit vector in the direction of the Z axis of Σ_i is denoted by $\boldsymbol{z}_i$. The ith motor, which drives the ith joint, is fixed to the $(i-1)$th link, and the origin of coordinate frame Σ_{mi} of the ith motor is set on the axis of rotation of the rotor in the ith motor, called the ith rotor. The Z axis of Σ_{mi} is selected in such a way that it aligns with the axis of rotation of the ith rotor, and the unit vector in the direction of the Z axis of Σ_{mi} is denoted by $\boldsymbol{z}_{mi}$. Let the moment of inertia of the ith rotor about the axis of rotation be I_{mi} and the viscous friction coefficient of the ith motor system be

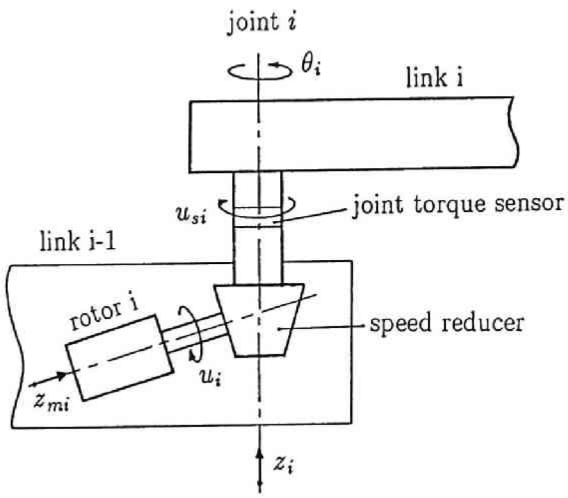

Figure 4.1: Joint model of a robot arm

b_{mi}. The output torque of the ith rotor is denoted by u_i, and u_{si} denotes the coupling force by the other motor systems and link systems, that is, the load exerted on the ith motor system that can be measured by joint torque sensor on the ith joint. Moreover, we use the following notations: $\boldsymbol{u} \triangleq [u_1, u_2, \ldots, u_n]^T \in \boldsymbol{R}^n$, $\boldsymbol{u}_s \triangleq [u_{s1}, u_{s2}, \ldots, u_{sn}]^T \in \boldsymbol{R}^n$, $\boldsymbol{\phi}_I \triangleq [I_{m1}, I_{m2}, \cdots, I_{mn}]^T$, $\boldsymbol{\phi}_B \triangleq [b_{m1}, b_{m2}, \cdots, b_{mn}]^T$, and $\boldsymbol{\Gamma} \triangleq \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, where $\gamma_i (\geq 1)$ represents the reduction ratio at the ith joint.

Now the following assumptions are made.

[Assumption 4.1] *Each rotor is symmetric with respect to the axis of rotation.* ∎

[Assumption 4.2] *The torsion at each joint due to the flexibility of the torque sensor is small enough that it can be ignored, so the joint axis is regarded as a rigid one.* ∎

[Assumption 4.3] *The transmitted force does not fail at the speed reducer, and the inertia between the torque sensor and the speed reducer is negligible.* ∎

Then the dynamic equation of the manipulator with joint torque sensors is derived as follows:

$$M(\phi_I, \theta)\ddot{\theta} + h(\phi_I, \phi_B, \theta, \dot{\theta}) + \Gamma^{-1} u_s = u \tag{4.1}$$

where $\theta \triangleq [\theta_1, \theta_2, \ldots, \theta_n]^T \in R^n$ is a joint angle vector, the (i, j) element of $M \in R^{n \times n}$, denoted by M_{ij}, is

$$M_{ij} \triangleq \begin{cases} I_{mi}\gamma_i & \text{if } i = j \\ I_{mi} z_{mi}^T z_j & \text{if } i > j \\ 0 & \text{if } i < j \end{cases} \tag{4.2}$$

and the ith element of $h \in R^n$, denoted by h_i, is

$$h_i \triangleq \begin{cases} b_{mi}\gamma_i\dot{\theta}_i & \text{if } i = 1, 2 \\ I_{mi} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} z_{mi}^T (z_k \times z_j)\dot{\theta}_k\dot{\theta}_j + b_{mi}\gamma_i\dot{\theta}_i & \text{if } i \geq 3 \end{cases} \tag{4.3}$$

Here $\times$ denotes the vector product. See Appendix for the proof of eq.(4.1).

Next we clarify some features of the dynamic equation given by eq.(4.1).

[Feature 4.1] *Nonlinear terms M and h contain only the physical parameters of the motor systems, ϕ_I and ϕ_B. On the other hand, the load u_s that acts on the motor system consists of the dynamics of link systems and external force.* ∎

[Feature 4.2] *The matrix M is nonsingular and lower triangular.* ∎

[Feature 4.3] *The first and second terms of the left-hand side of eq.(4.1) can be expressed as*

$$M(\phi_I, \theta)\ddot{\theta} + h(\phi_I, \phi_B, \theta, \dot{\theta}) = E(\phi_I, \phi_B) y(\theta, \dot{\theta}, \ddot{\theta}) \tag{4.4}$$

where $E(\phi_I, \phi_B)$ is an appropriate dimensional matrix consisting of physical parameters, and $y(\theta, \dot{\theta}, \ddot{\theta})$ is an appropriate dimensional vector whose elements are known functions of θ, $\dot{\theta}$, and $\ddot{\theta}$. ∎

[Feature 4.4] *The diagonal element M_{ii} of M is in proportion to reduction ratio γ_i. So if the reduction ratio is high, then the effect of the coupling term in the motor systems is small.* ∎

4.2.2 Advantages of the derived dynamic equation

We discuss some advantages of the dynamic equation (4.1) from the viewpoint of control system design. Let $\boldsymbol{\tau} \in \boldsymbol{R}^n$ be a new input. The following control law is considered.

$$\boldsymbol{u} = \boldsymbol{\Gamma}^{-1}\boldsymbol{u}_s + \boldsymbol{\tau} \tag{4.5}$$

Substituting eq.(4.5) into eq.(4.1), we obtain

$$\boldsymbol{M}(\boldsymbol{\phi}_I, \boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi}_I, \boldsymbol{\phi}_B, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{\tau} \tag{4.6}$$

This means that we do not have to consider the link dynamics at all when the joint torque information is available, and that the control problem of a robot manipulator can be reduced to that of the motor system. Moreover we easily see that the control system of eq.(4.6) can be designed in a similar way as the control method of the robot manipulator without joint torque sensors (For example, Luh [77]). Namely, the desired trajectory $\boldsymbol{\theta}_d(t)$ is assumed to be given whose derivatives $\dot{\boldsymbol{\theta}}_d$ and $\ddot{\boldsymbol{\theta}}_d$ exist and are bounded, and the control law $\boldsymbol{\tau}$ is given as follows:

$$\boldsymbol{\tau} = \boldsymbol{M}(\ddot{\boldsymbol{\theta}}_d - \boldsymbol{K}_d\dot{e} - \boldsymbol{K}_p e) + \boldsymbol{h} \tag{4.7}$$

where $\boldsymbol{e} \triangleq \boldsymbol{\theta} - \boldsymbol{\theta}_d$ is the control error, and $\boldsymbol{K}_p$ and $\boldsymbol{K}_d \in \boldsymbol{R}^{n\times n}$ are appropriate position and velocity gain matrices. Then from eqs.(4.6), (4.7), and 4.2, we obtain the error equation

$$\ddot{e} + \boldsymbol{K}_d\dot{e} + \boldsymbol{K}_p e = \mathbf{o} \tag{4.8}$$

Hence, if $\boldsymbol{K}_p = k_p\boldsymbol{I}$ and $\boldsymbol{K}_d = k_d\boldsymbol{I}$, where k_p and k_d are positive constants and $\boldsymbol{I}$ is a unit matrix, then $\boldsymbol{e}(t) \rightarrow \mathbf{o}(t \rightarrow \infty)$, and $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ are bounded. Furthermore, since $\boldsymbol{u}_s$ is a function of $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}$, and external force (note that the formulation of $\boldsymbol{u}_s$ can be explicitly expressed by Lagrangian method), the signal $\boldsymbol{u}_s$ is bounded when the external force is bounded. Therefore all the signals of the closed loop system given by eqs.(4.1), (4.5), and (4.7) are bounded.

Although the above discussion is about the joint servo system, it is straightforward that the control system can be designed even in the operational space. Now we consider the variable at the operational coordinates given by $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{\theta}) \in \boldsymbol{R}^n$. The relation between $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{\theta}}$ is given by $\dot{\boldsymbol{r}} = \boldsymbol{J}\dot{\boldsymbol{\theta}}$, where $\boldsymbol{J}$ is the Jacobian matrix. Then assuming that $\boldsymbol{J}$ is nonsingular in a certain region of $\boldsymbol{\theta}$, we consider the following control algorithm

$$\boldsymbol{\tau} = \boldsymbol{M}\boldsymbol{J}^{-1}(\ddot{\boldsymbol{r}}_d - \boldsymbol{K}_d\dot{\boldsymbol{e}}_r - \boldsymbol{K}_p\boldsymbol{e}_r - \dot{\boldsymbol{J}}\dot{\boldsymbol{\theta}}) + \boldsymbol{h} \tag{4.9}$$

where $\boldsymbol{r}_d$ is a desired trajectory expressed in the operational coordinates, and the control error is given by $\boldsymbol{e}_r \stackrel{\triangle}{=} \boldsymbol{r} - \boldsymbol{r}_d$. The error equation in this case is similar to eq.(4.8), and we get the same result as the joint servo system.

Moreover, when the force at the end effector, denoted by $\boldsymbol{f}$, is measurable, the control law $\boldsymbol{u}$ can be given as

$$\boldsymbol{u} = \boldsymbol{\Gamma}^{-1}(\boldsymbol{u}_s + \boldsymbol{J}^T\boldsymbol{f}) + \boldsymbol{\tau} \tag{4.10}$$

Substituting eq.(4.10) into eq.(4.1), we get

$$\boldsymbol{M}(\boldsymbol{\phi}_I, \boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi}_I, \boldsymbol{\phi}_B, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{\tau} + \boldsymbol{\Gamma}^{-1}\boldsymbol{J}^T\boldsymbol{f} \tag{4.11}$$

Therefore, the robust force control system with joint torque sensor feedback is designed in the same way as in the case of the robot manipulator without joint torque sensors.

As a result, "*we can design the control system of a robot manipulator with joint torque sensors based on the similar method used in the conventional case without joint torque sensors* ". Although this result is also pointed out by Kosuge [68], it is not clear because the dynamic equation given in a recursive form dose not clarify the explicit structure of the coupling term and whether $\boldsymbol{M}$ is nonsingular. On the other hand, our dynamic equation, where the structure of the robot manipulator with joint torque sensors is explicitly expressed, gives the above result more clearly. Furthermore, the proposed dynamic equation is effective for the design of robust control system against the uncertainty of the actuator system, which has never been considered in previous methods using joint torque information. It is stated in the next section.

4.3 Robust control design in the presence of motor system uncertainties

In this section, based on the dynamic equation in section 4.2, we take the uncertainties of the motor system into consideration to establish a more practical control system design, and propose a robust control method that achieves the specified tracking accuracy in the presence of the modeling error, where joint torque information is fully utilized to compensate for the uncertainty of link parameters etc.. Next we state some advantages of our method, compared with the conventional robust control methods without joint torque information.

We consider the following problem.

[Problem 4.1] *For the robot manipulator given by eq.(4.1), it is assumed that the desired trajectory* $\boldsymbol{\theta}_d$ *is given, and that the tracking precision* ε_P *and* ε_V *are given. Then find a control law such that*

$$\| \boldsymbol{e}(t) \| < \varepsilon_P, \quad \| \dot{\boldsymbol{e}}(t) \| < \varepsilon_V \tag{4.12}$$

holds for any $t \geq 0$. ∎

We assume that $\boldsymbol{e}(0) = \mathbf{o}$ and $\dot{\boldsymbol{e}}(0) = \mathbf{o}$ for simplicity, and that the following assumption is made.

[Assumption 4.4] *The values of the physical parameter vectors* $\boldsymbol{\phi}_I$ *and* $\boldsymbol{\phi}_B$ *are unknown, but it is known that* $\boldsymbol{\phi}_I$ *and* $\boldsymbol{\phi}_B$ *exist in known and bounded regions* Π_I *and* Π_B*, respectively. Moreover, the estimates of* $\boldsymbol{\phi}_I$ *and* $\boldsymbol{\phi}_B$*, denoted by* $\widehat{\boldsymbol{\phi}}_I$ *and* $\widehat{\boldsymbol{\phi}}_B$*, respectively, are given such that there exist bounded positive constants* α *and* β *that satisfy the following conditions for any non-zero vector* $\boldsymbol{x} \in \boldsymbol{R}^n$*, any non-zero and appropriate dimensional vector* $\boldsymbol{y}$*, any* $\boldsymbol{\theta} \in \boldsymbol{R}^n$*, any* $\boldsymbol{\phi}_I \in \Pi_I$*, and any* $\boldsymbol{\phi}_B \in \Pi_B$*:*

$$\alpha \| \boldsymbol{x} \|^2 < \boldsymbol{x}^T \widehat{\boldsymbol{I}} \boldsymbol{x}, \quad \widehat{\boldsymbol{I}} \triangleq \boldsymbol{M}^{-1}(\boldsymbol{\phi}_I, \boldsymbol{\theta}) \boldsymbol{M}(\widehat{\boldsymbol{\phi}}_I, \boldsymbol{\theta}) \tag{4.13}$$

$$\beta \| \boldsymbol{y} \| \geq \| \boldsymbol{M}^{-1}(\boldsymbol{\phi}_I, \boldsymbol{\theta}) \{ \boldsymbol{E}(\widehat{\boldsymbol{\phi}}_I, \widehat{\boldsymbol{\phi}}_B) - \boldsymbol{E}(\boldsymbol{\phi}_I, \boldsymbol{\phi}_B) \} \boldsymbol{y} \| \tag{4.14}$$

∎

Remark 4.1 *For simplicity, we use* $\beta \| \boldsymbol{y} \|$ *as the function that dominates the right-hand side of eq.(4.14). However, even in the case of* $\sum_i \beta_i g_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}})$*, where* β_i *are appropriate positive constants and* g_i *is a non-negative function, the argument here can be also applied.* ∎

Let $\widehat{\boldsymbol{M}} \triangleq \boldsymbol{M}(\widehat{\boldsymbol{\phi}}_I, \boldsymbol{\theta})$, $\widehat{\boldsymbol{h}} \triangleq \boldsymbol{h}(\widehat{\boldsymbol{\phi}}_I, \widehat{\boldsymbol{\phi}}_B, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, and $\widehat{\boldsymbol{E}} \triangleq \boldsymbol{E}(\widehat{\boldsymbol{\phi}}_I, \widehat{\boldsymbol{\phi}}_B)$. Note that α and β are obtained by calculating the smallest and largest singular values of $\widehat{\boldsymbol{I}}$ and $\boldsymbol{M}^{-1}\{\boldsymbol{E} - \widehat{\boldsymbol{E}}\}$, respectively, using the information on the bound of the uncertainty, that is, Π_I and Π_B.

Under the above assumption, we consider the same control law as given by eqs.(4.5) and (4.7) as follows.

$$\boldsymbol{u} = \boldsymbol{\Gamma}^{-1} \boldsymbol{u}_s + \widehat{\boldsymbol{M}} (\ddot{\boldsymbol{\theta}}_d - (\lambda + k) \dot{\boldsymbol{e}} - \lambda k \boldsymbol{e}) + \widehat{\boldsymbol{h}} \tag{4.15}$$

where λ is a constant gain and k is a time varying gain. Then substituting eq.(4.15) into eq.(4.1), we obtain the error equation:

$$\ddot{e} + (\lambda \boldsymbol{I} + k\widehat{\boldsymbol{I}})\dot{e} + \lambda k \widehat{\boldsymbol{I}} \boldsymbol{e} = \boldsymbol{\eta} \qquad (4.16)$$

where $\boldsymbol{\eta} \triangleq \boldsymbol{M}^{-1}(\widehat{\boldsymbol{E}} - \boldsymbol{E})\boldsymbol{y}_d$ and $\boldsymbol{y}_d \triangleq \boldsymbol{y}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}_d - \lambda\dot{e})$. If we have no modeling error, that is, if $\boldsymbol{\eta} = \mathbf{o}$ and $\widehat{\boldsymbol{I}} = \boldsymbol{I}$, then eq.(4.16) is almost the same as eq.(4.8). So $\boldsymbol{\eta}$ is treated as the disturbance that results from the modeling error, and $\widehat{\boldsymbol{I}}$ as one part of the feedback gain that contains uncertainty. The effect of these uncertainties, $\widehat{\boldsymbol{I}}$ and $\boldsymbol{\eta}$, on the control error $\boldsymbol{e}$ is evaluated by eqs.(4.13) and (4.14), respectively. That is to say, we can estimate the infimum value of the uncertain part, $\widehat{\boldsymbol{I}}$, of the feedback gain in terms of a positive number α in eq.(4.13), and the supremum value of the disturbance $\boldsymbol{\eta}$ by eq.(4.14) as follows:

$$\beta \parallel \boldsymbol{y}_d \parallel \geq \parallel \boldsymbol{\eta} \parallel \qquad (4.17)$$

Then we obtain the following result.

[Theorem 4.1] *Consider the manipulator eq.(4.1) that satisfies Assumptions 4.1 to 4.4. The desired trajectory $\boldsymbol{\theta}_d$ with twice partial derivatives and the specified tracking precision, ε_P and ε_V are given. Suppose that α in eq.(4.13) and β in eq.(4.14) are obtained from the knowledge of the regions Π_I and Π_B. If the control law eq.(4.15) whose feedback gains are given by*

$$\lambda = \frac{\varepsilon_V}{2\varepsilon_P}, \qquad k = \frac{\beta \parallel \boldsymbol{y}_d \parallel}{\alpha\lambda\varepsilon_P} \qquad (4.18)$$

is applied to the manipulator, then

$$\parallel \boldsymbol{e}(t) \parallel < \varepsilon_P, \quad \parallel \dot{\boldsymbol{e}}(t) \parallel < \varepsilon_V \qquad (4.19)$$

holds for any $t \geq 0$. ∎

A similar proof can be found in the proof of Theorem 2.1 in section 2.3. This theorem shows that it is possible to design a robust control system that achieves the specified tracking precision, through compensating for the uncertainty of the motor system by high gain feedback with eq.(4.18) and any disturbance to act on the link or the end effector etc. by joint torque sensor feedback.

The proposed robust control system based on joint torque information is constructed in the same way as the robust control method without joint torque information obtained in chapter 2. As shown in Table 4.1, however, the proposed method has some advantages, compared with the robust control method without joint torque informa-

Table 4.1: Advantage of joint torque information

	proposed method with torque information		robust control method without torque information	
uncertainty to be considered	motor system only		motor and link systems	
feedback gain	small		large	
computational load	add.	mul.	add.	mul.
	$\frac{9}{2}n^2 + \frac{11}{2}n - 17$	$\frac{13}{2}n^2 + \frac{39}{2}n - 23$	$96n - 83$	$122n - 92$
	$21n - 19$	$32n - 25$		

tion. Namely, the method proposed here has only to take account of the uncertainty of the motor system, because it compensates for the uncertainty of the link system by the joint torque sensor feedback. In the case without no joint torque information, on the other hand, the uncertainties of both the motor system and the link system are compensated for by high gain feedback of position and velocity control error, as you can see in section 2. Hence the robust control method without joint torque information requires the information on the bound of the uncertainties of not only the motor system but also the link system, and need a higher feedback gain than the method with joint torque information, in the condition of the same specified tracking precision (easily seen from eqs.(4.13) and (4.14)). High gain feedback often causes unexpected phenomena such as chattering in digital implementation.

Furthermore, as shown in Table 4.1, the computational load of the control law in the proposed method is about $1/2 \sim 1/5$ times smaller than in the robust control method without joint torque information. Note that the values in Table 4.1 is the computational amount for solving the inverse dynamics in the case that $z_{mi} = z_i (i = 1, 2, \cdots, n)$. The computational load in the robust control method without joint torque information is based on the formulation of Newton-Euler method [77]. The value of the upper section in the proposed method is the net computational load based on the dynamic equation (4.1), and the value of the lower section is the load based on the recursive formulation of the dynamic equation (4.1).

Compared with the former control methods with joint torque information [68, 42], the proposed method has quantitative evaluation of the effect of the uncertainty of motor systems and achieves the specified tracking precision. Furthermore, although we discussed the uncertainty of the motor system in this section, the proposed method can be applied even in the following case. In the high reduction ratio, the decoupling control law that $\widehat{\boldsymbol{M}}$ in eq.(4.15) is a diagonal matrix is effective because of 4.4 in section 4.2. Note that the obtained theorem holds even if $\widehat{\boldsymbol{M}}$ is any matrix to satisfy eq.(4.13). This decoupling control law is almost equivalent to that given by Hashimoto [42], but his method ignores the effect of the coupling term. On the other hand, the proposed method in this case can evaluate the effect of the coupling term that is not compensated, by calculating α in eq.(4.13) and β in eq.(4.14). Furthermore, if the maximum value of the measurement noise is known in the case of the torque information with measurement noise, the proposed method evaluates the effect on the control error.

4.4 Simulation

In this section, to verify the effectiveness of the proposed method with joint torque information, we show simulation results of trajectory control of a 2 d.o.f. direct-drive arm ($\gamma_i = 1.0, i = 1, 2$), where the desired trajectory is given by

$$\begin{aligned} \theta_{d1}(t) &= -1.8\cos(\pi t/3) \quad \text{(rad)} \\ \theta_{d2}(t) &= -1.8\cos(\pi t/3) - 1.0 \quad \text{(rad)} \\ &\qquad \text{for } 0 \le t \le 3.0 \quad \text{(sec)} \end{aligned} \tag{4.20}$$

The model of a 2 d.o.f. arm with joint torque sensors is given as follows, by using eq.(4.1)(see Fig.4.2).

$$\begin{bmatrix} I_{m1} & 0 \\ I_{m2} & I_{m2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} b_{m1}\dot{\theta}_1 \\ b_{m2}\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} u_{s1} \\ u_{s2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.21}$$

Here, we set real values of the physical parameters as shown in Table 4.2. In this situation, the only a priori information about the physical parameters of this arm is that $I_{mi}(i = 1, 2)$ is between 0.3 and 0.5, and b_{mi} is between 0.5 and 0.7. The estimate $\widehat{\phi}_I$ is given as $\widehat{\phi}_I = [0.4, 0.4]^T$ so as to satisfy the relation (4.13). The estimate $\widehat{\phi}_B$ is given as $\widehat{\phi}_B = [0.6, 0.6]^T$. Then the tracking precision is specified as $\varepsilon_P = 0.011$(rad) and $\varepsilon_V = 0.022$(rad/sec). Then $\lambda = 1.0$ by eq.(4.18). α is evaluated as $\alpha = 0.69$ by eq.(4.13), and k in eq.(4.18) is determined using eq.(4.14) as follows:

$$k = (0.45 \parallel \ddot{\boldsymbol{\theta}}_d - \lambda\dot{\boldsymbol{e}} \parallel + 0.54 \parallel \dot{\boldsymbol{\theta}} \parallel + 0.01)/(\alpha\lambda\varepsilon_p) \tag{4.22}$$

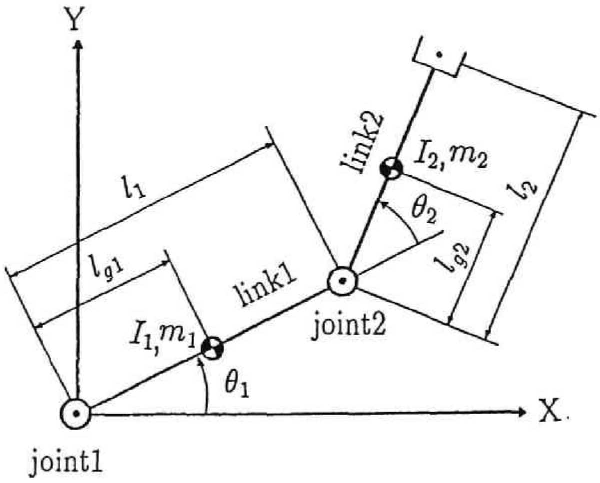

Figure 4.2: Model of 2 d.o.f. direct-drive arm

Table 4.2: Physical parameters

Moment of inertia of the ith link	$I_i = 0.4$ (kg·m^2), $i = 1, 2$
Length of the ith link	$l_i = 0.5$ (m), $i = 1, 2$
Distance between the ith joint and the center of the mass of the ith link	$l_{gi} = 0.25$ (m), $i = 1, 2$
Mass of the ith link	$m_i = \begin{cases} 5 \text{ (kg)} & \text{for } 0 \le t < 1.0 \text{ (sec)} \\ 8 \text{ (kg)} & \text{for } 1.0 \le t < 2.0 \text{ (sec)} \\ 2 \text{ (kg)} & \text{for } 2.0 \le t < 3.0 \text{ (sec)} \end{cases}$
Mass of the 2nd rotor	$m_{m2} = 2.0$ (kg)
Moment of inertia of the ith rotor	$I_{mi} = 0.3$ (kg·m^2), $i = 1, 2$
Viscous friction coefficient of the ith rotor	$b_{mi} = 0.5$, $i = 1, 2$

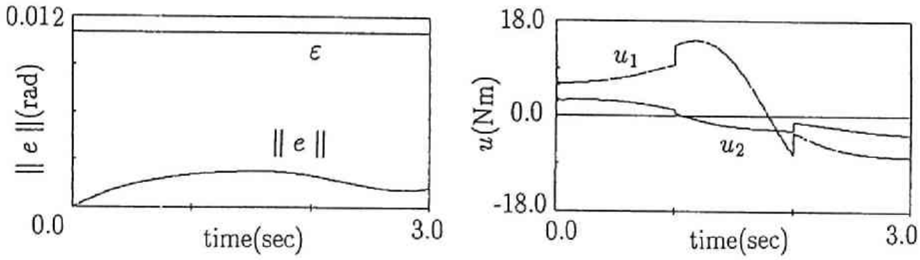

Figure 4.3: Simulation results for proposed robust control method using joint torque information

Note that, although a manipulator is a continuous-time system, this simulation uses the Euler method with the integral interval of 0.1(msec) as numerical integration, and the digital control with the sampling period of 1(msec). The measurable signals are assumed to be available at the moment, and the time delay for computing the control law is not considered. Fig.4.3 shows the results. The norm of the control error is smaller than the specified precision ε_P at any time, and the input torque $\boldsymbol{u}$ is smooth. A similar result has also been obtained concerning the velocity error $\dot{\boldsymbol{e}}$.

For comparison, we show the simulation results in the case of the robust control methods without joint torque information, which is treated in chapter 2. In this case, a priori information about the physical parameters is that $I_i = 0.4(\text{kg}\cdot\text{m}^2)$ and $m_{m2} = 2.0(\text{kg})$ (namely, the real values of $I_i (i = 1, 2)$ and m_{m2} are known), and that m_i is between 2.0 and 8.0. In addition, a priori information on the physical parameters of the rotors is assumed to be the same as the case of the proposed method, and the estimate of m_i is given as 4.0(kg). The specified tracking precision in this case is equal to the proposed case. The control parameters α and k are determined in the same way as the proposed case.

Simulation results are shown in Fig.4.4 . The norm of the control error is smaller than the specified precision. However, the chattering appears in the input torque. The smaller the specified precision is set, the larger the chattering becomes. This reason is as follows. In gen-

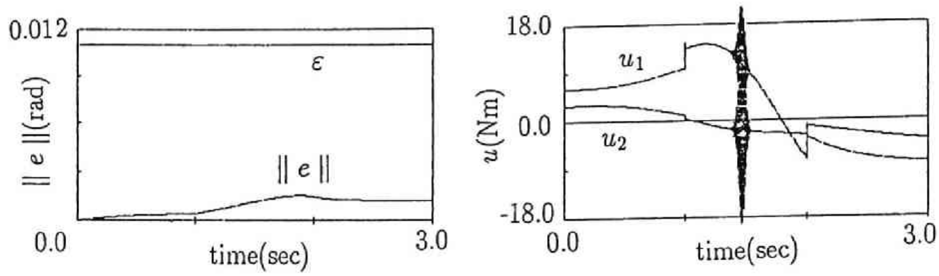

Figure 4.4: Simulation results for robust control method without joint torque information

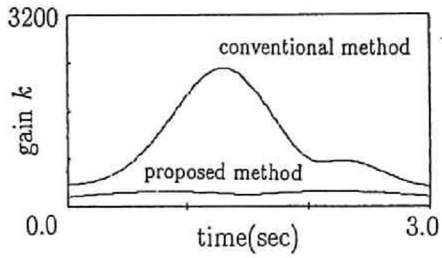

Figure 4.5: Comparison of feedback gain

eral, the robust control method without joint torque information tends to conservatively evaluate the feedback gain to achieve the specified precision, because there is a lot of uncertainty to be compensated for by high gain feedback. As a result, the feedback gains of position and velocity error are too high. So Fig.4.5 shows the comparison of the feedback gains between the method with joint torque information and the method without joint torque information. In spite of the same specified tracking precision, the feedback gain in the former method is about 0.10 times smaller than in the latter method. Consequently, the former method can reduce the chattering owing to the high gain feedback of position and velocity error more than can the latter method.

Furthermore, a simulation is made to take a time delay for generating the control input into account, and almost the same result as the case without considering the time delay are obtained there.

These results show the effectiveness of the proposed robust control method.

4.5 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A dynamic equation of the manipulator with joint torque sensors has been derived, which expresses explicitly the multivariable structure. As a result, the proposed dynamic equation clarifies that the robust control system of the manipulator with joint torque sensors can be designed as in the same way as the case of the manipulator without joint torque sensors.

(ii) It has been shown that the proposed dynamic equation is effective for the design of robust control system against the uncertainty of the motor system. The proposed robust control method achieves the specified tracking precision in the presence of the modeling error, where joint torque information is fully exploited to compensate for the uncertainty of the links and the load at the end-effector.

(iii) Although the proposed method requires the exact information on the joint torque, the following advantages has been clarified, compared with the conventional robust control without joint torque information; (a) we need less a priori information on uncertainties, (b) the computational load for the control law is smaller, and (c) the feedback gain to achieve the specified tracking precision is much lower. In addition, compared with the previous existing methods with joint torque information, (d) the proposed method is more systematic in the sense that it is possible to evaluate the effect on the uncertainty of the motor system and to achieve the specified tracking accuracy.

Appendix

Proof of Eq.(4.1) : Adding to the notations in section 4.2, we define the following notations. $\boldsymbol{n}_{mi}$ denotes the moment vector exerted on the ith rotor by the $i-1$th link, which is expressed in the reference frame. $\boldsymbol{n}_{si}$ denotes the moment vector exerted on the ith rotor through the torque sensor by the ith link, which is expressed in the reference frame. $\widetilde{\boldsymbol{I}}_{mi}$, expressed in the reference frame, is the tensor of the inertia of the ith rotor. $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_{mi}$, expressed in the reference frame, represent the angular velocity vectors of the ith link and the ith rotor, respectively. θ_{mi} is an angle about the axis of the rotation of the ith rotor.

Then from Euler's equation, we get

$$\boldsymbol{n}_{mi} = \widetilde{\boldsymbol{I}}_{mi}\dot{\boldsymbol{\omega}}_{mi} + \boldsymbol{\omega}_{mi} \times (\widetilde{\boldsymbol{I}}_{mi}\boldsymbol{\omega}_{mi}) + b_{mi}\dot{\theta}_{mi}\boldsymbol{z}_{mi} + \boldsymbol{n}_{si} \qquad (4.A1)$$

In addition, using Assumption 4.2 in section 4.2, we obtain

$$\dot{\theta}_{mi} = \gamma_i\dot{\theta}_i$$

$$\boldsymbol{\omega}_{mi} = \boldsymbol{\omega}_{i-1} + \dot{\theta}_{mi}\boldsymbol{z}_{mi}$$

$$\dot{\boldsymbol{\omega}}_{mi} = \dot{\boldsymbol{\omega}}_{i-1} + \gamma_i\ddot{\theta}_0\boldsymbol{z}_{mi} + \boldsymbol{\omega}_{i-1} \times (\gamma_i\dot{\theta}_i\boldsymbol{z}_{mi})$$

$$\boldsymbol{n}_{si} = u_{si}\boldsymbol{z}_{mi}/\gamma_i$$

Substituting these relations into eq.(4.A1), the following equation is obtained:

$$\begin{aligned} u_i &= \boldsymbol{z}_{mi}^T\boldsymbol{n}_{mi} \\ &= \boldsymbol{z}_{mi}^T\widetilde{\boldsymbol{I}}_{mi}\dot{\boldsymbol{\omega}}_{i-1} + (\boldsymbol{z}_{mi}^T\widetilde{\boldsymbol{I}}_{mi}\boldsymbol{z}_{mi})\gamma_i\ddot{\theta}_i \\ &\quad + \boldsymbol{z}_{mi}^T\widetilde{\boldsymbol{I}}_{mi}(\boldsymbol{\omega}_{i-1} \times \gamma_i\dot{\theta}_i\boldsymbol{z}_{mi}) + \boldsymbol{z}_{mi}^T[\boldsymbol{\omega}_{mi} \times (\widetilde{\boldsymbol{I}}_{mi}\boldsymbol{\omega}_{mi})] \\ &\quad + \gamma_i b_{mi}\dot{\theta}_i + u_{si}/\gamma_i \end{aligned} \qquad (4.A2)$$

Using Assumption 4.1, we prove that the third and fourth terms of the right-hand side of eq.(4.A2) is equal to 0. Moreover note that

$$\dot{\boldsymbol{\omega}}_{i-1} \triangleq \begin{cases} 0 & \text{if } i = 1 \\ \boldsymbol{z}_1\ddot{\theta}_1 & \text{if } i = 2 \\ \displaystyle\sum_{j=1}^{i-1}\boldsymbol{z}_j\ddot{\theta}_j + \sum_{j=2}^{i-1}\sum_{k=1}^{j-1}(\boldsymbol{z}_k \times \boldsymbol{z}_j)\dot{\theta}_k\dot{\theta}_j & \\ & \text{if } i \geq 3 \end{cases} \qquad (4.A3)$$

and $\boldsymbol{z}_{mi}^T \widetilde{\boldsymbol{I}}_{mi} = I_{mi}\boldsymbol{z}_{mi}^T$. Therefore from (4.A2), we get

$$u_i = I_{mi}\gamma_i\ddot{\theta}_i + I_{mi}\sum_{j=1}^{i-1} \boldsymbol{z}_{mi}^T \boldsymbol{z}_j \ddot{\theta}_j$$

$$+ I_{mi}\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} \boldsymbol{z}_{mi}^T(\boldsymbol{z}_k \times \boldsymbol{z}_j)\dot{\theta}_k\dot{\theta}_j + \gamma_i b_{mi}\dot{\theta}_i + u_{si}/\gamma_i \qquad (4.A4)$$

where the second and third terms are equal to 0 when $i = 1$, and the third term is equal to 0 when $i = 2$. This completes the derivation of eq.(4.1).

Chapter 5

DIGITAL ROBUST CONTROL OF ROBOT MANIPULATORS

5.1 Introduction

In chapters 2 to 4, various types of continuous-time robust controllers of robot manipulators have been discussed. However, since the above continuous-time robust controllers are nonlinear, we have to descretize them when it is implemented in practice. In other wards, we need a digital controller to control a robot manipulator in fact. If we descretize a continuous robust controller and implement it as a digital controller, the following unexpected phenomena may occur. When a feedback gain is too high, namely, the uncertainty is much large or the specified control precision is much small, the real control error is larger than the allowable control precision that is theoretically obtained in continuous-time robust control theory. In addition, although a continuous-time robust controller that is composed of a continuous function does not lead to chattering theoretically, even in such a case, the chattering often occurs if a digital control is used. These result from the fact that an input to a plant is constant in a sampling period. Thus there are few works on robust control premising a digital control [142, 78, 35]. However, these works treat the case of linear systems only, and there is no research in the nonlinear system case so far.

This chapter discusses a robust control method of robot manipulators premising a digital control, which is called a digital robust control. The effect of a sampling period on control performance is discussed theoretically. Based on the above analysis, a design procedure of the digital robust control system, that is to say, how to calculate the value of a feedback gain to achieve the specified tracking precision for a given sampling period is given. Moreover, a weighting function for a feedback gain is proposed to make the feedback gain small so as to decrease the chattering, and the effectiveness of this idea is shown by illustrative simulation results.

5.2 Problem statement

Consider a manipulator with n degrees of freedom whose dynamics is described by the following equation :

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{5.1}$$

where $\boldsymbol{\theta} \triangleq [\theta_1, \theta_2, \ldots, \theta_n]^T$ is the n-dimensional vector of joint displacements, $\boldsymbol{\phi}$ is the physical parameter vector with an appropriate dimension, $\boldsymbol{u}$ is the n-dimensional joint torque input vector, $\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})$ is the $n \times n$ manipulator inertia matrix, and $\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ is the n-dimensional vector that represents the nonlinear terms such as centrifugal, Coriolis, frictional, and gravitational forces.

This system usually has the following features.

[Feature 5.1] $\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})$ *is a positive definite matrix for any* $\boldsymbol{\theta}$. ∎

We begin with the definition of notations which express mechanical performance of a manipulator given here. Let θ_{imax} denote a maximum movable range of the ith joint angle, that is, $\theta_{imax} \triangleq \max \mid \theta_i \mid$. Let $\dot{\theta}_{imax}$ denote a maximum angular velocity of the ith joint, that is, $\dot{\theta}_{imax} \triangleq \max \mid \dot{\theta}_i \mid$. Let $\ddot{\theta}_{imax}$ denote a maximum angular acceleration of the ith joint, that is, $\ddot{\theta}_{imax} \triangleq \max \mid \ddot{\theta}_i \mid$. Also let

$$\boldsymbol{\Omega}_p \triangleq \{\boldsymbol{\theta} \mid \mid \theta_j \mid \leq \theta_{jmax},\ j = 1, 2, \ldots, n\} \tag{5.2}$$

$$\boldsymbol{\Omega}_v \triangleq \{\dot{\boldsymbol{\theta}} \mid \mid \dot{\theta}_j \mid \leq \dot{\theta}_{jmax},\ j = 1, 2, \ldots, n\} \tag{5.3}$$

$$\boldsymbol{\Omega}_a \triangleq \{\ddot{\boldsymbol{\theta}} \mid |\ \ddot{\theta}_j\ | \leq \ddot{\theta}_{jmax},\ j = 1, 2, \ldots, n\} \tag{5.4}$$

Let T denote a sampling period, which is assumed to be given in advance to depend on the degrees of freedom of the joints and mechanical performance such as computer performance. For given $\boldsymbol{\theta} \in \boldsymbol{\Omega}_p$ and $\dot{\boldsymbol{\theta}} \in \boldsymbol{\Omega}_v$, let

$$\boldsymbol{\Pi}_p(\boldsymbol{\theta}) \triangleq \{\boldsymbol{\xi} \mid |\ \xi_j - \theta_j\ | \leq \dot{\theta}_{jmax}T,\ j = 1, 2, \ldots, n\} \tag{5.5}$$

$$\boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}) \triangleq \{\boldsymbol{\xi} \mid |\ \xi_j - \dot{\theta}_j\ | \leq \ddot{\theta}_{jmax}T,\ j = 1, 2, \ldots, n\} \tag{5.6}$$

where $\boldsymbol{\xi} \triangleq [\xi_1, \xi_2, \ldots, \xi_n]^T$. $\boldsymbol{\Pi}_p(\boldsymbol{\theta})$ and $\boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}})$ express a set of joint angle and joint angular velocity which is reachable in 1 sampling period from $\boldsymbol{\theta}(t)$ and $\dot{\boldsymbol{\theta}}(t)$, respectively.

Then the following assumptions are made.

[Assumption 5.1] *The values of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ at each sampling point, that is, $\boldsymbol{\theta}(iT)$ and $\dot{\boldsymbol{\theta}}(iT)$ $(i = 0, 1, 2, \cdots)$ are known.* ∎

[Assumption 5.2] *θ_{imax}, $\dot{\theta}_{imax}$, and $\ddot{\theta}_{imax}$ are known.* ∎

[Assumption 5.3] *The values of a physical parameter vector $\boldsymbol{\phi}$ may be unknown, but it is known that $\boldsymbol{\phi}$ exists in a known bounded region $\boldsymbol{\Omega}_\phi$.* ∎

[Assumption 5.4] *Each element of $\boldsymbol{M}$ and $\boldsymbol{h}$ is continuous on $\boldsymbol{\phi}$, $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$.* ∎

In addition, let

$$\boldsymbol{M}(\boldsymbol{\theta}) \triangleq \begin{bmatrix} \boldsymbol{M}_1(\boldsymbol{\theta}) \\ \vdots \\ \boldsymbol{M}_n(\boldsymbol{\theta}) \end{bmatrix} \tag{5.7}$$

$$\boldsymbol{N}(\boldsymbol{\theta}) \triangleq \boldsymbol{M}^{-1}(\boldsymbol{\theta}) \triangleq \begin{bmatrix} \boldsymbol{N}_1(\boldsymbol{\theta}) \\ \vdots \\ \boldsymbol{N}_n(\boldsymbol{\theta}) \end{bmatrix} \tag{5.8}$$

For a given $\boldsymbol{\theta} \in \boldsymbol{\Omega}_p$, we define a vector $\widetilde{\boldsymbol{\theta}}^k \triangleq [\widetilde{\theta}_1^k, \widetilde{\theta}_2^k, \ldots, \widetilde{\theta}_n^k]^T \in \boldsymbol{\Pi}_p(\boldsymbol{\theta})$ $(k = 1, 2, \ldots, n)$. Using this, we also define

$$\widetilde{\boldsymbol{N}} \triangleq \begin{bmatrix} \boldsymbol{N}_1(\widetilde{\boldsymbol{\theta}}^1) \\ \vdots \\ \boldsymbol{N}_n(\widetilde{\boldsymbol{\theta}}^n) \end{bmatrix} \tag{5.9}$$

Then the following assumption is made.

[Assumption 5.5] *$\hat{\phi}$, a bounded estimate of ϕ, is given such that the following matrix is positive definite for all $\theta \in \Omega_p$, $\tilde{\theta}^k \in \Pi_p(\theta)$, $\phi \in \Omega_\phi$ and k,*

$$\bar{M} \triangleq \{\widetilde{N}\hat{M} + \hat{M}^T\widetilde{N}^T\}/2 \tag{5.10}$$

where $\hat{M} \triangleq M(\hat{\phi}, \theta)$. ∎

Assumption 5.4 implies that, for example, we do not consider here Coulomb's friction. In Assumption 5.5, although it is not easy to analytically show a condition of $\hat{\phi}$ satisfying that $\bar{M}$ is positive definite, we believe that Assumption 5.5 is satisfied mostly if the sampling period is small enough.

Note that, from Assumptions 5.3 and 5.5, there exist positive constants α_m and α_M such that the following conditions are satisfied.

$$0 < \alpha_m < \lambda_m(\widetilde{N}\bar{M} + \bar{M}^T\widetilde{N}^T)/2$$
$$\forall\theta \in \Omega_p,\ \forall\tilde{\theta}^k \in \Pi_p(\theta),\ \forall k \tag{5.11}$$

$$\alpha_M > \lambda_M(\widetilde{N}\bar{M})$$
$$\forall\theta \in \Omega_p,\ \forall\tilde{\theta}^k \in \Pi_p(\theta),\ \forall k \tag{5.12}$$

Now let

$$x = [\theta^T \quad \lambda\dot{\theta}^T]^T \tag{5.13}$$

where λ is a positive constant. Then we get the following state space equation of (5.1).

$$\dot{x} = A_c x + B_c M^{-1}(\theta)\{u - h(\theta, \dot{\theta})\} \tag{5.14}$$

$$A_c \triangleq \begin{bmatrix} 0 & \frac{1}{\lambda}I \\ 0 & 0 \end{bmatrix} \in R^{2n\times 2n} \tag{5.15}$$

$$B_c \triangleq \begin{bmatrix} 0 \\ \lambda I \end{bmatrix} \in R^{2n\times n} \tag{5.16}$$

where I is a unit matrix.

Remark 5.1 *When we estimate the bound of the control error in terms of the Euclidean norm, we can specify a ratio between θ and $\dot{\theta}$, by a positive constant λ.* ∎

For the above robot manipulator, we consider the following problem.

[Problem 5.1] *For a robot manipulator given by (5.1) or (5.14) that satisfies Assumptions 5.1 to 5.5, a desired trajectory $\theta_d(t)$ is given whose derivatives $\dot{\theta}_d$ and $\ddot{\theta}_d$ exist and are bounded. Consider a digital* *control system with a sampling period T. Then for given ε_P, find a control law such that*

$$\| \boldsymbol{e}(t) \| < \varepsilon_P \qquad \forall t \geq T \tag{5.17}$$

holds for all $t \geq 0$*, where* $\boldsymbol{e}(t) = \begin{bmatrix} \boldsymbol{\theta}(t) \\ \lambda\dot{\boldsymbol{\theta}}(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{\theta}_d(t) \\ \lambda\dot{\boldsymbol{\theta}}_d(t) \end{bmatrix}$ *and* t_0 *is an initial time.* ∎

5.3 Digital robust control

In this section, at first, we get a discrete-time state space equation from a continuous state space equation given by (5.14) and give a digital robust controller for a discrete-time system. Second, we estimate the bound of the control error in that case, and finally give a design procedure of a digital robust control system of a robot manipulator, which achieves a specified tracking control precision.

5.3.1 Discrete-time nonlinear systems

It is followed from (5.14) that

$$\begin{aligned} \boldsymbol{x}(t) &= e^{\boldsymbol{A}_c(t-t_0)}\boldsymbol{x}(t_0) \\ &\quad + \int_{t_0}^{t} e^{\boldsymbol{A}_c(t-\tau)}\boldsymbol{B}_c\boldsymbol{M}^{-1}(\boldsymbol{\theta}(\tau))\{\boldsymbol{u}(\tau) - \boldsymbol{h}(\boldsymbol{\theta}(\tau),\dot{\boldsymbol{\theta}}(\tau))\}d\tau \end{aligned} \tag{5.18}$$

Assume that the input $\boldsymbol{u}(\tau)$ is $\boldsymbol{u}_i$ for all $\tau \in [iT,(i+1)T)$, where $\boldsymbol{u}_i$ is a constant vector. Replacing t_0 and t in (5.18) by iT and $(i+1)T$, respectively, we get

$$\begin{aligned} \boldsymbol{x}_{i+1} &= \boldsymbol{A}\boldsymbol{x}_i + \int_{iT}^{(i+1)T} \boldsymbol{B}(\tau,i)\boldsymbol{M}^{-1}(\boldsymbol{\theta}(\tau))\{\boldsymbol{u}(\tau) \\ &\quad -\boldsymbol{h}(\boldsymbol{\theta}(\tau),\dot{\boldsymbol{\theta}}(\tau))\}d\tau \end{aligned} \tag{5.19}$$

$$\boldsymbol{A} \triangleq \begin{bmatrix} \boldsymbol{I} & \frac{T}{\lambda}\boldsymbol{I} \\ \mathbf{o} & \boldsymbol{I} \end{bmatrix} \in \boldsymbol{R}^{2n\times 2n} \tag{5.20}$$

$$\boldsymbol{B}(\tau,i) \triangleq \begin{bmatrix} \{(i+1)T-\tau\}\boldsymbol{I} \\ \lambda\boldsymbol{I} \end{bmatrix} \in \boldsymbol{R}^{2n\times n} \tag{5.21}$$

where $\boldsymbol{x}_i \triangleq \boldsymbol{x}(iT)$. Using (5.1), we can also express (5.19) by

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}\boldsymbol{x}_i + \int_{iT}^{(i+1)T} \boldsymbol{B}(\tau, i)\ddot{\boldsymbol{\theta}}(\tau)d\tau \tag{5.22}$$

Note that there exists a state in the integral term of the right-hand side of (5.19), comparing with the case of linear systems. So using the average-valued theory in the integral, we rewrite (5.19) or (5.22). Then let $\boldsymbol{f} \in \boldsymbol{R}^n$ be a function defined by

$$\boldsymbol{f}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \triangleq \boldsymbol{M}^{-1}\boldsymbol{h} \triangleq \begin{bmatrix} f_1(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \\ \vdots \\ f_n(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \end{bmatrix} \tag{5.23}$$

Let also $\boldsymbol{\theta}_i \triangleq \boldsymbol{\theta}(iT)$ and $\dot{\boldsymbol{\theta}}_i \triangleq \dot{\boldsymbol{\theta}}(iT)$. For given $\boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p$ and $\dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v$, we define

$$\widetilde{\boldsymbol{N}}_i \triangleq \begin{bmatrix} \boldsymbol{N}_1(\widetilde{\boldsymbol{\theta}}_i^1) \\ \vdots \\ \boldsymbol{N}_n(\widetilde{\boldsymbol{\theta}}_i^n) \end{bmatrix} \tag{5.24}$$

$$\widetilde{\boldsymbol{f}}_i \triangleq \begin{bmatrix} f_1(\widetilde{\boldsymbol{\theta}}_i^1, \widetilde{\dot{\boldsymbol{\theta}}}_i^1) \\ \vdots \\ f_n(\widetilde{\boldsymbol{\theta}}_i^n, \widetilde{\dot{\boldsymbol{\theta}}}_i^n) \end{bmatrix} \tag{5.25}$$

for any $\widetilde{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i)$ and $\widetilde{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i)$ $(k = 1, 2, \ldots, n)$. In addition, let

$$\widetilde{\ddot{\boldsymbol{\theta}}}_i \triangleq \widetilde{\boldsymbol{N}}_i\boldsymbol{u}_i - \widetilde{\boldsymbol{f}}_i \tag{5.26}$$

In the same way as the definition of $\widetilde{\ddot{\boldsymbol{\theta}}}_i$, for any $\widehat{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i)$ and $\widehat{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i)$ $(k = 1, 2, \ldots, n)$, let

$$\widehat{\ddot{\boldsymbol{\theta}}}_i \triangleq \widehat{\boldsymbol{N}}_i\boldsymbol{u}_i - \widehat{\boldsymbol{f}}_i \tag{5.27}$$

where $\widehat{\boldsymbol{N}}_i$ and $\widehat{\boldsymbol{f}}_i$ is defined in the same way as (5.24) and (5.25).

We use the following notations hereafter. For a vector $\boldsymbol{x}$ and a matrix $\boldsymbol{A}$, $\boldsymbol{x}_i$ and $\boldsymbol{A}_i$ express values at the ith sampling point. $\widetilde{\boldsymbol{x}}_i$, $\widehat{\boldsymbol{x}}_i$, $\widetilde{\boldsymbol{A}}_i$ and $\widehat{\boldsymbol{A}}_i$ express values at some time in the ith sampling interval.

Then we get the following lemma.

[Lemma 5.1] *For given $\boldsymbol{u}_i$, $\boldsymbol{\theta}_i$, and $\dot{\boldsymbol{\theta}}_i$, there exist $\widetilde{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i)$ and $\widetilde{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i)$ $(k = 1, 2, \ldots, n)$ satisfying*

$$\int_{iT}^{(i+1)T} \boldsymbol{B}(\tau, i)\ddot{\boldsymbol{\theta}}(\tau)d\tau = \boldsymbol{B}\tilde{\ddot{\boldsymbol{\theta}}}_i + \widetilde{\boldsymbol{B}}\Delta\tilde{\ddot{\boldsymbol{\theta}}}_i \tag{5.28}$$

where

$$\Delta\tilde{\ddot{\boldsymbol{\theta}}}_i \triangleq \hat{\ddot{\boldsymbol{\theta}}}_i - \tilde{\ddot{\boldsymbol{\theta}}}_i \tag{5.29}$$

$$\boldsymbol{B} \triangleq \begin{bmatrix} \frac{T^2}{2}\boldsymbol{I} \\ \lambda T\boldsymbol{I} \end{bmatrix} \in \boldsymbol{R}^{2n\times n}, \qquad \widetilde{\boldsymbol{B}} \triangleq \begin{bmatrix} \frac{T^2}{2}\boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \in \boldsymbol{R}^{2n\times n}$$

∎

Proof: Since an input is constant, $\ddot{\boldsymbol{\theta}}$ is continuous. Then using the average valued theory in the integral, we get

$$\int_{iT}^{(i+1)T} \boldsymbol{B}(\tau, i)\ddot{\boldsymbol{\theta}}(\tau)d\tau = \begin{bmatrix} \frac{T^2}{2}\hat{\ddot{\boldsymbol{\theta}}}_i \\ \lambda T\tilde{\ddot{\boldsymbol{\theta}}}_i \end{bmatrix} \tag{5.30}$$

Hence from (5.30) and (5.29), (5.28) follows. ∎

Applying Lemma 5.1 to (5.22), we obtain

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\tilde{\ddot{\boldsymbol{\theta}}}_i + \widetilde{\boldsymbol{B}}\Delta\tilde{\ddot{\boldsymbol{\theta}}}_i \tag{5.31}$$

which is a discrete-time expression of (5.14).

5.3.2 Digital robust controller

We consider the following reference model for a discrete-time system given by (5.31).

$$\boldsymbol{x}_{Mi+1} = \boldsymbol{A}_M\boldsymbol{x}_{Mi} + \boldsymbol{B}\boldsymbol{u}_{Mi} \tag{5.32}$$

$$\boldsymbol{A}_M = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} \tag{5.33}$$

$$\boldsymbol{u}_{Mi} \in \boldsymbol{\Omega}_M \tag{5.34}$$

where $\boldsymbol{x}_{Mi} \in \boldsymbol{R}^{2n}$ is the state of a reference model at the ith sampling point. $\boldsymbol{u}_{Mi}$ is a reference input which is given so as to hold

$$\boldsymbol{x}_{Mi} = [\boldsymbol{\theta}_{Mi}^T, \lambda\dot{\boldsymbol{\theta}}_{Mi}^T]^T \tag{5.35}$$

Note that we cannot always find $\boldsymbol{u}_{Mi}$ such that (5.35) holds. However for simplicity, it is assumed that there exists a reference input $\boldsymbol{u}_{Mi}$ such that (5.35) holds, because we can directly extend an approach obtained hereafter to the case that there exists no reference input such that (5.35) holds. In addition assume that $\boldsymbol{K} = [k_1\boldsymbol{I}, k_2\boldsymbol{I}]$ for simplicity, and $\boldsymbol{K}$ is selected such that the absolute value of the eigenvalue of $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is less than 1. $\boldsymbol{\Omega}_M$ expresses a set of reference inputs which are feasible, and will be defined later.

Let $\boldsymbol{e}_i$ be a control error at the ith sampling point denoted by $\boldsymbol{e}_i \triangleq \boldsymbol{x}_i - \boldsymbol{x}_{Mi}$. Then we get an error system between (5.31) and (5.32) as follows.

$$\boldsymbol{e}_{i+1} = \boldsymbol{A}_M \boldsymbol{e}_i + \boldsymbol{B}\{\widetilde{\boldsymbol{N}}_i \boldsymbol{u}_i - \widetilde{\boldsymbol{f}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i)\} + \widetilde{\boldsymbol{B}}\Delta\tilde{\ddot{\boldsymbol{\theta}}}_i \tag{5.36}$$

For an error system given by (5.36), consider the following controller.

$$\boldsymbol{u}_i = \boldsymbol{u}_{Li} + \boldsymbol{u}_{Ri} \tag{5.37}$$

where $\boldsymbol{u}_{Li}$ is a linearizing controller and $\boldsymbol{u}_{Ri}$ is a robust controller.

<Linearizing compensation$\boldsymbol{u}_{Li}$>

Noting (5.36), we consider

$$\boldsymbol{u}_{Li} = \bar{\boldsymbol{M}}_i\{\bar{\boldsymbol{f}}_i + (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i)\} \tag{5.38}$$

where $\bar{\boldsymbol{M}}_i \triangleq \bar{\boldsymbol{M}}(\boldsymbol{\theta}_i)$ and $\bar{\boldsymbol{f}}_i$ is the estimate value of $\widetilde{\boldsymbol{f}}_i$.

<Robust compensation$\boldsymbol{u}_{Ri}$>

Let $\boldsymbol{s}_i$ be an extended error given by

$$\boldsymbol{s}_i \triangleq \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{A}_M \boldsymbol{e}_i \tag{5.39}$$

where a matrix $\boldsymbol{P}$ is a positive definite solution to the Discrete Lyapunov equation

$$\boldsymbol{A}_M^T \boldsymbol{P} \boldsymbol{A}_M - \boldsymbol{P} = -\boldsymbol{Q} \tag{5.40}$$

for a positive definite matrix $\boldsymbol{Q}$. For simplicity, we assume

$$\boldsymbol{Q} = \begin{bmatrix} q_{l1}\boldsymbol{I} & q_{l2}\boldsymbol{I} \\ q_{l2}\boldsymbol{I} & q_{l3}\boldsymbol{I} \end{bmatrix} \in \boldsymbol{R}^{2n \times 2n} \tag{5.41}$$

$$\boldsymbol{P} = \begin{bmatrix} p_1\boldsymbol{I} & p_2\boldsymbol{I} \\ p_2\boldsymbol{I} & p_3\boldsymbol{I} \end{bmatrix} \in \boldsymbol{R}^{2n \times 2n} \tag{5.42}$$

In addition, let ψ be a switching function given by

$$\psi(\boldsymbol{s}_i) \triangleq \begin{cases} \frac{\boldsymbol{s}_i}{\|\boldsymbol{s}_i\|} & \text{if} \quad \| \boldsymbol{s}_i \| \neq \mathbf{o} \\ \mathbf{o} & \text{if} \quad \| \boldsymbol{s}_i \| = \mathbf{o} \end{cases} \tag{5.43}$$

Then noting that there exists a positive function $g(\boldsymbol{x}, \boldsymbol{u}_{Mi})$ such that

$$\begin{aligned} & g(\boldsymbol{x}_i, \boldsymbol{u}_{Mi}) \\ & \geq \| \widetilde{\boldsymbol{N}}_i \bar{\boldsymbol{M}}_i \bar{\boldsymbol{f}}_i - \widetilde{\boldsymbol{f}}_i + (\widetilde{\boldsymbol{N}}_i \bar{\boldsymbol{M}}_i - \boldsymbol{I})(\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i) \| \\ & \qquad \forall \boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p,\ \forall \dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v,\ \forall \boldsymbol{u}_{Mi} \in \boldsymbol{\Omega}_M \\ & \qquad \forall \tilde{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i),\ \forall \tilde{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i),\ \forall k \end{aligned} \tag{5.44}$$

we consider the following robust controller.

$$\boldsymbol{u}_{Ri} = -\frac{\bar{\boldsymbol{M}}_i}{\alpha_m} w g(\boldsymbol{x}_i,\ \boldsymbol{u}_{Mi}) \psi(\boldsymbol{s}_i) \tag{5.45}$$

where w (≥ 0) is a design parameter to specify the control error precision. We call $wg(\boldsymbol{x}_i, \boldsymbol{u}_{Mi})$ a switching gain hereafter. α_m is a positive number to satisfy (5.11).

5.3.3 Estimation of the bound of control error

In this subsection, we estimate the bound of the control error when a controller given by (5.37), (5.38), and (5.45) is applied to an error system (5.36).

First, note that there exists a positive number ν such that, for $\Delta\tilde{\ddot{\boldsymbol{\theta}}}_i$ given by (5.29),

$$\begin{aligned} &\nu > \lambda_M(\boldsymbol{A}_M^T\boldsymbol{P}\widetilde{\boldsymbol{B}}) \parallel \Delta\tilde{\ddot{\boldsymbol{\theta}}}_i \parallel \\ &\quad \forall\boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p,\ \forall\dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v,\ \forall\boldsymbol{u}_{Mi} \in \boldsymbol{\Omega}_M \\ &\quad\quad \forall\widetilde{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i),\ \forall\tilde{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i) \\ &\quad\quad \forall\widehat{\boldsymbol{\theta}}_i^k \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_i),\ \forall\hat{\dot{\boldsymbol{\theta}}}_i^k \in \boldsymbol{\Pi}_v(\dot{\boldsymbol{\theta}}_i) \end{aligned} \tag{5.46}$$

Let T_p be a positive constant denoted by

$$T_p \triangleq T^4 p_1/4 + \lambda T^3 p_2 + \lambda^2 T^2 p_3 \tag{5.47}$$

Then note that T_p has the following relation to $\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}$.

$$\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} = T_p\boldsymbol{I} \tag{5.48}$$

Furthermore, let $\bar{g}$ be a maximum value of $g(\boldsymbol{x}_i, \boldsymbol{u}_{Mi})$, that is,

$$\begin{aligned} &\bar{g} \triangleq \max\{g(\boldsymbol{x}_i,\ \boldsymbol{u}_{Mi})\} \\ &\quad \forall\boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p,\ \forall\dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v,\ \forall\boldsymbol{u}_{Mi} \in \boldsymbol{\Omega}_M \end{aligned} \tag{5.49}$$

Then we define the following function of w.

$$\delta(w) \triangleq \frac{\beta(w) + \sqrt{\beta(w)^2 + \lambda_m(\boldsymbol{Q})T_p\gamma(w)^2\bar{g}^2}}{\lambda_m(\boldsymbol{Q})} \tag{5.50}$$

where

$$\beta(w) \triangleq \begin{cases} \lambda_M(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}_M)\bar{g}(1-w)+\nu \\ \qquad\qquad \text{if} \quad 0 \leq w < 1 \\ \nu \qquad\quad \text{if} \quad w \geq 1 \end{cases} \tag{5.51}$$

$$\gamma(w) \triangleq \begin{cases} 1 + \frac{\alpha_M}{\alpha_m} & \text{if} \quad 0 \leq w < 1 \\ 1 + (\frac{\alpha_M}{\alpha_m})w & \text{if} \quad w \geq 1 \end{cases} \tag{5.52}$$

Then we get the following lemma.

[Lemma 5.2] *For a positive definite function*

$$V_i = \boldsymbol{e}_i^T \boldsymbol{P} \boldsymbol{e}_i \tag{5.53}$$

let $\Delta V_i \triangleq V_{i+1} - V_i$. *If a control input given by (5.37), (5.38), and (5.45) is applied to an error system (5.36), then*

$$\Delta V_i < 0 \tag{5.54}$$

when

$$\| \boldsymbol{e}_i \| \geq \delta \tag{5.55}$$

■

Proof: Substituting (5.36), (5.22), (5.39), and (5.40) into the equation

$$\Delta V_i = \boldsymbol{e}_{i+1}^T \boldsymbol{P} \boldsymbol{e}_{i+1} - \boldsymbol{e}_i^T \boldsymbol{P} \boldsymbol{e}_i \tag{5.56}$$

we get

$$\Delta V_i = -\boldsymbol{e}_i^T \boldsymbol{Q} \boldsymbol{e}_i + 2\boldsymbol{s}_i^T \{\widetilde{\boldsymbol{N}}_i \boldsymbol{u}_i - \widetilde{\boldsymbol{f}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i)\}$$
$$+2\boldsymbol{e}_i^T \boldsymbol{A}_M^T \boldsymbol{P} \widetilde{\boldsymbol{B}} \Delta \ddot{\widetilde{\boldsymbol{\theta}}}_i + \boldsymbol{y}_i^T \boldsymbol{P} \boldsymbol{y}_i \tag{5.57}$$

where

$$\boldsymbol{y}_i \triangleq \int_{iT}^{(i+1)} \boldsymbol{B}(\tau, i) \ddot{\boldsymbol{\theta}}(\tau) d\tau - \boldsymbol{B}(\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i) \tag{5.58}$$

We show the case of $0 \leq w < 1$ here. From (5.57), (5.38), (5.45), and

$$\boldsymbol{e}_i^T \boldsymbol{Q} \boldsymbol{e}_i \geq \lambda_m(\boldsymbol{Q}) \| \boldsymbol{e}_i \|^2 \tag{5.59}$$

$$\begin{aligned} \boldsymbol{s}_i^T \widetilde{\boldsymbol{N}}_i \bar{\boldsymbol{M}}_i \boldsymbol{s}_i &= \boldsymbol{s}_i^T (\widetilde{\boldsymbol{N}}_i \bar{\boldsymbol{M}}_i + \bar{\boldsymbol{M}}_i^T \widetilde{\boldsymbol{N}}_i^T) \boldsymbol{s}_i / 2 \\ &> \alpha_m \| \boldsymbol{s}_i \|^2 \end{aligned} \tag{5.60}$$

we get

$$\begin{aligned} \Delta V_i &< -\lambda_m(\boldsymbol{Q}) \| \boldsymbol{e}_i \|^2 + 2\nu \| \boldsymbol{e}_i \| \\ &\quad +2 \| \boldsymbol{s}_i \| (1-w) g + \boldsymbol{y}_i^T \boldsymbol{P} \boldsymbol{y}_i \\ &\leq -\lambda_m(\boldsymbol{Q}) \| \boldsymbol{e}_i \|^2 + 2\nu \| \boldsymbol{e}_i \| \\ &\quad +2\lambda_M(\boldsymbol{B}^T \boldsymbol{P} \boldsymbol{A}_M)(1-w)\bar{g} \| \boldsymbol{e}_i \| + \boldsymbol{y}_i^T \boldsymbol{P} \boldsymbol{y}_i \end{aligned} \tag{5.61}$$

It also follows from (5.51) and

$$\boldsymbol{y}_i^T \boldsymbol{P} \boldsymbol{y}_i < T_p \gamma(w)^2 \bar{g}^2 = T_p \gamma(1)^2 \bar{g}^2 \tag{5.62}$$

(See Appendix), that

$$\begin{aligned} \Delta V_i &< -\lambda_m(\boldsymbol{Q}) \| \boldsymbol{e}_i \|^2 + 2\beta(w) \| \boldsymbol{e}_i \| \\ &+ T_p \gamma(1)^2 \bar{g}^2 \end{aligned} \tag{5.63}$$

Therefore, eqs.(5.50) and (5.63) imply that (5.54) holds when $\| \boldsymbol{e}_i \| \geq \delta$ The case of $w \geq 1$ will be shown in the similar way. ■

The following result is concerned with the bound of the control error in each sampling term.

[Lemma 5.3] *Suppose a desired trajectory* $\boldsymbol{\theta}_M(t)$ *and* $\dot{\boldsymbol{\theta}}_M(t)$ *are continuous on* t. *Then if, for a positive constant* η,

$$\| \boldsymbol{e}_i \| < \eta \qquad \forall i \tag{5.64}$$

then

$$\| \boldsymbol{e}(t) \| < \eta + \varepsilon_t \tag{5.65}$$

for all $t > 0$, *where* ε_t *is a positive number defined by*

$$\varepsilon_t \triangleq T \sqrt{\sum_{j=1}^{n} (\dot{\theta}_{jmax}^2 + \lambda^2 \ddot{\theta}_{jmax}^2)} \tag{5.66}$$

■

Proof: For all $t \in [iT, (i+1)T]$, the following relation holds.

$$| \theta_j(t) - \theta_j(iT) | \leq \dot{\theta}_{jmax} T/2 \qquad \forall j \tag{5.67}$$

$$| \theta_{Mj}(t) - \theta_{Mj}(iT) | \leq \dot{\theta}_{jmax} T/2 \qquad \forall j \tag{5.68}$$

Then we get

$$| \{\theta_j(t) - \theta_{Mj}(t)\} - \{\theta_j(iT) - \theta_{Mj}(iT)\} | \leq \dot{\theta}_{jmax} T \\ j = 1, 2, \ldots, n \tag{5.69}$$

The same relation as (5.69) also holds with respect to the velocity. So letting $\Delta \boldsymbol{e}(t - iT) \triangleq \boldsymbol{e}(t) - \boldsymbol{e}(iT)$, we obtain

$$\| \Delta \boldsymbol{e}(t - iT) \| \leq \varepsilon_t \tag{5.70}$$

Hence noting

$$\| \boldsymbol{e}(t) \| \leq \| \boldsymbol{e}(iT) \| + \| \Delta \boldsymbol{e}(t - iT) \| \tag{5.71}$$

eq.(5.65) follows. ■

We get the following theorem using Lemmas 5.2 and 5.3

[Theorem 5.1] *Suppose a control input given by (5.37), (5.38), and (5.45) is applied to an error system (5.36). Then for any positive definite matrices* $\boldsymbol{P}$ *satisfying (5.42) and* $\boldsymbol{Q}$ *satisfying (5.41),*

$$\| \boldsymbol{e}(t) \| < \max\{\varepsilon, \varepsilon_0\} + \varepsilon_t \qquad \forall t > 0 \tag{5.72}$$

where

$$\varepsilon \triangleq \frac{\delta(w)}{c} \tag{5.73}$$

$$c \triangleq \sqrt{\frac{\lambda_m(\boldsymbol{P})}{\lambda_M(\boldsymbol{P}) + \lambda_m(\boldsymbol{Q})}} \tag{5.74}$$

$$\varepsilon_0 \triangleq \sqrt{\frac{V_0}{\lambda_m(\boldsymbol{P})}} \tag{5.75}$$

■

Proof: We complete the proof by considering two cases, namely, (i) $\| \boldsymbol{e}_0 \| < \delta$ and (ii) $\| \boldsymbol{e}_0 \| \geq \delta$.

(i) $\| e_0 \| < \delta$: From Lemma 5.2, we can show that there exists a i such that $\| e_i \| < \delta < \| e_{i+1} \|$ and V_{i+1} has a maximum value at i, that is,

$$V_{i+1} \geq \max_k \{V_k\} \tag{5.76}$$

Let such a i be i^*. Then

$$V_{i^*} \leq \lambda_M(\boldsymbol{P}) \| \boldsymbol{e}_{i^*} \|^2 < \lambda_M(\boldsymbol{P})\delta^2 \tag{5.77}$$

On the other hand, in the same way of the derivation of (5.63) of Lemma 5.2 we get

$$\Delta V_{i^*} < 2\beta\delta + T_p\gamma^2\bar{g}^2 \tag{5.78}$$

Hence it follows from (5.77) and (5.78) that

$$V_{i^*+1} < \lambda_M(\boldsymbol{P})\delta^2 + 2\beta\delta + T_p\gamma^2\bar{g}^2 \tag{5.79}$$

which means that , for all i,

$$\lambda_m(\boldsymbol{P}) \| \boldsymbol{e}_i \|^2 \leq V_i < \lambda_M(\boldsymbol{P})\delta^2 + 2\beta\delta + T_p\gamma^2\bar{g}^2 \tag{5.80}$$

Then we get

$$\| \boldsymbol{e}_i \| < \sqrt{\frac{\lambda_M(\boldsymbol{P})\delta^2 + 2\beta\delta + T_p\gamma^2\bar{g}^2}{\lambda_m(\boldsymbol{P})}} \tag{5.81}$$

for all i. Noting that

$$-\lambda_m(\boldsymbol{Q})\delta^2 + 2\beta\delta + T_p\gamma^2\bar{g}^2 = 0 \tag{5.82}$$

we get from (5.81)

$$\| \boldsymbol{e}_i \| < \frac{\delta}{c} = \varepsilon \qquad \forall i \tag{5.83}$$

(ii) $\| e_0 \| \geq \delta$: Let V_{max} be a maximum value of V_i with respect to i. Then V_{max} is equal to V_0 or V_{i^*+1} given in (i). Namely,

$$V_{max} \leq \max\{V_0, \lambda_M(\boldsymbol{P})\delta^2 + 2\beta\delta + T_p\gamma^2\bar{g}^2\} \tag{5.84}$$

Hence we get

$$\| \boldsymbol{e}_i \| < \max\{\varepsilon, \varepsilon_0\} \qquad \forall i \tag{5.85}$$

in the same way as the proof (i).

Noting that

$$\varepsilon > \varepsilon_0 \tag{5.86}$$

when $\| e_0 \| < \delta$, we conclude, from (i) and (ii),

$$\| \boldsymbol{e}_i \| < \max\{\varepsilon, \varepsilon_0\} \qquad \forall i \tag{5.87}$$

Finally using Lemma 5.3, we get (5.72). ∎

5.3.4 Design procedure

In this subsection, using Theorem 5.1, we give a design procedure, that is, how to determine design parameters, especially w based on the specified control precision. Then we define the following functions.

$$w_1(\varepsilon) \triangleq \left\{ \sqrt{\frac{(\lambda_m(\boldsymbol{Q})c\varepsilon - 2\nu)c\varepsilon}{T_p\bar{g}^2}} - 1 \right\} \frac{\alpha_m}{\alpha_M} \tag{5.88}$$

$$w_2(\varepsilon) \triangleq \frac{\{\lambda_m(\boldsymbol{Q})(\delta(1) + c\varepsilon) - 2\beta(1)\}\{\delta(1) - c\varepsilon\}}{2\{\beta(0) - \beta(1)\}c\varepsilon} + 1 \tag{5.89}$$

Note that these functions are obtained by solving (5.73) with respect to w.

First, we give a minimum value of feasible specified control precision by considering a class of feasible desired trajectories and also a maximum value of feasible control inputs which satisfy Assumption 5.2. A class of feasible desired trajectories is given as follows. $\boldsymbol{\theta}_M(t)$ is twice differentiable and satisfies, for a positive number ω_0,

$$| \theta_{Mj}(t) | \leq \theta_{jmax} - \omega_0 \quad j = 1, 2, \ldots, n \tag{5.90}$$

$$| \dot{\theta}_{Mj}(t) | \leq \dot{\theta}_{jmax} - \frac{\omega_0}{\lambda} \quad j = 1, 2, \ldots, n \tag{5.91}$$

$$| \ddot{\theta}_{Mj}(t) | \leq \ddot{\theta}_{jmax} \quad j = 1, 2, \ldots, n \tag{5.92}$$

where ω_0 will be specified later. According to the above desired trajectory, a set $\boldsymbol{\Omega}_M$ is given by

$$\begin{aligned} \boldsymbol{\Omega}_M \;=\; & \{\boldsymbol{u}_{Mi} \mid \ddot{\theta}_{jmax} > \{\boldsymbol{M}^{-1}(\boldsymbol{\theta}_i)(\boldsymbol{u}_{Li} - \boldsymbol{h}(\boldsymbol{\theta}_i, \dot{\boldsymbol{\theta}}_i))\}_j, \\ & j = 1, 2, \ldots, n, \quad \forall \boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p, \forall \dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v\} \end{aligned} \tag{5.93}$$

Thus, let $\boldsymbol{Q}_M(\omega_0)$ be a set of feasible desired trajectories satisfying (5.90), (5.91), (5.92), and (5.93).

Let also g_{max} be a maximum value of a switching gain which is determined by a control input satisfying $| \ddot{\theta}_j | \leq | \ddot{\theta}_{jmax} |$ (See Appendix). Then a maximum value of w, w_{max} is given by

$$w_{max} = \frac{g_{max}}{\bar{g}} \tag{5.94}$$

A ε in (5.73) is a function of w. Since $\delta(w)$ a monotonous decreasing function for $0 \leq w < 1$ and a monotonous increasing function for $w \geq 1$, ε has a minimum value at $w = 1$ or $w = w_{max}$. So let ε_{min} be a minimum value of ε. Then

$$\varepsilon_{min} \triangleq \begin{cases} \frac{\delta(1)}{c} & \text{if} \quad w_{max} \geq 1 \\ \frac{\delta(w_{max})}{c} & \text{if} \quad w_{max} < 1 \end{cases} \tag{5.95}$$

Therefore from Theorem 5.1, $\varepsilon_{min} + \varepsilon_t$ expresses a minimum value of the feasible specified tracking precision, provided $\varepsilon_{min} \geq \varepsilon_0$.

From the above discussion, ω_0 and ε_{min} must satisfy the following condition, if we use Theorem 5.1 to design a digital robust controller satisfying the specified tracking precision. Assume that, for a ω_0 satisfying

$$\omega_0 \geq \varepsilon_0 + \varepsilon_t \tag{5.96}$$

a desired trajectory which belongs to $\boldsymbol{Q}_M(\omega_0)$ is given. Then ε_{min} and ω_0 must have the relation

$$\varepsilon_{min} + \varepsilon_t \leq \omega_0 \tag{5.97}$$

A ω_0 given by (5.96) is an offset parameter to guarantee that a real trajectory always keeps within a feasible movable region even if control error exists. So (5.96) and (5.97) guarantee that ω_0 is larger than the tracking precision and the initial error. If, for a given ω_0, $\varepsilon_{min} + \varepsilon_t$ does not satisfy (5.97), we need to change the value of design parameters such as ω_0 or the sampling period. Thus if there exists a positive constant ω_0 such that (5.97) holds, we get the following result.

[Theorem 5.2] *For a robot manipulator given by (5.1), suppose Assumptions 5.1 to 5.4. A specified tracking precision ω_d is given so as to satisfy*

$$\varepsilon_{min} + \varepsilon_t \leq \omega_d \leq \omega_0 \tag{5.98}$$

Then maximum and minimum values of a allowable gain w are given by

$$\bar{w} = \min\{w_1(\omega_d - \varepsilon_t), w_{max}\} \tag{5.99}$$

$$\underline{w} = \max\{w_2(\omega_d - \varepsilon_t), 0\} \tag{5.100}$$

In addition, if, for a given w satisfying $\underline{w} \leq w \leq \bar{w}$, a control input given by (5.37), (5.38), and (5.45) is applied to a robot manipulator (5.1), then

$$\| \boldsymbol{e}(t) \| < \omega, \quad \forall t \geq 0 \tag{5.101}$$

where

$$\omega \triangleq \max\{\omega_d, \varepsilon_0 + \varepsilon_t\} \tag{5.102}$$

and we call ω a tracking precision, while ω_d a specified tracking precision. ∎

Proof: Noting $0 \leq w \leq w_{max}$, we have only to solve (5.73) with respect to w. ∎

If $\omega_d \leq \varepsilon_0 + \varepsilon_t$ in Theorem 5.2, then the tracking precision ω is characterized by the initial state error, so the specified tracking precision ω_d has no effect on ω. However, the norm of the control error converges to ω_d as time goes.

[Theorem 5.3] *Suppose a control input given by (5.37), (5.38), and (5.45) is applied to a robot manipulator (5.1). Then for all $\bar{\varepsilon}(> \omega_d - \varepsilon_t)$, there exists a finite number $I(\bar{\varepsilon})$ when $\| e_0 \| \geq \bar{\delta}$, and*

$$\| e(t) \| < \bar{\varepsilon} + \varepsilon_t, \quad \forall t \geq I(\bar{\varepsilon})T \tag{5.103}$$

where

$$\bar{\delta} \triangleq c\bar{\varepsilon} > \delta \tag{5.104}$$

$$I(\bar{\varepsilon}) \triangleq \left[\frac{\lambda_m(\boldsymbol{P})\bar{\delta}^2 - V_0}{-\lambda_m(\boldsymbol{Q})\bar{\delta}^2 + 2\beta\bar{\delta} + T_p\gamma^2\bar{g}^2} \right] \tag{5.105}$$

and $[\cdot]$ is a function which satisfies $[a] = b + 1$ when $b < a \leq b + 1$, for a real number a and a integer b. ∎

Proof: The proof can be shown in the same way as Theorem 3 in [78]. ∎

In general, when continuous time control theory is applied to digital control directly, we can frequently find that chattering becomes larger and so the specified tracking precision cannot be achieved as a switching gain becomes larger for a fixed sampling period. On the other hand, Theorem 5.2 gives an allowable bound of a switching gain to achieve the specified tracking precision in the case of digital control. Based on Theorem 5.2, a design procedure of a digital robust control system is shown in Fig.5.1. In addition, when a switching gain $wg(\boldsymbol{x}_i, \boldsymbol{u}_{Mi})$ and a specified tracking precision ω_d are given, we can estimate an allowable bound of the sampling period so as to achieve (5.101) by (5.73).

5.4 Discussion on chattering phenomenon

Since a robust controller proposed in section 5.3 has a discontinuous function on $\boldsymbol{s}_i$, i.e., $\psi(\boldsymbol{s}_i)$, chattering phenomena occur in the digital control system. In the case of continuous time control, the chattering

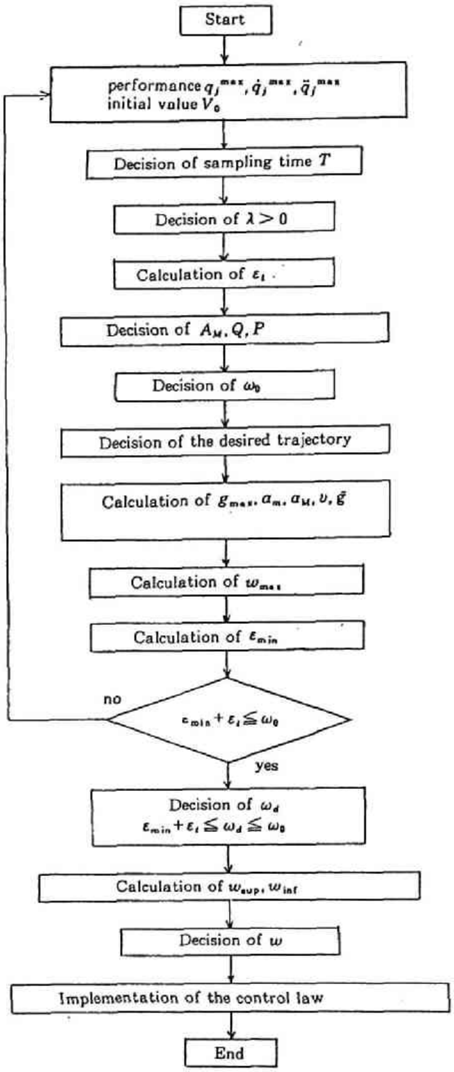

Figure 5.1: Flow chart of design procedure for digital robust control system

phenomenon does not theoretically yield by using a continuous function in place of a discontinuous function. However, in the digital control case, we believe that the chattering depends on the bound of a switching gain, rather than whether a robust control law has a continuous function or not, because a control law which has a too high gain tends to occur the chattering even if a continuous function is used. Thus in this section, we discuss how to decrease the chattering, by paying attention to the bound of a switching gain.

5.4.1 Use of weighting function for switching gain

We consider a weighting function that makes a switching gain smaller as the norm of the control error $\| \boldsymbol{e}_i \|$ becomes smaller. Thus we consider the following controller.

<Robust compensation $\boldsymbol{u}_{Ri}$>

$$\boldsymbol{u}_{Ri} = -\frac{\bar{\boldsymbol{M}}_i}{\alpha_m} z(\boldsymbol{e}_i) w g(\boldsymbol{x}_i,\ \boldsymbol{u}_{Mi}) \psi(\boldsymbol{s}_i) \tag{5.106}$$

where

$$z(\boldsymbol{e}_i) \triangleq \begin{cases} 1 & \text{if} \quad \| \boldsymbol{e}_i \| \geq \zeta \\ (\frac{\|\boldsymbol{e}_i\|}{\zeta})^{\kappa} & \text{if} \quad \| \boldsymbol{e}_i \| < \zeta \end{cases} \tag{5.107}$$

When $\kappa = 0$, (5.106) and (5.107) are equivalent to (5.45). When $\kappa > 0$, a switching gain in (5.106) is smaller than that of (5.45) if $\| \boldsymbol{e}_i \| < \zeta$. Note that a switching gain becomes smaller as κ becomes larger. Thus we expect some reduction of the chattering by using a weighting function $z(\boldsymbol{e}_i)$. Here we get the following lemma about a threshold ζ.

[Lemma 5.4] *Suppose a controller given by (5.37), (5.38), and (5.106) is applied to a robot manipulator (5.1), and there exists i such that* $\| \boldsymbol{e}_i \| < \zeta < \delta$. *If* ζ *satisfies*

$$\zeta \leq -\frac{\beta(0)}{\lambda_M(\boldsymbol{P})} + \sqrt{\left(\frac{\beta(0)}{\lambda_M(\boldsymbol{P})}\right)^2 + \frac{\lambda_m(\boldsymbol{P})\delta^2 - T_p\gamma^2\bar{g}^2}{\lambda_M(\boldsymbol{P})}} \tag{5.108}$$

then at the next sampling point $i+1$, $\| \boldsymbol{e}_{i+1} \| < \delta$. ■

Proof: Substituting (5.38) and (5.106) into (5.57), and using Appendix, we get

$$\Delta V_i < 2\beta(0)\zeta + T_p\gamma^2\bar{g}^2 \tag{5.109}$$

From $\| e_i \| < \zeta$, we also obtain

$$V_i \leq \lambda_m(\boldsymbol{P}) \| \boldsymbol{e}_i \|^2 < \lambda_M(\boldsymbol{P})\zeta^2 \tag{5.110}$$

Hence (5.109) and (5.110) imply

$$V_{i+1} < \lambda_M(\boldsymbol{P})\zeta^2 + 2\beta(0)\zeta + T_p\gamma^2\bar{g}^2 \tag{5.111}$$

Thus from (5.111) and

$$V_{i+1} \geq \lambda_m(\boldsymbol{P}) \| \boldsymbol{e}_{i+1} \|^2 \tag{5.112}$$

we conclude that if

$$\lambda_M(\boldsymbol{P})\zeta^2 + 2\beta(0)\zeta + T_p\gamma^2\bar{g}^2 \leq \lambda_m(\boldsymbol{P})\delta^2 \tag{5.113}$$

then $\| \boldsymbol{e}_{i+1} \| < \delta$. Eq.(5.108) follows from (5.113). ∎

From Lemma 5.4, we get the following theorem.

[Theorem 5.4] *Suppose a controller given by (5.37), (5.38), and (5.106) is applied to a robot manipulator (5.1). Then if ζ satisfies (5.108), then*

$$\| \boldsymbol{e}(t) \| < \max\{\varepsilon,\varepsilon_0\} + \varepsilon_t, \quad \forall t \geq 0 \tag{5.114}$$

which is the same result as that in the case of (5.38) and (5.45). ∎

Proof: From Lemma 5.4, a weighting function z always becomes 1 until $\| \boldsymbol{e}_i \|$ is larger than δ. Then Lemma 5.2 can be applied, and the proof is the same as Theorem 5.1. ∎

Therefore, from Theorem 4, the same results as Theorems 5.2 and 5.3 holds in the case of (5.106).

Remark 5.2 *Although $z(\boldsymbol{e}_i)$ is considered as (5.107) here, in general the above discussion holds if a weighting function z satisfies $0 \leq z \leq 1$ in the case of $\| \boldsymbol{e}_i \| \leq \zeta$. Therefore κ is independent on the other design parameters such as ω_d.* ∎

5.4.2 Estimation of the bound of uncertainty

In the right-hand side of (5.44), we use a set of the state which is determined by some mechanical performance to estimate the bound of the uncertainty. However, this estimation frequently becomes conservative. Thus we use here a set of the state which is within some distance from a given reference trajectory to estimate it. Using ω_0 in (5.90) and (5.91), we redefine $\boldsymbol{\Omega}_p$ and $\boldsymbol{\Omega}_v$ as follows.

$$\boldsymbol{\Omega}_p \triangleq \{\boldsymbol{\theta} \mid |\theta_j| \leq |\theta_{Mj}| + \omega_0, j = 1,2,\ldots,n\} \tag{5.115}$$

$$\boldsymbol{\Omega}_v \triangleq \{\dot{\boldsymbol{\theta}} \mid |\dot{\theta}_j| \leq |\dot{\theta}_{Mj}| + \frac{\omega_0}{\lambda}, j = 1,2,\ldots,n\} \tag{5.116}$$

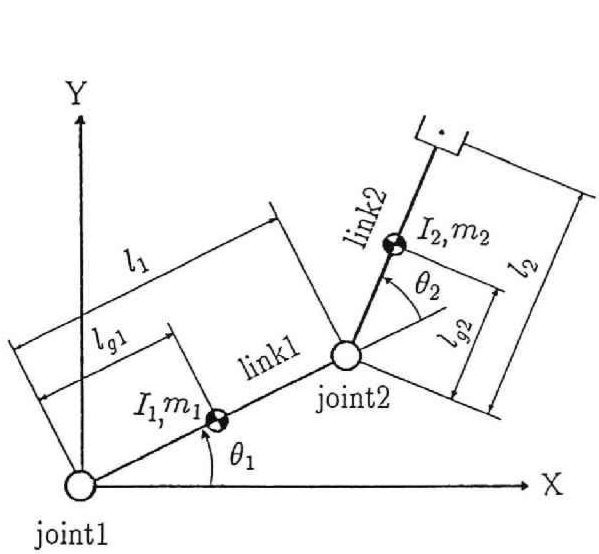

Figure 5.2: Model of 2 d.o.f. Manipulator

When the sets $\boldsymbol{\Omega}_p$ and $\boldsymbol{\Omega}_v$ are applied to estimate the bound of the uncertainty in the right-hand side of (5.44), the estimation becomes less conservative, so a part of a switching gain $g(\boldsymbol{x}_i, \boldsymbol{u}_{Mi})$ becomes relatively small. The estimation of (5.11), (5.12), and (5.46) can be treated in the same way.

5.5 Simulation

In order to verify the effectiveness of the proposed control method, we show some simulation results of a desired trajectory control of a 2 degree of freedom manipulator shown in Fig.5.2.

Let m_i, I_i, l_i, and l_{gi} denote the mass of link i, the moment of inertia of link i about the center of mass, the length of link i, and the distance between joint i and the center of mass of link i ($i = 1, 2$), respectively. The physical parameters ϕ_1, ϕ_2, and ϕ_3 are defined as $\phi_1 = m_1 l_{g1}^2 + I_1 + m_2 l_1^2$, $\phi_2 = m_2 l_1 l_{g2}$, and $\phi_3 = m_2 l_{g2}^2 + I_2$. Then a dynamic equation of a manipulator shown in Fig.5.2 is described by

$$\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{5.117}$$

Table 5.1: Unknown parameters of manipulator

Unknown parameter	Min. value	Max. value	Real value	Nominal value
m_2 (kg)	5.0	7.0	5.0	6.0
I_2 (kgm^2)	0.104	0.1456	0.104	0.125

Table 5.2: Known parameters of manipulator

Known parameter	Real value	Known parameter	Real value
I_1 (kgm^2)	0.104	m_1 (kg)	5.0
l_{n1} (m)	0.5	l_{n2} (m)	0.5
l_{g1} (m)	0.25	l_{g2} (m)	0.25

$$\boldsymbol{\theta} = [\theta_1, \theta_2]^T, \quad \boldsymbol{u} = [u_1, u_2]^T, \quad \boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^T$$
$$\boldsymbol{M}(\boldsymbol{\phi}, \boldsymbol{\theta}) = \begin{bmatrix} \phi_1 + \phi_3 + 2\phi_2 C_2 & \phi_3 + \phi_2 C_2 \\ \phi_3 + \phi_2 C_2 & \phi_3 \end{bmatrix}$$
$$\boldsymbol{h}(\boldsymbol{\phi}, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -\phi_3 S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \phi_3 S_2 \dot{\theta}_1^2 \end{bmatrix}$$

where $S_j \triangleq \sin\theta_j$ and $C_j \triangleq \cos\theta_j \quad (j = 1, 2)$. Assume that m_2 and I_2 are unknown, but a maximum value and a minimum value are known, which are shown together with real values and nominal values in Table 5.1. The other parameters are known as shown in Table 5.2. The mechanical performance is assumed as $\theta_{jmax} = \pi(rad)$, $\dot{\theta}_{jmax} = 0.7$ (rad/s), and $\ddot{\theta}_{jmax} = 3.0$ (rad/s^2) $(j = 1, 2)$.

Design parameters are given as follows. $T = 1(msec)$, $\lambda = 1.0$, $k_1 = k_2 = -1.0$, $\bar{q}_2 = 0.0$, and $\bar{q}_1 = \bar{q}_3 = 0.01$. Then from (5.66), $\varepsilon_t = 4.357 \times 10^{-3}$, and also from (5.96), $\omega_0 = 0.08$. Then a desired trajectory which belongs to $Q_M(\omega_0)$ is given by

$$\begin{aligned} \theta_{M1}(t) &= -0.5\cos(\pi t/3) \\ \theta_{M2}(t) &= -0.5\cos(\pi t/3) - 1.0 \\ &\text{for} \quad 0 \le t \le 6.0 \end{aligned} \tag{5.118}$$

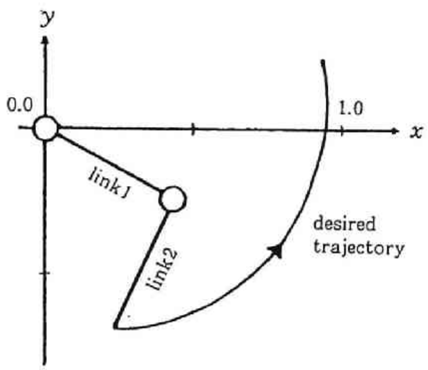

Figure 5.3: Desired trajectory of end effector

where a trajectory of an end effector is shown in Fig.5.3. The initial state errors are set as $e_1(0) = 0.02$ and $e_2(0) = 0.0$. In addition, $g_{max} = 1.0$ from (5.A12), $\alpha_m = 1.032$ from (5.11), $\alpha_M = 1.250$ from (5.12), $\nu = 3.0 \times 10^{-7}$ from (5.46), $T_p = 1.0 \times 10^{-5}$ from (5.47), $\bar{g} = 0.47$ from (5.49), $w_{max} = 2.12$ from (5.94), and $\varepsilon_{min} + \varepsilon_t = 0.0575$ from (5.95). Then we set a specified tracking precision as $\omega_d = 0.065$ ($\leq \omega_0$), so $\bar{w} = 1.252$ and $\underline{w} = 0.992$ from (5.99). Then we set w as $w = 0.992$. From (5.102), $\omega = 0.065$. A function $g(\boldsymbol{x}_i, \boldsymbol{u}_{Mi})$ is set as $g = \bar{g}$, and the sets $\boldsymbol{\Omega}_p$ and $\boldsymbol{\Omega}_v$ are given by (5.115) and (5.116), respectively.

Under the above situation, simulation results are shown in Figs.5.4 to 5.6. Note that, although a manipulator is a continuous-time system, we use in this simulation the Euler method with the integral interval of 0.1(msec) as numerical integration. Fig.5.4 shows a result in the case of (5.38) and (5.45), and Fig.5.4(a) shows a relation between the norm of the tracking control error $\| \boldsymbol{e}(t) \|$ and the specified tracking precision ω. As you can see in Fig.5.4, the norm of the control error is less than the specified tracking precision. Fig.5.4(b) shows an input. The chattering appears in an control input due to the use of a discontinuous function. Concerning u_2, the same result as u_1 is obtained, although it is not shown in figure. Fig.5.5 shows a result in the case of (5.38) and

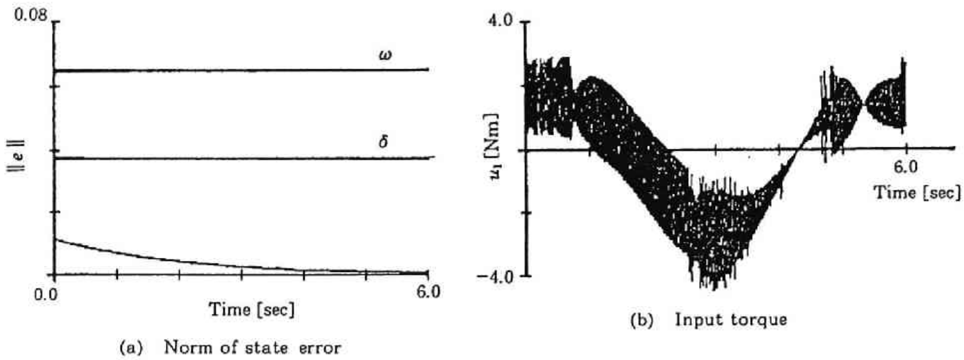

Figure 5.4: Simulation results using the proposed controller: (a) Norm of state error, (b) Input torque

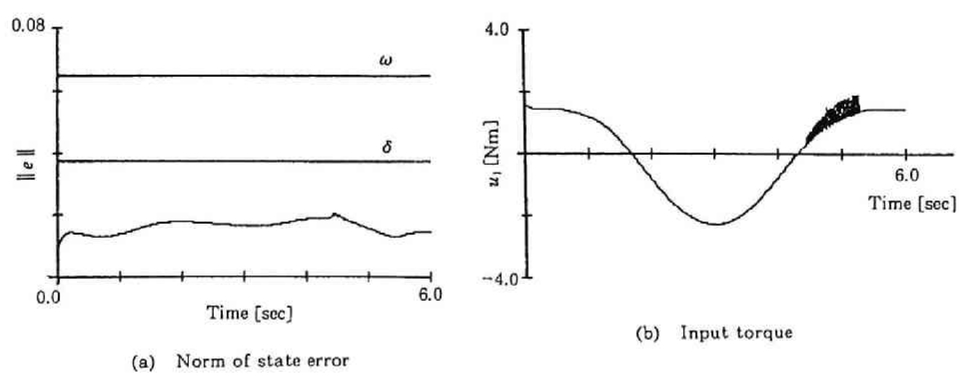

Figure 5.5: Simulation results using the proposed controller with a weighting function z: (a) Norm of state error, (b) Input torque

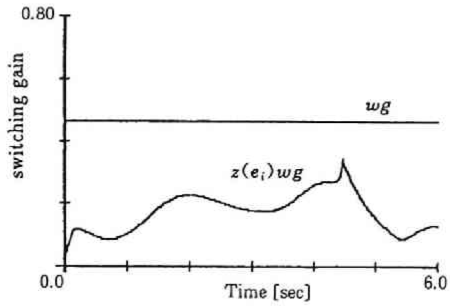

Figure 5.6: Comparison of both switching gains

(5.106). κ in (5.106) is given as $\kappa = 3$. In Fig.5.5(b), we can see that the chattering is much decreased, compared with Fig.5.4(b). Fig.5.6 shows a relation between time and a switching gain $z(\boldsymbol{e}_i)wg$. The use of a weighting function $z(\boldsymbol{e}_i)$ makes a switching gain much smaller.

These results shows the effectiveness of the proposed controller.

5.6 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) The relation between a feedback gain and a control error for a given sampling period has been clarified in the digital control of robot manipulators, by deriving some kind of discrete-time description of nonlinear systems.

(ii) Based on the above analysis, a digital robust control scheme of robot manipulators has been proposed, which gives a design procedure to find a feedback gain so as to achieve the specified tracking precision for a given sampling period.

(iii) A weighting function for a feedback gain has been proposed in order to decrease the chattering.

(iv) Simulation results have been given to illustrate the validity of the proposed method.

Appendix

Proof of (5.62) : From (5.30) in Lemma 5.1, we get

$$\boldsymbol{y}_i = \begin{bmatrix} \{\hat{\ddot{\boldsymbol{\theta}}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i)\}T^2/2 \\ \{\tilde{\ddot{\boldsymbol{\theta}}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i)\}\lambda T \end{bmatrix} \quad (5.A1)$$

Substituting (5.38) and (5.45)(or (5.106)) into (5.A1), we get

$$\| \tilde{\ddot{\boldsymbol{\theta}}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i) \| < \gamma(w)\bar{g} \quad (5.A2)$$

In the same way,

$$\| \hat{\ddot{\boldsymbol{\theta}}}_i - (\boldsymbol{u}_{Mi} + \boldsymbol{K}\boldsymbol{x}_i) \| < \gamma(w)\bar{g} \quad (5.A3)$$

Hence it follows from (5.A2), (5.A3), and (5.42) that

$$\boldsymbol{y}_i^T \boldsymbol{P} \boldsymbol{y}_i < T_p \gamma(w)^2 \bar{g}^2 \quad (5.A4)$$

Calculation of g_{max} : For a given desired trajectory $\boldsymbol{x}_M(t)$, we can estimate a maximum value of a linearizing input given by (5.38), and also a maximum value of each element of $\boldsymbol{M}^{-1}(\boldsymbol{u}_{Li} - \boldsymbol{h})$. Let $v_{jmax}(j = 1, 2, \ldots, n)$ be a maximum value of each element of $\boldsymbol{M}^{-1}(\boldsymbol{u}_{Li} - \boldsymbol{h})$, that is to say,

$$\begin{aligned} v_{jmax} \ &\triangleq\ \max\{\mid v_j \mid\} \\ &\qquad \forall \boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p, \forall \dot{\boldsymbol{\theta}}_i \in \boldsymbol{\Omega}_v, \forall \boldsymbol{u}_{Mi} \in \boldsymbol{\Omega}_M \end{aligned} \quad (5.A5)$$

where $v_j \triangleq \{\boldsymbol{M}^{-1}(\boldsymbol{u}_{Li} - \boldsymbol{h})\}_j$. In addition, let

$$\tilde{v}_{jmax} \triangleq \max\{\sum \mid m_{jk} \mid\} \qquad \forall \boldsymbol{\theta}_i \in \boldsymbol{\Omega}_p \quad (5.A6)$$

where $m_{jk} \triangleq \{\boldsymbol{M}^{-1}\bar{\boldsymbol{M}}_i\}_{jk}$

Then we get the following result.

[Lemma 5.5]

If a switching gain in (5.45) satisfies

$$wg(\boldsymbol{x}_i, \boldsymbol{u}_{Mi}) \leq \min_j\{\alpha_m(\ddot{\theta}_{jmax} - v_{jmax})/\tilde{v}_j\} \quad (5.A7)$$

then

$$\mid \ddot{\theta}_j \mid \leq \ddot{\theta}_{jmax} \quad \forall j \quad (5.A8)$$

∎

Proof: If a switching gain wg satisfies (5.A7), then

$$v_{jmax} + \tilde{v}_j wg/\alpha_m \leq \ddot{\theta}_{jmax}, \quad \forall j \tag{5.A9}$$

On the other hand, from

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}(\boldsymbol{u}_{Li} + \boldsymbol{u}_{Ri} - \boldsymbol{h}) \tag{5.A10}$$

we get

$$| \ddot{\theta}_j | \leq v_{jmax} + \tilde{v}_j wg/\alpha_m, \quad \forall j \tag{5.A11}$$

Hence (5.A8) follows from (5.A9) and (5.A11). ∎

From Lemma 5.5, we conclude that

$$g_{max} = \min_j \{\alpha_m(\ddot{\theta}_{jmax} - v_{jmax})/\tilde{v}_j\} \tag{5.A12}$$

Chapter 6

HIERARCHICAL ROBUST CONTROL OF ROBOT MANIPULATORS

6.1 Introduction

In the previous chapter, a digital robust control method has been discussed for tracking control of robot manipulators. This chapter treats a robust control system of robot manipulators which has a hierarchical structure composed of two loops, i.e., an upper level loop and a lower level loop. It is called a hierarchical robust control system.

Robust controllers of robot manipulators are usually composed of linearization and robust compensation. In general, the generation of the linearization requires much amount of calculation, so we may frequently be impossible to ignore the calculation time. On the other hand, the robust compensation is based on the high gain feedback, so it is desired that the computation period for the generation of the robust compensator is small as possible.

From the above point of view, in this chapter, a hierarchical robust control method of robot manipulators is proposed, where the control system has two loops; an upper level loop which works at a low sampling frequency and a lower level loop which works at a high sampling frequency. In the upper level loop, an input for the linearizing compensation, a desired trajectory, and a feedback gain are generated at a low sampling frequency. In the lower level loop, a switching input, which is one part of the robust compensator, is computed at a high sampling frequency. This scheme make the calculation for the generation of the robust compensator much faster, so we expect that the effect of the discretization of a robust controller on the control error is smaller. The hierarchical structure of the control system in the dynamic control of robot manipulators itself was proposed by Khatib et al. [63] in 1986, and the effectiveness of this method was shown by an experiment result by Yoshikawa et al. [144] in 1988. However, there was no theoretical discussion on the control performance in the system with the hierarchical structure. In the next sections, the control performance of this hierarchical system is analyzed under the consideration of a sampling period of an upper level loop and a modeling error. Finally, some simulation results are shown to verify the effectiveness of the hierarchical robust control systems.

6.2 Problem Statement

Consider a manipulator with n degrees of freedom whose dynamics is described by the following equation :

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}+\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\boldsymbol{u} \tag{6.1}$$

where $\boldsymbol{\theta} \triangleq [\theta_1,\theta_2,\ldots,\theta_n]^T$ is the n-dimensional vector of joint displacements, $\boldsymbol{\phi}$ is the physical parameter vector with an appropriate dimension, $\boldsymbol{u}$ is the n-dimensional joint torque input vector, $\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})$ is the $n \times n$ manipulator inertia matrix, and $\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$ is the n-dimensional vector that represents the nonlinear terms such as centrifugal, Coriolis, frictional, and gravitational forces.

This system usually has the following features.

[Feature 6.1] *$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})$ is a positive definite matrix for any $\boldsymbol{\theta}$.* ∎

[Feature 6.2] *The left-hand side of (6.1) can be expressed as*

$$\boldsymbol{M}(\boldsymbol{\phi},\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}+\boldsymbol{h}(\boldsymbol{\phi},\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\boldsymbol{E}(\boldsymbol{\phi})\boldsymbol{y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}}) \tag{6.2}$$

where $\boldsymbol{E}(\boldsymbol{\phi})$ is an appropriate dimensional matrix consisting of physical parameters, and $\boldsymbol{y}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\ddot{\boldsymbol{\theta}})$ is an appropriate dimensional vector whose elements are known functions of $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$. ∎

We begin with the definition of the notations which express mechanical performance of a manipulator given here. Let θ_{imax} denote maximum movable range of the ith joint angle, that is, $\theta_{imax} \stackrel{\triangle}{=} \max \mid \theta_i \mid$. Let $\dot{\theta}_{imax}$ denote maximum angular velocity of the ith joint, that is, $\dot{\theta}_{imax} \stackrel{\triangle}{=} \max \mid \dot{\theta}_i \mid$. Let $\ddot{\theta}_{imax}$ denote maximum angular acceleration of the ith joint, that is, $\ddot{\theta}_{imax} \stackrel{\triangle}{=} \max \mid \ddot{\theta}_i \mid$. We also define the following sets for θ_{jmax} and $\dot{\theta}_{jmax}$.

$$\boldsymbol{\Omega}_p \stackrel{\triangle}{=} \{\boldsymbol{\theta} \mid \mid \theta_j \mid \leq \theta_{jmax},\ j = 1, 2, \ldots, n \quad \} \tag{6.3}$$

$$\boldsymbol{\Omega}_d \stackrel{\triangle}{=} \{\dot{\boldsymbol{\theta}} \mid \mid \dot{\theta}_j \mid \leq \dot{\theta}_{jmax},\ j = 1, 2, \ldots, n \quad \} \tag{6.4}$$

Let T denote a sampling period in the upper level loop, which has a low sampling frequency. Note that, since T is dependent on degrees of freedom of joints and the performance of a hardware used in the control system, we assume here that T is a positive constant given in advance. Then for a given $\boldsymbol{\theta} \in \boldsymbol{\Omega}_p$, $\dot{\boldsymbol{\theta}} \in \boldsymbol{\Omega}_d$, we define the following sets.

$$\boldsymbol{\Pi}_p(\boldsymbol{\theta}) \stackrel{\triangle}{=} \{\boldsymbol{\xi} \mid \mid \xi_j - \theta_j \mid \leq 2\dot{\theta}_{jmax}T,\ j = 1, 2, \ldots, n\} \tag{6.5}$$

$$\boldsymbol{\Pi}_d(\dot{\boldsymbol{\theta}}) \stackrel{\triangle}{=} \{\boldsymbol{\xi} \mid \mid \xi_j - \dot{\theta}_j \mid \leq 2\ddot{\theta}_{jmax}T,\ j = 1, 2, \ldots, n\} \tag{6.6}$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]^T$. $\boldsymbol{\Pi}_p(\boldsymbol{\theta})$ and $\boldsymbol{\Pi}_d(\dot{\boldsymbol{\theta}})$ express a set of joint angle and joint angular velocity which is reachable in 2 sampling periods from $\boldsymbol{\theta}(t)$ and $\dot{\boldsymbol{\theta}}(t)$, respectively.

Then the following assumptions are made.

[Assumption 6.1] *$\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ are measurable.* ∎

[Assumption 6.2] *θ_{imax}, $\dot{\theta}_{imax}$, and $\ddot{\theta}_{imax}$ are known.* ∎

[Assumption 6.3] *The values of the physical parameter vector $\boldsymbol{\phi}$ may be unknown, but it is known that $\boldsymbol{\phi}$ exists in a certain bounded region $\boldsymbol{\Omega}_\phi$.* ∎

[Assumption 6.4] *$\hat{\boldsymbol{\phi}}$, a bounded estimate of $\boldsymbol{\phi}$, is given such that the following matrix is positive definite for all $\boldsymbol{\theta} \in \boldsymbol{\Omega}_p$, $\boldsymbol{\xi} \in \boldsymbol{\Pi}_p(\boldsymbol{\theta})$, and $\boldsymbol{\phi} \in \boldsymbol{\Omega}_\phi$.*

$$\bar{\boldsymbol{M}} \stackrel{\triangle}{=} \{\boldsymbol{M}^{-1}(\boldsymbol{\phi}, \boldsymbol{\xi})\hat{\boldsymbol{M}}(\hat{\boldsymbol{\phi}}, \boldsymbol{\theta}) + \hat{\boldsymbol{M}}^T(\hat{\boldsymbol{\phi}}, \boldsymbol{\theta})\boldsymbol{M}^{-T}(\boldsymbol{\phi}, \boldsymbol{\xi})\}/2 \tag{6.7}$$

∎

Note that, from Assumptions 6.3 and 6.4, there exist positive constants α and β such that the following conditions are satisfied.

$$\alpha < \lambda_m(\bar{\boldsymbol{M}}), \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Omega}_p,\ \forall \boldsymbol{\xi} \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}),\ \forall \boldsymbol{\phi} \in \boldsymbol{\Omega}_\phi \tag{6.8}$$

$$\beta > \lambda_M(\boldsymbol{M}^{-1}\hat{\boldsymbol{M}}) \quad \forall\boldsymbol{\theta} \in \boldsymbol{\Omega}_p,\ \forall\boldsymbol{\xi} \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}),\ \forall\boldsymbol{\phi} \in \boldsymbol{\Omega}_\phi \tag{6.9}$$

For the above robot manipulator, we consider the following problem.

[Problem 6.1] *For a robot manipulator given by (6.1) that satisfies Assumption 6.1 to 6.4, a desired trajectory $\boldsymbol{\theta}_d(t)$ is given whose derivatives $\dot{\boldsymbol{\theta}}_d$ and $\ddot{\boldsymbol{\theta}}_d$ exist and are bounded. Then we consider a control system, which has two hierarchical feedback loops, that is, an upper level loop to work at low sampling frequency T and a lower level loop to work at high sampling frequency. Then for given ε_P and ε_V, find a control law such that*

$$\| \boldsymbol{e}(t) \| < \varepsilon_P, \qquad \| \dot{\boldsymbol{e}}(t) \| < \varepsilon_V \qquad \forall t \geq T \tag{6.10}$$

holds for all $t \geq 0$, where $\boldsymbol{e}(t) \stackrel{\triangle}{=} \boldsymbol{\theta}(t) - \boldsymbol{\theta}_d(t)$ and t_0 is an initial time. ∎

For simplicity, we assume that $\boldsymbol{e}(t_0) = \mathbf{o}$ and $\dot{\boldsymbol{e}}(t_0) = \mathbf{o}$. In addition we assume hereafter that a sampling period in the lower level is small enough to be negligible, namely, a control signal in the lower level loop is continuous on time.

Remark 6.1 *If we assume that a control input in the lower level loop is digital, we can estimate the bound of the control error in such a case, using the technique developed in chapter 5.* ∎

6.3 Hierarchical robust control

In this section, we propose a hierarchical robust control system for a robot manipulator given by (6.1).

6.3.1 Approximated desired trajectory in lower level loop

For a given desired trajectory, we consider how to generate a desired value at each sampling point. In the lower level loop, we usually hope that the computational amount to generate a control input is as small as possible, because it is desirable to make a sampling period small in the lower level loop. If we use desired values stored in memory, we may need a huge memory because a sampling period in the lower level loop is very small. Hence we use the first order approximation of the desired values generated in the upper level loop as a desired value in the lower level loop, which requires no large computation. That is to say, we use the following as a desired value at $t_i \in [iT, (i+1)T)$ in the lower loop level,

$$\tilde{\boldsymbol{\theta}}_d(t_i) \triangleq \boldsymbol{\theta}_{di} + (\boldsymbol{\theta}_{di+1} - \boldsymbol{\theta}_{di})(t_i - iT)/T \tag{6.11}$$

$$\dot{\tilde{\boldsymbol{\theta}}}_d(t_i) \triangleq \dot{\boldsymbol{\theta}}_{di} + (\dot{\boldsymbol{\theta}}_{di+1} - \dot{\boldsymbol{\theta}}_{di})(t_i - iT)/T \tag{6.12}$$

where $\boldsymbol{\theta}_{di}$ and $\dot{\boldsymbol{\theta}}_{di}$ mean the values at the i sampling point, namley, $\boldsymbol{\theta}_{di} \triangleq \boldsymbol{\theta}_d(iT)$ and $\dot{\boldsymbol{\theta}}_{di} \triangleq \dot{\boldsymbol{\theta}}_d(iT)$

The value with the subscirpt i means the value at the ith sampling point, except for t_i which means the time at the ith sampling period, i.e., $t_i \in [iT, (i+1)T)$ hereafter.

Remark 6.2 *When an approximated desired trajectory is not used in the lower level loop, we have only to replace $\tilde{\boldsymbol{\theta}}_d(t_i)$ and $\dot{\tilde{\boldsymbol{\theta}}}_d(t_i)$ by $\boldsymbol{\theta}_d(t_i)$ and $\dot{\boldsymbol{\theta}}_d(t_i)$, respectively, for $t_i \in [iT, (i+1)T)$ in the next argument.* ∎

6.3.2 Hierarchical robust controller

We consider the following control input which is composed of a linearizing compensator $\boldsymbol{u}_L(t)$ and a robust compensator $\boldsymbol{u}_R(t)$.

$$\boldsymbol{u}(t) = \boldsymbol{u}_L(t) + \boldsymbol{u}_R(t) \tag{6.13}$$

Then a control input in the ith sampling period is given as follows

<Linearizing compensation>

$$\boldsymbol{u}_L(t_i) = \boldsymbol{E}(\hat{\boldsymbol{\phi}})\hat{\boldsymbol{y}}_{i-1} \tag{6.14}$$

where

$$\hat{\boldsymbol{y}}_{i-1} \triangleq \boldsymbol{y}(\boldsymbol{\theta}_{i-1}, \dot{\boldsymbol{\theta}}_{i-1}, \boldsymbol{r}_{i-1}) \tag{6.15}$$

$$\boldsymbol{r}_{i-1} \triangleq \ddot{\boldsymbol{\theta}}_{di} - \lambda(\dot{\boldsymbol{\theta}}_{i-1} - \dot{\boldsymbol{\theta}}_{di}) \tag{6.16}$$

and λ is a part of velocity gain and is a design parameter to specify the control error precision.

Note that a control law given by (6.14) has almost the same form as the conventional dynamic control method, and $\boldsymbol{r}_{i-1}$ means a modified desired trajectory.

<Robust compensation>

The parametric uncertainty which cannot be compensated for by the above linearizing compensation is give by

$$\triangle \boldsymbol{E} \triangleq \boldsymbol{E}(\boldsymbol{\phi}) - \boldsymbol{E}(\hat{\boldsymbol{\phi}}) \tag{6.17}$$

and the discrepancy in $\boldsymbol{y}$ between the measurement of the state in the upper level loop where a computation time lag exists and the real state is given by

$$\Delta \boldsymbol{y}_i \triangleq \boldsymbol{y}(\boldsymbol{\theta}(t_i), \dot{\boldsymbol{\theta}}(t_i), \ddot{\boldsymbol{\theta}}_d(t_i) - \lambda \dot{\boldsymbol{e}}(t_i)) - \hat{\boldsymbol{y}}_{i-1} \tag{6.18}$$

for all $t_i \in [iT, (i+1)T)$. Then from Assumption 6.3, there exists a function $g(\hat{\boldsymbol{y}}_{i-1})$ such that

$$\begin{aligned} g(\hat{\boldsymbol{y}}_{i-1}) > \| \boldsymbol{M}^{-1}(\Delta \boldsymbol{E} \hat{\boldsymbol{y}}_{i-1} + \boldsymbol{E} \Delta \boldsymbol{y}_i) \| \\ \forall \boldsymbol{\theta}_{i-1} \in \boldsymbol{\Omega}_p, \ \forall \dot{\boldsymbol{\theta}}_{i-1} \in \boldsymbol{\Omega}_d \\ \forall \boldsymbol{\theta}(t_i) \in \boldsymbol{\Pi}_p(\boldsymbol{\theta}_{i-1}), \ \forall \dot{\boldsymbol{\theta}}(t_i) \in \boldsymbol{\Pi}_d(\dot{\boldsymbol{\theta}}_{i-1}) \end{aligned} \tag{6.19}$$

because $\boldsymbol{M}^{-1}$, $\Delta \boldsymbol{E}$, and $\boldsymbol{E}$ are bounded, and for example, for all $t_i \in [iT, (i+1)T)$

$$| \theta_j(t_i) - \theta_j((i-1)T) | \leq 2\dot{\theta}_{jmax} T < \infty \tag{6.20}$$

Let $\tilde{\boldsymbol{s}}$ denote an extended error between the state and the approximated desired value given by

$$\tilde{\boldsymbol{s}} \triangleq (\dot{\boldsymbol{\theta}} - \tilde{\dot{\boldsymbol{\theta}}}_d) + \lambda(\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}_d) \tag{6.21}$$

and W denote a weighting function given by

$$W(\tilde{\boldsymbol{s}}, \delta, p) \triangleq \begin{cases} 1 & \text{if} \quad \| \tilde{\boldsymbol{s}} \| > \delta \\ (\frac{\|\tilde{\boldsymbol{s}}\|}{\delta})^p & \text{if} \quad \| \tilde{\boldsymbol{s}} \| \leq \delta \end{cases} \tag{6.22}$$

where δ is a design parameter to specify the control error. Then using (6.19), (6.21), and (6.22), we consider

$$\boldsymbol{u}_R(t_i) = -W(\tilde{\boldsymbol{s}}, \delta, p)(\hat{\boldsymbol{M}}_{i-1}/\alpha)\{g(\hat{\boldsymbol{y}}_{i-1}) + k\}\psi(\tilde{\boldsymbol{s}}) \tag{6.23}$$

where k is a positive constant, and

$$\hat{\boldsymbol{M}}_{i-1} \triangleq \hat{\boldsymbol{M}}(\boldsymbol{\theta}_{i-1}) \tag{6.24}$$

$$\psi(\tilde{\boldsymbol{s}}) \triangleq \begin{cases} \frac{\tilde{\boldsymbol{s}}}{\|\tilde{\boldsymbol{s}}\|} & \text{if} \ \| \tilde{\boldsymbol{s}} \| \neq \boldsymbol{0} \\ \boldsymbol{0} & \text{if} \ \| \tilde{\boldsymbol{s}} \| = \boldsymbol{0} \end{cases} \tag{6.25}$$

We call $(\hat{\boldsymbol{M}}_{i-1}/\alpha)(g+k)$ a switching gain, and $\psi(\tilde{\boldsymbol{s}})$ a switching input. g is a part of a switching gain to compensate for the parametric uncertainty given by (6.17) and the discrepancy due to the time lag given by (6.18). $W(\tilde{\boldsymbol{s}}, \delta, p)$ is a weighting function for a switching gain. A positive number p is a design parameter to specify the bound of a switching gain. If $p = 0$, then the robust compensator consists of a discontinuous function on $\tilde{\boldsymbol{s}}$. If $p > 0$, then it has a weighting function to make a switching gain smaller when $\| \tilde{\boldsymbol{s}} \|$ becomes smaller. If $p = 1$, then it corresponds to the conventional controller which has the first order weighting function. Therefore, when $\| \tilde{\boldsymbol{s}} \| \leq \delta$, the switching gain

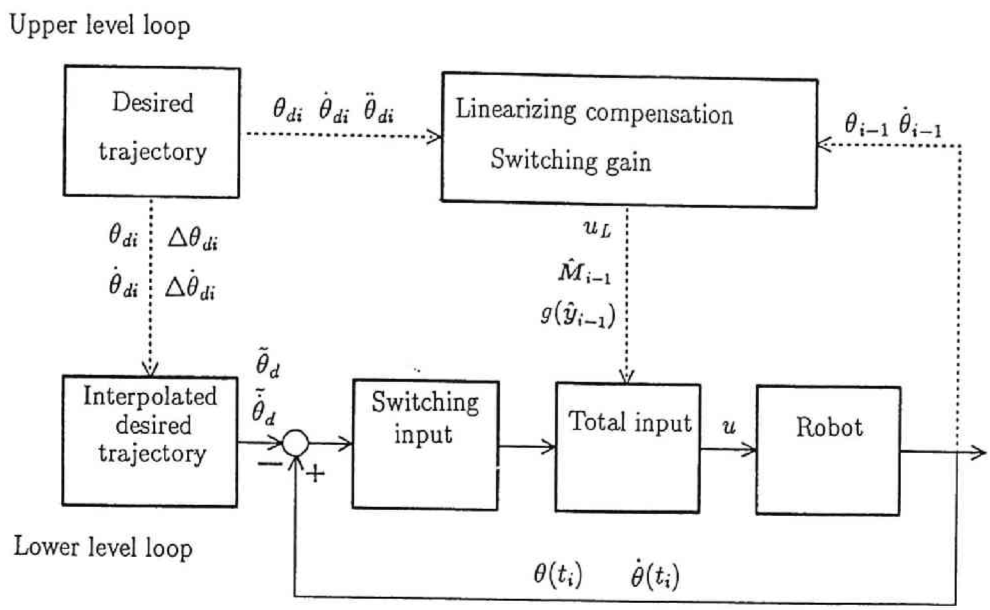

Figure 6.1: Hierarchical robust control system

in the case of $p \geq 2$ is smaller than that in the case of $p = 1$, which is the conventional case, and we can expect that a chattering phenomenon is decreased in the case of $p \geq 2$.

Remark 6.3 *We define W by (6.22). However if W satisfies $0 \leq W \leq 1$ in the case of $\| \tilde{s} \| \leq \delta$, the arguments in the next sections hold.* ∎

We show a hierarchical control system in Fig.6.1. The real line in Fig.6.1 expresses a signal of the lower level loop (high sampling frequency), and the solid line expresses a signal of the upper level loop (low sampling frequency). In the upper level loop, a desired value at the ith sampling point is generated, and a linearizing compensator and a switching gain are calculated using the ith desired value and the state at the $i-1$th sampling point. On the other hand, in the lower level loop, an approximated desired value is generated based on the desired value generated at each sampling point in the upper level loop. In addition, a switching input is calculated using the approximated desired value and the state at that time of a manipulator, which generates a control input $\boldsymbol{u}$, together with the linearizing compensator and the switching gain in the upper level loop. Note that we consider the time lag due to the computation in the upper level loop. As the computation time lag becomes larger, the switching gain becomes larger. Hence it is desirable to make the computational amount in the upper level loop as small as possible. In the computation of (6.14), we can use the computation method by Newton-Euler formulation, which is useful for the computation of the dynamics of a multiple d.o.f manipulator.

6.4 Estimation of the bound of control error

We estimate the bound of the control error when a controller given in section 6.3 is applied to a manipulator.

Let s denote an extended error between the state and the desired value given by

$$s \triangleq \dot{e} + \lambda e \tag{6.26}$$

Let ρ and g_{max} denote positive constants satisfying

$$\rho = \sup_t \| \dot{\tilde{\boldsymbol{\theta}}}_d(t) - \dot{\boldsymbol{\theta}}_d(t) + \lambda(\tilde{\boldsymbol{\theta}}_d(t) - \boldsymbol{\theta}_d(t)) \| \tag{6.27}$$

$$g_{max} > g(\hat{\boldsymbol{y}}_{i-1})$$
$$\forall \boldsymbol{\theta}_{i-1} \in \boldsymbol{\Omega}_p,\ \forall \dot{\boldsymbol{\theta}}_{i-1} \in \boldsymbol{\Omega}_d,\ \forall \dot{\boldsymbol{\theta}}_{di},\ \forall \ddot{\boldsymbol{\theta}}_{di} \tag{6.28}$$

Then for given ρ and g_{max}, define

$$\gamma \triangleq \rho(1 + \frac{\beta}{\alpha})(1 + \frac{g_{max}}{k}) \tag{6.29}$$

Then we get the following lemma.

[Lemma 6.1] *Suppose a control input given by (6.13) is applied to a manipulator (6.1). Consider*

$$V(t) = \frac{1}{2} s(t)^T s(t) \tag{6.30}$$

Then if δ *satisfies*

$$\delta > \gamma - \rho \tag{6.31}$$

then

$$\dot{V}(t) < 0, \qquad \forall t \geq 0 \tag{6.32}$$

when

$$\| s(t) \| \geq \delta + \rho \tag{6.33}$$

∎

Proof: We begin to consider $V(t_i)$, $iT \leq t_i < (i+1)T$. Differentiating $V(t_i)$ along the manipulator, we get

$$\begin{aligned} \dot{V}(t_i) &= s^T M^{-1}[u - M(\ddot{\theta}_d - \lambda \dot{e}) - h - g] \\ &= s^T M^{-1}[u_R + u_L - Ey(\theta, \dot{\theta}, r)] \end{aligned} \tag{6.34}$$

where $r = \ddot{\theta}_d - \lambda \dot{e}$. Substituting (6.14) into (6.34) leads to

$$\dot{V}(t_i) = s^T M^{-1}[u_R - \Delta E \hat{y}_{i-1} - E \Delta y_i] \tag{6.35}$$

From (6.27) and (6.33), we get

$$\| \tilde{s} \| \geq \delta \tag{6.36}$$

Noting that it follows from (6.7) and (6.8) that

$$\begin{aligned} \tilde{s}^T M^{-1} \hat{M} \tilde{s} &= \tilde{s}^T \bar{M} \tilde{s} \\ &\geq \lambda_m(\bar{M}) \| \tilde{s} \|^2 > \alpha \| \tilde{s} \|^2 \end{aligned} \tag{6.37}$$

we get, from (6.23) and(6.35)

$$\dot{V}(t_i) < -k \left[\| \tilde{s} \| - \rho \left\{ \frac{g_{max}}{k} \left(1 + \frac{\beta}{\alpha} \right) + \frac{\beta}{\alpha} \right\} \right] \tag{6.38}$$

On the other hand, it follows from (6.29) that

$$\delta > \gamma - \rho = \rho \left\{ \frac{g_{max}}{k} \left(1 + \frac{\beta}{\alpha} \right) + \frac{\beta}{\alpha} \right\} \tag{6.39}$$

This implies

$$\dot{V}(t_i) < 0 \qquad \forall t_i \in [iT, (i+1)T) \tag{6.40}$$

Hence noting that (6.40) holds for all i and $V(t)$ is continuous on t, we can show (6.32). ∎

From Lemma 6.1, we get the following result on the estimation of the bound of the control error.

[Lemma 6.2] *Suppose a control input given by (6.13) is applied to a manipulator (6.1). Then for all* $t \geq 0$,

$$\| s(t) \| < \delta + \rho \tag{6.41}$$

In addition, for all $t \geq 0$,

$$\| e(t) \| < \frac{\delta + \rho}{\lambda} \tag{6.42}$$

$$\| \dot{e}(t) \| < 2(\delta + \rho) \tag{6.43}$$

∎

Proof: We can show this in the same way as the latter part of the proof in Theorem 6.1 ∎

From Lemmas 6.1 and 6.2, we get the following theorem.

[Theorem 6.1] *For a manipulator given by (6.1) satisfying Assumptions 6.1 to 6.4, a desired trajectory is given. According to the desired trajectory, we calculate a positive number γ in (6.29). The parameters ε and λ that specify tracking precision are given to satisfy*

$$\varepsilon\lambda > \gamma \tag{6.44}$$

Also δ is given by

$$\delta = \varepsilon\lambda - \rho \tag{6.45}$$

Then if a control input (6.13) is applied to the manipulator, then

$$\| \boldsymbol{e}(t) \| < \varepsilon \tag{6.46}$$

$$\| \dot{\boldsymbol{e}}(t) \| < 2\lambda\varepsilon \tag{6.47}$$

for all $t \geq 0$. ∎

Proof: It follows from (6.44) and (6.45) that

$$\delta = \varepsilon\lambda - \rho > \gamma - \rho \tag{6.48}$$

and (6.32) of Lemma 6.1 holds. Then we get (6.42) and (6.43) from Lemma 6.2. Noting

$$\frac{\delta + \rho}{\lambda} = \varepsilon \tag{6.49}$$

we get (6.46) and (6.47). ∎

In Theorem 6.1, we have to note that the desired trajectory is realizable, because the mechanical performance of the given manipulator is restricted. Also we have to take account of the bound of the input, especially, the switching gain, when we determine the design parameters.

6.5 Comparison of computational amount

We discuss, in this section, how small the computational amount for the switching input is by using a hierarchical structure. We consider a n d.o.f. robot manipulator with rotational joints only, and use Denavit-Hartenberg notation. As for the upper level loop, we calculate the computational amount of the linearizing input (6.14) and the estimated value of the inertia matrix (6.24). As for the lower level loop, we calculate the computational amount of the approximated desired

Table 6.1: Computational amount for each method

Method		Multiplications	Additions
Hierarchical control	Upper level	$12n^2 + 190n - 71$	$7n^2 + 174n - 97$
	Lower level	$n^2 + 5n + 5 + p$	$n^2 + 9n + 2$
Non-hierarchical control		$137n - 15 + p$	$112n - 20$

Table 6.2: Number of arithmetic operations for n=3 and 6

Method			$n = 3$		$n = 6$	
Hierarchical control	Upper level	Mul.	607	1095	1501	2700
		Add.	488	(16.3)	1199	(16.5)
	Lower level	Mul.	$29 + p$	$67 + p$	$71 + p$	$163 + p$
		Add.	38	(1.0)	92	(1.0)
Non-hierarchical control		Mul.	$396 + p$	$712 + p$	$807 + p$	$1459 + p$
		Add.	316	(10.6)	652	(9.0)

( ) means the rate of the computational amount of the upper level (or Non-hierarchical control) for that of the low level.

trajectory (6.11) and (6.12), the switching input (6.25), the weighting function (6.22), and the total input (6.13). On the other hand, in the case of the conventional robust control which is not hierarchical, we calculate the computational amount of (6.14) and (6.23). Also we use the Newton-Euler method [136, 86] in the computation amount of the linearizing input and the estimated value of the inertia matrix in the upper level loop, and also in that of the non-hierarchical robust control. [1]. We do not calculate the computational amount of the desired trajectory and switching gain, because they are dependent on the trajectory or the function g.

The result is shown in Table 6.1. Especially, we show the case of $n = 3$ and $n = 6$ in Table 6.2. The notation $(\cdot)$ expresses the ratio of

[1]In the non-hierarchical robust control case, combining (6.14) and (6.23) in the following way, we can use Newton-Euler method.

$$\boldsymbol{u}(t_i) = \widehat{\boldsymbol{M}}_{i-1}[\boldsymbol{r}_{i-1} - (W/\alpha)(g+k)\psi] + \widehat{\boldsymbol{h}}_{i-1} + \widehat{\boldsymbol{g}}_{i-1}$$

where $\widehat{\boldsymbol{h}}$ and $\widehat{\boldsymbol{g}}$ is the estimate of $\boldsymbol{h}$ and $\boldsymbol{g}$.

the computational amount of the upper level loop (or Non-hierarchical control case) to the lower level loop. These tables show that the sampling period for computing the switching input in the hierarchical control case is about 1/9 times as small as that in the non-hierarchical control case. Hence we can expect the chattering is decreased by the use of the hierarchical structure. In addition, if we consider the computational amount of the desired trajectory and the switching gain in the upper loop level or the non-hierarchical control, the ratio of the computational amount of the upper level loop (or Non-hierarchical control case) to the lower level loop will be larger. Note that the computational amount in the upper level loop is about 2 times as large as that of the non-hierarchical control case. This is because of the computational amount of the estimates of the inertia matrix in the upper level loop.

6.6 Simulation

In order to verify the effectiveness of the proposed method, we show simulation results in the case of the trajectory tracking control of a 3 d.o.f. robot manipulator as shown in Fig.6.2.

Let m_j be the mass of the jth link ($j = 2, 3$). The center of the mass of the 1st link is in the Z axis, and the center of the mass of the jth link ($j = 2, 3$) is in the jth link. Let l_{gj} ($j = 2, 3$) denote the distance between the center of the mass of the jth link and the jth joint. The inertia tensor about the center of the mass of each link can be expressed by a diagonal matrix diag $[I_{jx}, I_{jy}, I_{jz}](j = 1, 2, 3)$, respectively. Let

$$\begin{aligned}
&P_{0k} = m_2 l_{g2}^2 + m_3 l_a^2 + I_{2k} \quad (k = y, z)\\
&P_{1k} = m_3 l_{g3}^2 + I_{3k} \quad (k = y, z)\\
&P_2 = m_3 l_a l_{g3}\\
&P_3 = m_2 l_{g2}^2 + m_3 l_a^2 + (I_{2y} - I_{2x})\\
&P_4 = m_3 l_{g3}^2 + (I_{3y} - I_{3x})
\end{aligned}$$

Then the dynamic equation in Fig.6.2 is given by

$$\boldsymbol{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{u} \tag{6.50}$$

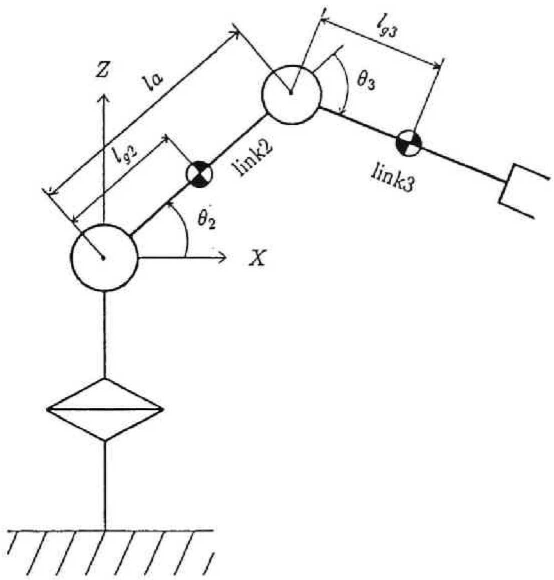

Figure 6.2: 3 d.o.f. robot manipulator model

$$
\begin{aligned}
&\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T \\
&\boldsymbol{u} = [u_1, u_2, u_3]^T \\
&\boldsymbol{M}(\boldsymbol{\theta}) = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & P_{1z} + P_2 C_3 \\ 0 & P_{1z} + P_2 C_3 & P_{1z} \end{bmatrix} \\
&\boldsymbol{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = [h_1, h_2, h_3]^T \\
&\quad M_{11} = I_{1z} + I_{2x} S_2^2 + P_{0y} C_2^2 \\
&\qquad + I_{3x} S_{23}^2 + P_{1y} C_{23}^2 + 2P_2 C_2 C_{23} \\
&\quad M_{22} = P_{0z} + P_{1z} + 2P_2 C_3 \\
&\quad h_1 = -2P_3 S_2 C_2 \dot{\theta}_1 \dot{\theta}_2 - 2P_4 S_{23} C_{23} \dot{\theta}_1 (\dot{\theta}_2 + \dot{\theta}_3) \\
&\qquad -2P_2 \{ (C_{23} S_2 + S_{23} C_2) \dot{\theta}_1 \dot{\theta}_2 + S_{23} C_2 \dot{\theta}_1 \dot{\theta}_3 \} \\
&\quad h_2 = P_3 S_2 C_2 \dot{\theta}_1^2 + P_4 S_{23} C_{23} \dot{\theta}_1^2 \\
&\qquad + P_2 \{ (C_{23} S_2 + S_{23} C_2) \dot{\theta}_1^2 - S_3 \dot{\theta}_3 (2\dot{\theta}_2 + \dot{\theta}_3) \} \\
&\quad h_3 = P_4 S_{23} C_{23} \dot{\theta}_1^2 + P_2 \{ S_{23} C_2 \dot{\theta}_1^2 + S_3 \dot{\theta}_2^2 \}
\end{aligned}
$$

where $S_j \triangleq \sin\theta_j$, $C_j \triangleq \cos\theta_j$ $(j = 1, 2)$, $S_{23} \triangleq \sin(\theta_2 + \theta_3)$, $C_{23} \triangleq \cos(\theta_2 + \theta_3)$, and we do not consider the gravity for simplicity. We assume that $\theta_{jmax} = \pi$ (rad), $\dot{\theta}_{jmax} = 2.0$ (rad/s), and $\ddot{\theta}_{jmax} = 5.0$ (rad/s^2) $(j = 1, 2, 3)$ as a mechanical performance. Also Define

$$\boldsymbol{E}_1 = \begin{bmatrix} I_{1z} & 0 & 0 \\ 0 & P_{0z} + P_{1z} & P_{1z} \\ 0 & P_{1z} & P_{1z} \end{bmatrix}, \qquad \boldsymbol{y}_1 = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}$$

$$\boldsymbol{E}_2 = \begin{bmatrix} 2P_2 & 0 & 0 \\ 0 & 2P_2 & P_2 \\ 0 & P_2 & 0 \end{bmatrix}, \qquad \boldsymbol{y}_2 = \begin{bmatrix} C_2C_{23}\ddot{\theta}_1 \\ C_3\ddot{\theta}_2 \\ C_3\ddot{\theta}_3 \end{bmatrix}$$

$$\boldsymbol{E}_3 = \begin{bmatrix} I_{2x} & P_{0y} & I_{3x} & P_{1y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{y}_3 = \begin{bmatrix} S_2^2\ddot{\theta}_1 \\ C_2^2\ddot{\theta}_1 \\ S_{23}^2\ddot{\theta}_1 \\ C_{23}^2\ddot{\theta}_1 \end{bmatrix}$$

$$\boldsymbol{E}_4 = \begin{bmatrix} P_3 & 0 & 0 \\ 0 & P_3 & 0 \\ 0 & 0 & P_3 \end{bmatrix}, \quad \boldsymbol{y}_4 = \begin{bmatrix} -2S_2C_2\dot{\theta}_1\dot{\theta}_2 \\ S_2C_2\dot{\theta}_1^2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{E}_5 = \begin{bmatrix} P_4 & 0 & 0 \\ 0 & P_4 & 0 \\ 0 & 0 & P_4 \end{bmatrix}, \boldsymbol{y}_5 = \begin{bmatrix} -2S_{23}C_{23}\dot{\theta}_1(\dot{\theta}_2 + \dot{\theta}_3) \\ S_{23}C_{23}\dot{\theta}_1^2 \\ S_{23}C_{23}\dot{\theta}_1^2 \end{bmatrix}$$

$$\boldsymbol{E}_6 = \begin{bmatrix} P_2 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_2 \end{bmatrix}$$

$$\boldsymbol{y}_6 = \begin{bmatrix} -2\{(C_{23}S_2 + S_{23}C_2)\dot{\theta}_1\dot{\theta}_2 + S_{23}C_2\dot{\theta}_1\dot{\theta}_3\} \\ (C_{23}S_2 + S_{23}C_2)\dot{\theta}_1^2 - S_3\dot{\theta}_3(2\dot{\theta}_2 + \dot{\theta}_3) \\ S_{23}C_2\dot{\theta}_1^2 + S_3\dot{\theta}_2^2 \end{bmatrix}$$

$$\boldsymbol{E} = [\boldsymbol{E}_1 \quad \boldsymbol{E}_2 \quad \boldsymbol{E}_3 \quad \boldsymbol{E}_4 \quad \boldsymbol{E}_5 \quad \boldsymbol{E}_6] \tag{6.51}$$

$$\boldsymbol{y} = [\boldsymbol{y}_1^T \quad \boldsymbol{y}_2^T \quad \boldsymbol{y}_3^T \quad \boldsymbol{y}_4^T \quad \boldsymbol{y}_5^T \quad \boldsymbol{y}_6^T]^T \tag{6.52}$$

Then we can verify that Feature 6.2 is satisfied. We show real values and estimated values of the physical parameters in Tables 6.3 and 6.4. Also g in (6.19) is given by

$$g(\hat{\boldsymbol{y}}_{i-1}) = \sum_{k=1}^{6} a_k \parallel \hat{y}_{k,i-1} \parallel + a_7 \tag{6.53}$$

where a_k $(k = 1, 2, \ldots, 7)$ is a positive constants satisfying

$$a_k > \lambda_M(\boldsymbol{M}^{-1}\Delta\boldsymbol{E}_k)$$

Table 6.3: Unknown parameters of manipulator

Unknown parameter	Inf. value	Sup. value	Real value	Nominal value
m_3 (kg)	6.0	12.0	6.0	9.0
I_{3x} (kgm^2)	0.3	0.9	0.3	0.6
I_{3y} (kgm^2)	0.3	0.9	0.3	0.6
I_{3z} (kgm^2)	0.3	0.9	0.3	0.6

Table 6.4: Known parameters of manipulator

Known parameter	Real value	Known parameter	Real value
I_{1z} (kgm^2)	0.4	m_2 (kg)	8.0
I_{2x} (kgm^2)	0.4	l_{g2} (m)	0.25
I_{2y} (kgm^2)	0.4	l_{g3} (m)	0.25
I_{2z} (kgm^2)	0.4	l_a (m)	0.5

Table 6.5: Design parameters

	CASE(i)	CASE(ii)	CASE(iii)
$T_{low}(msec)$	5.0	8.0	8.0
$T_{high}(msec)$		0.5	0.5
p	1	1	3
ε	0.0295	0.0045	0.002
a_1	1.518	a_2	0.794
a_3	0.478	a_4	3.399
a_5	3.569	a_6	1.700
a_7	1.194	k	0.3
α	0.58	γ	1.5×10^{-3}
β	2.22	ρ	8.0×10^{-6}

$a_7 > \| \boldsymbol{M}^{-1}\boldsymbol{E}\Delta\boldsymbol{y}_i \|$

The desired trajectory is given by

$$\begin{aligned} \theta_{d1} &= 0.5\cos(\pi t/3) - 1.0 \\ \theta_{d2} &= -0.5\cos(\pi t/3) \\ \theta_{d3} &= -0.5\cos(\pi t/3) - 1.0 \quad \text{for} \quad 0 \le t \le 3.0 \end{aligned} \tag{6.54}$$

We discuss how small the tracking precision can be specified in the following three cases.

(i) Non-hierarchical robust control method with $p = 1$

(ii) Hierarchical robust control **I** with $p = 1$

(iii) Hierarchical robust control **II** with $p = 3$

We consider $p = 1$ in the case of (i) and (ii), while $p = 3$ in the case of (iii). Design parameters for each case are shown in Table 6.5, where T_{low} is a sampling period in the upper level loop and T_{high} is a sampling period in the lower level loop. Note that ε is given as small as possible under the condition that the chattering does not appear almost. The switching gain a_j is given as shown in Table 6.5, although $a_7 = 0.0$ and $k = 0.0$ in the case (i). Also we set $\lambda = 1.0$. Note that, although a manipulator is a continuous-time system, we use in this simulation the Euler method with the integral interval of 0.1(msec) as numerical integration.

The simulation results are shown in Figs.6.3, 6.4, and 6.5. Fig.6.3 shows that the chattering occurs in the case (i) with $\varepsilon = 0.0295$. If ε becomes smaller, then the chattering becomes larger, and the control error cannot achieve the specified tracking precision. On the other hand, in the case (ii) in Fig.6.4, the chattering does not appear even in the case of $\varepsilon = 0.0045$, which is 1/6 times as small as $\varepsilon = 0.0295$. In addition, we can give the specified tracking precision ε in the case (iii) which is 1/15 times as small as $\varepsilon = 0.0295$ as shown in Fig.6.5. Also note that in the case (iii), the maximum value of the real control error is closer to the specified tracking precision than the case (i) or (ii), which means that the feedback gain is not so larger than necessary, by the effect of the weighting function W.

The above simulation results show the validity of the proposed control method.

6.7 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A hierarchical robust control method of robot manipulators has been proposed. A hierarchical control system enables us to generate a robust compensator much faster than the non-hierarchical case. By assuming that a control signal in the lower level loop is continuous on time, the effect of the uncertainty on the control error is theoretically analyzed. In addition, the part which cannot be linearized due to the computation time lag is theoretically compensated by the robust controller.

(ii) The proposal of some weighting function for a feedback gain enables us to make the feedback gain lower and to decrease chattering phenomena.

(iii) The simulation results have illustrated that the proposed hierarchical controller is effective.

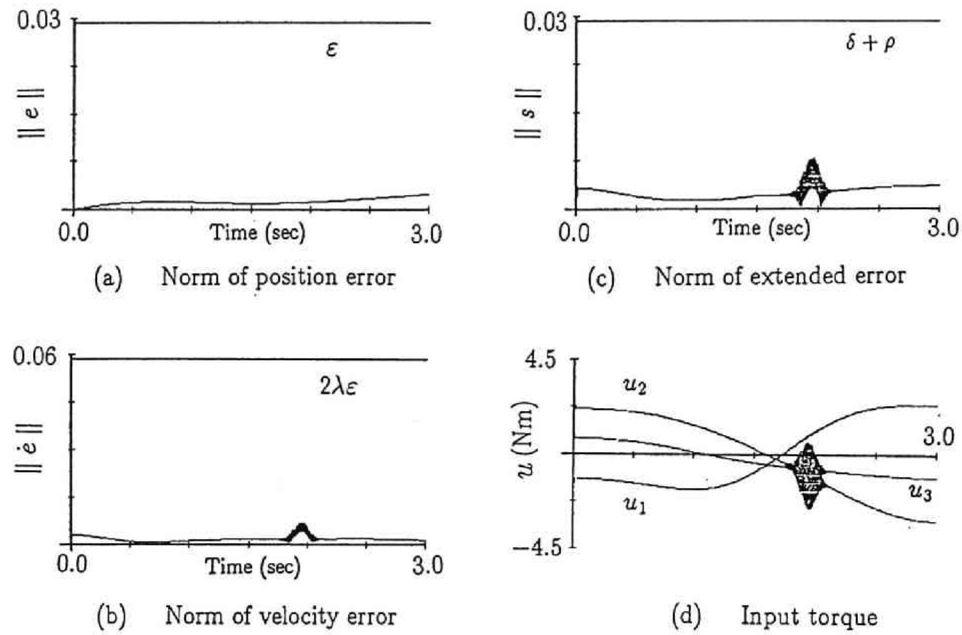

Figure 6.3: Simulation results of case (i): Non-hierarchical robust control with $p = 1$

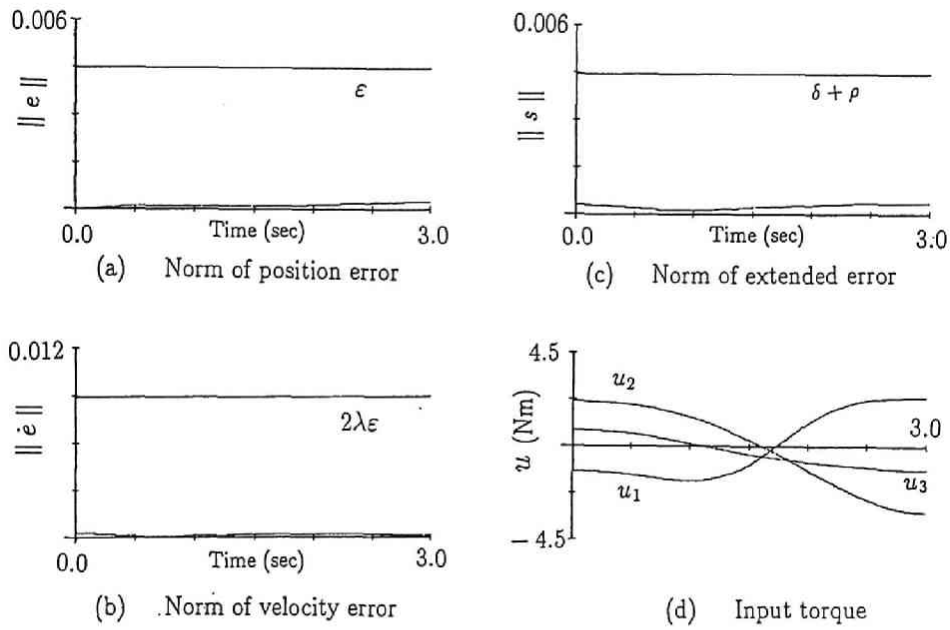

Figure 6.4: Simulation results of case (ii): Hierarchical robust control I with $p = 1$

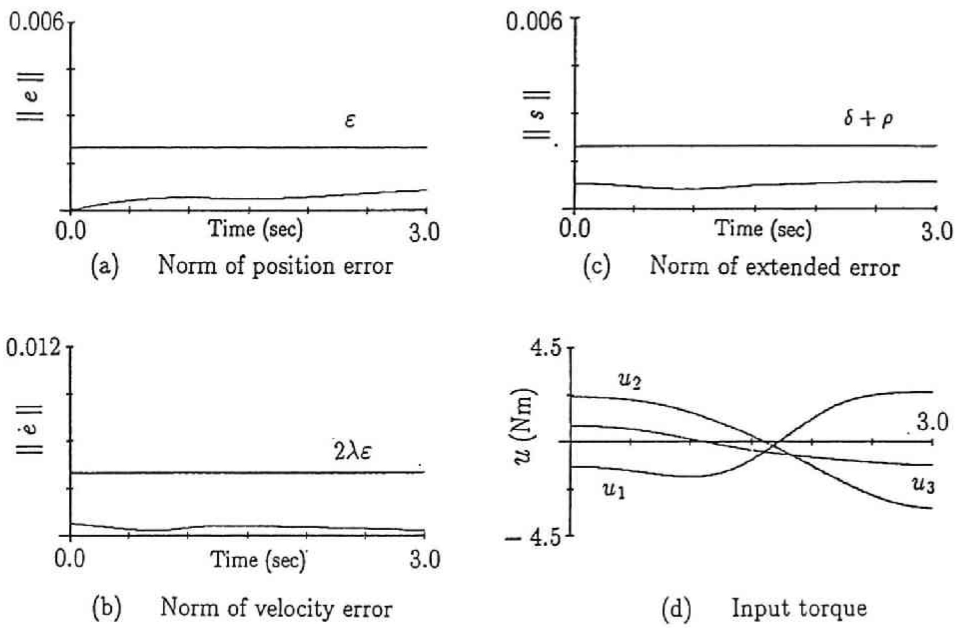

Figure 6.5: Simulation results of case (iii): Hierarchical robust control **II** with $p = 3$

Chapter 7

CHARACTERIZATION OF STRICT BOUNDED REAL CONDITION OF NONLINEAR SYSTEMS AND ITS APPLICATION TO NONLINEAR H_∞ CONTROL

7.1 Introduction

In the linear system control theory, the H_∞ control theory gives powerful tools for robust control theory. In other wards, various robust control problems such as robust stabilization can be solved by using the H_∞ control theory. Thus recently, some nonlinear extensions of the H_∞ control theory have been studied by several researchers, which is called nonlinear H_∞ control theory. However, since the conventional works about the nonlinear H_∞ theory strongly depend on the linearization or the linear H_∞ control techniques, it is not satisfactory in the sense that they do not give the answer to the following fundamental questions. (1) Can we treat the strict $\boldsymbol{H}_\infty$ problem of nonlinear systems in the case of asymptotic stability? (2) When does there exist a stabilizing solution of the Hamilton-Jacobi equation? (3) Do we really need a positive definite solution of the Hamilton-Jacobi-Isaacs equations rather than a positive semi-definite solution? (4) How do the $\boldsymbol{H}_\infty$ control (or $\boldsymbol{L}_2$ gain) results depend on the type of the stability (such as asymptotic stability or exponential stability)? (5) Can we extend the approach based on the Riccati strict inequality [149, 107] to nonlinear setting?

So there is a big gap between the linear $\boldsymbol{H}_\infty$ control theory and its nonlinear version obtained so far, and we can hardly say that the essence of $\boldsymbol{H}_\infty$ control of nonlinear systems was captured. Therefore we need a new different approach, which does not depend on the linearization of nonlinear systems, to capture the essential feature of the strict $\boldsymbol{H}_\infty$ control theory of nonlinear systems.

The main purpose of this chapter is to give answers to the above five questions by obtaining a nonlinear version of the bounded real lemma in a rigorous way. A characterization of the bounded real condition of nonlinear systems, which is a necessary and sufficient condition for nonlinear systems to be internally stable and to have the $\boldsymbol{L}_2$ gain less than a specified number γ is given via two approaches: an approach based on the Hamilton-Jacobi equation with a stabilizing solution and an approach based on the Hamilton-Jacobi strict inequality. In the former approach, a stabilizing solution plays an important role to develop the strict $\boldsymbol{H}_\infty$ control theory, while the latter is a nonlinear extension of a characterization based on the Riccati strict inequality and is useful for analyzing necessary conditions for the solvability of nonlinear H_∞ control problems.

The derived results on the $\boldsymbol{L}_2$ gain have the following properties, compared with the previous results. First, the necessity as well as the sufficiency are rigorously treated, though some natural assumptions are needed. Second, main results are not based on the linearization of the nonlinear system, and it is possible to treat the critical case and so on. Third, main results completely include the bounded real lemma of linear time-invariant systems. Finally, the relation between the internal stability of the system and the stabilizing solution of the Hamilton-Jacobi equation is clarified, which is particular to the nonlinear case.

The characterization by these two approaches will complete the strict bounded real condition of nonlinear systems to form a basis to solve the strict $\boldsymbol{H}_\infty$ control problem.

As an application of the above results, some sufficient (and necessary) conditions are given for the solvability of the strict $\boldsymbol{H}_\infty$ state (or output) feedback control problem. The derived results completely correspond to the case of linear systems, and are stronger than the sufficient condition derived by Isidori [55].

The following notations are used: For a function $\boldsymbol{u}(t) : \boldsymbol{R} \to \boldsymbol{R}^n$ on $[a, b]$, let $\boldsymbol{L}_2(a, b)$ be a set of measurable functions on $[a, b]$ with $\int_a^b \| \boldsymbol{u}(t) \|^2 \,\mathrm{d}t < \infty$, and $\| \boldsymbol{u} \|_2$ be $\left(\int_a^b \| \boldsymbol{u}(t) \|^2 \,\mathrm{d}t\right)^{1/2}$. Let $\boldsymbol{L}_{2e}(a)$ be an extended space of $\boldsymbol{L}_2(a, \infty)$, and

$$\boldsymbol{L}_{2e}(a) \triangleq \{\boldsymbol{u} : \boldsymbol{R} \to \boldsymbol{R}^n | \; \| \boldsymbol{u} \|_{2T} < \infty, \forall T \geq a\}$$

where $\| \boldsymbol{u} \|_{2T} \triangleq \left(\int_a^T \| \boldsymbol{u}(t) \|^2 \,\mathrm{d}t\right)^{1/2}$, and a constant a expresses an infimum value of the domain where a function $\boldsymbol{u}(t)$ in question is defined. For simplicity, $\boldsymbol{L}_{2e}(a)$ is denoted by $\boldsymbol{L}_{2e}$. Moreover, let $\boldsymbol{L}_2$ denote $\boldsymbol{L}_2(t_0, \infty)$ where t_0 is the initial time of the system considered in the next section, and $\boldsymbol{L}_2/\{0\}$ denote $\boldsymbol{L}_2$ with $\| \boldsymbol{x} \|_2 \neq 0$. Define $\boldsymbol{L}_{2e}/\{0\}$ in the same way. Let $\widehat{\boldsymbol{L}}_2$ denote $\boldsymbol{L}_2 \cap \boldsymbol{C}_0$, where $\boldsymbol{C}_0$ is the set of all functions which converge to 0 as the argument tends to ∞. Define $\widehat{\boldsymbol{L}}_{2e}$ in the same way. $\boldsymbol{L}_\infty$ expresses a set of bounded functions, and also $\boldsymbol{L}_\infty^c$ denotes a set of bounded functions with $\sup_t \| \boldsymbol{x}(t) \| \leq c$. $\boldsymbol{B}_r$ denotes a compact set on $\boldsymbol{R}^n$ such that $\boldsymbol{B}_r \triangleq \{\boldsymbol{x} \in \boldsymbol{R}^n | \; \| \boldsymbol{x} \| \leq r\}$, where r is a positive constant. Let $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ be scalar functions. Then we use $f = O(g)$ if $\lim_{\|\boldsymbol{x}\| \to 0} \frac{|f|}{|g|} < \infty$ holds, which means that there exists a positive constant k such that $| f | \leq k | g |$ in a neighborhood of $\boldsymbol{x} = \mathbf{o}$. A real-valued function $\psi(\cdot) : \boldsymbol{R} \to \boldsymbol{R}$ is said to belong to the class $\boldsymbol{K}$ (or $\psi \in \boldsymbol{K}$) if it is continuous and strictly increasing functions with $\psi(0) = 0$.

7.2 L_2 gain and the Hamilton-Jacobi equation

In this section, we state the relation between the $\boldsymbol{L}_2$ gain of nonlinear systems and the Hamilton-Jacobi equation which corresponds to the algebraic Riccati equation arising in the case of linear systems.

Consider the following nonlinear system whose input-output relation is given by an operator $\boldsymbol{S} : \boldsymbol{L}_{2e} \to \boldsymbol{L}_{2e}$.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \tag{7.1}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{7.2}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state, $\boldsymbol{u} \in \boldsymbol{R}^m$ is the input, $\boldsymbol{y} \in \boldsymbol{R}^p$ is the output, and t_0 is the initial time. $\boldsymbol{f}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $\boldsymbol{h}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^p$, and $\boldsymbol{g}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$ are sufficiently smooth known functions with $\boldsymbol{f}(\mathbf{o}) = \mathbf{o}$ and $\boldsymbol{h}(\mathbf{o}) = \mathbf{o}$. It is assumed that (7.1) has a unique solution for any $\boldsymbol{u} \in \boldsymbol{L}_{2e}$.

Define the $\boldsymbol{L}_2$ gain for the system $\boldsymbol{S}$ as follows.

[**Definition 7.1**]

$$\| \boldsymbol{S} \|_{L2} \triangleq \sup_{\boldsymbol{u}\in \boldsymbol{L}_2/\{0\}} \frac{\| \boldsymbol{S}\boldsymbol{u} \|_2}{\| \boldsymbol{u} \|_2} \tag{7.3}$$

subject to $\boldsymbol{x}_0 = \mathbf{o}$. ∎

The following assumptions are made.

[**Assumption 7.1**] *The system $\boldsymbol{S}$ is reachable from the origin ($\boldsymbol{x} = \mathbf{o}$). Namely, given any $\boldsymbol{x}_1$ and t_1, there exists a finite time $t_0 \le t_1$ and a control input $\boldsymbol{u} \in \boldsymbol{L}_2(t_0, t_1)$ such that the state can be driven from $\boldsymbol{x}(t_0) = \mathbf{o}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$.* ∎

[**Assumption 7.2**] *For the system $\boldsymbol{S}$, let $\phi_a(\boldsymbol{x})$ be the function defined by*

$$\phi_a(\boldsymbol{x}) \triangleq - \inf_{u\in \boldsymbol{L}_{2e}, T\ge t} \int_t^T (\gamma^2 \boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y})\, d\tau \tag{7.4}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}$ and γ is a given positive constant. When $\phi_a(\boldsymbol{x})$ exists, it is C^1. In addition, when $\phi_a(\boldsymbol{x})$ exists globally, there exists an optimal control input $\boldsymbol{u}_* \in \boldsymbol{L}_2(t_a, t_b)$ which minimizes the cost function given by

$$J(\boldsymbol{x}(t_a), \boldsymbol{u}(\cdot), t_a) = -\phi_a(\boldsymbol{x}(t_b)) + \int_{t_a}^{t_b} (\gamma^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \quad (7.5)$$

for a sufficiently small time period $[t_a, t_b]$. ∎

Assumption 7.1 implies that, when the system is linear, it is controllable. Assumption 7.2 holds whenever the system is linear time-invariant.

Then one gets the following theorem.

[Theorem 7.1] *Under Assumptions 7.1 and 7.2, let* γ *be a given positive constant. Then the following four statements are equivalent for the system* $\boldsymbol{S}$.

(i)

$$\frac{\| \boldsymbol{y} \|_{2T}}{\| \boldsymbol{u} \|_{2T}} < \gamma, \quad \forall T > t_0, \quad \forall \boldsymbol{u} \in \boldsymbol{L}_{2e}/\{0\}, \quad \boldsymbol{x}_0 = \mathbf{o} \quad (7.6)$$

(ii)

$$\frac{\| \boldsymbol{y} \|_{2}}{\| \boldsymbol{u} \|_{2}} < \gamma, \quad \forall \boldsymbol{u} \in \boldsymbol{L}_{2}/\{0\}, \quad \boldsymbol{x}_0 = \mathbf{o} \quad (7.7)$$

(iii)

$$\| \boldsymbol{S} \|_{L2} \leq \gamma \quad (7.8)$$

(iv) *There exists a positive semi-definite function* $\phi(\cdot) : \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ *such that, for all* $\boldsymbol{x} \in \boldsymbol{R}^n$

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{f} + \frac{1}{4\gamma^2} \frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{g} \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T \boldsymbol{h} = 0 \quad (7.9)$$

∎

Proof: See Appendix.

Using the methods of [84] and [46], van der Schaft [127] has already shown the equivalence between (iii) and (iv). However, the proof of (iii) → (iv) is not clear there, since we believe that Assumption 7.2 is necessary, but it is not explicitly made. In Theorem 7.1, we give a rigorous and alternative proof, which is convenient to prove the local setting of Theorem 7.1. The main feature of the proof is the derivation of (7.A1). In addition, the equivalence between (i), (ii), and (iii) is not clear in [127]. Note that it is not easy to show (i) → (ii) directly, because (7.6) at $T \rightarrow \infty$ means $\frac{\|\boldsymbol{y}\|_2}{\|\boldsymbol{u}\|_2} \leq \gamma$, not $\frac{\|\boldsymbol{y}\|_2}{\|\boldsymbol{u}\|_2} < \gamma$. The equivalence between (ii) and (iii) implies that there exists no input that maximizes the input-output rate of the system, and it is important that the condition (ii) does *not* necessarily mean the condition $\| \boldsymbol{S} \|_{L2} < \gamma$.

Remark 7.1 *We compare the result of Theorem 7.1 to the bounded real lemma of linear systems. For a linear system*

$$\tilde{S}\begin{cases} \dot{x} = Fx + Gu \\ y = Hx \end{cases} \tag{7.10}$$

where F, G, *and* H *are appropriate dimensional constant matrix, a necessary and sufficient condition for* $\| \tilde{S} \|_{L2} \leq \gamma$ *is that there exists a positive semi-definite solution* P *that satisfies the Riccati equation*

$$PF + F^T P + \frac{1}{\gamma^2} PGG^T P + H^T H = 0 \tag{7.11}$$

(See [4]). From this, one can see that the Hamilton-Jacobi equation of (7.9) in Theorem 7.1, when $\phi(x) = x^T P x$, *is equivalent to the Riccati equation given by (7.11).* ∎

Remark 7.2 *Let* $\phi_r(x)$ *be the function defined by*

$$\begin{aligned} \phi_r(x) \triangleq & \inf_{u \in L_{2e}, t_0 \leq t_1} \int_{t_0}^{t_1} (\gamma^2 u^T u - y^T y) d\tau \\ & x(t_0) = 0, \quad x(t_1) = x \end{aligned} \tag{7.12}$$

Assume that $\phi_r(x)$ *satisfies an assumption similar to Assumption 7.2. Then* $\phi_r(x)$ *is also a positive semi-definite solution of the Hamilton-Jacobi equation (7.9) when the condition (iii) of Theorem 7.1 holds.* ∎

In Theorem 7.1, we have discussed I/O relation only (i.e. L_2 gain of the system). However, the internal stability is important from the viewpoint of control system design. Therefore, we give a necessary and sufficient condition for the system to be internally stable with the specified L_2 gain in the following sections, where we consider three cases, namely asymptotic stability, exponential stability, and globally exponential stability as the internal stability.

7.3 Characterization via the Hamilton-Jacobi equation with a stabilizing solution

In this section, we give a nonlinear extension of the strict bounded real lemma of linear systems, based on a stabilizing solution of the

Hamilton-Jacobi equation. Next, as an application of the obtained result, we give some sufficient (and necessity) conditions for the solvability of a strict $\boldsymbol{H}_\infty$ control problem via state feedback.

7.3.1 Strict bounded real lemma including internal asymptotic stability

Here asymptotic stability is considered as internal stability.

[Definition 7.2] *For the system* $\boldsymbol{S}$ *with* $\boldsymbol{u} \equiv \mathbf{o}$, *namely,* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, *we call the system* $\boldsymbol{S}$ *internally asymptotically stable, if the origin* $(\boldsymbol{x} = \mathbf{o})$ *of the system is an asymptotically stable equilibrium.* ∎

Define the following input-output stability when the input belongs to $\boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$.

[Definition 7.3] *[134] The system* $\boldsymbol{S}$ *is said to be small signal* $\boldsymbol{L}_2$ *stable if there exist constants* k *and* c *such that* $\| \boldsymbol{y} \|_2 \leq k \| \boldsymbol{u} \|_2$, *for* $\boldsymbol{x}_0 = \mathbf{o}$ *and all* $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$. *The system* $\boldsymbol{S}$ *is also said to be small signal* $\widehat{\boldsymbol{L}}_2$ *stable if there exist constants* k *and* c *such that* $\| \boldsymbol{y} \|_2 \leq k \| \boldsymbol{u} \|_2$, *for* $\boldsymbol{x}_0 = \mathbf{o}$ *and all* $\boldsymbol{u} \in \widehat{\boldsymbol{L}}_2 \cap \boldsymbol{L}_\infty^c$. ∎

Define the $\boldsymbol{L}_2$ gain of the system $\boldsymbol{S}$ as follows.

[Definition 7.4]

$$\| \boldsymbol{S} \|_{L2c} \triangleq \sup_{\boldsymbol{u} \in \boldsymbol{L}_2/\{0\} \cap \boldsymbol{L}_\infty^c} \frac{\| \boldsymbol{S}\boldsymbol{u} \|_2}{\| \boldsymbol{u} \|_2} \tag{7.13}$$

where c *is an appropriate positive constant and* $\boldsymbol{x}_0 = \mathbf{o}$. *We say the system has the small signal* $\boldsymbol{L}_2$ *gain* $\| \boldsymbol{S} \|_{L2c}$. ∎

Note that the system which is small signal $\boldsymbol{L}_2$ stable has the finite small signal $\boldsymbol{L}_2$ gain.

In the case of linear systems, the internal stability automatically means globally exponential stability. So the state $\boldsymbol{x}(t)$ goes to the origin $(\boldsymbol{x} = \mathbf{o})$ as $t \to \infty$ for any input that belongs to $\boldsymbol{L}_2$. In the nonlinear case, since the internal stability considered in this section implies local asymptotic stability, we pay attention to $\boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$ as a class of input signals of the $\boldsymbol{L}_2$ gain, in order to guarantee that the state of the system $\boldsymbol{S}$ is always in the stability region. In addition, we need the fact that the state converges to the origin as t tends to ∞ for any input that belongs to $\widehat{\boldsymbol{L}}_2$, in order to prove the necessity on the bounded real lemma. Concerning this point, the following lemma is obtained.

[Lemma 7.1] *Assume that the system $\boldsymbol{S}$ is internally asymptotically stable on $\boldsymbol{B}_r$, namely, there exists a Lyapunov function in $\boldsymbol{B}_r$ to guarantee that the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is asymptotically stable at $\boldsymbol{x} = \mathbf{o}$. Then for any positive constant $r_1(< r)$, there exists a positive constant c such that $\boldsymbol{x} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^{r_1}$ holds for all $\boldsymbol{u} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^c$. There also exists a positive constant r_2 such that $\boldsymbol{y} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^{r_2}$ holds for all $\boldsymbol{u} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^c$. Furthermore, given any positive constant $c_1(< c)$, there exists a positive constant $r_3(< r_1)$ such that $\boldsymbol{x} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^{r_3}$ holds for all $\boldsymbol{u} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^{c_1}$.* ■

Proof: This lemma can be readily proven by using Theorem 68.2 in [40], page 344. ■

Now the following assumptions are needed, which are similar to those of section 7.2.

[Assumption 7.3] *The system $\boldsymbol{S}$ is locally reachable with a small input. Namely, given any $c > 0$, there exists an $r(c) > 0$ satisfying the following: for any $\boldsymbol{x}_1 \in \boldsymbol{B}_r$ and t_1, there exist finite time $t_0(\leq t_1)$ and a control input $\boldsymbol{u} \in \boldsymbol{L}_2(t_0, t_1) \cap \boldsymbol{L}_\infty^c$ such that the state is driven from $\boldsymbol{x}(t_0) = \mathbf{o}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$.* ■

[Assumption 7.4] *For the system $\boldsymbol{S}$, let $\widehat{\phi}_a(\boldsymbol{x})$ be the function defined by*

$$\widehat{\phi}_a(\boldsymbol{x}) \triangleq - \inf_{u \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^c, T \geq t} \int_t^T (\gamma^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \tag{7.14}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}$ and γ is a given positive constant. When $\widehat{\phi}_a(\boldsymbol{x})$ exists in a neighborhood of the origin, it is C^1. In addition, when $\widehat{\phi}_a(\boldsymbol{x})$ exists in a neighborhood of the origin, there exists an optimal control input $\boldsymbol{u}_* \in \boldsymbol{L}_2(t_a, t_b) \cap \boldsymbol{L}_\infty^c$ which minimizes the cost function given by

$$J(\boldsymbol{x}(t_a), \boldsymbol{u}(\cdot), t_a) = -\widehat{\phi}_a(\boldsymbol{x}(t_b)) + \int_{t_a}^{t_b} (\gamma^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \tag{7.15}$$

for a sufficiently small time period $t \in [t_a, t_b]$. Furthermore, let $\widehat{\phi}_{r0}(\boldsymbol{x})$ be the function defined by

$$\begin{aligned} \widehat{\phi}_{r0}(\boldsymbol{x}) \;\; &\triangleq \inf_{u \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^c, t_0 \leq t} \int_{t_0}^t (\gamma_0^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \\ & \qquad \boldsymbol{x}(t_0) = \mathbf{o}, \quad \boldsymbol{x}(t) = \boldsymbol{x} \end{aligned} \tag{7.16}$$

where $\gamma_0(<\gamma)$ is a positive constant which is sufficiently close to γ. When $\widehat{\phi}_{r0}(\boldsymbol{x})$ exists in a neighborhood of the origin, it is C^1. In addition, when $\widehat{\phi}_{r0}(\boldsymbol{x})$ exists in a neighborhood of the origin, there exists an optimal control input $\boldsymbol{u}_* \in \boldsymbol{L}_2(t_a, t_b) \cap \boldsymbol{L}_\infty^c$ which minimizes the cost function given by

$$J(\boldsymbol{x}(t_b), \boldsymbol{u}(\cdot), t_b) = \widehat{\phi}_{r0}(\boldsymbol{x}(t_a)) + \int_{t_a}^{t_b} (\gamma_0^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \tag{7.17}$$

for a sufficiently small time period $t \in [t_a, t_b]$. ∎

Define the following systems:

$$\begin{aligned} \boldsymbol{S}_v: \quad \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{v} &= -\frac{1}{2\gamma^2}\boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{u} \end{aligned} \tag{7.18}$$

$$\begin{aligned} \boldsymbol{S}_v^{-1}: \quad \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2\gamma^2}\boldsymbol{g}\boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{v} \\ \boldsymbol{u} &= \frac{1}{2\gamma^2}\boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{v} \end{aligned} \tag{7.19}$$

where $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ is an appropriate real function. Note the system $\boldsymbol{S}_v^{-1}$ is the inverse system of $\boldsymbol{S}_v$. It is assumed that the system $\boldsymbol{S}_v^{-1}$ has a unique solution for any $\boldsymbol{v} \in \boldsymbol{L}_{2e}$.

Then the following theorem is obtained.

[Theorem 7.2] *Under Assumptions 7.3 and 7.4, let γ be a given positive constant. Then the following statements are equivalent for the system $\boldsymbol{S}$.*

(i) The system $\boldsymbol{S}$ is internally asymptotically stable, and there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} < \gamma$.

(ii) There exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin which satisfies the following two conditions.

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \tag{7.20}$$

(B) The system $\boldsymbol{S}_v^{-1}$ is internally asymptotically stable, and small signal $\boldsymbol{L}_2$ stable. ∎

In [130] and [127], van der Schaft has shown that, under the assumption that there exists the first order term of Taylor series expansion at the origin of a nonlinear system, a sufficient condition for the nonlinear system $\boldsymbol{S}$ to be internally exponentially stable with $\| \boldsymbol{S} \|_{L2e} < \gamma$ is that the linearization $\widetilde{S}$ given by (7.10) is internally stable with $\| \widetilde{\boldsymbol{S}} \|_{L2} < \gamma$. It has also been shown that this sufficient condition guarantees the smoothness of $\phi_a(\boldsymbol{x})$. This result is useful for the evaluation of the $\boldsymbol{L}_2$ gain of a nonlinear system with internal stability, because one can evaluate it by checking the existence of the stabilizing solution of the Riccati equation. However, since this result is based on the one about the $\boldsymbol{L}_2$ gain of the linearization, it may not be sufficient for the analysis of the $\boldsymbol{L}_2$ gain of nonlinear systems. For example, the following simple system cannot be treated by his result.

$$\begin{aligned} \dot{x} &= -4x^3 + u \\ y &= \sqrt{3}x^3 \end{aligned} \qquad (7.21)$$

While, we can show that the small signal $\boldsymbol{L}_2$ gain of the system given by (7.21) is less than $\frac{1}{2}$, via simple calculation by using Theorem 7.2 (The function $\phi = \frac{1}{4}x^4$ satisfies the conditions (A) and (B)). Van der Schaft's result cannot also treat the critical case, that is, the system has the internally asymptotic stability at the origin (not exponential stability) and the I/O relation is small signal $\boldsymbol{L}_2$ stable, but Theorem 7.2 can. In addition, the result of Theorem 7.2 is necessary as well as sufficient for the nonlinear system $\boldsymbol{S}$ to be internally stable and have a kind of the $\boldsymbol{L}_2$ gain which is strictly less than a specified number.

Remark 7.3 *Theorem 7.2 exactly corresponds to the strict bounded real lemma of linear systems. In fact, for the linear system $\widetilde{S}$ given by (7.10), a necessary and sufficient condition for the system $\widetilde{S}$ to be internally stable with $\| \widetilde{\boldsymbol{S}} \|_{L2} < \gamma$ is that there exists a stabilizing solution $\boldsymbol{P} \geq 0$ for the Riccati equation (7.11), where $\boldsymbol{P}$ of (7.11) is said to be a stabilizing solution if $\boldsymbol{F} + \frac{1}{\gamma^2}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P}$ is exponentially stable (See [31]). In (B) of Theorem 7.2, the condition that the system $\boldsymbol{S}_v^{-1}$ is internally asymptotically stable corresponds to the requirement for $\boldsymbol{P}$ to be a stabilizing solution.*

It is important that Theorem 7.2 requires the additional condition that the system $\boldsymbol{S}_v^{-1}$ is small signal $\boldsymbol{L}_2$ stable, because the internal stability of $\boldsymbol{S}_v^{-1}$ in (B) is the asymptotic stability which is weaker than that of the linear case. ∎

Several lemmas are needed in order to prove Theorem 7.2. At first, we state a result corresponding to Theorem 7.1.

[Lemma 7.2] *Suppose Assumptions 7.3 and 7.4 for the system* $\boldsymbol{S}$ *and the system is internally asymptotically stable, and let* γ *be a given positive constant. Then the following statements are equivalent.*
(i) There exists a $c > 0$ *such that*

$$\frac{\| \boldsymbol{y} \|_2}{\| \boldsymbol{u} \|_2} < \gamma \quad \forall \boldsymbol{u} \in \boldsymbol{L}_2/\{0\} \cap \boldsymbol{L}_\infty^c \tag{7.22}$$

(ii) There exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *which satisfies (7.20) in a neighborhood of the origin.* ∎

Proof: See Appendix.

While Theorem 7.1 is for the global case, Lemma 7.2 is for the local case. As in Theorem 7.1, the condition (i) in Lemma 7.2 is equivalent to $\| \boldsymbol{S} \|_{L2c} \leq \gamma$, but does not mean $\| \boldsymbol{S} \|_{L2c} < \gamma$. In order to guarantee $\| \boldsymbol{S} \|_{L2c} < \gamma$, one has to pay attention to the existence of some specific solution satisfying the Hamilton-Jacobi equation.

Next the following lemma is given which is useful for the proof on the internal stability of the system $\boldsymbol{S}$ when $\phi(\boldsymbol{x})$ of (7.20) is positive semi-definite.

[Lemma 7.3] *Consider a system*

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{f}}(\boldsymbol{x}) + \tilde{g}(\boldsymbol{x})s(\boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{7.23}$$

where $\tilde{\boldsymbol{f}}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $s(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^m$, *and* $\tilde{g}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$ *are sufficiently smooth, with* $\tilde{\boldsymbol{f}}(\mathbf{o}) = \mathbf{o}$ *and* $s(\mathbf{o}) = \mathbf{o}$. *This system is assumed to satisfy the following conditions.*
(i) There exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *in a neighborhood of the origin such that, for a positive number* ρ,

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}[\tilde{\boldsymbol{f}} + \tilde{g}s] \leq -\rho s^T s \tag{7.24}$$

(ii) The system given by $\dot{\boldsymbol{x}} = \tilde{\boldsymbol{f}}(\boldsymbol{x})$ *has an asymptotic stable equilibrium at the origin.*

Then the system given by (7.23) is asymptotically stable at the origin.
∎

Proof: See Appendix.

Remark 7.4 *As you can easily see from the proof, the asymptotic stability property of (7.23) is guaranteed for all initial state* $\boldsymbol{x}_0$ *where the conditions (i) and (ii) hold simultaneously.* ∎

Isidori [55, 57] has shown, under the condition that $\phi(\boldsymbol{x})$ is the positive definite and asymptotic stabilizing solution of the Hamilton-

Jacobi equation, the asymptotic stability of the system given by (7.23), by using La Salle's Invariance Principle (See [55]). However, when ϕ is positive semi-definite, La Salle's Invariance Principle does not lead to the asymptotic stability. (Note that in the case of linear systems, the existence of the positive semi-definite stabilizing solution of the Riccati equation is enough to guarantee the internal stability of the system.) Thus, Lemma 7.3 is developed as a new tool to guarantee the asymptotic stability. The success in Lemma 7.3 is based on a kind of Lyapunov function obtained by fully exploiting the system structure, namely it is affine in $\boldsymbol{s}$.

Compared to the case of linear systems, Lemma 7.3 corresponds to the result that the internal stability of the linear system $\widetilde{\boldsymbol{S}}$ given by (7.10) is shown by the positive semi-definite solution of the Riccati equation (7.11) and the detectability of some suitable system. In order to clarify this relation, we define the following term.

[Definition 7.5] *For the system $\boldsymbol{S}$ given by (7.1) and (7.2), $(\boldsymbol{f}, \boldsymbol{h})$ is said to be asymptotically detectable, if the system with $\boldsymbol{u} = \mathbf{o}$ and $\boldsymbol{y} \equiv \mathbf{o}$ is asymptotically stable at the origin.* ∎

Note that this definition is different from the definition of zero-state detectability [19]. If $(\boldsymbol{f}, \boldsymbol{h})$ is asymptotically detectable, then it is zero-state detectable. But the converse is not true. In addition, if the system is internally asymptotically stable, then $(\boldsymbol{f}, \boldsymbol{h})$ is asymptotically detectable. The following corollary follows from Lemma 7.3 , which, as you can easily see, completely corresponds to the well known result in the case of linear systems.

[Corollary 7.1] *Suppose that, for the system $\boldsymbol{S}$ given by (7.1) and (7.2), $(\boldsymbol{f}, \boldsymbol{h})$ is asymptotically detectable. If there exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin such that*

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{f} \leq -\boldsymbol{h}^T \boldsymbol{h} \tag{7.25}$$

then the system $\boldsymbol{S}$ is internally asymptotically stable. ∎

Proof: Set $\boldsymbol{u} \equiv \mathbf{o}$. By an appropriate coordinate transformation, the system given by (7.1) and (7.2) can be transformed to the system

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \widehat{\boldsymbol{f}}(\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2) \\ \boldsymbol{y} &= \widehat{\boldsymbol{h}}(\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2) = \widehat{\boldsymbol{x}}_2 \end{aligned} \tag{7.26}$$

where $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{x}}_1^T \ \ \widehat{\boldsymbol{x}}_2^T]^T$, and $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{h}}$ are appropriate functions obtained by coordinate transformation. Then we get from (7.25)

$$\frac{\partial \widehat{\phi}}{\partial \widehat{\boldsymbol{x}}^T}\{\widehat{\boldsymbol{f}}(\widehat{\boldsymbol{x}}_1, \mathbf{o}) - \widehat{\boldsymbol{F}}(\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2)\widehat{\boldsymbol{x}}_2\} \leq -\widehat{\boldsymbol{x}}_2^T \widehat{\boldsymbol{x}}_2 \tag{7.27}$$

where $\widehat{\phi}(\widehat{\boldsymbol{x}})$ is an appropriate positive semi-definite function obtained by coordinate transformation of $\phi(\boldsymbol{x})$, and $\widehat{\boldsymbol{F}}(\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2)$ is an appropriate function. Then from Lemma 7.3, we conclude the internal asymptotic stability of the system given by (7.1) and (7.2). ∎

Third, concerning a stabilizing solution of the partial differential equation given by (7.20), the following result is obtained.

[Lemma 7.4] *Suppose Assumptions 7.3 and 7.4 for the system $\boldsymbol{S}$, and the system is internally asymptotically stable. Assume also that, given $\gamma > 0$, there exists a $c > 0$ such that $\| \boldsymbol{S} \|_{L2c} < \gamma$. Then $\widehat{\phi}_a(\boldsymbol{x})$ given by (7.14) satisfies (7.20) in a neighborhood of the origin, and the system $\dot{\boldsymbol{x}} = \boldsymbol{f} + \frac{1}{2\gamma^2}\boldsymbol{g}\boldsymbol{g}^T \frac{\partial \widehat{\phi}_a}{\partial \boldsymbol{x}}$ is asymptotically stable at the origin.* ∎

Proof: See Appendix.

In the case of the linear system $\widetilde{\boldsymbol{S}}$ given by (7.10), if the system is internally stable and satisfies $\| \widetilde{\boldsymbol{S}} \|_{L2} < \gamma$, then there exist solutions of the Riccati equation and the minimum solution is a stabilizing one (See [31, 4]). Lemma 7.4 is a nonlinear extension of this linear case. In Lemma 7.4, it is important that the existence of the stabilizing solution can be proven by the argument of time domain. There is no discussion on the above point in the former researches.

[Lemma 7.5] *Suppose Assumption 7.3 for the system $\boldsymbol{S}$, and the system is internally asymptotically stable. Then the system $\boldsymbol{S}$ is small signal $\boldsymbol{L}_2$ stable if and only if the system $\boldsymbol{S}$ is small signal $\widehat{\boldsymbol{L}}_2$ stable.* ∎

Proof See Appendix.

Now we are in the position to give the proof of Theorem 7.2 using Lemmas 7.1 to 7.4.

(Proof of Theorem 7.2)

(ii)→(i): (a) Internal stability of the system $\boldsymbol{S}$: In Lemma 7.3, let $\boldsymbol{s}$ be $\boldsymbol{s} = -\frac{1}{2\gamma^2}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}}$, $\widetilde{\boldsymbol{f}}$ be $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \frac{1}{2\gamma^2}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}}$, and $\widetilde{\boldsymbol{g}}$ be $\widetilde{\boldsymbol{g}} = \boldsymbol{g}$. Then $\widetilde{\boldsymbol{f}} + \widetilde{\boldsymbol{g}}\boldsymbol{s} = \boldsymbol{f}$. By the condition (A), one gets

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} \leq -\frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} \tag{7.28}$$

and the condition (i) of Lemma 7.3 holds when $\rho = \gamma^2$. Further the condition (ii) of Lemma 7.3 holds, because the system $\boldsymbol{S}_v^{-1}$ given by (7.19) is internally asymptotically stable. Therefore, the system $\boldsymbol{S}$ is internally asymptotically stable.

(b) $\| \boldsymbol{S} \|_{L2c} < \gamma$: Suppose the condition (A) holds for all $\boldsymbol{x} \in \boldsymbol{B}_{r_1}$. Since the system $\boldsymbol{S}$ is internally asymptotically stable, from Lemma 7.1, there exists a $c_1 > 0$ such that $\boldsymbol{x} \in \boldsymbol{L}_\infty^{r_1}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_\infty^{c_1}$. Thus using the condition (A), one obtains

$$\gamma^2 \| \boldsymbol{u} \|_2^2 - \| \boldsymbol{y} \|_2^2 \geq \gamma^2 \| \boldsymbol{v} \|_2^2 , \qquad \forall \boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_1},\ \boldsymbol{x}_0 = \mathbf{o} \tag{7.29}$$

Further there exists a positive constant c_2 such that $\boldsymbol{v} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_2}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_1}$. Since the system $\boldsymbol{S}_v^{-1}$ is small signal $\boldsymbol{L}_2$ stable, there exist $k > 1$ and $c_3 > 0$ such that

$$\| \boldsymbol{u} \|_2 \leq k \| \boldsymbol{v} \|_2 \quad \forall \boldsymbol{v} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_3} \tag{7.30}$$

Thus from (7.30), the internal stability of $\boldsymbol{S}_v^{-1}$, and Lemma 7.1, there exist $c_4 (\leq c_3)$ and $c_5 > 0$ such that $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_5}$ holds for all $\boldsymbol{v} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_4}$. Therefore one gets, for a sufficiently small positive number $c \leq \min\{c_1, c_5\}$,

$$\| \boldsymbol{u} \|_2 \leq k \| \boldsymbol{v} \|_2 \quad \forall \boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c} \tag{7.31}$$

From (7.29) and (7.31), one obtains

$$\gamma^2 - \frac{\| \boldsymbol{y} \|_2^2}{\| \boldsymbol{u} \|_2^2} \geq \gamma^2 \frac{\| \boldsymbol{v} \|_2^2}{\| \boldsymbol{u} \|_2^2} \geq \frac{\gamma^2}{k^2} \qquad \forall \boldsymbol{u} \in \boldsymbol{L}_2/\{0\} \cap \boldsymbol{L}_\infty^c,\ \boldsymbol{x}_0 = \mathbf{o} \tag{7.32}$$

This implies $\| \boldsymbol{S} \|_{L2c} < \gamma$.

(i)→(ii): **(a) Condition (A)**: From Lemma 7.2, one can see that there exists a $r_1 > 0$ such that $\widehat{\phi}_a(\boldsymbol{x})$ given by (7.14) satisfies (7.20) for all $\boldsymbol{x} \in \boldsymbol{B}_{r_1}$.

Then in the following proof, consider the systems $\boldsymbol{S}_v$ and $\boldsymbol{S}_v^{-1}$ given by (7.18) and (7.19) where $\phi = \widehat{\phi}_a$ is set.

(b) Internal stability of $\boldsymbol{S}_v^{-1}$: It follows from Lemma 7.4.

(c) Small signal $\boldsymbol{L}_2$ stability of $\boldsymbol{S}_v^{-1}$: If $\| \boldsymbol{S} \|_{L2c} < \gamma$, then there exists a positive number $\varepsilon (< \gamma^2)$ such that

$$\frac{\| \boldsymbol{y} \|_{2T}^2}{\| \boldsymbol{u} \|_{2T}^2} \leq \gamma^2 - \varepsilon \quad \forall T \geq t_0, \quad \forall \boldsymbol{u} \in \widehat{\boldsymbol{L}}_{2e}/\{0\} \cap \boldsymbol{L}_\infty^c \tag{7.33}$$

The internal stability of the system $\boldsymbol{S}$ and Lemma 7.1 yield that there exists a positive number $c_1 (\leq c)$ such that $\boldsymbol{x} \in \boldsymbol{C}_0 \cap \boldsymbol{L}_\infty^{r_1}$ holds for all

$u \in \widehat{L}_2 \cap L_\infty^{c_1}$. So using the condition (A), one gets

$$\gamma^2 \| u \|_{2T}^2 - \| y \|_{2T}^2 = \widehat{\phi}_a(x(T)) + \gamma^2 \| v \|_{2T}^2$$
$$\forall T \geq t_0, \quad \forall u \in \widehat{L}_{2e}/\{0\} \cap L_\infty^{c_1}, \ x_0 = o \qquad (7.34)$$

Eqs.(7.33) and (7.34) imply

$$\widehat{\phi}_a(x(T)) + \gamma^2 \| v \|_{2T}^2 \geq \varepsilon \| u \|_{2T}^2$$
$$\forall T \geq t_0, \quad \forall u \in \widehat{L}_{2e}/\{0\} \cap L_\infty^{c_1}, \ x_0 = o \qquad (7.35)$$

Further from the internal stability of S_v, there exists a $c_2 > 0$ such that $v \in \widehat{L}_{2e} \cap L_\infty^{c_2}$ holds for all $u \in \widehat{L}_{2e} \cap L_\infty^{c_1}$. Also from the internal stability of S_v^{-1}, there exist $c_3 > 0$ and $c_4 > 0$ such that $u \in \widehat{L}_{2e} \cap L_\infty^{c_4}$ holds for all $v \in \widehat{L}_{2e} \cap L_\infty^{c_3}$. Therefore, for a sufficiently small positive number $c_5 \leq \min\{c_2, c_3\}$, one gets from (7.35)

$$\widehat{\phi}_a(x(T)) + \gamma^2 \| v \|_{2T}^2 \geq \varepsilon \| u \|_{2T}^2$$
$$\forall v \in \widehat{L}_{2e}/\{0\} \cap L_\infty^{c_5}, \quad x_0 = o \qquad (7.36)$$

Noting that $x(\infty) = o$ holds for all $v \in \widehat{L}_2 \cap L_\infty^{c_5}$, because of the internal stability of S_v^{-1} and Lemma 7.1, and that $u \in \widehat{L}_2 \cap L_\infty^{c_4}$ holds for all $v \in \widehat{L}_2 \cap L_\infty^{c_5}$ because of (7.36), one obtains

$$\gamma^2 \| v \|_2^2 \geq \varepsilon \| u \|_2^2 \quad \forall v \in \widehat{L}_2 \cap L_\infty^{c_5}, \ x_0 = o \qquad (7.37)$$

This implies that S_v^{-1} is small signal $\widehat{L}_2$ stable. The system S_v^{-1} is locally reachable because of Assumption 7.3 and is internally asymptotically stable. Therefore, from Lemma 7.5, S_v^{-1} is small signal L_2 stable. ∎

7.3.2 Strict bounded real lemma including internal exponential stability

In this subsection, a necessary and sufficient condition is given for the system to be internally (globally) exponentially stable with the specified L_2 gain. In addition, a relation to linear approximation system is discussed.

At first, we consider the case of exponential stability. Thus several lemmas are given, which correspond to Lemmas 7.1, 7.3, and 7.4, respectively.

[Definition 7.6] *For the system S with $u \equiv o$, namely, $\dot{x} = f(x)$, we call the system S internally exponentially stable, if $x = o$ of the system is an exponentially stable equilibrium.* ∎

For the internal exponential stability, the following result which is similar to Lemma 7.1 is obtained.

[Lemma 7.6] *Suppose the system $\boldsymbol{S}$ has a unique solution for each $\boldsymbol{u} \in \boldsymbol{L}_\infty$ and is internally exponentially stable on $\boldsymbol{B}_r$, namely, there exists a Lyapunov function on $\boldsymbol{B}_r$ to guarantee that the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is exponentially stable at the origin. Then the system $\boldsymbol{S}$ is small signal $\boldsymbol{L}_2$ stable. Further, for any positive constant $r_1(< r)$, there exists a positive constant c such that $\boldsymbol{x} \in \hat{\boldsymbol{L}}_2 \cap \boldsymbol{L}_\infty^{r_1}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$. There also exists a positive constant r_2 such that $\boldsymbol{y} \in \hat{\boldsymbol{L}}_2 \cap \boldsymbol{L}_\infty^{r_2}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$. Furthermore, given any positive constant $c_1(< c)$, there exists a positive constant $r_3(< r_1)$ such that $\boldsymbol{x} \in \hat{\boldsymbol{L}}_2 \cap \boldsymbol{L}_\infty^{r_3}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^{c_1}$.* ∎

Proof: It is straightforward from [134, 44]. ∎

Lemma 7.6 shows that the internal exponential stability guarantees $\boldsymbol{x}(t) \to \mathbf{o}$ as $t \to \infty$ for all $\boldsymbol{u} \in \boldsymbol{L}_2$, as long as the state remains in the stability region.

We give a result corresponding to Lemma 7.3.

[Lemma 7.7] *Consider the system given by (7.23). Suppose the following two conditions hold.*
(i)' There exists a C^2 positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin such that, for a positive number ρ, eq.(7.24) holds.
(ii)' The system given by $\dot{\boldsymbol{x}} = \tilde{\boldsymbol{f}}(\boldsymbol{x})$ has an exponential stable equilibrium at the origin.

Then the system given by (7.23) is exponentially stable at the origin. ∎

Proof: See Appendix.

Now in Assumption 7.4, C^1 is replaced by C^2 with respect to the smoothness of $\widehat{\phi}_a$ and $\widehat{\phi}_{r0}$. Then we call this Assumption 7.4'. The following result corresponding to Lemma 7.4 is given.

[Lemma 7.8] *Suppose Assumptions 7.3 and 7.4' for the system $\boldsymbol{S}$, and the system is internally exponentially stable. Assume also that, given $\gamma > 0$, there exists a $c > 0$ such that $\| \boldsymbol{S} \|_{L2c} < \gamma$. Then $\widehat{\phi}_a(\boldsymbol{x})$ satisfies (7.20) in a neighborhood of the origin, and the system given by $\dot{\boldsymbol{x}} = \boldsymbol{f} + \frac{1}{2\gamma^2}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial\widehat{\phi}_a}{\partial\boldsymbol{x}}$ is exponentially stable at the origin.* ∎

Proof: The proof is almost the same as Lemma 7.4, except for the use of Lemmas 7.2, 7.6, and 7.7. ∎

Then the following theorem is obtained, which corresponds to Theorem 7.2.

[Theorem 7.3] *Under Assumptions 7.3 and 7.4', let* γ *be a given positive constant. Then the following statements are equivalent for the system* $\boldsymbol{S}$.
(i) The system $\boldsymbol{S}$ *is internally exponentially stable, and there exists a positive constant* c *such that* $\| \boldsymbol{S} \|_{L2c} < \gamma$.
(ii) There exists a C^2 *positive semi-definite function* $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *in a neighborhood of the origin which satisfies the following two conditions (A) and (B).*
(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \tag{7.38}$$

(B) The system $\boldsymbol{S}_v^{-1}$ *is internally exponentially stable.* ∎

Proof: It can be proven in the same way as Theorem 7.2, by utilizing Lemmas 7.2, and 7.6 to 7.8. ∎

Theorem 7.3 is concerned with the small signal $\boldsymbol{L}_2$ gain, since $\boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$ is treated as a class of input signals. By the fact that if the system is internally exponentially stable, then it is small signal $\boldsymbol{L}_2$ stable (see Lemma 7.6), Theorem 7.3 requires only the exponential stabilizing solution, while Theorem 7.2 requires the small signal $\boldsymbol{L}_2$ stability of $\boldsymbol{S}_v^{-1}$ as well as the asymptotic stabilizing solution. Note that the result of Theorem 7.3 is necessary as well as sufficient for the nonlinear system $\boldsymbol{S}$ to be internally exponential stable with $\| \boldsymbol{S} \|_{L2c} < \gamma$.

Next, let us consider the case of global exponential stability. The following assumption is made.

[Assumption 7.5] *Define a function* $\phi_{r0}(\boldsymbol{x})$ *where* γ *in (7.12) is replaced with* $\gamma_0 (< \gamma)$. *When* $\phi_{r0}(\boldsymbol{x})$ *exists, it is* C^1. *In addition when* $\phi_{r0}(\boldsymbol{x})$ *globally exists, there exists an optimal control input* $\boldsymbol{u}_* \in \boldsymbol{L}_2(t_a, t_b)$ *which minimizes the cost function defined by*

$$J(\boldsymbol{x}(t_b), \boldsymbol{u}(\cdot), t_b) = \phi_{r0}(\boldsymbol{x}(t_a)) + \int_{t_a}^{t_b} (\gamma_0^2 \boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y})d\tau \tag{7.39}$$

for a sufficiently small time period $t \in [t_a, t_b]$. ∎

Now a global version of Theorem 7.3 is given.

[Theorem 7.4] *Under Assumptions 7.1, 7.2, and 7.5, let γ be a any given positive constant. Assume that $\boldsymbol{f}$ is globally Lipschitz in $\boldsymbol{x}$, and that $\frac{\partial \phi_a}{\partial \boldsymbol{x}}$ given by (7.4) and $\frac{\partial \phi_{r0}}{\partial \boldsymbol{x}}$, when they exists globally, are globally Lipschitz in $\boldsymbol{x}$. Suppose also $\sup_{\boldsymbol{x}} \| \boldsymbol{g}(\boldsymbol{x}) \| < \infty$. Then the following statements are equivalent for the system $\boldsymbol{S}$.*

(i) The system $\boldsymbol{S}$ is globally and internally exponentially stable, and satisfies $\| \boldsymbol{S} \|_2 < \gamma$.

(ii) There exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ which satisfies the following conditions (A), (B) and (C).

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0, \qquad \forall \boldsymbol{x} \tag{7.40}$$

(B) The system $\boldsymbol{S}_v^{-1}$ is globally and internally exponentially stable.

(C) $\frac{\partial \phi}{\partial \boldsymbol{x}}(\boldsymbol{x})$ is globally Lipschitz in $\boldsymbol{x}$. ∎

Proof: See Appendix.

This result is a global extension of Theorem 7.3. Theorem 7.4 completely includes the bounded real lemma of controllable linear systems, since the linear systems satisfy the assumptions of Theorem 7.4. As a result, the approach derived here enables us to naturally extend the idea of the $\boldsymbol{H}_\infty$ norm of linear systems to the $\boldsymbol{L}_2$ gain of nonlinear systems. Theorems 7.2 to 7.4 also clarify the relation between the internal stability of the system and the stabilizing solution of the Hamilton-Jacobi equation. We believe that Theorems 7.2 to 7.4 give essential results for the $\boldsymbol{L}_2$ gain of nonlinear systems.

Finally, combining Theorem 7.3 with van der Schaft's results [130, 127], we show the relation on $\boldsymbol{L}_2$ gain between the nonlinear system $\boldsymbol{S}$ and the linearization $\widetilde{\boldsymbol{S}}$ given by (7.10) where $\boldsymbol{F} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^T}(\mathbf{o})$, $\boldsymbol{G} = \boldsymbol{g}(\mathbf{o})$, and $\boldsymbol{H} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}^T}(\mathbf{o})$.

Then the following result is obtained.

[Corollary 7.2] *Under Assumptions 7.3 and 7.4', let γ be a given positive constant. Then the following statements are equivalent for the nonlinear system $\boldsymbol{S}$ and the linearization $\widetilde{\boldsymbol{S}}$.*

(i) The system $\widetilde{\boldsymbol{S}}$ is internally stable with $\| \widetilde{\boldsymbol{S}} \|_{L2} < \gamma$.

(ii) The system $\boldsymbol{S}$ is internally exponentially stable, and there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} < \gamma$.

(iii) There exists a C^2 positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$

in a neighborhood of the origin which satisfies the following two conditions (A) and (B).

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \tag{7.41}$$

(B) *The system $\boldsymbol{S}_v^{-1}$ is internally exponentially stable.* ∎

Proof: The proof of (i)→(ii)(or (iii)) is given by van der Schaft [130, 127]. So it is enough to show (ii)→(iii) and (iii) → (i). From Theorem 7.3, the former is obvious. Concerning to the latter, the second order linearization of the partial differential equation leads to the Riccati equation of (7.11), and by the fact that $\widehat{\phi}_a(\boldsymbol{x})$ is the exponentially stabilizing solution, the linearization of $\widehat{\phi}_a(\boldsymbol{x})$ is a stabilizing solution of (7.11). The above argument implies that the condition (i) holds. ∎

Van der Schaft has also shown the result similar to Corollary 7.2 in [129], which has a different proof from that of Corollary 7.2. Corollary 7.2 is utilized to prove a necessary condition for the existence of $\boldsymbol{H}_\infty$ state feedback control in the next section.

Remark 7.5 *The equivalence between (i) and (ii) can be shown under the assumption that $\phi_{r0}(\boldsymbol{x})$ is C^2, in the similar way to Corollary 7.2.* ∎

7.3.3 Strict H_∞ state feedback control problem

In this subsection, based on the above results about the strict bounded real lemma, we give some conditions for the solvability of a strict $\boldsymbol{H}_\infty$ state feedback control problem.

Consider the following nonlinear systems:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})\boldsymbol{w} + \boldsymbol{g}_2(\boldsymbol{x})\boldsymbol{u} \tag{7.42}$$

$$\boldsymbol{z} = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{j}(\boldsymbol{x})\boldsymbol{u} \tag{7.43}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state, $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control input, $\boldsymbol{w} \in \boldsymbol{R}^p$ is the disturbance, $\boldsymbol{z} \in \boldsymbol{R}^q$ is the controlled output, and t_0 is the initial time. $\boldsymbol{f}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^n$, $\boldsymbol{h}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^q$, $\boldsymbol{g}_1(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times p}$, $\boldsymbol{g}_2(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{n\times m}$, and $\boldsymbol{j}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{q\times m}$ are sufficiently smooth known functions with $\boldsymbol{f}(\mathbf{o}) = \mathbf{o}$ and $\boldsymbol{h}(\mathbf{o}) = \mathbf{o}$. It is assumed that (7.42) has a unique solution for any $\boldsymbol{u} \in \boldsymbol{L}_{2e}$ and $\boldsymbol{w} \in \boldsymbol{L}_{2e}$. For simplicity, assume that the following condition holds.

$$[\boldsymbol{h}^T \quad \boldsymbol{j}^T]\boldsymbol{j} = [\mathrm{o} \quad \boldsymbol{I}] \qquad \forall \boldsymbol{x} \in \boldsymbol{R}^n$$

Let $\boldsymbol{S}_{zw}$ define an operator which expresses the relation between $\boldsymbol{w}$ and $\boldsymbol{z}$ in the closed loop system given by (7.42), (7.43), and $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$, where $\boldsymbol{k}(\cdot) : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^m$ is an appropriate function.

Then the following problem is considered.

[Problem 7.1] (Strict $\boldsymbol{H}_\infty$ state feedback control problem) *For the system given by (7.42) and (7.43), find a state feedback control $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ which satisfies the following conditions.*
(S1) $\| \boldsymbol{S}_{zw} \|_{L2c} < \gamma$
(S2) *The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_2(\boldsymbol{x})\boldsymbol{k}(\boldsymbol{x})$ is asymptotically stable (or exponentially stable) at $\boldsymbol{x} = \mathbf{o}$.* ∎

The following system is defined.

$$\begin{aligned} \boldsymbol{S}_{\tilde{z}}^{-1} : \quad \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2\gamma^2}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} - \frac{1}{2}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{g}_1\tilde{\boldsymbol{z}} \\ \boldsymbol{w} &= \frac{1}{2\gamma^2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \tilde{\boldsymbol{z}} \end{aligned} \tag{7.44}$$

where $\phi(\cdot) : \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is an appropriate real function. Then we get the following result.

[Theorem 7.5] *For the system by (7.42) and (7.43), a positive constant γ is given. If there exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ in a neighborhood of the origin which satisfies the following two conditions*
(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} - \frac{1}{4}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \tag{7.45}$$

(B) $\boldsymbol{S}_{\tilde{z}}^{-1}$ is internally asymptotically stable and is small signal $\boldsymbol{L}_2$ stable then the strict $\boldsymbol{H}_\infty$ state feedback control problem is solvable in the case of asymptotic stability.

In addition, when there exists a $\phi(\boldsymbol{x})$ which satisfies conditions (A) and (B), one of nonlinear state feedback controllers which satisfy conditions (i) and (ii) can be given by $\boldsymbol{k}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$. ∎

Proof: Consider a system with $\boldsymbol{u} = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$ in (7.42) and (7.43). Note that the condition (A) implies

$$\begin{aligned} &\tfrac{\partial \phi}{\partial \boldsymbol{x}^T}(\boldsymbol{f} - \tfrac{1}{2}\boldsymbol{g}_2\boldsymbol{g}_2^T\tfrac{\partial \phi}{\partial \boldsymbol{x}}) + \tfrac{1}{4\gamma^2}\tfrac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\tfrac{\partial \phi}{\partial \boldsymbol{x}} \\ &+(\boldsymbol{h} - \tfrac{1}{2}\boldsymbol{j}\boldsymbol{g}_2^T\tfrac{\partial \phi}{\partial \boldsymbol{x}})^T(\boldsymbol{h} - \tfrac{1}{2}\boldsymbol{j}\boldsymbol{g}_2^T\tfrac{\partial \phi}{\partial \boldsymbol{x}}) = 0 \end{aligned} \tag{7.46}$$

Then conditions (i) and (ii) follow from conditions (A) and (B), applying the result on (ii) → (i) of Theorem 7.2 to the system with $\boldsymbol{u} = -\frac{1}{2}\boldsymbol{g}_2^T \frac{\partial \phi}{\partial \boldsymbol{x}}$. ∎

In the same way as Theorem 7.5, one can get the result in the case of exponential stability.

[Theorem 7.6] *For the system by (7.42) and (7.43), a positive constant γ is given. If there exists a C^2 positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ in a neighborhood of the origin which satisfies the following two conditions (A)*

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} - \frac{1}{4}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \quad (7.47)$$

(B) $\boldsymbol{S}_{\tilde{z}}^{-1}$ is internally exponentially stable

then the strict $\boldsymbol{H}_\infty$ state feedback control problem is solvable in the case of exponential stability.

In addition, when there exists a $\phi(\boldsymbol{x})$ with conditions (A) and (B), one of nonlinear state feedback controllers which satisfy conditions (i) and (ii) can be given by $\boldsymbol{k}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$. ∎

Remark 7.6 *Under appropriate assumptions, the result in the global case is also obtained in the same way as Theorem 7.6.* ∎

The above results, although they are the sufficient conditions, completely correspond to those in linear systems: the derived conditions require a positive semi-definite solution of the Hamilton-Jacobi-Isaacs equation given by (7.45) (or (7.47)) as the linear case, and the existence of the stabilizing solutions leads to $\| \boldsymbol{S}_{zw} \|_{L2c} < \gamma$ ($\| \boldsymbol{S}_{zw} \|_{L2} < \gamma$) as well as $\| \boldsymbol{S}_{zw} \|_{L2c} \leq \gamma$ ($\| \boldsymbol{S}_{zw} \|_{L2} \leq \gamma$). Note that the results by Isidori [55] do not clarify that the existence of the stabilizing solution implies the $\boldsymbol{L}_2$ gain strictly less than the specified number. In addition, the sufficient conditions derived here are stronger than that of Isidori in the sense that the condition by Isidori requires a positive definite (not positive semi-definite) solution in order to guarantee the asymptotic stability of the closed loop system.

Next, the sufficient condition of Theorem 7.6 is proven to be a necessary condition for the existence of the strict $\boldsymbol{H}_\infty$ state feedback control under a certain assumption.

Assume $\boldsymbol{F} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^T}(\mathbf{o})$, $\boldsymbol{G}_i = \boldsymbol{g}_i(\mathbf{o})$ $(i = 1, 2)$, $\boldsymbol{H} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}^T}(\mathbf{o})$, and $\boldsymbol{J} = \boldsymbol{j}(\mathbf{o})$, and consider the linearization of the system given by (7.42) and (7.43):

$$\dot{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{G}_1\boldsymbol{w} + \boldsymbol{G}_2\boldsymbol{u} \tag{7.48}$$

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{J}\boldsymbol{u} \tag{7.49}$$

Further let $\widetilde{\boldsymbol{S}}_{zw}$ be the operator which expresses the relation between $\boldsymbol{w}$ and $\boldsymbol{z}$ in the closed loop system given by (7.48), (7.43a), and $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$, where $\boldsymbol{K}$ is an appropriate matrix.

Then the following result is obtained.

[Corollary 7.3] *Let γ be a given positive constant. Assume that, for any C^1 function $\boldsymbol{k}(\boldsymbol{x})$ satisfying (S1) and (S2), the system $\boldsymbol{S}_{zw}$ satisfies the assumptions in Corollary 7.2, and also $(\boldsymbol{H}, \boldsymbol{F})$ is detectable. Then the following statements are equivalent for the nonlinear system $\boldsymbol{S}_{zw}$ and the linearization $\widetilde{\boldsymbol{S}}_{zw}$.*

(i) There exists a linear state feedback controller $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$ such that the system $\widetilde{\boldsymbol{S}}_{zw}$ is internally stable with $\| \widetilde{\boldsymbol{S}}_{zw} \|_{L2} < \gamma$.

(ii) There exists a nonlinear state feedback controller $\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x})$ which solves the strict $\boldsymbol{H}_\infty$ control problem in the case of exponential stability.

(iii) There exists a C^2 positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin which satisfies conditions (A) and (B) in Theorem 7.6. ∎

Proof: The proof of (i)→(iii) has already been proven by van der Schaft [130, 127]. So we prove (iii) → (ii) and (ii) → (i). The former is shown by Theorem 7.6. Concerning to the latter, assume that $\boldsymbol{u} = \widehat{\boldsymbol{k}}(\boldsymbol{x})$ satisfies condition (ii), and that $\widehat{\boldsymbol{K}}$ is the linearization of $\widehat{\boldsymbol{k}}(\boldsymbol{x})$. Then Corollary 7.2 is applied to the closed loop system $\boldsymbol{S}_{zw}$ with $\boldsymbol{u} = \widehat{\boldsymbol{k}}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{S}}_{zw}$ with $\boldsymbol{u} = \widehat{\boldsymbol{K}}\boldsymbol{x}$. Thus (i) follows from (ii). ∎

7.4 Characterization via the Hamilton-Jacobi strict inequality

In this section, we characterize the strict bounded real condition of nonlinear systems via the Hamilton-Jacobi strict inequality. In addition, based on the obtained bounded real lemma, we give a necessary and sufficient condition for the solvability of a strict $\boldsymbol{H}_\infty$ control problem via state feedback.

7.4.1 Strict bounded real lemma including internal asymptotic stability

In this subsection, we discuss the relation between the $\boldsymbol{L}_2$ gain of nonlinear systems and the Hamilton-Jacobi strict inequality.

The following assumption is made.

[Assumption 7.6] *Let γ_0 be a given positive constant. Then for the system $\boldsymbol{S}$, a function $\widehat{\phi}_{a0}(\boldsymbol{x})$ defined by*

$$\widehat{\phi}_{a0}(\boldsymbol{x}) \triangleq - \inf_{u \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_{\infty}^{c}, T \geq t} \int_t^T (\gamma_0^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}) d\tau \tag{7.50}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}$ and γ_0 is a given positive constant, is C^1, when it exists in a neighborhood of the origin. ∎

Then the following theorem is obtained.

[Theorem 7.7] *For the system $\boldsymbol{S}$ given by (7.1) and (7.2) which is locally reachable, let γ be a given positive constant, and assume Assumption 7.6 for a positive constant $\gamma_0 < \gamma$ which is sufficiently close to γ. Then the following statements are equivalent.*

(i) The system $\boldsymbol{S}$ is internally asymptotically stable, and there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} < \gamma$.

(ii) There exist positive definite functions $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\psi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin which satisfy the following two conditions.

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{f} + \frac{1}{4\gamma^2} \frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{g}\boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T \boldsymbol{h} + \psi(\boldsymbol{x}) \leq 0 \tag{7.51}$$

(B) $\| \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} \|^2 = O(\psi)$ ∎

The following lemmas are needed in order to prove Theorem 2.1. First we need local version of the Bounded real lemma of nonlinear systems [46].

[Lemma 7.9] *For the system $\boldsymbol{S}$ which is locally reachable, let γ be a given positive constant, and assume A1 for a positive constant $\gamma_0 = \gamma$. Then the following statements are equivalent.*

(i) For the system $\boldsymbol{S}$, there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} \leq \gamma$.

(ii) There exists a positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin satisfying

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} \le 0$$

▮

The following lemma is concerned with the relation between the internal asymptotic stability and the $\boldsymbol{L}_2$ stability, which is much useful for the proof of Theorem 7.7.

[Lemma 7.10] *The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is assumed to be asymptotically stable at $\boldsymbol{x} = \mathbf{o}$, and $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are sufficiently smooth in $\boldsymbol{x}$. Then there exist sufficiently smooth and positive definite functions $W(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\delta(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin such that*

$$\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4k}\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial W}{\partial \boldsymbol{x}} + \delta \le 0 \tag{7.52}$$

holds for some $k > 0$. ▮

Proof: See Appendix.

Remark 7.7 *This lemma implies that if the system is asymptotically stable at the origin, then there exists some output function such that the system is small signal $\boldsymbol{L}_2$ stable. Namely, let $\tilde{y} = \sqrt{\delta(\boldsymbol{x})}$ in Lemma 7.10. Then it follows from (7.52) that the system*

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \\ \tilde{y} &= \sqrt{\delta(\boldsymbol{x})} \end{aligned}$$

is small signal $\boldsymbol{L}_2$ stable. ▮

Now we are in the position to give the proof of Theorem 7.7 using Lemmas 7.9 amd 7.10.

(Proof of Theorem 7.7) The condition (i) implies that there exist positive constants $\gamma_i (i = 0, 1)$ such that

$$\frac{\| \boldsymbol{y} \|_2^2}{\| \boldsymbol{u} \|_2^2} \le \gamma_1^2 < \gamma_0^2 < \gamma^2$$

It also follows from the internal stability of the system $\boldsymbol{S}$ and Lemma 7.10 that there exist sufficiently smooth and positive definite functions $W(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$, $\delta(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$, and positive constant k in a neighborhood of the origin such that

$$\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4k}\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial W}{\partial \boldsymbol{x}} + \delta \le 0 \tag{7.53}$$

In addition, under Assumption 7.6 and Lemma 7.9, since $\| S \|_{L2c} \le \gamma_1$, there exists a positive semi-definite function $\widehat{\phi}_{a0}$ in (7.50) such that

$$\frac{\partial \widehat{\phi}_{a0}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma_1^2}\frac{\partial \widehat{\phi}_{a0}}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \widehat{\phi}_{a0}}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} \leq 0 \tag{7.54}$$

holds in a neighborhood of the origin.

Consider $W'(\boldsymbol{x}) = lW(\boldsymbol{x})$, using a positive number l. Then, from (7.53), we get

$$\frac{\partial W'}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4lk}\frac{\partial W'}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial W'}{\partial \boldsymbol{x}} + l\delta \leq 0 \tag{7.55}$$

Let $\phi_* = W' + \widehat{\phi}_{a0}$. Then using (7.54) and (7.53), we can show that the following relation holds in a neighborhood of the origin, for a sufficiently small number l.

$$\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma_0^2}\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} + l\delta \leq 0 \tag{7.56}$$

Let $\psi_*(\boldsymbol{x})$ be

$$\psi_* = \frac{1}{4}\left(\frac{1}{\gamma_0^2} - \frac{1}{\gamma^2}\right)\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} + l\delta$$

Then one gets

$$\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} + \psi_* \leq 0$$

from (7.54). It is obvious that $\psi_*(\boldsymbol{x})$ is positive definite and satisfies the condition (B). Therefore, it was shown that there exist positive definite functions ϕ_* and ψ_* which satisfy the conditions (A) and (B).

(ii)$\rightarrow$(i): Noting that ϕ and ψ are positive definite functions, it follows from the condition (A) that the system $\boldsymbol{S}$ is internally asymptotically stable. The condition (B) implies that there exists a positive constant ε such that $\varepsilon \parallel \frac{1}{2}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} \parallel^2 \leq \psi$ holds in a neighborhood of $\boldsymbol{x} = \mathbf{o}$. From this fact and the condition (A), there exists a positive constant $\gamma_1(< \gamma)$ such that

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma_1^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} \leq 0$$

Using Lemma 7.9, this implies that there exists a positive constant c such that $\parallel \boldsymbol{S} \parallel_{L2c} \leq \gamma_1 < \gamma$. ∎

Remark 7.8 *Let us compare Theorem 7.7 with the non-strict bounded real condition of nonlinear systems i.e., Lemma 7.9 given by Hill and Moylan [46]. Lemma 7.9 takes no account of the internal stability of the system as well as the* $\boldsymbol{L}_2$ *gain, while Theorem 7.7 does. As a result, Theorem 7.7 requires the positive definiteness of* ϕ*, not the positive semi-definiteness, the existence of a positive definite function* ψ*, and the* *condition (B) which expresses the relation on the boundedness between the term* $\boldsymbol{g}\frac{\partial \phi}{\partial \boldsymbol{x}}$ *and* ψ. *Since most of previous works on the nonlinear* $\boldsymbol{H}_\infty$ *control theory are based on Lemma 7.9, they are concerned with the non-strict case, and do not discuss the necessary condition on the internal stability in both cases of state feedback and output feedback, although the strict* $\boldsymbol{H}_\infty$ *control problem can be solved in the case of linear systems. On the other hand, since Theorem 7.7 treats the internal stability, Theorem 2.1 will be more useful than Lemma 7.9 in developing the strict* $\boldsymbol{H}_\infty$ *control theory. We will apply Theorem 7.7 to the strict* $\boldsymbol{H}_\infty$ *control problem via state feedback or output feedback in the next sections. As you can easily see in the proof of Theorem 7.7, the positive definiteness of* ϕ *and* ψ *guarantees the internal stability of the system* $\boldsymbol{S}$, *and the condition (B) guarantees that* $\| \boldsymbol{S} \|_{L2c}$ *is strictly less than* γ. ∎

Remark 7.9 *We compare the result of Theorem 7.7 to the strict bounded real lemma of linear systems [149]. Consider the controllable linear system* $\tilde{\boldsymbol{S}}$ *given by (7.10). Then the strict bounded real condition for the linear system, i.e., a necessary and sufficient condition for the system* $\boldsymbol{S}$ *to be internal stable and has* $\| \boldsymbol{S} \|_{L2} < \gamma$ *is that there exist a positive definite solution* $\boldsymbol{P}$ *and positive number* ε *which satisfies*

$$\boldsymbol{P}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{P} + \frac{1}{\gamma^2}\boldsymbol{P}\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{P} + \boldsymbol{H}^T\boldsymbol{H} + \varepsilon\boldsymbol{I} = \mathbf{o} \tag{7.57}$$

Now set $\boldsymbol{f} = \boldsymbol{F}\boldsymbol{x}$, $\boldsymbol{g} = \boldsymbol{G}$, *and* $\boldsymbol{h} = \boldsymbol{H}\boldsymbol{x}$ *in (7.1) and (7.2). Then if we consider* $\phi(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ *and* $\psi(\boldsymbol{x}) = \varepsilon\boldsymbol{x}^T\boldsymbol{x}$ *in the condition (A), respectively, the positive definiteness of* ϕ *and* ψ *is satisfied. Here note that Theorem 7.7 holds even when the inequality in the condition (A) is replaced by the equality. In addition, one can see that* $\psi = \varepsilon\boldsymbol{x}^T\boldsymbol{x}$ *automatically satisfies the condition (B) in Theorem 7.7. Therefore, Theorem 7.7 consistently corresponds to the case of linear systems. Comparing to the linear case, however, the main feature of the nonlinear case is the explicit requirement of the condition (B).* ∎

Remark 7.10 *Note that Assumption 7.6 is required to prove the necessity in Theorem 7.7, rather than the sufficiency.* ∎

Next we will show another characterization of the bounded real condition via Hamilton-Jacobi strict inequality. Namely, it will be proven that the condition (B) in Theorem 7.7 can be replaced by the condition between $\boldsymbol{h}$ and ψ.

[Theorem 7.8] *Suppose the same assumption as Theorem 7.7 for the system $\boldsymbol{S}$. Then the following statements are equivalent.*
(i) The system $\boldsymbol{S}$ is internally asymptotically stable, and there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} < \gamma$.
(ii) There exist positive definite functions $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\psi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin that satisfy the following two conditions.
(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} + \psi(\boldsymbol{x}) \le 0$$

(B) The following condition holds.
(B1) $\| \frac{1}{2}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} \|^2 = O(\psi)$
or
(B2) $\| \boldsymbol{h} \|^2 = O(\psi)$
or
(B3) $\| \frac{1}{2}\boldsymbol{g}^T\frac{\partial \phi}{\partial \boldsymbol{x}} \|^2 = O(\psi)$ *and* $\| \boldsymbol{h} \|^2 = O(\psi)$ ∎

Proof: First, we show (i) $\to$ (ii)(A) and (B3). From $\| \boldsymbol{S} \|_{L2c} < \gamma$, there exist γ_1 and γ_2 such that $\gamma_1 < \gamma_2 < \gamma$ and $\| \boldsymbol{S} \|_{L2c} \le \gamma_1$. Let $\boldsymbol{y}_1 \triangleq \frac{\gamma_2}{\gamma_1}\boldsymbol{h}$. Then

$$\| \boldsymbol{y}_1 \|_2 \le \gamma_2 \| \boldsymbol{u} \|_2$$

which means that there exists a positive semi-definite function $\tilde{\phi}$ such that

$$\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma_2^2}\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}} + \frac{\gamma_2^2}{\gamma_1^2}\boldsymbol{h}^T\boldsymbol{h} \le 0 \tag{7.58}$$

Hence noting $\frac{\gamma_2}{\gamma_1} > 1$, we can show that there exists a ψ satisfying $\| \boldsymbol{h} \|^2 = O(\psi)$ as well as the conditions (A) and (B1) in the same way as the proof of Theorem 7.7.
Concerning (ii)(A) and (B2) $\to$ (i), the proof is straightforward using the the above technique and the proof of Theorem 7.7. ∎

Remark 7.11 *In Theorem 7.8, the conditions (A) and (B2) is more useful than the other cases in the sense that we can easily specify the form of ψ because the condition (B2) does not include the unknown function ϕ. On the other hand, the condition (B1) will plays a important role to derive a necessary and sufficient condition for the solvability of the strict $\boldsymbol{H}_\infty$ control problem via state feedback. The condition (B3) will also be used in deriving a necessary condition of the solvability of* *the nonlinear H_∞ problem via output feedback.* ∎

7.4.2 Strict bounded real lemma including internal exponential stability

Next we show a result about the exponential stability case. So in Assumption 7.6, C^1 is replaced by C^2, which is called Assumption 7.6'.

[Theorem 7.9] *For the system $\boldsymbol{S}$ which is locally reachable, let γ be a given positive constant, and assume Assumption 7.6' for a positive constant $\gamma_0 = \gamma$. Then the following statements are equivalent.*
(i) The system $\boldsymbol{S}$ is internally exponentially stable, and there exists a positive constant c such that $\| \boldsymbol{S} \|_{L2c} < \gamma$.
(ii) There exists a C^2 positive definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and a positive constant ε in a neighborhood of the origin which satisfies the following condition.

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{f} + \frac{1}{4\gamma^2} \frac{\partial \phi}{\partial \boldsymbol{x}^T} \boldsymbol{g} \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T \boldsymbol{h} + \varepsilon \boldsymbol{x}^T \boldsymbol{x} \leq 0 \tag{7.59}$$

∎

Proof: (i)→(ii). It can be shown in the same way of the proof of Theorem 7.7. (ii) → (i). For a C^2 positive definite function ϕ satisfying (7.59), there exists an appropriate positive definite matrix $\boldsymbol{P}$ such that $\phi(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \phi_1(\boldsymbol{x})$, where $\phi_1(\boldsymbol{x})$ is a function vanishing at the origin together with all the second order derivatives. Then since $\phi(\boldsymbol{x})$ is a Lyapunov function, the system $\boldsymbol{S}$ is internally exponentially stable. In addition, there exists a positive constant ε_0 such that

$$\varepsilon_0 \| \frac{1}{2} \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} \|^2 \leq \varepsilon \| \boldsymbol{x} \|^2 \tag{7.60}$$

holds in a neighborhood of the origin. It follows from (7.59) and (7.60) that $\| \boldsymbol{S} \|_{L2c} < \gamma$ holds, in the same way as the proof of Theorem 7.13. ∎

Remark 7.12 *In the case of exponential stability, a positive definite function ψ to satisfy the condition (B) can be chosen as $\psi = \varepsilon \boldsymbol{x}^T \boldsymbol{x}$, in the same way as the linear case.* ∎

Remark 7.13 *We can treat a global exponential stability case in the same way as Theorems 7.7 and 7.9, under some assumptions such as a global Lipschitz condition.* ∎

7.4.3 Strict H_∞ state feedback control problem

In this subsection, we give a necessary and sufficient condition for the solvability of the strict $\boldsymbol{H}_\infty$ state feedback control problem, based on the results obtained in the above subsections.

Consider the same problem as in section 7.3. We say a state feedback controller which satisfies the conditions (S1) and (S2) in the strict $\boldsymbol{H}_\infty$ state feedback control problem an admissible controller. Then the following assumption is made.

[Assumption 7.7] *For all admissible controller, the system given by (7.42) is locally reachable by $\boldsymbol{w}$, and the assumption such as Assumption 7.6 holds for the system $\boldsymbol{S}_{zw}$ for all admissible controllers.* ∎

Then we obtain the following result.

[Theorem 7.10] *Let γ be a given positive constant, and assume Assumption 7.7 for a positive constant $\gamma_0 < \gamma$ which is sufficiently close to γ. Then for the system given by (7.42) and (7.43), the strict $\boldsymbol{H}_\infty$ state feedback control problem is solvable if and only if there exist positive definite functions $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\psi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin which satisfy the following two conditions.*
(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f}+\frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}}-\frac{1}{4}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}+\boldsymbol{h}^T\boldsymbol{h}+\psi \le 0 \quad (7.61)$$

(B) $\| \boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} \|^2 = O(\psi)$

In addition, when there exist ϕ and ψ which satisfy the conditions (A) and (B), one of admissible controllers is given by $\boldsymbol{k}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$. ∎

Proof: "If": Set $\boldsymbol{k}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$. Then it is shown that the conditions (S1) and (S2) hold, by using Theorem 7.7. "Only if": Suppose $\boldsymbol{u} = \boldsymbol{u}_*$ satisfies the condition (S1) and (S2). Then under Assumption 7.7, by Theorem 7.7, there exist positive definite functions ϕ_* and ψ_* satisfying

$$\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}(\boldsymbol{f}+\boldsymbol{g}_2\boldsymbol{u}_*)+\frac{1}{4\gamma^2}\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi_*}{\partial \boldsymbol{x}}+\boldsymbol{h}^T\boldsymbol{h}+\boldsymbol{u}_*^T\boldsymbol{u}_*+\psi_* \le 0 \quad (7.62)$$

and $\| \boldsymbol{g}_1^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} \|^2 = O(\psi_*^2)$. It follows from (7.62) that

$$\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} - \frac{1}{4}\frac{\partial \phi_*}{\partial \boldsymbol{x}^T}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi_*}{\partial \boldsymbol{x}} + h^T h + \psi_* \le 0$$

Therefore, ϕ_* and ψ_* satisfy the conditions (A) and (B). ∎

Theorem 7.10 gives a necessary and sufficient condition in the case of the asymptotic stability, although Assumption 7.7 is required. This success is based on the characterization of the strict bounded real condition by the Hamilton-Jacobi strict inequality.

Remark 7.14 *Assumption 7.7 is needed to prove the necessity in Theorem 7.10 rather than the sufficiency. If $\boldsymbol{g}_1 = \boldsymbol{g}_2$ and $(\boldsymbol{f}, \boldsymbol{g}_1)$ is reachable, then the assumption of reachability in Assumption 7.7 is satisfied.* ∎

Theorem 7.10 can be extended to the case of the exponential stability and even the global exponential stability under appropriate assumptions. We show the exponential stability case only as follows. In Assumption 7.7, Assumption 7.6 is replaced by Assumption 7.6', which is called Assumption 7.7'.

[Theorem 7.11] *Let γ be a given positive constant, and assume Assumption 7.7' for a positive constant $\gamma_0 < \gamma$ which is sufficiently close to γ. Then for the system given by (7.42) and (7.43), the strict $\boldsymbol{H}_\infty$ state feedback control problem in the exponential stability case is solvable if and only if there exists a C^2 positive definite function $\phi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and a positive number ε in a neighborhood of the origin which satisfy*

$$\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}} - \frac{1}{4}\frac{\partial \phi}{\partial \boldsymbol{x}^T}\boldsymbol{g}_2\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} + \varepsilon\boldsymbol{x}^T\boldsymbol{x} \leq 0 \quad (7.63)$$

In addition, when there exist ϕ and ψ which satisfy the above condition, one of admissible controllers is given by $\boldsymbol{k}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{g}_2^T\frac{\partial \phi}{\partial \boldsymbol{x}}$. ∎

7.4.4 Strict H_∞ output feedback control problem

Consider

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})\boldsymbol{w} + \boldsymbol{g}_2(\boldsymbol{x})\boldsymbol{u} && (7.64)\\
\boldsymbol{z} &= \boldsymbol{h}_1(\boldsymbol{x}) \qquad\qquad + \boldsymbol{j}_{12}(\boldsymbol{x})\boldsymbol{u} && (7.65)\\
\boldsymbol{y} &= \boldsymbol{h}_2(\boldsymbol{x}) + \boldsymbol{j}_{21}(\boldsymbol{x})\boldsymbol{w} && (7.66)
\end{aligned}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{w} \in \boldsymbol{R}^{m_1}$, $\boldsymbol{u} \in \boldsymbol{R}^{m_2}$, $\boldsymbol{z} \in \boldsymbol{R}^{p_1}$, $\boldsymbol{y} \in \boldsymbol{R}^{p_2}$. $\boldsymbol{f}$, $\boldsymbol{h}_i$, $\boldsymbol{g}_i$ $(i = 1, 2)$, $\boldsymbol{j}_{12}$, and $\boldsymbol{j}_{21}$ are sufficiently smooth functions, and $\boldsymbol{f}(\mathbf{o}) = \mathbf{o}$, $\boldsymbol{h}_i(\mathbf{o}) = \mathbf{o}$. It is assumed that (7.64) has a unique solution for any $\boldsymbol{u} \in \boldsymbol{L}_{2e}$ and $\boldsymbol{w} \in \boldsymbol{L}_{2e}$. Assume $[\boldsymbol{h}_1^T \ \ \boldsymbol{j}_{12}^T]\boldsymbol{j}_{12} = [\mathbf{o} \ \ \boldsymbol{I}]$ and $\boldsymbol{j}_{21}[\boldsymbol{g}_1^T \ \ \boldsymbol{j}_{21}^T] = [\mathbf{o} \ \ \boldsymbol{I}]$ for all $\boldsymbol{x} \in \boldsymbol{R}^n$.

Now we consider as a output feedback controller

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \boldsymbol{f}_c(\boldsymbol{\xi}) + \boldsymbol{g}_c(\boldsymbol{\xi})\boldsymbol{y} \\ \boldsymbol{u} &= \boldsymbol{k}(\boldsymbol{\xi}) \end{aligned} \tag{7.67}$$

where $\boldsymbol{\xi} \in \boldsymbol{R}^{\nu}$, and functions $\boldsymbol{k}$, $\boldsymbol{f}_c$, and $\boldsymbol{g}_c$ are sufficiently smooth and satisfy $\boldsymbol{k}(\mathbf{o}) = \mathbf{o}$ and $\boldsymbol{f}_c(\mathbf{o}) = \mathbf{o}$. Let $\boldsymbol{S}_{zw}$ define the operator which expresses the relation between $\boldsymbol{w}$ and $\boldsymbol{z}$ in the closed loop system given by (7.64) to (7.67). Then the following problem is considered.

[Problem 7.2] (Strict $\boldsymbol{H}_\infty$ output feedback control problem) *For the system given by (7.64) to (7.66), find an output feedback control given by (7.67) which satisfies the following conditions.*
(S1) $\| \boldsymbol{S}_{zw} \|_{L2c} < \gamma$
(S2) *The system $\boldsymbol{S}_{zw}$ is internally asymptotically stable.* ∎

We say an output feedback controller which satisfies the conditions (S1) and (S2) an admissible controller. Then the following assumption is made.

[Assumption 7.8] *For all admissible controller, the system given by (7.64) to (7.66) is locally reachable by $\boldsymbol{w}$, and the assumption such as Assumption 7.6 holds for the system $\boldsymbol{S}_{zw}$. In addition concerning a function $\widehat{\phi}_{a0}(\boldsymbol{x}, \boldsymbol{\xi})$ in Assumption 7.6 for the closed loop system $\boldsymbol{S}_{zw}$, there exists a function $\rho(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^{\nu}$ satisfying*

$$\frac{\partial \widehat{\phi}_{a0}}{\partial \boldsymbol{\xi}}(\boldsymbol{x}, \rho(\boldsymbol{x})) = 0, \qquad \rho(\mathbf{o}) = \mathbf{o}$$

in a neighborhood of the origin [129]. ∎

Then the following result is concerned with a necessity condition of the strict $\boldsymbol{H}_\infty$ control problem via output feedback.

[Theorem 7.12] *Let γ be a given positive constant, and suppose Assumption 7.8 for a positive constant $\gamma_0 < \gamma$ which is sufficiently close to γ. Then for the system given by (7.64) to (7.66), the strict $\boldsymbol{H}_\infty$ control problem via output feedback is solvable only if the following conditions hold.*
(A) There exist positive definite functions $\widetilde{\phi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\widetilde{\psi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ in a neighborhood of the origin that satisfy

$$\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4}\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}^T}\{\frac{1}{\gamma^2}\boldsymbol{g}_1\boldsymbol{g}_1^T - \boldsymbol{g}_2\boldsymbol{g}_2^T\}\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\psi} \le 0 \tag{7.68}$$

$$\| \boldsymbol{g}_1^T \frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}} \|^2 = O(\widetilde{\psi}) \tag{7.69}$$

(B) There exist positive definite functions $\widehat{\phi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *and* $\widehat{\psi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *in a neighborhood of the origin that satisfy*

$$\frac{\partial\widehat{\phi}}{\partial\boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial\widehat{\phi}}{\partial\boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial\widehat{\phi}}{\partial\boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 - \gamma^2\boldsymbol{h}_2^T\boldsymbol{h}_2 + \widehat{\psi} \le 0 \quad (7.70)$$

$$\| \boldsymbol{g}_1^T\frac{\partial\widehat{\phi}}{\partial\boldsymbol{x}} \|^2 = O(\widehat{\psi}) \quad (7.71)$$

(C) A function $\widehat{\phi} - \widetilde{\phi}$ *is positive definite in a neighborhood of the origin.* ∎

Proof: Using Theorem 7.8 and the same technique as [10] and [129], we can show that there exists a positive function $\widetilde{\phi}$ such that

$$\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}^T}(\boldsymbol{f} + \boldsymbol{g}_2\widetilde{\boldsymbol{c}}) + \frac{1}{4\gamma^2}\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}} + \widetilde{\psi} \le 0 \quad (7.72)$$

$$\| \boldsymbol{g}_1^T\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}} \|^2 = O(\widetilde{\psi}) \quad (7.73)$$

$$\boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}} = O(\widetilde{\psi}) \quad (7.74)$$

where $\widetilde{\boldsymbol{c}}(\boldsymbol{x}) \triangleq \boldsymbol{c}(\rho(\boldsymbol{x})$ for some function $\boldsymbol{\xi} = \rho(\boldsymbol{x})$ in Assumption 7.8. Now let $\widetilde{\phi}_1 \triangleq (1-\varepsilon)\widetilde{\phi}$, where $0 < \varepsilon < 1$. Then, it will be shown that $\widetilde{\phi}_1$ satisfies three conditions (7.72) to (7.74). Let

$$T(\varepsilon) \triangleq \frac{\partial\widetilde{\phi}_1}{\partial\boldsymbol{x}^T}(\boldsymbol{f} + \boldsymbol{g}_2\widetilde{\boldsymbol{c}}) + \frac{1}{4\gamma^2}\frac{\partial\widetilde{\phi}_1}{\partial\boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial\widetilde{\phi}_1}{\partial\boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}}$$

Then noting that $\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}^T}(\boldsymbol{f} + \boldsymbol{g}_2\widetilde{\boldsymbol{c}}) \le 0$,

$$\begin{aligned} T(\varepsilon) &\le (1-\varepsilon)^2\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}^T}(\boldsymbol{f} + \boldsymbol{g}_2\widetilde{\boldsymbol{c}}) + (1-\varepsilon)^2\frac{1}{4\gamma^2}\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial\widetilde{\phi}}{\partial\boldsymbol{x}} \\ &\quad + \boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}} \\ &= (1-\varepsilon)^2 T(0) + \varepsilon(2-\varepsilon)(\boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}}) \end{aligned} \quad (7.75)$$

Noting that $T(0) \le -\bar{\varepsilon}\widetilde{\psi} - (1-\bar{\varepsilon})\widetilde{\psi}$ where $0 < \bar{\varepsilon} < 1$, we get

$$\begin{aligned} T(\varepsilon) &\le -(1-\varepsilon)^2\bar{\varepsilon}\widetilde{\psi} - (1-\varepsilon)^2(1-\bar{\varepsilon})\widetilde{\psi} \\ &\quad + \varepsilon(2-\varepsilon)(\boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}}) \end{aligned} \quad (7.76)$$

In addition, from (7.74), there exists a positive constant k such that

$$\boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\boldsymbol{c}}^T\widetilde{\boldsymbol{c}} \le k\bar{\varepsilon}\widetilde{\psi} \quad (7.77)$$

Therefore it follows from (7.30) to (7.31) that, for $\varepsilon > 0$ which is sufficiently close to 0,

$$T(\varepsilon) \le -\widetilde{\psi}_1 \quad (7.78)$$

where $\widetilde{\psi}_1 = (1-\varepsilon)^2(1-\bar{\varepsilon})\widetilde{\psi}$. This means that $\widetilde{\phi}_1$ satisfies (7.72) to (7.74) where $\widetilde{\psi}$ is replaced by $\widetilde{\psi}_1$. In addition, it follows from (7.72) to (7.74) that $\widetilde{\phi}_1$ and $\widetilde{\psi}_1$ satisfy the condition (A). Concerning the condition (B), we can show that there exists a positive definite functions $\widehat{\phi}$ and $\widehat{\psi}$ satisfying the condition (B) in a similar way to the above proof.

Now, $\widehat{\phi} - \widetilde{\phi} \geq 0$, which is shown in [129]. Then

$$\widehat{\phi} - \widetilde{\phi}_1 > \widehat{\phi} - \widetilde{\phi} \geq 0, \quad \boldsymbol{x} \neq \mathbf{o}$$

These mean that the functions $\widetilde{\phi}_1$, $\widetilde{\psi}$, $\widehat{\phi}$, and $\widehat{\psi}$ satisfy the conditions (A), (B), and (C). ∎

Remark 7.15 *Ball et al. and van der Schaft have derived the results on the necessity condition of the solvability of nonlinear $\boldsymbol{H}_\infty$ control problems, which are concerned with the non-strict case, and also take no account of the internal stability of the closed loop systems. On the other hand, Theorem 7.12 is concerned with the strict case, and takes account of the internal stability. So Theorem 7.12 completely corresponds to the necessary condition obtained in the linear case [107], and is a natural extension of the linear case to the nonlinear setting.* ∎

Using Theorem 7.9, we can also obtain a necessary condition in the exponential stability case, in the same way as Theorem 7.12. In addition, in the case of exponential stability, we can show that the necessary condition is sufficient for the solvability of the strict $\boldsymbol{H}_\infty$ control problem via output feedback, using van der Schaft's method.

[Theorem 7.13] *Let γ be a given positive constant, and suppose Assumption 7.8 where Assumption 7.6 is replaced by Assumption 7.6', for a positive constant $\gamma_0 < \gamma$ which is sufficiently close to γ. Then for the system given by (7.64) to (7.66), the strict $\boldsymbol{H}_\infty$ control problem via output feedback is solvable if and only if the following conditions hold.*
(A) There exist a C^2 positive definite function $\widetilde{\phi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and a positive number $\widetilde{\varepsilon}$ in a neighborhood of the origin that satisfy

$$\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4}\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}^T}\{\frac{1}{\gamma^2}\boldsymbol{g}_1\boldsymbol{g}_1^T - \boldsymbol{g}_2\boldsymbol{g}_2^T\}\frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 + \widetilde{\psi} = 0 \quad (7.79)$$

where $\widetilde{\psi} \overset{\Delta}{=} \widetilde{\varepsilon}\boldsymbol{x}^T\boldsymbol{x} + \widetilde{\psi}_3$, and $\widetilde{\psi}_3$ is an appropriate function vanishing at the origin together with all the second order derivatives.
(B) There exist a C^2 positive definite function $\widehat{\phi}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and $\widehat{\varepsilon}$ in a neighborhood of the origin that satisfy

$$\frac{\partial \widehat{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma^2}\frac{\partial \widehat{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \widehat{\phi}}{\partial \boldsymbol{x}} + \boldsymbol{h}_1^T\boldsymbol{h}_1 - \gamma^2\boldsymbol{h}_2^T\boldsymbol{h}_2 + \widehat{\psi} = 0 \quad (7.80)$$

where $\widehat{\psi} \triangleq \widehat{\varepsilon}\boldsymbol{x}^T\boldsymbol{x} + \widehat{\psi}_3$, *and* $\widehat{\psi}_3$ *is an appropriate function vanishing at the origin together with all the second order derivatives.*
(C) A function $\widehat{\phi} - \widetilde{\phi}$ *is positive definite in a neighborhood of the origin and* $\widehat{\varepsilon} - \widetilde{\varepsilon}$ *is positive.*

In addition, when there exist $\widetilde{\phi}$ *and* $\widehat{\phi}$ *satisfying the above conditions, one of admissible controllers is given by*

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \boldsymbol{f}(\boldsymbol{\xi}) + \frac{1}{2\gamma^2}\boldsymbol{g}_1^T(\boldsymbol{\xi})\boldsymbol{g}_1(\boldsymbol{\xi})\frac{\partial\widetilde{\phi}}{\partial \boldsymbol{x}}(\boldsymbol{\xi}) - \frac{1}{2}\boldsymbol{g}_2^T(\boldsymbol{\xi})\boldsymbol{g}_2(\boldsymbol{\xi})\frac{\partial\widetilde{\phi}}{\partial \boldsymbol{x}}(\boldsymbol{\xi}) \\ &\quad + \boldsymbol{L}(\boldsymbol{\xi})\{\boldsymbol{h}_2(\boldsymbol{\xi}) - \boldsymbol{y}\} \\ \boldsymbol{u} &= -\frac{1}{2}\boldsymbol{g}_2^T(\boldsymbol{\xi})\frac{\partial\widetilde{\phi}}{\partial \boldsymbol{x}}(\boldsymbol{\xi}) \end{aligned} \tag{7.81}$$

where $\boldsymbol{L}$ *satisfies*

$$\frac{1}{2}\{\frac{\partial\widehat{\phi}}{\partial \boldsymbol{x}^T}(\boldsymbol{x}) - \frac{\partial\widetilde{\phi}}{\partial \boldsymbol{x}^T}(\boldsymbol{x})\}\boldsymbol{L}(\boldsymbol{x}) = -\gamma^2\boldsymbol{h}_2^T(\boldsymbol{x})$$

∎

Proof: "Only if": It is obtained from Theorem 7.9. "If": It can be shown based on the linearization ([129, 57, 75]). Namely, we can show that the linearization of the closed loop system solves the strict $\boldsymbol{H}_\infty$ control problem in almost the same way as the proof of the sufficiency of Theorem 1 in [107]. Note also that $\boldsymbol{L}$ exists locally, because the Hessian matrix of $\widehat{\phi} - \widetilde{\phi}$ is positive definite by the conditions (A) to (C). ∎

Remark 7.16 *A sufficient condition for the solvability of the strict* $\boldsymbol{H}_\infty$ *control problem via output feedback has been given by van der Schaft in [129], based on the linearization. However, it has not been shown that the sufficient condition is necessary there. On the other hand, Theorem 7.13 shows the necessary and sufficient condition. This success is based on the approach via the Hamilton-Jacobi strict inequality, that is, Theorem 7.9* ∎

7.5 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A new approach for nonlinear $\boldsymbol{H}_\infty$ control theory has been given, which does not depend on the Linearization and the linear $\boldsymbol{H}_\infty$

control techniques.

(ii) Some strict bounded real conditions of nonlinear systems have been characterized via two approaches: One is based on the Hamilton-Jacobi equation with a stabilizing solution and another is based on the Hamilton-Jacobi strict inequality. The former has an important role to analyze the internal stability of nonlinear systems, and the latter has an advantage that it can simply be applied to the strict $\boldsymbol{H}_\infty$ control problem. Each characterization of the strict bounded real condition is much significant, and the total use of both characterizations forms a more useful foundation to develop the strict $\boldsymbol{H}_\infty$ control theory of nonlinear systems. The derived results completely include the strict bounded real lemma of linear systems, and are also stronger and applicable to more general nonlinear systems, compared to the former results.

(iii) The relations between internal stability of nonlinear systems and the stabilizing solution of the Hamilton-Jacobi equation have been clarified, which are peculiar to nonlinear systems.

(iv) Based on results of (ii), several sufficient (and necessary) conditions for the solvability of the strict $\boldsymbol{H}_\infty$ state feedback control problem have been derived.

(v) Based on results of (ii), a necessary condition for the solvability of the strict $\boldsymbol{H}_\infty$ output feedback control problem has been given, and also a necessary and sufficient condition in the case of exponential stability has been given.

Appendix

Proof of Theorem 7.1: We prove (a)(i)→(iii) and (ii)→(iii), (b)(iii)→(iv), and (c)(iv)→(i) and (iv)→(ii), subsequently.
(a): Obvious.
(b): From (iii), one can show the global existence of $\phi_a(\boldsymbol{x})$ given by (7.4) (See [84]). Simple calculation shows

$$-\phi_a(\boldsymbol{x}_1) = \inf_{\boldsymbol{u}\in \boldsymbol{L}_2(t_1,t_2)}[-\phi_a(\boldsymbol{x}(t_2)) + \textstyle\int_{t_1}^{t_2} L(\boldsymbol{u},\boldsymbol{y})\mathrm{d}\tau]$$

$$\forall \boldsymbol{x}_1 = \boldsymbol{x}(t_1) \tag{7.A1}$$

where $L(\boldsymbol{u}, \boldsymbol{y}) \triangleq \gamma^2 \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{y}^T \boldsymbol{y}$. If the time period $[t_1, t_2]$ is small enough, under Assumption 7.2, there exists an optimal control input which minimizes the right hand side of (7.A1). Thus one obtains

$$-\phi_a(\boldsymbol{x}_1) = \min_{\boldsymbol{u} \in \boldsymbol{L}_{2(t_1,t_2)}} [-\phi_a(\boldsymbol{x}(t_2)) + \int_{t_1}^{t_2} L(\boldsymbol{u}, \boldsymbol{y}) \mathrm{d}\tau]$$
$$\forall \boldsymbol{x}_1 = \boldsymbol{x}(t_1) \tag{7.A2}$$

Applying the Hamilton-Jacobi Theory (see e.g. [3]) to (7.A2) yields

$$\min_{\boldsymbol{u}(t)} [\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{y}) - \frac{\partial \phi_a}{\partial \boldsymbol{x}^T} (\boldsymbol{f} + \boldsymbol{g}\boldsymbol{u})] = 0 \tag{7.A3}$$

which implies (7.9).

(c): At first, (iv)→(i) is shown. Substituting (7.1), (7.2) and (7.9) to (7.6), one gets

$$\begin{aligned} & \gamma^2 \parallel \boldsymbol{u} \parallel_{2T}^2 - \parallel \boldsymbol{y} \parallel_{2T}^2 \\ = \; & \phi(\boldsymbol{x}(T)) - \phi(\boldsymbol{x}(t_0)) + \gamma^2 \parallel \boldsymbol{u} - \frac{1}{2\gamma^2} \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}} \parallel_{2T}^2 \end{aligned} \tag{7.A4}$$

by completion of the square. Since the third term of the right hand side of (7.A4) is non-negative, and $\phi(\boldsymbol{x}) \geq 0 (\phi(\boldsymbol{x}_0) = 0,\ \boldsymbol{x}_0 = \mathbf{o})$, one obtains

$$\gamma^2 \parallel \boldsymbol{u} \parallel_{2T}^2 - \parallel \boldsymbol{y} \parallel_{2T}^2 \geq 0 \qquad \forall \boldsymbol{u} \in \boldsymbol{L}_{2e} \tag{7.A5}$$

Now it is enough to show that the equality in (7.A5) holds only if $\boldsymbol{u} \equiv \mathbf{o}$. Let $\boldsymbol{u}^*$ be the input satisfying the equality, then it must be $\boldsymbol{u}^* = \frac{1}{2\gamma^2} \boldsymbol{g}^T \frac{\partial \phi}{\partial \boldsymbol{x}}$ from (7.A4). However, $\boldsymbol{u}^*(t_0) = \mathbf{o}$ from (7.9). This implies $\boldsymbol{x}(t) \equiv \mathbf{o}$, i.e., $\boldsymbol{u}^*(t) \equiv \mathbf{o}$. (vi)→ (ii) can be proven in the same way as the proof of (iv) → (i). ∎

Proof of Lemma 7.2: At first, (ii) → (i). From (ii), there exists a positive constant r such that both eq.(7.22) and the internal stability of $\boldsymbol{S}$ hold for all $\boldsymbol{x} \in \boldsymbol{B}_r$. Since $\boldsymbol{S}$ is internally asymptotically stable, using Lemma 7.1, for a positive number $r_1 (< r)$, there exists a positive number c_1 such that $\boldsymbol{x} \in \boldsymbol{L}_\infty^{r_1}$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^{c_1}$. Thus, the condition (ii) holds in the presence of $\boldsymbol{u} \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^{c_1}$. Therefore, one can prove (i) in the same way as the proof of Theorem 7.1.

Second, (i)→(ii). One can prove this in the same way as the proof of Theorem 7.1. Note that the minimizing control input $\boldsymbol{u}(t)$ of (7.A3) is in $\boldsymbol{B}_{c1}$ and that a function $\widehat{\phi}_a(\boldsymbol{x})$ exists only in a neighborhood of the origin. In order to cope with these conditions, one has only to consider the optimal control problem given by (7.A2) for some sufficiently small time period $[t_1, t_2]$ and the initial state $\boldsymbol{x}_1$ that is sufficiently close to the origin. ∎

Proof of Lemma 7.3 : At first, it is shown that the system given by (7.23) is stable in the sense of Lyapunov. From the condition (ii), by the converse theorem of Lyapunov (see [40, 145, 133]), there exist a continuous differentiable and positive definite function $W(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$, functions $\psi_i(\cdot) \in \boldsymbol{K}(i = 1, 2, 3)$, and a positive constant r_1 such that

$$\begin{aligned} \psi_1(\| \boldsymbol{x} \|) &\leq W(\boldsymbol{x}) \leq \psi_2(\| \boldsymbol{x} \|) \\ \dot{W}|_{\dot{x}=\tilde{f}} &= \frac{\partial W}{\partial \boldsymbol{x}^T} \tilde{\boldsymbol{f}} \leq -\psi_3(\| \boldsymbol{x} \|) \quad \forall \boldsymbol{x} \in \boldsymbol{B}_{r_1} \end{aligned} \tag{7.A6}$$

Also assume that the condition (i) holds for all $\boldsymbol{x} \in \boldsymbol{B}_{r_2}$. Therefore, when $r \triangleq \min\{r_1, r_2\}$, it is enough for us to show that, given any positive number $\varepsilon(< r)$, there exists a $\delta > 0$ such that, whenever $\| \boldsymbol{x}_0 \| < \delta$,

$$\| \boldsymbol{x}(t) \| < \varepsilon \quad \forall t \geq t_0 \tag{7.A7}$$

As a preparation, it is shown that the function W in (7.A6) can be chosen to satisfy

$$\| \frac{\partial W}{\partial \boldsymbol{x}} \|^2 = O(\psi_3(\| \boldsymbol{x} \|)) \tag{7.A8}$$

When $W_i \triangleq W^i$(i is a positive integer), one obtains

$$\psi_1^i(\| \boldsymbol{x} \|) \leq W_i(\boldsymbol{x}) \leq \psi_2^i(\| \boldsymbol{x} \|) \tag{7.A9}$$

$$\begin{aligned} \dot{W}_i|_{\dot{x}=\tilde{f}} &= iW^{i-1} \frac{\partial W}{\partial \boldsymbol{x}^T} \tilde{\boldsymbol{f}} \leq -iW^{i-1} \psi_3(\| \boldsymbol{x} \|) \\ & \quad \forall \boldsymbol{x} \in \boldsymbol{B}_{r_1} \end{aligned} \tag{7.A10}$$

Now define $\mathrm{ord}(y)$ as

$$\mathrm{ord}(y) \triangleq \max\{s | \lim_{\|\boldsymbol{x}\| \to 0} \frac{y}{\| \boldsymbol{x} \|^s} < \infty\} \tag{7.A11}$$

Assume $\mathrm{ord}(\psi_3(\| \boldsymbol{x} \|)) = s_1$, where $s_1(< \infty)$ is a positive number. Since W is continuously differentiable and $W(\boldsymbol{x}) = O(\| x \|)$, one gets $\mathrm{ord}(W) = s_2$, where $1 \leq s_2 < \infty$. From this, $\mathrm{ord}(W_i) = s_2 i$, and $\mathrm{ord}(W^{i-1}\psi_3) = s_1 + s_2(i-1)$. Let $\psi_3' \in \boldsymbol{K}$ be a function to be less than $W^{i-1}\psi_3$ for each $\boldsymbol{x}$ in a neighborhood of the origin and to satisfy $\mathrm{ord}(\psi_3') = s_1 + s_2(i-1) + 1$. Since $\mathrm{ord}(\| \frac{\partial W_i}{\partial \boldsymbol{x}} \|^2) \geq 2s_2(i-1)$, there exists an i such that, for sufficiently large i, $\| \frac{\partial W_i}{\partial \boldsymbol{x}} \|^2 = O(\psi_3'(\| \boldsymbol{x} \|))$. If W_i is redefined as W, W given by (7.A6) satisfies (7.A8), without loss of generality.

Now define, given $\varepsilon_1 > 0$,

$$k(\varepsilon_1) \triangleq \max_{\varepsilon_1 \leq \| \boldsymbol{x} \| \leq r} \{ \frac{\| \boldsymbol{g} \|^2 \| \frac{\partial W}{\partial \boldsymbol{x}} \|^2}{4\psi_3(\| \boldsymbol{x} \|)} \} \tag{7.A12}$$

and a function $\psi_4 \in \boldsymbol{K}$ satisfying $\phi(\boldsymbol{x}) \leq \psi_4(\| \boldsymbol{x} \|)$ for all $\boldsymbol{x} \in \boldsymbol{B}_r$. Then consider a positive definite function V given by $V(\boldsymbol{x}) = \frac{k}{\rho}\phi(\boldsymbol{x}) + W(\boldsymbol{x})$. Differentiating V along the system given by (7.23), one gets, from (7.24) and (7.A6)

$$\begin{aligned} \dot{V} \ \leq \ & -k(\| s \| - \frac{\| \boldsymbol{g} \| \| \frac{\partial W}{\partial \boldsymbol{x}} \|}{2k})^2 \\ & + \frac{\| \boldsymbol{g} \|^2 \| \frac{\partial W}{\partial \boldsymbol{x}} \|^2}{4k} - \psi_3(\| \boldsymbol{x} \|) \leq 0 \\ & \text{for} \quad \varepsilon_1 \leq \| \boldsymbol{x} \| < r \end{aligned} \tag{7.A13}$$

Using this, one can show that, whenever $\| \boldsymbol{x}_0 \| < \varepsilon_1$, $\| \boldsymbol{x}(t) \| < \psi_1^{-1}(\frac{1}{\rho}k(\varepsilon_1)\psi_4(\varepsilon_1) + \psi_2(\varepsilon_1))$ for all $t(\geq t_0)$. Therefore, if there exists an $\varepsilon_1 > 0$ such that

$$\frac{1}{\rho}k(\varepsilon_1)\psi_4(\varepsilon_1) + \psi_2(\varepsilon_1) < \psi_1(\varepsilon) \tag{7.A14}$$

holds for any given positive number $\varepsilon(< r)$, it is concluded that (7.A7) holds whenever $\| \boldsymbol{x}_0 \| < \varepsilon_1(= \delta)$. Since $\sup_{0<\varepsilon_1<r} k(\varepsilon_1) < \infty$ by (7.A8), there exists an $\varepsilon_1 > 0$ satisfying (7.A14) for any positive number $\varepsilon(< r)$.

Next it is shown that $\boldsymbol{x}(t) \to \mathbf{o}$ as $t \to \infty$. The stability in the sense of Lyapunov implies $\boldsymbol{x} \in \boldsymbol{L}_\infty$, for all $\boldsymbol{x}_0$ in a neighborhood of $\boldsymbol{x} = \mathbf{o}$. Then $\boldsymbol{s} \in \boldsymbol{L}_\infty$. Integrating the both side hands of (7.24), one gets

$$\phi(\boldsymbol{x}(t_0)) - \phi(\boldsymbol{x}(t)) \geq \rho \int_{t_0}^{t} \| s \|^2 \, \mathrm{d}\tau \tag{7.A15}$$

From $\phi(\boldsymbol{x}(t)) < \infty$ for all $t \geq t_0$ and (7.A15), $\boldsymbol{s} \in \boldsymbol{L}_2$. Further since $\boldsymbol{x}(t)$ is continuous and $\dot{\boldsymbol{x}} \in \boldsymbol{L}_\infty$, $\boldsymbol{s}$ is uniformly continuous with respect to t. From the uniform continuity of $\boldsymbol{s}$ and $\boldsymbol{s} \in \boldsymbol{L}_2$, $\boldsymbol{s} \to \mathbf{o}$ as $t \to \infty$ (See [30]). Therefore $\boldsymbol{s} \in \boldsymbol{L}_\infty \cap \boldsymbol{C}_0$. If $\boldsymbol{s}$ is regarded as an input of the system given by (7.23) and Lemma 7.1 is applied, then $\boldsymbol{x} \in \boldsymbol{L}_\infty \cap \boldsymbol{C}_0$.

The stability in the sense of Lyapunov and the attractivity mean that the system given by (7.23) is asymptotically stable at $\boldsymbol{x} = \mathbf{o}$. ∎

Proof of Lemma 7.4: From Lemma 7.2, it is clear that $\widehat{\phi}_a(\boldsymbol{x})$ given by (7.14) satisfies (7.20) in a neighborhood of the origin. Further from $\| \boldsymbol{S} \|_{L2c} < \gamma$, there exists a positive number $\gamma_0(< \gamma)$ such that $\| \boldsymbol{S} \|_{L2c} < \gamma_0$. Therefore one can show that, under Assumptions 7.3 and 7.4, there exists a positive semi-definite function $\widehat{\phi}_{r0}(\boldsymbol{x})$ given by (7.16) such that it satisfies

$$\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}^T}\boldsymbol{f} + \frac{1}{4\gamma_0^2}\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}} + \boldsymbol{h}^T\boldsymbol{h} = 0 \tag{7.A16}$$

in a neighborhood of the origin.

Now we show $\bar{\phi}(\boldsymbol{x}) \triangleq \widehat{\phi}_{r0}(\boldsymbol{x}) - \widehat{\phi}_a(\boldsymbol{x}) \geq 0$ in a neighborhood of the origin. So suppose that there exist $\widehat{\phi}_a(\boldsymbol{x})$ and $\widehat{\phi}_{r0}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{B}_\delta$ to satisfy each Hamilton-Jacobi equation. From $\| \boldsymbol{S} \|_{L2e} < \gamma_0$, one gets

$$\int_{t_0}^{T}(\gamma_0^2\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau \geq 0$$

$$\forall \boldsymbol{u} \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^c, \quad \forall T \geq t_0, \quad \boldsymbol{x}(t_0) = \mathbf{o} \tag{7.A17}$$

In (7.A17), consider a time t_1 ($t_0 \leq t_1 \leq T$), and define $\boldsymbol{u}_1 : [t_0, t_1) \rightarrow \boldsymbol{R}^m$ and $\boldsymbol{u}_2 : [t_1, \infty) \rightarrow \boldsymbol{R}^m$. Then one obtains

$$\int_{t_0}^{t_1}(\gamma_0^2\boldsymbol{u}_1^T\boldsymbol{u}_1 - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau + \int_{t_1}^{T}(\gamma_0^2\boldsymbol{u}_2^T\boldsymbol{u}_2 - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau \geq 0$$

$$\forall T \geq t_1, \quad \forall t_0 \leq t_1, \quad \boldsymbol{x}(t_0) = \mathbf{o}$$

$$\forall \boldsymbol{u}_1 \in \boldsymbol{L}_2(t_0, t_1) \cap \boldsymbol{L}_\infty^c, \quad \forall \boldsymbol{u}_2 \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^c \tag{7.A18}$$

Adding $\int_{t_1}^{T}(\gamma^2 - \gamma_0^2)\boldsymbol{u}_2^T\boldsymbol{u}_2\mathrm{d}\tau \geq 0$ to the left side hand of (7.A18), one gets

$$\int_{t_0}^{t_1}(\gamma_0^2\boldsymbol{u}_1^T\boldsymbol{u}_1 - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau + \int_{t_1}^{T}(\gamma^2\boldsymbol{u}_2^T\boldsymbol{u}_2 - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau \geq 0$$

$$\forall T \geq t_1, \quad \forall t_0 \leq t_1, \quad \boldsymbol{x}(t_0) = \mathbf{o}$$

$$\forall \boldsymbol{u}_1 \in \boldsymbol{L}_2(t_0, t_1) \cap \boldsymbol{L}_\infty^c, \quad \forall \boldsymbol{u}_2 \in \boldsymbol{L}_{2e} \cap \boldsymbol{L}_\infty^c \tag{7.A19}$$

Therefore it follows from (7.A19) that $\widehat{\phi}_{r0}(\boldsymbol{x}_1) - \widehat{\phi}_a(\boldsymbol{x}_1) \geq 0$ for all $\boldsymbol{x}_1 \in \boldsymbol{B}_\delta$, where $\boldsymbol{x}_1 \triangleq \boldsymbol{x}(t_1)$.

Now from (7.20) with $\phi = \widehat{\phi}_a$ and (7.A16), one gets

$$\frac{\partial \bar{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f}_a = -\frac{1}{4\gamma^2}\frac{\partial \bar{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \bar{\phi}}{\partial \boldsymbol{x}} - \frac{1}{4}\left(\frac{1}{\gamma_0^2} - \frac{1}{\gamma^2}\right)\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}^T}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}}$$

$$\forall \boldsymbol{x} \in \boldsymbol{B}_\delta \tag{7.A20}$$

where $\bar{\phi} \triangleq \widehat{\phi}_{r0} - \widehat{\phi}_a$ and $\boldsymbol{f}_a \triangleq \boldsymbol{f} + \frac{1}{2\gamma^2}\boldsymbol{g}\boldsymbol{g}^T\frac{\partial \widehat{\phi}_a}{\partial \boldsymbol{x}}$. Now $\boldsymbol{a} \triangleq \boldsymbol{g}^T\frac{\partial \widehat{\phi}_{r0}}{\partial \boldsymbol{x}}$, $\boldsymbol{b} \triangleq \boldsymbol{g}^T\frac{\partial \widehat{\phi}_a}{\partial \boldsymbol{x}}$, and $\boldsymbol{c} \triangleq [\boldsymbol{a}^T\boldsymbol{b}^T]^T$. Then one gets $\frac{\partial \bar{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f}_a = -\boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}$, where

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \mathbf{o} & -\boldsymbol{I} \end{bmatrix}\begin{bmatrix} \frac{1}{4}(\frac{1}{\gamma_0^2} - \frac{1}{\gamma^2})\boldsymbol{I} & \mathbf{o} \\ \mathbf{o} & \frac{1}{4\gamma^2}\boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \mathbf{o} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \tag{7.A21}$$

, and $\boldsymbol{M} > 0$ since $\frac{1}{\gamma_0^2} - \frac{1}{\gamma^2} > 0$. Therefore let α be $4\gamma^4\lambda_{min}(\boldsymbol{M})$($\lambda_{min}(\cdot)$ expresses a minimum singular value), so one obtains

$$\frac{\partial \bar{\phi}}{\partial \boldsymbol{x}^T}\boldsymbol{f}_a \leq -\alpha \left\| \frac{1}{2\gamma^2}\boldsymbol{g}^T\frac{\partial \widehat{\phi}_a}{\partial \boldsymbol{x}} \right\|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{B}_\delta \tag{7.A22}$$

From (7.A22) and the internal stability of the system $\boldsymbol{S}$, using Lemma 7.3, one can show that the system given by $\dot{\boldsymbol{x}} = \boldsymbol{f}_a(\boldsymbol{x})$ is asymptotically stable at the origin. ∎

Proof of Lemma 7.5 : The necessity is obvious. Consider the sufficiency. Assume that the system $\boldsymbol{S}$ is small signal $\widehat{\boldsymbol{L}}_2$ stable. Then the following function exists in a neighborhood of the origin, from the reachability of the system.

$$\widetilde{\phi}_a(\boldsymbol{x}) \triangleq -\inf_{u\in\widehat{\boldsymbol{L}}_{2e}\cap\boldsymbol{L}_\infty^c, T\geq t} \int_t^T (k^2\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau \tag{7.A23}$$

where $\boldsymbol{x}(t) = \boldsymbol{x}$. Then since the system $\boldsymbol{S}$ is internally asymptotically stable, one can show (see [139])

$$\widetilde{\phi}_a(\boldsymbol{x}) = -\inf_{u\in\boldsymbol{L}_{2e}\cap\boldsymbol{L}_\infty^c} \int_t^\infty (k^2\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y})\mathrm{d}\tau \triangleq \bar{\phi}_a(\boldsymbol{x}) \tag{7.A24}$$

Then this implies $k^2 \parallel \boldsymbol{u} \parallel_2 - \parallel \boldsymbol{y} \parallel_2 \geq 0$ for all $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$. ∎

Proof of Lemma 7.7 : From the condition (ii)', by the converse theorem of Lyapunov (see [40, 145, 133]), there exist a smooth positive definite function $W(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$, and positive constants $a_i (i = 1, 2, 3, 4)$ and r_1 such that

$$\begin{aligned} a_1 \parallel \boldsymbol{x} \parallel^2 &\leq W(\boldsymbol{x}) \leq a_2 \parallel \boldsymbol{x} \parallel^2 \\ \dot{W}|_{\dot{x}=\widetilde{f}} &= \frac{\partial W}{\partial \boldsymbol{x}^T}\widetilde{\boldsymbol{f}} \leq -a_3 \parallel \boldsymbol{x} \parallel^2 \\ \parallel \frac{\partial W}{\partial \boldsymbol{x}} \parallel &\leq a_4 \parallel \boldsymbol{x} \parallel \quad \forall \boldsymbol{x} \in \boldsymbol{B}_{r_1} \end{aligned} \tag{7.A25}$$

It is also assumed that the condition (i)' holds for all $\boldsymbol{x} \in \boldsymbol{B}_{r_2}$. Define $r \triangleq \min\{r_1, r_2\}$ and $\bar{g} \triangleq \max_{\boldsymbol{x}\in\boldsymbol{B}_r} \parallel \boldsymbol{g} \parallel$. Then consider a positive definite function $V \triangleq \frac{k}{\rho}\phi + W$, where k is selected as

$$k > \frac{a_4^2\bar{g}^2}{4a_3} \tag{7.A26}$$

Differentiating V along the system given by (7.23) and completing the squares, one gets

$$\begin{aligned} \dot{V} &\leq -a_3 \parallel \boldsymbol{x} \parallel^2 + \bar{g}a_4 \parallel \boldsymbol{x} \parallel\parallel \boldsymbol{s} \parallel - k \parallel \boldsymbol{s} \parallel^2 \\ &\leq -(a_3 - \frac{\bar{g}^2 a_4^2}{4k}) \parallel \boldsymbol{x} \parallel^2 \quad \forall \boldsymbol{x} \in \boldsymbol{B}_r \end{aligned} \tag{7.A27}$$

Therefore from (7.A26), there exists a $b_3 > 0$ such that $\dot{V} \leq -b_3 \parallel \boldsymbol{x} \parallel^2$. Further noting that $\frac{\partial\phi}{\partial\boldsymbol{x}}(\mathbf{o}) = \mathbf{o}$ and that ϕ is C^2, there exist $b_i > 0$ $(i = 1, 2)$ such that, for all $\boldsymbol{x} \in \boldsymbol{B}_r$, $b_1 \parallel \boldsymbol{x} \parallel^2 \leq V(\boldsymbol{x}) \leq b_2 \parallel \boldsymbol{x} \parallel^2$. The existence of $b_i > 0 (i = 1, 2, 3)$ implies that the system given by (7.23) is exponential stable at the origin. ∎

Proof of Theorem 7.4: (i)→(ii): Since ϕ_a defined by (7.4) exists globally and satisfies (7.40) by Theorem 7.1, the condition (A) is satisfied. In order to prove that ϕ_a satisfies the condition (B), it is enough to show that Lemmas 7.7 and 7.8 hold globally. A global version of Lemma 7.7 can be proven in the same way as Lemma 7.7, under the assumption that $\frac{\partial \phi_a}{\partial \boldsymbol{x}}$ is globally Lipschitz and that $\sup \| \boldsymbol{g}(\boldsymbol{x}) \| < \infty$.

A global version of Lemma 7.8 can also be proven by using a global Lipschitz condition of $\frac{\partial \phi_a}{\partial \boldsymbol{x}}$ and $\frac{\partial \phi_r}{\partial \boldsymbol{x}}$. It is trivial that the condition (C) is necessary.

(ii)→(i): The global and internal exponential stability of the system $\boldsymbol{S}$ can be easily shown by the condition (A) and the global version of Lemma 7.7. Now it is shown that $\| \boldsymbol{S} \|_{L2} < \gamma$. At first, it is shown that, since the system $\boldsymbol{S}$ is globally and internally exponentially stable, $\boldsymbol{x} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty \cap \boldsymbol{C}_0$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2$ as follows. It is obvious that $\boldsymbol{x} \in \boldsymbol{L}_2$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2$, using the result of [44] from the global Lipschitz condition of $\boldsymbol{f}$ and $\sup \| \boldsymbol{g}(\boldsymbol{x}) \| < \infty$. Further one can easily show $\dot{\boldsymbol{x}} \in \boldsymbol{L}_2$. Therefore from $\boldsymbol{x} \in \boldsymbol{L}_2$ and $\dot{\boldsymbol{x}} \in \boldsymbol{L}_2$, $\boldsymbol{x} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty \cap \boldsymbol{C}_0$ holds. Using the above fact and the condition (A), one gets

$$\gamma^2 \| \boldsymbol{u} \|_2^2 - \| \boldsymbol{y} \|_2^2 = \gamma^2 \| \boldsymbol{v} \|_2^2 \quad \forall \boldsymbol{u} \in \boldsymbol{L}_2 \tag{7.A28}$$

where $\boldsymbol{v}$ is defined by (7.18).

Using the global and internal exponential stability of the system $\boldsymbol{S}_v^{-1}$ given by (7.19), the global Lipschitz condition of $\boldsymbol{f}$ and $\frac{\partial \phi}{\partial \boldsymbol{x}}$, and $\sup \| \boldsymbol{g}(\boldsymbol{x}) \| < \infty$, it is obtained that the system $\boldsymbol{S}_v^{-1}$ is $\boldsymbol{L}_2$ stable, that is, there exists $k > 0$ such that

$$\| \boldsymbol{u} \|_2 \leq k \| \boldsymbol{v} \|_2 \quad \forall \boldsymbol{v} \in \boldsymbol{L}_2 \tag{7.A29}$$

In the system $\boldsymbol{S}_v$, from (7.A28), $\boldsymbol{v} \in \boldsymbol{L}_2$ holds for all $\boldsymbol{u} \in \boldsymbol{L}_2$, and in the system $\boldsymbol{S}_v^{-1}$, from (7.A29), $\boldsymbol{u} \in \boldsymbol{L}_2$ holds for all $\boldsymbol{v} \in \boldsymbol{L}_2$. Then one gets

$$\| \boldsymbol{u} \|_2 \leq k \| \boldsymbol{v} \|_2 \quad \forall \boldsymbol{u} \in \boldsymbol{L}_2 \tag{7.A30}$$

Eqs. (7.A28) and (7.A30) imply $\| \boldsymbol{S} \|_{L2} < \gamma$. ∎

Proof of Lemma 7.10: Since the $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is asymptotically stable at $\boldsymbol{x} = \boldsymbol{o}$, by the converse theorem of Lyapunov, there exist a sufficiently smooth and positive definite function $W(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ and C^∞ functions $\psi_i(\cdot) \in \boldsymbol{K} (i = 1, 2, 3)$ satisfying in a neighborhood of $\boldsymbol{x} = \boldsymbol{o}$

$$\psi_1(\| \boldsymbol{x} \|) \leq W(\boldsymbol{x}) \leq \psi_2(\| \boldsymbol{x} \|) \tag{7.A31}$$

$$\dot{W}|_{\dot{\boldsymbol{x}}=\boldsymbol{f}} = \frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f} \leq -\psi_3(\| \boldsymbol{x} \|) \tag{7.A32}$$

In addition, we can show that, without loss of generality, W satisfies

$$\| \frac{\partial W}{\partial \boldsymbol{x}} \|^2 = O(\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f}) \tag{7.A33}$$

Now noting that $-\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f}$ is a positive definite function, there exists a sufficiently smooth positive definite function $\delta(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ such that $\frac{1}{\varepsilon}\delta \leq -\frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f}$ in a neighborhood of $\boldsymbol{x} = \mathbf{o}$, for a ε satisfying $0 < \varepsilon < 1$.

Let k_1 and k_2 be positive constants satisfying on some local region U

$$\| \frac{\partial W}{\partial \boldsymbol{x}} \|^2 = k_1 \mid \frac{\partial W}{\partial \boldsymbol{x}^T}\boldsymbol{f} \mid$$

$$k_2 = \max_{\boldsymbol{x} \in U} \lambda_{max}(\boldsymbol{g}\boldsymbol{g}^T)$$

Then if k satisfies

$$k > \frac{k_1 k_2}{1 - \varepsilon}$$

we can show that eq.(7.52) holds locally. ∎

Chapter 8

ROBUST STABILIZATION OF NONLINEAR SYSTEMS BY H_∞ STATE FEEDBACK

8.1 Introduction

Among stabilization problems of control systems, it is important that parametric and/or unstructured uncertainty is taken into account. Although there are many researches about robust stabilization of nonlinear systems with parametric or structured uncertainty [52, 26, 122],there are few researches about robust stabilization of nonlinear systems with unstructured uncertainty.

Recently, several researchers have attempted to extend the $\boldsymbol{H}_\infty$ control theory to the case of nonlinear systems, and the solutions of $\boldsymbol{H}_\infty$ state or output feedback control problems are given as shown in the Introduction 1.3.1. In addition, some new results were given in the previous chapter 7. For linear systems, $\boldsymbol{H}_\infty$ control theory combined with the small gain theorem solves the robust stabilization problem for unstructured uncertainty directly. However, this is not the case for nonlinear systems. In order to solve the problem, we need to discuss not only the $\boldsymbol{L}_2$ gain property (with I/O stability) but also internal stability rigorously. So far several researchers discussed the relation between the L_2 stability and the internal stability [138, 46, 47, 44, 134], but the result seems to be too restrictive to apply to the robust stabilization problem.

In this chapter, a robust stabilization problem by state feedback for nonlinear systems with unstructured uncertainty is considered. First, based on some results in chapter 7, the robust stability condition is given. The obtained condition completely corresponds to the well-known robust stability condition for linear systems. Second, a sufficient condition for the existence of a robust stabilizing controller is given, based on nonlinear H_∞ state feedback control theory in chapter 7. The obtained approach allows us to treat various types of stability, i.e. asymptotic, exponential, and global exponential stability (which includes linear system case), in a unified way in solving the robust stabilization problem. In this sense, the result obtained here is a natural nonlinear extension of the robust stabilization of linear systems. In addition, some numerical examples show the effectiveness of the obtained nonlinear robust stabilizing controller.

The same notations are used as chapter 7.

8.2 Robust stabilization problem

Consider the following nonlinear system whose input-output relation is given by an operator $G : L_{2e} \to L_{2e}$.

$$G \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) + j(x)u \\ x(t_0) & = & x^0 \end{array} \right.$$

where $x \in R^n$ is the state, $u \in R^m$ is the input, $y \in R^m$ is the output, and t_0 is the initial time. $f(\cdot) : R^n \to R^n$, $g(\cdot) : R^n \to R^{n\times m}$, $h(\cdot) : R^n \to R^p$, and $j(\cdot) : R^n \to R^{p\times m}$ are sufficiently smooth functions with $f(o) = o$ and $h(o) = o$.

Then three kinds of internal stability are defined as follows.

[Definition 8.1] *The system G is said to be internally asymptotically, internally exponentially, and internally globally exponentially stable, if the origin ($x = o$) of the system G with $u \equiv o$, namely*

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, *is an asymptotically, exponentially, and globally exponentially stable equilibrium, respectively.* ∎

We consider input-output stability in the case that the input belongs to $\boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$ as follows.

[Definition 8.2] *The system* $\boldsymbol{G}$ *is said to be small signal* $\boldsymbol{L}_2$ *stable if there exist constants* k *and* c *such that* $\| \boldsymbol{y} \|_2 \leq k \| \boldsymbol{u} \|_2$, *for* $\boldsymbol{x}^0 = \boldsymbol{o}$ *and all* $\boldsymbol{u} \in \boldsymbol{L}_2 \cap \boldsymbol{L}_\infty^c$ *[134]. Furthermore the system* $\boldsymbol{G}$ *is said to be strongly small signal* $\boldsymbol{L}_2$ *stable if there exist a positive semi-definite function* $\psi(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ *and a positive constant* k *satisfying, in a neighborhood of the origin,*

$$\frac{\partial \psi}{\partial \boldsymbol{x}^T}\boldsymbol{f} + (\frac{1}{2}\boldsymbol{g}^T\frac{\partial \psi}{\partial \boldsymbol{x}} + \boldsymbol{j}^T\boldsymbol{h})^T(k^2\boldsymbol{I} - \boldsymbol{j}^T\boldsymbol{j})^{-1}(\frac{1}{2}\boldsymbol{g}^T\frac{\partial \psi}{\partial \boldsymbol{x}} + \boldsymbol{j}^T\boldsymbol{h})$$
$$+\boldsymbol{h}^T\boldsymbol{h} \leq 0$$

∎

Note that if the system is strongly small signal $\boldsymbol{L}_2$ stable, then it is small signal $\boldsymbol{L}_2$ stable. Under some assumptions such as the smoothness of an available storage function, the strong small signal $\boldsymbol{L}_2$ stability is equivalent to the small signal $\boldsymbol{L}_2$ stability [46]. The strong small signal $\boldsymbol{L}_2$ stability also implies that the system is dissipative with a C^1 storage function when a supply rate is $k^2\boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{y}^T\boldsymbol{y}$ as defined in [18].

We also define the $\boldsymbol{L}_2$ gain for the system $\boldsymbol{G}$ as follows.

[Definition 8.3] *The system* $\boldsymbol{G}$ *is said to have a small signal* $\boldsymbol{L}_2$ *gain, if there exists a positive constant* c *such that* $\| \boldsymbol{G} \|_{L2c}$ *is finite subject to* $\boldsymbol{x}^0 = \boldsymbol{o}$, *where*

$$\| \boldsymbol{G} \|_{L2c} \triangleq \sup_{\boldsymbol{u} \in \boldsymbol{L}_2/\{0\} \cap \boldsymbol{L}_\infty^c} \frac{\| \boldsymbol{y} \|_2}{\| \boldsymbol{u} \|_2}$$

If $c = \infty$, $\| \boldsymbol{G} \|_{L2\infty}$ *is denoted by* $\| \boldsymbol{G} \|_{L2}$, *which is* $\boldsymbol{L}_2$*-induced norm.* ∎

Now let us state a robust stabilization problem. Consider the state feedback system as shown in Fig.8.1. Let $\boldsymbol{P}$ be the nominal plant and $\boldsymbol{\Delta}$ be the uncertain plant, which have the following state space realizations.

$$\boldsymbol{P}\left\{\begin{array}{rcl} \dot{\boldsymbol{x}}_1 &=& \boldsymbol{f}_1(\boldsymbol{x}_1) + \boldsymbol{g}_1(\boldsymbol{x}_1)\boldsymbol{u} \\ \boldsymbol{y} &=& \boldsymbol{h}_1(\boldsymbol{x}_1) \\ \boldsymbol{x}_1(t_0) &=& \boldsymbol{x}_1^0 \end{array}\right.$$

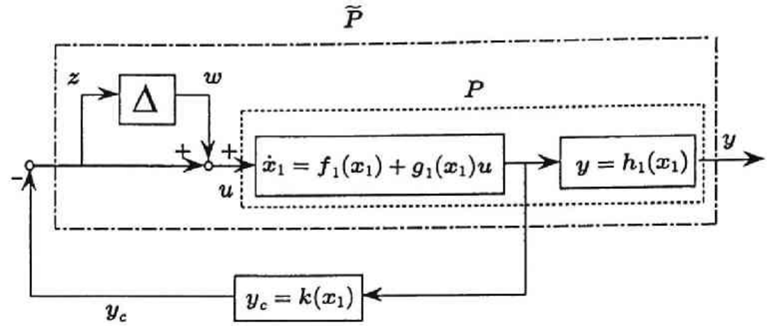

Figure 8.1: Closed loop system

$$\boldsymbol{\Delta} \begin{cases} \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}_2(\boldsymbol{x}_2) + \boldsymbol{g}_2(\boldsymbol{x}_2)\boldsymbol{z} \\ \boldsymbol{w} &= \boldsymbol{h}_2(\boldsymbol{x}_2) \\ \boldsymbol{x}_2(t_0) &= \boldsymbol{x}_2^0 \end{cases}$$

where $\boldsymbol{x}_1 \in \boldsymbol{R}^{n_1}$ and $\boldsymbol{x}_2 \in \boldsymbol{R}^{n_2}$ are the state vector, $\boldsymbol{u} \in \boldsymbol{R}^{m_1}$ and $\boldsymbol{z} \in \boldsymbol{R}^{m_1}$ are the input, $\boldsymbol{y} \in \boldsymbol{R}^{m_2}$ and $\boldsymbol{w} \in \boldsymbol{R}^{m_1}$ is the output, respectively. Functions $\boldsymbol{f}_i$, $\boldsymbol{g}_i$, and $\boldsymbol{h}_i$ are sufficiently smooth with $\boldsymbol{f}_i(\mathbf{o}) = \mathbf{o}$, and $\boldsymbol{h}_i(\mathbf{o}) = \mathbf{o}$ $(i = 1, 2)$. It is assumed that $\boldsymbol{f}_1$, $\boldsymbol{g}_1$, and $\boldsymbol{h}_1$ are known, but $\boldsymbol{f}_2$, $\boldsymbol{g}_2$, and $\boldsymbol{h}_2$ are unknown. A function $\boldsymbol{k}(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}^{m_1}$ in Fig.8.1 expresses a state feedback controller.

Then the following assumptions are made.

[Assumption 8.1] *Concerning the nominal plant $\boldsymbol{P}$, $\boldsymbol{P}$ is locally reachable with small input. Namely, given any $c > 0$, there exists an $r(c) > 0$ satisfying the following: for any $\boldsymbol{x}_1^1 \in \boldsymbol{B}_r$ and t_1, there exist finite time $t_0 (\leq t_1)$ and a control input $\boldsymbol{u} \in \boldsymbol{L}_2(t_0, t_1) \cap \boldsymbol{L}_\infty^c$ such that the state is driven from $\boldsymbol{x}_1(t_0) = \mathbf{o}$ to $\boldsymbol{x}_1(t_1) = \boldsymbol{x}_1^1$.* ∎

[Assumption 8.2] *Concerning to the uncertain plant $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}$ is locally reachable with small input. Furthermore, the uncertain plant $\boldsymbol{\Delta}$ is internally asymptotically (exponentially, or globally exponentially) stable at the origin, and for a positive constant γ, there is a positive constant c such that $\| \boldsymbol{\Delta} \|_{L2c} \leq \frac{1}{\gamma}$.* ∎

Let the real plant which is composed of $\boldsymbol{P}$ and $\boldsymbol{\Delta}$ be $\widetilde{\boldsymbol{P}}$, and let the set of the systems which are composed of $\boldsymbol{P}$ and the set of the plant $\boldsymbol{\Delta}$ with Assumption 8.2 be $\mathcal{A}$. Then the closed loop system in Fig.8.1

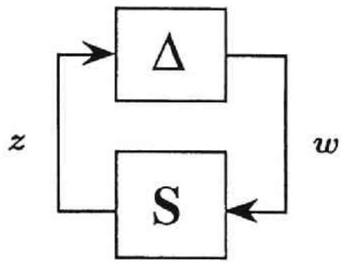

Figure 8.2: Equivalent system

is said to be robustly asymptotically (exponentially, or globally exponentially) stable, if the closed loop system is internally asymptotically (exponentially, or globally exponentially) stable for all the systems $\widetilde{\boldsymbol{P}}$ which belong to the set $\mathcal{A}$. Then we consider the following problem.

[Problem 8.1] **(Robust stabilization problem)** *Find a state feedback controller which robustly asymptotically (exponentially, or globally exponentially) stabilizes the closed loop system in Fig.8.1.* ∎

8.3 Robust stabilization

In the first part of this section, a sufficient condition for the closed loop system in Fig.8.1 to be robustly asymptotically stable is derived. Second, a sufficient condition for the existence of the robust asymptotic stabilizing controller, that is, the robust stabilizability condition is given. Finally, we give some results about the robust exponential stability case.

8.3.1 Robust asymptotic stability condition

The system in Fig.8.2 is equivalent to the system in Fig.8.1. $\boldsymbol{S}$ is an operator which expresses the input-output relation from $\boldsymbol{w}$ to $\boldsymbol{z}$, and has the following state space realization.

$$S\left\{\begin{array}{rcl}\dot{\boldsymbol{x}}_1 &=& \boldsymbol{f}_1(\boldsymbol{x}_1)-\boldsymbol{g}_1(\boldsymbol{x}_1)\boldsymbol{k}(\boldsymbol{x}_1)+\boldsymbol{g}_1(\boldsymbol{x}_1)\boldsymbol{w}\\ \boldsymbol{z} &=& -\boldsymbol{k}(\boldsymbol{x}_1)\end{array}\right.$$

We define the following functions for the system $\boldsymbol{S}$ and $\boldsymbol{\Delta}$, respectively.

$$\widehat{\phi}_1(\boldsymbol{x}_1(t)) \triangleq -\inf_{w\in \boldsymbol{L}_{2e}(t)\cap \boldsymbol{L}_\infty^{c_1}} \int_t^\infty ({\gamma_1}^2\boldsymbol{w}^T\boldsymbol{w}-\boldsymbol{z}^T\boldsymbol{z})\mathrm{d}\tau$$

$$\widehat{\phi}_2(\boldsymbol{x}_2(t)) \triangleq -\inf_{z\in \boldsymbol{L}_{2e}(t)\cap \boldsymbol{L}_\infty^{c}} \int_t^\infty (\frac{1}{\gamma^2}\boldsymbol{z}^T\boldsymbol{z}-\boldsymbol{w}^T\boldsymbol{w})\mathrm{d}\tau$$

where γ, $\gamma_1(<\gamma)$, c_1, and c are positive constants. Then the following assumptions are made.

[Assumption 8.3] *When $\widehat{\phi}_1(\boldsymbol{x}_1)$ exists in a neighborhood of the origin, it is C^1.* ∎

[Assumption 8.4] *When $\widehat{\phi}_2(\boldsymbol{x}_2)$ exists in a neighborhood of the origin, it is C^1.* ∎

The following result is obtained.

[Theorem 8.1] *For positive constants γ and c, the nominal plant $\boldsymbol{P}$ and the uncertain plant $\boldsymbol{\Delta}$ are assumed to satisfy Assumptions 8.1 to 8.4 ($\boldsymbol{\Delta}$ is internally asymptotically stable). Then the closed loop system in Fig.8.2 is robustly asymptotically stable, if the following two conditions hold simultaneously.*

(i) *The system $\boldsymbol{S}$ is internally asymptotically stable.*

(ii) *There exists a positive constant c_1 such that $\| \boldsymbol{S} \|_{L2c_1} < \gamma$.*

∎

Proof: The proof is based on Lemma 7.3 in chapter 7: we show that the conditions (i) and (ii) in 7.3 hold for the closed loop system which consists of $\boldsymbol{S}$ and $\boldsymbol{\Delta}$ in Fig.8.2.

From $\| \boldsymbol{S} \|_{L2c_1} < \gamma$, there exists a positive constant $\gamma_1(<\gamma)$ such that $\| \boldsymbol{S} \|_{L2c_1} \leq \gamma_1$. Under Assumptions 8.1 and 8.2, $\| \boldsymbol{S} \|_{L2c_1} \leq \gamma_1$ and $\| \boldsymbol{\Delta} \|_{L2c} \leq \frac{1}{\gamma}$ imply that there exist positive semi-definite functions $\widehat{\phi}_1(\boldsymbol{x}_1)$ and $\widehat{\phi}_2(\boldsymbol{x}_2)$ with $\widehat{\phi}_1(\mathbf{o}) = 0$ and $\widehat{\phi}_2(\mathbf{o}) = 0$ in a neighborhood of the origin, respectively [84, 46]. In addition, under Assumptions 8.3 and 8.4, $\| \boldsymbol{S} \|_{L2c_1} \leq \gamma_1$ and $\| \boldsymbol{\Delta} \|_{L2c} \leq \frac{1}{\gamma}$ imply that the following relations hold locally.

$$\dot{\widehat{\phi}}_1(\boldsymbol{x}_1) \triangleq \frac{\partial \widehat{\phi}_1}{\partial \boldsymbol{x}_1{}^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \leq \gamma_1{}^2\boldsymbol{w}^T\boldsymbol{w} - \boldsymbol{z}^T\boldsymbol{z} \quad (8.1)$$

$$\dot{\widehat{\phi}}_2(\boldsymbol{x}_2) \triangleq \frac{\partial \widehat{\phi}_2}{\partial \boldsymbol{x}_2{}^T}(\boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{z}) \leq \frac{1}{\gamma^2}\boldsymbol{z}^T\boldsymbol{z} - \boldsymbol{w}^T\boldsymbol{w} \quad (8.2)$$

Then we define a function $\widehat{\phi}(\boldsymbol{x})$ by

$$\widehat{\phi}(\boldsymbol{x}) \triangleq \frac{1}{\gamma_2{}^2}\widehat{\phi}_1(\boldsymbol{x}_1) + \widehat{\phi}_2(\boldsymbol{x}_2)$$

where $\boldsymbol{x} = [\boldsymbol{x}_1{}^T, \boldsymbol{x}_2{}^T]^T \in \boldsymbol{R}^{n_1+n_2}$ and γ_2 is a positive constant satisfying $\gamma_1 < \gamma_2 < \gamma$. Differentiating $\widehat{\phi}$ along the closed loop system which consists of $\boldsymbol{S}$ and $\boldsymbol{\Delta}$, we obtains, from (8.1) and (8.2)

$$\begin{aligned}\dot{\widehat{\phi}}(\boldsymbol{x}) &= \frac{1}{\gamma_2{}^2}\dot{\widehat{\phi}}_1(\boldsymbol{x}_1) + \dot{\widehat{\phi}}_2(\boldsymbol{x}_2)\\ &\leq \frac{1}{\gamma_2{}^2}(\gamma_1{}^2\boldsymbol{w}^T\boldsymbol{w} - \boldsymbol{z}^T\boldsymbol{z}) + \frac{1}{\gamma^2}\boldsymbol{z}^T\boldsymbol{z} - \boldsymbol{w}^T\boldsymbol{w}\\ &= -(\frac{1}{\gamma_2{}^2} - \frac{1}{\gamma^2})\boldsymbol{z}^T\boldsymbol{z} - (1 - \frac{\gamma_1{}^2}{\gamma_2{}^2})\boldsymbol{w}^T\boldsymbol{w}\end{aligned}$$

Therefore, noting that $\frac{1}{\gamma_2^2} - \frac{1}{\gamma^2} > 0$ and $1 - \frac{\gamma_1^2}{\gamma_2^2} > 0$, the following relation holds in a neighborhood of the origin.

$$\dot{\widehat{\phi}}(\boldsymbol{x}) \leq -\rho(\boldsymbol{k}^T(\boldsymbol{x}_1)\boldsymbol{k}(\boldsymbol{x}_1) + \boldsymbol{h}_2^T(\boldsymbol{x}_2)\boldsymbol{h}_2(\boldsymbol{x}_2)) \quad (8.3)$$

where

$$\rho \triangleq \min\left[\frac{1}{\gamma_2{}^2} - \frac{1}{\gamma^2}, 1 - \frac{\gamma_1{}^2}{\gamma_2{}^2}\right]$$

Let $\widetilde{\boldsymbol{f}}$, $\widetilde{\boldsymbol{g}}$, and $\boldsymbol{s}$ in Lemma 7.3 in chapter 7 be

$$\widetilde{\boldsymbol{f}} \triangleq \begin{bmatrix} \boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} \\ \boldsymbol{f}_2 \end{bmatrix}, \quad \widetilde{\boldsymbol{g}} \triangleq \begin{bmatrix} \boldsymbol{g}_1 & 0 \\ 0 & \boldsymbol{g}_2 \end{bmatrix}, \quad \boldsymbol{s} \triangleq \begin{bmatrix} \boldsymbol{h}_2 \\ -\boldsymbol{k} \end{bmatrix}$$

Then from (8.3), the condition (i) in Lemma 7.3 hold for the closed loop system with $\boldsymbol{S}$ and $\boldsymbol{\Delta}$. Furthermore, the condition (i) and the internal asymptotic stability of $\boldsymbol{\Delta}$ imply the condition (ii) in Lemma 7.3. Consequently, by Lemma 7.3, the closed loop system with $\boldsymbol{S}$ and $\boldsymbol{\Delta}$ is internally asymptotically stable. This completes the proof. ∎

Remark 8.1 *Note that Theorem 8.1 cannot be derived directly from the small gain theorem, since the small gain theorem is concerned with the input-output stability, not the internal stability. Willems [138] and Hill and Moylan [47] showed that the closed loop system in Fig.8.2* *is internally asymptotically stable, if both systems* $\boldsymbol{S}$ *and* $\boldsymbol{\Delta}$ *have a kind of observable property and the* $\boldsymbol{L}_2$ *gain of the system composed of* $\boldsymbol{S}$ *and* $\boldsymbol{\Delta}$ *is less than 1. However this result cannot be applied to the robust stability condition considered here, because, in [138, 47], the assumption of the observability for the system is crucial to prove the internal stability of the closed loop system. On the other hand, Theorem 8.1 is an extension of the previous results such as Corollary 2 in [47] to the case that the observability is not assumed. Theorem 8.1 shows that if both systems* $\boldsymbol{S}$ *and* $\boldsymbol{\Delta}$ *are internally asymptotically stable and* $\| \boldsymbol{S} \|_{L2c} \| \boldsymbol{\Delta} \|_{L2c} < 1$, *then the closed loop system in Fig.8.2 is internally asymptotically stable.* ∎

In the proof of Theorem 8.1, the idea of the derivation of (8.3) is the same as Corollary 2 in [47]. Hill and Moylan showed in Corollary 2 of [47] that, using the equation corresponding to (8.3) and a kind of observability, the storage function is positive definite, and then the La Salle invariance principle proves the asymptotic stability of the closed loop system. However, we cannot use the principle in Theorem 8.1 where the observability is not assumed, because the storage function is not necessarily positive definite (When the storage function is positive semi-definite, not positive definite, the La Salle invariance principle implies that the system is attractive, not asymptotically stable.). Thus we need the different approach, which is Lemma 7.3 in chapter 7. Lemma 7.3 has an important role in our paper.

Although Theorem 8.1 is concerned with the sufficiency of robust stability, it naturally corresponds to the well-known robust stability condition for linear systems.

8.3.2 Robust asymptotic stabilizability condition

Consider the following system:

$$\tilde{\boldsymbol{S}} \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_1 & = & \boldsymbol{f}_1(\boldsymbol{x}_1) + \frac{1}{2}(\frac{1}{\gamma^2} - 1)\boldsymbol{g}_1\boldsymbol{g}_1^T \frac{\partial \phi}{\partial \boldsymbol{x}_1} + \boldsymbol{g}_1 \boldsymbol{v} \\ \boldsymbol{w} & = & \frac{1}{2\gamma^2}\boldsymbol{g}_1^T \frac{\partial \phi}{\partial \boldsymbol{x}_1} + \boldsymbol{v} \end{array} \right.$$

where $\phi(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ is an appropriate real function. Then we get the following result, based on Theorem 8.1.

[Theorem 8.2] *For positive constants* γ *and* c, *the uncertain plant* $\boldsymbol{\Delta}$ *is assumed to satisfy Assumptions 8.2 and 8.4 (*$\boldsymbol{\Delta}$ *is inter-*

nally asymptotically stable). Then the closed loop system in Fig.8.1 is robustly asymptotically stabilizable by state feedback $\boldsymbol{y}_c = \boldsymbol{k}(\boldsymbol{x}_1) \in \boldsymbol{R}^{m_1}$, *if there exist a* C^1 *positive semi-definite function* $\phi(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ *and a vector-valued function* $\boldsymbol{l}(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}^l$ *(l is a positive integer) in a neighborhood of the origin satisfying the following two conditions simultaneously.*

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}_1^T}\boldsymbol{f}_1 + \frac{1}{4}(\frac{1}{\gamma^2} - 1)\frac{\partial \phi}{\partial \boldsymbol{x}_1^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1} + \boldsymbol{l}^T\boldsymbol{l} = 0$$

(B) The system $\widetilde{\boldsymbol{S}}$ *is internally asymptotically stable and strongly small signal* $\boldsymbol{L}_2$ *stable.*

Then a robust stabilizing controller is given by $\boldsymbol{k}(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$. ∎

Proof: Consider $\boldsymbol{k} = \frac{1}{2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$ as a state feedback controller. First, we show that the system $\boldsymbol{S}$ is internally asymptotically stable, by using Lemma 7.3 in chapter 7. Let $\widetilde{\boldsymbol{f}}$, $\widetilde{\boldsymbol{g}}$, and $\boldsymbol{s}$ in Lemma 7.3 be $\widetilde{\boldsymbol{f}} \triangleq \boldsymbol{f}_1 - \frac{1}{2}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1} + \frac{1}{2\gamma^2}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$, $\widetilde{\boldsymbol{g}} \triangleq \boldsymbol{g}_1$, and $\boldsymbol{s} \triangleq -\frac{1}{2\gamma^2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$, respectively. Then the condition (A) implies (7.24), namely, the condition (i) in Lemma 7.3. In addition, the internal asymptotic stability of the system $\widetilde{\boldsymbol{S}}$ implies the asymptotic stability of $\dot{\boldsymbol{x}}_1 = \widetilde{\boldsymbol{f}}(\boldsymbol{x}_1)$, namely, the condition (ii) in Lemma 7.3. Therefore, the system $\boldsymbol{S}$ is internally asymptotically stable.

Next, it is shown that the relation which corresponds to (8.1) holds under the condition (A) and the $\boldsymbol{L}_2$ stability of $\widetilde{\boldsymbol{S}}$. The condition (A) implies that the following relation holds locally.

$$\dot{\phi} \triangleq \frac{\partial \phi}{\partial \boldsymbol{x}_1^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \le \gamma^2\boldsymbol{w}^T\boldsymbol{w} - \boldsymbol{z}^T\boldsymbol{z} - \gamma^2\boldsymbol{v}^T\boldsymbol{v} \qquad (8.4)$$

where $\boldsymbol{v} = \boldsymbol{w} - \frac{1}{2\gamma^2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$. Furthermore, the strong small signal $\boldsymbol{L}_2$ stability of $\widetilde{\boldsymbol{S}}$ means, by Definition 8.2, that there exists a positive semi-definite function $\psi(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ and a positive constant $k(> 1)$ satisfying, in a neighborhood of the origin,

$$\begin{aligned} & \frac{\partial \psi}{\partial \boldsymbol{x}_1^T}\{\boldsymbol{f}_1 + \frac{1}{2}(\frac{1}{\gamma^2} - 1)\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}\} \\ + \ & \frac{1}{4(k^2-1)}(\frac{\partial \psi}{\partial \boldsymbol{x}_1} + \frac{1}{\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}_1})^T\boldsymbol{g}_1\boldsymbol{g}_1^T(\frac{\partial \psi}{\partial \boldsymbol{x}_1} + \frac{1}{\gamma^2}\frac{\partial \phi}{\partial \boldsymbol{x}_1}) \\ + \ & \frac{1}{4\gamma^4}\frac{\partial \phi}{\partial \boldsymbol{x}_1^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1} \le 0 \end{aligned} \qquad (8.5)$$

From (8.5), we get

$$\dot{\psi} \triangleq \frac{\partial \psi}{\partial \boldsymbol{x}_1^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \leq k^2\boldsymbol{v}^T\boldsymbol{v} - \boldsymbol{w}^T\boldsymbol{w} \tag{8.6}$$

Now consider $\widetilde{\phi} \triangleq \phi + \frac{\gamma^2}{k^2}\psi$. Then by (8.4) and (8.6), there exists a positive constant $\gamma_1(<\gamma)$ such that

$$\dot{\widetilde{\phi}} \triangleq \frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}_1^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \leq \gamma_1^2\boldsymbol{w}^T\boldsymbol{w} - \boldsymbol{z}^T\boldsymbol{z} \tag{8.7}$$

Eq.(8.7) corresponds to (8.1) in the proof of Theorem 8.1.

Under the above preparation, we can prove that the closed loop system which is composed of $\boldsymbol{S}$ and $\boldsymbol{\Delta}$ is internally asymptotically stable, in a similar way to the part after (8.2) in the proof of Theorem 8.1. This completes the proof. ∎

Remark 8.2 *Theorem 8.2 shows that the robust stabilizability condition is given in terms of the solvability of nonlinear $\boldsymbol{H}_\infty$ state feedback control, in the same way as linear system case. Note that a solution satisfying the conditions (A) and (B) corresponds to a stabilizing solution of the Riccati equation appeared in linear system case.* ∎

Remark 8.3 *Note that the assumptions with respect to the system $\boldsymbol{S}$ (or $\boldsymbol{P}$), namely Assumptions 8.1 and 8.3, are not made in Theorem 8.2. Thus, we do not need to check Assumptions 8.1 and 8.3.* ∎

Remark 8.4 *Theorem 8.2, as you can see from the proof and Lemma 7.3, shows if there exist positive semi-definite functions ϕ and ψ on $\boldsymbol{B}_r$ satisfying the conditions (A) and (B), and there exists a Lyapunov function on $\boldsymbol{B}_r$ which ensures the internal asymptotic stability of the system $\widetilde{\boldsymbol{S}}$, then there exists a Lyapunov function on $\boldsymbol{B}_r$ which guarantees the internal asymptotic stability of the closed loop system with $\boldsymbol{k}(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$. Thus using the Lyapunov function on $\boldsymbol{B}_r$, we can estimate the stability region of the closed loop system.* ∎

Remark 8.5 *If we consider the case of $\| \boldsymbol{\Delta} \|_{L2c} < \frac{1}{\gamma}$, not $\| \boldsymbol{\Delta} \|_{L2c} \leq \frac{1}{\gamma}$ in Theorem 8.1, then the robust stabilizability condition is the condition (A) and the internal asymptotic stability of $\widetilde{\boldsymbol{S}}$. As a result, it is not required that the system $\widetilde{\boldsymbol{S}}$ is strongly small signal $\boldsymbol{L}_2$ stable, because we have only to show that the system $\boldsymbol{S}$ satisfies $\| \boldsymbol{S} \|_{L2c_1} \leq \gamma$, not $\| \boldsymbol{S} \|_{L2c_1} < \gamma$.* ∎

8.3.3 Robust exponential stabilization

In this section, we discuss a robust exponential stabilization problem. In Assumptions 8.3 and 8.4, C^1 is replaced by C^2 with respect to the smoothness of $\widetilde{\phi}_i$ $(i = 1, 2)$, and these are called Assumptions 8.3' and 8.4', respectively. Then concerning to the robust exponential stability condition, the following result corresponding to Theorem 8.1 is obtained.

[Theorem 8.3] *For positive constants γ and c, the nominal plant $\boldsymbol{P}$ and the uncertain plant $\boldsymbol{\Delta}$ are assumed to satisfy Assumptions 8.1, 8.2, 8.3', and 8.4' ($\boldsymbol{\Delta}$ is internally exponentially stable). Then the closed loop system in Fig.8.2 is robustly exponentially stable, if the following two conditions hold simultaneously.*

(i) *The system $\boldsymbol{S}$ is internally exponentially stable.*

(ii) *There exists a positive constant c_1 such that $\| \boldsymbol{S} \|_{L2c_1} < \gamma$.*

∎

Proof: We can prove Theorem 8.3, by using Lemma 7.7 in chapter 7 in a similar way to the proof of Theorem 8.1. ∎

In addition, we get a result corresponding to Theorem 8.2.

[Theorem 8.4] *For positive constants γ and c, the uncertain plant $\boldsymbol{\Delta}$ is assumed to satisfy Assumptions 8.2 and 8.4' ($\boldsymbol{\Delta}$ is internally exponentially stable). Then the closed loop system in Fig.8.1 is robustly exponentially stabilizable by state feedback $\boldsymbol{y}_c = \boldsymbol{k}(\boldsymbol{x}_1) \in \boldsymbol{R}^{m_1}$, if there exist a C^2 positive semi-definite function $\phi(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ and a vector-valued function $\boldsymbol{l}(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}^l$ (l is a positive integer) in a neighborhood of the origin satisfying the following two conditions simultaneously.*

(A)

$$\frac{\partial \phi}{\partial \boldsymbol{x}_1^T}\boldsymbol{f}_1 + \frac{1}{4}(\frac{1}{\gamma^2} - 1)\frac{\partial \phi}{\partial \boldsymbol{x}_1^T}\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1} + \boldsymbol{l}^T\boldsymbol{l} = 0$$

(B) *The system $\widetilde{\boldsymbol{S}}$ is internally exponentially stable.*
Then a robust stabilizing controller is given by $\boldsymbol{k}(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$. ∎

Proof: We prove Theorem 8.4 in a similar way to the proof of Theorem 8.2. Consider $\boldsymbol{k}(\boldsymbol{x}_1) = \frac{1}{2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$ as a state feedback controller. First by

Lemma 7.7, the internal exponential stability of the system $\boldsymbol{S}$ follows from the conditions (A) and (B). Second, the condition (A) yields (8.4). Third, we show the relation which corresponds to (8.6). From the condition (B), by the Converse Theorem of Lyapunov, there exists a positive definite function $V(\cdot) : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ and positive constants $a_i (i = 1, 2)$ in a neighborhood of the origin such that

$$\frac{\partial V}{\partial \boldsymbol{x}_1^T}\{\boldsymbol{f}_1 + \frac{1}{2}(\frac{1}{\gamma^2} - 1)\boldsymbol{g}_1\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}\} \le -a_1 \parallel \boldsymbol{x}_1 \parallel^2 \tag{8.8}$$

$$\parallel \frac{\partial V}{\partial \boldsymbol{x}_1} \parallel \le a_2 \parallel \boldsymbol{x}_1 \parallel \tag{8.9}$$

Differentiating V along the system $\boldsymbol{S}$, we get, from (8.8) and (8.9)

$$\begin{aligned} \dot{V} &\triangleq \frac{\partial V}{\partial \boldsymbol{x}_1^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \\ &\le -a_1 \parallel \boldsymbol{x}_1 \parallel^2 + a_2 a_3 \parallel \boldsymbol{x}_1 \parallel\parallel \boldsymbol{v} \parallel \end{aligned} \tag{8.10}$$

where a_3 is an appropriate positive constant, and $\boldsymbol{v} = \boldsymbol{w} - \frac{1}{2\gamma^2}\boldsymbol{g}_1^T\frac{\partial \phi}{\partial \boldsymbol{x}_1}$ By simple computation, (8.10) yields

$$\begin{aligned} \dot{V} &\triangleq \frac{\partial V}{\partial \boldsymbol{x}_1^T}(\boldsymbol{f}_1 - \boldsymbol{g}_1\boldsymbol{k} + \boldsymbol{g}_1\boldsymbol{w}) \\ &\le a_4 \parallel \boldsymbol{v} \parallel^2 - a_5 \parallel \boldsymbol{w} \parallel^2 \end{aligned} \tag{8.11}$$

where $a_i (i = 4, 5)$ are appropriate positive constants and satisfy $a_4 > a_5$. Eq.(8.11) corresponds to (8.6). From (8.4) and (8.11), we get the relation which corresponds to (8.7).

Under the above preparation, we can prove that the closed loop system with $\boldsymbol{S}$ and $\boldsymbol{\Delta}$ is internally exponentially stable, in a similar way to the proof of Theorem 8.2. ∎

Remark 8.6 *While the case of asymptotic stabilization in Theorem 8.2 requires some kind of $\boldsymbol{L}_2$ stability for the system $\tilde{\boldsymbol{S}}$, that is, the latter part of the condition (B), the case of exponential stabilization explicitly does not. This is because the internal exponential stability of $\tilde{\boldsymbol{S}}$ means the strong small signal $\boldsymbol{L}_2$ stability of $\tilde{\boldsymbol{S}}$.* ∎

Remark 8.7 *Theorems 8.1 to 8.4 are concerned with the local stability. Global exponential stability case can be treated, under some assumptions such as global Lipschitz condition, in a similar way to the exponential stability case. These global results to correspond to Theorems 8.3 and 8.4 completely include the well-known results of the linear system case. Therefore, our results are natural nonlinear extensions of the linear* *system case.* ∎

8.4 Numerical examples

In this section, two numerical examples concerning to Theorems 8.2 and 8.4 are given. In addition, we discuss the feature of the obtained nonlinear controllers, comparing to linear controllers.

8.4.1 Example 1 : Robust asymptotic stabilization

In Fig.8.1, consider the following system as a nominal plant $\boldsymbol{P}$.

$$\boldsymbol{P}\left\{\begin{array}{lcl} \dot{x_1} & = & 2{x_1}^3 + u \\ y_1 & = & h(x_1) \end{array}\right.$$

where $h(x_1)$ is an arbitrary function. Concerning to an uncertain plant $\boldsymbol{\Delta}$, it is assumed that $\| \boldsymbol{\Delta} \|_{L2c} \leq \frac{1}{\sqrt{2}}$(This means $\gamma = \sqrt{2}$). Then we find a state feedback controller which robustly stabilizes the closed loop system in Fig.8.1, by using Theorem 8.2. When the function ϕ, and $\boldsymbol{l}$ are given as

$$\phi(x_1) = 4x_1^4 \ , \ l(x_1) = 0,$$

then the condition (A) is satisfied. Since the system $\widetilde{\boldsymbol{S}}$ is given as

$$\widetilde{\boldsymbol{S}}\left\{\begin{array}{lcl} \dot{x}_1 & = & -2x_1^3 + v \\ w & = & 4x_1^3 + v \end{array}\right.$$

the system $\widetilde{\boldsymbol{S}}$ is internally asymptotically stable. In addition, a positive semi-definite function $\psi(x_1)$ and a positive constant k satisfying (8.5) are given as $\psi(x_1) = 4x_1^4$ and $k = \sqrt{17}$. Thus the system $\widetilde{\boldsymbol{S}}$ is strongly small signal $\boldsymbol{L}_2$ stable.

Therefore, this closed loop system can be robustly asymptotically stabilized by

$$\boldsymbol{k}(x_1) = \frac{1}{2} g_1^T \frac{\partial \phi}{\partial x_1} = 8x_1^3$$

if the uncertain plant is internally stable and satisfies Assumptions 8.2 and 8.4.

8.4.2 Example 2 : Robust exponential stabilization

In Fig.8.1, consider the following system as a nominal plant $\boldsymbol{P}$.

$$\boldsymbol{P}\left\{\begin{array}{rcl} \dot{x_{p1}} &=& 4x_{p2}+16{x_{p2}}^3 \\ \dot{x_{p2}} &=& -\frac{1}{4}x_{p1}+\frac{3}{4}x_{p2}-16{x_{p1}}^3+3{x_{p2}}^3+u \\ y_1 &=& h(\boldsymbol{x}_1) \end{array}\right.$$

where h is an arbitrary function, and $\boldsymbol{x}_1=[x_{p1},x_{p2}]^T$. Concerning to an uncertain plant $\boldsymbol{\Delta}$, it is assumed that $\|\ \boldsymbol{\Delta}\ \|_{L2c}\leq\frac{1}{2}$(This means $\gamma=2$). Then we find a state feedback controller which robustly stabilizes the closed loop system in Fig.8.1, by using Theorem 8.4.

When the functions ϕ and l are given as

$$\begin{array}{rcl} \phi(\boldsymbol{x}_1) &=& \frac{1}{8}{x_{p1}}^2+2{x_{p2}}^2+4({x_{p1}}^4+{x_{p2}}^4) \\ l(\boldsymbol{x}_1) &=& 0 \end{array}$$

then the condition (A) is satisfied. Since the system $\widetilde{\boldsymbol{S}}$ is given as

$$\widetilde{\boldsymbol{S}}\left\{\begin{array}{rcl} \dot{x_{p1}} &=& 4x_{p2}+16{x_{p2}}^3 \\ \dot{x_{p2}} &=& -\frac{1}{4}x_{p1}-\frac{3}{4}x_{p2}-16{x_{p1}}^3-3{x_{p2}}^3+v \\ w &=& \frac{1}{2}x_{p2}+2{x_{p2}}^3+v \end{array}\right.$$

the condition (B) is satisfied.

Therefore, this closed loop system can be robustly exponentially stabilized by

$$\boldsymbol{k}(\boldsymbol{x}_1)=\frac{1}{2}\boldsymbol{g}_1^T\frac{\partial\phi}{\partial\boldsymbol{x}_1}=2x_{p2}+8{x_{p2}}^3 \tag{8.12}$$

if the uncertain plant is internally exponentially stable and satisfies Assumptions 8.2 and 8.4'.

Now we compare the nonlinear controller given by (8.12) to a linear controller obtained by the robust stabilization of the linearization of the nonlinear plant $\boldsymbol{P}$. Since the linearization of $\boldsymbol{P}$ is given as

$$\dot{\boldsymbol{x}}_1=\left[\begin{array}{cc} 0 & 4 \\ -\frac{1}{4} & \frac{3}{4} \end{array}\right]\boldsymbol{x}_1+\left[\begin{array}{c} 0 \\ 1 \end{array}\right]u$$

we get $k(\boldsymbol{x}_1)=2x_{p2}$ as a robust stabilizing controller. Then as an uncertain plant which has $\|\ \Delta\ \|_{L2}=\frac{1}{2}$, let us consider, for simplicity, the following linear system.

$$\boldsymbol{\Delta}\left\{\begin{array}{rcl} \dot{x}_2 &=& -\frac{5}{3}x_2+z \\ w &=& -\frac{5}{6}x_2 \end{array}\right.$$

It can be easily checked that Assumptions 8.2 and 8.4' are satisfied.

Fig.8.3 show the initial state response when $\boldsymbol{x}_1(0)=[0,\ 0.65]^T, x_2(0)=$

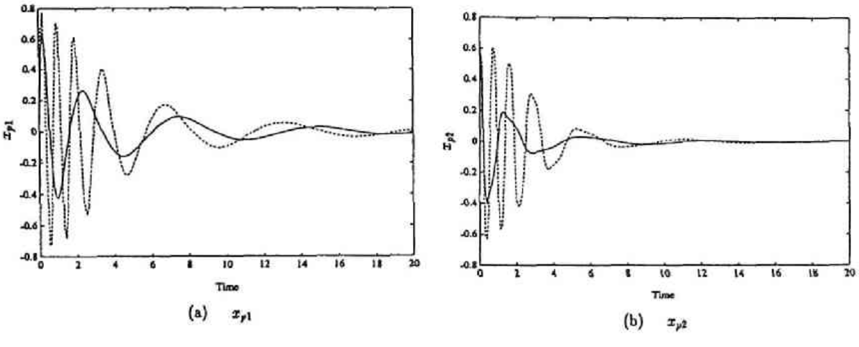

Figure 8.3: Initial state response: (a) x_{p1}, (b) x_{p2}, Solid lines express the nonlinear case, while dashed lines express the linear case

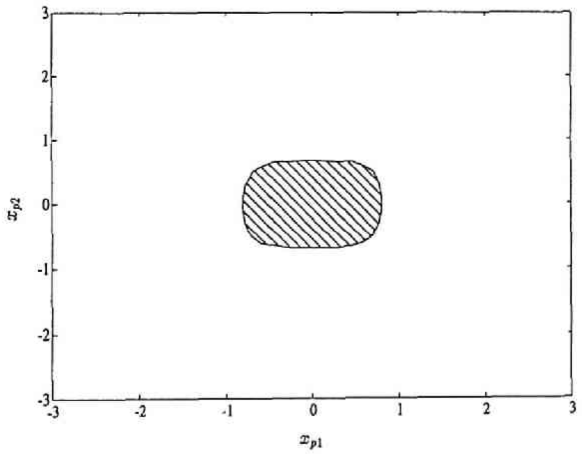

Figure 8.4: Stability region in linear controller case

0. We can see that the state response in the nonlinear controller case is better than that in the linear controller case. In addition, we numerically estimated the stability region in each case. Fig.8.4 shows the stability region in the linear controller case. The stability region may not be large. In the case of the nonlinear controller, on the other hand, the stability region is much larger than that of the linear case (it may be everywhere, although we cannot show it analytically).

8.5 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A robust stability condition has been given for nonlinear systems with unstructured uncertainty. Furthermore, a robust stabilizability condition has been derived in terms of the solvability of some partial differential equation and a robust stabilizing controller has been given, which is based on the nonlinear $\boldsymbol{H}_\infty$ state feedback control theory developed in section 7.3.

(ii) The obtained approach in the robust stabilization problem with unstructured uncertainty allows us to treat various types of stability, i.e. asymptotic stability, exponential stability, and global exponential stability, in a unified way in solving the robust stabilization problem of nonlinear systems.

(iii) Some numerical examples show the validity of the proposed nonlinear controller, compared to a robust stabilizing linear controller for the linearization of the original nonlinear system.

Chapter 9

GLOBAL ROBUST STABILIZATION OF NONLINEAR CASCADED SYSTEM

9.1 Introduction

Global stabilization of nonlinear systems is one of fundamental problems, but it is a very difficult problem. Recently, such a problem has been attacked as a stabilization problem for a nonlinear system which has the so-called "normal form" [15, 79, 17, 18, 67, 123, 105] or a class of nonlinear cascaded systems [117, 116, 120, 115, 111, 91, 121, 110, 19]. Needless to say, the important next step is to discuss the stabilization of the nonlinear cascaded system in the presence of uncertainty. This will be the first step to the robust stabilization of general nonlinear systems. However, the above approach cannot be straightforwardly extended to robust setting. Although there are a few researches [109, 108, 14, 34] about robust control of uncertain systems with a normal form, these are concerned with robust output tracking control problem, not global stabilization one.

From a quite different viewpoint, various stabilization techniques have been developed for nonlinear systems in the presence of uncertainty. Most of them treat the case where the uncertainty satisfies the so-called matching condition [37, 28, 2], though some of them tackled the mismatched uncertainty case such as the cone-bounded [89, 140, 25, 106] or singular perturbation case [104, 72, 26] which has local nature essentially. However, in order to discuss the robust stabilization of nonlinear cascaded systems, the matching condition is too restrictive. In addition, it is difficult to apply the latter methods to the global stabilization problem.

The purpose of this chapter is to give a sufficient condition for the global robust stabilization, via state feedback, of a class of nonlinear cascaded systems in the presence of uncertainty which does not necessarily satisfy the so-called matching condition. In addition, considering a specified class of the systems, a more practical condition for global robust stabilization is derived. The obtained results extend a condition of global stabilization for nonlinear cascade systems without uncertainty, which has been derived recently by [15, 18, 19], in the sense that the system uncertainty is taken into consideration. Further the obtained results show that, under a certain condition, a class of systems with the uncertainty that is acted on by input through a strictly positive real linear system (of course this uncertainty does not satisfy the matching condition and is not cone bounded) is globally stabilizable.

The following notations are used: The Euclidean norm and its induced norm are denoted by $\|\cdot\|$. A function $\boldsymbol{f}(\boldsymbol{x})$ is referred to as C^∞ if its partial derivatives of any order with respect to $\boldsymbol{x} \in \boldsymbol{R}^n$ exist and are continuous. Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$, $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^T}$, is denoted by $D_x \boldsymbol{f}$.

9.2 Problem statement

Consider a nonlinear cascaded system $\boldsymbol{S}_1$ given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \tag{9.1}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{A}\boldsymbol{\eta} + \boldsymbol{B}\boldsymbol{u} \tag{9.2}$$

$$\boldsymbol{\xi} = \boldsymbol{C}\boldsymbol{\eta} \tag{9.3}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ and $\boldsymbol{\eta} \in \boldsymbol{R}^m$ are the state of $\boldsymbol{S}_1$, $\boldsymbol{u} \in \boldsymbol{R}^l$ is the input, and $\boldsymbol{\xi} \in \boldsymbol{R}^l$ is the output of the subsystem given by (9.2) and (9.3). $\boldsymbol{p} \in \boldsymbol{R}^q$ expresses a vector composed of uncertain parameters. $\boldsymbol{f}(\cdot)$:

$\boldsymbol{R}^n \times \boldsymbol{R}^l \rightarrow \boldsymbol{R}^n$ and $\Delta \boldsymbol{f}(\cdot) : \boldsymbol{R}^n \times \boldsymbol{R}^l \times \boldsymbol{R}^q \rightarrow \boldsymbol{R}^n$ represent the nominal and perturbed part of the system, respectively. The system given by (9.1) is referred to as an upper system and the system given by (9.2) and (9.3) as a lower system. The following assumptions are made.

[Assumption 9.1] *A $\boldsymbol{p}$ belongs to a known compact set given by $\boldsymbol{\Omega}$.* ∎

[Assumption 9.2] *Functions $\boldsymbol{f}(\cdot)$ and $\Delta \boldsymbol{f}(\cdot)$ are known, and C^∞ in $\boldsymbol{x}$ and $\boldsymbol{\xi}$ for all $\boldsymbol{p} \in \boldsymbol{\Omega}$. $\Delta \boldsymbol{f}(\cdot)$ is continuous in $\boldsymbol{p}$.* ∎

[Assumption 9.3] *Constant matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are known, and rank$\boldsymbol{B} = l$. For the lower system given by (9.2) and (9.3), there are positive definite matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ that satisfy the following condition.*

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} = -\boldsymbol{Q} \tag{9.4}$$

$$\boldsymbol{B}^T\boldsymbol{P} = \boldsymbol{C} \tag{9.5}$$

∎

The system $\boldsymbol{S}_1$ has almost the same structure as considered in the former researches [15, 18, 19]. However, it includes parametric uncertainty which does not satisfy the matching condition.

Remark 9.1 *In the case that $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is minimal, the condition given by (9.4) and (9.5) of Assumption 9.3 is equivalent to a strictly positive real condition [88]. One can weaken Assumption 9.3 by considering input transformation, such as [67].* ∎

The following term is defined in order to state the stabilization problem.

[Definition 9.1] *A function $\boldsymbol{g}(\cdot) : \boldsymbol{R}^n \times \boldsymbol{R}^q \rightarrow \boldsymbol{R}^n$ and a compact set $\boldsymbol{\Omega} \subset \boldsymbol{R}^q$ are assumed to be given. Then consider a nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{p})$ with a parametric uncertainty $\boldsymbol{p} \in \boldsymbol{\Omega}$. Let the initial state $\boldsymbol{x}(t_0)$ be $\boldsymbol{x}_0$, where $t_0 \in \boldsymbol{R}$ is the initial time. Suppose also that the system has an unique solution. Then if the system has a globally asymptotically stable equilibrium at $\boldsymbol{x} = \boldsymbol{x}_e$ for all $\boldsymbol{p} \in \boldsymbol{\Omega}$ and all $\boldsymbol{x}_0 \in \boldsymbol{R}^n$, the system is said to be globally robustly asymptotically stable (or, simply, GRA stable) at $\boldsymbol{x} = \boldsymbol{x}_e$.* ∎

Then the following problem is considered here.

[Problem 9.1] *For the nonlinear cascaded system $\boldsymbol{S}_1$ which satisfies Assumptions 9.1 to 9.3, find a sufficient condition for GRA stabilization at an equilibrium $(\boldsymbol{x}, \boldsymbol{\eta}) = (\boldsymbol{x}_e, \boldsymbol{\eta}_e)$ via appropriate continuous state feedback.* ∎

Now define terms about stability to be required in the following sections.

[Definition 9.2] *Consider the same system as in Definition 1. For some positive constants a and b, and $\boldsymbol{x}_e \in \boldsymbol{R}^n$, if*

$$\| \boldsymbol{x}(t) - \boldsymbol{x}_e \| \leq a \| \boldsymbol{x}_0 - \boldsymbol{x}_e \| e^{-b(t-t_0)} \quad \forall \boldsymbol{p} \in \boldsymbol{\Omega}, \forall \boldsymbol{x}_0 \in \boldsymbol{R}^n, \forall t (\geq t_0)$$

then the system is said to be globally robustly exponentially stable (or, simply, GRE stable) at $\boldsymbol{x} = \boldsymbol{x}_e$. ∎

[Definition 9.3] *Consider the same system as in Definition 9.1. If there exist certain positive constants a_i $(i = 1, 2, 3)$ and a real-valued function $V(\boldsymbol{x}, \boldsymbol{p})$ that satisfies the following condition for all $\boldsymbol{x} \in \boldsymbol{R}^n$ and all $\boldsymbol{p} \in \boldsymbol{\Omega}$, and that is C^∞ in $\boldsymbol{x}$ and continuous in $\boldsymbol{p}$, then we say the system is globally robustly exponentially stable at $\boldsymbol{x} = \boldsymbol{x}_e$ by Lyapunov Stability Theorem (or LGRE stable).*

$$a_1 \| \boldsymbol{x} \|^2 \leq V(\boldsymbol{x}, \boldsymbol{p}) \leq a_2 \| \boldsymbol{x} \|^2 \tag{9.6}$$

$$D_x V\{\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{p})\} \leq -a_3 \| \boldsymbol{x} \|^2 \tag{9.7}$$

∎

Remark 9.2 *Note that LGRE stability is sufficient for GRE stability, not necessary. Furthermore for the system without uncertainty, we define the globally exponential stability by Lyapunov Stability Theorem (or LGE stability) in a similar way.* ∎

9.3 Sufficient condition for robust stabilization

At first, we derive a robust stabilization condition for systems of the form $\boldsymbol{S}_1$ which satisfy additional requirements. Next, based on the obtained robust stabilization condition, we give a robust stabilization condition for the general system $\boldsymbol{S}_1$.

9.3.1 Special case of the cascaded system

Consider the system that satisfies $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) = (\mathbf{o}, \boldsymbol{I}, \boldsymbol{I})$ in the system $\boldsymbol{S}_1$, that is,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \tag{9.8}$$

$$\dot{\boldsymbol{\xi}} = \boldsymbol{u} \tag{9.9}$$

where $\boldsymbol{I}$ is a unit matrix. This system is referred to as $\boldsymbol{S}_2$. Then the following result is obtained.

[Theorem 9.1] *Assume that any $\boldsymbol{x}_e$ and $\boldsymbol{\xi}_e$ are given. If, for the nonlinear system (9.8) which satisfies Assumptions 9.1 and 9.1, there exists a C^∞ function $\boldsymbol{k}(\cdot): \boldsymbol{R}^n \to \boldsymbol{R}^l$ such that*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{p}) \tag{9.10}$$

is LGRE stable at $\boldsymbol{x} = \boldsymbol{x}_e$, and $\boldsymbol{k}(\boldsymbol{x}_e) = \boldsymbol{\xi}_e$, then the nonlinear system $\boldsymbol{S}_2$ which satisfies Assumptions 9.1 and 9.2 is GRA stabilizable at $(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{x}_e, \boldsymbol{\xi}_e)$ via an appropriate continuous state feedback law $\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{x}, \boldsymbol{\xi})$. In addition, if $\boldsymbol{k}(\cdot)$ satisfies that $\| \boldsymbol{k}(\boldsymbol{x}) - \boldsymbol{k}(\boldsymbol{x}_e) \| < k \| \boldsymbol{x} - \boldsymbol{x}_e \|$ for some positive constant k, then the system $\boldsymbol{S}_2$ is GRE stabilizable at $(\boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{x}_e, \boldsymbol{\xi}_e)$. ∎

Proof: Suppose, without loss of generality, that $\boldsymbol{x}_e = \mathbf{o}$ and $\boldsymbol{\xi}_e = \mathbf{o}$. If we transform the coordinates in the state space of the system $\boldsymbol{S}_2$ by global diffeomorphism

$$\psi\left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\xi} \end{bmatrix}\right) \triangleq \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} \tag{9.11}$$

where $\boldsymbol{s} \triangleq \boldsymbol{\xi} - \boldsymbol{k}(\boldsymbol{x})$, then one gets

$$\boldsymbol{S}_2' \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}} &=& \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \\ \dot{\boldsymbol{s}} &=& \boldsymbol{u} - D_x \boldsymbol{k}\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})\} \end{array} \right.$$

If the system $\boldsymbol{S}_2'$ is GRE stabilizable at $(\boldsymbol{x}, \boldsymbol{s}) = (\mathbf{o}, \mathbf{o})$ via continuous state feedback, then the system $\boldsymbol{S}_2$ is GRA stabilizable at $(\boldsymbol{x}, \boldsymbol{\xi}) = (\mathbf{o}, \mathbf{o})$. So we show that there exist a feedback law $\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{x}, \boldsymbol{\xi})$, a real function $W(\cdot): \boldsymbol{R}^{n+l} \times \boldsymbol{R}^q \to \boldsymbol{R}$ and positive constants α_i $(i = 1, 2, 3)$ that satisfy the following condition for all $\boldsymbol{p} \in \Omega$.

$$\alpha_1 \| \boldsymbol{z} \|^2 \leq W(\boldsymbol{z}, \boldsymbol{p}) \leq \alpha_2 \| \boldsymbol{z} \|^2 \tag{9.12}$$

$$\dot{W}(\boldsymbol{z}, \boldsymbol{p})|_{S_2'} \leq -\alpha_3 \| \boldsymbol{z} \|^2 \tag{9.13}$$

where $\boldsymbol{z} \triangleq [\boldsymbol{x}^T \quad \boldsymbol{s}^T]^T$ and $\dot{W}(\boldsymbol{z}, \boldsymbol{p})|_{S_2'}$ expresses the time derivative of $W(\boldsymbol{z}, \boldsymbol{p})$ along the solution of $\boldsymbol{S}_2'$ with $\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{x}, \boldsymbol{\xi})$. Furthermore by (9.11), (9.12) and (9.13), the additional condition $\| \boldsymbol{k}(\boldsymbol{x}) \| < k \| \boldsymbol{x} \|$ implies that the system $\boldsymbol{S}_2$ is GRE stabilizable at $(\boldsymbol{x}, \boldsymbol{\xi}) = (\mathbf{o}, \mathbf{o})$.

Since the system (9.10) is LGRE stabilizable, there exist a real-valued function $V(\boldsymbol{x}, \boldsymbol{p})$, which is C^∞ in $\boldsymbol{x}$ and continuous in $\boldsymbol{p}$, and positive constants β_i $(i = 1, 2, 3)$ such that, for all $\boldsymbol{p} \in \boldsymbol{\Omega}$ and $\boldsymbol{x} \in \boldsymbol{R}^n$,

$$\beta_1 \parallel \boldsymbol{x} \parallel^2 \leq V(\boldsymbol{x}, \boldsymbol{p}) \leq \beta_2 \parallel \boldsymbol{x} \parallel^2 \tag{9.14}$$

$$D_x V\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{p})\} \leq -\beta_3 \parallel \boldsymbol{x} \parallel^2 \tag{9.15}$$

From Assumption 9.2, there exist $\Delta \boldsymbol{F}_i(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})$ $(i = 1, 2, 3)$ and $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\xi})$ such that

$$\Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) = -\boldsymbol{f}(\mathbf{o}, \mathbf{o}) + \Delta \boldsymbol{F}_1 \boldsymbol{x} + \Delta \boldsymbol{F}_2 \boldsymbol{s} \tag{9.16}$$

$$\Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) = \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{p}) + \Delta \boldsymbol{F}_3 \boldsymbol{s} \tag{9.17}$$

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) + \boldsymbol{F}\boldsymbol{s} \tag{9.18}$$

where $\Delta \boldsymbol{F}_i$ are C^∞ in $\boldsymbol{x}$ and $\boldsymbol{\xi}$, and continuous in $\boldsymbol{p}$, and $\boldsymbol{F}$ is C^∞ in $\boldsymbol{x}$ and $\boldsymbol{\xi}$.

Based on the above preparation, we show that there exist a feedback law $\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{x}, \boldsymbol{\xi})$ and a real function $W(\boldsymbol{z}, \boldsymbol{p}) \triangleq V(\boldsymbol{x}, \boldsymbol{p}) + \boldsymbol{s}^T \boldsymbol{s}/2$ which satisfies (9.12) and (9.13).

It is obvious that the function $W(\boldsymbol{z}, \boldsymbol{p})$ satisfies (9.12). Consider (9.13). There exist continuous functions $\phi_i(\cdot) : \boldsymbol{R}^n \times \boldsymbol{R}^l \to \boldsymbol{R}$ $(i = 1, 2, 3)$ such that, for all $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{\xi} \in \boldsymbol{R}^m$, and $\boldsymbol{p} \in \boldsymbol{\Omega}$,

$$\parallel \Delta \boldsymbol{F}_i(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \parallel \leq \phi_i(\boldsymbol{x}, \boldsymbol{\xi}) \qquad i = 1, 2 \tag{9.19}$$

$$\parallel \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{F}_3(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \parallel \leq \phi_3(\boldsymbol{x}, \boldsymbol{\xi}) \tag{9.20}$$

because of Assumption 9.1 and the continuity of $\Delta \boldsymbol{F}_i$ in $\boldsymbol{p}$. Furthermore from (9.14) and (9.15), it is obtained that $D_x V^T(\mathbf{o}, \boldsymbol{p}) = \mathbf{o}$, and that there exists a continuous function $\phi_4(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ such that

$$\parallel D_x V^T(\boldsymbol{x}, \boldsymbol{p}) \parallel \leq \phi_4(\boldsymbol{x}) \parallel \boldsymbol{x} \parallel \qquad \forall \boldsymbol{p} \in \boldsymbol{\Omega},\ \forall \boldsymbol{x} \in \boldsymbol{R}^n \tag{9.21}$$

Then consider the following control law

$$\boldsymbol{u} = D_x \boldsymbol{k}\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) - \boldsymbol{f}(\mathbf{o}, \mathbf{o})\} - g(\boldsymbol{x}, \boldsymbol{\xi})\boldsymbol{s} \tag{9.22}$$

where $g(\cdot) : \boldsymbol{R}^n \times \boldsymbol{R}^l \to \boldsymbol{R}$ is given by, for a positive constant $\varepsilon (< \beta_3)$.

$$g = \frac{(\parallel D_x \boldsymbol{k} \parallel \phi_1 + \phi_4 \phi_3)^2}{4(\beta_3 - \varepsilon)} + \parallel D_x \boldsymbol{k} \parallel \phi_2 + \varepsilon \tag{9.23}$$

Now differentiating W along the system $\boldsymbol{S}_2'$, one can get, by using (9.16), (9.17), and (9.18),

$$\begin{aligned} \dot{W} &= D_x V\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{p})\} \\ &+ D_x V\{\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{F}_3(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})\}\boldsymbol{s} \\ &+ \boldsymbol{s}^T \boldsymbol{u} \\ &- \boldsymbol{s}^T D_x \boldsymbol{k}\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) - \boldsymbol{f}(\mathbf{o}, \mathbf{o})\} \\ &- \boldsymbol{s}^T D_x \boldsymbol{k}\{\Delta \boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})\boldsymbol{x} + \Delta \boldsymbol{F}_2(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})\boldsymbol{s}\} \end{aligned} \tag{9.24}$$

Then by using (9.22) and completing the squares with respect to $\boldsymbol{x}$ and $\boldsymbol{s}$, one gets

$$\begin{aligned}\dot{W} &\leq -\varepsilon \parallel \boldsymbol{z} \parallel^2 \\ &- (\beta_3-\varepsilon)\{\parallel \boldsymbol{x} \parallel - \frac{\parallel D_x\boldsymbol{k} \parallel \phi_1 + \phi_4\phi_3}{2(\beta_3-\varepsilon)} \parallel \boldsymbol{s} \parallel\}^2 \\ &+ \{\parallel D_x\boldsymbol{k} \parallel \phi_2 + \varepsilon + \frac{(\parallel D_x\boldsymbol{k} \parallel \phi_1 + \phi_4\phi_3)^2}{4(\beta_3-\varepsilon)} - g\} \parallel \boldsymbol{s} \parallel^2\end{aligned} \tag{9.25}$$

Therefore, (9.23) and (9.25) imply (9.13) if $\alpha_3 = \varepsilon$. This completes the proof. ∎

This theorem shows that the cascaded system $\boldsymbol{S}_2$ is GRA stabilizable via continuous state feedback, if the upper system (9.8) is LGRE stabilizable via C^∞ state feedback in the case that $\boldsymbol{\xi}$ in (9.8) is regarded as the input. We explain the main difference from the former methods by using (9.24) in the proof. If there is no uncertainty in (9.24) as in the former researches, then one can straightforwardly cancel the terms on the right hand side of (9.24) except for the first term by an appropriate $\boldsymbol{u}$, and make $\dot{W}$ negative definite. However in the presence of the uncertainty, these terms cannot be directly canceled. This implies that the former methods cannot be applied immediately to the problem in the presence of the uncertainty. One point of our method is the combination of (a) completing the squares with respect to $\boldsymbol{x}$ and $\boldsymbol{s}$ and (b) nonlinear high gain feedback, which makes $\dot{W}$ negative definite.

9.3.2 General case of the cascaded system

The following result is obtained by using Theorem 9.1.

[Theorem 9.2] *Assume that any $\boldsymbol{x}_e$ and $\boldsymbol{\xi}_e$ are given. If, for the nonlinear system (9.1) which satisfies Assumptions 9.1 and 9.2, there exists a C^∞ function $\boldsymbol{k}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^l$ such that*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x}), \boldsymbol{p}) \tag{9.26}$$

is LGRE stable at $\boldsymbol{x} = \boldsymbol{x}_e$, and $\boldsymbol{k}(\boldsymbol{x}_e) = \boldsymbol{\xi}_e$, then the nonlinear system $\boldsymbol{S}_1$ which satisfies Assumptions 9.1 to 9.3 is GRA stabilizable at $(\boldsymbol{x}, \boldsymbol{\eta}) = (\boldsymbol{x}_e, \boldsymbol{\eta}_e)$, where $\boldsymbol{\eta}_e \triangleq \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{\xi}_e$, via an appropriate continuous state feedback law $\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{x}, \boldsymbol{\eta})$. ∎

Proof: Suppose, without loss of generality, that $\boldsymbol{x}_e = \mathbf{o}$ and $\boldsymbol{\xi}_e = \mathbf{o}$ (automatically $\boldsymbol{\eta}_e = \mathbf{o}$). Since $\boldsymbol{CB}$ is nonsingular from Assumption 9.3, there exists a matrix $\widetilde{\boldsymbol{C}}$ such that $\begin{bmatrix} \boldsymbol{C} \\ \widetilde{\boldsymbol{C}} \end{bmatrix}$ is nonsingular and $\widetilde{\boldsymbol{C}}\boldsymbol{B} = \mathbf{o}$. So transforming the coordinates of the system $\boldsymbol{S}_1$ by

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{C} \\ \widetilde{\boldsymbol{C}} \end{bmatrix} \eta - \begin{bmatrix} \boldsymbol{k}(\boldsymbol{x}) \\ \mathbf{o} \end{bmatrix} \tag{9.27}$$

one can obtain the following system, which is referred to as $\boldsymbol{S}_1^{'}$.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p}) \tag{9.28}$$

$$\dot{s}_1 = \boldsymbol{C}\boldsymbol{A}\eta + \boldsymbol{C}\boldsymbol{B}\boldsymbol{u} - D_x\boldsymbol{k}\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}) + \Delta \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{p})\} \tag{9.29}$$

$$\dot{s}_2 = \widetilde{\boldsymbol{C}}\boldsymbol{A}\eta \tag{9.30}$$

Thus we show that the system $\boldsymbol{S}_1^{'}$ is GRA stabilizable at $(\boldsymbol{x}, s_1, s_2) = (\mathbf{o}, \mathbf{o}, \mathbf{o})$ via continuous state feedback. The proof is completed in the following three steps.

Step 1: One can show the system given by (9.28) and (9.29) is globally robustly stabilized at $(\boldsymbol{x}, s_1) = (\mathbf{o}, \mathbf{o})$ by an appropriate control law. In fact, this is reduced to the stabilization problem considered in Theorem 9.1, by appropriate input transformation: if we consider $\boldsymbol{u} = (\boldsymbol{CB})^{-1}(\tilde{\boldsymbol{u}} - \boldsymbol{CA}\eta)$, where $\tilde{\boldsymbol{u}}$ is a new input, then there exists a GRA stabilizing control law for a new system by input transformation, by the same way as Theorem 9.1.

Step 2: Here consider the subsystem (9.30). By (9.27), (9.30), and

$$[\boldsymbol{D}_1\boldsymbol{D}_2] \triangleq \begin{bmatrix} \boldsymbol{C} \\ \widetilde{\boldsymbol{C}} \end{bmatrix}^{-1}$$

one gets

$$\dot{s}_2 = \widetilde{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{D}_2 s_2 + \widetilde{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{D}_1\{s_1 + \boldsymbol{k}(\boldsymbol{x})\} \tag{9.31}$$

Now consider a positive definite function $V_1 = \boldsymbol{\eta}^T \boldsymbol{P} \boldsymbol{\eta}$, using a positive definite matrix $\boldsymbol{P}$ in Assumption 9.3. Differentiating V_1 along the system (9.2) with any input that $\boldsymbol{x}(t) \equiv \mathbf{o}$ and $s_1(t) \equiv \mathbf{o}$, one obtains that $\dot{V}_1 = -\eta^T \boldsymbol{Q} \eta$ by noting $\boldsymbol{C}\eta = \boldsymbol{B}^T \boldsymbol{P} \eta = \mathbf{o}$, and so $\eta \to \mathbf{o}$ as $t \to \infty$. This means $s_2 \to \mathbf{o}$ by (9.27), when $\boldsymbol{x}(t) \equiv \mathbf{o}$ and $s_1(t) \equiv \mathbf{o}$ in (9.31). Consequently, $\widetilde{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{D}_2$ in (9.31) is asymptotically stable, and there exists a positive definite matrix $\widetilde{\boldsymbol{P}}$ such that

$$\widetilde{\boldsymbol{P}}(\widetilde{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{D}_2) + (\widetilde{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{D}_2)^T \widetilde{\boldsymbol{P}} = -\boldsymbol{I} \tag{9.32}$$

Step 3: Finally, the GRA stability of the total system given by (9.28), (9.29), and (9.31) with the control law derived in Step 1 can be proved using the result shown by Sontag [116] (See Appendix 9.5). Now let

$$\boldsymbol{x}_1 \triangleq \begin{bmatrix} \boldsymbol{x} \\ s_1 \end{bmatrix}, \qquad \boldsymbol{x}_2 \triangleq s_2$$

and suppose, using $\widetilde{\boldsymbol{P}}$ in (9.32)

$$V_2 = \boldsymbol{x}_2^T \widetilde{\boldsymbol{P}} \boldsymbol{x}_2$$

From this, one can show that, by differentiating V_2 along (9.31), for each positive constant c_1, there exists some positive constant c_2 such that

$$\begin{aligned} \dot{V}_2 \ \leq \ & - \parallel \boldsymbol{x}_2 \parallel (\parallel \boldsymbol{x}_2 \parallel - c_2) \leq 0 \\ & \forall \parallel \boldsymbol{x}_1 \parallel \leq c_1, \quad \forall \parallel \boldsymbol{x}_2 \parallel \geq c_2 \end{aligned} \tag{9.33}$$

Hence by Lemma in Appendix 9.5, (9.33) in addition to the facts of Steps 1 and 2 implies that the closed loop system given by (9.28), (9.29), and (9.31) with the control law derived in Step 1 is GRA stable at $(\boldsymbol{x}, s_1, s_2) = (\mathbf{o}, \mathbf{o}, \mathbf{o})$. This completes the proof. ∎

Theorem 9.2 is an extension of the former results [15, 19] in the sense that the uncertainty of the system is considered. Furthermore compared to the former robust control methods in the presence of the matching condition [37, 28, 2], Theorem 9.2 shows that it is possible that the system with the mismatched uncertainty is globally and robustly stabilized, if the input acts on the uncertainty through a strictly positive real linear system.

Remark 9.3 *In the field of adaptive control, there also are some researches (see, e.g., [60, 61]) as an extension of global stabilization of nonlinear cascaded systems. However the adaptive control methods assume that the parametric uncertainty has a special form, and for example, can not treat an uncertainty such as* $\cos(px)$, *where* p *is an unknown parameter and* x *is the state. More detailed comparison to the adaptive control will be an interesting topic in a future research.* ∎

Remark 9.4 *We indeed consider a continuous robust stabilizing controller here, but our approach can treat the case of a smooth controller: it can be obtained by simple modification of the obtained continuous controller.* ∎

Remark 9.5 *The local robust exponential stabilization problem is trivial, if we do not request to specify the bound of the stability region.*

This stabilization can be achieved by a robust stabilizing controller for the linearization of the original nonlinear system. However, the approach proposed here can be straightforwardly extended to the local case (rigorously, semi-global case) where the robust exponential stability is achieved on any specified compact set. ∎

Theorem 9.2 also gives the following result for the system $\boldsymbol{S}_1$ without the uncertainty.

[Corollary 9.1] *Assume that any* $\boldsymbol{x}_e$ *and* $\boldsymbol{\xi}_e$ *are given. If, for the nonlinear system (9.1) which satisfies* $\Delta \boldsymbol{f}(\cdot) \equiv \mathbf{o}$ *and Assumption 9.2, there exists a* C^∞ *function* $\boldsymbol{k}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^l$ *such that*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(\boldsymbol{x})) \tag{9.34}$$

has a globally asymptotically stable equilibrium at $\boldsymbol{x} = \boldsymbol{x}_e$ *and* $\boldsymbol{k}(\boldsymbol{x}_e) = \boldsymbol{\xi}_e$*, then the nonlinear cascaded system* $\boldsymbol{S}_1$ *which satisfies* $\Delta \boldsymbol{f}(\cdot) \equiv \mathbf{o}$*, and Assumptions 9.2 and 9.3 is globally asymptotically stabilizable at* $(\boldsymbol{x}, \boldsymbol{\eta}) = (\boldsymbol{x}_e, \boldsymbol{\eta}_e)$*, where* $\boldsymbol{\eta}_e \triangleq \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{\xi}_e$*, via* C^∞ *state feedback.* ∎

Proof: Noting the fact that if the system given by (9.34) is globally asymptotically stable, there exists some Lyapunov function by the Converse Lyapunov Theorem, set $\Delta \boldsymbol{f}(\cdot) \equiv \mathbf{o}$ in the proof of Theorem 9.2. ∎

Tsinias [123] and Byrnes [15] have shown that the system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi})\\ \dot{\boldsymbol{\xi}} &= \boldsymbol{u}\end{aligned}$$

is globally asymptotically stabilizable if the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi})$ is globally asymptotically stabilizable when the input is $\boldsymbol{\xi}$. On the other hand, Kokotovic et al. [67, 105] have considered the stabilization problem of the system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\xi})\\ \dot{\boldsymbol{\eta}} &= \boldsymbol{A}\boldsymbol{\eta} + \boldsymbol{B}\boldsymbol{u}\\ \boldsymbol{\xi} &= \boldsymbol{C}\boldsymbol{\eta}\end{aligned} \tag{9.35}$$

as a more general form, and shown that the system is globally asymptotically stabilizable, if (i) $\boldsymbol{A}$ is stable and $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is positive real, and (ii) the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mathbf{o})$ is globally asymptotically stable at the origin. Moreover, the condition (i) is strengthened in [91, 19]. However when the condition (ii) is replaced by the condition (ii') that the system $\dot{x} = f(x, \xi)$ is globally asymptotically stabilizable provided the input is ξ, it is difficult to extend their approach straightforwardly, as you can easily see from their proof in [67]. Corollary 9.1 shows that, if the condition (i) is replaced by Assumption 9.3 which is stronger than (i), the system given by (9.35) is globally asymptotically stabilizable under the condition (ii'). Therefore Corollary 9.1 is some extension of the result by Kokotovic et. al. This success is based on the factorization of the system such as (9.28) to (9.30) and the full exploitation of the result by Sontag [116]. However it should be noted that our result cannot be straightforwardly extended to stabilization problem of the nonlinear cascaded system whose lower system is nonlinear, which is treated in [91, 19].

9.4 Robust stabilization for a certain class of nonlinear cascaded systems

In Theorem 9.2, we have derived the stabilization condition that there exists a C^∞ function $k(x)$ such that the system (9.26) is LGRE stable at $x = x_e$. When does such a function $k(\cdot)$ exist? So in this section, we discuss this problem for a certain class of nonlinear systems.

Consider the following system whose input satisfies a matching condition for uncertainty.

$$\dot{x} = \hat{f}(x) + \widehat{G}(x)\{\xi + \Delta\hat{f}(x, p)\} \tag{9.36}$$

where $x \in R^n$ and $p \in R^q$ are defined in the same way as in the system S_1 of section 9.2, and $\xi \in R^l$ is regarded as a input. The following assumptions are made.

[Assumption 9.4] $\hat{f}(\cdot) : R^n \to R^n$, $\Delta\hat{f}(\cdot) : R^n \times R^q \to R^l$, *and* $\widehat{G}(\cdot) : R^n \to R^n \times R^l$ *are known functions, which are* C^∞ *in* x *for all* $p \in \Omega$. $\Delta\hat{f}$ *is continuous in* p. *Further* $\hat{f}(x_e) = o$. ∎

[Assumption 9.5] *rank*$\{\widehat{G}(x)\} = l$, *for all* $x \in R^n$. ∎

Then the following result is obtained.

[Theorem 9.3] *A nonlinear system (9.36) which satisfies Assumptions 9.1, 9.4, and 9.5 is LGRE stabilizable at* $x = x_e$ *by an appropriate* C^∞ *state feedback control law* $\xi = \hat{\tau}(x)$ *with* $\hat{\tau}(x_e) = \xi_e$,

if and only if the following conditions are satisfied.
(i) There exists a C^∞ *function* $\widehat{\boldsymbol{k}}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^l$ *such that a system*

$$\dot{\boldsymbol{x}} = \widehat{\boldsymbol{f}}(\boldsymbol{x}) + \widehat{\boldsymbol{G}}(\boldsymbol{x})\widehat{\boldsymbol{k}}(\boldsymbol{x}) \tag{9.37}$$

is LGE stable at $\boldsymbol{x} = \boldsymbol{x}_e$.
(ii) $\Delta\widehat{\boldsymbol{f}}(\boldsymbol{x}_e, \boldsymbol{p}) + \boldsymbol{\xi}_e = \mathbf{o}, \quad \forall \boldsymbol{p} \in \boldsymbol{\Omega}$. ∎

Proof: Assume, without loss of generality, that $\boldsymbol{x}_e = \mathbf{o}$ and $\boldsymbol{\xi}_e = \mathbf{o}$. First, the necessity is proved. Now one of the elements that belong to the known set $\boldsymbol{\Omega}$ is denoted by $\boldsymbol{p}^*$. Since the closed loop system by (9.36) and $\boldsymbol{\xi} = \widehat{\boldsymbol{\tau}}(\boldsymbol{x})$ is LGRE stable, the closed loop system with $\boldsymbol{p} = \boldsymbol{p}^*$ is also LGE stable. So the condition (i) is obtained by

$$\widehat{\boldsymbol{k}}(\boldsymbol{x}) = \widehat{\boldsymbol{\tau}}(\boldsymbol{x}) + \Delta\widehat{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{p}^*) \tag{9.38}$$

because $\widehat{\boldsymbol{\tau}}$ and $\Delta\widehat{\boldsymbol{f}}$ are C^∞. Further Assumptions 9.4 and 9.5 and $\widehat{\boldsymbol{\tau}}(\mathbf{o}) = \mathbf{o}$ obviously mean the condition (ii).

Second, the sufficiency is proved. The condition (i) implies that there exist a C^∞ function $\widehat{V}$ and positive constants $\widehat{\alpha}_i$ $(i = 1, 2, 3)$ such that

$$\widehat{\alpha}_1 \parallel \boldsymbol{x} \parallel^2 \leq \widehat{V}(\boldsymbol{x}) \;\leq\; \widehat{\alpha}_2 \parallel \boldsymbol{x} \parallel^2 \tag{9.39}$$

$$D_x\widehat{V}\{\widehat{\boldsymbol{f}}(\boldsymbol{x}) + \widehat{\boldsymbol{G}}(\boldsymbol{x})\widehat{\boldsymbol{k}}(\boldsymbol{x})\} \;\leq\; -\widehat{\alpha}_3 \parallel \boldsymbol{x} \parallel^2 \tag{9.40}$$

Assumption 9.4 and the condition (ii) also imply that there exists a C^∞ function $\Delta\widehat{\boldsymbol{F}}$ such that

$$\Delta\widehat{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{p}) = \Delta\widehat{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{p})\boldsymbol{x} \tag{9.41}$$

Note that $\Delta\widehat{\boldsymbol{F}}$ is continuous in $\boldsymbol{p}$ from Assumption 9.4.

Now differentiating $\widehat{V}$ along the system (9.36), one gets

$$\dot{\widehat{V}} = D_x\widehat{V}\{\widehat{\boldsymbol{f}} + \widehat{\boldsymbol{G}}(\boldsymbol{\xi} + \Delta\widehat{\boldsymbol{F}}\boldsymbol{x})\} \tag{9.42}$$

Then consider the following control law:

$$\boldsymbol{\xi} = \widehat{\boldsymbol{k}}(\boldsymbol{x}) - \widehat{g}(\boldsymbol{x})\boldsymbol{v} \tag{9.43}$$

where $\boldsymbol{v} \triangleq \widehat{\boldsymbol{G}}^T D_x\widehat{V}^T$, $\widehat{g}(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}$ is a C^∞ function which satisfies, for a positive constant $\widehat{\varepsilon}(< \widehat{\alpha}_3)$,

$$\widehat{g}(\boldsymbol{x}) \geq \frac{\parallel \Delta\widehat{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{p}) \parallel^2}{4(\widehat{\alpha}_3 - \widehat{\varepsilon})} \quad \forall \boldsymbol{x} \in \boldsymbol{R}^n, \quad \forall \boldsymbol{p} \in \boldsymbol{\Omega} \tag{9.44}$$

The control law given by (9.43) is composed of C^∞ functions, because $\widehat{\boldsymbol{k}}$, $\widehat{\boldsymbol{G}}$, $\widehat{V}$, and $\widehat{g}$ is C^∞. Note also that the control law (9.43) satisfies $\widehat{\tau}(\mathbf{o}) = \mathbf{o}$, since $\widehat{\boldsymbol{k}}(\mathbf{o}) = \mathbf{o}$ and $D_x\widehat{V}^T(\mathbf{o}) = \mathbf{o}$.

Substituting (9.40) and (9.43) into (9.42) and completing the squares, one can get

$$\dot{\hat{V}} \leq -\hat{\varepsilon} \| \boldsymbol{x} \|^2 \quad \forall \boldsymbol{x} \in \boldsymbol{R}^n \tag{9.45}$$

Eqs. (9.39) and (9.45) mean that the system (9.36) is LGRE stabilizable. This completes the proof. ∎

In Theorem 9.3, the stabilization condition is much simpler, because the condition about the uncertain part and the condition about the known part are independently derived. For example, if the known part is a linear system, that is, $\widehat{\boldsymbol{f}}(\boldsymbol{x}) = \widehat{\boldsymbol{A}}\boldsymbol{x}$ and $\widehat{\boldsymbol{G}}(\boldsymbol{x}) = \widehat{\boldsymbol{B}}$, where $\widehat{\boldsymbol{A}}$ and $\widehat{\boldsymbol{B}}$ are constant matrices, the condition (i) of Theorem 9.3 is equivalent to the condition that $(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{B}})$ is stabilizable. Moreover it is important that Theorem 9.3 gives, where of course the matching condition is satisfied, the condition for the robust asymptotic stabilization via C^∞ state feedback, because the conventional researches have developed robust asymptotic stabilization via discontinuous feedback [37] or practical stabilization via continuous feedback [28, 2].

Now the combination of Theorems 9.2 and 9.3 gives the following result directly.

[Corollary 9.2] *Suppose that the following cascaded system $\boldsymbol{S}_3$ consists of the upper system (9.36) with Assumptions 9.1, 9.4, and 9.5, and the lower system (9.2) and (9.3) with Assumption 9.3.*

$$\dot{\boldsymbol{x}} = \widehat{\boldsymbol{f}}(\boldsymbol{x}) + \widehat{\boldsymbol{G}}(\boldsymbol{x})\{\boldsymbol{\xi} + \Delta\widehat{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{p})\} \tag{9.36}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{A}\boldsymbol{\eta} + \boldsymbol{B}\boldsymbol{u} \tag{9.2}$$

$$\boldsymbol{\xi} = \boldsymbol{C}\boldsymbol{\eta} \tag{9.3}$$

Then the system $\boldsymbol{S}_3$ is GRA stabilizable at $(\boldsymbol{x}, \boldsymbol{\eta}) = (\boldsymbol{x}_e, \boldsymbol{\eta}_e)$, where $\boldsymbol{\eta}_e \triangleq \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{\xi}_e$, via continuous state feedback, if the conditions (i) and (ii) in Theorem 9.3 are satisfied. ∎

Corollary 9.2 clarifies that, if the input acts on the uncertainty through a strictly positive real linear system and the uncertainty satisfies the condition (ii), the system is globally robust stabilizable under the condition (i). Note that Corollary 9.2 permits a broader class of uncertainty than the cone-bounded case, while the conventional results for global robust stabilization [89, 140, 25, 106] are concerned only with the case of the cone-bounded uncertainty. Furthermore, the stabilization condition in Corollary 9.2 is more practical than that in Theorem 9.2.

9.5 Conclusion

The main results obtained in this chapter are summarized as follows.

(i) A sufficient condition has been given for global robust stabilization of a class of nonlinear cascaded systems with uncertainty, which does not necessarily satisfy the matching condition. The obtained result is an extension of the conventional researches about global stabilization of nonlinear cascade systems without uncertainty, in the sense that system uncertainty is considered. The obtained result also clarifies that the system that includes the uncertainty without the matching condition is globally stabilizable, if the input acts on the uncertainty through a strictly positive real linear system.

(ii) A sufficient condition for global stabilization of a class of nonlinear cascaded system without uncertainty has been derived, which is stronger than the previous existing results and can be applied to a more large class of nonlinear systems.

(iii) For a specified class of the systems, a more practical condition for global robust stabilization has been derived.

Appendix

Lemma by Sontag: For the readers' convenience, we show the result obtained by Sontag [116]

[Lemma] Consider a nonlinear cascaded system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1) \tag{9.A1}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{9.A2}$$

where $\boldsymbol{x}_1 \in \boldsymbol{R}^{n_1}$ and $\boldsymbol{x}_2 \in \boldsymbol{R}^{n_2}$ are state vectors. Then this nonlinear cascaded system has a globally asymptotically stable equilibrium at $(\boldsymbol{x}_1, \boldsymbol{x}_2) = (\mathbf{o}, \mathbf{o})$, if the following three conditions are satisfied.

(i) A subsystem given by (9.A1) has a globally asymptotically stable equilibrium at $\boldsymbol{x}_1 = \mathbf{o}$.

(ii) A subsystem $\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\mathbf{o}, \boldsymbol{x}_2)$ has a globally asymptotically stable equilibrium at $\boldsymbol{x}_2 = \mathbf{o}$.

(iii) For each positive constant c_1, there exists a positive constant c_2, and a positive definite and radically unbounded function $V(\cdot) : \boldsymbol{R}^{n_2} \rightarrow \boldsymbol{R}$ such that

$$\begin{aligned} \dot{V}(\boldsymbol{x}_2) &= \frac{\partial V}{\partial \boldsymbol{x}_2^T} \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \leq 0 \\ & \text{for } \ ^\forall \parallel \boldsymbol{x}_1 \parallel \leq c_1, \quad ^\forall \parallel \boldsymbol{x}_2 \parallel \geq c_2 \end{aligned} \tag{9.A3}$$

Chapter 10

CONCLUDING REMARKS

In this paper, robust control problems of nonlinear systems including robot manipulators have been investigated from the viewpoint of Lyapunov-based approach and H_∞-type approach. In chapters 2 to 6, some new results on the robust trajectory control of robot manipulators have been given, from the practical viewpoint, based on the Lyapunov-based approach. In chapters 7 to 9, fundamental problems on nonlinear H_∞ control theory and (global) robust stabilization have been discussed and some new and useful results have been derived.

In chapter 2, a new robust trajectory control scheme of robot manipulators with uncertainty has been proposed, which is almost as simple as that of the dynamic control method, and has a less conservative evaluation in determining the feedback gain, fully exploiting the effective expression of the dynamics of the robot manipulator. Based on the above robust control, a new adaptive robust trajectory control scheme of robot manipulators with uncertainty has been proposed, in addition to the above merits, where the tracking precision is explicitly specified and, as a result, it is possible to evaluate if the feedback gain is small enough for the specified tracking precision. By an experiment of the trajectory control of a 2 link DD arm, it has been verified that the feedback gain of the adaptive robust control method is much smaller than that of the robust control method, and is almost necessary and minimum for the specified tracking precision.

In chapter 3, it has first been pointed out that the conventional acceleration feedback system compensates for the uncertainty by high gain feedback essentially, and the use of acceleration feedback gain matrix which is diagonal reduces a multivariable control problem to a decoupled control problem. Second, the disadvantages of the conventional acceleration feedback methods have been clarified. Finally, a robust tracking control methods using acceleration information for a robot manipulator with uncertainties has been proposed, where the acceleration information is fully exploited and the disadvantages of the conventional control methods are overcome.

In chapter 4, a robust control problem of robot manipulators where joint torque sensor information is available has been discussed. First, a dynamic equation of the manipulator with joint torque sensors has been derived, which expresses explicitly the multivariable structure. As a result, the proposed dynamic equation clarifies that the robust control system of the manipulator with joint torque sensors can be designed as in the same way as the case of the manipulator without joint torque sensors. It has also been shown that the proposed dynamic equation is effective for the design of the robust control system against the uncertainty of the motor system. The proposed robust control method achieves the specified tracking precision in the presence of the modeling error.

In chapter 5, the relation between a feedback gain and a control error for a given sampling period has been clarified in the digital control of robot manipulators, by deriving some kind of discrete-time description of nonlinear systems. Based on the above analysis, a new digital robust control scheme of robot manipulators has been proposed, which gives a systematic design procedure to find a feedback gain so as to achieve the specified tracking precision for a given sampling period. A weighting function for a feedback gain has also been proposed in order to decrease the chattering.

In chapter 6, a hierarchical robust control method of robot manipulators has been proposed. A hierarchical control system makes the sampling period to generate a robust compensator much smaller than that of the non-hierarchical case. By assuming a control signal in the lower level loop is continuous on time, the effect of the uncertainty on the control error is theoretically analyzed. In addition, the part which cannot be linearized due to the computation time lag is theoretically compensated by the robust controller.

In chapter 7, a new approach for nonlinear $\boldsymbol{H}_\infty$ control theory has been given, which does not depend on the Linearization or the linear $\boldsymbol{H}_\infty$ control techniques. First, some strict bounded real conditions of nonlinear systems have been characterized via two approaches: One is based on the Hamilton-Jacobi equation with a stabilizing solution and another is based on the Hamilton-Jacobi strict inequality. The former has an important role to analyze the internal stability of nonlinear systems, and the latter has an advantage that it can simply be applied to the strict $\boldsymbol{H}_\infty$ control problem. Both will form a useful foundation to develop the strict $\boldsymbol{H}_\infty$ control theory of nonlinear systems. The obtained results completely include the strict bounded real lemma of linear systems, and are also stronger and applicable to more general nonlinear systems, compared with the former results. Based on the above results, several sufficient (and necessary) conditions for the solvability of the strict $\boldsymbol{H}_\infty$ state feedback control problem have been derived. In addition, a necessary condition for the solvability of the strict $\boldsymbol{H}_\infty$ output feedback control problem has been given and it has also been shown that the obtained necessary condition is sufficient in the case of exponential stability.

In chapter 8, a robust stability condition has been given for nonlinear systems with unstructured uncertainty. Furthermore, a robust stabilizability condition has been derived in terms of the solvability of some partial differential equation and a robust stabilizing controller has been given, which is based on the nonlinear $\boldsymbol{H}_\infty$ state feedback control theory developed in chapter 7. The obtained approach in the robust stabilization problem with unstructured uncertainty allows us to treat various types of stability, i.e. asymptotic stability, exponential stability, and global exponential stability, in a unified way.

In chapter 9, a sufficient condition has been given for global robust stabilization of a class of nonlinear cascaded systems with uncertainty, which does not necessarily satisfy the matching condition. The obtained result is an extension of the conventional researches about global stabilization of nonlinear cascaded systems without uncertainty, in the sense that system uncertainty is considered. The obtained result also clarifies that the system that includes the uncertainty without the matching condition is globally stabilizable, if the input acts on the uncertainty through a strictly positive real linear system. A sufficient condition for global stabilization of a class of nonlinear cascaded system without uncertainty has been derived, which is stronger than the previous existing results and can be applied to a more large class of nonlinear systems. For a specified class of the systems, a more practical condition for global robust stabilization has been derived.

Robust control of nonlinear systems is one of attractive and important control problems. However there are many open problems in this field. Most of the results obtained in this paper are for basic problems in this field. We will need to work more deeply and widely to establish a systematic robust control design method of nonlinear systems. The author believes that the results obtained here will contribute to the development of this field.

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