STUDIES ON CONTROL PROBLEMS FOR STOCHASTIC SYSTEMS

BY

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ABSTRACT

The problems of controllability and optimal control for a wide class of nonlinear stochastic lumped parameter systems and linear distributed parameter systems with stochastic coefficients are studied in the framework of stochastic calculas and functional analysis. The purpose of this dissertation is to give mathematical developments for the theory of controllability and optimal control.

This dissertation is divided into two major parts. Part 1 is devoted to the theory of stochastic controllability and the methods of generating the practical control signal for nonlinear stochastic lumped parameter systems described by the Itô-stochastic differential equation, and Part 2 to give the new mathematical models of distributed parameter systems with stochastic coefficients and to study the optimal control and controllability problems.

In Part 1, defining the stochastic controllability, based on notions from stochastic Lyapunov stability theory, sufficient conditions and/or necessary and sufficient conditions are studied. Part 1 is divided into three categories: the first is to establish the new definitions of stochastic controllability, the second to derive the new conditions for the stochastic controllability by using the Lyapunov function like approach, and the third concerned with finding the feasible method of generating the practical control signal which transfers the system state to the neighborhood of a given point for stochastic nonlinear lumped parameter systems within the preassigned time interval.

In Part 2, establishing the mathematical models of distributed

parameter systems with stochastic coefficients with the aid of the theory of functional analysis, the optimal distributed control, optimal boundary control and controllability problems are solved for various types of stochastic distributed parameter systems. Part 2 is divided mainly into four categories: the first is concerned with the mathematical aspects of the newly established system models, the second the optimal distributed control problem, the third the optimal boundary control one and the final the stochastic controllability for stochastic distributed parameter systems with stochastic coefficients.

Throughout Parts 1 and 2, various kinds of numerical computations are performed in order to show the feasible computer implementation.

GENERAL NOTATION

According to many published researches in the field of the stochastic systems, the most familiar symbolic conventions are used. In Part 1, we use the standard notations, referring to the well-known stochastic system theory. On the other hand, symbols used in Part 2 are basically due to those appearing in the Lions' excellent book "*Optimal Control of Systems Governed by Partial Differential Equations*". Consequently, it shoud be noted that there happen many cases in Parts 1 and 2 where the same symbol has different meanings. For example, in spite of the fact that the symbol T expresses the time interval $[t_0, t_f]$ in Part 1, this is turned into the open time interval $]t_0, t_f[$ in Part 2. Furthermore, the symbol x is the n-diminsional state vector in Part 1 but, in Part 2, this denotes the spatial variable. The symbol u(t) denotes the m-dimensional control vector in Part 1 and the infinite dimensional state variable in Part 2, respectively. In order to avoid confusion, the mathematical preliminaries in Parts 1 and 2 involve respectively clear definitions of all the symbols so that the reader can carefully follow the symbolic conventions in Parts 1 and 2.

In Chapter n, Section n.m,the theorems, conditions and hypotheses are indexed by n.m.k. When we refer to such a Condition (Theorem, Hypothesis)-n.m.k, we sometimes designate it by C (T,H)-n.m.k.

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General Introduction

One of the most challenging areas in the field of modern system theory is the optimal state estimation and control taking into account the fact that the system exhibits various kinds of nonlinearities and operates in the random environments. From this point of view, it is the opinion of great majority of researchers in the field that the following two important aspects in the dynamical characteristics of systems and the nature of environment must be developed as fundamental studies: (1) The first aspect in considering the dynamical characteristics of systems is that actual systems are not linear. In fact, we may see many nonlinear characteristics in practical systems such as saturation, relay and etc. There are no general methods for the analysis and synthesis of nonlinear systems. It is, in fact, only a trick of semantics which causes one to assume that there is any unity at all in the field. (2) The second which comes up in considering nonlinear systems is the fact that actual behaviors are usually random due to changes of environment and/or system parameters.

The method of organization of this thesis consists of two parts: In Part 1, the stochastic controllability problem is studied for nonlinear stochastic lumped parameter systems. In analogy with the concept of deterministic controllability, new definitions of stochastic controllability are estabilshed corresponding to various kinds of stochastic measures. By using the Lyapunov function like approach, sufficient conditions and/or necessary conditions are derived for the stochastic controllability. Finally, for the purpose of showing roles of stochastic controllability theorems in practical applications, a feasible algorithm is established for finding the generating method of the practical control signal which transfers the system state to a neighborhood of the given terminal point within the preassigned control time interval, including various kinds of numerical examples. In Part 2, our attention is focused on distributed parameter systems with stochastic coefficients contained in the partial differential operators. Within the framework of functional analysis, distributed parameter systems with the Markov chain and white Gaussian coefficients are respectively well modeled by the stochastic evolution equation in Hilbert spaces. The first half of Part 2 is concerned with the optimal distributed and boundary control problems. The optimal distributed and boundary control signals for the parabolic and hyperbolic type systems are derived by the Dynamic Programming and Stochastic Maximum Principle, respectively. The remainder is devoted to the extension of the theory of stochastic controllability established in Part 1 to the case of distributed parameter systems with white Gaussian coefficients by using the Lyapunov function like approach in Hilbert space.

PART 1

CONTROLLABILITY AND CONTROL FOR NONLINEAR STOCHASTIC LUMPED PARAMETER SYSTEMS

CHAPTER 1. INTRODUCTION

One of the important problems that arise often in control system is to answer a question: is it possible to find the control signal so as to transfer the system state to the desired one within the preassigned final time? This belongs to the context of controllability. The controllability concept of a class of deterministic system is introduced in order to answer the question: Does any control policy exist that will permit the desired terminal state to be achieved? If we can give an affirmative answer, then the system considered is said to be controllable. For deterministic linear systems, since the system trajectry can be determined provided that the initial state is given, it is easy to examine whether a controllable control signal exists or not. However, recognizing that many physical systems exhibit various kinds of nonlinearities and operate in random environment, the system considered is reasonably modeled by a class of nonlinear differential equations and the system state becomes a stochastic process. For stochastic systems, because of the existence of random perturbations, the exact trajectory of stochastic system is not available in advance. For this reason, in order to extend the concept of deterministic controllability to the stochastic version, we must introduce a new measure to evaluate stochastic system behaviors. With random circumstance, a possible analog to the controllability problem is to transfer the initial state to a target state within the final time under a prescribed probability or mean square sense. The most important consideration of stochastic controllability is that " In what stochastic sense is the system state guaranteed to be at the desired state?", while the deterministic system state is perfectly guaranteed.

Furthermore, due to nonlinearities of system dynamics, the target region relates strongly to the states of the system and the preassigned time interval. Thus, it is important to investigate the explicit relation between the initial region and the desired target one with some help of probability appraisal. To do this, first of all the computational algorithm should be established for realizing the control signal for nonlinear stochastic systems whose states can be transferred to the desired target region.

1.1. Historical Background

For convenience of the present description, the historical backgrounds are separately retrospected in two versions; one is the controllability of deterministic systems and the other is its stochastic version.

1.1.A. Controllability and optimal control of deterministic systems

The optimal control problem for deterministic systems has been discussed for many years and various optimization techniques related to some performance criterion have been developed by many investigators,[B1] [B2] [D1]. According to the development of optimization theory, the basic problem in control theory was generated by Kalman, [K1], [K2], with the terminology of the controllability. During the past decade, the controllability theory for deterministic linear systems has already been established including an extensive principle of duality between observability and controllability, [K3], [K4].

In analogy with the concept of controllability for linear systems, Lee and Markus studied the local controllability of nonlinear systems in the neighborhood of the critical point, and discussed the existence of optimal controls, [L1], [M1], [M2]. The global controllability for nonlinear systems was investigated by Hermes [H1]. The mutual relation between global controllability and local one was also studied by Davison [D1] and Lukes [L2]. Using the Grownwall's inequality, Aronsson studied the bang-bang controllability theorems, [A1]. In recent advance of the Lie algebra theory of vector fields, various types of global controllability conditions for a class of nonlinear systems were investigated by Brockett [B3], Haynes [H2], Hirshorn [H3], and many other researchers.

On the other hand, from the practical point of view, by using the Lyapunov stability approach [L3], the practical controllability theory for nonlinear systems was presented with the method of constructing the practical control signal and the explicit relation between the initial state and the nonlinearity of system dynamics, [G1].

1.1.B. Controllability and optimal control of stochastic systems

According to the theoretical development of stochastic differential equation [I1], the deterministic control theory was extended to the stochastic control version mainly in the field of linear optimal control problems, i.e., Dynamic programming technique has been applied to linear stochastic systems for quadratic performance criterion, [W1]. For nonlinear stochastic systems, various kinds of optimization techniques have been discussed by Kushner, [K5], [K6], Flemming, [F1],[F2] and Stratonovich,[S1]. Despite the increasing interest in controllability problem, little interest has been shown in stochastic controllability of nonlinear systems. Aoki, [A2] and Connors, [C1],[C2] studied the stochastic controllability in the mean-square sense for linear discrete systems with the aid of Dynamic programming concept. By using the same approch as Connors, Bertsekas discussed the stochastic controllability problem for nonlinear dynamical systems, [B4]. In [B4] he obtained for one special case that the unknown disturbances of system dynamics belong to the known bounded set. Applying Itô-Dynkin's formula, the stochastic controllability of nonlinear systems was discussed by Gershwin, [G2]. The results of Gershwin are direct extensions of deterministic nonlinear controllability theorems,[G1] to stochastic ones. In spite of his success in stochastic nonlinear controllability theorem, the relation between the preassigned target state and the stochastic appraisal of achieving the target state is not explicitly studied.

Recently, taking into account both the preassigned desired target state and finite time interval, the concepts of complete controllability and ε-controllability were established. Sufficient conditions were given for a general class of nonlinear stochastic systems, [S2],[S3]. The motivation stated in [S2] is somewhat similar to that in [B4], [D2] and [K7]. However in addition to mathematical characteristics of stochastic ε-controllability of nonlinear dynamical systems, a number of qualitative studies have been developed, regarding the hitting probability which guarantees a given system state be transferred into the desired target domain within the preassigned terminal time. The proposed technique in [S2] is applicable to the study of the stochastic observability, [S4],[S5],[S6] and furthermore the mutual relation between stochastic observability and stochastic controllability, [S3].

1.2. Problem Statement

We consider the problem of the stochastic controllability for a class of nonlinear stochastic dynamical systems and derive the control signal which transfers the given initial state to the desired target domain within the preassigned terminal time. A dynamical system under consideration is described by the n-dimensional vector nonlinear differential equation:

$$(1.2.1)\quad \left| \begin{array}{l} \dfrac{dx(t,\omega)}{dt} = f(t,x,u) + G(t,x)\gamma(t,\omega) \\ x(t_0) = x_0 \end{array} \right.$$

In Eq.(1.2.1), $x(t,\omega)$ is an n-dimensional vector stochastic process representing state variable, $f(t,x,u)$ and $G(t,x)$ are respectively an n-vector and an $n\times p$-matrix nonlinear functions, $\gamma(t,\omega)$ is a p-vector white Gaussian noise, and $u(t,x)$ is an m-dimensional control vector where $n \geq m$.

We assume that the system state can completely be observed. As will be pointed out in Chap. 2, the system model (1.2.1) is mathematically formal because of the existence of white Gaussian noise term. According to the theory of stochastic differential equation [I1],[S1], we rewrite Eq.(1.2.1) by the stochastic differential equation of Itô-type:†

$$(1.2.2)\quad \left| \begin{array}{l} dx(t,\omega) = f(t,x,u)dt + G(t,x)dw(t) \\ x(t_0) = x_0 \end{array} \right.$$

† The revised term due to the existence of state dependent noise term is neglected, [W2].

where $w(t)$ is a p-vector Brownian motion process.

Important aspects in stochastic controllability are as follows:

(i) The system state $x(t)$ is completely observed.

(ii) Only a finite time control interval $[t_0,t_f]$ is considered.

(iii) A set of the initial states is preassigned.

(iV) Sets of the desired target domain are preassigned in some stochastic sense.

The aspect (iii) is highly important with respect to the controllability for nonlinear systems because this aspect keeps in touch with inherent characteritics of nonlinearities i.e., input dependence. It is thus desirable to know the relation between the region of initial sets which can be transfered to the desired target domain and the nonlinearity of system dynamics. The problem of "What stochastic sense can we adopt?" depends on the situation of system designers. In this thesis, we adopt two stochastic sense i.e., the mean square sense and in probability one.

1.3. Summary of Part 1

Description in Part 1 is outlined in the sequel. In Chapter 2, some of general fundamental works required in the context of controllability are reviewed as mathematical preliminaries. The mathematical model of the system is also established by the theory of Itô stochastic differential equations with the Itô stochastic calculas. In Chapter 3, new definitions of stochastic controllability are given. As the stochastic sense of controllability we adopt first in the mean square sense and next in probability sense. From the mathematical point of view, the concept of ε-controllability is first introduced and extended to the complete controllability. Furthermore the mutual relations between the mean square sense and in probability one, and between ε-controllability and complete one are also discussed. Applying the stochastic Lyapunov stability theory, [K8], sufficient conditions for stochastic controllability for a general class of nonlinear systems are demonstrated, and choosing the quadratic Lyapunov function, explicit sufficient conditions for stochastic controllability reflecting precisely nonlinearites of system dynamics are also given. For linear systems, it is found that sufficient conditions for nonlinear systems fall into necessary and sufficient conditions for linear systems with showing the explicit relation between stochastic controllability and deterministic one. Chapter 5 is devoted to an algorithm of deriving the control signal which transfers the given initial state to the preassigned desired target domain in the sense of stochastic hitting problem including numerical examples for the purpose of interpreting the general theory.

The remainder of Part 1 is devoted to the discussion of the summary of results and to the suggestions of possibility of showing the duality theorem for stochastic controllability and stochastic observability.

CHAPTER 2. MATHEMATICAL PRELIMINARIES

2.1. Symbolic Conventions and Basic Definitions

Let $R^{(n)}$ denote an n-dimensional Euclidean space. If x is an element of $R^{(n)}$ ($x \varepsilon R^{(n)}$), then x' denotes the transpose of the vector x with the norm $\| x \| = x'x$. Similarly, if M is a matrix, then M' denotes its transpose with the norm $\| M \|^2 = tr.[M'M]$. According to the standard notation and terminology, lower case letters a,b and c,... denote column vectors with i-th real components a_i, b_i and c_i,.... Capital letters A,B and C,... denote matrices with elements a_{ij}, b_{ij} and c_{ij}..., respectively. Certain algebraic quantities such as algebras, fields, etc. are expressed by F, Y,...etc.

We collect some of standard knowledges of the probability theory.

Probability space: Let P be a probability measure on Ω, where Ω denotes the set of all events. Let F be the smallest σ-algebra of subsets of Ω. The triplet (Ω, F, P) is called a probability space.

Random variable: A real-valued function $x(\omega)$ ($\omega \varepsilon \Omega$) defined on Ω is called a random variable if, for every Borel set B in the Euclidean space $R^{(n)}$, the set $\{\omega; x(\omega) \varepsilon B\}$ is in F.

Expectation: If $x(\cdot)$ is integrable on Ω, then the expectation of x, denoted by $E\{x\}$, is given by

$$E\{x\} = \int_{\Omega} x dP .$$

Conditional expectation: Let C be a Borel field with $C \subset F$ and let $x(\cdot)$ be integrable on Ω. The conditional expectation of x relative to C, denoted by $E\{x|C\}$, is a random variable such that

$$\int_{\Lambda} x dP = \int_{\Lambda} E\{x|C\} dP$$

for all $\Lambda \varepsilon C$.

Stochastic process: A stochastic process $\{x(t,\omega),\ t_0 \le t \le t_f\}$ is a family of random variables, with a real parameter t and defined on the probability space (Ω, F, P).

For economy of description, we omit to write the symbol ω in the sequel because no confusion will result.

When a probability statement is true almost sure on Ω or true with probability 1, then the abbreviation a.s. or w.p.1 is used. A limit in the mean square sense is denoted by l.i.m.

The principal symbols used here are listed below;

t: Time variable, particularly present time

t_0: The initial time at which control action starts

t_f: A preassigned terminal time

$x(t)$: An n-dimensional vector stochastic process representing the system state $x \varepsilon R^{(n)}$

$u(t,\cdot)$: An m-dimensional control vector taking its value in a convex compact subset $U \subset R^{(m)}$

$w(t)$: A p-dimensional standard Brownian motion process

$f(t,x)$: An n-dimensional vector-valued nonlinear function

$G(t,x)$: An $n \times p$ dimensional parameter matrix whose components depend on t and x

2.2. Stochastic Differential Equation of Itô-type

In this section we summarize important properties of the solution process of Itô-stochastic differential equations [I1],[W3]. Important phases of the relation between system dynamics and Itô-stochastic differential equation will be stated in the next section.

2.2.1. Itô-stochastic integral: Before describing the precise statement of Itô-equation, we need the following definition of Itô-stochastic integral.

[Definition-2.2.1] (Itô-stochastic integral)

Let $w(t)$ be a scalar standard Brownian motion process, and let $\phi(t,\omega)$ be a scalar function such that

(i) $\phi = \{\phi(t,\omega)\}_{t \geq t_0}$ is measurable to $\mathcal{F}_t$, where $\mathcal{F}_t$ is a minimum σ-algebra generated by $\{w(t)\}_{t \geq t_0}$.

(ii) $E\{\int_{t_0}^{t_f} \| \phi(t,\omega) \|^2 dt \} < \infty$.

The stochastic integral $I(\phi)$ is defined by

$$
\begin{aligned}
(2.2.1) \quad I(\phi)(t) &= \int_{t_0}^{t} \phi(s,\omega) dw(s) \\
&= \lim_{n \to \infty} [\sum_{i=0}^{n-1} \phi(t_i,\omega)(w(t_{i+1},\omega) - w(t_i,\omega)) \\
&\qquad + \phi(t_n,\omega)(w(t,\omega) - w(t_n,\omega))] \text{ a.s.}
\end{aligned}
$$

The generalized definition of Itô-stochastic integral for the vector valued Brownian motion process is easily obtained from the definition-2.2.1 (see Watanabe [W3])

[Proposition 2.2.1]: Let $I(\phi)(t)$ be the Itô-stochastic integral defined by Eq.(2.2.1). Then we have the following properties;

(i) $I(\phi)(t_0) = 0$

(ii) For any t and s, $(t_0 \leq s < t)$, we have

$$(2.2.2) \quad E\{ I(\phi)(t) - I(\phi)(s) | \mathcal{F}_s \} = 0 \quad \text{a.s.}$$

and

(2.2.3) $E\{(I(\phi)(t) - I(\phi)(s))^2 | F_s\} = E\{\int_s^t \phi^2(\tau,\omega)d\tau | F_s\}$ a.s.

2.2.2. Itô-stochastic differential equation: Let $\{w(t)\}_{t \geq t_0}$ be a p-dimensional Brownian motion process defined on (Ω, F, P).

Consider the following integral equation;

(2.2.4) $$x(t) = x(t_0) + \int_{t_0}^{t} f(s,x(s))ds + \int_{t_0}^{t} G(s,x(s))dw(s) \quad \text{for } t_0 \leq t \leq t_f .$$

It is well-known that, with the following conditions, there exists a unique continuous solution satisfying Eq.(2.2.4).

(C-2.2.1) There exists a positive constant K_1, such that

(2.2.5) $$\| f(t,x) \|^2 + \| G(t,x) \|^2 \leq K_1(1 + \| x \|^2) \quad \text{for any } x \varepsilon R^{(n)} \text{ and any } t.$$

(C-2.2.2) $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ satisfy a uniformly Lipschitz condition, that is, for any t and any $x_1, x_2 \varepsilon R^{(n)}$

(2.2.6) $$\| f(t,x_1) - f(t,x_2) \|^2 + \| G(t,x_1) - G(t,x_2) \|^2 \leq K_2 \| x_1 - x_2 \|^2 .$$

A stochastic process $\{x(t)\}_{t \geq t_0}$ which satisfies Eq.(2.2.4) is called an Itô-process with respect to the Brownian motion process $\{w(t)\}_{t \geq t_0}$.

[Proposition 2.2.2]: Let $\{x(t)\}_{t \geq t_0}$ be the solution process of Eq.(2.2.4). Then, $\{x(t)\}_{t \geq t_0}$ has the following properties;

(i) If the initial condition satisfies $E\{\| x(t_0) \|^{2m}\} < \infty$ for $m \geq 1$, we have for any t, (2.2.7) $E\{\| x(t) \|^{2m}\} < \infty.$

(ii) $x(t)$ is sample continuous w.p.1.

(iii) $x(t)$ is a Markov process.

In remainder of this section, the analytic tool of studying the control problem will be presented. From the definition and propositions mentioned above, the Itô's stochastic differential rule (so called "Itô-calculus") is directly derived.

2.2.3. Itô-stochastic differential rule: Let $x(t)$ be the solution of Eq.(2.2.4) and let $V(t,x(t))$ be a scalar-valued real function, which is continuously differentiable in x. Then the continuous stochastic process $V(t,x(t))$ becomes an Itô-process and satisfies;

(2.2.8)
$$V(t,x(t)) - V(t_0,x(t_0))$$
$$= \int_{t_0}^{t} \Big[\frac{\partial V(s,x(s))}{\partial s} + f'(s,x(s))\frac{\partial V(s,x(s))}{\partial x}$$
$$+ \frac{1}{2}\mathrm{tr.}\Big[G'(s,x(s))\frac{\partial}{\partial x}\Big[\frac{\partial V(s,x(s))}{\partial x}\Big]'G(s,x(s))\Big]\Big]ds$$
$$+ \int_{t_0}^{t} \Big[\frac{\partial V(s,x(s))}{\partial x}\Big]'G(s,x(s))dw(s) \quad \text{w.p.1}$$

2.2.4. Itô-Dynkin's formula: From Eq.(2.2.8), the averaged value of $V(t,x(t))$ conditioned by $x(t_0)$, satisfies the following equation:

(2.2.9) $E_{x_0}\{V(t,x)\} - V(t_0,x_0) = E_{x_0}\{\int_{t_0}^{t} LV(s,x)\, ds\},$

where $L(\cdot)$ is the differential generator;

(2.2.10) $L(\cdot) = \frac{\partial(\cdot)}{\partial s} + f'(s,x)\frac{\partial(\cdot)}{\partial x} + \frac{1}{2}\,\mathrm{tr.}[G'(s,x)\frac{\partial}{\partial x}[\frac{\partial(\cdot)}{\partial x}]'G(s,x)]$.

In this section a brief summary has been given of the Itô-theory of stochastic differential equations and this will be one of main analytical tools for studying the stochastic control problem.

2.3. Mathematical Modeles of Dynamical Systems

Guided by the well known state space representation concept, the dynamics of an important class of dynamical systems can be described by a nonlinear vector differential equation,

$$(2.3.1)\quad \left| \begin{array}{l} \dfrac{dx(t,\omega)}{dt} = f(t,x(t,\omega),u(t)) + G(t,x(t,\omega))\gamma(t,\omega), \\ x(t_0,\omega) = x_0(\omega) \ , \end{array} \right.$$

where $f(t,x,u)$ is an n-vector-valued nonlinear function, $u(t)$ is an m-vector control signal to be specified later, and $\gamma(t,\omega)$ is a p-vector system noise.

From the fact that the spectral density of the physical system noise has the finite frequency domain, the model of system noise must be identified by the colored noise. However, by introducing the shaping filter technique, the situation of colored noise is easily converted into that of white noise case. Then, we assume that $\gamma(t,\omega)$ is a p-vector white Gaussian noise process with zero mean and covariance martix

$$(2.3.2)\quad E\{\gamma(t,\omega)\gamma'(s,\omega)\} = I\delta(t - s) \ .$$

Noting that the white noise process $\{\gamma(t,\omega)\}_{t\geq t_0}$ has a delta correlation, Eq.(2.3.1) has no mathematically precise meaning. Recalling the white Gaussian noise related to [G3],[S7],[H4]

$$(2.3.3)\quad w(t) = \int_{t_0}^{t}\gamma(s,\omega)ds,$$

Eq.(2.3.1) can be integrated and replaced by the following integral equation;

$$(2.3.4)\quad x(t) = x_0 + \int_{t_0}^{t} f(s,x,u)ds + \int_{t_0}^{t} G(s,x)dw(s) \ .$$

Extending the conditions (C-2.2.1) and (C-2.2.2) to Eq.(2.3.4), it is fairly stated that Eq.(2.3.4) has a unique continuous solution, if the following conditions are satisfied,

(C-2.3.1) There exists a positive constant K_3 such that, for any $x\varepsilon R^{(n)}$ and any $u\varepsilon U$ (convex closed subset of $R^{(n)}$),

$$(2.3.5)\quad \| f(t,x,u) \|^2 + \| G(t,x) \|^2 \leq K_3(1 + \| x \|^2 + \| u \|^2) \quad \text{for any } t\varepsilon[t_0,t_f] \ .$$

(C-2.3.2) $f(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot)$ satisfy a uniformly Lipschitz condition, that is, for any $x_1,x_2\varepsilon R^{(n)}$ and $u_1,u_2\varepsilon U$,

$$(2.3.6)\quad \| f(t,x_1,u_1) - f(t,x_2,u_2) \|^2 + \| G(t,x_1) - G(t,x_2) \|^2 \leq K_4(\| x_1 - x_2 \|^2 + \| u_1 - u_2 \|^2) \ , \quad \text{for any } t\varepsilon[t_0,t_f] \ .$$

With the statement mentioned above, we formally write Eq.(2.3.4) by

$$(2.3.7)\quad \left| \begin{array}{l} dx(t) = f(t,x,u)dt + G(t,x)dw(t) \\ x(t_0) = x_0. \end{array} \right.$$

Then, in what follows, we adopt Eq.(2.3.4) as the mathematical model of dynamical systems.

Remark: For a general class of control problems, the restriction of the nonlinear function f(t,x,u) cannot always be satisfied in the Lipschitz condition (C-2.3.2). (For example, in the case where u(t) becomes the Bang-Bang form.) However, it has been proved that, with the condition (C-2.3.1), Eq.(2.3.4) has an only one weak solution, w.p.l. (For details, see Refs.[W3] and [B5])

CHAPTER 3. DEFINITIONS OF STOCHASTIC CONTROLLABILITY FOR NONLINEAR LUMPED PARAMETER SYSTEMS

3.1. Introductory Remarks

In deterministic dynamical systems, the concept of controllability may roughly be stated as the answer for the basic question, "Can the initial state of a dynamical system under consideration be transferred to any desired state within a preassigned time interval by some control operation?" For stochastic dynamical systems, the concept of stochastic controllability becomes different from that of deterministic controllability. That is, the most important difference is to show, "With what stochastic sense can the initial state be transferred to the desired state?"

In this chapter, valuable definitions of stochastic controllability will be presented. First we adopt the "mean square sense" as the stochastic measure, and then the concept of controllability in probability is introduced towards useful applications.

3.2. Definitions of Stochastic Controllability

Before presenting precise definitions of stochastic controllability, the admissible control class must be defined. Noting that the considered dynamical system is nonlinear, the admissible control class is defined, which includes the feed-back control low.

Admissible Control Class: Let F_t be the σ-algebra generated by the solution process $\{x(t)\}_{t \geq t_0}$ of Eq.(2.3.7). We denote the admissible control class U_{ad} by

$U_{ad} = \{u(t,x)\varepsilon R^{(m)}$, and is F_t measurable for $\forall t(t_0 \le t \le t_f)$

and $E\{\int_{t_0}^{t_f} \| u(t,x) \|^2 dt\} < \infty\}$

In analogy with the concept of deterministic controllability, definitions of stochastic controllability are listed below;

[Definition-3.2.1](ε-controllability in the mean square sense)

An initial state x_0 of the system is said to be stochastically ε-controllable in the mean square sense with respect to the specified target domain $\varepsilon(t_f,x_0)$ within the time interval $[t_0,t_f]$, if there exists a control $u(t,x)\epsilon U_{ad}$ such that

$$E_{x_0}\{\| x(t_f) \|^2\} \le \varepsilon(t_f,x_0) .$$

[Definition-3.2.2](Complete controllability in the mean square sense)

The system under consideration is said to be completely controllable in the mean square sense within the time interval $[t_0,t_f]$, if there exists a control $u(t,x)\epsilon U_{ad}$ such that for any $\varepsilon>0$, and any $x_0 \epsilon X_0 \subset R^{(n)}$ ($x_0 \ne 0$ and X_0 denotes the uniformly bounded subset in $R^{(n)}$)

$$E_{x_0}\{\| x(t_f) \|^2\} \le \varepsilon .$$

In the definitions -3.2.1 and -3.2.2 , as a representative of measures corresponding to the expression, "stochastic measure", the mean square sense, $E_{x_0}\{\| x(t_f) \|^2\} \le \varepsilon$ was reasonably taken into account from the mathematical viewpoint. For convenience of practical treatments, we slightly modified the above definitions by using the "another stochastic measure", that is, "in probability".

[Definition-3.2.3](ε-controllability in probability)

An initial state x_0 of the system is said to be stochastically ε-controllable in probability ρ, in the normed square sense, with respect to a specified target domain with the norm $\sqrt{\varepsilon}$ within the time interval $[t_0,t_f]$, if there exists a control $u(t,x)\in U_{ad}$ such that

$$\Pr\{\| x(t_f) \|^2 \geq \varepsilon | \ x(t_0) = x_0\} \leq 1 - \rho(t_f,x_0)$$

where $0<\rho<1$.

[Definition-3.2.4](Complete controllability in probability)

The system under consideration is said to be completely controllable in probability ρ, with respect to a specified target domain with the norm $\sqrt{\varepsilon}$, within the time interval $[t_0,t_f]$, if there exists a control $u(t,x)\in U_{ad}$ such that, for any $\varepsilon>0$, any $\rho(0<\rho<1)$ and any $x_0\in X_0\subset R^{(n)}(x_0\neq 0)$

$$\Pr\{\| x(t_f) \|^2 \geq \varepsilon | \ x(t_0) = x_0\} \leq 1 - \rho.$$

Mutual relation between definitions -3.2.1 , -3.2.2 , -3.2.3 , and -3.2.4 can be easily obtained, i.e.,

1) If the system under consideration is completely controllable in the mean square sense, then the initial state of the system is ε-controllable in the mean square sense. In other words, the definition -3.2.2 includes the definition-3.2.1 .
2) If the system under consideration is completely controllable in probability, then the initial state x_0 of the system is ε-controllable in probability. In other words, the definition-3.2.4 includes the definition-3.2.3 .
3) If the system under consideration is completely controllable in the mean square sense, then the system is completely controllable in

probability. In other words, definition-3.2.4 includes the definition-3.2.2.

From the relations (1) to (3) and noting that the conditions based on the mean square sense are involved in those in probability, we may find that the stochastic measure " in the mean square sense " plays an important role to study the stochastic structure of dynamical systems in the version of stochastic controllability from the mathematical point of view, while the stochastic measure " in probability " shows various kinds of advantages for the purpose of generating the control signal, because the concept of " in probability " directly relates to the sample behavior. Furthermore, from the conclusion that the conditions for complete controllability are involved in those for ε-controllability, the definition of complete controllability requires the stronger conditions than those of ε-controllability with respect to the system dynamics, in particulary, its nonlinearity. However, from the fact that the definition of ε-controllability depends on the initial state x_0 and the terminal time t_f, we can show the explicit relation between the initial state x_0 and the system nonlinearity, that is, if the ε-controllability is adopted, we may find the controllable region with respect to both the initial state and the desired target domain. In Chap. 5, the hitting problem as one of applications of the ε-controllability in probability will be discussed.

CHAPTER 4. CONDITIONS ON CONTROLLABILITY FOR NONLINEAR LUMPED PARAMETER SYSTEMS

4.1 Introductory Remarks

In this chapter, according to the new definitions of stochastic controllability introduced in the previous chapter, various sufficient conditions of stochastic controllability for a general class of stochastic nonlinear lumped parameter systems are first derived by using the Lyapunov-like approach [K8],[G2]. Furthermore, concrete forms of valuable sufficient conditions for semi-linear stochastic lumped parameter systems are also demonstrated. In order to compare the stochastic controllability theory with the deterministic one, sufficient conditions for stochastic nonlinear systems are shown to fall into the necessary and sufficient conditions in the case of stochastic linear systems.

Finally, the relation is investigated between stochastic controllability theory and deterministic controllability one [G2], for nonlinear systems.

4.2. Stochastic Controllability Theorems

In this section, the dynamical system Eq.(2.3.7) is considered and the mathematical development follows on the basis of discussions in Sec.2.3, Chap.2, that is

$$(4.2.1)\quad \left|\begin{array}{l} dx(t) = f(t,x(t),u(t,x))dt + G(t,x)dw(t) \\ x(t_0) = x_0. \end{array}\right.$$

The following theorem gives sufficient conditions for ε-controllability in probability.

[Theorem-4.2.1](ε-controllability in probability): The initial state x_0 of the system(4.2.1) is ε-controllable in probability $\rho(t_f,x_0)$

with respect to the terminal state $\sqrt{\varepsilon}$, within the time interval $[t_0,t_f]$, if the following conditions are satisfied;

(Condition-4.2.1) In the time interval $[t_0,t_f]$, a scalar function $V(t,x)$ is defined and has bounded continuous first and second derivatives with respect to every component of x and a first derivative with respect to t.

(Condition-4.2.2) $V(t,x)$ satisfies the terminal and initial conditions for given ρ and ε ,

(4.2.2) $V(t_f,x) \geq \frac{1}{\alpha} x'x$

and

(4.2.3) $V(t_0,x(t_0)) \leq (1 - \rho(t_f,x_0))\frac{\varepsilon}{\alpha}$.

(Condition-4.2.3) A control $u(t,x) \in U_{ad}$ exists such that, along the trajectory obtained by the solution of Eq.(4.2.1), the following inequality holds,

(4.2.4) $LV(t,x) \leq 0$,

where $L(\cdot)$ is the differential generator defined in Sec.2.2, Chap.2, such that

(4.2.5) $L(\cdot) = \frac{\partial(\cdot)}{\partial t} + f'(t,x,u)\frac{\partial(\cdot)}{\partial x} + \frac{1}{2}\mathrm{tr.}[G'(t,x)\frac{\partial}{\partial x}(\frac{\partial(\cdot)}{\partial x})'G(t,x)]$.

Proof: Using Conditions-4.2.1 and-4.2.3, and applying the Itô-Dynkin's formula defined by Eq.(2.2.9) in Sec.2.2,Chap.2, we have

(4.2.6) $E\{V(t_f,x(t_f))|x(t_0) = x_0\} - V(t_0,x_0)$

$$= E\{\int_{t_0}^{t_f} LV(s,x)ds|x(t_0) = x_0\} \leq 0 .$$

It follows that

(4.2.7) $E\{V(t_f,x(t_f))|x(t_0) = x_0\} \leq V(t_0,x_0)$.

Equation(4.2.7) implies that

(4.2.8) $\Pr\{V(t_f,x(t_f)) \geq \lambda | x(t_0) = x_0\} \leq \dfrac{V(t_0,x_0)}{\lambda}$.

From Condition-4.2.2 and replacing $\lambda=\varepsilon/\alpha$, we obtain

(4.2.9) $\Pr\{x'(t_f)x(t_f) \geq \varepsilon | x(t_0) = x_0\} \leq \frac{\alpha}{\varepsilon}V(t_0,x_0)$

From Eq.(4.2.3), the inequality (4.2.9) becomes

(4.2.10) $\Pr\{\| x(t_f) \|^2 \geq \varepsilon | x(t_0) = x_0\} \leq 1 - \rho(t_0,x_0)$.

The proof has been completed.

[Corollary-4.2.1](ε-controllability in the mean square sense): If in Theorem-4.2.1, in stead of Eq.(4.2.3), the initial condition x_0 satisfies for given ε

(4.2.11) $\alpha V(t_0,x_0) < \varepsilon$,

the initial condition x_0 becomes ε-controllable in the mean square sense.

The proof can be carried out similarly as that of Theorem-4.2.1

In order to compare the sufficient conditions stated in Teorem-4.2.1 with conditions for linear stochastic systems, we shall consider a class of linear systems determined by

(4.2.12) $\left|\begin{array}{l} dx(t) = A(t)x(t)dt + B(t)u(t,x)dt + x(t)\sigma(t)dw(t) \\ x(t_0) = x_0 \end{array}\right.$

where $B(t)$ is an $n\times m$ parameter matrix and $\sigma(t)$ is a p-dimensional row vector.

Regarding the ε- controllability in probability of the system (4.2.12), the following theorem is useful.

[Theorem-4.2.2](ε-controllability in probability): The initial state of the system (4.2.12) is ε-controllable in probability $\rho(t_f,x_0)$

with respect to the terminal state with the norm $\sqrt{\varepsilon}$, within the time interval $[t_0,t_f]$, if the following conditions are satisfied:

(Condition-4.2.4) There exists a bounded, symmetric and positive definite $P(t)$ defined in the time interval $[t_0,t_f]$, which satisfies

$$(4.2.13)\quad \frac{dP(t)}{dt} + A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t) + tr.[\sigma'(t)\sigma(t)]P(t) = 0 ,$$

with the terminal condition $(\alpha>0)$

$$(4.2.14)\quad P(t_f) = I/\alpha .$$

(Condition-4.2.5) The initial state x_0 of the system (4.2.2) and the initial condition $P(t_0)$ defined by Eqs.(4.2.13) and (4.2.14) satisfies

$$(4.2.15)\quad x_0{}'P(t_0)x_0 \leq (1 - \rho(t_f,x_0))\frac{\varepsilon}{\alpha} ,$$ for given ρ and ε.

Proof: Let a scalar function $V(t,x)$ be

$$(4.2.16)\quad V(t,x) = x'(t)P(t)x(t).$$

Suppose that relations (4.2.4), (4.2.5) and (4.2.12) hold and noting that, in this case, the differential generator $L(\cdot)$ is given by

$$(4.2.17)\quad L(\cdot) = \frac{\partial(\cdot)}{\partial t} + [A(t)x + B(t)u(t,x)]'\frac{\partial(\cdot)}{\partial x} + \frac{1}{2}tr.[[x\sigma(t)]'\frac{\partial}{\partial x}(\frac{\partial(\cdot)}{\partial x})'[x\sigma(t)]] ,$$

and choosing $u(t,x) = -\frac{1}{2}B'(t)P(t)x$, we have

$$(4.2.18)\quad LV(t,x) = x'[\frac{dP(t)}{dt} + A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t) + tr.[\sigma'(t)\sigma(t)]P(t)]x .$$

Applying Condition-4.2.4 to Eq.(4.2.18), it follows that

$$(4.2.19)\quad LV(t,x) = 0 .$$

Then, from Condition-4.2.5, all conditions stated in Theorem-4.2.1 are satisfied. The proof has been completed.

[Corollary-4.2.2](Complete controllability in probability): The system (4.2.12) is completely controllable in probability, if the following condition is satisfied:

(Condition-4.2.6) There exists a positive definite matrix $\tilde{W}(t_0,t_f)$ such that

$$(4.2.20)\quad \tilde{W}(t_0,t_f) \triangleq [\int_{t_0}^{t_f}\tilde{\Phi}(t_f,s)B(s)B'(s)\tilde{\Phi}'(t_f,s)ds]^{-1}$$

where, for $t \geq s$

$$(4.2.21)\quad \frac{\partial\tilde{\Phi}(t,s)}{\partial t} = [A(t) + \frac{1}{2}tr.[\sigma'(t)\sigma(t)]]\tilde{\Phi}(t,s)$$

and

$$(4.2.22)\quad \tilde{\Phi}(s,s) = I .$$

Proof: For an arbitraly constant $\alpha>0$, define

$$(4.2.23)\quad P(t) \triangleq \tilde{\Phi}'(t_f,t)[\alpha + \int_t^{t_f}\tilde{\Phi}(t_f,s)B(s)B'(s)\tilde{\Phi}'(t_f,s)ds]^{-1}\tilde{\Phi}(t_f,t).$$

It is easy to show that $P(t)$ is a solution of Eq.(4.2.13) with the same terminal condition as given by Eq.(4.2.14). Through the proof of Theorem-4.2.2, for any x_0, ε and ρ, we must show Eq.(4.2.15). Then, from Eqs.(4.2.20) and (4.2.23), for any $\alpha>0$ we have

$$(4.2.24)\quad x_0'\tilde{\Phi}'(t_f,t_0)[\alpha + \tilde{W}^{-1}(t_0,t_f)]^{-1}\tilde{\Phi}(t_f,t_0)x_0$$
$$\leq x_0'\tilde{\Phi}'(t_f,t_0)\tilde{W}(t_0,t_f)\tilde{\Phi}(t_f,t_0)x_0 .$$

From Condition-4.2.6 and noting that α is an arbitrary constant, we can choose α such that

$$(4.2.25)\quad x_0'\tilde{\Phi}'(t_f,t_0)\tilde{W}(t_0,t_f)\tilde{\Phi}(t_f,t_0)x_0 \leq (1 - \rho)\varepsilon/\alpha ,$$

for any x_0, ε and ρ. The proof has been completed.

Theorem-4.2.2 and Corollary-4.2.2 may easily be extended to the following theorem giving conditions for the complete controllability in the mean square sense.

[Theorem-4.2.3](Complete controllability in the mean square sense):
The system (4.2.12) is completely controllable in the mean square sense, if and only if the following condition is satisfied;

(Condition-4.2.7) There exists a positive definite matrix $W(t_0,t_f)$ such that

$$(4.2.26)\quad W(t_0,t_f) \stackrel{\Delta}{=} \left[\int_{t_0}^{t_f}\Phi(t_f,s)B(s)B'(s)\Phi'(t_f,s)ds\right]^{-1}$$

where for $t\geq s$

$$(4.2.27)\quad \frac{\partial\Phi(t,s)}{\partial t} = A(t)\Phi(t,s)$$

and

$$(4.2.28)\quad \Phi(s,s) = I.$$

Proof: (Sufficiency) Define

$$(4.2.29)\quad P(t) \stackrel{\Delta}{=} \Phi'(t_f,t)\left[\alpha + \int_{t_0}^{t_f}\Phi(t_f,s)B(s)B'(s)\Phi'(t_f,s)ds\right]^{-1}\Phi(t_f,t),$$

where α is an arbitrary positive constant.

According to the results of Theorem-4.2.2 and Eq.(4.2.29), define

$$(4.2.30)\quad V(t,x)= \exp\left[-\int_{t_0}^{t}[\sigma'(s)\sigma(s)]ds\right]x'P(t)x.$$

It is easily shown that $P(t)$ defined by Eq.(4.2.29) satisfies the following differential equation,

$$(4.2.31a)\quad \frac{dP(t)}{dt} + A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t) = 0$$

with the terminal condition

(4.2.31b) $P(t_f) = I/\alpha$.

Suppose that, for the preassigned terminal time t_f, a control signal is

(4.2.32) $u(t,x) = -\frac{1}{2}B'(t)P(t)x$.

By using the same approach as that in Theorem-4.2.2, we have

(4.2.33) $$\begin{aligned} LV(t,x) &= \exp[-\int_{t_0}^{t} tr.[\sigma'(s)\sigma(s)]ds]x'[\frac{dP(t)}{dt} + A'(t)P(t) \\ &\quad + P(t)A(t) - P(t)B(t)B'(t)P(t) + tr.[\sigma'(t)\sigma(t)]P(t)]x \\ &\quad - \exp[-\int_{t_0}^{t} tr.[\sigma'(s)\sigma(s)]ds]tr.[\sigma'(t)\sigma(t)]x'P(t)x \\ &= \exp[-\int_{t_0}^{t} tr.[\sigma'(s)\sigma(s)]ds]x'[\frac{dP(t)}{dt} + A'(t)P(t) \\ &\quad + P(t)A(t) - P(t)B(t)B'(t)P(t)]x . \end{aligned}$$

From Eq.(4.2.31a), it follows that

(4.2.34) $LV(t,x) = 0$.

Applying the Itô-Dynkin's formula, we have

(4.2.35) $$\exp[-\int_{t_0}^{t_f}\sigma'(s)\sigma(s)ds]\frac{1}{\alpha}E\{\|x(t_f)\|^2|x(t_0) = x_0\} = x_0'P(t_0)x_0.$$

From Eq.(4.2.29), the initial condition $P(t_0)$ becomes

(4.2.36) $$P(t_0) = \Phi'(t_f,t_0)[\alpha + \int_{t_0}^{t_f}\Phi(t_f,s)B(s)B'(s)\Phi'(t_f,s)ds]^{-1}\Phi(t_f,t_0).$$

Noting that $W(t_0,t_f)$ is a positive definite matrix from Condition-4.2.7, for any $x_0 \in X_0 \subset R^{(n)}$, we have

(4.2.37) $x_0'P(t_0)x_0 \le x_0'\Phi'(t_f,t_0)W(t_0,t_f)\Phi(t_f,t_0)x_0$.

Then, Eq.(4.2.35) becomes

$$(4.2.38)\quad E\{\|x(t_f)\|^2 | x(t_0) = x_0\} \leq \alpha\cdot\exp[\int_{t_0}^{t_f}\sigma'(s)\sigma(s)ds]$$
$$\times x_0'\Phi'(t_f,t_0)W(t_0,t_f)\Phi(t_f,t_0)x_0 \to 0\ (\alpha\to 0).$$

Bearing in mind that α is an arbitrary positive constant and that W is positive definite, we can choose α such that, for any $\varepsilon>0$,

$$(4.2.39)\quad \alpha = \frac{\varepsilon\cdot\exp[-\int_{t_0}^{t_f}tr.[\sigma'(s)\sigma(s)]ds]}{x_0'\Phi'(t_f,t_0)W(t_0,t_f)\Phi(t_f,t_0)x_0}\ .$$

From Eqs.(4.2.38) and (4.2.39), we have

$$(4.2.40)\quad E\{\|x(t_f)\|^2 | x(t_0) = x_0\} \leq \varepsilon.$$

Then, we find that Condition-4.2.7 is a sufficient condition of complete controllability in the mean square sense.

The next step is to show the necessity version of the proof.

(Necessity) From Eq.(4.2.12), we can easily show that the averaged process $E\{x(t)|x(t_0)=x_0\}$ conditioned by x_0 satisfies

$$(4.2.41)\quad \left| \begin{array}{l} \dfrac{dE\{x(t)|x(t_0)=x_0\}}{dt} = A(t)E\{x(t)|x(t_0)=x_0\} \\ \qquad\qquad + B(t)E\{u(t,x)|x(t_0)=x_0\} \\ E\{x(t_0)|x(t_0)=x_0\} = x_0 \end{array} \right.$$

Now suppose that Condition-4.2.7 is not satisfied and the averaged control $E\{u(t,x)|x(t_0)=x_0\}$ is an arbitrary element of $R^{(m)}$. Then, from the deterministic controllability theory [K1], Condition-4.2.7 is a necessary and sufficient condition of deterministic controllability for the averaged system(4.2.41). In the version of necessity, as we assume that Condition-4.2.7 is not satisfied, that is, the averaged system (4.2.41) is not controllable, there exists some positive constant

δ, such that, for any $E\{u(t,x)|x(t_0)=x_0\}\epsilon R^{(m)}$,

(4.2.42) $\quad \| E\{x(t_f)|x(t_0)=x_0\} \|^2 \geq \delta > 0.$

By using the Schwartz's inequality, it follows that

(4.2.43) $\quad E\{\| x(t_f) \|^2 | x(t_0)=x_0\} \geq \| E\{x(t_f)|x(t_0)=x_0\} \|^2 \geq \delta > 0.$

Then, Eq.(4.2.43) means that, if Condition-4.2.7 is not satisfied, the system (4.2.12) is not completely controllable in the mean square sense. The necessity version of the proof has been completed.

Theorem-4.2.3 may be extended to the following nonlinear stochastic lumped parameter system:

(4.2.44) $\quad \left| \begin{array}{l} dx(t) = [A(t)x(t) + h(t,x)]dt + B(t)u(t,x)dt + x(t)\sigma(t)dw(t) \\ x(t_0) = x_0, \end{array} \right.$

where h(t,x) is an n-dimensional vector valued nonlinear function.

[Theorem-4.2.4](Complete controllability in the mean square sense):†

The system (4.2.44) is completely controllable in the mean square sense, if in addition to Condition-4.2.7 stated in Theorem-4.2.3 (or Condition-4.2.6) the following condition is satisfied;

(Condition-4.2.8) A nonlinear function h(t,x) satisfies

(4.2.45) $\quad B(t)q(t,x) + h(t,x) = -p(x)R(t)x$,

where q(t,x) is an m-dimensional vector valued nonlinear function for $t\epsilon[t_0,t_f]$, p(x) is a nonnegative scalar-valued function and R(t) is an $n\times n$ matrix such that

(4.2.46) $\quad P(t)R(t) + R'(t)P(t) \geq 0$, for $^{\forall}t\epsilon[t_0,t_f]$,

where P(t) is defined by Eq.(4.2.31) (or Eq.(4.2.22)).

Proof: Let a scalar function V(t,x) be given by Eq.(4.2.30). Suppose that, for the preassigned terminal time t_f, a control signal u(t,x) is

† It is obvious that if the conditions of Theorem-4.2.4 are satisfied, the system (4.2.44) is completely controllable in probability.

$$(4.2.47)\quad u(t,x) = -\frac{1}{2}B'(t)P(t)x + q(t,x).$$

Using the relations (4.2.4), (4.2.5) and (4.2.44) and noting that, in this case, the differential generator L is given by

$$(4.2.48)\quad L(\cdot) = \frac{\partial(\cdot)}{\partial t} + [A(t)x + B(t)u(t,x) + h(t,x)]'\frac{\partial(\cdot)}{\partial x} + \frac{1}{2}\mathrm{tr.}[[x\sigma(t)]'\frac{\partial}{\partial x}(\frac{\partial(\cdot)}{\partial x})'[x\sigma(t)]],$$

we have

$$(4.2.49)\quad LV(t,x) = \exp[-\int_{t_0}^{t}\mathrm{tr.}[\sigma'(s)\sigma(s)]ds]x'[\frac{dP(t)}{dt} + A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t)]x + 2\exp[-\int_{t_0}^{t}\mathrm{tr.}[\sigma'(s)\sigma(s)]ds]x'P(t)[B(t)q(t,x) + h(t,x)] .$$

Applying Eq.(4.2.31) and Condition-4.2.8 to Eq.(4.2.49), it follows that

$$(4.2.50)\quad LV(t,x) = -\exp[-\int_{t_0}^{t}\mathrm{tr.}[\sigma'(s)\sigma(s)]ds]p(x)x'[P(t)R(t) + R'(t)P(t)]x \leq 0.$$

The proof has been completed.

Instead of Condition-4.2.8, the following condition is considered with respect to the nonlinear function $h(t,x)$.

[Corollary-4.2.3] The system (4.2.44) is completely controllable in the mean square sense, if Condition-4.2.7 and the following condition hold:

(Condition-4.2.9) There exists a constant μ such that $\beta(t,x)\leq\mu/2$, where $\beta(t,x)$ is a scalar function given by $h(t,x) = \beta(t,x)x$.

Proof: We shall suppose that

$$(4.2.51) \quad u(t,x) = -\frac{1}{2}B'(t)P(t)x ,$$

where P(t) is defined by Eq.(4.2.31). Define

$$(4.2.52) \quad V(t,x) \triangleq \exp[-\int_{t_0}^{t}(\mu + \mathrm{tr.}[\sigma'(s)\sigma(s)])ds]x'P(t)x .$$

From Eqs.(4.2.31) and (4.2.48), we have

$$(4.2.53) \quad LV(t,x) = \exp[-\int_{t_0}^{t}(\mu + \mathrm{tr.}[\sigma'(s)\sigma(s)])ds]$$
$$\times\{2x'P(t)h(t,x) - \mu x'P(t)x\}.$$

Applying Condition-4.2.9 to the equality (4.2.53), we have

$$(4.2.54) \quad LV(t,x) \le 0 .$$

The proof has been completed.

[Corollary-4.2.4] The system(4.2.44) is completely controllable in the mean square sense, if Condition-4.2.7 and the following condition hold:

(Condition-4.2.10) There exists a scalar function $\gamma(t,x)$ and a vector valued function $q(t,x)$ such that

$$(4.2.55) \quad B(t)q(t,x) + h(t,x) = \gamma(t,x)x .$$

and

$$(4.2.56) \quad \gamma(t,x) \le \frac{\mu}{2} .$$

Proof: Let the control signal $u(t,x)$ be given by Eq.(4.2.47) and define the scalar function $V(t,x)$ by Eq.(4.2.52).
From Condition-4.2.10, we have the inequality (4.2.54). The proof has been completed.

4.3. Stochastic Uncontrollability Theorem

In this section, a theorem is presented, stating sufficient conditions of uncontrollability with probability one for the following dynamical system ,

$$(4.3.1)\quad \left|\begin{array}{l} dx(t) = f(t,x,u)dt + G(t,x)dw(t) \\ x(t_0) = x_0 \ . \end{array}\right.$$

[Theorem-4.3.1] If the scalar function exists satisfying the following conditions:

(Condition-4.3.1) Within the time interval $[t_0,t_f)$, the scalar function $V(t,x)$ has bounded continuous first and second derivatives with respect to every component of x and a first derivative with respect to t and $V(t,x) > 0$ for any $x\epsilon R^{(n)}$ and $x\neq 0$.

(Condition-4.3.2) $V(t,x)$ satisfies, for any $u\epsilon R^{(m)}$†,

$$(4.3.2)\quad LV(t,x) \leq 0,$$

where $L(\cdot)$ is the differential generator defined by Eq.(4.2.5).

(Condition-4.3.3) For all continuous n-vector functions $a(t)$ such that $\lim_{t\to t_f} a(t) = 0$,

$$(4.3.3)\quad \lim_{t\to t_f} V(t,a(t)) = \infty ,$$

then the system(4.3.1) is not stochastically controllable with probability one, that is, $Pr\{\lim_{t\to t_f} \| x(t) \|^2 \neq 0 | x(t_0)=x_0\} = 1.$

Proof: From Condition-4.3.2, it is easy to show that [K8],

$$(4.3.4)\quad Pr\{ \sup_{t_0\leq t<\infty} V(t,x) \geq \lambda | x(t_0)=x_0\} \leq \frac{V(t_0,x_0)}{\lambda} \ .$$

On the other hand, the following relation is obvious:

$$(4.3.5)\quad Pr\{ \sup_{t_0\leq t<t_f} V(t,x) \geq \lambda | x(t_0)=x_0\} \leq Pr\{ \sup_{t_0\leq t<\infty} V(t,x) \geq \lambda | x(t_0)=x_0\} \ .$$

† In this chapter, we don't restrict the admissible control class and in the sequel assume the control signal takes any value in $R^{(m)}$.

From Eqs.(4.3.4) and (4.3.5),it follows that

$$(4.3.6)\quad \Pr\{\sup_{t_0 \le t < t_f} V(t,x) \ge \lambda \mid x(t_0)=x_0\} \le \frac{V(t_0,x_0)}{\lambda}.$$

The inequality (4.3.6) can be written as

$$(4.3.7)\quad \Pr\{\lim_{t \to t_f} V(t,x) < \infty \mid x(t_0)=x_0\} = 1 .$$

Consequently, from Condition-4.3.4, we have

$$(4.3.8)\quad \Pr\{\lim_{t \to t_f} \| x(t) \|^2 \ne 0 \mid x(t_0)=x_0\} = 1.$$

This completes the proof.

[Example-4.3]: Consider the system given by

$$(4.3.9)\quad dx(t) = Ax(t)dt + Bx(t)u(t,x)dt + h(x)dt + x(t)\sigma dw(t)$$

where

$$(4.3.10)\quad A = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

$$(4.3.11)\quad B = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$$

$$(4.3.12)\quad h(x) = \begin{bmatrix} x_1^3 \\ x_2^5 \end{bmatrix}$$

and

$$(4.3.13)\quad \sigma = [e,f].$$

Letting $V(x)=(x_1^2 + x_2^2)^{-1}$, then we have

$$(4.3.14)\quad LV(x) = (x_1^2 + x_2^2)^{-3}\{(3e^2 + 3f^2 - 2a)x_1^4 + (6e^2 + 6f^2 - 2a - 2d)x_1^2x_2^2 + (3e^2 + 3f^2 - 2d)x_2^4 - 2(b + c)(x_1^2 + x_2^2) \times x_1x_2 - 2x_1^4 - 2x_2^6\} .$$

Consequently, if

$$(4.3.15)\quad 3(e^2 + f^2) \le 2a,\ 3(e^2 + f^2) \le 2d,\ b = c ,$$

then we have $LV(x)\leq 0$. The system is thus not stochastically controllable with probability one.

4.4. Relation between Stochastic Controllability and Deterministic Controllability

In this section, the influence of random perturbation on the controllability of nonlinear systems is considered. It is sufficient to establish the relation between sufficient conditions for stochastic controllability presented here and those for deterministic controllability stated in [G1].

$$(4.4.1)\quad \left|\begin{array}{l} \dfrac{dx(t)}{dt} = f(t,x,u) \\ x(t_0) = x_0 \,. \end{array}\right.$$

[Theorem-4.4.1](Deterministic Controllability)[G1]: If a scalar function $V(t,x)$ exists with the following properties:

(Condition-4.4.1) $\dfrac{\partial V(t,x)}{\partial x}$ and $\dfrac{\partial V(t,x)}{\partial t}$ exist for all x and all $t\in [t_0,t_f)$.

(Condition-4.4.2) For all continuous, n-dimensional vector functions $a(t)$ such that $\lim_{t\to t_f} a(t) \neq 0$, $\lim_{t\to t_f} V(t,a(t)) = \infty$.

(Condition-4.4.3) A control $u(t,x)$ exists such that, along the trajectory obtained by the solution of Eq.(4.4.1), for all $t\in[t_0,t_f)$ the following inequality holds:

$$(4.4.2)\quad \frac{dV(t,x)}{dt} = \frac{\partial V(t,x)}{\partial t} + f'(t,x,u)\frac{\partial V(t,x)}{\partial x} \leq M < \infty \,.$$

then the system (4.4.1) is controllable in the deterministic sense, that is, $\lim_{t\to t_f} x(t) = 0$.

Proof: For all $t\in[t_0,t_f)$, we have the identity, (4.4.3) $$V(t,x) = V(t_0,x_0) + \int_{t_0}^{t} \frac{dV(s,x)}{ds}\, ds.$$

The inequality (4.4.2) implies

(4.4.4) $$V(t,x) \le V(t_0,x_0) + M(t-t_0).$$

Taking limits of both sides of Eq.(4.4.4), we have

(4.4.5) $$\lim_{t\to t_f} V(t,x) \le V(t_0,x_0) + M(t_f - t_0) < \infty.$$

Condition-4.4.3 implies that $\lim_{t\to t_f} x(t) = 0$ and the theorem is proved.

The following theorem stated the precise relation between Theorem-4.2.1(ε-controllability in probability) and Theorem-4.4.1(Deterministic controllability).

[Theorem-4.4.2] If the system (4.4.1) is controllable in the sense of Theorem-4.4.1, and if there exists an $n\times p$ matrix $G(t,x)$ and a quadratic scalar function $V(t,x)$ such that

(4.4.6) $$\frac{dV(t,x)}{dt} = \frac{\partial V(t,x)}{\partial t} + f'(t,x,u)\frac{\partial V(t,x)}{\partial x} \le -\frac{1}{2}\mathrm{tr.}[G'(t,x)\frac{\partial}{\partial x}(\frac{\partial V(t,x)}{\partial x})'G(t,x)] ,$$

then the system described by Eq.(4.2.1) is ε-controllable in probability in the sense of Theorem-4.2.1.

Proof: From Eq.(4.4.6), it follows that

(4.4.7) $$\frac{\partial V(t,x)}{\partial t} + f'(t,x,u)\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\mathrm{tr.}[G'(t,x)\frac{\partial}{\partial x}(\frac{\partial V(t,x)}{\partial x})'G(t,x)] \le 0 .$$

Hence

(4.4.8) $$LV(t,x) \le 0 .$$

Let the terminal time of deterministic systems denoted by t_f^d which corresponds to the symbol t_f in Theorem-4.4.1[G1]. As shown in Fig.4.4.1, taking the ε-controllable region into account, the terminal time of stochastic systems t_f defined by the Sec.4.2 is considered to be

(4.4.9) $t_f^d - \delta \triangleq t_f^s$,

where δ is an arbitrary, and this can be determined by preassigning the controllable region.

From Eq.(4.2.2), set as

(4.4.10) $$\alpha = \frac{\| x_{t_f}^s \|^2}{V(t_f^s, x)} \quad .$$

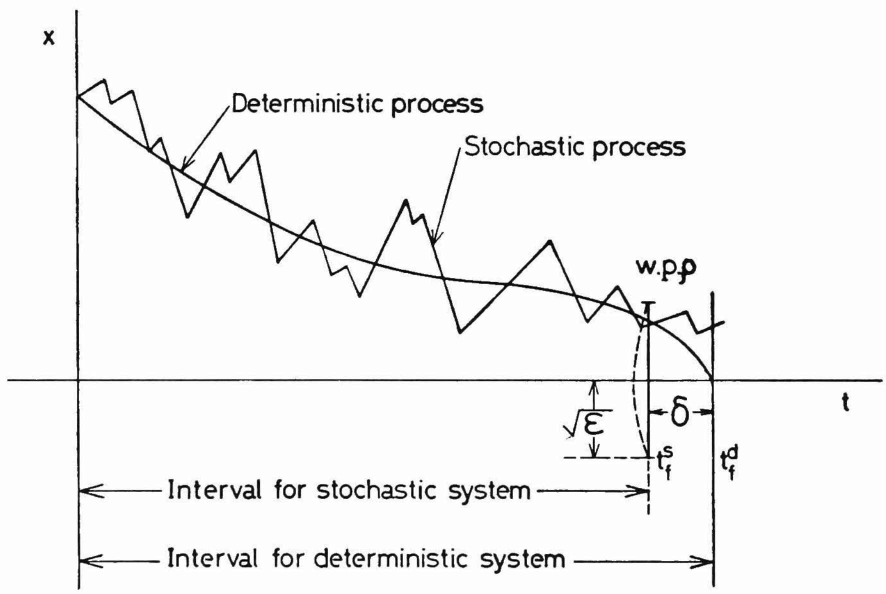

Fig. 4.4.1 Relation between t_f^d and t_f^s .

Since, from Eq.(4.4.6),

$$(4.4.11)\quad V(t_f^s,x) = V(t_0,x_0) + \int_{t_0}^{t_f^s}\frac{dV(s,x)}{ds}ds < V(t_0,x_0) < \infty ,$$

the value of α is nonzero. Thus the conditions in Theorem-4.2.1 hold and the proof has been completed.

The explicit relation between complete controllability in the mean square sense and deterministic controllability for linear systems is listed in Table-4.4.1.

Table 4.4.1 Comparision of stochastic controllability and deterministic controllability for linear systems.

	Deterministic Controllability	Stochastic Controllability
System dynamics	$\dot{x}(t) \quad A(t)x(t) + B(t)u(t)$ $x(t_0) \quad x_0$	$dx(t) = A(t)x(t)dt + B(t)u(t,x)dt + x(t)\sigma(t)dw(t)$ $x(t_0) = x_0$
Necessary and sufficient condition	$\int_{t_0}^{t_f}\Phi(t_f,s)B(s)B'(s)\Phi'(t_f,s)ds>0$ $\frac{\partial\Phi(t,s)}{\partial t} = A(t)\Phi(t,s)$ $\Phi(s,s) = I$	$\int_{t_0}^{t_f}\Phi(t_f,s)B(s)B'(s)\Phi'(t_f,s)ds>0$ $\frac{\partial\Phi(t,s)}{\partial t} = A(t)\Phi(t,s)$ $\Phi(s,s) - I$
Control signal	open-loop / feed-back	feed-back

From Theorem-4.4.1[G1], we can easily obtain the same sufficient conditions as stated in Theorem-4.2.4, Corollary-4.2.3 and Corollary-4.2.4 for the following deterministic nonlinear system,

$$(4.4.12)\quad \left| \begin{array}{l} \dfrac{dx(t)}{dt} = A(t)x(t) + h(t,x) + B(t)u(t,x) \\ x(t_0) = x_0 \end{array} \right.$$

4.5. Examples for Nonlinear Controllable Systems

In this section, consider the 2-dimensional nonlinear stochastic systems with constant coefficients,

$$(4.5.1)\quad \left| \begin{array}{l} dx(t) = Ax(t)dt + h(x)dt + Bu(t,x)dt + x(t)\sigma dw(t) \\ x(t_0) = x_0 \ , \end{array} \right.$$

where the matrices A and B are respectively assumed to satisfy Condition-4.2.7. Then, if the nonlinear term h(x) is vanished, the system (4.5.1) is completely controllable in the mean square sense.

[Example-4.5.1] Consider the following forms of nonlinear function h(x) and coefficient matrix B given by

$$(4.5.2)\quad h(x) = \begin{bmatrix} 0 \\ -x_2^3 \end{bmatrix}$$

and

$$(4.5.3)\quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ,$$

respectively. In this case, choosing the q(t,x)-function stated in Condition-4.2.8 of Theorem-4.2.4 by

$$(4.5.4)\quad q(t,x) = -x_1 \cdot x_2^2 ,$$

we have

$$(4.5.5)\quad Bq(t,x) + h(x) = -x_2^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Then, we may find

$$(4.5.6)\quad p(x) = x_2^2 \geq 0$$

and

(4.5.7) $R = I$.

From Eqs.(4.5.6) and (4.5.7), Condition-4.2.8 is satisfied. Hence it is apparent that the system (4.5.1) is completely controllable in the mean square sense and in probability.

[Example-4.5.2] Consider the following nonlinear function h(x) with the same coefficient matrix B as in Eq.(4.5.3),

(4.5.8) $$h(x) = \begin{bmatrix} 0 \\ -x_2(\tanh(x_2) + 1) \end{bmatrix}.$$

Choosing the q(t,x) function as

(4.5.9) $q(t,x) = -x_1(\tanh(x_2) + 1)$,

we have

(4.5.10) $$Bq(t,x) + h(x) = -(\tanh(x_2) + 1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It follows that $R(t) = I$ and

(4.5.11) $p(x) = (\tanh(x_2) + 1) \geq 0$.

Then, from Eq.(4.5.11), all conditions stated in Theorem-4.2.4 are satisfied.

[Example-4.5.3] With the same coefficient matrix B as in Example-4.5.1, the following nonlinear function is considered,

(4.5.12) $$h(x) = \tanh(x_1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It is apparent that the nonlinear function h(x) given by Eq.(4.5.12) satisfies Condition-4.2.9 in Corollary-4.2.3, that is, in this case, we find

(4.5.13) $\beta(t,x) = \tanh(x_1) \leq 1$.

Then, the system (4.5.1) with the nonlinear function h(x) defined by (4.5.12) is completely controllable in the mean square sense and in probability.

4.6. Discussions and Summary

In analogy with the deterministic version of deterministic controllability, the stochastic complete controllability and the stochastic ε-controllability have respectivrly been defined and necessary and /or sufficient conditions for stochatic controllability defined here have also been exploited including comparative discussion on mutual relations between those two controllability concepts. Four theorems of the above mentioned stochastic controllability were established which gave the sufficient conditions for nonlinear stochastic systems and the necessary and sufficient condition for linear stochastic systems. Furthermore, a uncontrollability theorem w.p.1 has been presented, showing sufficient conditions for nonlinear stochastic systems. In order to show the influence of system noise disturbances to the stochastic controllability, a theorem has also been established stating the relation between the stochastic controllability and deterministic one, by introducing Lyapunov like function approach.

For the purpose of better understanding, three examples of completely controllable nonlinear systems in the mean square sense were demonstrated.

CHAPTER 5. APPLICATION OF STOCHASTIC CONTROLLABILITY THEOREM TO PRACTICAL CONTROL PROBLEM

5.1. Introductory Remarks

From the stochastic complete controllability theorems given in Chap.4, Sec.4.2, we may learn that there exists some control signal which transfers the initial state into the arbitrary small target region with the norm $\sqrt{\varepsilon}$. However, different from deterministic systems, we have to examine the problem "Can we generate an exact control signal which transfers the initial system state to the desired target domain?". The ε-controllability theorems stated in Chap.4, Sec.4.2 give the profitable answer for the above mentioned problem. In this chapter, from the notion of deterministic reachability[D2], the stochastic hitting problem of control systems is discussed.

5.2. Hitting Problem for Nonlinear Lumped Parameter Systems

In this section, for the purpose of better understanding, the system dynamics is limitted to

$$(5.2.1)\quad \left|\begin{array}{l} dx(t) = A(t)x(t)dt + h(t,x)dt + B(t)u(t,x)dt + x(t)\sigma(t)dw(t) \\ x(t_0) = x_0. \end{array}\right.$$

Stochastic Hitting Problem: We assume that the *a priori* given parameters are the initial condition x_0, the terminal time t_f and the target domain ε. Furthermore, the hitting probability ρ, defined by

$$(5.2.2)\quad \rho = \Pr\{\|x(t_f)\|^2 < \varepsilon | x(t_0)=x_0\} ,$$

is preassigned. The stochastic hitting problem is to generate a control signal which transfers the initial state x_0 into the desired target domain with the radious $\sqrt{\varepsilon}$ at the terminal time t_f with a hitting probability larger than ρ given by Eq.(5.2.2).

Recalling the results given in Theorem-4.2.4, if the system (5.2.1) is completely controllable in the mean square sense, we may find that there exists a control signal satisfying the stochastic hitting problem. Then, from the relation between complete controllability in the mean square sense and ε-controllability in probability, the results of Theorem-4.2.4 are applicable to the hitting problem.

Under the assumption that "The considered system (5.2.1) is completely controllable in the mean square sense.", we discuss how to design the practical control signal which satisfies the hitting problem.

With Theorems-4.2.1, -4.2.2 and the assumption that the system (5.2.1) is completely controllable in the mean square sense, we choose the control signal,

$$u(t,x) = -\frac{1}{2}B'(t)P(t)x + q(t,x), \tag{5.2.3}$$

where

$$\left| \begin{aligned} &\frac{dP(t)}{dt} + A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t) \\ &\qquad\qquad\qquad + tr.[\sigma'(t)\sigma(t)]P(t) = 0 \\ &P(t_f) = I/\alpha \end{aligned} \right. \tag{5.2.4}$$

and where the value of α must be determined so as to satisfy the hitting problem.

From the results of Theorems-4.2.1, -4.2.2 and -4.2.4, we have

$$x_0'P(t_0)x_0 \le (1 - \rho)\frac{\varepsilon}{\alpha} . \tag{5.2.5}$$

It should be noted that the control function $u(t,x)$ determined by Eq. (5.2.3) depends on $P(t)$ whose terminal state is given by a choice of α. Consequentry, the value of α may be determined in terms of the values ρ, x_0, ε and t_f, satisfying the inequality (5.2.5).

(With the completely controllable assumption, it is obvious that there exists the value of α which satisfies Eq.(5.2.5).)

Use of the following graphical procedure is practical and convenient in determing the value of α.

(i) Given the initial state x_0,the target domain ε and the terminal time t_f.

(ii) Compute the solution process $P(t)$ of Eq.(5.2.4) with the various terminal values of α and obtain the initial value of $P(t_0)$.

(iii) Compute the value of ρ which satisfies

(5.2.6) $\rho = 1 - \frac{\alpha}{\varepsilon} x_0{}' P(t_0) x_0$

and drow the relation between ρ and α with the value of t_f as the parameter.

The procedure stated above will be applied to practical examples in Sec.5.3 .

5.3. Numerical Examples

In this section, we shall consider the 2-dimensional stochastic system whose sample path is determined by

$$(5.3.1) \quad \left| \begin{array}{l} dx(t) = Ax(t)dt + h(x)dt + Bu(t,x)dt + x(t)\sigma dw(t) \\ x(t_0) = x_0 \ . \end{array} \right.$$

[Example-5.3.1] We set all parameters of Eq.(5.3.1) as:

$$(5.3.2) \quad A = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$$

$$(5.3.3) \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(5.3.4) \quad \sigma = [0.6 \ 0.6]$$

and

(5.3.5) $$h(x) = \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix}.$$

From Eq.(5.2.3) in Sec.5.2, the control is chosen as

(5.3.6) $$u(t,x) = -\frac{1}{2}(p_{11}(t)x_1 + p_{12}(t)x_2) - x_1x_2^2,$$

where

(5.3.7) $$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$$

and from Eq.(5.2.4), each element is obtained by solving the following differential equations;

(5.3.8) $$\dot{p}_{11}(t) - 1.28p_{11}(t) - 2p_{12}(t) - p_{12}(t)^2 = 0, \; p_{11}(t_f) = \alpha^{-1}$$

(5.3.9) $$\dot{p}_{12}(t) - 2.28p_{12}(t) - p_{22}(t) - p_{11}(t)p_{12}(t) = 0, \; p_{12}(t_f) = 0$$

and

(5.3.10) $$\dot{p}_{22}(t) - 2.28p_{22}(t) - p_{12}(t)^2 = 0, \; p_{22}(t_f) = \alpha^{-1}.$$

From Eqs.(5.3.2) and (5.3.3), we have

(5.3.11) $$\mathrm{rank}[B, AB] = 2.$$

Eqs.(5.3.11) and (5.3.5) imply that the system (5.3.1) is completely controllable in the mean square sense, that is, there exists a value of α which satisfies the hitting problem.

Following the general procedure mentioned in Section 5.2, the values of ρ and α can be determined by the following manner:

(i) Set the initial state variables as, for instance, $x_1(0)=x_2(0) = 10$ and the circle with the radius $\sqrt{\varepsilon} = \sqrt{0.8}$ as the target domain, where $t_0 = 0$.

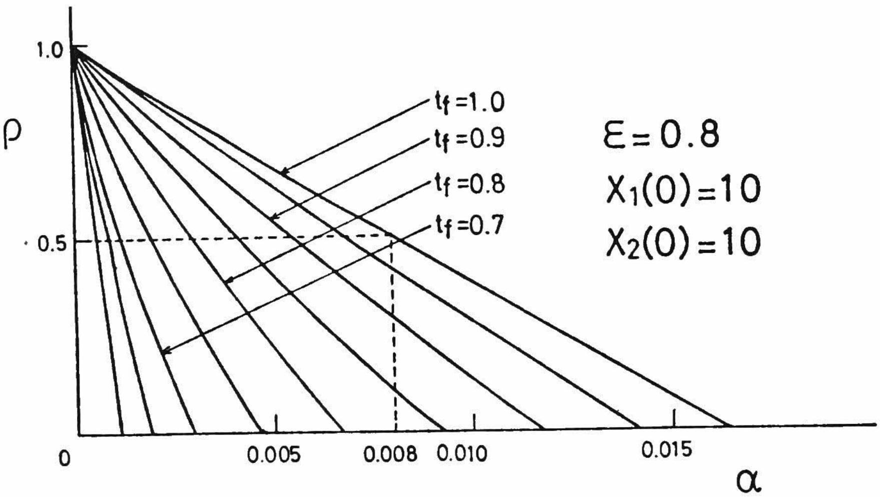

Fig. 5.3.1 Relation between ρ and α in Example-5.3.1

(ii) Preassign the value of t_f as $t_f = 1.0$. Solve the differential equations (5.3.8), (5.3.9) and (5.3.10) simultaneously and obtain the initial values $p_{11}(0)$, $p_{12}(0)$ and $p_{22}(0)$ with respect to the various values of α.

(iii) Applying the relation between α and

(5.3.12) $$P(0) = \begin{bmatrix} p_{11}(0) & p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{bmatrix}$$

to the equality

(5.3.13) $$\rho = 1 - \frac{\alpha}{\varepsilon} x_0{}' P(t_0) x_0 ,$$

the relation between ρ and α is obtained.

Figure 5.3.1 shows the relation between ρ and α with the value of t_f as a parameter. Thus, the value of α may easily be found in

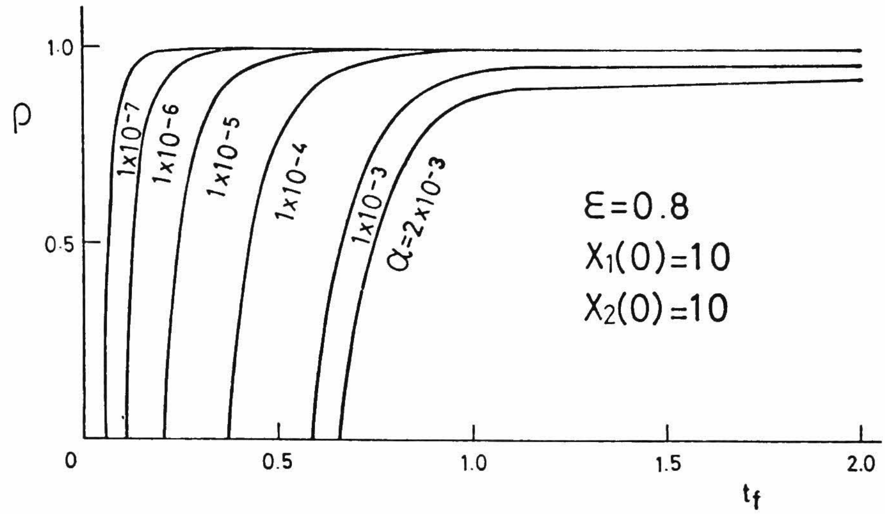

Fig. 5.3.2 Relation between ρ and t_f in Example-5.3.1

terms of the hitting probability ρ. For convenience of discussions the relation between ρ and t_f is shown in Fig. 5.3.2 with the value of α as a parameter.

By using Eq.(5.3.6) , the controllable regions are determined on the (x_1, x_2)-plane as shown by the shaded area in Fig. 5.3.3 with the value of α as a parameter, in Fig. 5.3.4 with the value of ρ as a parameter, in Fig. 5.3.5 with the value of ε as a parameter and in Fig. 5.3.6 with the value of t_f as a parameter.

A variety of sample runs is simulated. The results presented below are representative of the simulation experiments. In all experiments, the simulation procedure followed by the method in [S8] with a constant partition time 0.001s. The result of a sample trajectory of the system (5.3.1) driven by the control signal (5.3.6) is shown in Fig. 5.3.7 where the value of ρ and α are, respectively,

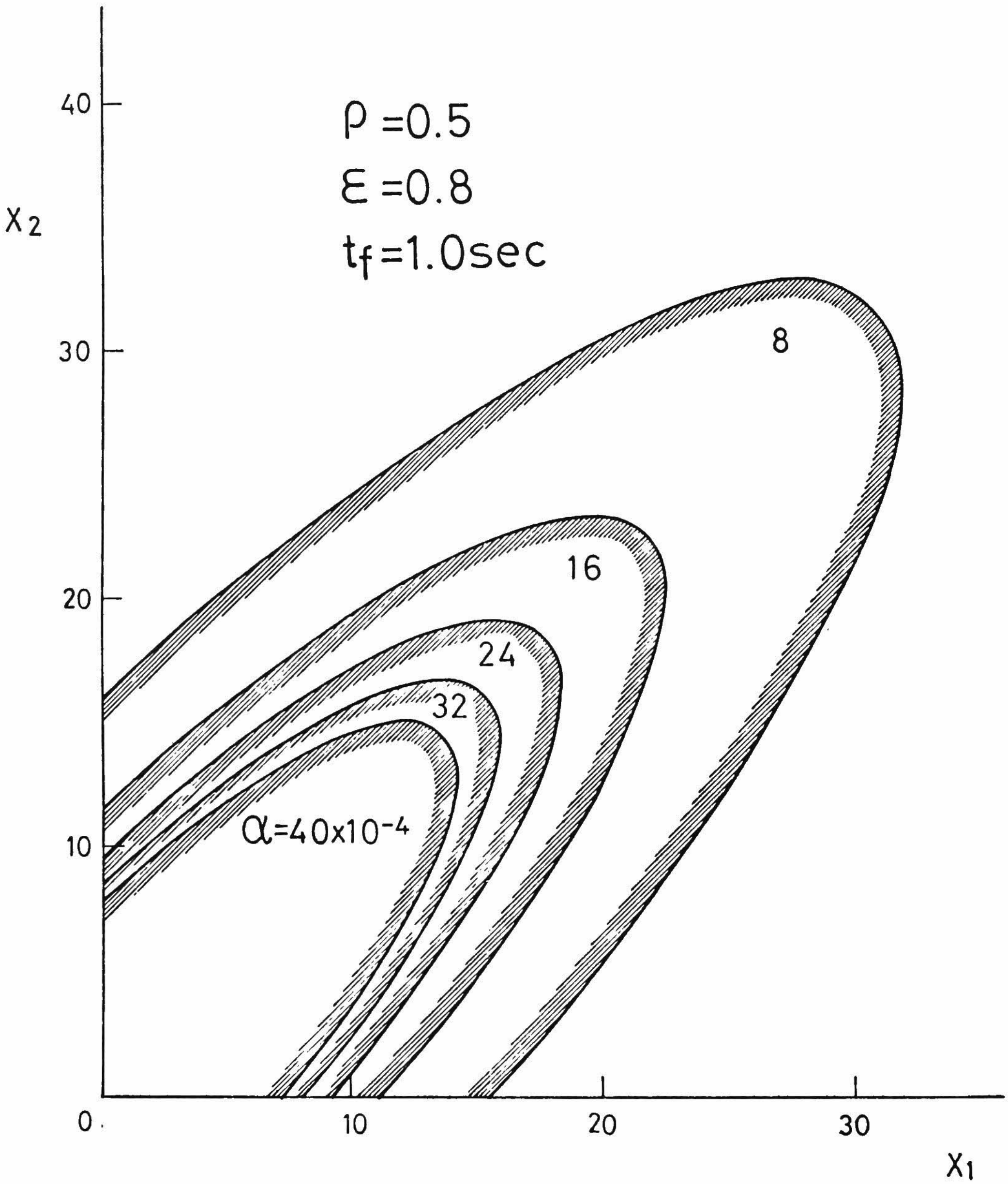

Fig. 5.3.3 Controllable region with the value of α as a parameter in Example-5.3.1

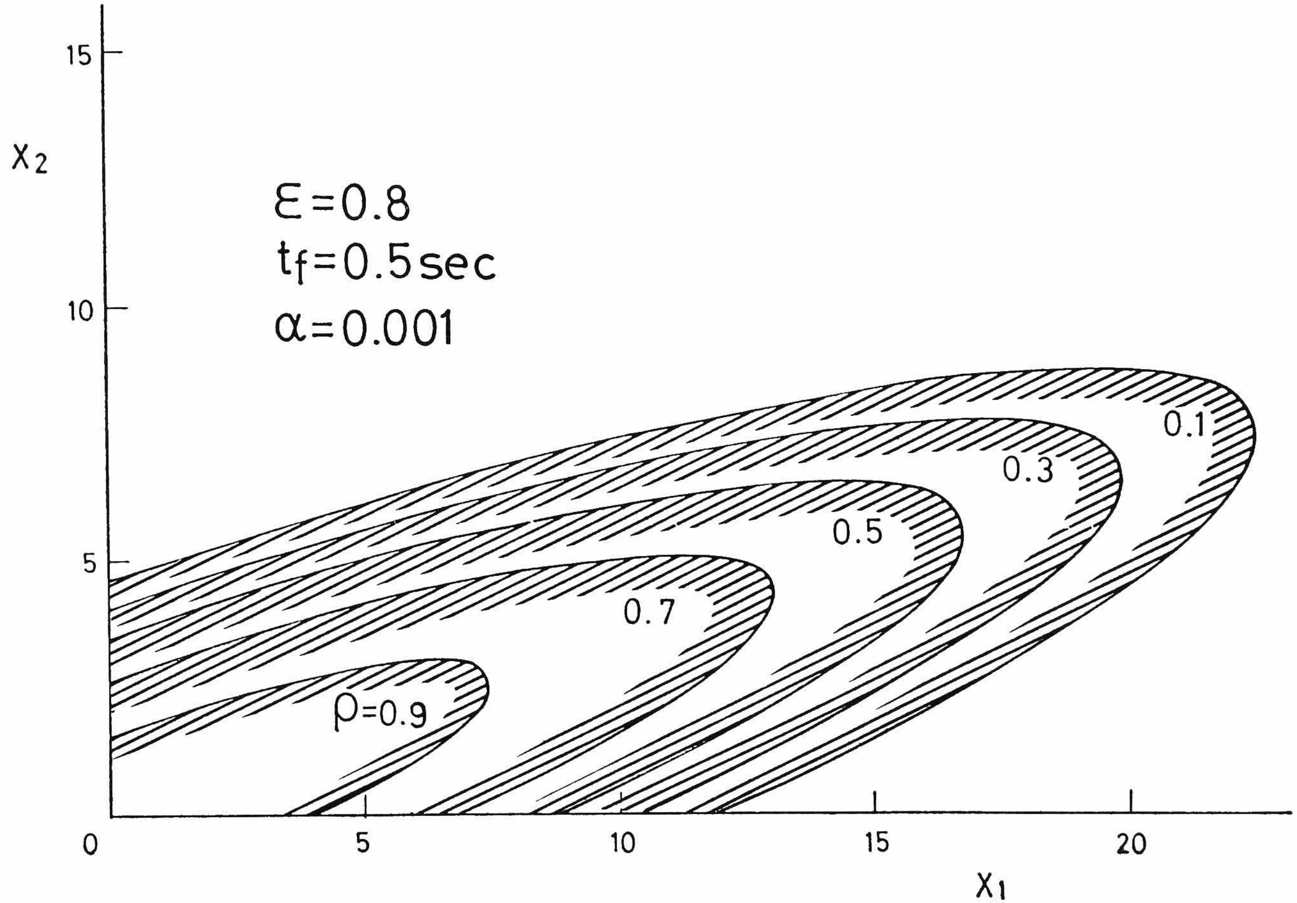

Fig. 5.3.4 Controllable region with the value of ρ as a parameter in Example-5.3.1

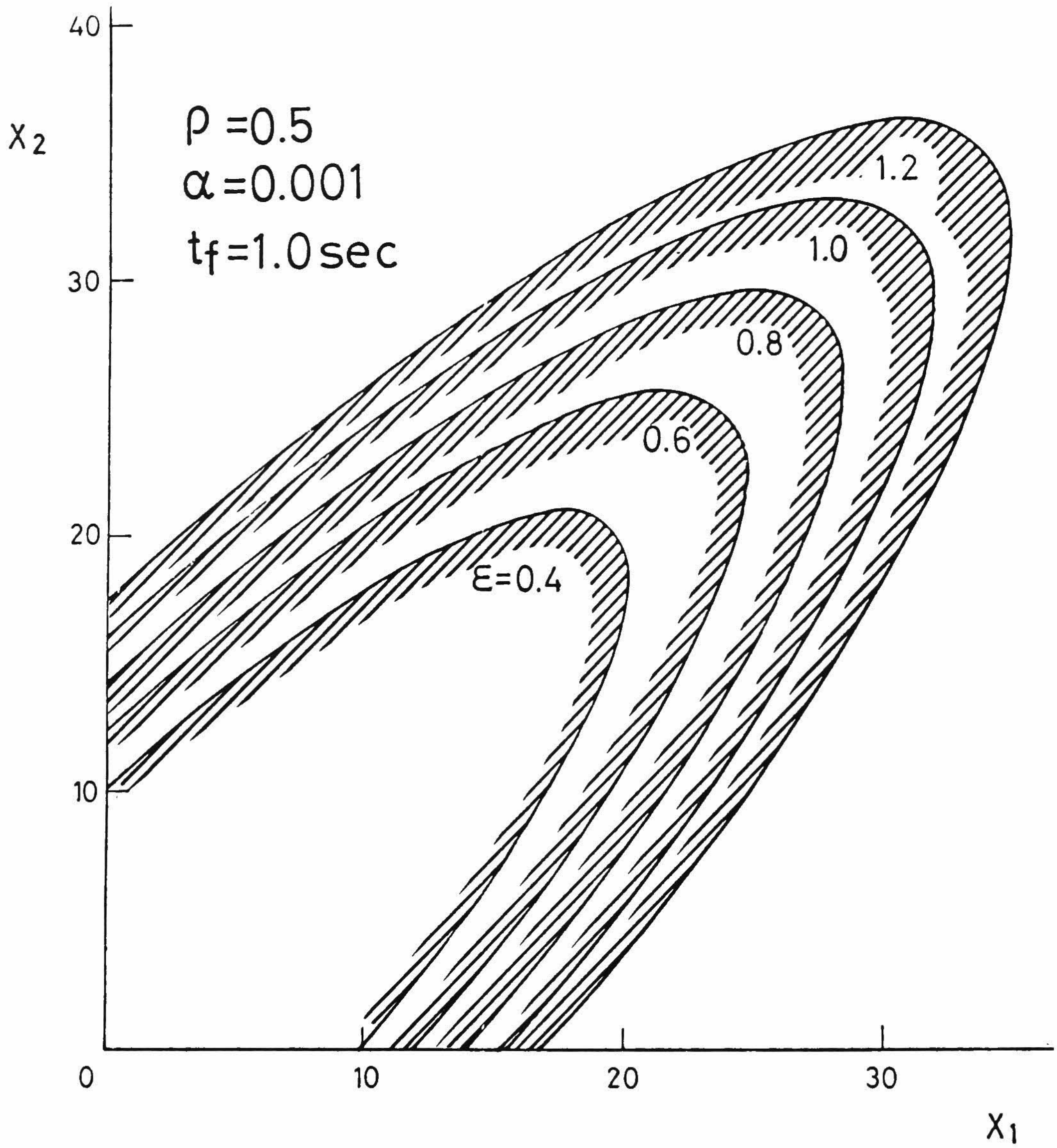

Fig. 5.3.5 Controllable region with the value of ε as a parameter in Example-5.3.1

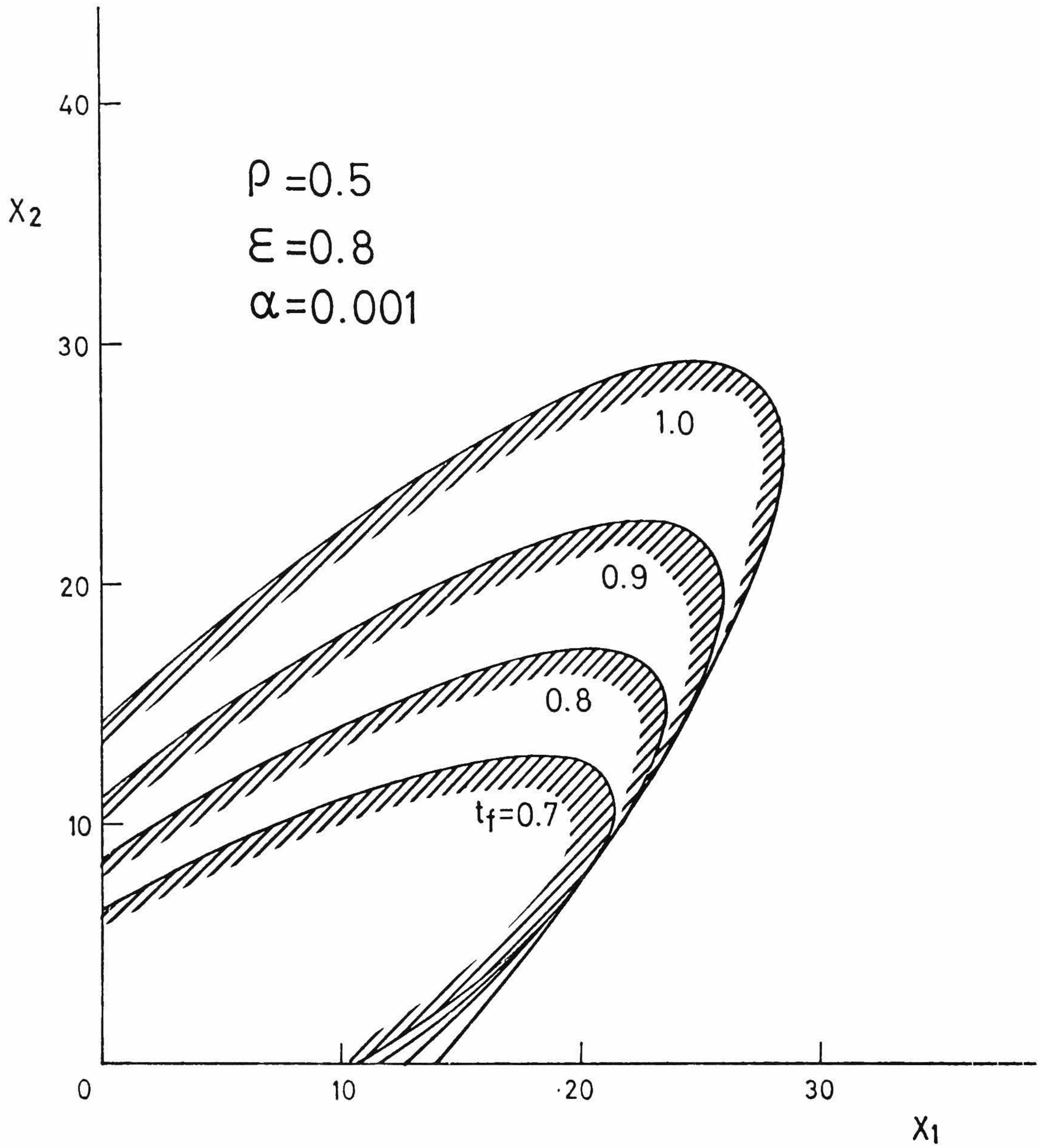

Fig. 5.3.6 Controllable region with the value of t_f as a parameter in Example-5.3.1

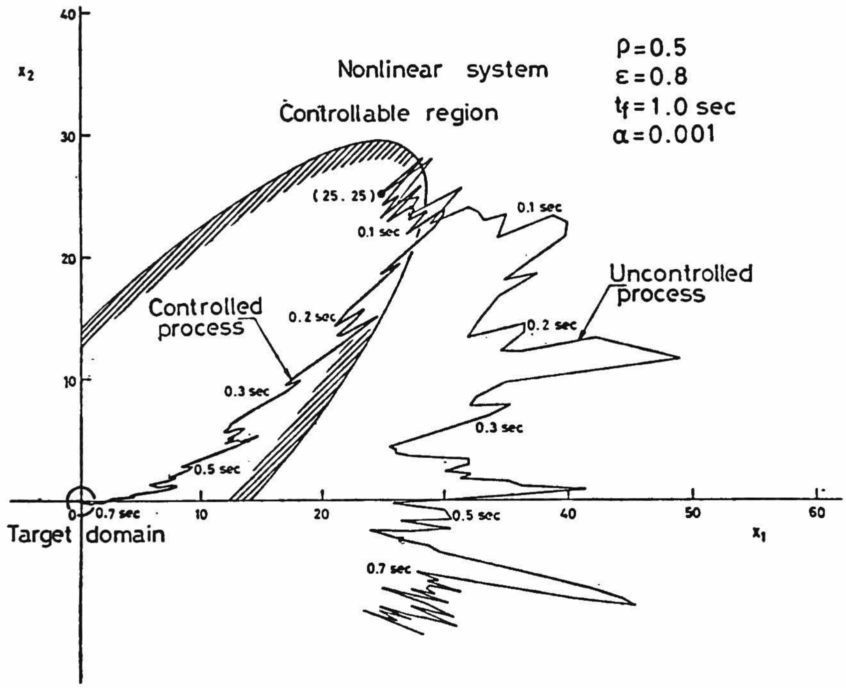

Fig. 5.3.7 Sample trajectories of nonlinear systems in Example-5.3.1 (ε-controllable initial state)

0.5 and 0.001. From this result, we may find that a trajectory whose initial state was (25,25) reached the target domain with the radius $\sqrt{\varepsilon} = \sqrt{0.8}$ within the preassigned time $t_f = 1.0$s. Thus the initial state (25,25) is ε-controllable with probability 0.5. Although it is difficult to examine the ε-controllability with probability 0.5, many simulation experiments brought corrobolating results. From Fig.5.3.8, it is apparent that both the initial state (26,7) and (2,26) are not controllable. If h(x) = 0, then the following control is adopted

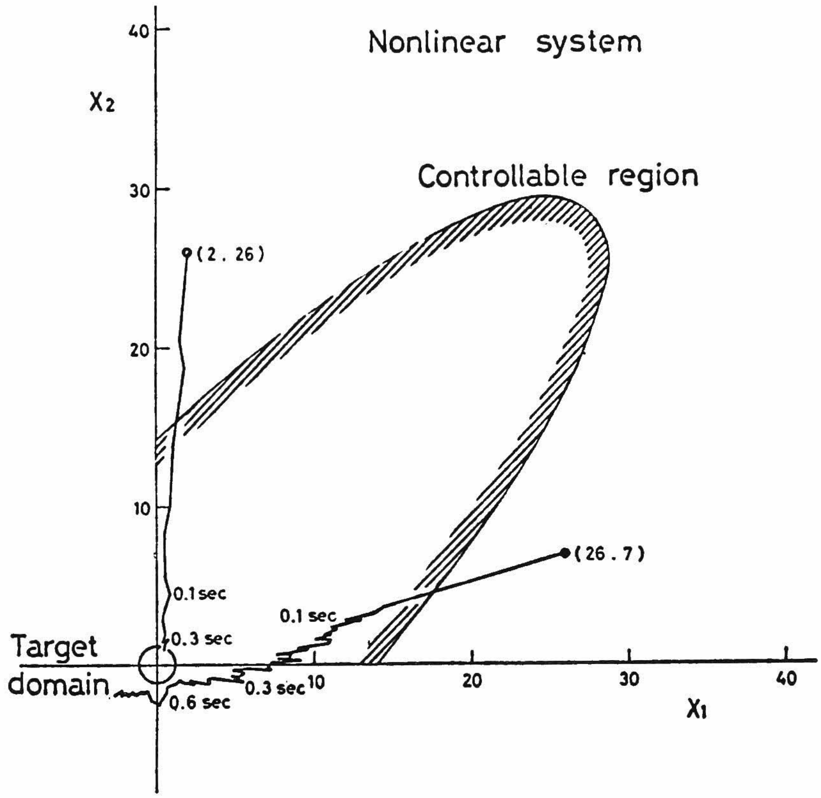

Fig. 5.3.8 Sample trajectories of nonlinear systems in Example-5.3.1 (ε-uncontrollable initial states)

$$u(t,x) = -\frac{1}{2}(p_{11}(t)x_1 + p_{12}(t)x_2), \tag{5.3.14}$$

where both $p_{11}(t)$ and $p_{12}(t)$ are the solutions to Eqs.(5.3.8) and (5.3.9). Two sample runs are shown of linear systems with controllable initial states (25,25) and (10,10) driven by the control signal u(t,x) given by Eq.(5.3.14) in Fig.5.3.9. In Fig.5.3.10, two sample runs are shown in the case of ε-uncontrollable initial states (2,26) and (26,7), respectively.

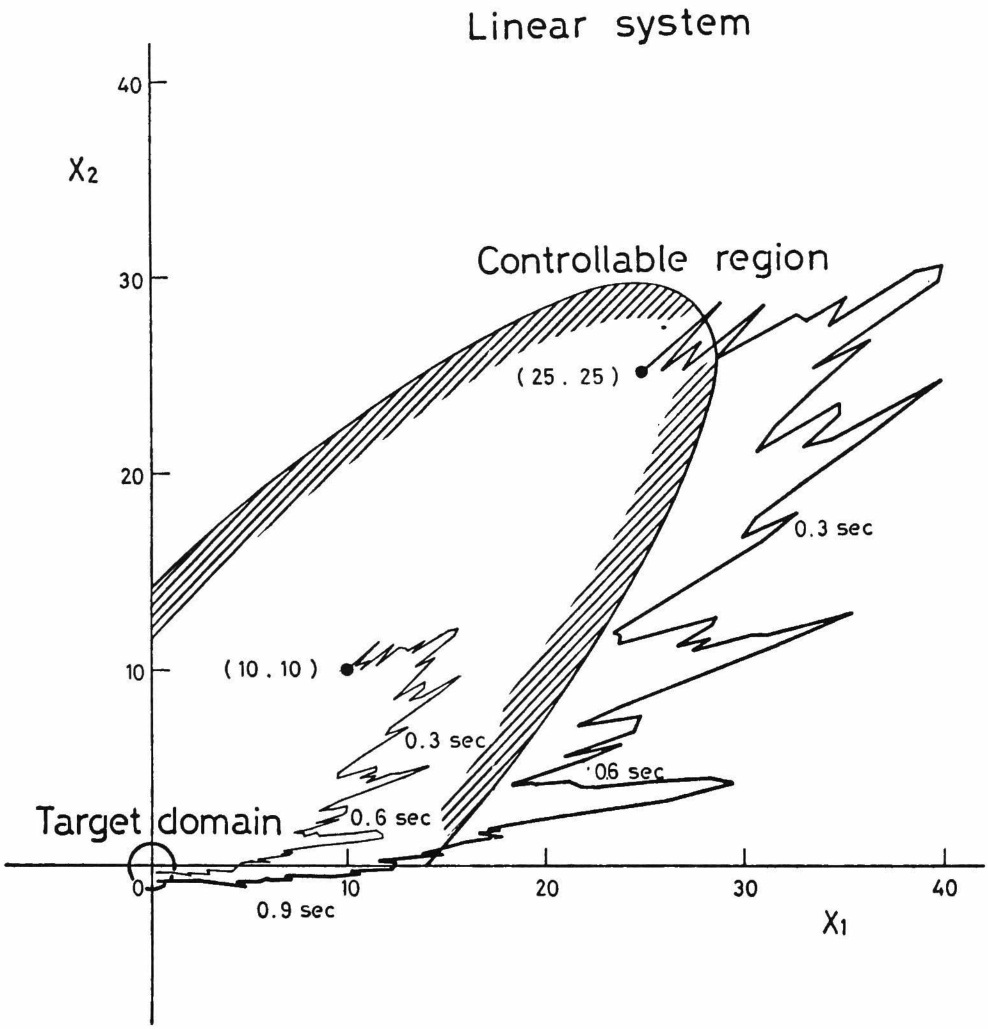

Fig. 5.3.9 Sample trajectories of linear systems in Example-5.3.1 (ε-controllable initial states)

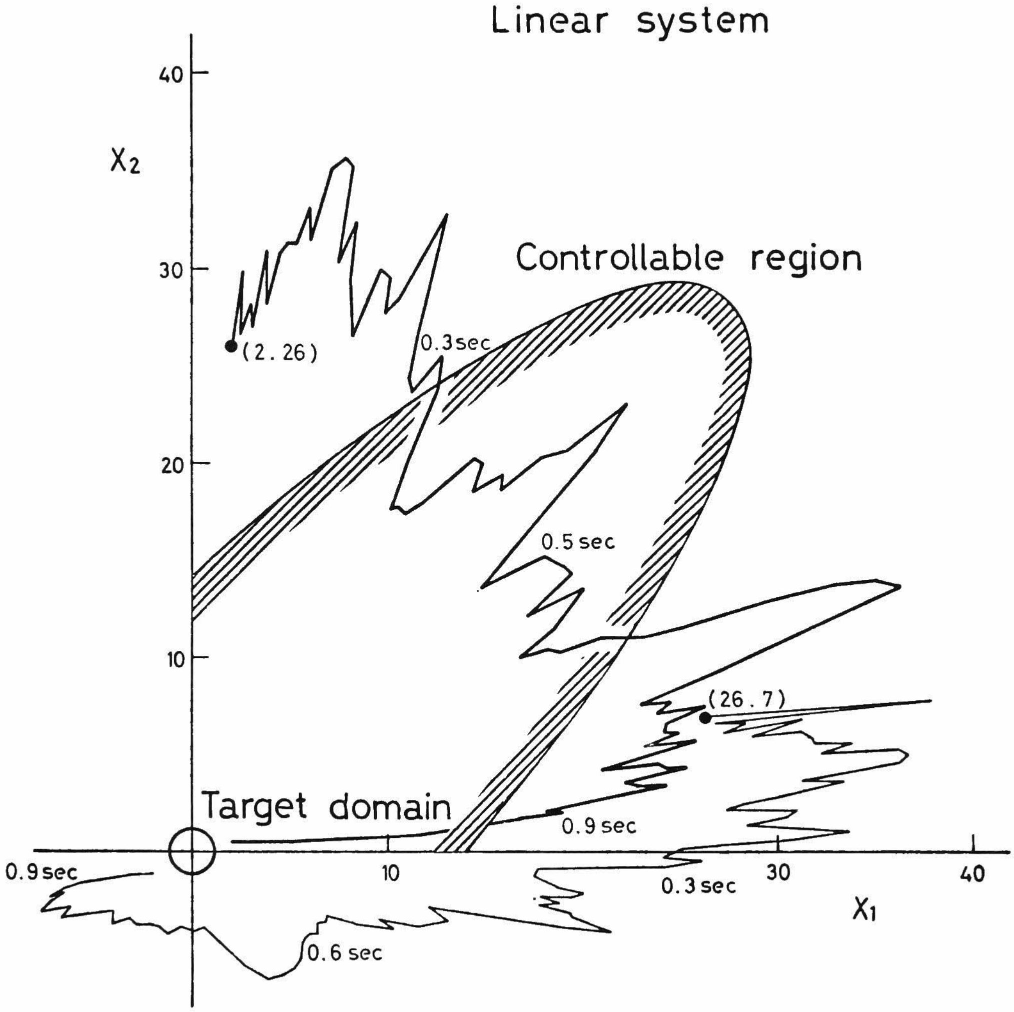

Fig. 5.3.10 Sample trajectories of linear systems in Example-5.3.1 (ε-uncontrollable initial states)

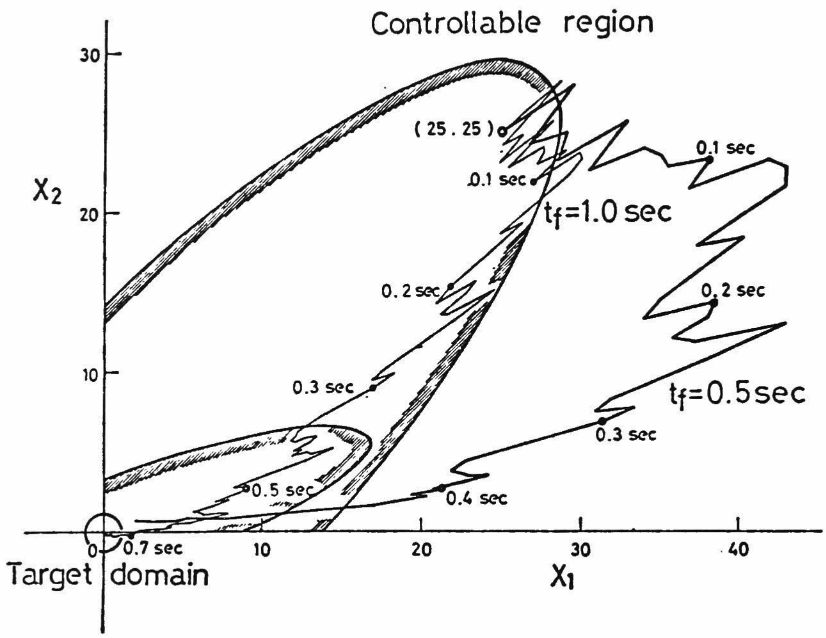

Fig. 5.3.11 Sample trajectories of nonlinear systems in Example-5.3.2

[Example-5.3.2] Consider once again the system given by Eq.(5.3.1), where all parameters are the same as in Example-5.3.1 but

$$(5.3.15) \quad h(x) = \begin{bmatrix} 0 \\ - x_2(\tanh(x_2) + 1) \end{bmatrix}.$$

The control signal adopted is

$$(5.3.16) \quad u(t,x) = -\frac{1}{2}(p_{11}(t)x_1 + p_{12}(t)x_2) - x_1(\tanh(x_2) + 1)$$

where $p_{11}(t)$ and $p_{12}(t)$ are respectively determined by solving Eqs. (5.3.8) and (5.3.9).

A result of the digital simulation studies is shown in Fig.5.3.11. The principal purpose of this experiment is to show the influence of the preassigned control interval t_f on the ε-controllability. Two sample trajectories are shown with the same initial state (25,25) and $P(0.5) = I/0.001$, where $\rho = 0.5$, $\varepsilon = 0.8$ and $\alpha = 0.001$. In the case where $t_f = 0.5$sec, the initial state was not controllable.

5.4. Discussions and Summary

With the aid of ε-controllability theorems in probability presented in Chap.4, the stochastic hitting problem has been formulated, guaranteeing a transfer of a system state into the target domain within the preassigned terminal time t_f. In Secs.5.2 and 5.3, the computer aided numerical algorithm for generating the exact control signal has also been presented with a wide variety of digital simulation studies.

The proposed approach in this chapter gives a feasible way to solve the inequality constrained control problem, that is, to find the control signal which satisfies the following constraint, for the preassigned function M(t),

$$(5.4.1)\quad E_{x_0}\{\| x(t) \|^2\} \le M(t) \quad , \text{ for } \forall t \in [t_0, t_f] \; .$$

In order to solve the problem mentioned above, using the same control signal as in Eq.(5.2.3), we have

$$(5.4.2)\quad E_{x_0}\{\| \; x(t) \; \|^2\} \le \frac{x_0{}'P(t_0)x_0}{\| P(t) \|_{min}} ,$$

where $\| P(t) \|_{min}$ denotes the minimal eigenvalue of P(t). Then, with the aid of the numerical algorithm presented in this chapter, we can easily find the value of α which satisfies

$$(5.4.3)\quad \frac{x_0{}'P(t_0)x_0}{\| P(t) \|_{min}} \le M(t) \; , \text{ for } \forall t \in [t_0, t_f] \; .$$

In this chapter, for the purpose of better understanding, the considered system is assumed to be completely controllable in the mean square sense. However, the algorithm stated in Sec.5.2 is applicable for the dynamical systems which are completely controllable in the mean square sense or not. However the numerical method of determing the controllable region of initial states presented in numerical examples gives the powerful tool to generate a practical control signal.

CHAPTER 6. CONCLUSIONS

6.1. Discussions

It is well known that, in linear deterministic systems, there is an interesting principle of the duality between the controllability and observability. In stochastic systems we can examine the relation between the stochastic controllability and stochastic observability. For example, consider the following dynamical systems;

$$(6.1.1)\quad \begin{cases} \dot{x}(t) = f(t,x) \\ dy(t) = h(t,x)dt + R(t)dw(t). \end{cases}$$

The problem of stochastic observability, which has already been defined by the auther [S3],[S4],[S5],[S6] and [A2], for the system described by Eq.(6.1.1), can be solved by using the same approach presented as in Part I. For linear stochastic systems, the duality like relation has also been obtained in [S3].

Although Part I will contribute to the study of stochastic controllability for nonlinear systems, the study of stochastic controllability under noisy observations will be one of the topics of current reserches.

6.2. Concluding Remarks

New definitions of the ε-controllability and the complete controllability has, in Part I, been established for a general class of nonlinear stochastic systems. Theorems were also stated giving sufficient conditions and/or necessary and sufficient conditions for the stochastic complete controllability. In particular, influences of nonlinearities exhibited in stochastic systems on the stochastic controllability have been exploited by the stochastic Lyapunov function-like approach. However, we pause for a moment to remark that a large body of stochastic controllability results depends on a choise of the $V(t,x)$ function based on a type of nonlinearities as in the deterministic situation.

We shall close Part I by discussing briefly some results on a rather significant probrem of stochastic controllability theory. Which of all the controllability concepts we have mentioned is most useful, or most significant? This must depend on the problem on a possibility of establishing a feasible algorithm for generating the control signal which transfers the initial system to the target domain within the preassigned time interval. From this point of views, the stochastic ε-controlability concept may be concluded as a useful concept in connection with the hitting probrem described in Chap. 5.

PART 2

OPTIMAL CONTROL AND CONTROLLABILITY

FOR

STOCHASTIC DISTRIBUTED PARAMETER SYSTEMS

CHAPTER 1. INTRODUCTION

In view of the recent trend of rapidly advanced technology in particular, industrial, biological environmental and social sciences, it is more adequate that partial differential equations are introduced to describe the dynamical behaviors of system rather than ordinary differential equations. In practical problems, one or more coefficients of distributed parameter systems are random because measurement of physical properties of the system considered intrinsically exhibits various kinds of uncertainties. Thus, in order to solve problems of finding the optimal control and controllability of stochastic distributed parameter systems, our attention should be placed on a general class of stochastic partial differential equations whose differential operator contains random coefficients. Considering the distributed parameter systems modeled by a stochastic differential equation of Parabolic type and Hyperbolic type, in Part 2, the optimal control and controllability problems are solved by using the notion of the well-known function space analysis, where statistics of random coefficients are considered to be the Markov chain process and the white Gaussian noise process.

The main body of Part 2 is concerned with the distributed optimal control, the boundary optimal control and controllability of stochastic distributed parameter systems.

1.1. Historical Background

The historical background is divided into the following two aspects.

1.1.1. Optimal Control

A valuable work for distributed parameter systems was initiated by Butkovskii [B6]. His work was concentrated on problem formulation and the derivation of a maximum principle for a certain class of distributed parameter systems governed by a set of nonlinear integral equations. Subsequently, Brogan [B7] and Wang [W4] studied in detail the optimal control problem with various performance indices. Recently, using the function space representation technique, Lions [L4] developed a general discussion of various aspects of optimal control of linear deterministic distributed parameter systems with the aid of variational inequality. In analogy with the stochastic lumped parameter systems, significant contributions to the stochastic distributed parameter systems were made by Kushner [K9] and Tzafestas [T1]. In both [K9] and [T1], only the additive noise disturbances were considered and very complex calculations were demanded. From the recent advance of semi-group and generalized function theory [Y1],[K10],[K11],[G3], Balakrishnan [B8], Bensoussan [B9]∿[B11] and Curtain [C3],[C4] have been succeeded in establishing the total theory of optimal control and estimation problem for stochastic distributed parameter systems with additive noise disturbances. However, in the excellent works by Bensoussan and Curtain, the randomness of coefficients in the partial differential operator was not considered. For a class of stochastic distributed parameter systems with random coefficients, first Boyce challenged to study the stochastic properties of solution process behaviours by introducing the honest and dishonest methods [B12] and Tsokos [T2] and Sunahara [S9] discussed about the optimal control problem with the aid of random eigenvalue problems. In these works, the randomness of coefficients was restricted to the class of random constants. Recently, Sunahara and the author made an effort to formulate the mathematical system with stochastically varying coefficients and studied various types of optimal control problems, [S10] ∿ [S15],[P1].

1.1.2. Controllability Problem

For the deterministic controllability problem of distributed parameter systems, first, Wang has established in analogy with the well-known Kalman's theory,[W4]. However, since the fundamental solution stemmed by the semigroup associated with the system operator can not easily be obtained, it is difficult to examine the controllability conditions by Wang. Fattorinii considered another controllability definition and obtained the necessary and sufficient condition for complete controllability with the aid of spectral representation theory, [F3]∿[F5],[T3]. Recently, many investigators pay their attentions to generalize the Fattorinii's works. Russell [R1]∿[R4] and Trigianii [T4]∿[T6] have studied for various types of deterministic distributed parameter systems. For stochastic distributed parameter systems, Sunahara and the author [S16] presented a new definition of stochastic controllability and obtained the sufficient conditions for nonlinear stochastic distributed parameter systems with additive noise disturbances.

1.2. Problem Considered

In Part 2, we consider problems of optimal control and controllability for a class of distributed parameter systems with stochastic coefficients.

The dynamic behaviour of a large number of distributed parameter systems can be described by the following two types of stochastic partial differential equations:

i) Parabolic type

$$(1.2.1)\quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} + A(t,x,\omega;D_x)u(t,x) = B(t,x)f(t,x) \\ \qquad\qquad\qquad\qquad \text{for } (t,x)\in T\times G \\ u(t_0,x) = u_0(x) \qquad \text{for } x\in G \\ \mathcal{B}_j(t,x;D_x)u(t,x) = g_j(t,x) \quad \text{for } (t,x)\in T\times\partial G \text{ and } j = 1,2,\cdots\frac{n}{2} \end{cases}$$

ii) Hyperbolic type

$$(1.2.2)\quad \begin{cases} \dfrac{\partial^2 v(t,x)}{\partial t^2} + A(t,x,\omega;D_x)v(t,x) = B(t,x)f(t,x) \\ \qquad\qquad\qquad\qquad \text{for } (t,x)\in T\times G \\ v(t_0,x) = v_0(x) \qquad \text{for } x\in G \\ \mathcal{B}_j(t,x;D_x)v(t,x) = g_j(t,x) \quad \text{for } (t,x)\in T\times\partial G \text{ and } j = 1,2,\cdots\frac{n}{2} \end{cases}$$

In Eqs.(1.2.1) and (1.2.2), the operator $A(t,x,\omega;D_x)$ contains the stochastic coefficients whose principal part is of order n, $\{\mathcal{B}_j(t,x;D_x)\}_{j=1}^{n/2}$ are deterministic boundary operators and f(t,x) and $\{g_j(t,x)\}_{j=1}^{n/2}$ are respectively distributed and boundary control signals.

Characterizing the stochastic coefficients by Markov chain and white Gaussian process, Eqs.(1.2.1) and (1.2.2) are represented as stochastic evolution equations in the Hilbert space. In Part 2, we consider the following three important problems, i.e.,

(1) Optimal distributed control problem

(2) Optimal boundary control problem

and

(3) Stochastic controllability problem.

1.3. Summary of Contents

For the purpose of studying the optimal control problem, first the precise mathematical model of dynamic systems is formulated by knowledge of function space representation. The optimal control problems and stochastic controllability problems are discussed in view of various kinds of stochastic coefficients and related information.

Part two is outlined as follows:

In Chap.2, establishing the operator valued stochastic integral, the precise mathmatical model of dynamical systems is costructed on the well-known Sobolev spaces.

In Chap.3, the optimal control for the system with Markov chain coefficients is first derived under sample information of stochastic coefficients. Secondly, introducing the stochastic eigenvalue problem, the suboptimal control scheme is exploited without sample information of stochastic coefficients.

Chapter 4 contains the boundary optimal control for distributed parameter systems with Markov chain coefficients. From the Green's formula, the infulence of boundary control to the interior domain of state variable is precisely investigated.

In Chap.5, for systems of Parabolic type with white Gaussian noise coefficients, the optimal control problem under quadratic performance criteria is studied by using the dynamic programming approach, and various kinds of numerical examples are also demonstrated.

In Chap.6, by invoking Maximum principle, the optimal control under perfect and noisy observations for systems of Hyperbolic type with white Gaussian noise coefficients is derived.

Chapter 7 involves the general theory of boundary optimal control for systems of Parabolic type with white Gaussian noise coefficients. Corresponding to the boundary operators, the optimal boundary control gain is obtained associated with various types of operator Riccati equations. In the final section, giving precise form of boundary operators, i.e., Dirichlet, Neumann and mixed type boundary conditions, the exact optimal boundary controls are shown by an numerical example.

In Chap.8, in analogy with the results of Part 1, the new definitions of stochastic controllability are presented. Sufficient conditions of stochastic controllability are derived for systems of Parabolic type with white Gaussian noise coefficients. Using the eigenvalue expansion method, easily checked conditions are also demonstrated.

CHAPTER 2. MATHEMATICAL PRELIMINARIES

2.1. Basic Definitions and Symbolic Conventions

In order to study the optimal control of distributed parameter systems, it may not be allowed to bypass the basic knowldge of the functional analisis. Consequently, several important results established already in the framework of functional analysis, are summarized, including related symbolic conventions.

Let G be an arbitrary open set in $R^{(n)}$ $(x=\{x_1,x_2,\cdots x_n\}\epsilon G$, $dx=dx_1\cdot dx_2\cdots dx_n)$ and ∂G be the boundary of G. D_x^p denotes a linear partial differential operator on G such that

(2.1.1) $$D_x^p = \frac{\partial^{|p|}}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}} ,$$

where $|p| = p_1 + p_2 + \cdots + p_n$.

We need the following background knowledges of the well-known Sobolev space,[L5].

(1) $L^2(G)$-*Spaces:* We denote by $L^2(G)$ the space of (classes of) functions u which are square integrable on G, i.e., measurable and

(2.1.2) $$||u||_{L^2(G)} = \left(\int_G u^2 \, dx \right)^{1/2} < \infty .$$

We shall often set as

(2.1.3) $L^2(G) = H(G)$.

It is a classical result that $L^2(G)$ is a Hilbert Space for the scalar product,

(2.1.4) $$(u,v)_{L^2(G)} = \int_G u(x)v(x) \, dx ,$$

associated with the norm (2.1.2).

(2) $H^m(G)$-*Spaces:* Let m be an integer† ≥ 1. Briefly speaking the Sobolev space $H^m(G)$ of order m on G is defined by

$$(2.1.5)\quad H^m(G) = \{u \mid D_x^\alpha u \in L^2(G) \text{ for } \forall\alpha \,,\ |\alpha| \leq m\} .$$

We provide $H^m(G)$ with the norm:

$$(2.1.6)\quad ||u||_{H^m(G)} = \Big(\sum_{|\alpha|\leq m} ||D_x^\alpha u||^2_{L^2(G)}\Big)^{1/2} .$$

[Proposition 2.1]: With the norm (2.1.6), $H^m(G)$ is a Hilbert space, the scalar product of two elements $u,v \in H^m(G)$ being given by

$$(2.1.7)\quad (u,v)_{H^m(G)} = \sum_{|\alpha|\leq m} (D_x^\alpha u, D_x^\alpha v)_{L^2(G)} .$$

Remark 2.1.1: If $m_1>m>0$, we have the strict inclusions

$$(2.1.8)\quad H^{m_1}(G) \subset H^m(G) \subset L^2(G) = H(G) .$$

(3) $H^s(\partial G)$-*Spaces*††: Let θ_j, $j=1,2,\cdots\upsilon$ be a family of open bounded sets in $R^{(n)}$, covering ∂G, such that, for each j, there exists an infinitely differentiable mapping

$$x \to \psi_j(x) = y$$

of $\theta_j \to \P=\{y \mid y=\{y', y_n\},\ |y'|<1,\ -1<y_n<1\}$ such that ψ_j has an inverse

$$y \to \psi_j^{-1}(y) = x$$

which is also an infinitely differentiable mapping of $\P \to \theta_j$, ψ_j mapping $\theta_j \cap G \to \P_+ = \{y \mid y \in \P,\ y_n>0\}$.

---

† The hypothesis "m is an integer" is not essential. For the definition of Non-Integer order Sobolev space, see Ref.[L5].

†† We assume that the boundary ∂G of G is a (n-1) dimensional infinitely differentiable variety, G being locally on one side of G. (i.e. we consider G a variety with boundary of class C^∞.)

Furthermore, let the following compatibility conditions hold: if $\theta_j \cap \theta_i \neq \phi$, there exists an infinitely differentiable homeomorphism J_{ij} of $\psi_i(\theta_i \cap \theta_j)$ onto $\psi_j(\theta_i \cap \theta_j)$, with positive Jacobian, such that

$$\psi_j(x) = J_{ij}(\psi_i(x)) \qquad \forall x \in \theta_i \cap \theta_j \quad .$$

Let $\{\alpha_j\}$ be a partition of unity on ∂G having the properties:

$$\begin{cases} \alpha_j \in \mathcal{D}(\partial G) = \text{ space of infinitely differentiable functions on } \partial G, \\ \alpha_j \text{ with compact support in } \theta_j \cap \partial G, \sum_{j=1}^{\nu} \alpha_j = 1 \text{ on } \partial G. \end{cases}$$

If u is a function on ∂G, then we decompose

$$u = \sum_{j=1}^{\nu} (\alpha_j u)$$

and define

$$\psi_j^*(\alpha_j u)(y',0) = (\alpha_j u)(\psi_j^{-1}(y',0)), \quad y' \in \P \cap \{y_n = 0\} \ .$$

Since α_j has compact support in $\partial G \cap \theta_j$, the function $\psi_j^*(\alpha_j u)$ has compact support in $\P \cap \{y_n = 0\}$ and therefore we may also consider $\psi_j^*(\alpha_j u)$ to be defined in $R_{y'}^{(n)}$ by extending it to zero out of $\P \cap \{y_n = 0\}$. A mapping

$$u \to \psi_j^*(\alpha_j u)$$

is a continuous linear mapping of $L^1(\partial G) \to L^1(R_{y'}^{(n-1)})$, of $\mathcal{D}(\partial G) \to \mathcal{D}(R_{y'}^{(n-1)})$ and extends by continuity to a continuous linear mapping of $\mathcal{D}'(\partial G) \to \mathcal{D}'(R_{y'}^{(n-1)})$.

Now, we define

$$\text{(2.1.9)} \quad H^s(\partial G) = \{u \mid \psi_j^*(\alpha_j u) \in H^s(R_{y'}^{(n-1)}),\ j=1,2,\cdots \nu \}$$

which is valid for any real s.

By invoking the local character of $H^s(R_{y'}^{(n-1)})$, we see that the definition of (2.1.9) is independent of the choice of the system of local maps $\{\theta_j, \psi_j\}$ and of the partition of unity $\{\alpha_j\}$.

We may take the norm

$$(2.1.10) \quad \| u \|_{H^s(\partial G)} = \Big(\sum_{j=1}^{\nu} \| \psi_j^*(\alpha_j u) \|^2_{H^s(R_{y'}^{(n-1)})}\Big)^{1/2},$$

which, of course, depends on the system $\{\theta_j, \psi_j, \alpha_j\}$. We easily verify that with Eq.(2.1.10), $H^s(\partial G)$ is a Hilbert space and that the different norms (2.1.10) with respect to α_j are equivalent.

(4) $H_0^s(G)$-*Spaces:* Since the mapping

$$u \to \{\frac{\partial^j u}{\partial \nu^j} \mid 0 \le j < s - \frac{1}{2}\}$$

vanishes on $\mathcal{D}(G)$ and is a surjection of

$$H^s(G) \to \prod_{0 \le j < s-\frac{1}{2}} H^{s-j-\frac{1}{2}}(\partial G),$$

where $\frac{\partial^j u}{\partial \nu^j}$ = normal j-order derivative on ∂G, oriented toward the injector of G, it follows that, if $s > \frac{1}{2}$, the space $\mathcal{D}(G)$ is not dense in $H^s(G)$. In general, we shall set:

(2.1.11) $H_0^s(G)$ = closure of $\mathcal{D}(g)$ in $H^s(G)$.

(5) $H^{-s}(G)$-*Spaces:* For any real $s>0$, we define

(2.1.12) $H^{-s}(G)$ = dual of $H_0^s(G)$,

and furthermore if m is a positive integer, then every $f \in H^{-m}(G)$ may, in non-unique fashion, be represented by

$$(2.1.13) \quad f = \sum_{|p| \le m} D_x^p f_p, \quad f_p \in L^2(G).$$

For convenience of the present description, the principal symbols used here are listed below:

t: time variable, particulary present time

T: open time interval, $]t_0,t_f[$

$u(t,x)$, $v(t,x)$: scalar functions representing the state variable of the system, respectively

$L(X;Y)$: family of bounded linear mappings from X to Y

$(\cdot,\cdot)_*$, $\|\cdot\|_*$: the inner product and the norm in the space "$_*$", respectively

$(\overset{*}{\cdot})$: adjoint of $(\cdot)$

We shall consider a topological probability space $(\Omega,F,P;F_t)$ where F is the minimal σ-algebra of Borel sets of Ω, P the stochastic measure defined on Ω and $\{F_t\}_{t\geq t_0}$ the monotone increasing and right continuous family of sub-Borel sets of F. For the purpose of studying the stochastic process, based on the notion of field, we need the following new concepts of Hilbert spaces.

Let X and Y be two Sobolev spaces.

(6) $L^2(\Omega,P;X)$-*Spaces:* In analogy with the definition of $L^2(G)$ space, we define

(2.1.14) $$L^2(\Omega,P;X) = \{u|\ E\{\|u\|_X^2\}<\infty\}.$$

(7) $L^2(T;L^2(\Omega,P;Y))$-*Spaces:* From Eq.(2.1.14), we define

(2.1.15) $$L^2(T;L^2(\Omega,P;Y)) = \{u=\{u(t)\}_{t\geq t_0} \mid \int_T E\{\|u(t)\|_Y^2\}dt<\infty \} .$$

(8) $L^2(\Omega,P;X|F_t)$-*Spaces:* The space of functions $u(t)$ is denoted by $L^2(\Omega,P;X|F_t)$, whose elements belong to $L^2(\Omega,P;X)$ and furthermore F_t-measurable.

(9) $L^2(T;L^2(\Omega,P;Y)|F_t)$-*Spaces:* All functions $u(t)$ are elements of $L^2(T;L^2(\Omega,P;Y))$ and F_t-measurable.

2.2. Mathematical Models of Stochastic Distributed Parameter Systems

In analogy with the lumped parameter system, a class of the physical systems is modeled by the stochastic partial differential equations with additive noise disterbances [C3],[B10],[K9]. For nonlinear stochastic distributed parameter systems with additive noise, Sunahara and the author established the mathematical model in Hilbert space with the aid of the method of functional analysis, [S16].

In this chapter, we concentrate our attention on more practical distributed parameter systems, that is, stochastic distributed parameter systems with stochastic coefficients.

i) *Parabolic type:* Consider a distributed parameter system described by

(2.2.1a) $$\frac{\partial u(t,x)}{\partial t} + A(t,x,\omega;D_x)u(t,x) = 0 \quad \text{for } (t,x)\in T\times G$$

with the initial condition,

(2.2.1b) $$u(t_0,x) = u_0(x) \qquad \text{for } x \in G$$

and the boundary conditions

(2.2.1c) $$B_j(t,x;D_x)u(t,x) = 0, \quad \text{for } (t,x) \in T\times\partial G \text{ and } j=1,2,\cdots,\frac{n}{2}$$

where $B_j(t,x;D_x)$ is a deterministic boundary operator and $A(t,x,\omega;D_x)$ is a linear elliptic partial differential operator with stochastic coefficients such that

(2.2.2) $$A(t,x,\omega;D_x) = A^M(t,x,\omega;D_x) + A^S(t,x,\omega;D_x)$$

where

(2.2.3) $$A^M(t,x,\omega;D_x) = \sum_{|p|\le n} a^M_{|p|}(t,x,\omega)D^p_x$$

and

(2.2.4) $$A^S(t,x,\omega;D_x) = \sum_{|p|\le\frac{n}{2}} a^S_{|p|}(t,x)\dot{w}_{|p|}(t)D^p_x$$

and where $a^M_{|p|}(t,x,\omega)$ $(|p|=1,2,\cdots,n)$ are right continuous and almost sure uniformly bounded Markov chain processes and $\dot{w}_{|p|}(t)$ $(|p|=1,2,\cdots,\frac{n}{2})$ are white Gaussian processes. In order to guarantee the existence and uniqueness of the solution process to Eq.(2.2.1), we need the following hypotheses.

(Hypothesis-2.2.1): $a^{M}_{|p|}(t,x,\omega)$ $(|p|= 1,2,\cdots,n)$ are right continuous Markov chain fields in t.

(Hypothesis-2.2.2): $a^{s}_{|p|}(t,x)$ $(|p|= 1,2,\cdots,\frac{n}{2})$ are deterministic and sufficiently smooth functions in t and x.

(Hypothesis-2.2.3): $w_{|p|}(t)$ $(|p|= 1,2,\cdots,\frac{n}{2})$ are mutually independent standard Brownian motion processes.

(Hypothesis-2.2.4): The Brownian motion processes $w_{|p|}(t)$ $(|p|=1,2,\cdots,\frac{n}{2})$ are independent of the Markov chain processes $a^{M}_{|p|}(t,x,\omega)$ $(|p|= 1,2,\cdots,n)$.

Guided by the notion of Sobolev spaces, $H^{\frac{n}{2}}(G)$ and $H^{-\frac{n}{2}}(G)$, we shall denote V_n by†

(2.2.5) $V = H^{\frac{n}{2}}_{0}(G)$ or $H^{\frac{n}{2}}_{0}(G) \subset V \subseteq H^{\frac{n}{2}}(G)$

and V' is the dual of V. Furthermore, according to definitions stated in the previous section, we summarize two spaces which are useful to define a new operator valued stochastic integral.

(1) $L^2_{step}(T;L^2(\Omega,P;V)|F_t)$

$$\triangleq \{\ v=\{v(t)\}_{t\geq t_0}\ |v(t)\in L^2(T;L^2(\Omega,P;v)|F_t),\ v(t) \text{ is a t-step function on T and } \int_T E\{||v(s)||^2_V\}ds < M\ \}$$

---

† If the boundary conditions $\{B_j(t,x;D_x)\}^{n/2}_{j=1}$ are stable,[L5], we set $V=H^{n/2}_0(G)$ (i.e. $H^{n/2}_0(G)$ contains the boundary conditions $\{B_j(t,x;D_x)\}^{n/2}_{j=1}$), and on the other hand, we set $H^{n/2}_0 \subset V \subseteq H^{n/2}(G)$. The precise selection of the space V will be shown in numerical examples.

(2) $L^2_{mar}(\Omega,P;H|F_t)$

$\triangleq$ { $\tilde{v}=\{\tilde{v}(t)\}_{t\geq t_0}$ | $\tilde{v}(t)$ is measurable relative to F_t for all $t\in T$, $E\{||\tilde{v}(t)||^2_H\} < \infty$ and $E\{\tilde{v}(t)|F_\tau\}=\tilde{v}(\tau)$ for $t > \tau$ }

By the knowledge of Sobolev spaces mentioned previously, we can formulate the precise mathematical description of considered dynamical systems. Before stating the detailed concept, the operator valued stochastic integral is defined.

[Definition-2.2.1]: For $v_{n'}(t) \in L^2_{step}(T;L^2(\Omega,P;V)|F_t)$, we define

$$(2.2.6)\quad \int_{t_0}^{t} dA^s(s,\omega)[v_{n'}(s)] \triangleq \sum_{i=1}^{n'-1}\sum_{|p|\leq\frac{n}{2}} a^s_{|p|}(t,x)D^p_x v_{n'}(t_i,x)[w_{|p|}(t_{i+1}) - w_{|p|}(t_i)] + \sum_{|p|\leq\frac{n}{2}} a^s_{|p|}(t,x)D^p_x v_{n'}(t_{n'},x)[w_{|p|}(t) - w_{|p|}(t_{n'})] .$$

The following proposition is a direct consequence of the strong measurability of all elements of $L^2(T;L^2(\Omega,P;V)|F_t)$.

[Proposition 2.2.1]: If $v(t)$ is an element of $L^2(T;L^2(\Omega,P;V)|F_t)$, there exists a subsequence $v_{n'}(t) \in L^2_{step}(T;L^2(\Omega,P;V)|F_t)$, $n'= 1,2,$ $\cdots$ such that for any $|\alpha|\leq\frac{n}{2}$ and any $z \in L^2(\Omega,P;V)$,

$$(2.2.7)\quad \lim_{n'\to\infty}\int_{t_0}^{t} E\{(D^\alpha_x v(s) - D^\alpha_x v_{n'}(s),z)^2_H\}ds = 0.$$

From Proposition 2.2.1, we have the following definition:

[Definition-2.2.2]: The stochastic integral $\int_{t_0}^{t} dA^s(s,\omega)[v(s)]$ is defined by the mapping from $v \in L^2(T;L^2(\Omega,P;V)|F_t)$ to $L^2_{mar}(\Omega,P;C(T;H|F_t))$, as follows:

$$(2.2.8)\quad \int_{t_0}^{t} dA^s(t,\omega)[v(s)] = \lim_{n'\to\infty}\int_{t_0}^{t} dA^s(s,\omega)[v_{n'}(s)] .$$

From Definition-2.2.2, we obtain the following proposition.

[Proposition 2.2.2]: For $t>\tau$, it follows that

(2.2.9) $E\{\int_{t_0}^{t} dA^s(s,\omega)[v(s)]|F_\tau\} = \int_{t_0}^{\tau} dA^s(s,\omega)[v(s)]$,

and

(2.2.10) $E\{(\int_{t_0}^{t} dA^s(s,\omega)[v(s)], \int_{t_0}^{t} dA^s(s,\omega)[v(s)])_H\}$

$$= E\{\int_{t_0}^{t} \sum_{|p|\le\frac{n}{2}} (a^s_{|p|}(s,x)D_x^p v(s,x), a^s_{|p|}(s,x)D_x^p v(s,x))_H ds\}$$

$$\triangleq E\{\int_{t_0}^{t} (\hat{A}^s(s)v(s), \hat{A}^s(s)v(s))ds\} .$$

For convenience of descriptions we set

$$\hat{A}^{s^2}(t) = \sum_{|p|\le\frac{n}{2}} (-1)^{|p|} D_x^p(a^s_{|p|}(t,x))^2 D_x^p(\cdot) \in L(V;V') .$$

The Proposition 2.2.2 can easily be obtained from Definition-2.2.1.

For the purpose of describing the existence and uniqueness theorem for the dynamical system considered, we need the condition for the partial differential operators $A^M(t,x,\omega;D_x)$ and $A^s(t,x,\omega;D_x)$, i.e., the so-called stochastic Coercivity condition. We assume that $A^M(t,\omega) = A^M(t,x,\omega;D_x) \in L(V;V')$ w.p.1 .

Coercivity condition 2.2.1: For any $v \in V$, and all $t \in T$, there exists a positive constant α such that

(2.2.11) $2\langle A^M(t,\omega)v,v\rangle - \langle\hat{A}^{s^2}(t)v,v\rangle \ge \alpha\|v\|_V^2$ w.p.1,

where $\langle\cdot,\cdot\rangle$ denotes the duality between V and V'.

According to the new operator valued stochastic integral in Definition-2.2.2, the existence and uniqueness of solution u(t) to

Eq.(2.2.1) is stated in the following theorem.

[Theorem-2.2.1]: Under the Coercivity condition-2.2.1, Eq.(2.2.1) can be rewritten by,

$$(2.2.12)\quad u(t) + \int_{t_0}^{t} A^M(s,\omega)u(s)ds + \int_{t_0}^{t} dA^S(s,\omega)[u(s)] = u_0, \text{ a.s.}$$

Furthermore, Eq.(2.2.12) has a unique solution with the initial condition $u(t_0) = u_0 \in H$, such that

$$(2.2.13)\quad u \in L^2(\Omega,P;L^2(T;V)) \cap L^2(\Omega,P;C(\overline{T};H)),$$

and we have the energy inequality which will be useful to prove the stochastic property of the solution $u(t)$,

$$(2.2.14)\quad E\{\| u(t) \|_H^2\} + \alpha\int_{t_0}^{t} E\{\| u(s) \|_V^2\}ds \le E\{\| u(0)\|_H^2\}.$$

Proof will be shown in Appendix-A.

ii) *Hyperbolic type:* We shall consider a physical system described by

$$(2.2.15a)\quad \frac{\partial^2 v(t,x)}{\partial t^2} + A(t,x,\omega;D_x)v(t,x) = 0, \text{ for } (t,x) \in T\times G$$

with the initial and boundary conditions

$$(2.2.15b)\quad v(t_0,x) = v_0(x) \qquad \text{for } x \in G$$

$$(2.2.15c)\quad \dot{v}(t_0,x) = \dot{v}_0(x) \qquad \text{for } x \in G$$

and

$$(2.2.15d)\quad B_j(t,x,;D_x)v(t,x) = 0, \text{ for } (t,x) \in T\times\partial G \text{ and } j=1,2,\cdots,\frac{n}{2},$$

where $B_j(t,x,D_x)$ $(j=1,2,\cdots,\frac{n}{2})$ are deterministic boundary operators and $A(t,x,\omega;D_x)$ is a linear elliptic partial differential operator such that

$$(2.2.16)\quad A(t,x,\omega;D_x) = A^D(t,x;D_x) + A^S(t,x,\omega;D_x)$$

where

$$(2.2.17)\quad A^{D}(t,x;D_x) = \sum_{|p|\le n} a^{D}_{p}(t,x)D_x^{p} \quad \dagger$$

and

$$(2.2.18)\quad A^{S}(t,x,\omega;D_x) = \sum_{|p|\le \frac{n}{2}} a^{S}_{|p|}(t,x)\dot{w}_{|p|}(t)D_x^{p} \quad .$$

We need the following hypotheses on the above mentioned operators.

(Hypothesis-2.2.5): $a^{D}_{p}(t,x)$ ($|p| = 1,2,\cdots,n$) are sufficiently smooth deterministic functions in t and x, and differentiable in t and x.

(Hypothesis-2.2.6): The operator defined by Eq.(2.2.18) satisfies the Hypotheses-2.2.2 and -2.2.3.

As in the Parabolic type model, we shall work with the Sobolev spaces, $H^{n/2}(G)$ and $H_0^{n/2}(G)$. Let V and V' be respectively††

$$(2.2.19)\quad V = H_0^{\frac{n}{2}}(G) \quad \text{or} \quad H_0^{\frac{n}{2}}(G) \subset V \subseteq H^{\frac{n}{2}}(G)$$

and

$$(2.2.20)\quad V' = \text{dual of } V \ .$$

Guided by the notion of Sobolev spaces V and V', Definition-2.2.2 and Hypothesis-2.2.6, it is easily found that the following stochastic integral is defined:

$$(2.2.21)\quad \int_{t_0}^{t} dA^{S}(s,\omega)[v(s)] \triangleq \int_{t_0}^{t} \sum_{|p|\le\frac{n}{2}} a^{S}_{|p|}(s,x)D_x^{p}v(s,x)dw_{|p|}(s)$$

$$\text{for } \forall v \in L^2(T;L^2(\Omega,P;V)|F_t)$$

and

---

† For convenience of the description, we assume the principal part of $A^{D}(t,x;D_x)$ is symmetric.

†† See comments in pp.71.

$$(2.2.22)\quad E\{(\int_{t_0}^{t} dA^{s}(s,\omega)[v(s)], \int_{t_0}^{t} dA^{s}(s,\omega)[v(s)])_H\}$$
$$\triangleq E\{\int_{t_0}^{t} (\hat{A}^{s}(s)v(s), \hat{A}^{s}(s)v(s))_H ds\}.$$

We shall further assume that

$$(2.2.23)\quad A^{D}(t) = A^{D}(t,x;D_x) \in L(V;V') \quad \text{for } \forall t \in T$$

and

$$(2.2.24)\quad \hat{A}^{s^2}(t) = \sum_{|p|\le\frac{n}{2}} (-1)^{|p|} D_x^{p}(a^{s}_{|p|}(t,x))^2 D_x^{p} \in L(V;V') .$$

In order to support the existence and uniqueness of the solution process $v(t)$, the following Coercivity conditions are useful.

Coercivity condition 2.2.2: There exist $\alpha_1>0$ and $\beta_1 \in R^{(1)}$ such that

$$(2.2.25a)\quad \langle A^{D}(t)v,v\rangle + \beta_1 \| v \|_H^2 \ge \alpha_1 \| v \|_V^2 , \text{ for } \forall v\in V \text{ and } \forall t\in T,$$

and furthermore $\gamma_1>0$,

$$(2.2.25b)\quad \langle \dot{A}^{D}(t)v,v\rangle \le \gamma_1 \| v \|_V^2 .^{\dagger}$$

Coercivity condition 2.2.3: There exist $\alpha_2 \in R^{(1)}$ $(\alpha_2>0)$ and $\gamma_2>0$ such that

$$(2.2.26)\quad \alpha_2 \| v \|_V^2 \le \langle \hat{A}^{s^2}(t)v,v\rangle \le \gamma_2 \| v \|_V^2 , \text{ for } \forall v\in V \text{ and } \forall t\in T.$$

For convenience of theoretical development, Eq.(2.2.15) is represented by the vector notation in the following theorem.

[Theorem-2.2.2]: With Coercivity conditions-2.2.2 and -2.2.3, Eq.(2.2.15) can be rewritten by

$$(2.2.27)\quad z(t) + \int_{t_0}^{t} \tilde{A}^{D}(s)z(s)ds + \int_{t_0}^{t} d\tilde{A}^{s}(s,\omega)[z(s)] = z_0, \text{ a.s.},$$

where $z(t)=[v(t), \dot{v}(t)]'$ and

---

† $\dot{A}^{D}(t)$ denotes $\sum_{|p|\le n} \dot{a}^{D}_{p}(t,x)D_x^{p}$ and we assume $\dot{A}^{D}(t) \in L(V;V')$.

$$(2.2.28)\quad \int_{t_0}^{t} d\tilde{A}^{S}(s,\omega)[z(s)] = \begin{bmatrix} 0 \\ \int_{t_0}^{t} dA^{S}(s,\omega)[v(s)] \end{bmatrix}, \qquad A^{D}(t) = \begin{bmatrix} 0 & -I \\ A^{D}(t) & 0 \end{bmatrix} .$$

Furthermore, Eq.(2.2.27) has a unique solution with the initial condition $z(t_0) = z_0 \in V\times H$ such that

$$(2.2.29)\quad z \in L^2(\Omega,P;\ C(\overline{T};V)) \times L^2(\Omega,P;\ C(\overline{T};H)) .$$

Proof will be demonstrated in Appendix-B.

2.3. Mathematical Models of Dynamical Systems

In this section, several types of mathematical models for the dynamical systems to be controlled are established:

i) *Parabolic type systems with stochastic coefficients*

[Definition-2.3.1](System Σ_1) Markov chain coefficients model with distributed control signals

Let $u(t,x)$ be a scalar process of dynamical system given by

$$(2.3.1)\quad \frac{\partial u(t,x)}{\partial t} + A^{M}(t,x,\omega;D_x)u(t,x) = B(t,x)f(t,x), \text{ for } (t,x)\in T\times G$$

with the initial condition $u(t_0,x) = u_0(x)$ for $x\in G$ and the boundary conditions $\mathcal{B}_j(t,x;D_x)u(t,x) = 0$ for $(t,x)\in T\times\partial G$ and $j = 1,2,\cdots,\frac{n}{2}$.

For the partial differential operator $A^{M}(t,x,\omega;D_x)$ with Markov chain coefficients, Coercivity condition-2.2.1 in Sec.2.2 is considered.

(Condition-2.3.1) The control signal $f(t,x)$ takes the value in a convex subset U of a fixed Hilbert space, that is, $E\{\int_T \| f(s)\|_U^2 ds\} < \infty$.

(Condition-2.3.2) $B(t,x)$ is restricted as follows:

(2.3.2) $B(t) = B(t,x) \in L^{\infty}(T;\mathcal{L}(U;H))$.

With Theorem-2.2.1, the dynamical system (2.3.1) is represented by†

$$(2.3.3) \quad \Sigma_1 \left| \begin{array}{l} \dfrac{du(t)}{dt} + A^M(t,\omega)u(t) = B(t)f(t) \ , \ \text{for } t\in T \\ u(t_0) = u_0 \end{array} \right.$$

in the spaces V,H and V'.

[Definition-2.3.2](System Σ_2) <u>Markov chain coefficients model with boundary control signals</u>

Let $u(t,x)$ be a solution process of dynamical system given by

$$(2.3.4a) \quad \frac{\partial u(t,x)}{\partial t} + A^M(t,x,\omega;D_x)u(t,x) = 0, \text{ for } (t,x)\in T\times G$$

with the initial condition $u(t_0,x) = u_0(x)$ for $x\in G$ and the boundary conditions

$$(2.3.4b) \quad \mathcal{B}_j(t,x;D_x)u(t,x) = g_j(t,x), \text{ for } (t,x)\in T\times\partial G \text{ and } j= 1,2,\cdots,\frac{n}{2},$$

where $g_j(t,x)$ $(j= 1,2,\cdots,\frac{n}{2})$ are boundary control signals.

Coercivity condition-2.2.1 for the partial differential operator $A^M(t,x,\omega;D_x)$ is considered:

(Condition-2.3.3) The boundary controls $g_j(t,x)$ $(j= 1,2,\cdots,\frac{n}{2})$ belong to $L^2(T;\tilde{W})$, where

$$(2.3.5) \quad \tilde{W} = \bigcap_j L^2(\Omega,P;H^{n-n_j-\frac{1}{2}}(\partial G)) \text{ for } j= 1,2,\cdots,\frac{n}{2} \text{ and } n_j\leq n-1 \ .$$

---

† As Eq.(2.3.1) does not contain the partial differential operator with Gaussian coefficients, the differential equation form Eq.(2.3.3) has the precise mathematical meaning.

(i) Natural Boundary Condition Case [order$\{\mathcal{B}_j(t)\}\geq n/2+1$ for any j]

From Theorem-2.2.1, Eq.(2.3.4) can be represented by

$$\text{(2.3.6a)}\quad \textstyle\sum_2 \left|\begin{array}{l} \dfrac{du(t)}{dt} + A^M(t,\omega)u(t) = 0, \text{ for } t\in T \\ u(t_0) = u_0 \end{array}\right.$$

with the boundary conditions $\mathcal{B}_j(t)u(t)|_{\partial G} = g_j(t)$ $(j=1,2,\cdots,n/2)$ in the spaces H,V and V'.

(ii) Stable (Mixed) Boundary Condition Case [See Ref.[L5]]

In this case, the solution of system $\sum_2$ is considered as a weak solution of the corresponding stochastic equation;

$$\text{(2.3.6b)}\quad \textstyle\sum_2' \;\Big|\; \displaystyle\int_{t_0}^{t_f}(u(t), [-\frac{\partial\psi}{\partial t} + A^{M*}(t,\omega)\psi])dt = (u_0, \psi(t_0)) - \sum_{j=1}^{n/2}\int_{t_0}^{t_f}\langle g_j(t), T_j^M(t,\omega)\psi\rangle_{\partial G}dt$$

for any $\psi\in\{\psi|\ \psi\in H^n(G)$ and $\mathcal{B}_j^*(t)\psi = 0$ on ∂G for $j=1,2,\cdots,n/2\}$, $\psi\in C(\overline{T};H)$† and $\psi(t_f) = 0$ and where $\{T_j^M(t,\omega)\}_{j=1}^{n/2}$ is selected as the system $\{\mathcal{B}_j^*(t), T_j^M(t,\omega)\}_{j=1}^{n/2}$ becomes a Dirichlet system. (The precise form of $\{T_j^M(t,\omega)\}_{j=1}^{n/2}$ will be shown in Chap. 4 with the aid of Green's formula.)

[Definition-2.3.3](System Σ_3) White Gaussian coefficients model with distributed control signals

Let u(t,x) be a scalar process determined by

$$\text{(2.3.7)}\quad \frac{\partial u(t,x)}{\partial t} + [A^D(t,x;D_x) + A^S(t,x,\omega;D_x)]u(t,x) = B(t,x)f(t,x) \quad \text{for } (t,x)\in T\times G\ ,$$

with the initial condition $u(t_0,x) = u_0(x)$ for $x\in G$ and the boundary conditions $\mathcal{B}_j(t,x;D_x)u(t,x) = 0$, for $(t,x)\in T\times\partial G$ and $j= 1,2,\cdots,\frac{n}{2}$, where the partial differential operators $A^D(t,x;D_x)$ and $A^S(t,x,\omega;D_x)$ are defined by Eqs.(2.2.17) and (2.2.18) respectively.

---

† $\overline{T}$ denotes the closure of T, i.e. $\overline{T} = [t_0,t_f]$.

In stead of $A^M(t,x,\omega;D_x)$, the operators $A^D(t)\triangleq A^D(t,x;D_x)\in L(V;V')$ and $\hat{A}^{s^2}(t)$ associated with $A^s(t,x,\omega;D_x)$ are assumed to satisfy Coercivity condition-2.2.1 and furthermore f(t,x) and B(t,x) to satisfy Conditions -2.3.1 and -2.3.2, respectively.

Theorem-2.2.1 guarantees that, with the precise mathematical meaning, for $t\in T$, Eq.(2.3.7) can be represented by

$$(2.3.8)\quad \Sigma_3| \ u(t) + \int_{t_0}^{t} A^D(s)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)]$$
$$= u_0 + \int_{t_0}^{t} B(s)f(s)ds ,$$

in the spaces V,H and V'.

[Definition-2.3.4](System Σ_4) <u>White Gaussian noise coefficients model with boundary control signals</u>

Let u(t,x) be a scalar process of dynamical systems given by

$$(2.3.9a)\quad \frac{\partial u(t,x)}{\partial t} + [A^D(t,x;D_x) + A^s(t,x,\omega;D_x)]u(t,x) = 0 \qquad \text{for } (t,x)\in T\times G ,$$

with the initial condition $u(t_0,x) = u_0(x)$, for $x\in G$ and the boundary conditions :

$$(2.3.9b)\quad B_j(t,x;D_x)u(t,x) = g_j(t,x), \text{ for } (t,x)\in T\times\partial G \text{ and } j= 1,2,\cdots,\frac{n}{2}$$

where $g_j(t,x)$ $(j= 1,2,\cdots,\frac{n}{2})$ are boundary control signals.

For the operators $A^D(t,x;D_x)$ and $A^s(t,x,\omega;D_x)$, Coercivity condition stated in Definition-2.3.3 is considered and for the boundary control signals $g_j(t,x)$ $(j= 1,2,\cdots,\frac{n}{2})$, Condition-2.3.3 is assumed.

(i) Natural Boundary Condition Case [order$\{B_j(t)\}\geq n/2+1$ for any j]

With the same procedure as in Definition-2.3.3, we can rewrite Eq.(2.3.9) by the following stochastic evolution equation built on the spaces V, H and V';

$$\text{(2.3.10a)}\quad \Sigma_4 \mid u(t) + \int_{t_0}^{t} A^D(s)u(s)ds + \int_{t_0}^{t} dA^S(s,\omega)[u(s)] = u_0, \text{ for } t\in T,$$

with the boundary conditions $\mathcal{B}_j(t)u(t) = g_j(t)$ $(j=1,2,\cdots,n/2)$.

(ii) Stable Boundary Condition Case [order$\{\mathcal{B}_j(t)\}\leq n/2$ for any j]

In this case, the weak solution to Σ_4 is defined by

$$\text{(2.3.10b)}\quad \Sigma_4' \mid \int_{t_0}^{t_f} (u(t), [-\frac{\partial\psi}{\partial t} + A^{D*}(t)\psi])dt + \int_{t_0}^{t_f}(u(t), dA^{S*}(t,\omega)[\psi]) = (u(t_0), \psi(t_0)) + \sum_{j=1}^{n/2} \int_{t_0}^{t_f} (g_j(t), \mathcal{T}_j(t)\psi)_{\partial G}dt$$

for any $\psi\in\{\psi \mid \psi\in H^n(G)$ and $\mathcal{B}_j^*(t)\psi = 0$ on ∂G for $j=1,2,\cdots,n/2\}$, $\psi\in C(\overline{T};H)$ and $\psi(t_f) = 0$ and where $\{\mathcal{T}_j(t)\}_{j=1}^{n/2}$ is a subset of a Dirichlet system $\{\mathcal{B}_j^*(t), \mathcal{T}_j(t)\}_{j=1}^{n/2}$.

(iii) Mixed Boundary Condition Case

As is mentioned in Definition-2.3.2, a weak solution must be considered. The precise formulation is demonstrated in Appendix C.

Furthermore from Definitions-2.3.1 to -2.3.4, we can define the dynamical systems with both Markov chain and white Gaussian noise coefficients, which are called here <u>mixed coefficients model</u>.

[Definition-2.3.5](System Σ_5) <u>Mixed coefficients model with distributed control signals</u>.

Let $u(t,x)$ be a scalar stochastic process represented by

$$(2.3.11)\quad \Sigma_5 \Big| \; u(t) + \int_{t_0}^{t} A^M(s,\omega)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)] = u_0 + \int_{t_0}^{t} B(s)f(s)ds \; ,$$

in the spaces V,H and V'.

For Eq.(2.3.11), Coercivity condition-2.2.1 and Conditions-2.3.1 and -2.3.2 are considered:

Equation (2.3.11) with Conditions-2.2.1, -2.3.1 and -2.3.2 is specified by Σ_6.

[Definition-2.3.6](System Σ_6) Mixed coefficients model with boundary control signals (Natural Boundary Condition Case)

Let u(t,x) be a scalar stochastic process represented by

$$(2.3.12)\quad \Sigma_6 \Big| \; u(t) + \int_{t_0}^{t} A^M(s,\omega)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)] = u_0$$

with the boundary conditions $\mathcal{B}_j(t)u(t)|_{\partial G} = g_j(t)$ $(j= 1,2,\cdots,\frac{n}{2})$ in the spaces V,H,$\tilde{W}$ and V', where Coercivity condition-2.2.1 and Condition-2.3.3 are considered. Equation (2.3.12) with Conditions-2.2.1 and -2.3.3 is specified by Σ_6. In this model, the boundary control signals $\{g_j(t,x)\}_{j=1}^{n/2}$ are contained in the space $L^2(T;L^2(\Omega,P;\tilde{W}))$.

ii) *Hyperbolic type systems with stochastic coefficients*

As we may easily observe from the theoretical development in the preceding section, it is almost impossible to assert precisely the existence and uniqueness of a solution process to Hyperbolic type partial differential equation with stochastic coefficients in the partial differential operator whose statistics of stochastic coefficients are specified by Markov chain process. Regarding the mathematical version mentioned above, we are limited ourselves to consider a distributed parameter system modeled by the stochastic partial differential equation of Hyperbolic type with white Gaussian noise coefficients.

[Definition-2.3.7](System Σ_7) White Gaussian noise coefficients model with distributed control signals

Let v(t,x) be a scalar stochastic process given by

$$(2.3.13)\quad \frac{\partial^2 v(t,x)}{\partial t^2} + [A^D(t,x;D_x) + A^S(t,x,\omega;D_x)]v(t,x) = B(t,x)f(t,x)$$

$$\text{for } (t,x)\in T\times G$$

with the initial conditions $v(t_0,x)=v_0(x)$ and $\dot{v}(t_0,x)=\dot{v}_0(x)$, for $x\in G$ and the boundary conditions $B_j(t,x;D_x)v(t,x)=0$ $(j=1,2,\cdots,n/2)$, where the partial differential operators $A^D(t,x;D_x)$ and $A^S(t,x,\omega;D_x)$ are defined by Eqs.(2.2.17) and (2.2.18), respectively.

Considering Coercivity conditions 2.2.2 and 2.2.3, and Conditions-2.3.1 and-2.3.2, Theorem-2.2.2 allows us to write Eq.(2.3.13) in the form,

$$(2.3.14)\quad \Sigma_7\Big|\ z(t) + \int_{t_0}^{t} \tilde{A}^D(s)z(s)ds + \int_{t_0}^{t} d\tilde{A}^S(s,\omega)[z(s)]$$

$$= z_0 + \int_{t_0}^{t} \tilde{B}(s)f(s)ds$$

in the spaces, V×H, V×V and V', where all terms of the left hand side of Eq.(2.3.14) are determined by Eqs.(2.2.28) and (2.2.29) and, furthermore,

$$(2.3.15)\quad \int_{t_0}^{t} \tilde{B}(s)f(s)ds \overset{\Delta}{=} \int_{t_0}^{t} \begin{bmatrix} 0 \\ B(s)f(s) \end{bmatrix} ds \quad .$$

[Definition-2.3.8](System Σ_8) White Gaussian noise coefficients model with boundary control signals (Natural Boundary Control Case)

Let v(t,x) be a scalar stochastic process given by

$$(2.3.16a)\quad \frac{\partial^2 v(t,x)}{\partial t^2} + [A^D(t,x;D_x) + A^S(t,x,\omega;D_x)]v(t,x) = 0 ,$$

for $(t,x)\in T\times G$

and the initial conditions $v(t_0,x)=v_0(x)$ and $\dot{v}(t_0,x)=\dot{v}_0(x)$, for $x\in G$ and the boundary conditions

$$(2.3.16b)\quad \mathcal{B}_j(t,x;D_x)v(t,x) = g_j(t,x) , \quad \text{for } (t,x)\in T\times\partial G \text{ and } j=1,2,\cdots,\frac{n}{2}$$

where $g_j(t,x)$ $(j=1,2,\cdots,\frac{n}{2})$ are boundary control signals which satisfy Condition-2.3.3 in Sec.2.3. From Coercivity conditions-2.3.1 and-2.3.2, Eq.(2.3.16) can be rewritten by the following stochastic evolution equation built on the spaces V×H, V×H, $\tilde{W}$ and V',

$$(2.3.17)\quad \Sigma_8|\ z(t) + \int_{t_0}^t \tilde{A}^D(s)z(s)ds + \int_{t_0}^t d\tilde{A}^S(s,\omega)[z(s)] = z_0 , \text{ for } t\in T$$

with the boundary conditions $\mathcal{B}_j(t)v(t)=g_j(t)$ $(j=1,2,\cdots,\frac{n}{2})$.

CHAPTER 3. THE OPTIMAL CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS WITH MARKOV CHAIN COEFFICIENTS

3.1. Introductory Remarks

Various aspects of the problem of determining the optimal control for stochastic linear distributed parameter systems have been treated with some degree of success. However, many physical examples of distributed parameter systems in engineering, biological and environmental sciences require to consider uncertainties of system parameters. In such problems, from the differential operator with stochastic coefficients, there arises a difficulty in finding stochastic properties of a solution process to the partial differential equation. Given the mathematical description of the system and stochastic properties of system parameters, it is desired to determine precisely or approximately the stochastic properties of the system state. In this chapter, a type of finite Markov chain processes is introduced to describe the stochastic property of system coefficients. Even though this is a more general form than the system treated in [S9] and great difficulty would be encountered in handling certain problem of estimating the system behaviour, it is not necessary for making any further restriction on the mathematical model of the system.

In Sec. 3.2, we shall begin with the mathematical model of system parameters considered here which is specified by Markov chain with finite stages. Preassigning the transition probability of Markov chain process, the original partial differential equation is converted into the equation of stochastic evolution equation in the Sobolev spaces stated in Definition-2.3.1. In Sec. 3.3, the differential rule which plays an important role to study the optimal control problem is derived from the process determined by the solution of a partial differential equation with Markov chain coefficients. Section 3.4 is devoted to the derivation of optimal control given in terms of a known sample value of the stochastic system parameter. In practice, since the exact information of system parameters may not be obtained, it may be desirable to generate the suboptimal control on the information of parameter estimate and the entire past of the generated suboptimal control signal. In Sec.3.5, with the three basic assumptions on the eigenvalues and eigenfunctions reflecting stochastic properties of system parameters, the computational algorithm of the suboptimal control is shown. As an illustrative example, the suboptimal control scheme is considered for a class of stochastic distributed parameter systems modeled by the one-dimensional heat equation with a Markov chain coefficient in Sec.3.6.

3.2. Model Formulation and Problem Statement

Let the state variable $u(t,x)$ of a stochastic distributed parameter system be determined by

(3.2.1a) $$\frac{\partial u(t,x)}{\partial t} + A^M(t,x,\omega;D_x)u(t,x) = B(t,x)f(t,x) \quad \text{for } (t,x) \in T\times G$$

with the initial condition

(3.2.1b) $$u(t_0,x) = u_0(x) \quad \text{for } x \in G$$

and the boundary conditions

(3.2.1c) $$\mathcal{B}_j(t,x;D_x)u(t,x) = 0 \quad \text{for } (t,x) \in T\times\partial G \text{ and } j = 1,2,\cdots,\frac{n}{2},$$

where $\mathcal{B}_j(t,x;D_x)$ $(j = 1,2,\cdots,\frac{n}{2})$ are deterministic boundary operators and $A^M(t,x,\omega,D_x)$ is the partial differential operator as is defined in Definition-2.3.1,

$$(3.2.2) \quad A^M(t,x,\omega;D_x) = \sum_{|p|\le n} a^M_{|p|}(t,x,\omega)D^p_x \ .$$

In Eq.(3.2.2), we assume that the coefficients $a^M_{|p|}(t,x,\omega)$ ($|p|= 1, 2,\cdots,n$) are modeled by the right-continuous Markov chain processes with finite stages defined on (Ω,F,P). Associated with $a^M_{|p|}(t,x,\omega)$, we shall define two vectors,

$$(3.2.3) \quad \alpha(t,x,\omega) = [a^M_1(t,x,\omega),a^M_2(t,x,\omega),\cdots,a^M_n(t,x,\omega)]'$$

and

$$(3.2.4) \quad M = [\alpha_1(x),\alpha_2(x),\cdots,\alpha_n(x)]'$$

where "'" expresses the transpose as usual and where

$$(3.2.5) \quad \alpha_i(x) = [a^i_1(x),a^i_2(x),\cdots,a^i_n(x)]$$

and $a^i_j(x)$ ($i= 1,2,\cdots,m$, $j= 1,2,\cdots,n$) are assumed to be sufficiently smooth functions. By two kinds of vectors $\alpha(t,x,\omega)$ and M defined above, realizations are right-continuous piecewise-constant functions over T defined on the probability space (Ω,F,P) and those are elements of the vector process M. We denote realizations of $\alpha(t,x,\omega)$ at time t and $t + \Delta t$ by $\alpha_j(x)$ and $\alpha_k(x)$ ($j,k= 1,2,\cdots,n$) respectively. Thus, for a sufficiently small Δt,

$$(3.2.6) \quad P_r[\ \alpha(t+\Delta t,x,\omega) = \alpha_k(x) \mid \alpha(t,x,\omega) = \alpha_j(x)\ ] = \begin{cases} q_{jk}(t)\Delta t + o(\Delta t), & \text{for } j \neq k \\ 1 + q_{jj}(t)\Delta t + o(\Delta t) & \end{cases}$$

where the initial probability is $P_r[\ \alpha(t_0,x,\omega) = \alpha_j(x)\ ] = p_j$.

All realizations $\alpha_i(x)$ (i= 1,2,···,n) of $\alpha(t,x,\omega)$ are assumed to satisfy Coercivity condition-2.2.1, i.e.,

(3.2.7) $\langle A_i^M(t)v,v\rangle \geq \alpha\|v\|_V^2$ for $\forall v \in V$ and $\forall i$ (i= 1,2,···,m)

where $A_i^M(t) = \sum_{|p|\leq n} a^i_{|p|}(x)D_x^p \in \mathcal{L}(V;V')$.

Based on the conditions of Definition 2.3.1, Eq.(3.2.1) can be rewritten by the following equation of random evolution in the spaces V,H and V';

$$(3.2.8) \quad \Sigma_1 \left| \begin{array}{l} \dfrac{du(t)}{dt} + A^M(t,\omega)u(t) = B(t)f(t) \quad \text{for } t\in T \\ u(t_0) = u_0 \quad . \end{array} \right.$$

Before considering the cost functional, we must describe the precise statement of admissible control class.

Admissible Control Class: We denote the admissible control class W_{ad}, if all elements of W_{ad} satisfy the following conditions:

i) f(t) is $\mathcal{F}_t^M$-measurable for all $t\in T$, where $\mathcal{F}_t^M$ is a σ-algebra generated by $\{u(s)\}_{s\geq t_0}$ and $\{\alpha(t,x,\omega)\}_{s\geq t_0}$, i.e.,

(3.2.9) $\mathcal{F}_t^M = \sigma\{ u(s), \alpha(s,x,\omega), t_0\leq s\leq t \}$.†

ii) $f \in L^2(\Omega,P;L^2(T;U))$, where U is a convex subset of H.

Consider the quadratic cost functional

(3.2.10) $L(t,x,f) = (M(t)u(t),u(t))_H + (Q(t)f(t),f(t))_H$

where M(t) and Q(t) are bounded semipositive and positive self adjoint operators, respectively. The problem is to find the feedback optimal control $f^o(t)$ in such a way that the functional

† If $\mathcal{F}_t^M$ becomes $\sigma\{u(s), t_0\leq s\leq t\}$, i.e., the exact information of system parameters is lacked, the optimal control problem is fallen into the version of suboptimal control. (See Sec.3.4)

$$(3.2.11)\quad J(f) = E\{\int_{t_0}^{t_f} L(t,x,u)\,dt \mid u(t_0) = u_0,\ \alpha(t_0,x,\omega) = \alpha_0(x)\}$$

becomes minimal with respect to $f \in W_{ad}$.

3.3. Differential Rule for Quadratic Functional

To simplify the mathematical derivation, without loss of generality, set $f(t) = 0$ in Eq.(3.2.8) and define the quadratic functional $V(t,u,\alpha)$ by

$$(3.3.1)\quad V(t,u,\alpha) \triangleq (u(t),P(t,\alpha)u(t))_H$$

where $P(t,\alpha)$ is a bounded operator from V to V w.p.1 which depends on both time t and the realization of $\alpha(t,x,\omega)$.

Let realizations of $u(t)$ and $\alpha(t,x,\omega)$ be $\bar{u}(t)$ and $\alpha_i(x)$,respectively. An expression of the differential rule associated with the quadratic functional $V(t,u,\alpha)$ defined by Eq.(3.3.1) is obtained in the following theorem.

[Theorem-3.3.1]: With the conditions of Definition-2.3.1, the following limit exists and this is given by

$$(3.3.2)\quad \lim_{\delta\to 0} \frac{E\{V(t+\delta,u,\alpha) \mid u(t)=\bar{u},\alpha(t,x,\omega)=\alpha_i(x)\} - V(t,\bar{u},\alpha_i)}{\delta}$$

$$= (\bar{u},[\dot{\tilde{P}}_i(t) + \Sigma q_{ij}(t)\tilde{P}_j(t)]\bar{u})_H - \langle A_i^M(t)\bar{u},\tilde{P}_i(t)\bar{u}\rangle - \langle \tilde{P}_i^*(t)\bar{u},A_i^M(t)\bar{u}\rangle$$

where the superscript "*" denotes the conjugate and $\tilde{P}_i(t)$ and $A_i^M(t)$ are respectively defined by

$$(3.3.3)\quad \tilde{P}_i(t) \triangleq E\{P(t,\alpha) \mid \alpha(t,x,\omega)=\alpha_i(x)\}$$

and

$$(3.3.4)\quad A_i^M(t) \triangleq E\{A^M(t,\omega) \mid \alpha(t,x,\omega)=\alpha_i(x)\} = \sum_{|p|\le n} a^i_{|p|}(x)D_x^p \ .$$

Proof: From Eqs.(3.2.8) and (3.3.1), it follows that

$$(3.3.5)\quad V(t+\delta\ ,u,\alpha) = \langle u(t) + \frac{du(t)}{dt}\delta,[P(t,\alpha) + \frac{dP(t,\alpha)}{dt}\delta](u(t) + \frac{du(t)}{dt}\delta)\rangle$$

$$= V(t,u,\alpha) + (u(t),[\frac{dP(t,\alpha)}{dt}]u(t))_H$$

$$- 2\langle A^M(t,\omega)u(t),P(t,\alpha)u(t)\rangle\delta + o(\delta)$$

Noting that the time evolution of $\alpha(t,x,\omega)$ is independent of the system state $u(t)$, we have

$$(3.3.6)\quad E\{V(t+\delta,u,\alpha)\,|\,u(t)=\bar{u},\ \alpha(t,x,\omega)=\alpha_i\} - V(t,\bar{u},\alpha_i)$$

$$= (\bar{u},[E\{\frac{dP(t,\alpha)}{dt}\,|\,\alpha(t,x,\omega)=\alpha_i\}]\bar{u})_H$$

$$- [\langle A_i^M(t)\bar{u},\tilde{P}_i(t)\bar{u}\rangle + \langle\tilde{P}_i(t)\bar{u},A_i^M(t)\bar{u}\rangle]\delta + o(\delta).$$

The conditional expectation in the right hand side of Eq.(3.3.6) is

$$(3.3.7)\quad E\{\frac{dP(t,\omega)}{dt}\,|\,\alpha(t,x,\omega)=\alpha_i\} = \lim_{\delta\to 0}\frac{1}{\delta}[E\{P(t+\delta,\alpha)\,|\,\alpha(t,x,\omega)=\alpha_i\}$$

$$- E\{P(t,\alpha)\,|\,\alpha(t,x,\omega)=\alpha_i\}]\ .$$

Recalling Eq.(3.2.6), it follows that

$$(3.3.8)\quad E\{P(t+\delta,\alpha)\,|\,\alpha(t,x,\omega)=\alpha_i\}$$

$$= \sum_{k=1}^{m}\tilde{P}_k(t+\delta)P_r\{\ \alpha(t+\delta,x,\omega)=\alpha_k(x)\ |\ \alpha(t,x,\omega)=\alpha_i(x)\ \}$$

$$= \tilde{P}_i(t+\delta) + \sum_{k=1}^{m}\tilde{P}_k(t+\delta)q_{ik}(t)\delta + o(\delta)\ .$$

Substituting Eq.(3.3.8) into Eq.(3.3.7), we obtain

$$(3.3.9)\quad E\{\frac{dP(t,\alpha)}{dt} \mid \alpha(t,x,\omega)=\alpha_i(x)\} = \frac{d\tilde{P}_i(t)}{dt} + \sum_{k=1}^{m} \tilde{P}_k(t)q_{ik}(t) .$$

With the results (3.3.6) and (3.3.9), the proof has been completed.

3.4. Derivation of Optimal Control

The optimal control problem will be solved by using the method of dynamic programming. For Eq.(3.2.11), define the functional $V(t,\alpha,\omega)$ by

$$(3.4.1)\quad V(t,\alpha,\omega) = \min_{f\in W_{ad}} E\{\int_t^{t_f} L(s,u,f)\, ds \mid F_t^M\}$$

Then, from Eq.(3.2.9), it is apparent that

$$(3.4.2)\quad V(t,\alpha,\omega) = \min_{f\in W_{ad}} E\{\int_t^{t_f} L(s,u,f)\, ds \mid u(t)=\bar{u},\ \alpha(t,x,\omega)=\alpha_i\} .$$

Since Theorem-3.3.1 may be assumed to extend to the case where $f(t)\neq 0$, we shall write a bounded operator $\Pi(t,\alpha)$ for $P(t,\alpha)$ in Eq.(3.3.1) and apply Theorem-3.3.1 to find the optimal control. Bearing Eq.(3.3.1) in mind, we have

$$(3.4.3)\quad V(t,\alpha,\omega) = (\bar{u},\hat{\Pi}_i(t)\bar{u})_H ,$$

where

$$(3.4.4)\quad \hat{\Pi}_i(t) = E\{\Pi(t,\alpha,\omega) \mid \alpha(t,x,\omega)=\alpha_i\}$$

and $\Pi(t,\alpha,\omega)$ is a stochastic operator depending on α. From Eq.(3.4.2), by applying the principle of optimality and by Theorem-3.3.1, the following basic equation is derived;

$$(3.4.5)\quad \min_{f\in W_{ad}} [(M(t)\bar{u},\bar{u})_H + (Q(t)f,f)_H + (\bar{u},[\dot{\hat{\Pi}}_i(t) + \sum_{k=1}^{m} q_{ik}(t)\hat{\Pi}_k(t)]\bar{u})_H$$

$$- \langle A_i^M(t)\bar{u}, \hat{\Pi}_i(t)\bar{u}\rangle - \langle \hat{\Pi}_i^*(t)\bar{u}, A_i^M(t)\bar{u}\rangle$$

$$+ (\bar{u}, \hat{\Pi}_i(t)B(t)f)_H + (f, B^*(t)\hat{\Pi}_i(t)\bar{u})_H] = 0.$$

The optimal control is thus given by

$$(3.4.6)\quad f^o(t) = - Q^{-1}(t)B^*(t)\hat{\Pi}_i(t)\bar{u}$$

provided that $\alpha(t,x,\omega) = \alpha_i(x)$ $(i = 1,2,\cdots,m)$, where $\hat{\Pi}_i(t)$ satisfies

$$(3.4.7a)\quad \dot{\hat{\Pi}}_i(t) - A_i^{M*}(t)\hat{\Pi}_i(t) - \hat{\Pi}_i(t)A_i^M(t) - \hat{\Pi}_i(t)B(t)Q^{-1}(t)B^*(t)\hat{\Pi}_i(t)$$
$$+ M(t) + \sum_{k=1}^{m} q_{ik}(t)\hat{\Pi}_k(t) = 0$$

and

$$(3.4.7b)\quad \hat{\Pi}_i(t_f) = 0 \ .$$

We now obtain the equation for the kernel $\hat{\pi}_i(t,x,y)$ of $\hat{\Pi}_i(t)$ by using the Schwartz Kernel-Theorem [L4].

In Eq.(3.4.7a), assuming that $Q(t) = cI$, where c is a constant and I an identity mapping, the version of $\frac{\partial\hat{\pi}_i(t,x,y)}{\partial t}$ yields that

$$(3.4.8a)\quad \frac{\partial\hat{\pi}_i(t,x,y)}{\partial t} - (A_i^{M*}(x;D_x) + A_i^{M*}(y;D_y))\hat{\pi}_i(t,x,y)$$
$$- \frac{1}{c}\int_{G_z} \hat{\pi}_i(t,x,z)B(t,z)B(t,z)\hat{\pi}_i(t,z,y)dz$$
$$+ m(t,x,y) + \sum_{k=1}^{m} q_{ik}(t)\hat{\pi}_k(t,x,y) = 0 \ ,$$
$$\text{for } (t,x,y) \in T\times G_x\times G_y \ ,$$

with the terminal condition

$$(3.4.8b)\quad \hat{\pi}_i(t_f,x,y) = 0, \quad \text{for } (x,y) \in G_x\times G_y$$

and the boundary conditions,

(3.4.8c) $C_j(t,x;D_x)\hat{\pi}_i(t,x,y) = 0$, for $(t,x,y) \in T\times\partial G_x\times G_y$

(3.4.8d) $C_j(t,y;D_y)\hat{\pi}_i(t,x,y) = 0$, for $(t,x,y) \in T\times G_x\times\partial G_y$

where $\{C_j\}_{j=1}^{n/2}$ is an adjoint boundary system of $\{B_j\}_{j=1}^{n/2}$ and $\hat{\pi}_i(t,x,y) = \hat{\pi}_i(t,y,x)$.

3.5. Estimate for Markov Chain Process and Suboptimal Control Scheme

As shown in the previous section, in order to generate the optimal control $f^o(t)$ given by Eqs.(3.4.6) and (3.4.7), it is required to introduce a relization $\alpha_i(x)$ of $\alpha(t,x,\omega)$ which is a set of the stochastic coefficients in the differential operator $A^M(t,\omega)$. Unfortunately, it is, in practice, seldom possible to measure precisely realizations of stochastic coefficients. In this section, an approximate method is developed for a sample estimate of Markov chain process under the past value of the state variable $u(t,x)$ and control signal $f^o(t)$. For this purpose, mathematical aspects of stochastic eigenvalues and eigenfunctions are first developed by invoking the knowledge of stochastic eigenvalue problems.

3.5.1. Stochastic Eigenvalues and Model of Markov Chain Processes

For convenience of discussions, suppose that the uncertainty of coefficients contained in $A^M(t,\omega)$ depends only on t, i.e.,

(3.5.1) $$A^M(t,\omega) = \sum_{|p|\le n} a^M_{|p|}(t,\omega)D_x^p \quad .$$

We need the following assumptions:

(A-3.5.1): The Hilbert space V is separable. Then the orthonormal basis of H may be made up with elements of V. We denote these by ϕ_1, $\phi_2,\cdots$.

(A-3.5.2); There exists a sequence $\{\lambda_i(t,\omega), \phi_i;\ i = 1,2,\cdots\}$ of stochastic eigenvalues and deterministic eigenvectors such that, for any regular function ψ,

(3.5.2a) $(A^{M*}(t,\omega)\phi_i, \psi)_H = (\lambda_i(t,\omega)\phi_i, \psi)_H$

(3.5.2b) $(C_j(t)\phi_i, \psi)_{L^2(\partial G)} = 0$.

(A-3.5.3): The relation between stochastic eigenvalues $\lambda_i(t,\omega)$ and Markov chain coefficients in $A^M(t,\omega)$ is linear. Thus

(3.5.3) $\lambda_i(t,\omega) = \tilde{\lambda}_i(t) + \mu_i'(t)N(t,\omega)$,

where $\tilde{\lambda}_i(t)$ is a scalar deterministic function, $\mu_i(t)$ is a r-dimensional vector ($r \leq n$) and $N(t,\omega)$ corresponds to the Markov chain coefficients $a^M_{|p|}(t,\omega)$,([S17])

(A-3.5.4): For all $t \in T$, the eigenvalues $\lambda_i(t,\omega)$ has the following relations;

(3.5.4) $0 < \lambda_1(t,\omega) \leq \lambda_2(t,\omega) \leq \cdots \leq \lim_{i\to\infty} \lambda_i(t,\omega) = \infty$ a.s.

With the assumptions (A-3.5.1) to (A-3.5.4), the weak solution to Eq.(3.2.8) is given by

(3.5.5) $u(t) = \sum_{i=1}^{\infty} h_i(t,\omega)\phi_i$

and $E\{\|u(t)\|_H^2 \mid u(t_0) = u_0\} < \infty$, where

(3.5.6) $h_i(t,\omega) = (u(t), \phi_i)_H$.

From (A-3.5.3), it can easily be seen that the $h_i(t,\omega)$-process is determined by

(3.5.7a) $\dfrac{dh_i(t,\omega)}{dt} + \tilde{\lambda}_i(t)h_i(t,\omega) + \mu_i{}'(t)N(t,\omega)h_i(t,\omega) = f_i(t)$

and

(3.5.7b) $h_i(t_0,\omega) = (u(t_0),\phi_i)_H$

where $f_i(t) = (B(t)f^o(t),\phi_i)_H$.

3.5.2 Observations of Markov Chain Coefficients

Since the solution given by Eq.(3.5.5) is computed in terms of deterministic eigenvalues $\phi_j, \phi_{j+\tilde{m}}, \phi_{j+2\tilde{m}}, \cdots$ and $\phi_{j+(r-1)\tilde{m}}$ in such a way that the r×r matrix,

(3.5.8) $\tilde{\mu}(t) = [\ \mu_j'(t), \mu_{j+\tilde{m}}'(t), \mu_{j+2\tilde{m}}'(t), \cdots, \mu_{j+(r-1)\tilde{m}}'(t)]'$

becomes positive definite, where $\tilde{m}$ denotes the multiplicity of eigen-values. From the given information u(t,x) and $f^o(t,x)$, the observation data are acquired in the form,

(3.5.9) $y(t) = [y_j(t), y_{j+\tilde{m}}(t), \cdots, y_{j+(r-1)\tilde{m}}(t)]'$

and

(3.5.10) $\tilde{y}(t) = [\tilde{y}_j(y), \tilde{y}_{j+\tilde{m}}(t), \cdots, \tilde{y}_{j+(r-1)\tilde{m}}(t)]'$

where

(3.5.11) $y_{j+r\tilde{m}}(t) = \int_G u(t,x)\phi_{j+r\tilde{m}}(x)dx$

and

(3.5.12) $\tilde{y}_{j+r\tilde{m}}(t) = \int_G B(t,x)f^o(t,x)\phi_{j+r\tilde{m}}(x)dx$.

From Eq.(3.5.7a), it is easy to show that the y(t)-process is a solution to the differential equation,

$$(3.5.13)\quad \frac{dy(t)}{dt} + \{\bar{\lambda}(t)y(t) + \bar{y}(t)\tilde{\mu}(t)N(t,\omega)\} = \tilde{y}(t) ,$$

where

$$(3.5.14)\quad \bar{\lambda}(t) = \begin{bmatrix} \tilde{\lambda}_j(t) & 0 & \cdots & 0 \\ 0 & \tilde{\lambda}_{j+\tilde{m}}(t) & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots 0 & \tilde{\lambda}_{j+(r-1)\tilde{m}}(t) \end{bmatrix}$$

and

$$(3.5.15)\quad \bar{y}(t) = \begin{bmatrix} y_j(t) & 0 & \cdots & 0 \\ 0 & y_{j+\tilde{m}}(t) & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots 0 & y_{j+(r-1)\tilde{m}}(t) \end{bmatrix} .$$

Furthermore, the deterministic eigenfunctions $\phi_{j+\tilde{m}}$ may be chosen in such a way that $\bar{y}(t)$ becomes invertible. Consequently, from Eq.(3.5.13), it follows that

$$(3.5.16)\quad N(t,\omega) = \tilde{\mu}^{-1}(t)\bar{y}^{-1}(t)\{ - \frac{dy(t)}{dt} - \bar{\lambda}(t)y(t) + \tilde{y}(t) \} .$$

Let the time interval $]t_0,t_f[$ be discretized by $t_0<t_1<t_2<,\cdots,<t_k<\cdots t_n = t_f$. In the right hand side of Eq.(3.5.16), we shall set t as t_{k-1} and try to compute Eq.(3.5.16). An assumption that, for a sufficiently small interval $\Delta t_k = t_k - t_{k-1}$, we set as

$$(3.5.17)\quad \frac{dy(t)}{dt} \cong \frac{y(t_k) - y(t_{k-1})}{\Delta t_k}$$

allows us to compute Eq.(3.5.16) in an approximated form expressed by

$$(3.5.18)\quad \frac{\tilde{\mu}(t_{k-1})\bar{y}^{-1}(t_{k-1})}{\Delta t_k}\Big[- y(t_k) + y(t_{k-1}) - \Delta t_k\{\bar{\lambda}(t_{k-1})y(t_{k-1}) - \tilde{y}(t_{k-1})\}\Big] = \bar{N}(t_k).$$

By computing $\bar{N}(t_k)$, the following decision is made at time t_k†:

$$(3.5.19)\quad N(t,\omega) = N_i \quad \text{if } |\bar{N}(t_k) - N_i|^2 = \min.\{|\bar{N}(t_k) - N_1|^2, |\bar{N}(t_k) - N_2|^2, \cdots, |\bar{N}(t_k) - N_m|^2\}$$

where $|\cdot|$ denotes the Euclidean norm.

3.5.3 Configuration of Suboptimal Control Scheme

In this section, by using the estimation method of Markov chain coefficients, in what follows, the suboptimal control scheme is proposed.

i) With the terminal condition (3.4.7b), the gain operator equations defined by Eq.(3.4.7a) for all i are *a priori* solved by using the digital computer.

ii) By using the estimation method of Markov chain process stated in Sec. 3.5.2, we can obtain the approximated value for the realization of $\alpha(t,x,\omega)$ as $\bar{N}_i$.

iii) If from the above mentioned decision rule, we obtain the approximated value $\bar{N}_i$ for the realization of $\alpha(t,x,\omega)$, we take the gain operator $\hat{\Pi}_i(t)$ at every time $t_k (k= 1,2,\cdots)$.

The configuration of the suboptimal control system is shown in Fig. 3.5.1 .

† At initial time $t=t_0$, we assume that the information of $N(t_0,\omega)$ is given.

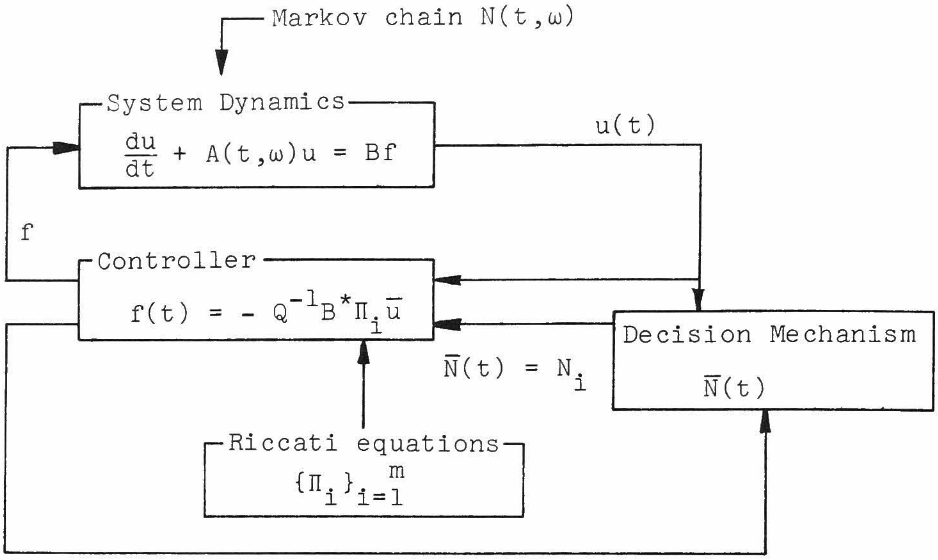

Fig. 3.5.1 Configuration of the suboptimal control system

3.6. An Illustrative Example

Consider the one dimensional heat equation:

$$(3.6.1a)\quad \frac{\partial u(t,x)}{\partial t} - \{a(t) + g(t)\alpha(t,\omega)\}\frac{\partial^2 u(t,x)}{\partial x^2} = b(t)f(t,x)$$

$$\text{for } (t,x) \in T\times]0,1[$$

with the initial and boundary conditions

$$(3.6.1b)\quad u(t_0,x) = u_0(x) = \sin^2 \pi x\,, \qquad \text{for } x \in\ ]0,1[$$

and

$$(3.6.1c)\quad u(t,0) = u(t,1) = 0\,, \qquad \text{for } t\in T\,,$$

where $\alpha(t,\omega)$ is a Markov chain coefficient with finite states whose stochastic behavior is specified by

$$(3.6.2a)\quad P_r\{\alpha(t+\Delta t,\omega)=\alpha_k \mid \alpha(t,\omega)=\alpha_j\} = \begin{cases} q_{jk}(t)\Delta t + o(\Delta t) & \text{for } j\neq k \\ 1 + q_{jj}(t)\Delta t + o(\Delta t) \end{cases}$$

and

(3.6.2b) $P_r\{\ \alpha(t_0,\omega)=\alpha_j\} \ = p_j.$

In this example, choosing $V = H_0^1(G)$, from Coercivity condition-2.2.1, we assume that, for all $j= 1,2,\cdots,m$ and $t\in\overline{T}$, the following inequality holds:

$$(3.6.3)\quad a(t) + g(t)\alpha_j > 0\ .$$

The quadratic cost functional is preassigned by

$$(3.6.4)\quad J(f) = E\{\int_{t_0}^{t_f}\int_0^1 [u^2(t,x) + f^2(t,x)]\ dxdt \mid u(t_0,x)=\sin^2\pi x,$$
$$\alpha(t_0,\omega)=\alpha_j\}\ .$$

Using Eq.(3.4.6), the optimal feedback control $f^o(t)$ minimizing Eq. (3.6.4) is obtained in the form,

$$(3.6.5)\quad f^o(t,x) = -\int_0^1 b(t)\hat{\pi}_j(t,x,y)\bar{u}(t,y)dy$$

where from Eq.(3.4.7), the $\hat{\Pi}_i(t)$-process is determined by

$$(3.6.6)\quad \frac{\partial\hat{\pi}_j(t,x,y)}{\partial t} + \{a(t) + g(t)\alpha_j\}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\hat{\pi}_j(t,x,y)$$
$$- b^2(t)\int_0^1\hat{\pi}_j(t,x,z)\hat{\pi}_j(t,z,y)dz + \delta(x-y)$$
$$+ \sum_{k=1}^{m}\hat{\pi}_k(t,x,y)q_{jk}(t) = 0,\ (j= 1,2,\cdots,m)$$

with the terminal and boundary conditions

$$(3.6.6b)\quad \hat{\pi}_j(t_f,x,y) = 0$$

and

$$(3.6.6c)\quad \hat{\pi}_j(t,x,y)\Big|_{\substack{at\ x=0,x=1\\ y\in]0,1[}} = \hat{\pi}_j(t,x,y)\Big|_{\substack{at\ y=0,y=1\\ x\in]0,1[}} = 0\ .$$

Taking into account the result in Sec.3.5, in this example, it is easily seen that $\alpha(t,\omega)=N(t,\omega)$. Furthermore, it is a simple exercise to show that the deterministic eigenfunction and eigenvalues are respectively given by

(3.6.7) $\phi_i(x) = \sqrt{2}\sin(i\pi x)$

and

(3.6.8) $\lambda_i(t,\omega) = -(i\pi)^2\{a(t) + g(t)N(t,\omega)\}$.

Noting the assumption that the multiplicity of eigenvalues is limited to only one, we choose one eigenfunction in Eq.(3.6.7). The following observation data $y(t)$ and $\tilde{y}(t)$ can thus be obtained by

(3.6.9) $y(t) = \sqrt{2}\int_0^1 u(t,x)\sin\pi x dx$

and

(3.6.10) $\tilde{y}(t) = \sqrt{2}\int_0^1 b(t)f^o(t,x)\sin\pi x dx$,

respectively. From Eqs.(3.6.9) and (3.6.10), the estimate of realizations of the Markov chain coefficients $\bar{N}_k=\bar{\alpha}_k$ is performed by

(3.6.11) $$\bar{N}_k = \frac{1}{\Delta t_k\pi^2 g(t_{k-1})y(t_{k-1})}[y(t_{k-1}) - y(t_k) + \Delta t_k\{\pi^2 a(t_{k-1})y(t_{k-1}) - y(t_{k-1})\}] .$$

We set the four stages for the Markov chain process $\alpha(t,\omega) = N(t,\omega)$ as $N_1=-2,-1,1$ and 2 and the transition matrix $Q=[q_{ij}]$ is assumed to be

(3.6.12) $$Q = \begin{bmatrix} -4 & 1 & 2 & 1 \\ 1 & -3 & 1 & 1 \\ 2 & 1 & -5 & 2 \\ 1 & 1 & 2 & -4 \end{bmatrix}$$

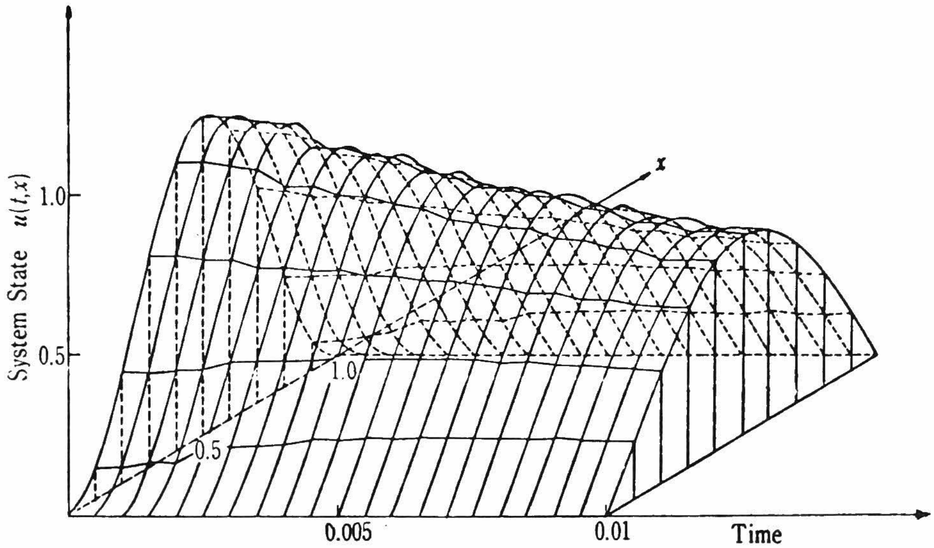

Fig. 3.6.2 A sample run of the system state u(t,x) without control

Simulation experiments follow the following steps:

Step 1: Equation (3.6.6a) is simulated on a digital computer and solved together with Eqs.(3.6.6b) and (3.6.6c).

Step 2: By using Eqs.(3.6.9) and (3.6.11), the observation data sequences $y(t_k)$ and $\bar{y}(t_k)$ are respectively generated and these are used for computing $\bar{N}_k$ by Eq.(3.6.11). The estimate of realizations of the Markov chain coefficient is performed by using the values of $\bar{N}_k$ at time t_k. Based on the estimate for N_i, one of the control gain $\hat{\pi}_j(t,x,y)$ $(j= 1,2,\cdots,m)$ is selected and the suboptimal control signal $f^o(t)$ is generated by Eq.(3.6.5). The state variable u(t,x) is simultaneously obtained by solving Eq.(3.6.1a) with Eqs.(3.6.1b) and (3.6.1c) and $f^o(t)$.

Throughout the simulation experiments, parameter values were set as a(t)=3, g(t)=1 and b(t)=5. The partitioned time interval and spa-

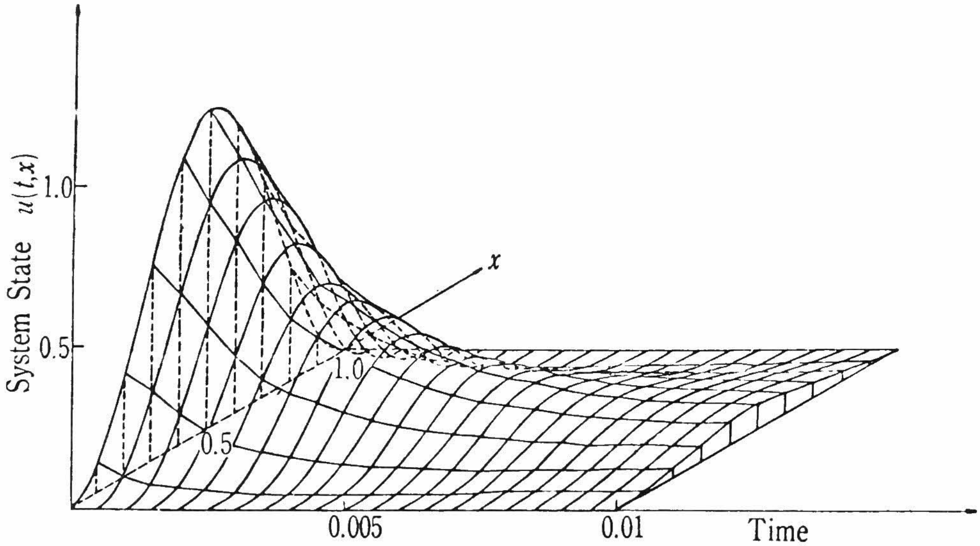

Fig. 3.6.3 A sample run of the system state u(t,x) with suboptimal control

tial variables were taken to be $\Delta t_k = 0.0001$ and $\Delta x = 0.05$ respectively. Among them, Fig. 3.6.2 shows a representative of system state runs u(t,x) without control and,in Fig. 3.6.3, the u(t,x)-run derived by the suboptimal control is plotted. In this experiment, the true value of $N(t,\omega)$ is exactly same as $\overline{N}_k$.

3.7. Discussions and Summary

In this chapter, the optimal control problem for the distributed parameter systems with Markov chain coefficients has been studied for two types of observed information, i.e., $F_t^M = \{u(s,x),\ \alpha(s,x,\omega),\ t_0 \le s \le t\}$ and $F_t = \{u(s,x),\ t_0 \le s \le t\}$.

The estimate for the true value of Markov chain coefficients introduced in Sec. 3.5 represents only one procedure for generating the suboptimal control signal associated. Although this technique is in fact, feasible to realize our final goal in the version of suboptimal control, its usefulness for particular applications must be compared with alternatives that may be considered by other observation mechanism.

The method presented here is directly applicable with a slight modification to solve the problem of assuming a linear relation between the stochastic system coefficients and the stochastic eigenvalue.

CHAPTER 4. THE OPTIMAL BOUNDARY CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS WITH MARKOV CHAIN COEFFICIENTS

4.1. Introductory Remarks

It is well-known that, in the optimal control problem for distributed parameter systems, it is a more practical and applicable situation that control inputs are applied at the boundary of the spatial reagion than distributed control inputs stated in Chap.3. In this chapter, we study the optimal boundary control problem for distributed parameter systems with Markov chain coefficients. As mentioned in the previous chapter 3, if a realization of Markov chain coefficients is not precisely known, we must adopt the suboptimal control version with an estimation mechanism for the true value of stochastic coefficients. Then, in this chapter, the stochastic eigenvalue problem is introduced in advance.

In Sec.4.2, the mathematical model of the system with boundary control signals is re-formulated with the aid of the stochastic eigenvalue problem. By using the Green's formula, the differential rule stated in Sec.3.3 of Chap.3 is extended to the case of the system with boundary inputs, and the infulence of boundary inputs to the interior domain of the system state is investigated in Sec.4.3. Section 4.4 is devoted to derive the optimal and suboptimal boundary control with the estimation scheme of the true value of Markov chain coefficients.

4.2. Model Formulation and Stochastic Eigenvalue Problem

Consider a dynamical system described by

$$(4.2.1a) \quad \frac{\partial u(t,x)}{\partial t} + A^{M}(t,x,\omega;D_x)u(t,x) = 0 \ , \ \text{for } (t,x) \in T\times G$$

with the initial condition

$$(4.2.1b) \quad u(t_0,x) = u_0(x) \ , \quad \text{for } x \in G$$

and the boundary conditions

$$(4.2.1c) \quad B_j(t;D_x)u(t,x) = g_j(t,x) \ , \quad \text{for } (t,x) \in T\times\partial G \text{ and } j=1,2,\cdots,n.$$

In this chapter, for the purpose of better understanding of the problem, we assume that G is a open domain of $R^{(1)}$ and the operators $A^{M}(t,x,\omega;D_x)$ and $\{B_j(t;D_x)\}_{j=1}^{n}$ are self-adjoint, i.e.,

$$(4.2.2) \quad A^{M}(t,x,\omega;D_x) \overset{\Delta}{=} \sum_{p=0}^{n} a_{2p}^{M}(t,\omega)D_x^{2p}$$

and

$$(4.2.3) \quad B_j(t;D_x) \overset{\Delta}{=} \sum_{h=0}^{n_j} b_{j,2h}(t)D_x^{2h} \quad , \quad \text{for } x \in G \text{ and } j=1,2,\cdots,n$$

where $a_{2p}^{M}(t,\omega)(p= 0,1,2,\cdots,n)$ are Markov chain processes as defined in Sec.3.2 in the previous chapter, and the deterministic operator $\{B_j\}_{j=1}^{n}$ is assumed to be a Dirichlet system of order n on ∂G. As we assume the boundary controls $g_j(t,x)$ $(j=1,2,\cdots,n)$ satisfy Condition-2.3.3 in Sec.2.3 of Chap.2, from the results of Definition-2.3.2 in Sec.2.3 of Chap.2, we define the solution process u to Eq.(4.2.1) as the unique element in $L^2(\Omega;L^2(T;H))$ such that

$$(4.2.4a)\Sigma_2' \mid \int_{t_0}^{t_f}(u(t), [-\frac{\partial\psi}{\partial t} + A^{M*}(t,\omega)\psi])dt = (u_0, \psi(t_0)) + \sum_{j=1}^{n}\int_{t_0}^{t_f}(g_j(t), \mathfrak{T}_j^{M}(t,\omega)\psi)_{L^2(\partial G)}dt$$

for any $\psi\in\{\psi|\ \Psi\in H^{2n}(G)$ and $B_j^{*}(t)\psi=0$ on ∂G for $j=1,2,\cdots,n\},\psi\in C(\overline{T};H)$ and $\psi(t_f)=0$ and where

(4.2.4b) $\tilde{T}^M_j(t,\omega) = \sum_{i=j}^{n}\sum_{k=1}^{i+j-1} a^M_{2i}(t,\omega)\alpha^*_{(2i-2k),j}D_x^{2k-1}$

and where $\alpha^*_{(2i-2k),j}$ satisfies $D_x^{2k} = \sum_{i=1}^{n}\alpha^*_{2n,i}B_i(t;D_x)$.

In order to examine the relation between the boundary control signal and the system state u(t), we introduce the stochastic eigenvalue problem. From the results of Sec.3.5 of Chap.3, we need the following assumptions;

(A-4.2.1): The Hilbert space H^{2n} is separable.Then, the orthonormal basis of H may be made up with elements of H^{2n}. We denote these by $\phi_1, \phi_2, \cdots$.

(A-4.2.2): There exists a sequence $\{\lambda_i(t,\omega), \phi_i;\ i= 1,2,\cdots\}$ of stochastic eigenvalues and deterministic eigenvectors such that, for any regular function ψ,

(4.2.5a) $(A^{M*}(t,\omega)\phi_i, \psi)_H = (\lambda^d_i(t,\omega)\phi_i, \psi)_H$

and

(4.2.5b) $(C_j(t)\phi_i, \psi)_{L^2(\partial G)} = 0$

where, in this chapter, from Eq.(4.2.3) it is easily found that $\{B_j\} = \{C_j\}$ for $j= 1,2,\cdots,n$.

(A-4.2.3): The relation between stochastic eigenvalues $\{\lambda^d_i(t,\omega)\}_{j=1}^{\infty}$ and Markov chain coefficients in $A^M(t,\omega)$ is linear. Thus

(4.2.6) $\lambda^d_i(t,\omega) = \tilde{\lambda}^d_i(t) + \mu^{d\prime}_i(t)N(t,\omega)$,

where $\tilde{\lambda}^d_i(t)$ is a scalar deterministic function, $\mu^d_i(t)$ is a r-dimensional vector ($r \le 2n$) and $N(t,\omega)$ corresponds to the Markov chain coefficients $a^M_{2p}(t,\omega)(p=0,1,\cdots,n)$. $\lambda^d_i(t,\omega)$ satisfies Assumption (A-3.5.4) in Sec.3.5 of Chap.3.

(A-4.2.4): There exists a sequence $\{\lambda^b_{l,j,k};\ l=1,2,\cdots,n,\ k=1,2,\cdots, n+1-l,\ l=1,2,\cdots\}$ of stochastic eigenvalues for same ϕ_j defined by

Eq.(4.2.5) such that,

$$(4.2.7)\quad (g_l(t,x), \sum_{i=l}^{n} \sum_{k=1}^{i+1-l} a_{2i}^{M}(t,\omega)\alpha^{*}_{(2i-2k),l}\tilde{S}_k(D_x)\phi_j(x))_{L^2(\partial G)}$$

$$= \sum_{k=1}^{n+1-l} \lambda^{b}_{l,j,k}(t,\omega) g_{l,j,k}(t) ,$$

where $\tilde{S}_k(D_x) = D_x^{2k-1}$ and $\alpha^{*}_{(2i-2k),l}$ is a deterministic function which satisfies

$$(4.2.8)\quad D_x^{2k} = \sum_{l=1}^{n} \alpha^{*}_{2n,l} B_l(t;D_x) .$$

(A-4.2.5): The stochastic eigenvalue $\lambda^{b'}_{l,j,k}(t,\omega)$ can be rewritten as†

$$(4.2.9)\quad \lambda^{b}_{l,j,k}(t,\omega) = \tilde{\lambda}^{b}_{l,j,k}(t) + \mu^{b'}_{l,j,k}(t)N(t,\omega) ,$$

where $\tilde{\lambda}^{b}_{l,j,k}(t)$ is a scalar deterministic function, and $\mu^{b'}_{l,j,k}(t)$ is a r-dimensional vector function.

Before constructing the weak solution of the system equation (4.2.4), we need the following lemma.

[Lemma-4.2.1]: For any $u \in C^{2n}(\overline{G})$ and $v \in \{ \xi;\ \xi \in C^{2n}(\overline{G})\ B_j(t;D_x)\xi = 0,$ for $j = 1,2,\cdots,n \}$, we have

$$(4.2.10)\quad (A^{M}(t,\omega)u,v)_H = (u,A^{M*}(t,\omega)v)_H$$

$$- \sum_{l=1}^{n} (B_l(t;D_x)u, \sum_{i=l}^{n} \sum_{k=1}^{i+1-l} a_{2i}^{M}(t,\omega)\alpha^{*}_{(2i-2k),l}\tilde{S}_k(D_x)v)_{L^2(\partial G)}$$

where $\tilde{S}_k(D_x) = D_x^{2k-1}$.

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† It is evident from Eqs.(4.2.6) and (4.2.7) that $\lambda^{b}_{l,j,k}(t,\omega)$ satisfies Eq.(4.2.9).

Proof: Integrating by parts, we have

$$(4.2.11)\quad (A^{M}(t,\omega)u,v)_{H} = (u,A^{M*}(t,\omega)v)_{H} + \sum_{i=1}^{n} a_{2i}^{M}(t,\omega)\sum_{k=1}^{2i}(-1)^{k-1}(D_x^{2i-k}u,D_x^{k-1}v)_{L^2(\partial G)}.$$

On the other hand, from the definition of the operator $\tilde{S}_k(D_x)$, it is easy to find that $\{B_1(t;D_x),B_2(t;D_x),\cdots,B_n(t;D_x),\tilde{S}_1(D_x),\cdots,\tilde{S}_n(D_x)\}$ is a Dirichlet system. Then, for $0\le k\le n-1$, we have

$$(4.2.12)\quad D_x^{2k} = \sum_{l=1}^{m} \alpha_{2k,l}^{*}B_l(t;D_x)$$

and

$$(4.2.13)\quad D_x^{2k+1} = \tilde{S}_{k+1}(D_x) .$$

The second term of the right hand side of Eq.(4.2.11) becomes

$$(4.2.14)\quad \sum_{i=1}^{n} a_{2i}^{M}(t,\omega)\sum_{k=1}^{2i}(-1)^{k-1}(D_x^{2i-k}u,D_x^{k-1}v)_{L^2(\partial G)}$$

$$= \sum_{i=1}^{n} a_{2i}^{M}(t,\omega)[\sum_{k=1}^{i}(-1)^{2k-1}(D_x^{2i-2k}u,D_x^{2k-1}v)_{L^2(\partial G)} + \sum_{k'=1}^{i}(-1)^{2k'-2}(D_x^{2i-2k'+1}u,D_x^{2k'-2}v)_{L^2(\partial G)}].$$

Noting that v is an element of $\{\ \xi;\ \xi\in C^{2n}(\overline{G}),\ B_j(t;D_x)\xi=0$ for $j=1,2,\cdots,n$ on $\partial G\}$, we have

$$(4.2.15)\quad \sum_{k'=1}^{n}(-1)^{2k'-2}(D_x^{2i-2k'+1}u,D_x^{2k'-2}v)_{L^2(\partial G)} = 0.$$

Then, from Eqs.(4.2.12) and (4.2.13), Eq.(4.2.14) becomes

$$(4.2.16)\quad \sum_{i=1}^{n} a_{2i}^{M}(t,\omega)\sum_{k=1}^{i}(-1)^{2k-1}(D_x^{2i-2k}u, D_x^{2k-1}v)_{L^2(\partial G)}$$

$$= \sum_{i=1}^{n} a_{2i}^{M}(t,\omega)\sum_{k=1}^{i}(-1)^{2k-1}\Big(\sum_{l=1}^{i-k+1}\alpha_{(2i-2k),l}\mathcal{B}_l(t;D_x)u,\ \tilde{S}_k(D_x)v\Big)_{L^2(\partial G)}$$

$$= -\sum_{l=1}^{n}\sum_{k=1}^{n+1-l}\Big(\mathcal{B}_l(t;D_x)u,\ \sum_{i=k+l-1}^{n} a_{2i}^{M}(t,\omega)\alpha^{*}_{(2i-2k),l}\times\tilde{S}_k(D_x)v\Big)_{L^2(\partial G)}.$$

The proof has been completed.

[Theorem-4.2.1]: From the Assumptions (A-4.2.1) to (A-4.2.5), and Lemma-4.2.1, the weak solution of Eq.(4.2.4) becomes

$$(4.2.17)\quad u(t) = \sum_{i=1}^{\infty} h_i(t,\omega)\phi_i\ ,$$

where the Fourier coefficients $\{h_i(t,x)\}$ satisfy,

$$(4.2.18)\quad \left|\begin{array}{l} \dfrac{dh_i(t,\omega)}{dt} + \tilde{\lambda}_i^{d}(t)h_i(t,\omega) - \displaystyle\sum_{l=1}^{n}\sum_{k=1}^{n+1-l}\tilde{\lambda}^{b}_{l,i,k}(t)g_{l,i,k}(t) \\ \quad + [h_i(t,\omega)\mu_i^{d'}(t) - \displaystyle\sum_{l=1}^{n}\sum_{k=1}^{n+1-l} g_{l,i,k}(t)\mu^{b'}_{l,i,k}(t)]N(t,\omega) = 0 \\ h_i(t_0,\omega) = (u(t_0),\phi_i)_H \end{array}\right.$$

Proof: The variational form of Eq.(4.2.4) is

$$(4.2.19)\quad (\frac{du(t)}{dt},\phi_i)_H + (A^{M}(t,\omega)u(t),\phi_i)_H = 0.$$

By using the Green's formula [L5] and Lemma-4.2.1, Eq.(4.2.19) becomes

$$(4.2.20)\quad \left(\frac{du(t)}{dt},\phi_i\right)_H + \left(u(t),A^{M*}(t,\omega)\phi_i\right)_H - \sum_{l=1}^{n}\left(B_l(t;D_x)u(t),\sum_{i=l}^{n}\sum_{k=1}^{i+1-l} a_{2i}^{M}(t,\omega)\alpha_{(2l-2k),l} \times\tilde{S}_k(D_x)\phi_i\right)_{L^2(\partial G)} = 0$$

From the Assumptions (A-4.2.1) and (A-4.2.2), we have

$$(4.2.21)\quad \left(\sum_{i=1}^{\infty}\frac{dh_i(t,\omega)}{dt}\phi_i,\phi_j\right)_H + \left(\sum_{i=1}^{\infty}h_i(t,\omega)\phi_i,\lambda_j^d(t,\omega)\phi_j\right)_H - \sum_{l=1}^{n}\left(g_l(t,x),\sum_{i=l}^{n}\sum_{k=1}^{i+1-l} a_{2i}^{M}(t,\omega)\alpha^{*}_{(2l-2k),l}\,\tilde{S}_k(D_x)\phi_j\right)_{L^2(\partial G)} = 0 .$$

By using the orthogonality of eigenvector, and the assumption (A-4.2.4), $h_j(t,\omega)$ satisfies,

$$(4.2.22)\quad \frac{dh_j(t,\omega)}{dt} + \lambda_j^d(t,\omega)h_j(t,\omega) - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l}\lambda^b_{l,j,k}(t,\omega)g_{l,j,k}(t) = 0.$$

Then, from the assumptions (A-4.2.3) and (A-4.2.5), it is easily found that Eq.(4.2.21) can be rewritten as Eq.(4.2.18). The proof has been completed.

4.3. Differential Rule for Quadratic Functional

In this section, with the aid of stochastic eigenvalue problem, we assume $N(t,\omega)$ is a set of Markov chain coefficients defined by Eq.(4.2.6) and has the same transition probability low as stated in Sec.3.2 of Chap.3.

Define the quadratic functional $V(t,u,N)$ by

$$(4.3.1) \quad V(t,u,N) \triangleq (u(t),P(t,N)u(t))_H$$

where $P(t,N)$ is a bounded operater from H to H, w.p.1. For convenience of the present description, for any $\psi\epsilon H$, the following partial differential operators are assumed to be constructed;

$$(4.3.2) \quad (A^{M*}(t)\phi_j,\psi)_H = (\tilde{\lambda}_j^d(t)\phi_j,\psi)_H$$

$$(4.3.3) \quad (G_i^*(t)\phi_j,\psi)_H = (\mu_j^{d'}(t)N_i\phi_j,\psi)_H$$

$$(4.3.4) \quad (\tilde{B}_l^*(t)\phi_j,\psi)_{L^2(\partial G)} = (\sum_{k=1}^{n+1-l}\tilde{\lambda}^{b'}_{l,j,k}(t)\tilde{S}_k(D_x)\phi_j,\psi)_{L^2(\partial G)}$$

and

$$(4.3.5) \quad (H_l^{i*}(t)\phi_j,\psi)_{L^2(\partial G)} = (\sum_{k=1}^{n+1-l}\mu^{b'}_{l,j,k}N_i\tilde{S}_k(D_x)\phi_j,\psi)_{L^2(\partial G)}$$

where N_i denotes the i-th stage of the Markov chain $N(t,\omega)$.

[Theorem-4.3.1]: With the conditions of Definition-2.3.2 in Sec.2.3 of Chap.2, the following limit exists and this is given by

$$(4.3.6) \quad \lim_{\delta\to 0}\frac{E\{V(t+\delta,u,N)|u(t)=\bar{u},\ N(t,\omega)=N_i\} - V(t,\bar{u},N_i)}{\delta}$$

$$= (\bar{u},\{\dot{\tilde{P}}_i(t) - [\tilde{A}^{M*}(t) + G_i{}^*(t)]\tilde{P}_i(t)$$

$$- ([\tilde{A}^{M*}(t) + G_i^*(t)]\tilde{P}_i(t))^* + \sum_{k=1}^{m}q_{ij}(t)\tilde{P}_k(t)\}\bar{u})_H$$

$$+ \sum_{l=1}^{n}\{(E\{g(t,x)|u(t)=\bar{u},N(t,\omega)=N_i\},[\tilde{B}_l^*(t) - H_l^{i*}(t)]$$

$$\times\tilde{P}_i(t)\bar{u})_{L^2(\partial G)} + ([\tilde{B}_l^*(t) - H_l^{i*}(t)]\tilde{P}_i(t),$$

$$E\{g(t,x)|u(t)=\bar{u},N(t,\omega)=N_i\})_{L^2(\partial G)}\}$$

where $\tilde{P}_i(t)$ is defined by

(4.3.7) $\tilde{P}_i(t) = E\{P(t,N)|N(t,\omega)=N_i\}$.

Remark-4.3.1: Without using the stochastic eigenvalue problem, the differential rule for quadratic functional can be reduced for the general form of system dynamics with boundary inputs. Detailed discussions will be stated in Chap.7.

Remark-4.3.2: In spite of the fact that the considered boundary operators are deterministic, the influence of the boundary controls to the interior domain of the system state is perturbed by the Markov chain coefficients. From this fact, the term $H_l^{i*}(t)$ appears in Eq. (4.3.6) .

Proof: From Eq.(4.2.17), Eq.(4.3.1) can be rewritten as

(4.3.8) $$V(t,u,N) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_i(t,\omega)p_{ij}(t,N)h_j(t,\omega),$$

where $p_{ij}(t,N)$ is defined by

(4.3.9) $$p_{ij}(t,N) = (\phi_i,P(t,N)\phi_j)_H.$$

By using the same procedure as in Sec.3.3 of Chap.3 and noting that the $h_i(t,\omega)$-process satisfies Eq.(4.2.18), we have

(4.3.10) $$E\{V(t+\delta,u,N)|u(t)=\bar{u},N(t,\omega)=N_i\} - V(t,\bar{u},N_i)$$

$$= (\bar{u},E\{\frac{dP(t,N)}{dt}|N(t,\omega)=N_i\}\bar{u})_H\delta + (\sum_{i'=1}^{\infty}\bar{h}_{i'}\phi_{i'},\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\{-[\tilde{\lambda}_j^d(t)$$

$$+ \mu_j^{d'}N_i]E\{p_{jk}(t,N)|N(t,\omega)=N_i\} - E\{p_{jk}(t,N)|N(t,\omega)=N_i\}$$

$$\times[\tilde{\lambda}_j^d(t) + \mu_j^{d'}N_j]\}\phi_i(\phi_k,\sum_{l'=1}^{\infty}\bar{h}_{l'}\phi_{l'})_H)_H\delta$$

$$+ \sum_{l=1}^{n} \sum_{k'=1}^{n+1-l} \Big(\sum_{i'=1}^{\infty} E\{g_{l,i',k'}(t) \mid u(t)=\bar{u},\ N(t,\omega)=N_i\}\phi_{i'},$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\{\tilde{\lambda}^{b}_{l,j,k'}(t) - \mu^{b'}_{l,j,k'}(t)N_i\}E\{p_{ij}(t,N) \mid N(t,\omega)=N_i\}$$

$$\times\phi_i(\phi_k, \sum_{l'=1}^{\infty}\bar{h}_{l'}\phi_{l'})_H\Big)_H\delta + \sum_{l=1}^{n}\sum_{k'=1}^{n+1-l}\Big(\sum_{i'=1}^{\infty}\bar{h}_{i'}\phi_{i'},$$

$$\sum_{j=1}^{\infty}\sum_{k=i}^{\infty}E\{p_{ij}(t,N) \mid N(t,\omega)=N_i\}[\tilde{\lambda}^{b}_{l,k,k'}(t) - \mu^{b'}_{l,k,k'}(t)N_i]$$

$$\times\phi_j(\phi_k, \sum_{l'=1}^{\infty}E\{g_{l,l',k'}(t) \mid u(t)=\bar{u},\ N(t,\omega)=N_i\}\phi_{l'})_H\Big)_H\delta$$

$$+\ o(\delta)\ ,$$

where $\sum_{i=1}^{\infty}\bar{h}_i\phi_i = (\bar{u},\phi_i)_H$.

From Eq.(3.3.7) in Sec.3.3 of Chap.3 and Eq.(4.2.7), we have

$$(4.3.11)\quad E\{\frac{dP(t,N)}{dt} \mid N(t,\omega)=N_i\} = \frac{d\tilde{P}_i(t)}{dt} + \sum_{k=1}^{m}\tilde{P}_k(t)q_{ik}(t)$$

and

$$(4.3.12)\quad \sum_{k'=1}^{n+1-l}\Big(\sum_{i'=1}^{\infty}E\{g_{l,i'k'}(t) \mid u(t)=\bar{u},N(t,\omega)=N_i\}\phi_{i'}, \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\{\tilde{\lambda}^{b}_{l,j,k'}(t)$$

$$- \mu^{b'}_{l,j,k'}(t)N_i\}E\{p_{jk}(t,N) \mid N(t,\omega)=N_i\}\phi_i(\phi_k, \sum_{l'=1}^{\infty}\bar{h}_{l'}\phi_{l'})_H\Big)_H$$

$$= (E\{g_l(t,x) \mid u(t)=\bar{u},\ N(t,\omega)=N_i\}, \sum_{k=1}^{\infty}(\mathcal{B}^{*}_{l}(t) - H^{i*}_{l}(t))\phi_k$$

$$\times\sum_{j=1}^{\infty}E\{p_{jk}(t,N) \mid N(t,\omega)=N_i\}(\phi_j,\bar{u})_H\Big)_{L^2(\partial G)}$$

$$= (E\{g_l(t,x) \mid u(t)=\bar{u},N(t,\omega)=N_i\},[\mathcal{B}^{*}_{l} - H^{i*}_{l}(t)]\tilde{P}_i(t)\bar{u})_{L^2(\partial G)}.$$

With the results of Eqs.(4.3.11) and (4.3.12) and the relations between operators and eigenvalue defined by Eqs.(4.3.2), (4.3.3), (4.3.4) and (4.3.5), it follows that

$$(4.3.13)\quad E\{V(t+\delta,u,N)\,|\,u(t)=\bar{u},\ N(t,\omega)=N_i\} - V(t,\bar{u},N_i)$$
$$= (\bar{u},[\frac{d\tilde{P}_i(t)}{dt} - [\tilde{A}^{M*}(t) + G_i^*(t)]\tilde{P}_i(t)$$
$$- ([\tilde{A}^{M*}(t) + G_i^*(t)]\tilde{P}_i^*(t))^* + \sum_{k=1}^{m} q_{ik}(t)\tilde{P}_k(t)]\bar{u})_H\delta$$
$$+ \sum_{l=1}^{m}\{(E\{g(t,x)\,|\,u(t)=\bar{u},\ N(t,\omega)=N_i\},[\tilde{B}_l^*(t) - H_l^{i*}(t)]$$
$$\times\tilde{P}_i(t)\bar{u})_{L^2(\partial G)} + ([B_l^*(t) - H_l^{i*}(t)\tilde{P}_i(t)\bar{u},$$
$$E\{g(t,x)\,|\,u(t)=\bar{u},N(t,\omega)=N_i\})_{L^2(\partial G)}\}\delta + o(\delta)\ .$$

The proof has been completed.

4.4. Derivation of Optimal and Suboptimal Boundary Controls

4.4.1. Optimal Boundary Control

We define the following admissble control class W^b_{ad} by

$$(4.4.1)\quad W^b_{ad} = \{g_j;\ g_j\in L^2(T;\tilde{W}),\ g_j(t) \text{ is } F_t^M\text{-measurable atthe present time } t \text{ for } j= 1,2,\cdots,n\}\ ,$$

where $\tilde{W}$-space is defined by Eq.(2.3.5) in Sec.2.3 of Chap.2, and F_t^M is the σ-algebra generated by the random variables $u(s)$ and $N(s,\omega)$ for $t_0\le s\le t$.

Consider the quadratic cost functional

$$(4.4.2)\quad L(t,u,g) = (M(t)u(t),u(t))_H + \sum_{l=1}^{n}(Q_l(t)g_l(t),g_l(t))_{L^2(\partial G)},$$

where $M(t)$ and $\{Q_l(t)\}_{l=1}^n$ are bounded semipositive and positive self adjoint operators, respectively. The problem is to find the feedback optimal boundary controls $\{g_l^o(t)\}_{l=1}^n$ in such a way that the functional

$$(4.4.3)\quad J(g(t)) = E\{\int_{t_0}^{t_f} L(t,u,g)\, dt \,|\, u(t_0)=u_0,\ N(t_0,\omega)=N_0\}$$

becomes minimal with respect to $\{g_l(t)\}_{l=1}^n \in W_{ad}^b$.

In this case, by using the same approach as stated in Sec.3.4 of Chap.3, and the result of Theorem-4.3.1, the following basic equation of optimal boundary control is derived;

$$(4.4.4)\quad \min_{g\in W_{ad}^b} [(M(t)\bar{u},\bar{u})_H + \sum_{l=1}^n (Q_l(t)g_l(t),g_l(t))_{L^2(\partial G)} + (\bar{u},[\dot{\hat{\Pi}}_i(t)$$

$$- [\mathfrak{A}^{M*}(t) + G_i^*(t)]\hat{\Pi}_i(t) - ([\mathfrak{A}^{M*}(t) + G_i^*(t)]\hat{\Pi}_i(t))^*$$

$$+ \sum_{k=1}^m q_{ik}(t)\hat{\Pi}_k(t)]\bar{u})_H + \sum_{l=1}^n (g_l,[\mathfrak{B}_l^*(t) - H_l^{i*}(t)]\hat{\Pi}_i(t)\bar{u})_{L^2(\partial G)}$$

$$+ \sum_{l=1}^n ([\mathfrak{B}_l^*(t) - H_l^{i*}(t)]\hat{\Pi}_i(t),g_l)_{L^2(\partial G)}] = 0.$$

With vector notation, Eq.(4.4.4) becomes

$$(4.4.5)\quad \min_{g\in W_{ad}^b} [(\bar{u},[\dot{\hat{\Pi}}_i(t) - [\mathfrak{A}^{M*}(t) + G_i^*(t)]\hat{\Pi}_i(t) - ([\mathfrak{A}^{M*}(t) + G_i^*(t)]$$

$$\times\hat{\Pi}_i(t))^* + \sum_{k=1}^m q_{ik}(t)\hat{\Pi}_k(t)]\bar{u})_H + (M(t)\bar{u},\bar{u})_H$$

$$+ (g'(t),\mathfrak{Q}(t)g(t))_{L^2(\partial G)} + (g'(t),\mathfrak{C}_i^*(t)\hat{\Pi}_i(t)\bar{u})_{L^2(\partial G)}$$

$$+ ((\mathfrak{C}_i^*\hat{\Pi}_i^b(t))^* g'(t),\bar{u})_H] = 0 .$$

where $\mathcal{Q}(t)$ and $\mathcal{C}(t)$ are the following n-dimensional vector operators,

$$(4.4.6)\quad \mathcal{Q}(t) = \begin{bmatrix} Q_1(t) & 0 & & 0 \\ 0 & Q_2(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Q_n(t) \end{bmatrix}$$

$$(4.4.7)\quad \mathcal{C}_i^*(t) = [\mathcal{B}_1^*(t) - H_1^{i*}(t), \mathcal{B}_2^*(t) - H_2^{i*}(t), \cdots, \mathcal{B}_n^*(t) - H_n^{i*}(t)]'$$

and

$$(4.4.8)\quad g(t) = [g_1(t), g_2(t), \cdots, g_n(t)]'$$

and where $\hat{\Pi}_i^b(t)$ denotes the operator,

$$(4.4.9)\quad \hat{\Pi}_i^b(t) = \int_{\partial G_y} \hat{\pi}_i(t,x,y)(\cdot)dy, \text{ for } x \in G_x \text{ and } y \in \partial G_y .$$

The optimal boundary control $g^o(t)$ is given by

$$(4.4.10)\quad g^o(t) = -\mathcal{Q}^{-1}(t)\mathcal{C}^*(t)\hat{\Pi}_i(t)\bar{u} , \quad \text{for } N(t,\omega) = N_i$$

where for $i = 1,2,\cdots,m$, $\hat{\Pi}_i(t)$ satisfies

$$(4.4.11a)\quad \dot{\hat{\Pi}}_i(t) - [\mathcal{A}^{M*}(t) + G_i^*(t)]\hat{\Pi}_i(t) - ([\mathcal{A}^{M*}(t) + G_i^*(t)]\hat{\Pi}_i(t))^*$$

$$+ M(t) + \sum_{k=1}^{m} q_{ik}(t)\hat{\Pi}_k(t) - (\mathcal{C}_i'^*(t)\hat{\Pi}_i(t))^* Q^{-1}(t)\mathcal{C}_i^*(t)\hat{\Pi}_i^b(t) = 0$$

with the terminal condition,

$$(4.4.11b)\quad \hat{\Pi}_i(t_f) = 0 .$$

4.4.2. Observation of Markov Chain Coefficients and Suboptimal Control

As shown in the previous section in Sec.3.5 of Chap.3, for the purpose of generating the optimal boundary control $g^o(t)$ given by Eqs. (4.4.10) and (4.4.11), it is required to obtain a realization N_i of the $N(t,\omega)$ process. By using the same procedure as in the Sec.3.5 of Chap.3 and the result of Theorem-4.2.1, we can construct the observation mechanism of Markov chain coefficients.

By using the same approach of Sec.3.5 of Cha.3, we select the deterministic eigenfunctions $\phi_j, \phi_{j+\tilde{m}}, \phi_{j+2\tilde{m}}, \cdots$ and $\phi_{j+(r-1)\tilde{m}}$ in such a way that r×r matrices

$$(4.4.12)\quad \bar{\lambda}^d = \begin{bmatrix} \tilde{\lambda}_j^d(t) & 0 & \cdots\cdots & 0 \\ 0 & \tilde{\lambda}^d_{j+\tilde{m}}(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots\cdots & 0 & \tilde{\lambda}^d_{j+(r-1)\tilde{m}}(t) \end{bmatrix}$$

$$(4.4.13)\quad \bar{\lambda}^b_{l,k} = \begin{bmatrix} \tilde{\lambda}^b_{l,j,k}(t) & 0 & \cdots\cdots & 0 \\ 0 & \tilde{\lambda}^b_{l,j+\tilde{m},k}(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots\cdots & 0 & \tilde{\lambda}^b_{l,j+(r-1)\tilde{m},k}(t) \end{bmatrix}$$

$$(4.4.14)\quad \bar{\mu}^d = [\mu_j^{d'}(t), \mu^{d'}_{j+\tilde{m}}(t), \cdots, \mu^{d'}_{j+(r-1)\tilde{m}}(t)]'$$

and

$$(4.4.15)\quad \bar{\mu}^b_{l,k} = [\mu^{b'}_{l,j,k}(t), \mu^{b'}_{l,j+\tilde{m},k}(t), \cdots, \mu^{b'}_{l,j+(r-1)\tilde{m},k}(t)]'$$

become positive definite, where $\tilde{m}$ denotes the multiplicity of eigenvalues. From the given information $u(t,x)$ and $g^o(t,x)$, the observation data are acquired in the form,

$$(4.4.16)\quad y(t) = [y_j(t), y_{j+\tilde{m}}(t), \cdots, y_{j+(r-1)\tilde{m}}(t)]'$$

and

$$(4.4.17)\quad z_{l,k}(t) = [z_{l,j,k}(t), z_{l,j+\tilde{m},k}(t), \cdots, z_{l,j+(r-1)\tilde{m},k}(t)]'$$

where

$$(4.4.18)\quad y_{j+k\tilde{m}}(t) = \int_G u(t,x)\phi_{j+k\tilde{m}}(x)dx\ , \quad \text{for } k = 0,1,2,\cdots,r-1$$

and

$$(4.4.19)\quad z_{l,j+l\tilde{m},k}(t) = \int_G g_l(t,x)\tilde{S}(D_x^k)\phi_{j+l\tilde{m}}(x)dx \quad ,\text{for } l=0,1,\cdot\cdot,r-1.$$

From Eq.(4.2.21) in Sec.4.2, it is easy to show that the y(t)-process is a solution to the r-dimensional vector ordinary differential equation,

$$(4.4.20)\quad \frac{dy(t)}{dt} + [\bar{\lambda}^d y(t) - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \bar{\lambda}^b_{l,k}\tilde{z}_{l,k}]$$

$$+ [\bar{y}(t)\bar{\mu}^d - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \bar{z}_{l,k}\bar{\mu}^b_{l,k}]N(t,\omega) = 0 \ ,$$

where

$$(4.4.21)\quad \bar{y}(t) = \begin{bmatrix} y_j(t) & 0 & \cdots & 0 \\ 0 & y_{j+\tilde{m}}(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & y_{j+(r-1)\tilde{m}}(t) \end{bmatrix}$$

and

$$(4.4.22)\quad \bar{z}_{l,k} = \begin{bmatrix} z_{l,j,k}(t) & 0 & \cdots & 0 \\ 0 & z_{l,j+\tilde{m},k}(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & z_{l,j+(r-1)\tilde{m},k}(t) \end{bmatrix}.$$

Consequently, from Eq.(4.4.20), it follows that

$$(4.4.23)\quad N(t,\omega) = [\bar{y}(t)\bar{\mu}^d - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \bar{z}_{l,k}\bar{\mu}^b_{l,k}]^{-1}$$

$$\times\{ -\frac{dy(t)}{dt} - [\bar{\lambda}^d y(t) - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \lambda^b_{l,k}\tilde{z}_{l,k}] \}.$$

Let the time interval $]t_0,t_f[$ be discretized by $t_0<t_1<t_2<\cdots<t_k<$

$\cdots < t_n = t_f$, where the time interval $\Delta t_k = t_k - t_{k-1}$ is assumed to be sufficiently small. Then, we set as

$$(4.4.24)\quad \frac{dy(t)}{dt} \cong \frac{y(t_k) - y(t_{k-1})}{\Delta t_k} .$$

Equation (4.4.23) is approximately computed from Eq.(4.4.24) as

$$(4.4.25)\quad [\bar{y}(t_{k-1})\bar{\mu}^d(t_{k-1}) - \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \bar{z}_{l,k}(t_{k-1})\bar{\mu}^b_{l,k}(t_{k-1})]^{-1}$$

$$\times\{[- y(t_k) + y(t_{k-1})]/\Delta t_k - [\bar{\lambda}^d(t_{k-1})y(t_{k-1})$$

$$- \sum_{l=1}^{n}\sum_{k=1}^{n+1-l} \bar{\lambda}^b_{l,k}(t_{k-1})\tilde{z}_{l,k}(t_{k-1})]\} = \bar{N}(t_k) .$$

By computing $\bar{N}(t_k)$ at time t_k †, the following decision is made;

$$(4.4.26)\quad N(t,\omega) = N_i ,$$

$$\text{if } |\bar{N}(t_k) - N_i|^2 = \min\{|\bar{N}(t_k) - N_1|^2, |\bar{N}(t_k) - N_2|^2, \cdots |\bar{N}(t_k) - N_m|^2\}$$

where $|\cdot|$ denotes the Euclidean norm.

From the decision mechanism defined by Eq.(4.4.26), we can construct the suboptimal control proposed in Sec.3.5 of Chap.3.

4.5. Discussions and Summary

In this chapter, the suboptimal boundary control problem has been studied with the aid of the stochastic eigenvalue problem. The estimation procedure for Markov chain coefficients is the same as those defined by Sec.3.5 of Chap.3. However, since the randomness of Markov chain coefficients strongly affects the boundary control signal, the estimation mechanism given by Eqs.(4.4.25) and (4.4.26)

† At the initial time $t=t_0$, we assume that the information $N(t_0,\omega)$ is given.

becomes complexed.

Without using the stochastic eigenvalue problem, the stochastic differential rule and the optimal boundary control can be derived from the well-known Green's formula. The general form of optimal boundary control for the stochastic partial differential equation with white Gaussian noise coefficients will be presented in Chap.7.

CHAPTER 5. THE OPTIMAL CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS GOVERNED BY PARABOLIC EQUATIONS WITH WHITE GAUSSIAN NOISE COEFFICIENTS

5.1. Introductory Remarks

In Chaps. 3 and 4, we identify the stochastic charactor of random coefficients with Markov chain process with finite stages. The Markov chain assumption is easily applicable to various kinds of physical systems with random coefficients without considering the precise modeling of random coefficients. On the other hand, the state variable of the system dynamics with Markov chain coefficients is out of the framework of the theory of Markov process. From this fact, we must construct the estimation mechanism of Markov chain processes.

Based on the fact that there are also many physical examples whose uncertainties of random coefficients can be well modeled by the white Gaussian noise, in this chapter, the mathematical model of the system considered is given in a form of partial differential equation of parabolic type whose differential operator contains the white Gaussian noise.

In Section 5.2, the formulation of the system model considered here is reviewed within the framework of the functional analysis. In Section 5.3, a differential rule in Hilbert space is derived which plays an important role to determine the optimal control. The optimal control low is derived within the framework of dynamic programming under the criterion that a quadratic cost functional becomes minimal in Sec.5.4. Section 5 is devoted to show two examples for the purpose of interpreting the general theory.

5.2. Review of Mathematical Model and Problem Statement

We shall consider a dynamical system described by

$$(5.2.1a)\quad \frac{\partial u(t,x)}{\partial t} + [A^D(t,x;D_x) + A^s(t,x,\omega;D_x)]u(t,x) = B(t,x)f(t,x)$$

$$\text{for } (t,x)\in T\times G$$

with the initial condition,

$$(5.2.1b)\quad u(t_0,x) = u_0(x), \quad \text{for } x\in G$$

and the boundary conditions,

$$(5.2.1c)\quad \mathcal{B}_j(t,x;D_x)u(t,x) = 0\ , \quad \text{for } (t,x)\in T\times\partial G, \text{ and } j=1,2,\cdots,\frac{n}{2}$$

where we assume the operators $A^D(t,x;D_x)$, $A^s(t,x,\omega;D_x)$, $\{\mathcal{B}_j(t,x;D_x)\}_{j=1}^{n/2}$ and $B(t,x)$ satisfy the conditions described in Definition-2.3.4 in Sec.2.3 of Chap.2.

From the Definition-2.2.2 in Sec.2 of Chap.2, the stochastic integral is constructed by the partial differential operator with white Gaussian coefficients $A^s(t,x,\omega;D_x)$. Then, for $t\in T$, Eq.(5.2.1) can be represented by

$$(5.2.2)\quad \Sigma_3 \mid u(t) + \int_{t_0}^{t} A^D(s)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)]$$

$$= u_0 + \int_{t_0}^{t} B(s)f(s)ds$$

in the spaces V, H and V'.(For detail, see Chap.2.)

In Eq.(5.2.2), the control signal is an element of the admissible control class W_{ad}.

Admissible Control Class: We denote the admissible control class W_{ad}, if all elements of W_{ad} satisfy

i) $f(t)$ is F_t-measurable for all $t\in T$, where F_t is a σ-algebra generated by $\{u(s)\}_{s\geq t_0}$,i.e.,

(5.2.3) $\quad F_t = \sigma\{\, u(s)\ ;\ t_0 \leq s \leq t\}$.

ii) $f \in L^2(\Omega,P;L^2(T;U))$, where U is a convex subset of H.

In this chapter, the quadratic cost is given by

(5.2.4) $\quad L(t,u,f) = (u(t),M(t)u(t))_H + (f(t),N(t)f(t))_H$

where M(t) and N(t) are bounded semipositive and positive definite and self adjoint operators, respectively. The problem is to find the optimal feedback control in such a way that

(5.2.5) $\quad J(f) = E\{\int_{t_0}^{t_f} L(t,u,f)dt \mid u(t_0)=u_o\}$

becomes minimal with respect to $f \in W_{ad}$.

5.3. <u>Differential Rule for Quadratic Functional</u>

Define the quadratic functional $V(t,u)$ by

(5.3.1) $\quad V(t,u) = (u(t),P(t)u(t))_H$,

where P(t) is a deterministic symmetric bounded operator.

[Theorem-5.3.1]: With the conditions of Definition-2.3.3 in Sec.2.3 of Chap.2, we have

(5.3.2)
$$V(t,u) - V(t_0,u_0) = \int_{t_0}^{t_f} \langle u(s),[\dot{P}(s) - A^{D*}(s)P(s) - P(s)A^D(s) + \hat{A}^{s*}(s)P(s)\hat{A}^s(s)]u(s)\rangle ds + 2\int_{t_0}^{t_f}(B(s)f(s),P(s)u(s))_H ds + 2\int_{t_0}^{t_f}(dA^s(s,\omega)[u(s)],P(s)u(s))_H .$$

Proof: Let $V_m = [e_1, e_2, \cdots, e_m]$ and $A_m^D(t)$ be defined by

$$(5.3.3)\quad A_m^D(t) \triangleq \sum_{i=1}^{m} \langle A^D(t)(\cdot), e_i \rangle e_i ,$$

where e_i is an orthonormal basis of H.

Furthermore, define

$$(5.3.4)\quad dA_m^S(t,\omega) \triangleq \sum_{i=1}^{m} (dA^S(t,\omega)(\cdot), e_i)_H e_i$$

and

$$(5.3.5)\quad u_{0m} = \sum_{i=1}^{m} (u_0, e_i)_H e_i \ .$$

We can approximate the stochastic equation (5.2.2) by

$$(5.3.6)\quad u_m(t) + \int_{t_0}^{t} A_m^D(s) u_m(s) ds + \int_{t_0}^{t} dA_m^S(s,\omega)[u_m(s)] = u_{0m} + \int_{t_0}^{t} (B(s)f(s))_m ds$$

where

$$(5.3.7)\quad \int_{t_0}^{t} (B(s)f(s))_m ds = \int_{t_0}^{t} \sum_{i=1}^{m} (B(s)f(s), e_i)_H e_i ds \ .$$

By using the solution process $u_m(t)$ of Eq.(5.3.6), we approximate the quadratic functional by

$$(5.3.8)\quad V(t, u_m(t)) = (u_m(t), P(t) u_m(t))_H \ .$$

Noting that Eq.(5.3.6) is an ordinary differential equation of Itô-type [I1] and applying the Itô-stochastic differential rule, we have

$$(5.3.9)\quad V(t, u_m(t)) - V(t_0, u_{0m}) = \int_{t_0}^{t} \{(u_m(s), \dot{P}(s) u_m(s))_H$$

$$- 2\langle A_m^D(s)u_m(s),P(s)u_m(s)\rangle + (u_m(s),\hat{A}_m^{s*}(s)P(s)\hat{A}_m^s(s)u_m(s))_H$$

$$+ 2((B(s)f(s))_m,P(s)u_m(s))_H\}ds$$

$$+ 2\int_{t_0}^{t} (dA_m^s(s,\omega)[u_m(s)],P(s)u_m(s))_H \,.$$

From the results of Appendix-A, passing to the limit, it follows that

$$(5.3.10)\quad V(t,u) - V(t_0,u_0) = \int_{t_0}^{t} \langle u(s),[\dot{P}(s) - A^{D*}(s)P(s) - P(s)A^D(s)$$

$$+ \hat{A}^{s*}(s)P(s)\hat{A}^s(s)]u(s)\rangle ds + 2\int_{t_0}^{t} (B(s)f(s),P(s)u(s))_H ds$$

$$+ 2\int_{t_0}^{t} (dA^s(s,\omega)[u(s)],P(s)u(s))_H \,.$$

The proof has been completed.

5.4. Derivation of Optimal Control

The optimal control problem is solved by using the method of dynamic programming.

For Eq.(5.2.5), define a minimal cost functional,

$$(5.4.1)\quad V(t,\bar{u}) = \min_{f\in W_{ad}} E\{\int_{t}^{t_f} L(s,u,f)ds \mid u(t)=\bar{u}\} \,.$$

Bearing in mind the feedback optimal control, the minimal cost functional $V(t,\bar{u})$ is defined,†

$$(5.4.2)\quad V(t,\bar{u}) = (\bar{u},\Pi(t)\bar{u})_H \,.$$

Applying the principle of optimality to the cost functional and using Theorem-5.3.1, the following basic equation is derived,

† The justification of Eq.(5.4.2) will be demonstrated in Chap.6 by using the stochastic maximum principle.

$$(5.4.3)\quad \min_{f\in W_{ad}} \{(\bar{u},M(t)\bar{u})_H + (f(t),N(t)f(t))_H + \langle \bar{u},[\dot{\Pi}(t) - A^{D*}(t)\Pi(t) - \Pi(t)A^D(t) + \hat{A}^{S*}(t)\Pi(t)\hat{A}^S(t)]\bar{u}\rangle + (B(t)f(t),\Pi(t)\bar{u})_H + (\bar{u},\Pi(t)B(t)f(t))_H\} = 0 .$$

Minimization in the left-hand side of Eq.(5.4.3) with respect to f gives the optimal control

$$(5.4.4)\quad f^o(t) = - N^{-1}(t)B^*(t)\Pi(t)\bar{u} ,$$

where $\Pi(t)$ satisfies the following operator Riccati equation,

$$(5.4.5a)\quad \dot{\Pi}(t) - A^{D*}(t)\Pi(t) - \Pi(t)A^D(t) + \hat{A}^{S*}(t)\Pi(t)\hat{A}^S(t) + M(t) - \Pi(t)B(t)N^{-1}(t)B^*(t)\Pi(t) = 0$$

with the terminal condition,

$$(5.4.5b)\quad \Pi(t_f) = 0 .$$

For the purpose of application to the practical control problem, the control signal and the associated gain operator equation are re-written in the original form.

In Eq.(5.4.4), assuming that $N(t)= cI$, where c is a constant and I an identity mapping, tne version of $f^o(t,x)$ yields

$$(5.4.6)\quad f^o(t,x) = - \frac{1}{c}B\ (t,x)\int_{G_y}\pi(t,x,y)\bar{u}(t,y)dy$$

where $\pi(t,x,y)$ is a kernel of the operator $\Pi(t)$, which satisfies for $(t,x,y)\in T\times G_x\times G_y$,

$$(5.4.7a)\quad \frac{\partial\pi(t,x,y)}{\partial t} - (A^{D*}(t,x;D_x) + A^{D*}(t,y;D_y))\pi(t,x,y) - \frac{1}{c}\int_{G_z}\pi(t,x,z)B(t,z)B^*(t,z)\pi(t,z,y)dz + \hat{A}^{S*}(t,y;D_y)\hat{A}^{S*}(t,x;D_x)\pi(t,x,y) + m(t,x,y) = 0$$

with the terminal condition,

(5.4.7b) $\pi(t_f,x,y) = 0$, for $(x,y)\in G_x\times G_y$

and the boundary conditions,

(5.4.7c) $C_j(t,x;D_x)\pi(t,x,y) = 0$, for $(t,x,y)\in T\times\partial G_x\times G_y$ and $j=1,2,\cdots,\frac{n}{2}$

(5.4.7d) $C_j(t,y;D_y)\pi(t,x,y) = 0$, for $(t,x,y)\in T\times G_x\times\partial G_y$ and $j=1,2,\cdots,\frac{n}{2}$

where $\{C_j\}_{j=1}^{n/2}$ is an adjoint boundary system of $\{B_j\}_{j=1}^{n/2}$ and $\pi(t,x,y) = \pi(t,y,x)$.

5.5. Digital Simulation Studies and Illustrative Examples

In this section, first we shall consider the one dimensional stochastic heat equation.

[Example-5.5.1] Consider the following system of parabolic type described by,

$$(5.5.1a)\quad \frac{\partial u(t,x)}{\partial t} + a_2(t)\frac{\partial^2 u(t,x)}{\partial x^2} + a_1(t)\eta(t,\omega)\frac{\partial u(t,x)}{\partial x} = b(t)f(t,x)$$

$$\text{for } (t,x)\in T\times G =]0,1[$$

with the initial condition,

(5.5.1b) $u(t_0,x) = u_0(x)$, for $x\in G$

and the boundary condition,

(5.5.1c) $u(t,0) = u(t,1) = 0$,

where $\eta(t,\omega)$ is a white Gaussian noise with zero mean and unit variance. From the Coercivity condition-2.2.1 in Sec.2.2 of Chap.2, setting as $V = H_0^1(G)$, the deterministic coefficients $a_2(t)$ and $a_1(t)$ are assumed to satisfy

(5.5.2) $2a_2(t) + a_1^2(t) < 0$,for $^\forall t\in\bar{T}$,

where $(\bar{\cdot})$ denotes the closure of $(\cdot)$.

For instance, consider a particle which is being transferred in a moving stream. If a particle is heavy, i.e., the ratio of its density to that of the density of the fluid is much less than unity, gravitational forces play an important role in its transport, and we may consider a random field which is expressed by the term of $a_1(t)\eta(t,\omega)\frac{\partial u(t,x)}{\partial x}$. From the fact mentioned above, its randomness in the coefficient of $\frac{\partial u(t,x)}{\partial x}$ exhibits the force field.

In this example, we can easily find

$$(5.5.3) \quad A^D(t) = a_2(t)D_x^2$$

and

$$(5.5.4) \quad \hat{A}^s(t) = a_1(t)D_x^1 \ .$$

The quadratic functional is preassigned by

$$(5.5.5) \quad J(f) = E\{\int_{t_0}^{t_f}\int_0^1 [u^2(t,x) + f^2(t,x)]dxdt\,|\,u(t_0,x)=\bar{u}_0(x)\}$$

By using the results of Sec.5.4, the optimal feedback control $f^o(t)$ is obtained;

$$(5.5.6) \quad f^o(t,x) = -\int_0^1 b(t)\pi(t,x,y)\bar{u}(t,y)dy$$

where, from Eq.(5.4.7), the kernel $\pi(t,x,y)$ is determined by

$$(5.5.7a) \quad \frac{\partial\pi(t,x,y)}{\partial t} - a_2(t)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\pi(t,x,y) + a_1^2(t)\frac{\partial^2}{\partial x\partial y}\pi(t,x,y)$$

$$+ \delta(x-y) - b^2(t)\int_0^1 \pi(t,x,z)\pi(t,z,y)dy = 0 \ ,$$

$$\text{for } (t,x,y)\in T\times]0,1[\times]0,1[$$

with the terminal condition,

$$(5.5.7b) \quad \pi(t_f,x,y) = 0 \ , \quad \text{for } (x,y)\in]0,1[\times]0,1[$$

and the boundary conditions for $t\in T$

$$(5.5.7c)\quad \pi(t,x,y)\Big|_{\substack{\text{at } x=0 \text{ or } 1\\ y\in]0,1[}} = \pi(t,x,y)\Big|_{\substack{\text{at } y=0 \text{ or } 1\\ x\in]0,1[}} = 0.$$

[Example-5.5.2] We shall consider a somewhat artificial but important class of the 4-th order system of parabolic type described by

$$(5.5.8a)\quad \frac{\partial u(t,x)}{\partial t} + a_4(t)\frac{\partial^4 u(t,x)}{\partial x^4} + (a_2(t) + c(t)\eta(t,\omega))\frac{\partial^2 u(t,x)}{\partial x^2}$$

$$= b(t)f(t,x) \ , \quad \text{for } (t,x)\in T\times G=]0,1[\ ,$$

with the initial condition,

$$(5.5.8b)\quad u(t_0,x) = u_0(x) \ , \quad \text{for } x\in G$$

and the boundary conditions,

$$(5.5.8c)\quad u(t,x) = 0 \ , \quad \text{for } (t,x)\in T\times\partial G$$

$$(5.5.8d)\quad \frac{\partial^2 u(t,x)}{\partial x^2} = 0 \ , \quad \text{for } (t,x) \ T\times\partial G$$

where $\eta(t,\omega)$ is a white Gaussian noise process with zero mean and unit variance, and $a_4(t)$, $a_2(t)$, $c(t)$ and $b(t)$ are scalar functions, respectively. Furthermore, from Coercivity condition-2.2.1 in Sec.2.2 of Chap.2, setting $V=H_0^1(G)\cap H^2(G)$, it is assumed that

$$(5.5.9)\quad 2a_4(t) - c^2(t) > 0 \text{ and } a_2(t) < 0 \ , \quad \text{for } {}^\forall t\in\overline{T} \ .$$

It follows that

$$(5.5.10)\quad A^D(t) = a_4(t)D_x^4 + a_2(t)D_x^2$$

and

$$(5.5.11)\quad \hat{A}^s(t) = c(t)D_x^2 \ .$$

In this example, the cost functional is given by Eq.(5.5.5). From the results of Sec.5.4, the optimal control $f^o(t,x)$ is given by

$$(5.5.12)\quad f^o(t,x) = -\int_0^1 b(t)\pi(t,x,y)\bar{u}(t,y)dy$$

where $\pi(t,x,y)$ satisfies

$$(5.5.13a)\quad \frac{\partial\pi(t,x,y)}{\partial t} - a_4(t)\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)\pi(t,x,y)$$

$$- a_2(t)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\pi(t,x,y) + c^2(t)\frac{\partial^4\pi(t,x,y)}{\partial x^2\partial y^2}$$

$$+ \delta(x-y) - b^2(t)\int_0^1 \pi(t,x,z)\pi(t,z,y)dz = 0\ ,$$

$$\text{for } (t,x,y)\in T\times]0,1[\times]0,1[$$

with the terminal condition

$$(5.5.13b)\quad \pi(t_f,x,y) = 0\ ,\quad \text{for } (x,y)\in]0,1[\times]0,1[$$

and the boundary conditions,

$$(5.5.13c)\quad \pi(t,x,y)\Big|_{\substack{\text{at } x=0 \text{ or } 1\\ y\in]0,1[}} = \pi(t,x,y)\Big|_{\substack{\text{at } y=0 \text{ or } 1\\ x\in]0,1[}} = 0$$

$$(5.5.13d)\quad \frac{\partial^2\pi(t,x,y)}{\partial x^2}\Bigg|_{\substack{\text{at } x=0 \text{ or } 1\\ y\in]0,1[}} = \frac{\partial^2\pi(t,x,y)}{\partial y^2}\Bigg|_{\substack{\text{at } y=0 \text{ or } 1\\ x\in]0,1[}} = 0.$$

In this example, Eq.(5.5.8) was simulated on a digital computer and the optimal control $f^o(t,x)$ was determined by Eq.(5.5.12) with the solution to Eq.(5.5.13). A wide variety of sample runs was simulated. The results presented below are representative of simulation experiments. In all experiments, the values of a_4, a_2, c and b were respectively set as a_4=0.002, a_2=-0.1, c=0.015 and b=0.5. The initial condition (5.5.8b) was given by $u_0(x)=\sin^2\pi x$. Throughout the experiments, the partitioned time interval and spatial variable were taken

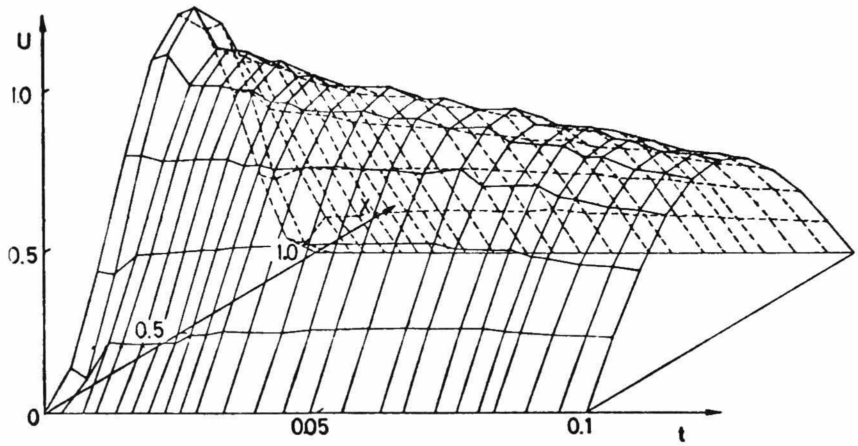

Fig. 5.5.1 A sample run of the system state u(t,x) without control

as $\Delta t=2.5\times10^{-4}$ and $\Delta x=0.05$. respectively.

Figure 5.5.1 shows a representative of sample runs of the system without control. A sample run of the system deriven by the optimal control signal $f^{o}(t,x)$ is shown in Fig.5.5.2.

5.6. Discussions and Summary

In this chapter, the optimal control problem for distributed parameter systems governed by Parabolic equation with white Gaussian noise coefficients has been stidied. First, establishing the stochas-- tic differential rule in Hilbert spaces, the optimal feedback control was derived by using the well-known dynamic programming approach. Although, in illustrative examples, the simulated control

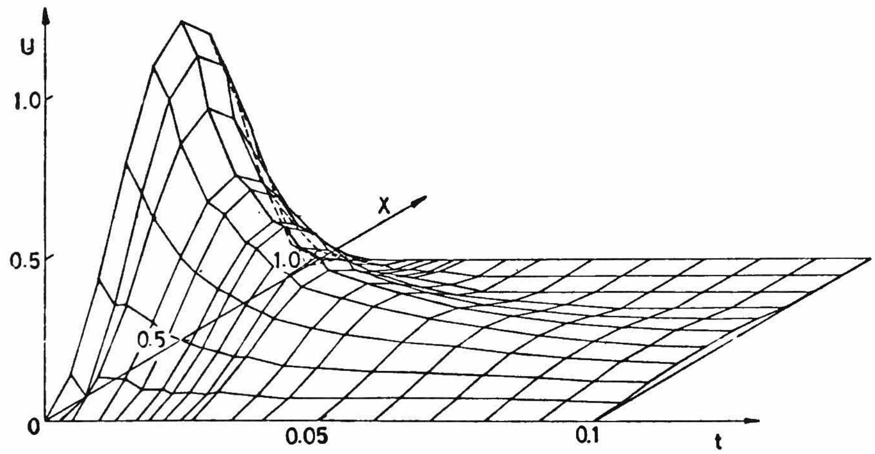

Fig. 5.5.2 A sample run of the system state u(t,x) with optimal control.

signals are spatially distributed, the theory presented here may be applicable to the case of point wise controlers,i.e., $B(t)(\cdot) = \int_G \delta(x-a)(\cdot)dx$.

CHAPTER 6. THE OPTIMAL CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS GOVERNED BY HYPERBOLIC EQUATION WITH WHITE GAUSSIAN COEFFICIENTS

6.1. Introductory Remarks

In the preceding chapters, the distributed parameter system has been considered modeled by a general class of stochastic partial differential equation of Parabolic type. However, in practice, we may find that many physical systems can be described by a class of partial differential equation of Hyperbolic type. For a wave travelling in a random mediam and a vibration of prismatic bar with a randomly varying axial compressive force, the second derivative term with respect to time variable must be taken into account the system dynamics, which we briefly call the Hyperbolic system. In particular, a stochastic Hyperbolic system is receiving considerable attention in the field of the structure response analysis of building subjected to the earthquake force. As mentioned in Chap.2, by invoking the theory of stochastic evolution equation, the original Hyperbolic system is represented by the two dimensional stochastic equation in Hilbert space, and the method for deriving the optimal control presented in the previous chapters is directly extended to Hyperbolic case.

In Sec.6.2, the mathematical model of Hyperbolic system is given and the associated differential rule is also derived. By using the stochastic maximum principle and the decoupling theory, the optimal feedback control is derived under the quadratic cost functional in Sec.6.3. Section 6.4 is devoted to show the possibility of finding the optimal control under noisy observations.

6.2. Review of Mathematical Model and Stochastic Differential Rule

Let the state variable $v(t,x)$ of a stochastic distributed parameter system satisfy

$$(6.2.1a)\quad \frac{\partial^2 v(t,x)}{\partial t^2} + [A^D(t,x;D_x) + A^S(t,x,\omega;D_x)]v(t,x) = B(t,x)f(t,x) \quad \text{for } (t,x)\in T\times G$$

with the initial conditions,

$$(6.2.1b)\quad v(t_0,x) = v_0(x)\ , \qquad \text{for } x\in G$$

$$(6.2.1c)\quad \dot{v}(t_0,x) = \dot{v}_0(x)\ , \qquad \text{for } x\in G$$

and the boundary conditions,

$$(6.2.1d)\quad \mathcal{B}_j(t,x;D_x)v(t,x) = 0\ , \quad \text{for } (t,x)\in T\times\partial G \text{ and } j=1,2,\cdots,\tfrac{n}{2}$$

where $\mathcal{B}_j(t,x;D_x)$ $(j=1,2,\cdots,\frac{n}{2})$ are deterministic boundary operators, $A^D(t,x;D_x)$ is the deterministic partial differential operator which satisfies Coercivity condition-2.2.2, and $A^S(t,s,\omega;D_x)$ is the partial differential operator with white Gaussian coefficients whose variance operator satisfies Coercivity condition-2.2.3.

From Theorem-2.2.2, Eq.(6.2.1) can be rewritten as the two dimensional stochastic evolution equation,

$$(6.2.2)\quad \Sigma_7|\ z(t) + \int_{t_0}^{t} \tilde{A}^D(s)z(s)ds + \int_{t_0}^{t} d\tilde{A}^S(s,\omega)[z(s)] = z_0 + \int_{t_0}^{t} \tilde{B}(s)f(s)ds\ ,$$

where $z(t)=[v(t),\dot{v}(t)]'$,

$$(6.2.3)\quad \tilde{A}^D(t) = \begin{bmatrix} 0 & -I \\ A^D(t) & 0 \end{bmatrix}$$

$$(6.2.4)\quad \int_{t_0}^{t} d\tilde{A}^s(s,\omega)[z(s)] = [\ 0\ ,\ \int_{t_0}^{t} dA^s(s,\omega)[v(s)]\ ]'$$

and

$$(6.2.5)\quad \int_{t_0}^{t} \tilde{B}(s)f(s)ds = [\ 0\ ,\ \int_{t_0}^{t} B(s)f(s)ds\ ]' \ .$$

Noting that the system Σ_7 contains the stochastic integral term given by Eq.(6.2.4), the stochastic differential rule for the quadratic functional can be easily derived.

[Theorem-6.2.1]: Define

$$(6.2.6)\quad V(t,z) = [\ z(t),\ Q(t)z(t)\ ]_H$$

where $Q(t)$ is a 2×2 dimensional deterministic bounded self-adjoint operator, and $[\vec{\psi}_1,\ Q(t)\vec{\psi}_2]_H$ denotes

$$(6.2.7)\quad [\vec{\psi}_1,\ Q(t)\vec{\psi}_2]_H = (\psi_1^1,\ Q_{11}(t)\psi_2^1)_H + (\psi_1^2,\ Q_{22}(t)\psi_2^2)_H + \langle\psi_1^1,\ Q_{12}(t)\psi_2^2\rangle + \langle\psi_1^2,\ Q_{21}(t)\psi_2^1\rangle$$

$$\text{for } \vec{\psi}_1=[\psi_1^1,\ \psi_1^2]' \text{ and } \vec{\psi}_2=[\psi_2^1,\ \psi_2^2]' \ .$$

With the conditions of Definition-2.3.7, we have

$$(6.2.8)\quad V(t,z) - V(t_0,z_0) = \int_{t_0}^{t} [z(s),\ (\dot{Q}(s) - \tilde{A}^{D*}(s)Q(s) - Q(s)\tilde{A}^D(s) + \hat{G}^{s*}(s)Q(s)\hat{G}^s(s))z(s)\ ]_H ds$$

$$+ 2\int_{t_0}^{t} [\tilde{B}(s)f(s),\ Q(s)z(s)]_H ds$$

$$+ 2\int_{t_0}^{t} [d\tilde{A}^s(s,\omega)[z(s),\ Q(s)z(s)]_H \ ,$$

where

(6.2.9) $[z(t),\ \hat{G}^{s*}(t)Q(t)\hat{G}^s(t)z(t)]_H = (v(t),\ \hat{A}^{s*}(t)Q_{22}(t)\hat{A}^s(t)v(t))_H$.

Proof: By using the same approach as in the proof of Theorem-5.3.1, we first assume Sobolev space V is separable. Then, the orthonormal basis of H can be made up with element of V. We denote these by e_1, $e_2, \cdots$.

Let $V_m=[e_1,e_2,\cdots,e_m]$, $A_m^D(t)\in \mathcal{L}(V_m,V_m')$ and

(6.2.10) $d\tilde{A}_m^s(t,\omega)[\cdot] = \sum_{i=1}^{m} (dA^s(t,\omega)[\cdot],\ e_i)_H e_i$.

Then, the system Σ_7 can be approximated by

(6.2.11) $$z_m(t) + \int_{t_0}^{t} \tilde{A}_m^D(s)z_m(s)ds + \int_{t_0}^{t} d\tilde{A}_m^s(s,\omega)[z_m(s)]$$

$$= z_{0m} + \int_{t_0}^{t} [\tilde{B}(s)f(s)]_m ds \quad ,$$

where $z_{0m}=[\sum_{i=1}^{m}(v_0,e_i)_H e_i,\ \sum_{i=1}^{m}(\dot{v}_0,e_i)_H e_i]'$, $z_m(t)=[z_m^1(t),\ z_m^2(t)]'$

(6.2.12) $$\tilde{A}_m^D(t) = \begin{bmatrix} 0 & \sum_{i=1}^{m}(\cdot,e_i)_H e_i \\ A_m^D(t) & 0 \end{bmatrix}$$

(6.2.13) $d\tilde{A}_m^s(t,\omega)[z_m(t)] = [\ 0\ ,\ \ dA_m^s(t,\omega)[z_m^1(t)]]'$

and

(6.2.14) $[\tilde{B}(t)f(t)]_m = [\ 0\ ,\sum_{i=1}^{m}(B(t)f(t),\ e_i)_H e_i]'$.

By using the solution process $z_m(t)$ to Eq.(6.2.11), the quadratic functional given by Eq.(6.2.6) is also approximated by

$$(6.2.15)\quad V(t,z_m(t)) = [z_m(t),\ Q(t)z_m(t)]_H .$$

From the fact that the Sobolev space V_m is finite dimensional, Eq.(6.2.11) is an ordinary differential equation of Itô-type. Then, we have

$$(6.2.16)\quad V(t,z_m(t)) - V(t_0,z_{0m}) = \int_{t_0}^{t} [z_m(s),\ [\dot{Q}(s) - \tilde{A}_m^{D*}(s)Q(s)$$

$$- Q(s)\tilde{A}_m^{D}(s) + \hat{G}_m^{S*}(s)Q(s)\hat{G}_m^{S}(s)]z_m(s)]_H ds$$

$$+ 2\int_{t_0}^{t} [[\tilde{B}(s)f(s)]_m,\ Q(s)z_m(s)]_H ds$$

$$+ 2\int_{t_0}^{t} [d\tilde{A}_m^{S}(s,\omega)[z_m(s)],\ Q(s)z_m(s)]_H ,$$

where

$$(6.2.17)\quad \hat{G}_m^{S}(t) = \begin{bmatrix} 0 & 0 \\ \sum_{i=1}^{m}(\hat{A}^S(t)(\cdot),\ e_i)e_i & 0 \end{bmatrix}$$

From Eq.(6.2.16), passing to the limit $(m\to\infty)$, we can derive Eq.(6.2.8). The proof has thus been completed.

6.3. Derivation of Optimal Control (Stochastic Maximum Principle)

In this section, consider the following quadratic cost:

$$(6.3.1)\quad L(t,v,f) = \tfrac{1}{2}(M(t)v(t),\ v(t))_H + \tfrac{1}{2}(N(t)f(t),\ f(t))_H ,$$

where $M(t)$ and $N(t)$ are deterministic bounded semipositive and positive definite self-adjoint operators, respectively, and $v(t)$ is a solution process to Eq.(6.2.1), i.e., $v(t)$ is the first component of $z(t)$-process. The problem is to find the optimal feedback control in such a way that the functional,

$$(6.3.2)\quad J(f) = E\{\int_{t_0}^{t_f} L(t,v,f)dt\},$$

becomes minimal with respect to $f \in W_{ad}$, where W_{ad} denotes the following admissible control class,

Admissible Control Class:

i) $f(t)$ is F_t-measurable for all $t \in T$, where F_t is a σ-algebra with respect to which $\{v(s), \dot{v}(s)\}$ becomes measurable for $s \geq t_0$, i.e.,

$$(6.3.3)\quad F_t = \sigma\{[v(s), \dot{v}(s)]'; t_0 \leq s \leq t\}$$
$$= \sigma\{z(s); t_0 \leq s \leq t\}$$

ii) $f \in L^2(\Omega,P;L^2([t_0,t_f];U))$, where U is a convex subset of $V \times H$.

For the purpose of theoretical developments, the cost functional is rewritten in the following form with the aid of the representation of the system Σ_7,

$$(6.3.4)\quad L(t,v,f) = \tilde{L}(t,z,f)$$
$$= \frac{1}{2}[\tilde{M}(t)z(t), z(t)]_H + \frac{1}{2}(N(t)f(t), f(t))_H$$

where

$$(6.3.5)\quad \tilde{M}(t) = \begin{bmatrix} M(t) & 0 \\ 0 & 0 \end{bmatrix} .$$

Then, the optimal control problem at hand is converted into that of finding the control so as to minimizing the new functional $\mathcal{J}$

$$(6.3.6)\quad \tilde{J}(f) = E\{\int_{t_0}^{t_f} \tilde{L}(t,z,f)dt\}$$

becomes minimal with respect to $f \in W_{ad}$.

By using the variational inequality theorem[L4], we can easily find that it is necessary and sufficient that f^o is the optimal control which satisfies

$$(6.3.7)\quad (\delta\tilde{J}(f^o),\ f - f^o)_H \geq 0\ ,\quad \text{for all } f \in W_{ad}$$

where $\delta(\cdot)$ denotes the Gateaux-differential in $L^2([t_0,t_f];U)$.

In this chapter, we set

$$(6.3.8)\quad E\{\int_{t_0}^{t_f}([M(t)z(t),\ \tilde{z}(t;f - f^o)]_H + (N(t)f^o(t),\ f(t) - f^o(t))_H)dt\} \geq 0\ ,\quad \text{for all } f \in W_{ad}$$

where $\tilde{z}(t;f - f^o)$ satisfies

$$(6.3.9)\quad \tilde{z}(t;f - f^o) + \int_{t_0}^{t}\tilde{A}^D(s)\tilde{z}(s;f - f^o)ds + \int_{t_0}^{t} d\tilde{A}^S(s,\omega)[\tilde{z}(s;f - f^o)] = \int_{t_0}^{t}\tilde{B}(s)(f(s) - f^o(s))ds\ .$$

Let us now introduce $\vec{\phi}(t)$ as the solution of stochastic adjoint system equation†:

$$(6.3.10)\quad \vec{\phi}(t) + \int_{t}^{t_f}\tilde{A}^{D*}(s)\vec{\phi}(s)ds = -\int_{t}^{t_f}\tilde{M}(s)z(s)ds + \int_{t}^{t_f}\vec{\Pi}(s)dA^S(s,\omega)[z(s)] + \int_{t}^{t_f}\hat{G}^{S*}(s)\vec{\Pi}(s)\hat{G}^S(s)z(s)ds\ ,$$

† The precise meaning of adjoint system is given by Ref.[B13].

where, with a 2×2 dimensional operator $\vec{\Pi}(t)$, $\vec{\phi}(t)=[\phi_1(t),\ \phi_2(t)]$, and $\vec{\phi}(t)=\vec{\Pi}(t)z(t)$.

Define

(6.3.11) $V(t,z) = [\vec{\phi}(t),z(t)]_H$.

Using the stochastic differential rule given in Theorem-6.2.1, Eq.(6.3.11) becomes

$$(6.3.12)\ E\{[\vec{\phi}(t_f),\tilde{z}(t_f;f - f^o)]_H\} - E\{[\vec{\phi}(t_0),\tilde{z}(t_0;f - f^0)]_H\}$$

$$= - E\{\int_{t_0}^{t_f}[\tilde{M}(t)z(t),\tilde{z}(t;f - f^o)]_H dt\}$$

$$+ E\{\int_{t_0}^{t_f}[\vec{\phi}(t),B(t)(f(t) - f^o(t))]_H dt\}.$$

Noting that, from Eqs.(6.3.9) and (6.3.10), $\vec{\phi}(t_f)=0$ and $\tilde{z}(t_0;f - f^o)=0$, Eq.(6.3.12) becomes

$$(6.3.13)\ E\{\int_{t_0}^{t_f}[\tilde{M}(t)z(t),\tilde{z}(t;f - f^o)]_H dt\}$$

$$= E\{\int_{t_0}^{t_f}[\vec{\phi}(t),\tilde{B}(t)(f(t) - f^o(t))]_H dt\}.$$

From Eq.(6.3.13), Eq.(6.3.8) is rewritten as

$$(6.3.14)\ E\{\int_{t_0}^{t_f}([\vec{\phi}(t),B(t)(f(t) - f^o(t))]_H$$

$$+ (N(t)f^o(t),f(t) - f^o(t))_H)dt\} \geq 0 \text{ for all } f\epsilon W_{ad} .$$

Noting that

$$(6.3.15)\ [\vec{\phi}(t),\tilde{B}(t)(f(t) - f^o(t))]_H = (\phi_2(t),B(t)(f(t) - f^o(t))_H ,$$

it follows that

$$(6.3.16)\quad E\{\int_{t_0}^{t_f} (B^*(t)\phi_2(t) + N(t)f^o(t), f(t) - f^o(t))_H dt\} \geq 0$$

for all $f \epsilon W_{ad}$.

Since all elements of W_{ad} are F_t measurable, Eq.(6.3.16) can be re-written as

$$(6.3.17)\quad E\{\int_{t_0}^{t_f} (B^*(t)E\{\phi_2(t)|F_t\} + N(t)f^o(t), f(t) - f^o(t))_H dt\} \geq 0$$

for all $f \epsilon W_{ad}$.

Then, the optimal control is given by

$$(6.3.18)\quad f^o(t) = - N^{-1}(t)B^*(t)E\{\phi_2(t)|F_t\} .$$

In order to generate the optimal control signal, the conditional expectation $E\{\phi_2(t)|F_t\}$ must be calculated. Noting that the Proposition 2.2.2 of stochastic integral,

$$(6.3.19)\quad E\{\int_{t}^{t_f} d\tilde{A}^s(s,\omega)[z(s)]|F_t\} = 0$$

and

$$(6.3.20)\quad E\{\int_{t}^{t_f} \vec{\Pi}(s)d\tilde{A}^s(s,\omega)[z(s)]|F_t\} = 0 ,$$

we have

$$(6.3.21a)\quad \frac{d\hat{z}_t(\tau)}{d\tau} + \tilde{A}^D(\tau)\hat{z}_t(\tau) = - \tilde{B}(\tau)N^{-1}(\tau)B^*(\tau)\hat{\phi}_2^t(\tau)$$

$$(6.3.21b)\quad \hat{z}_t(t) = z(t)$$

and

$$(6.3.22a)\quad \frac{d\hat{\vec{\phi}}_t(\tau)}{d\tau} - \tilde{A}^{D*}(\tau)\hat{\vec{\phi}}_t(\tau) = - \tilde{M}(\tau)\hat{z}_t(\tau) - \hat{G}^{s*}(\tau)\vec{\Pi}(\tau)\hat{G}^s(\tau)\hat{z}_t(\tau)$$

(6.3.22b) $\hat{\vec{\phi}}_t(t) = \vec{\phi}(t)$ with $\vec{\phi}_t(t_f) = 0$,

where we denote

(6.3.23) $\hat{z}_t(\tau) \triangleq E\{z(\tau)|F_t\}$

and

(6.3.24) $\hat{\vec{\phi}}_t(\tau) \triangleq E\{\vec{\phi}(\tau)|F_t\} = [E\{\phi_1(\tau)|F_t\},E\{\phi_2(\tau)|F_t\}]'$.

From Eq.(6.3.10), we have $\hat{\vec{\phi}}_t(\tau) = \Pi(\tau)\hat{z}_t(\tau)$. Then it is easy to show that $\vec{\Pi}(\tau)$ satisfies,

$$\text{(6.3.25a)}\quad \frac{d\vec{\Pi}(\tau)}{d\tau} - \tilde{A}^{D*}(\tau)\vec{\Pi}(\tau) - \vec{\Pi}(\tau)\tilde{A}^{D}(\tau) + \tilde{M}(\tau)$$

$$+ \hat{G}^{S*}(\tau)\vec{\Pi}(\tau)\hat{G}^{S}(\tau) - \vec{\Pi}(\tau)\begin{bmatrix} 0 & 0 \\ 0 & B(\tau)N^{-1}(\tau)B^{*}(\tau)\end{bmatrix}\vec{\Pi}(\tau) = 0$$

with the terminal condition,

(6.3.25b) $\vec{\Pi}(t_f) = 0$.

Therefore, the optimal control signal $f^{o}(t)$ becomes

(6.3.26) $f^{o}(t) = - N^{-1}(t)B^{*}(t)[\Pi_{21}(t)z_1(t) + \Pi_{22}(t)z_2(t)]$

where $\Pi_{21}(t)$ and $\Pi_{22}(t)$ are elements of the 2×2 dimensional operator $\Pi(t)$, which satisfy

$$\text{(6.3.27)}\quad \dot{\Pi}_{11}(t) - A^{D*}(t)\Pi_{21}(t) - \Pi_{12}(t)A^{D}(t) + M(t)$$

$$+ \hat{A}^{S*}(t)\Pi_{22}(t)\hat{A}^{S}(t) - \Pi_{12}(t)B(t)N^{-1}(t)B^{*}(t)\Pi_{21}(t) = 0$$

$$\text{(6.3.28)}\quad \dot{\Pi}_{12}(t) - A^{D}(t)\Pi_{22}(t) + \Pi_{11}(t) - \Pi_{12}(t)B(t)N^{-1}(t)B^{*}(t)\Pi_{22}(t)$$

$$= 0$$

and

$$(6.3.29)\quad \dot{\Pi}_{22}(t) + \Pi_{12}(t) + \Pi_{21}(t) - \Pi_{22}(t)B(t)N^{-1}(t)B^*(t)\Pi_{22}(t) = 0$$

with the terminal conditions

$$(6.3.30)\quad \Pi_{11}(t_f) = \Pi_{12}(t_f) = \Pi_{21}(t_f) = \Pi_{22}(t_f) = 0 .$$

Remark 6.3: The Riccati equation given by Eq.(6.3.27) has not been studied except for the case of losing the term $\hat{A}^{s*}(t)\Pi_{22}(t)\hat{A}^s(t)$ by Lions [L5]. However, with the aid of computer calculation, we can solve the operator Riccati equation (6.3.24), and construct the feedback control.

6.4. On the Optimal Control under Noisy Observations

In this chapter, in order to derive the optimal control, a use is made of the method of stochastic maximum principle. Different from the dynamic programming approach, the stochastic maximum principle demands the complex calculation to derive the explicit feedback control gain. In spite of this fact, an extension to the control problem under noisy observation is possible.

From the results of the previous section 6.3., in order to design the optimal feedback control, we must obtain the information about $z_1(t)$ and $z_2(t)$-processes. From the practical point of view, there are many situations that only the v(t) process information is obtained, and furthermore we must consider

$$(6.4.1)\quad dy(t) = \int_G h(t,x)v(t,x)dxdt + R(t)de(t) ,$$

where e(t) denotes the observation noise, which is assumed to be modeled by the Brownian motion process independent of the white noise coefficients.

Based on the σ-algebra $F_{y_t}=\sigma\{y(s);t_0\le s\le t\}$ with respect to which $y(s)$ $(t_0\le s\le t_f)$ becomes measurable, consider the optimal control problem that the functional defined by Eq.(6.3.1) becomes minimal with respect to $f\in W^y_{ad}$, where W^y_{ad} is an admissible control class. In this case we assume all elements of W^y_{ad} are F_{y_t}- measurable. Applying the stochastic maximum principle, we have from Eq.(6.3.17),

$$(6.4.2)\quad E\{\int_{t_0}^{t_f}(B^*(t)E\{\phi_2(t)|F_{y_t}\} + N(t)f^o(t),\ f(t) - f^o(t))_H\}dt \ge 0 \qquad \text{for all } f\in W^y_{ad}$$

where $\phi_2(t)$ satisfies Eq.(6.3.10). Then, the optimal control under noisy observations becomes

$$(6.4.3)\quad f^o(t) = - N^{-1}(t)B^*(t)E\{\phi_2(t)|F_{y_t}\} .$$

Using the same approach stated in Sec.6.3, we may find that the calculation of the conditional expectation of $\phi_2(t)$ is equivalent to calculate the minimal variance estimate $E\{z(t)|F_{y_t}\}$. Hence, the stochastic maximum principle shows that if the minimal variance estimate $E\{z(t)|F_{y_t}\}$ is obtained, we can construct the optimal control signal under noisy observations. The estimation problem is out of consideration. (For estimation problem for distributed parameter systems with stochastic coefficients, please refer Refs.[S12] and [S14].)

6.5. Discussions and Summary

In this chapter, based on the stochastic maximum principle, the method has been developed for finding the optimal control for the Hyperbolic type partial differential equation with white Gaussian noise coefficients. Comparing with the dynamic programming approach the stochastic maximum principle requires the complex calculation to derive the optimal control. However, an extention is straightforward to the version of the optimal control under noisy observations. The optimal state estimate and control are performed by using the sufficient statistics, $E\{z(t)|F_{y_t}\}$ in the configuration of feedback control system.

The construction of the minimal variance estimation mechanism becomes difficult problem because of uncertainties exhibited in coefficients. The Radon-Nikodym theorem in the Hilbert space, [D3], [B14] plays a key role to explore the estimation problem.

CHAPTER 7. THE OPTIMAL BOUNDARY CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS OF PARABOLIC TYPE WITH WHITE GAUSSIAN NOISE COEFFICIENTS

7.1. Introductory Remarks

Considering the control problem of distributed parameter systems, the boundary control problem is a peculiar one of distributed parameter systems. Such a problem does not have an analogy with a class of ordinary differential equations. Because of the randomness of coefficients in the partial differential operator modeled by the white Gaussian noise process, the influence of the boundary control to the interior domain of the spatially distributed state variable becomes random. As mentioned in Chap.4, the boundary control is being perturbed by stochastic coefficients contained in the partial differential operator and transmitted to the interior domain of the system state. Generalizing the well-known Green's formula to the case of partial differential operator with stochastic coefficients, the aspect mentioned above is figured out. The purpose of this chapter is to find the optimal boundary control for the distributed parameter systems with white Gaussian noise coefficients circumventing difficulties due to the stochastic eigenvalue problem.

In Section 7.2, the mathematical model of the system considered here is given through a new definition of stochastic integral defined in Sec.2.2. The differential rule given in Chap.5, in Sec.7.3., is extended to the process determined by the solution process to the partial differential equation with white Gaussian noise coefficients and with boundary inputs. Furthermore, by using the Green's formula, the influence of boundary inputs to the interior domain of the solution process is investigated for various types of boundary conditions. Section 7.4 is devoted to the optimal boundary control in a form of depending the stochastic coefficients with the aid of dynamic programming approach. In Section 7.5, the optimal control strategy developed in Sec.7.4 is shown for a class of stochastic distributed parameter systems. Finally, in order to examine contributions of the boundary control to the interior domain of the system, results of simulation experiments are demonstrated.

7.2. Review of Mathematical Model

Consider a dynamical system described by

$$\text{(7.2.1a)} \quad \frac{\partial u(t,x)}{\partial t} + [A^D(t,x;D_x) + A^S(t,x,\omega;D_x)]u(t,x) = 0 \qquad \text{for } (t,x) \in T \times G ,$$

with the initial condition,

$$\text{(7.2.1b)} \quad u(t_0,x) = u_0(x) , \qquad \text{for } x \in G$$

and the boundary conditions,

$$\text{(7.2.1c)} \quad B_j(t,x;D_x)u(t,x) = g_j(t,x) , \text{ for } (t,x) \in T \times \partial G \text{ and } j=1,2,\cdots,\tfrac{n}{2},$$

where $B_j(t,x;D_x)$ $(j=1,2,\cdots,\frac{n}{2})$ are deterministic boundary operators and $A^D(t,x;D_x)$ and $A^S(t,x,\omega;D_x)$ are deterministic and stochastic partial differential operators which satisfy Coercivity condition-2.2.1, respectively and boundary control signals $g_j(t,x)$ $(j=1,2,\cdots,\frac{n}{2})$ are elements of admissible control class W^b_{ad}.

Admissible Control Class: We denote the admissible control class W^b_{ad}, if all elements of W^b_{ad} satisfy the following conditions:

i) $g_j(t)$ is F_t-measurable for all $t\in T$, where F_t is a σ-algebra generated by $\{u(s)\}_{t\geq s\geq t_0}$, i.e.,

$$(7.2.2) \quad F_t = \sigma\{u(s);\ t_0\leq s\leq t\}.$$

ii) $g_j \in L^2(T;\ U_b)$, where U_b is a convex subset of $\tilde{W} = \underset{j}{\cap} L^2(\Omega,P;\ H^{n-n_j-\frac{1}{2}}(\partial G))$ for $j=1,2,\cdots,\frac{n}{2}$, $n_j\leq n-1$.

From Theorem-2.2.1 and Definition-2.3.4, Eq.(7.2.1) can be rewritten as

$$(7.2.3) \quad \Sigma_4| \ u(t) + \int_{t_0}^{t} A^D(s)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)] = u_0 \qquad \text{for } t\in T$$

with $B_j(t)u(t)=g_j(t)$ on $T\times\partial G$. From now on, in order to treat the stable and mixed types of boundary conditions, we formally assume that $u\in L^2(\Omega,P;L^2(T;V\cap H^n(G)))$. (The precise treatment of weak solution is shown in Appendix-C.)

7.3. Stochastic Differential Rule and Green's Formula

Theorem-5.3.1 in Chap.5 may easily be extended to the boundary control class. Our principal purpose in this section is to examine contributions of boundary inputs to the interior domain of the solution process. The following theorem states the Itô-Dynkin's formula in Hilbert space.

[Theorem-7.3.1]: Define the quadratic functional,

$$(7.3.1) \quad V(t,u) \triangleq (u(t),\ P(t)u(t))_H ,$$

where $P(t)$ is a deterministic bounded self-adjoint operator and is differentiable in t.

It follows that

(7.3.2) $$E\{V(t,u)|u(s)=\bar{u}_s\} - V(s,\bar{u}_s) = E\{\int_s^t \Lambda V(\tau,u)d\tau|u(s)=\bar{u}_s\} ,$$

where Λ is the differential generator such that

(7.3.3) $$\Lambda V(t,u) = (u(t), \frac{dP(t)}{dt}u(t))_H - (A^D(t)u(t), P(t)u(t))_H - (u(t), P(t)A^D(t)u(t))_H + (\hat{A}^s(t)u(t), P(t)\hat{A}^s(t)u(t))_H .$$

Proof; By using the same approach as in Theorem-5.3.1, Eq.(7.3.2) can easily be obtained. In this section, the intuitive proof is presented.

Formally we write

(7.3.4) $$du(t) = - A^D(t)u(t)dt - dA^s(t,\omega)[u(t)] .$$

Noting that

(7.3.5) $$\begin{aligned} dV(t,u) &= (u(t+dt), P(t+dt)u(t+dt))_H - (u(t), P(t)u(t))_H \\ &= (u(t),dP(t)u(t))_H + (du(t), P(t)u(t))_H \\ &\quad + (u(t), P(t)du(t))_H + (du(t), P(t)u(t))_H + o(dt) , \end{aligned}$$

and taking into consideration that the order of $dA^s(t,\omega)$ is equal to $(dt)^{1/2}$, from Proposition 2.2.2, we have

(7.3.6) $$\begin{aligned} dV(t,u) &= (u(t), dP(t)u(t))_H - (A^D(t)u(t)dt, P(t)u(t))_H \\ &\quad - (dA^s(t,\omega)[u(t)], P(t)u(t))_H - (u(t), P(t)A^D(t)u(t)dt)_H \\ &\quad - (u(t), P(t)dA^s(t,\omega)[u(t)])_H \\ &\quad - (dA^s(t,\omega)[u(t)], P(t)dA^s(t,\omega)[u(t)])_H + o(dt) . \end{aligned}$$

From Eq.(7.3.6), neglecting higher order terms than $(dt)^{1/2}$, the following equality is formally obtained,

$$(7.3.7)\quad \lim_{dt\to 0} \frac{E\{V(t+dt,u)|u(t)=\bar{u}_t\} - E\{V(t,u)|u(t)=\bar{u}_t\}}{dt}$$

$$= \lim_{dt\to 0} \frac{E\{dV(t,u)|u(t)=\bar{u}_t\}}{dt}$$

$$= E\{(u(t), \frac{dP(t)}{dt}u(t))_H - (A^D(t)u(t), P(t)u(t))_H$$

$$- (u(t), P(t)A^D(t)u(t))_H$$

$$+ (\hat{A}^s(t)u(t), P(t)\hat{A}^s(t)u(t))_H | u(t)=\bar{u}_t\}.$$

Thus, the proof has been completed.

In Theorem-7.3.1, boundary controls in Eq.(7.2.1) are not appeared explicitly in Eq.(7.3.3). In order to study the boundary optimal controls, the following corollary is useful.

[Corollary-7.3.1]: Using the Green's formula, Eq.(7.3.3) becomes

$$(7.3.8)\quad \Lambda V(t,u) = (u(t), \frac{dP(t)}{dt} u(t))_H - (u(t), A^{D*}(t)P(t)u(t))_H$$

$$- \sum_{j=1}^{n} (S_j(t)u(t), T_j(t)P(t)u(t))_{L^2(\partial G)}$$

$$+ (u(t), \hat{A}^{s*}(t)P(t)\hat{A}^s(t)u(t))_H - (u(t), P(t)A^D(t)u(t))_H$$

$$+ \sum_{j=1}^{n/2} (G_j(t)u(t), H_j(t)\cdot P(t)u(t))_{L^2(\partial G)}$$

$$+ \sum_{j=1}^{n/2} \sum_{k=1}^{n/2} (G_j(t)u(t), \Xi_{jk}(t)\cdot P_b(t)G_k(t)u(t))_{L^2(\partial G)} ,$$

where the symbol $P_b(t)$ denotes

$$(7.3.9)\quad P_b(t) = \int_{\partial G_y} p(t,x,y)(\cdot)dy ,$$

and where $p(t,x,y)$ is the kernel of the operator $P(t)$. In Eq.(7.3.8), $S_j(t)$, $T_j(t)$ and $G_j(t)$ are partial differential operators formally defined on ∂G. The system $\{S_j(t)\}_{j=1}^n$ is a Dirichlet system of order n on ∂G and the system $\{T_j(t)\}_{j=1}^n$ becomes the adjoint system of $\{S_j(t)\}_{j=1}^n$ relative to the operator $A^D(t)$. Furthermore, $\{G_j(t)\}_{j=1}^{n/2}$ is one of subsets of a Dirichlet system of order n/2 on ∂G, and $\{H_j(t)\cdot(\ )\}_{j=1}^{n/2}$ and $\{\Xi_{jk}(t)\cdot(\ )\}_{j=1\ k=1}^{n/2\ n/2}$ are boundary systems which depend on the determination of the system $\{G_j(t)\}_{j=1}^{n/2}$.

Proof: Since a complete mathematical generality is not the object of this section, we assume the spacial domain G is R^1. The following results can easily be extended to the case of the n-dimensional Euclidean domain. (See Ref.[L 5], for more details) For any ϕ and ψ $\in H^n(G)$, by integrating by parts, the second term of the right hand side of Eq.(7.3.3) becomes

$$(7.3.10)\quad (A^D(t)\phi,\ P(t)\psi)_H = (\sum_{i\le n} a_i^D(t,x)D_x^i\phi,\ P(t)\psi)_H$$

$$= \sum_{i\le n} (\phi,\ (-1)^i D_x^i a_i^D(t,x)P(t)\psi)_H$$

$$+ \sum_{j\le n}\sum_{j=1}^{i} (D_x^{i-j}\phi,\ (-1)^{j-1}D_x^{j-1}a_i^D(t,x)P(t)\psi)_{L^2(\partial G)}$$

$$= (\phi,\ A^{D*}(t)P(t)\psi)_H$$

$$+ \sum_{j=1}^{n}\sum_{i=j}^{n} (D_x^{j-1}\phi,\ (-1)^{i-j+1}D_x^{i-j+1}a_i^D(t,x)P(t)\psi)_{L^2(\partial G)}.$$

Noting that $\{S_j(t)\}_{j=1}^n$ is a Dirichlet system, we can always express the partial differential operator D_x^j for any j by the linear combinations of $\{T_j(t)\}_{j=1}^n$. (See Ref.[L5]) Then, in the Corollary-

7.3.1, choosing the operator $S_j(t)$ such that

$$(7.3.11)\quad (S_j(t)\phi,\ T_j(t)\psi)_{L^2(\partial G)} = (D_x^{j-1}\phi,\ \sum_{i=j}^{n}(-1)^{i-j+1}D_x^{i-j+1}a_i^D(t,x) \times\psi)_{L^2(\partial G)},$$

we have

$$(7.3.12)\quad (A^D(t)\phi,\ P(t)\psi)_H = (\phi,\ A^{D*}(t)P(t)\psi)_H + \sum_{j=1}^{n}(S_j(t)\phi,\ T_j(t)P(t)\psi)_{L^2(\partial G)}.$$

By using a similar method mentioned above, we obtain

$$(7.3.13)\quad (\hat{A}^s(t)\phi,\ P(t)\hat{A}^s(t)\psi)_H = \sum_{h\le n/2}(a_h^s(t,x)D_x^h\phi,\ P(t)a_h^s(t,y)D_y^h\psi)_H$$

$$= \sum_{h\le n/2}(\phi,\ (-1)^hD_x^ha_h^s(t,x)P(t)s_h^s(t,y)D_y^h\psi)_H$$

$$+ \sum_{h\le n/2}\sum_{j=1}^{h}(D_x^{h-j}\phi,\ (-1)^{j-1}D_x^{j-1}a_h^s(t,x)P(t)a_h^s(t,y)D_y^h\psi)_{L^2(\partial G)}$$

$$= (\phi,\ \hat{A}^{s*}(t)P(t)\hat{A}^s(t)\psi)_H$$

$$+ \sum_{j=1}^{n/2}\sum_{h=j}^{n/2}(D_x^{h-j}\phi,\ (-1)^{j-1}D_x^{j-1}a_h^s(t,x)P(t)a_h^s(t,y)D_y^h\psi)_{L^2(\partial G)}.$$

Noting that the operator $P(t)$ can be expressed by

$$(7.3.14)\quad P(t) = \int_G p(t,x,y)(\cdot)dy,$$

from the second term of the right hand side of Eq.(7.3.13), by integrating by parts with respect to y, it follows that

$$(7.3.15)\quad \sum_{j=1}^{n/2}\sum_{h=1}^{n/2}(D_x^{h-j}\phi,\ (-1)^{j-1}D_x^{j-1}D_x^{j-1}a_h^s(t,x)P(t)a_h^s(t,y)D_y^h\psi)_{L^2(\partial G)}$$

$$= \sum_{j=1}^{n/2} \sum_{h-j}^{n/2} (D_x^{h-j}\phi, \int_G (-1)^{j-1} D_x^{j-1} a_h^s(t,x)(-1)^h D_y^h a_h^s(t,y)p(t,x,y) \times\psi dy)_{L^2(\partial G)}$$

$$+ \sum_{j=1}^{n/2} \sum_{h=j}^{n/2} (D_x^{h-j}\phi, \int_{\partial G} (-1)^{j-1} D_x^{j-1} D_x^{j-1} a_h^s(t,x)(\sum_{k=j}^{h} (-1)^{k-1} D_y^{k-1} \times a_h^s(t,y)p(t,x,y) D_y^{h-k}\psi dy)_{L^2(\partial G)} .$$

Defining,

$$G_j(t) \triangleq D_x^{j-1}$$

$$H_j(t)\cdot P(t) \triangleq \int_G \sum_{h=1}^{n/2} (-1)^{2h-j} D_x^{h-j} a_h^s(t,x) D_y^h a_h^s(t,y) p(t,x,y)(\cdot)dy$$

$$K_{h-j}^{k+1}(t,D_x) \triangleq (-1)^{k-j+1} D_x^{k-j+1}(a_h^s(t,x)(\cdot))$$

and

$$\Xi_{ij}(t)\cdot P_b(t) \triangleq \int_{\partial G} \sum_{h=\max(i,j)}^{\frac{n}{2}-1} K_{k-i}^{k+1}(t,D_x) K_{k-j}^{k+1}(t,D_y) \times p(t,x,y)(\cdot)dy ,$$

from Eqs.(7.3.13), (7.3.14) and (7.3.15), we have

$$(7.3.16) \quad (\hat{A}^s(t)\phi,\ P(t)\hat{A}^s(t)\psi)_H = (\phi,\ \hat{A}^{s*}(t)\hat{A}^s(t)\psi)_H + \sum_{j=1}^{n/2} (G_j(t)\phi,\ H_j(t)\cdot P(t)\psi)_{L^2(\partial G)} + \sum_{j=1}^{n/2} \sum_{k=1}^{n/2} (G_j(t)\phi,\ \Xi_{jk}(t)\cdot P_b(t) G_k(t)\psi)_{L^2(\partial G)} .$$

The proof has been completed.

If concrete forms of the boundary operators defined by Eq.(7.2.1c) are given, then we can easily find the relation between the abstractly defined operators $S_j(t)$, $T_j(t)$ and $G_j(t)$, and boundary operators $\{B_j(t)\}_{j=1}^{n/2}$ given in Eq.(7.2.1c) from Corollary-7.3.1. However, in this chapter, because of the fact that boundary operators given in Eq.(7.2.1c) are abstract fashions, the relation between the operators appearing in Eq.(7.3.8) can not precisely be discussed. Roughly speaking, we can only say that some of $G_j(t)$ $(j=1,2,\cdots,\frac{n}{2})$ are equal to boundary operators $\{B_j(t)\}_{j=1}^{n/2}$, and furthermore there are many cases such that

i) All of $\{G_j(t)\}_{j=1}^{n/2}$ are equal to boundary operators $\{B_j(t)\}_{j=1}^{n/2}$.

ii) Some of $\{G_j(t)\}_{j=1}^{n/2}$ are equal to some of $\{B_j(t)\}_{j=1}^{n/2}$.

iii) Any of $\{G_j(t)\}_{j=1}^{n/2}$ is not equal to boundary operators $\{B_j(t)\}_{j=1}^{n/2}$.

In the following corollary, the most interesting situation is considered, that is, some of $\{G_j(t)\}_{j=1}^{n/2}$ are equal to some of $\{B_j(t)\}_{j=1}^{n/2}$.

[Corollary-7.3.2]: With boundary operators $\{B_j(t)\}_{j=1}^{n/2}$ given by Eq.(7.2.1c) and their adjoint operators $\{C_j(t)\}_{j=1}^{n/2}$, operators $\{S_j(t)\}_{j=1}^{n}$ and $\{T_j(t)\}_{j=1}^{n}$ are assumed to be represented respectively by†

$$(7.3.17)\quad \{S_j(t)\}_{j=1}^{n} = \{-B_1(t),-B_2(t),\cdots,-B_{\frac{n}{2}}(t),\tilde{S}_1(t),\tilde{S}_2(t),\quad,\tilde{S}_{\frac{n}{2}}(t)\}$$

and

$$(7.3.18)\quad \{T_j(t)\}_{j=1}^{n} = \{\tilde{T}_1(t),\tilde{T}_2(t),\cdots,\tilde{T}_{\frac{n}{2}}(t),C_1(t),C_2(t),\cdots,C_{\frac{n}{2}}(t)\}.$$

Futhermore we assume the system $\{G_j(t)\}_{j=1}^{n/2}$ relative to the operator

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† The concrete forms of operators $\{\tilde{S}_j(t)\}_{j=1}^{n/2}$ and $\{\tilde{T}_j(t)\}_{j=1}^{n/2}$ will be shown in examples in Sec.7.5.

$\hat{A}^s(t)$ becomes

(7.3.19) $\{G_j(t)\}_{j=1}^{n/2} = \{\mathcal{B}_1(t),\mathcal{B}_2(t),\cdots,\mathcal{B}_\ell(t),\alpha_1(t),\alpha_2(t),\cdots,\alpha_{\frac{n}{2}-\ell}(t)\}.$

Equation (7.3.8) can be changed into

(7.3.20)
$$\begin{aligned}\Lambda V(t,u) = {} & (u(t),\ [\frac{dP(t)}{dt} - A^{D*}(t)P(t) - P(t)A^D(t) \\ & + \hat{A}^{s*}(t)P(t)\hat{A}^s(t)]u(t))_H \\ & + \sum_{j=1}^{n/2}(g_j(t),\tilde{T}_j(t)P(t)u(t))_{L^2(\partial G)} \\ & - \sum_{j=1}^{n/2}(\tilde{S}_j(t)u(t),C_j(t)P(t)u(t))_{L^2(\partial G)} \\ & + \sum_{j=1}^{\ell}(g_j(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)} \\ & + \sum_{j=\ell+1}^{n/2}(\alpha_{j-\ell}(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)} \\ & + \sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(g_j(t),\ \Xi_{jk}(t)\cdot P_b(t)g_k(t))_{L^2(\partial G)} \\ & + \sum_{j=1}^{\ell}\sum_{k=\ell+1}^{n/2}(g_j(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)} \\ & + \sum_{j=\ell+1}^{n/2}\sum_{k=1}^{\ell}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)g_k(t))_{L^2(\partial G)} \\ & + \sum_{j=\ell+1}^{n/2}\sum_{k=\ell+1}^{n/2}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}.\end{aligned}$$

Proof: From Eqs.(7.3.17) to (7.3.19), we have

(7.3.21)
$$\sum_{j=1}^{n}(S_j(t)u(t),\ T_j(t)P(t)u(t))_{L^2(\partial G)} = -\sum_{j=1}^{n/2}(\mathcal{B}_j(t)u(t),\ \tilde{T}_j(t)P(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=1}^{n/2}(\tilde{S}_j(t)u(t),\ C_j(t)P(t)u(t))_{L^2(\partial G)}$$

(7.3.22) $$\sum_{j=1}^{n/2}(G_j(t)u(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)}$$

$$=\sum_{j=1}^{\ell}(\mathcal{B}_j(t)u(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{i=\ell+1}^{n/2}(\alpha_{j-\ell}(t)u(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)}$$

and

(7.3.23) $$\sum_{j=1}^{n/2}\sum_{k=1}^{n/2}(G_j(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)G_k(t)u(t))_{L^2(\partial G)}$$

$$=\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\mathcal{B}_j(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\mathcal{B}_k(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=1}^{\ell}\sum_{k=\ell+1}^{n/2}(\mathcal{B}_j(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=\ell+1}^{n/2}\sum_{k=1}^{\ell}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\mathcal{B}_k(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=\ell+1}^{n/2}\sum_{k=\ell+1}^{n/2}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}.$$

From Eqs.(7.3.21) to (7.3.23) and Corollary-7.3.1, the differential generator defined by Eq.(7.3.3) is rewritten by

(7.3.24) $$\Lambda V(t,u) = (u(t),\ [\frac{dP(t)}{dt} - A^{D*}(t)P(t) - P(t)A^{D}(t)$$

$$+ \hat{A}^{S*}(t)P(t)\hat{A}^{S}(t)]u(t))_H$$

$$+\sum_{j=1}^{n/2}(\mathcal{B}_j(t)u(t),\ \tilde{T}_j(t)P(t)u(t))_{L^2(\partial G)}$$

$$-\sum_{j=1}^{n/2}(\tilde{S}_j(t)u(t),\ \mathcal{C}_j(t)P(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=1}^{\ell}(\mathcal{B}_j(t)u(t),\ H_j(t)\cdot P(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=\ell+1}^{n/2}(\alpha_{j-\ell}(t)u(t),\ H_j(t)\cdot P(t))_{L^2(\partial G)}$$

$$+\sum_{j=1}^{\ell}\sum_{k=1}^{\ell}(\mathcal{B}_j(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\mathcal{B}_k(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=1}^{\ell}\sum_{k=\ell+1}^{n/2}(\mathcal{B}_j(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=\ell+1}^{n/2}\sum_{k=1}^{\ell}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}$$

$$+\sum_{j=\ell+1}^{n/2}\sum_{k=\ell+1}^{n/2}(\alpha_{j-\ell}(t)u(t),\ \Xi_{jk}(t)\cdot P_b(t)\alpha_{k-\ell}(t)u(t))_{L^2(\partial G)}.$$

Then, noting that $\mathcal{B}_j(t,x;D_x)u(t,x) = g_j(t,x)$ $(j=1,2,\cdots,n/2)$, the proof has been completed.

7.4. Derivation of Optimal Boundary Control

Consider the following quadratic cost functional,

(7.4.1) $L(t,u,g) = (u(t),\ P(t)u(t))_H + (g(t),\ R(t)g(t))_{L^2(\partial G)}$

where $P(t)$ is a bounded semi-positive definite self-adjoint operator and $R(t)$ and $g(t)$ denote

$$(7.4.2)\quad R(t) = \begin{bmatrix} \tilde{R}_1(t) & 0 & \cdots & 0 \\ 0 & \tilde{R}_2(t) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \tilde{R}_{n/2}(t) \end{bmatrix}$$

and

$$(7.4.3)\quad g(t) = [g_1(t), g_2(t), \cdots, g_{\frac{n}{2}}(t)] ,$$

where $\{\tilde{R}_i(t)\}_{i=1}^{n/2}$ are bounded positive definite self-adjoint operators, and $\{g_i(t)\}_{i=1}^{n/2}$ are boundary control signals. The problem is to find the feedback optimal boundary control signal $g^o(t)$ in such a way that the functional

$$(7.4.4)\quad J(g) = E\{\int_{t_0}^{t_f} L(t,u,g)dt \mid u(t_0)=u_0\}$$

becomes minimal with respect to $g \in W_{ad}^b$.

In this case, from the results of Chaps.3, 4 and 5, we may find the minimal cost functional $V(t,\bar{u}_t)$ is expressed by

$$(7.4.5)\quad V(t,\bar{u}_t) = (\bar{u}_t, \Pi(t)\bar{u}_t)_H .$$

From Eq.(7.4.5), by applying the principle of optimality and by Corollary-7.3.2, the following basic equation of optimal boundary control is derived,

$$(7.4.6)\quad \min_{g \in W_{ad}^b} [(\bar{u}_t, P(t)\bar{u}_t)_H + (g(t), R(t)g(t))_{L^2(\partial G)}$$

$$+ (\bar{u}_t, [\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - \Pi(t)A^D(t) + \hat{A}^{S*}(t)\Pi(t)\hat{A}^S(t)\bar{u}_t)_H$$

$$+\sum_{j=1}^{n/2} (g_j(t), C_j(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)} - \sum_{j=1}^{n/2} (\tilde{T}_j(t)\bar{u}_t, \tilde{S}_j(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ \sum_{j=1}^{\ell} (g_j(t), H_j(t)\cdot\Pi(t)u_t)_{L^2(\partial G)}$$

$$+ \sum_{j=\ell+1}^{n/2} (\alpha_{j-\ell}(t)\bar{u}_t, H_j(t)\cdot\Pi(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} (g_j(t), \Xi_{jk}(t)\cdot\Pi_b(t)g_k(t))_{L^2(\partial G)}$$

$$+ \sum_{j=1}^{\ell} \sum_{k=\ell+1}^{n/2} (g_j(t), \Xi_{jk}(t)\cdot\Pi_b(t)\alpha_{k-\ell}(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ \sum_{j=\ell+1}^{n/2} \sum_{k=1}^{\ell} (\alpha_{j-\ell}(t)\bar{u}_t, \Xi_{jk}(t)\cdot\Pi_b(t)g_k(t))_{L^2(\partial G)}$$

$$+ \sum_{j=\ell+1}^{n/2} \sum_{k=\ell+1}^{n/2} (\alpha_{j-\ell}(t)\bar{u}_t, \Xi_{jk}(t)\cdot\Pi_b(t)\alpha_{k-\ell}(t)\bar{u}_t)_{L^2(\partial G)}] = 0$$

or in a vector form

$$\min_{g\in W^b_{ad}} [(\bar{u}_t, P(t)\bar{u}_t)_H + (g(t), R(t)g(t))_{L^2(\partial G)} \tag{7.4.7}$$

$$+ (\bar{u}_t, [\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^*$$

$$+ \hat{A}^{S*}(t)(\hat{A}^{S*}(t)\Pi(t))^*]\bar{u}_t)_H$$

$$+ 2(g(t), \tilde{T}(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)} - 2(\tilde{S}(t)\bar{u}_t, \mathcal{C}(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ 2(Kg(t), H(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)} + 2(QG(t)\bar{u}_t, H(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ (Kg(t), \Xi(t)\Pi_b(t)Kg(t))_{L^2(\partial G)}$$

$$+ (Kg(t), \Xi(t)\Pi_b(t)QG(t)\bar{u}_t)_{L^2(\partial G)}$$

$$+ (QG(t)\bar{u}_t, \Xi(t)\Pi_b(t)Kg(t))_{L^2(\partial G)}$$

$$+ \left(QG(t)\bar{u}_t,\ \Xi(t)\Pi_b(t)QG(t)\bar{u}_t\right)_{L^2(\partial G)}] = 0 \ ,$$

where $C(t)$, $S(t)$, $T(t)$, $H(t)$, $G(t)$ and $\Xi(t)$ are the following $\frac{n}{2}$-dimensional vectors and $\frac{n}{2}\times\frac{n}{2}$ matrix operators, whose components are boundary partial differential operators, respectively,

$$(7.4.8)\quad C(t) = [\mathcal{C}_1(t),\ \mathcal{C}_2(t),\cdots,\ \mathcal{C}_{\frac{n}{2}}(t)]'$$

$$(7.4.9)\quad T(t) = [\tilde{T}_1(t),\ \tilde{T}_2(t),\cdots,\ \tilde{T}_{\frac{n}{2}}(t)]'$$

$$(7.4.10)\quad S(t) = [\tilde{S}_1(t),\ \tilde{S}_2(t),\quad ,\ \tilde{S}_{\frac{n}{2}}(t)]'$$

$$(7.4.11)\quad H(t) = [H_1(t)\cdot(\ ),\ H_2(t)\cdot(\ ),\cdots,\ H_{\frac{n}{2}}(t)\cdot(\ )]'$$

$$(7.4.12)\quad \Xi(t) = \begin{bmatrix} \Xi_{11}(t)\cdot(\ ) & \Xi_{12}(t)\cdot(\ ) & \cdots & \Xi_{1\frac{n}{2}}(t)\cdot(\ ) \\ \Xi_{21}(t)\cdot(\ ) & \Xi_{22}(t)\cdot(\ ) & & \vdots \\ \vdots & & \ddots & \vdots \\ \Xi_{\frac{n}{2}1}(t)\cdot(\ ) & \cdots & \cdots & \Xi_{\frac{n}{2}\frac{n}{2}}(t)\cdot(\ ) \end{bmatrix}$$

and

$$(7.4.13)\quad G(t) = [\mathcal{B}_1(t),\ \mathcal{B}_2(t),\cdots,\mathcal{B}_\ell(t),\alpha_1(t),\alpha_2(t),\cdots,\alpha_{\frac{n}{2}-\ell}(t)]'$$

and K and Q are respectively $\frac{n}{2}\times\frac{n}{2}$ square matrices,

$$(7.4.14a)\qquad K = \begin{array}{cc} & \overbrace{\hphantom{1\ 0\ \cdots\ 0}}^{\ell} \\ \ell\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. & \left[\begin{array}{cccc|ccc} 1 & 0 & & 0 & & & \\ 0 & 1 & & \cdot & & 0 & \\ \cdot & & \cdot & \cdot & & & \\ \cdot & & \cdot & 0 & & & \\ 0 & \cdots & 0 & 1 & & & \\ \hline & & & & 0 & & \\ & 0 & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & 0 \end{array}\right] \end{array}$$

and

$$(7.4.14b)\quad Q = \left[\begin{array}{cccccc} 0 & & & & & \\ & 0 & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & \\ & 0 & & & & \underbrace{\left.\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \cdot \\ \cdot & & \cdot & 0 \\ 0 & \cdots & 0 & 1 \end{matrix}\right\}\frac{n}{2}-\ell}_{\frac{n}{2}-\ell} \end{array}\right] .$$

From Eq.(7.4.7), it follows that

$$\begin{aligned}(7.4.15)\quad \min_{g\in W^b_{ad}} \; & [(\{R(t) + K^*\Xi(t)\Pi_b(t)K\}g(t) + \{T(t)\Pi(t) + K^*H(t)\Pi(t) \\ & + K^*\Xi(t)\Pi_b(t)QG(t)\}\bar{u}_t,\ \{R(t) + K^*\Xi(t)\Pi_b(t)K\}^{-1}[\{R(t) \\ & + K^*\Xi(t)\Pi_b(t)K\}g(t) + \{T(t)\Pi(t) + K^*H(t)\Pi(t) \\ & + K^*\Xi(t)\Pi_b(t)QG(t)\}\bar{u}_t])_{L^2(\partial G)} \\ & - (\{T(t)\Pi(t) + K^*H(t)\Pi(t) + K^*\Xi(t)\Pi_b(t)QG(t)\}\bar{u}_t, \\ & \{R(t) + K^*\Xi(t)\Pi_b(t)K\}^{-1} \\ & \times \{T(t)\Pi(t) + K^*H(t)\Pi(t) + K^*\Xi(t)\Pi_b(t)QG(t)\}\bar{u}_t)_{L^2(\partial G)} \\ & - 2(S(t)\bar{u}_t,\ C(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)} \\ & + 2(QG(t)\bar{u}_t,\ H(t)\Pi(t)\bar{u}_t)_{L^2(\partial G)} \\ & + (QG(t)\bar{u}_t,\ \Xi(t)\Pi_b(t)QG(t)\bar{u}_t)_{L^2(\partial G)} \\ & + (\bar{u}_t,\ \{\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^* \\ & \qquad + \hat{A}^{S*}(t)(\hat{A}^{S*}(t)\Pi(t))^* + P(t)\}\bar{u}_t)_H] = 0\end{aligned}$$

Then, noting that the first term of the left hand side of Eq.(7.4.15) is a quadratic form, the optimal boundary control $g^o(t)$ is given by

$$g^o(t) = -\{R(t) + K^*\Xi(t)\Pi_b(t)K\}^{-1}\{T(t)\Pi(t) + K^*H(t)\Pi(t) + K^*\Xi(t)\Pi_b(t)QG(t)\}\bar{u}_t \tag{7.4.16}$$

where, for $\forall\phi\in H^n(G)$ and $\forall\psi\in H^n(G)$, $\Pi(t)$ satisfies

$$\begin{aligned}(\phi, \{\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^* \\ + \hat{A}^{s*}(t)(\hat{A}^{s*}(t)\Pi(t))^* + P(t)\}\psi)_H \\ - (\{T(t)\Pi(t) + K^*H(t)\Pi(t) + K^*\Xi(t)\Pi_b(t)QG(t)\}\phi, \\ \{R(t) + K^*\Xi(t)\Pi_b(t)K\}^{-1} \\ \times \{T(t)\Pi(t) + K^*H(t)\Pi(t) + K^*\Xi(t)\Pi_b(t)QG(t)\}\psi)_{L^2(\partial G)} \\ - 2(S(t)\phi, C(t)\Pi(t)\psi)_{L^2(\partial G)} + 2(QG(t)\phi, H(t)\Pi(t)\psi)_{L^2(\partial G)} \\ + (QG(t)\phi, \Xi(t)\Pi_b(t)QG(t)\psi)_{L^2(\partial G)} = 0 ,\end{aligned} \tag{7.4.17}$$

with the terminal condition

$$(\phi, \Pi(t_f)\psi) = 0. \tag{7.4.18}$$

7.5. Illustrative Examples

In this section, the explicit form of the optimal boundary control is shown for the one dimensional stochastic heat equation with the non-homogeneous Dirichlet and Neumann conditions.

[Example-7.5.1]:(Non-homogeneous Dirichlet condition)

Consider the same equation as Example-5.5.1 in Sec.5.5 of Chap. 5, that is, (7.5.1a) $$\frac{\partial u(t,x)}{\partial t} + a_2(t)\frac{\partial^2 u(t,x)}{\partial x^2} + a_1(t)\eta(t,\omega)\frac{\partial u(t,x)}{\partial x} = 0$$

for $(t,x)\in T\times G(]t_0,t_f[\times]0,1[)$

with the initial condition

(7.5.1b) $u(t_0,x) = u_0(x)$ for $x\in G$

and the boundary condition

(7.5.1c) $u(t,x) = g(t,x)$ for $(t,x)\in T\times\partial G$

where $g(t,x)$ is a boundary control input and $\eta(t,\omega)$ is a white Gaussian noise with zero mean and unit variance, which denotes the random velocity of the moving rod in the x-direction. From Coersivity condition-2.2.1, the deterministic coefficients $a_2(t)$ and $a_1(t)$ are assumed to satisfy

(7.5.2) $2a_2(t) + a_1^2(t) < 0$ for $\forall t\in\bar{T}$.

In this example, we can easily find

(7.5.3) $A^D(t) = a_2(t)D_x^2$

(7.5.4) $\hat{A}^s(t) = a_1(t)D_x^1$

and, for $x\in\partial G$, the boundary operator $\mathcal{B}(t)$ can be rewritten by

(7.5.5) $\mathcal{B}(t) = D_x^0$.

Using the Green's formula, for any $\phi\in H^2(G)$, we have

(7.5.6) $$(A^D(t)\phi,\ \Pi(t)\phi)_H = (\phi,\ A^{D*}(t)\Pi(t)\phi)_H + (D_x^1\phi,\ a_2(t)D_x^0\ (t)\phi)_{L^2(\partial G)} - (D_x^0\phi,\ a_2(t)D_x^1\Pi(t)\phi)_{L^2(\partial G)} .$$

Noting that the boundary operator of the system considered here is $\mathcal{B}(t)=D_x^0$, it is sufficient to choose another boundary operator $S(t)$ defined by Eq.(7.4.9) as follows:

$$(7.5.7)\quad \tilde{S}(t) = D_x^1 .$$

From Eqs.(7.5.5) and (7.5.7), we can easily choose boundary operators $C(t)$ and $\tilde{T}(t)$ defined by Eqs.(7.4.8) and (7.4.10) such that

$$(7.5.8)\quad C(t) = a_2(t)D_x^0$$

and

$$(7.5.9)\quad \tilde{T}(t) = a_2(t)D_x^1 .$$

Then, Eq.(7.5.6) can be rewritten by

$$(7.5.10)\quad (A^D(t)\phi,\ \Pi(t)\phi)_H = (\phi,\ A^{D*}(t)\Pi(t)\phi)_H + (\tilde{S}(t)\phi,\ C(t)\Pi(t)\phi)_{L^2(\partial G)} - (\mathcal{B}(t)\phi,\ \tilde{T}(t)\Pi(t)\phi)_{L^2(\partial G)} .$$

Furthermore, for any $\phi\in H^2(G)$, we have the following relation†

$$(7.5.11)\quad (\hat{A}^s(t)\phi,\ \Pi(t)\hat{A}^s(t)\phi)_H = (\phi,\ \hat{A}^{s*}(t)\Pi(t)\hat{A}^s(t)\phi)_H$$
$$+ (D_x^0\phi,\ a_1(t)D_x^0 a_1(t)D_y^1\Pi(t)\phi)_{L^2(\partial G)}$$
$$+ (D_x^0\phi,\ a_1(t)D_x^0 a_1(t)D_y^0\Pi_b(t)D_y^0\phi)_{L^2(\partial G)}$$

In this example, for $x\in\partial G$, boundary operators defined by Eqs.(7.4.12) and (7.4.13) become

† In this section we use the symbol

$$D_x^n D_y^m \Pi(t)\phi = \int_G D_x^n D_y^m \pi(t,x,y)\phi dy .$$

(7.5.12) $G(t) = \mathcal{B}(t) = D_x^0$

(7.5.13) $\alpha(t) = 0$

(7.5.14a) $H(t)\cdot\Pi(t) = a_1^2(t)\int_G D_y^1\pi(t,x,y)(\cdot)dy$

and

(7.5.14b) $\Xi(t)\cdot\Pi(t) = a_1^2(t)\int_{\partial G}\pi(t,x,y)(\cdot)dy$.

Then, Eq.(7.5.11) can be rewritten in the form,

(7.5.15) $$(\hat{A}^s(t)\phi,\ \Pi(t)\hat{A}^s(t)\phi)_H = (\phi,\ \hat{A}^{s*}(t)\Pi(t)\hat{A}^s(t)\phi)_H$$
$$+ (\mathcal{B}(t)\phi,\ H(t)\cdot\Pi(t)\phi)_{L^2(\partial G)}$$
$$+ (\mathcal{B}(t)\phi,\ \Xi(t)\cdot\Pi_b(t)\mathcal{B}(t)\phi)_{L^2(\partial G)} .$$

From Eqs.(7.5.11) and (7.5.15), the matrices K and Q defined by Eq.(7.4.14) are K=1 and Q=0, respectively.

Considering the relation between boundary operators in Corollary-7.3.2 and those of the system considered in this example, the optimal boundary control given by Eq.(7.4.16) is obtained to be

(7.5.16) $$g^o(t) = -\{R(t) + \Xi(t)\cdot\Pi_b(t)\}^{-1}\{\tilde{T}(t)\Pi(t) + H(t)\cdot\Pi(t)\}\bar{u}_t$$

and furthermore, from Eq.(7.4.17), for $\forall\phi,\psi\in H^2(G)$, $\Pi(t)$ satisfies

(7.5.17a) $$(\phi,\ \{\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^* + \hat{A}^{s*}(t)(\hat{A}^{s*}(t)\Pi(t))^*$$
$$\times P(t)\}\psi)_H - (\{\tilde{T}(t)\Pi(t) + H(t)\cdot\Pi(t)\}\phi,$$
$$\{R(t) + \Xi(t)\cdot\Pi_b(t)\}^{-1}\{\tilde{T}(t)\Pi(t) + H(t)\cdot\Pi(t)\}\psi)_{L^2(\partial G)}$$
$$- 2(\tilde{S}(t)\ ,\ C(t)\Pi(t)\psi)_{L^2(\partial G)} = 0$$

with the terminal condition,

(7.5.17b) $(\phi,\ \Pi(t_f)\psi)_H = 0$.

By using the kernel theorem, the optimal control $g^o(t)$ given by Eq.(7.5.16) can be expressed by the original notation,

$$(7.5.18)\quad g^o(t,x) = -\left[r^{-1}(t,x,y)\int_0^1 a_2(t)\frac{\partial\pi(t,y,z)}{\partial y}\bar{u}_t(z)dz\right]_{y=0}^{y=1}$$

where $r^{-1}(t,x,y)$ is the kernel of the inverse operator $R^{-1}(t)$, and the kernel equation of the operator $\Pi(t)$ defined by Eq.(7.5.17) becomes

$$(7.5.19a)\quad \frac{\partial\pi(t,x,y)}{\partial t} - a_2(t)\left\{\frac{\partial^2\pi(t,x,y)}{\partial x^2} + \frac{\partial^2\pi(t,x,y)}{\partial y^2}\right\}$$

$$+ a_1^2(t)\frac{\partial^2\pi(t,x,y)}{\partial x\partial y} + p(t,x,y) - \left[a_2(t)\frac{\partial\pi(t,x,z_1)}{\partial z_1}\right.$$

$$\left.\times r^{-1}(t,z_1,z_2)a_2(t)\frac{\partial\pi(t,z_2,y)}{\partial z_2}\right]_{z_2=0}^{z_2=1}\Big]_{z_1=0}^{z_1=1} = 0$$

with the terminal and boundary conditions,

(7.5.19b) $\pi(t_f,x,y) = 0,\quad$ for $(x,y)\in G_x\times G_y$

and

(7.5.19c) $\pi(t,x,y) = 0,\quad$ for $(t,x,y)\in T\times\partial G_x\times G_y$

(7.5.19d) $\pi(t,x,y) = 0,\quad$ for $(t,x,y)\in T\times G_x\times\partial G_y$,

where $p(t,x,y)$ denotes the kernel of the operator $P(t)$.

In the Dirichlet boundary condition, the terms $\Xi(t)\cdot\Pi_b(t)$ and

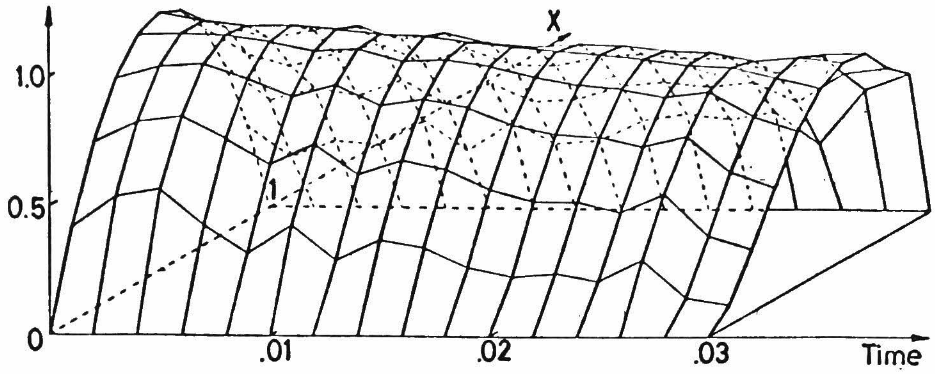

Fig. 7.5.1 A sample run of the system state u(t,x) with homogenious boundary in Example-7.5.1

$H(t)\cdot\Pi(t)$ become zero from the boundary condition of the kernel $\pi(t,x,y)$ of $\Pi(t)$, i.e., Eqs.(7.5.19c) and (7.5.19d).

In simulation studies, Eq.(7.5.1) was simulated on a digital computer and the control gain $\pi(t,x,y)$ was determined in advance by solving Eq.(7.5.19) with the help of the implicit formura of difference method. Then, the optimal boundary control $g^o(t)$ was given by Eq.(7.5.18) with the control gain $\pi(t,x,y)$. The results presented below are representatives of simulation experiments. In all experiments, the value of a_1 and a_2 were respectively set as $a_1=-0.5$, $a_2=0.9$. The initial condition (7.5.1b) was given by $u_0(x)=13.56\times(\sqrt{3-(x-0.5)^2}-\sqrt{2.75})$ and P(t) and R(t) were respectively given by

$$(7.5.20)\quad P(t) = \int_0^1 p(t,x,y)(\cdot)dy = \int_0^1 35\exp(-2((x-0.5)^2+(y-0.5)^2))(\cdot)dy$$

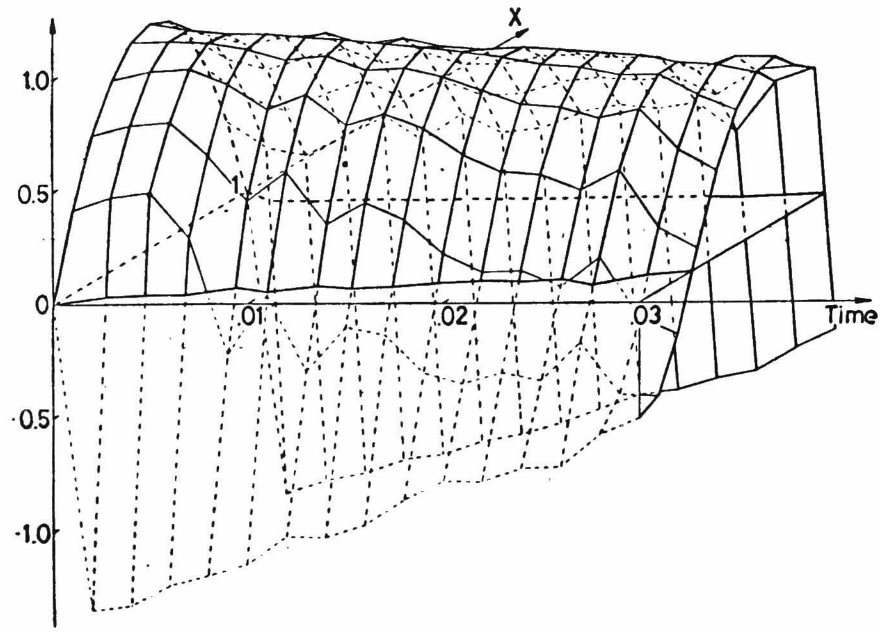

Fig. 7.5.2 A sample run of the system state u(t,x) with optimal boundary control in Example-7.5.1

(7.5.21) $R(0,0)=1,\ R(0,1)=R(1,0)=0,\ R(1,1)=1$.

Throughout the experiments, the partitioned time interval and spacial variable were set as $\Delta t=0.0001$ and $\Delta x=0.1$, respectively.

Figure-7.5.1 shows a representative of sample runs of the system with homogenious boundary ($g(t,x)=0$). A sample run of the system deriven by the optimal boundary control signal $g^{o}(t,x)$ is shown in Fig. -7.5.2. In order to show the difference between the controlled process and the uncontrolled process, the system state $u(t,x)$ at the fixed spatial points is shown in Fig.-7.5.3.

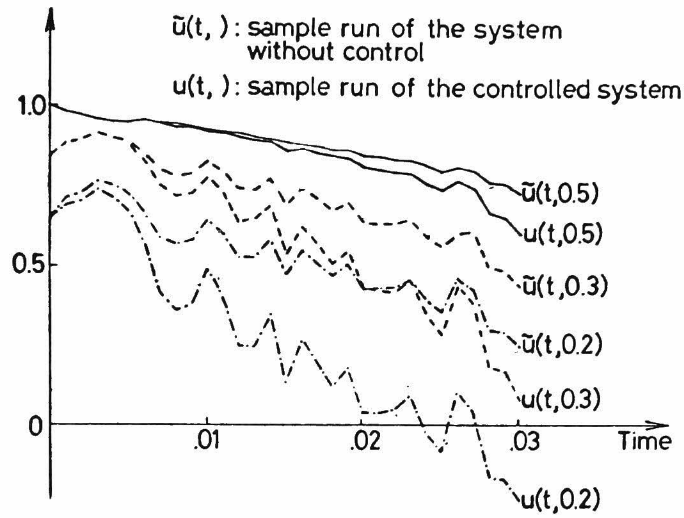

Fig. 7.5.3 Sample runs of the system state u(t,x) at the fixed spacial points in Example-7.5.1

[Example-7.5.2]:(Non-homogeneous Neumann condition)

Consider the Neumann problem in stead of Dirichlet condition (7.5.1c) in the same system as in Example-7.5.1. (In this example we choose $V=H^1(G)$.)

(7.5.22) $\dfrac{\partial u(t,x)}{\partial x} = g(t,x)$, For $(t,x)\epsilon T\times\partial G$.

Noting that the boundary operator of the system considered here is $B(t)=D_x^1$, it is sufficient to choose $\tilde{S}(t)=D_x^0$. By using the same procedure as in Example-7.5.1, a concrete form of boundary operators $C(t)$, $\tilde{T}(t)$, $G(t)$, $H(t)$ and $\Xi(t)$ can be shown by

(7.5.23) $\tilde{T}(t) = -a_2(t)D_x^0$,

$$(7.5.24)\quad \mathcal{C}(t) = -a_2(t)D_x^1 \ ,$$

$$(7.5.25)\quad G(t) = \alpha(t) = D_x^0 \ ,$$

$$(7.5.26a)\quad H(t)\cdot\Pi(t) = a_1^2(t)\int_G D_y^1\pi(t,x,y)(\cdot)dy$$

and

$$(7.5.26b)\quad \Xi(t)\cdot\Pi_b(t) = a_1^2(t)\int_{\partial G}\pi(t,x,y)(\cdot)dy \ .$$

Then, from Eqs.(7.5.6) and (7.5.11) for $^\forall\phi\epsilon H^2(G)$, we have

$$(7.5.27)\quad (A^D(t)\phi,\ \Pi(t)\phi)_H = (\phi,\ A^{D*}(t)\Pi(t)\phi)_H - (\mathcal{B}(t)\phi,\ \tilde{T}(t)\Pi(t)\phi)_{L^2(\partial G)} + (\tilde{S}(t)\phi,\ \mathcal{C}(t)\Pi(t)\phi)_{L^2(\partial G)}$$

and

$$(7.5.28)\quad (\hat{A}^S(t)\phi,\ \Pi(t)\hat{A}^S(t)\phi)_H = (\phi,\ \hat{A}^{S*}(t)\Pi(t)\hat{A}^S(t)\phi)_H + (\alpha(t)\phi,\ H(t)\cdot\Pi(t)\phi)_{L^2(\partial G)} + (\alpha(t)\phi,\ \Xi(t)\cdot\Pi_b(t)\alpha(t)\phi)_{L^2(\partial G)} \ .$$

Then, Eqs.(7.4.14) and (7.5.28) yield that K=0 and Q=1, and, from Eq. (7.4.16), we have

$$(7.5.29)\quad g^o(t) = -\ R^{-1}(t)\mathcal{C}(t)\Pi(t)\bar{u}_t \ ,$$

where we may find from Eq.(7.4.17) that, for $^\forall\phi,\psi\epsilon H^2(G)$, $\Pi(t)$ satisfies

$$(7.5.30)\quad (\phi,\ \{\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^* + \hat{A}^{S*}(t)(\hat{A}^{S*}(t)\Pi(t))^* + P(t)\}\psi)_H - (\tilde{T}(t)\Pi(t)\phi,\ R^{-1}(t)\tilde{T}(t)\Pi(t)\psi)_{L^2(\partial G)}$$

$$- 2(S(t)\phi,\ C(t)\Pi(t)\psi)_{L^2(\partial G)} + 2(\alpha(t)\phi,\ H(t)\cdot\Pi(t)\psi)_{L^2(\partial G)}$$

$$+ (\alpha(t)\phi,\ \Xi(t)\cdot\Pi_b(t)\alpha(t)\psi)_{L^2(\partial G)} = 0$$

with the terminal condition,

(7.5.31) $(\phi,\ \Pi(t_f)\psi)_H = 0.$

With the aid of the kernel theorem, the original form of optimal control $g^o(t)$ can be expressed by

(7.5.32) $$g^o(t,x) = [r^{-1}(t,x,y)\int_0^1 a_2(t)\pi(t,y,z)\bar{u}_t(z)dz]_{y=0}^{y=1} ,$$

where

(7.5.33) $$\frac{\partial\pi(t,x,y)}{\partial t} - a_2(t)\{\frac{\partial^2\pi(t,x,y)}{\partial x^2} + \frac{\partial^2\pi(t,x,y)}{\partial y^2}\} + a_1^2(t)\frac{\partial^2\pi(t,x,y)}{\partial x\partial y}$$

$$+ p(t,x,y) - [a_2(t)\pi(t,x,z_1)[r^{-1}(t,z_1,z_2)$$

$$\times a_2(t)\pi(t,z_2,y)]_{z_2=0}^{z_2=1}]_{z_1=0}^{z_1=0} = 0 ,$$

with the terminal condition,

(7.5.34) $\pi(t_f,x,y) = 0$, for $(x,y)\in G_x\times G_y$

and the boundary conditions,

(7.5.35a) $a_2(t)\frac{\partial\pi(t,x,y)}{\partial x} + a_1^2(t)\frac{\partial\pi(t,x,y)}{\partial y} = 0$, for $(t,x,y)\in T\times\partial G_y\times G_x$

(7.5.35b) $a_2(t)\frac{\partial\pi(t,x,y)}{\partial x} + a_1^2(t)\frac{\partial\pi(t,x,y)}{\partial y} = 0$, for $(t,x,y)\in T\times G_x\times\partial G_y$

and

(7.5.36) $\pi(t,x,y) = 0$, for $(t,x,y)\in T\times\partial G_x\times\partial G_y$.

In the Neumann boundary condition, it should be noted that the randomness of the coefficients causes the boundary conditions (7.5.35) and (7.5.36).

In the following example, we obtain the interesting result where the control signal directly depends on random coefficients of the partial differential operator.

[Example-7.5.3]: We shall consider a somewhat artificial but important class of the 4-th order system of parabolic type described by

$$(7.5.37)\quad \frac{\partial u(t,x)}{\partial t} + a_4(t)\frac{\partial^4 u(t,x)}{\partial x^4} + (a_2(t) + c(t)\eta(t,\omega))\frac{\partial^2 u(t,x)}{\partial x^2} = 0 ,$$

$$\text{for } (t,x)\in T\times G\ (]t_0,t_f[\times]0,1[)$$

with the initial condition,

$$(7.5.38)\quad u(t_0,x) = u_0(x) ,\quad \text{for } x\in G ,$$

and the boundary conditions,

$$(7.5.39)\quad b_{1,1}\frac{\partial u(t,x)}{\partial x} = g(t,x) ,$$

and

$$(7.5.40)\quad b_{3,1}\frac{\partial u(t,x)}{\partial x} + b_{3,3}\frac{\partial^3 u(t,x)}{\partial x^3} = 0 \text{ for } (t,x)\in T\times\partial G ,$$

where $b_{1,1}\cdot b_{3,3}\neq 0$, and $g(t,x)$ is a boundary input, and $\eta(t,\omega)$ is a white Guassian process with unit variance and zero mean, and $a_4(t)$, $a_2(t)$ and $c(t)$ are scalar functions, respectively. Furthermore, from Coercivity condition-2.2.1 in Sec. 2.2 of Chap. 2, it is assumed that

$$(7.5.41)\quad 2a_4(t) - c^2(t) > 0,\quad \text{for } \forall t\in\overline{T}.$$

From Eq.(7.5.37), we find that

$$(7.5.42)\quad A^D(t) = a_4(t)D_x^4 + a_2(t)D_x^2 \quad ,$$

$$(7.5.32)\quad \hat{A}^s(t) = c(t)D_x^2 \ ,$$

and, for $x \in \partial G$,

$$(7.5.44)\quad \mathcal{B}_1(t) = b_{1,1}D_x^1 \quad ,$$

and

$$(7.5.45)\quad \mathcal{B}_2(t) = b_{3,1}D_x^1 + b_{3,3}D_x^3 \quad ,$$

from which we have

$$(7.5.46)\quad D_x^1 = \gamma_{1,1}\mathcal{B}_1(t) \ ,$$

and

$$(7.5.47)\quad D_x^3 = \gamma_{3,3}\mathcal{B}_2(t) + \gamma_{3,1}\mathcal{B}_1(t) \ , \quad \text{for } (t,x)\in T\times\partial G \quad ,$$

where

$$(7.5.48)\quad \gamma_{1,1} = \frac{1}{b_{1,1}} \ , \ \gamma_{3,3} = \frac{1}{b_{3,3}} \ \text{ and } \ \gamma_{3,1} = \frac{-b_{3,1}}{b_{3,3}\cdot b_{1,1}} \ .$$

Using the Green's formula and Eq.(7.5.48), for $^\forall\phi\in H^4(G)$, we have

$$(7.5.49)\quad (A^D(t)\phi,\ \Pi(t)\phi)_H = (\phi,\ A^{D*}(t)\Pi(t)\phi)_H$$
$$+ (D_x^3\phi,\ a_4(t)D_x^0\Pi(t)\phi)_{L^2(\partial G)}$$
$$- (\mathcal{B}_2(t)\phi,\ \gamma_{3,3}^* a_4(t)D_x^1\Pi(t)\phi)_{L^2(\partial G)}$$
$$- (\mathcal{B}_1(t)\ ,\ ((\gamma_{3,1}a_4(t) + \gamma_{1,1}a_2(t))D_x^1 + \gamma_{1,1}a_4(t)D_x^3)$$

$$\times\Pi(t)\phi)_{L^2(\partial G)} + (D_x^1\phi,(a_2(t)D_x^0 + a_4(t)D_x^2)\Pi(t)\phi)_{L^2(\partial G)}.$$

From Corollary-7.4.2 and Eq.(7.5.49), we can define another set of boundary operators,

(7.5.50a) $\tilde{T}_1(t) =-(\gamma_{3,1}^* a_4(t) + \gamma_{1,1}^* a_2(t))D_x^0 - \gamma_{1,1}^* a_4(t)D_x^2$,

(7.5.50b) $\tilde{T}_2(t) =- a_4(t)D_x^1$,

and from Eqs.(7.5.46) and (7.5.47), we set

(7.5.51a) $\tilde{S}_1 = D_x^0$

(7.5.51b) $\tilde{S}_2 = D_x^2$

(7.5.52a) $\mathcal{C}_1(t) =-a_2(t)D_x^1 - a_4(t)D_x^3$,

and

(7.5.52b) $\mathcal{C}_2(t) =-a_4(t)D_x^1$.

Consequently, from Eqs.(7.5.49) to (7.5.52), for $\forall\phi\epsilon H^4(G)$, we have

(7.5.53) $$(A^D(t)\phi, \Pi(t)\phi)_H = (\phi, A^{D*}(t)\Pi(t)\phi)_H + (\tilde{S}_2\phi, \mathcal{C}_2(t)\Pi(t)\phi)_{L^2(\partial G)} + (\tilde{S}_1\phi, \mathcal{C}_1(t)\Pi(t)\phi)_{L^2(\partial G)}$$
$$- (\mathcal{B}_2(t)\phi, \tilde{T}_2(t)\Pi(t)\phi)_{L^2(\partial G)} - (\mathcal{B}_1(t)\phi, \tilde{T}_1(t)\Pi(t)\phi)_{L^2(\partial G)}.$$

The term $(\hat{A}^S(t)\phi,\Pi(t)\hat{A}^S(t)\phi)_H$ can also be represented by the Green's formula in the form

(7.5.54) $$(\hat{A}^S(t)\phi, \Pi(t)\hat{A}^S(t)\phi)_H = (\phi, \hat{A}^{S*}(t)\Pi(t)\hat{A}^S(t)\phi)_H + (D_x^1\phi, c(t)D_x^0 c(t)D_y^2\Pi(t)\phi)_{L^2(\partial G)}$$

$$+ (D_x^1\phi,\ c(t)D_x^0 c(t)D_y^0 \Pi_b(t)D_y^1\phi)_{L^2(\partial G)}$$

$$- (D_x^1\phi,\ c(t)D_x^0 c(t)D_y^1 \Pi_b(t)D_y^0\phi)_{L^2(\partial G)}$$

$$- (D_x^0\phi,\ c(t)D_x^1 c(t)D_y^2 \Pi(t)\phi)_{L^2(\partial G)}$$

$$- (D_x^0\phi,\ c(t)D_x^1 c(t)D_y^0 \Pi_b(t)D_y^1\phi)_{L^2(\partial G)}$$

$$+ (D_x^0\phi,\ c(t)D_x^1 c(t)D_y^1 \Pi_b(t)D_y^0\phi)_{L^2(\partial G)} .$$

In this example, the concrete forms of boundary operators defined by Eq.(7.3.19), become

(7.5.55a) $G_1(t) = \mathcal{B}_1(t) = b_{1,1}D_x^1$,

(7.5.55b) $G_2(t) = \alpha_1(t) = D_x^0$,

(7.5.56a) $H_1(t)\cdot\Pi(t) = \gamma_{1,1}^* c^2(t)\int_G D_y^2\pi(t,x,y)(\cdot)dy$

(7.5.56b) $H_2(t)\cdot\Pi(t) = -c^2(t)\int_G D_x^1 D_y^2\pi(t,x,y)(\cdot)dy$

and

(7.5.56c) $\Xi_{\ell k}(t)\cdot\Pi_b(t) = (-1)^{\ell+k}c^2(t)\gamma_{1,1}^{-4+\ell+k}(t)\int_{\partial G} D_x^{\ell-1}D_y^{k-1}\pi(t,x,y)(\cdot)dy.$

From Eqs.(7.5.55) and (7.5.56), Eq.(7.5.54) can be rewritten by

(7.5.57) $(\hat{A}^s(t)\phi,\ \Pi(t)\hat{A}^s(t)\phi)_H = (\phi,\ \hat{A}^{s*}(t)\Pi(t)\hat{A}^s(t)\phi)_H$

$$+ (\mathcal{B}_1(t)\phi,\ H_1(t)\cdot\Pi(t)\phi)_{L^2(\partial G)}$$

$$+ (\alpha_1(t)\phi,\ H_2(t)\cdot\Pi(t)\phi)_{L^2(\partial G)}$$

$$+ (\mathcal{B}_1(t)\phi,\ \Xi_{11}(t)\cdot\Pi_b(t)\mathcal{B}_1(t)\phi)_{L^2(\partial G)}$$

$$+ (\mathcal{B}_1(t)\phi,\ \Xi_{12}(t)\cdot\Pi_b(t)\alpha_1(t)\phi)_{L^2(\partial G)}$$

$$+ (\alpha_1(t)\phi,\ \Xi_{21}(t)\cdot\Pi_b(t)\mathcal{B}_1(t)\phi)_{L^2(\partial G)}$$

$$+ (\alpha_1(t)\phi,\ \Xi_{22}(t)\cdot\Pi_b(t)\alpha_1(t)\phi)_{L^2(\partial G)}\ .$$

Then, from Eqs.(7.5.54) to (7.5.57), it is easy to show that

$$(7.5.58)\quad K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\ .$$

By using the boundary operators (7.5.5) and (7.5.6), the optimal boundary control defined by Eqs.(7.4.16) and (7.4.17) becomes

$$(7.5.59)\quad g^o(t) = -\ \{R(t) + \Xi_{11}(t)\cdot\Pi_b(t)\}^{-1}\{\tilde{T}_1(t)\Pi(t) + H_1(t)\cdot\Pi(t) + \Xi_{12}(t)\cdot\Pi_b(t)\alpha_1(t)\}\bar{u}_t\ ,$$

where, for $\forall\phi,\psi\in H^4(G)$, $\Pi(t)$ satisfies

$$(7.5.60)\quad (\phi,\ \{\frac{d\Pi(t)}{dt} - A^{D*}(t)\Pi(t) - (A^{D*}(t)\Pi(t))^* + \hat{A}^{s*}(t)(\hat{A}^{s*}(t)\Pi(t))^* + P(t)\}\psi)_H - (\{\tilde{T}_1(t)\Pi(t) + H_1(t)\cdot\Pi(t) + \Xi_{12}(t)\cdot\Pi_b(t)\alpha_1(t)\}\phi,\ \{R(t) + \Xi_{11}(t)\cdot\Pi_b(t)\}^{-1} \times\{\tilde{T}_1(t)\Pi(t) + H_1(t)\cdot\Pi(t) + \Xi_{12}(t)\cdot\Pi_b(t)\alpha_1(t)\}\psi)_{L^2(\partial G)} - 2(S_1\phi,\ \mathcal{C}_1(t)\Pi(t)\psi)_{L^2(\partial G)} - 2(S_2\phi,\ \mathcal{C}_2(t)\Pi(t)\psi)_{L^2(\partial G)}$$

$$+ (\alpha_1(t)\phi,\ \{H_2(t)\cdot\Pi(t) + \Xi_{22}(t)\cdot\Pi_b(t)\alpha_1(t)\}\psi)_{L^2(\partial G)} = 0..$$

with the terminal condition,

(7.5.61) $(\phi,\ \Pi(t_f)\psi) = 0$.

Equation (7.5.59) can be expressed by the following kernel equation,

(7.5.62)
$$g^o(t,x) = \Big[e(t,x,y)\int_0^1 \{(\gamma^*_{3,1}a_4(t) + \gamma^*_{1,1}a_2(t))\pi(t,y,z)$$
$$+ \gamma^*_{1,1}a_4(t)\frac{\partial^2\pi(t,y,z)}{\partial y^2}$$
$$- \gamma_{1,1}c^2(t)\frac{\partial^2\pi(t,y,z)}{\partial z^2}\}\bar{u}(t,z)dz\Big]_{y=0}^{y=1}$$

where $e(t,x,y)$ is the kernel of the inverse operator $E(t)$,

(7.5.63) $E(t) = [e(t,x,y)(\cdot)]_{y=0}^{y=1}$

and for $\forall\phi\epsilon H^4(G)$, $E(t)$ satisfies the following relation:

(7.5.64) $(\phi,\ (R(t) + \Xi_{11}(t)\cdot\Pi_b(t))E(t)\phi)_{L^2(\partial G)} = (\phi,\ \phi)_{L^2(\partial G)}$

and the kernel $\pi(t,x,y)$ of the operator $\Pi(t)$ satisfies

(7.5.65)
$$\frac{\partial\pi(t,x,y)}{\partial t} - a_4(t)(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4})\pi(t,x,y) - a_2(t)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$
$$\times\pi(t,x,y) + c^2(t)\frac{\partial^4\pi(t,x,y)}{\partial x^2\partial y^2} + p(t,x,y)$$
$$- [\{ -(\gamma^*_{3,1}a_4(t) + \gamma^*_{1,1}a_2(t))\pi(t,x,z_1)$$
$$- \gamma_{1,1}a_4(t)\frac{\partial^2\pi(t,x,z_1)}{\partial z_1^2} + \gamma_{1,1}c^2(t)\frac{\partial^2\pi(t,x,z_1)}{\partial z_1^2}\}$$

$$\times[e(t,z_1,z_2)\{-(\gamma_{3,1}a_4(t)+\gamma_{1,1}a_2(t))\pi(t,z_2,y)$$

$$-\gamma_{1,1}a_4(t)\frac{\partial^2\pi(t,z_2,y)}{\partial z_2^2}+\gamma_{1,1}c^2(t)\frac{\partial^2\pi(t,z_2,y)}{\partial y^2}\}]_{z_2=0}^{z_2=1}]_{z_1=0}^{z_1=1}=0$$

with the terminal condition,

(7.5.66) $\pi(t_f,x,y) = 0$

and the boundary conditions,

$$(7.5.67)\quad \left|\begin{array}{l}\pi(t,x,y)=0\\[2ex] \dfrac{\partial^2\pi(t,x,y)}{\partial x^2}=0\end{array}\right. \qquad \text{for } (t,x,y)\in T\times\partial G_x\times G_y$$

and

$$(7.5.68)\quad \left|\begin{array}{l}\pi(t,x,y)=0\\[2ex] \dfrac{\partial^2\pi(t,x,y)}{\partial y^2}=0\end{array}\right. \qquad \text{for } (t,x,y)\in T\times G_x\times\partial G_y$$

where $p(t,x,y)$ is the following integral kernel of $P(t)$,

(7.5.69) $P(t) = \int_0^1 p(t,x,y)(\cdot)dy$.

It should be noted that the terms $\Xi_{12}(t)\cdot\Pi_b(t)$, $H_2(t)\cdot\Pi(t)$ and $\Xi_{22}(t)\cdot\Pi_b(t)$ in Eqs.(7.5.59) and (7.5.60) become zero, respectively, due to the boundary conditions (7.5.67) and (7.5.68) of $\pi(t,x,y)$.

7.6. Discussions and Summary

In this chapter, the optimal boundary control problem for a general class of partial differential equation of Parabolic type with white Gaussian noise coefficients has been solved by using the dynamic programming approach and the well-known Green's formula.

The results obtained in this chapter, can easily be extended to the case of Hyperbolic equation,i.e., System Σ_8 in Definition-2.3.8 of Sec.2.3 of Chap.2, by using the stochastic differential rule given in Chap.6. In order to study the boundary control problem, it should be noted that the influence of boundary control to the interior domain of the system state is disturbed by the randomness of coefficients.

CHAPTER 8. CONTROLLABILITY FOR STOCHASTIC DISTRIBUTED PARAMETER SYSTEMS GOVERNED BY THE PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE

8.1. Introductory Remarks

In analogy with lumped parameter systems, the controllability problem for distributed parameter systems has, up to the present time, been studied by many researchers. [W4], [S16],[T4],[R2]. Comparing with the lumped parameter systems, it is difficult to check conditions of the controllability for distributed parameter systems because of the infinite dimension of system states. Recently, by using the generalized canonical representation technique, Triggiani derived the easy-to-check conditions for deterministic distributed parameter systems. For stochastic distributed parameter systems, Sunahara and the author present sufficient conditions of stochastic controllability for distributed parameter systems with additive noise disturbances by using the same procedure as stated in Part 1,[S2].

In this chapter, we pay an attension to the stochastic controllability problem for distributed parameter systems with white Gaussian noise coefficients. (The stochastic controllability for distributed parameter systems with additive noise disturbances is included in the case of this chapter.)

In Section 8.2, first of all, the mathematical model of system dynamics is given by a form of evolution equation in the Sobolev spaces, and two definitions for the stochastic controllability are presented in analogy with the lumped parameter systems. In Section 8.3, four theorems are stated, giving sufficient conditions, which are useful to check the stochastic controllability. Section 8.4 is devoted to show two examples for the purpose of verifying the sufficient conditions obtained in Sec.8.3. Furthermore, in the third example, the boundary controllability is also examined.

8.2. System Model and Definitions of Stochastic Controllability

Consider the following stochastic evolution equation in the spaces V,H and V'

$$(8.2.1)\quad \Sigma_3 \Big|\ u(t) + \int_{t_0}^{t} A^D(s)u(s)ds + \int_{t_0}^{t} dA^s(s,\omega)[u(s)] = u(t_0) + \int_{t_0}^{t} B(s)f(s)ds,$$

where the conditions for the operators $A^D(t)$ and $dA^s(t,\omega)$ are the same as those stated in Chap.5, B(t) is a linear operator from U (Some Hilbert space) to H and f(t) is a F_t-measurable† control signal which is an element of U.

As mentioned in Part 1, in order to study the stochastic controllability, we must consider the following two important items:

i) What stochastic measure shall we adopt?

ii) How to construct an exact control signal.

In this chapter, for convenience of discussions, the following definitions are first stated:

[Definition-8.2.1](Stochastic ε-controllability)††

An initial state u_0 of the system Σ_3 is said to be stochastically ε-controllable in probability ρ with respect to the specified target

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† F_t denotes the σ-algebra generated by the u(s) process for $t_0 \le s \le t$.

†† The definition of Controllability in this chapter is the same as the stochastic controllability in probability. For convenience of descriptions, we omit, in the sequel, to write the term "in probability".

domain ε within the time interval $\overline{T}=[t_0,t_f]$, if there exists a control signal $f\in W_{ad}$ † such that

$$(8.2.2)\quad Pr\{\ \|u(t_f)\|_H^2 \geq \varepsilon \mid u(t_0)=u_0\ \} \leq 1-\rho\ ,$$

where $0<\rho<1$.

[Definition-8.2.2](Stochastic Controllability)

An initial state u_0 of the system Σ_3 is said to be stochastically controllable within the time interval $\overline{T}$, if the inequality (8.2.2) holds for any $\varepsilon>0$ and any ρ, where $0<\rho<1$.

Remark-8.2.1: If the system considered is stochastically controllable, then the system becomes stochastic ε-controllable.

8.3. Sufficient Conditions of Stochastic Controllability

In this section, a theorem is first stated, giving sufficient conditions for the stochastic ε-controllability.

[Theorem-8.3.1](Stochastic ε-Controllability): An initial state u_0 of the system Σ_3, is stochastically ε-controllable in probability ρ, with respect to the specified target domain ε within the time interval $\overline{T}$, if the following conditions are satisfied:

(Condition-8.3.1) There exists a partial differential operator $K(t)$ such that, for any $\psi\in H^n(G)$ and any symmetric positive operator Q

$$(8.3.1)\quad (\psi,\ \hat{A}^{S*}(t)Q(t)\hat{A}^{S}(t)\psi)_H \leq (\psi,\ [K^*(t)Q(t)+Q(t)K(t)]\psi)_H \qquad \text{for } {}^\forall t\in T,$$

where $A^D(t)-K(t)$ satisfies Coercivity condition-2.2.1.††

(Condition-8.3.2) For the given initial state u_0 and the preassigned parameters ε,ρ, there exists a positive constant $\alpha_f>0$ such that

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† The symbol W_{ad} denotes the admissible control class defined in Chap.5.

†† See Eq.(8.3.3) of Condition-8.3.2.

$$(8.3.2)\quad (\Phi(t_f,t_0)u_0,\ [\alpha_f I + \int_{t_0}^{t_f} \Phi(t_f,t)B(t)B^*(t)\Phi^*(t_f,t)dt]^{-1} \times\Phi(t_f,t)u_0)_H \le \frac{\varepsilon}{\alpha_f}(1 - \rho)\ ,$$

where, for $t>s$,

$$(8.3.3)\quad \frac{\partial\Phi(t,s)}{\partial t} = -(A^D(t) - K(t))\Phi(t,s) \text{ and } \Phi(s,s) = I.$$

Proof: Define

$$(8.3.4)\quad P(t) = \Phi\ (t_f,t)[\alpha_f I + \int_{t}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,s)ds]^{-1} \times\Phi(t_f,t)\ .$$

From Eq.(8.3.4), we can easily find that $P(t)$ satisfies

$$(8.3.5)\quad \dot{P}(t) - (A^D(t) - K(t))^*P(t) - P(t)(A^D(t) - K(t)) - - P(t)B(t)B^*(t)P(t) = 0\ ,$$

with the terminal condition $P(t_f) = I/\alpha_f$.

Setting $f(t) = -\frac{1}{2}B^*(t)P(t)u(t)$ and applying the stochastic differential rule stated in Theorem-5.3.1 in Sec.5.3 of Chap.5, we have

$$\begin{aligned}(8.3.6)\quad & E\{(u(t_f),\ P(t_f)u(t_f))_H|u(t_0)=u_0\} - (u_0,\ P(t_0)u_0)_H \\ & \le E\{\int_{t_0}^{t_f}(u(s),\ [\dot{P}(s) - (A^D(s) - K(s))^*P(s) - \\ & \quad - P(s)(A^D(s) - K(s)) - P(s)B(s)B^*(s)P(s)] \\ & \quad u(s))_H ds|\ u(t_0)=u_0\} = 0.\end{aligned}$$

From Eq.(8.3.4), the initial state $P(t_0)$ becomes

$$(8.3.7)\quad P(t_0) = \Phi^*(t_f,t_0)[\alpha_f I + \int_{t_0}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,s)ds]^{-1} \times\Phi(t_f,t_0).$$

From Eqs.(8.3.6), (8.3.7) and Condition-8.3.2, we have

$$(8.3.8)\quad E\{(u(t_f),\ P(t_f)u(t_f))_H \mid u(t_0)=u_0\}$$

$$\le (\Phi(t_f,t_0)u_0,\ [\alpha_f I + \int_{t_0}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,s)ds]^{-1} \times\Phi(t_f,t_0)u_0)_H$$

$$\le \frac{\varepsilon}{\alpha_f}(\ 1 - \rho)\ .$$

Using the terminal condition $P(t_f)=I/\alpha_f$, Eq.(8.3.8) becomes

$$(8.3.9)\quad E\{\ \|u(t_f)\|_H^2 \mid u(t_0)=u_0\} \le \varepsilon(1 - \rho)\ .$$

Equation (8.3.9) implies that

$$(8.3.10)\quad Pr\{\|u(t_f)\|_H^2 \ge \varepsilon \mid u(t_0)=u_0\} \le 1 - \rho.$$

The proof has been completed.

Theorem-8.3.1 is easily extended to the following stochastic controllability.

[Theorem-8.3.2](Stochastic controllability): In addition to Condition-8.3.1, the initial state u_0 of the system Σ_3 is stochastically controllable within the time interval $\overline{T}$, if the following condition is satisfied,

(Condition-8.3.3) For the given initial state u_0, there exists a constant M such that

$$(8.3.11)\quad (\Phi(t_f,t_0)u_0,\ [\int_{t_0}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,s)ds]^{-1}\Phi(t_f,t_0)u_0)_H \le M\ .$$

Proof: With Eq.(8.3.4), setting $f(t)=-\frac{1}{2}B^*(t)P(t)u(t)$ and applying Condition-8.3.1, we obtain,

$$(8.3.12)\quad E\{(u(t_f),\ P(t_f)u(t_f))_H \mid u(t_0)=u_0\} \le (u_0,\ P(t_0)u_0)_H$$

$$\leq (\Phi(t_f,t_0)u_0,\ [\alpha_f I + \int_{t_0}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,s)ds]^{-1} \times\Phi(t_f,t_0)u_0)_H$$

$$\leq (\Phi(t_f,t_0)u_0,[\int_{t_0}^{t_f} \Phi(t_f,s)B(s)B^*(s)\Phi^*(t_f,t_0)ds]^{-1}\Phi(t_f,t_0)u_0)_H.$$

Using Condition-8.3.3, it follows that

$$(8.3.13)\quad E\{(u(t_f),\ P(t_f)u(t_f))_H\} \leq M.$$

Since M does not depend on α_f, we can choose α_f such that, for $\forall\varepsilon>0$ and $\forall\rho$ $(0<\rho<1)$,

$$(8.3.14)\quad E\{(u(t_f),\ u(t_f))_H |\ u(t_0)=u_0\} \leq \varepsilon(1 - \rho).$$

The proof has been completed.

In the following theorem, we assume that the control signal is an element of $H^{n/2}(G)$ for each $t\in T$.

[Theorem-8.3.3](Stochastic controllability): The initial condition u_0 of the system Σ_3 is stochastically controllable within the time interval $\overline{T}$, if the following conditions are satisfied,

(Condition-8.3.4) There exists a constant $\tilde{M}>0$ such that, for the given initial state u_0,

$$(8.3.15)\quad (\tilde{\Phi}(t_f,t_0)u_0,\ [\int_{t_0}^{t_f}\tilde{\Phi}(t_f,s)B(s)B^*(s)\tilde{\Phi}^*(t_f,s)ds]^{-1}\tilde{\Phi}(t_f,t_0)u_0)_H \leq \tilde{M},$$

where, for $t>s$,

$$(8.3.16)\quad \frac{\partial\tilde{\Phi}(t,s)}{\partial t} = - A^D(t)\Phi(t,s) \text{ and } \Phi(s,s) = I .$$

(Condition-8.3.5) For $\forall\psi\in H$ $(\psi\neq0)$, the operator B(t) satisfies

$$(8.3.17)\quad (\psi,\ B(t)B\ (t)\psi)_H \neq 0 \text{ for } \forall t\in T .$$

Proof: Define, for $\forall\alpha>0$

$$(8.3.18)\quad \tilde{P}(t) = \tilde{\Phi}^*(t_f,t)[\alpha I + \int_t^{t_f}\tilde{\Phi}(t_f,s)B(s)B^*(s)\tilde{\Phi}^*(t_f,s)ds]^{-1} \times\tilde{\Phi}(t_f,t)\ .$$

Then, it can easily be shown that

$$(8.3.19)\quad \frac{d\tilde{P}(t)}{dt} - A^{D*}(t)\tilde{P}(t) - \tilde{P}(t)A^D(t) - \tilde{P}(t)B(t)B^*(t)\tilde{P}(t) = 0\ ,$$

with the terminal condition $\tilde{P}(t_f) = I/\alpha$. Since, from Condition-8.3.5, for $u(t)\neq 0$, we have

$$(8.3.20)\quad (u(t),\ \tilde{P}(t)B(t)B^*(t)\tilde{P}(t)u(t))_H \neq 0\ ,$$

we set

$$(8.3.21)\quad f(t) = -\frac{1}{2}B^*(t)\tilde{P}(t)u(t)[1 + \frac{(u(t),\ \hat{A}^{s*}(t)\tilde{P}(t)\hat{A}^s(t)u(t))_H}{(u(t),\ \tilde{P}(t)B(t)B^*(t)\tilde{P}(t)u(t))_H}]\ .$$

By using the same approach as in Theorem-8.3.1, we have

$$(8.3.22)\quad E\{(u(t_f),\ \tilde{P}(t_f)u(t_f))_H|u(t_0)=u_0\} - (u_0,\ \tilde{P}(t_0)u_0)_H$$
$$= E\{\int_{t_0}^{t_f}(u(s),\ [\dot{\tilde{P}}(s) - A^{D*}(s)\tilde{P}(s) - \tilde{P}(s)A^D(s) - \tilde{P}(s)B(s)B^*(s)\tilde{P}(s)]u(s))_H ds|u(t_0)=u_0\}$$
$$= 0\ .$$

Using Condition-8.3.4 and the terminal condition $\tilde{P}(t_f) = I/\alpha$, it follows that

$$(8.3.23)\quad E\{\ \|u(t_f)\|_H^2|u(t_0)=u_0\} \le \alpha\tilde{M}\ .$$

Noting that α is an arbitrary positive constant and $\tilde{M}$ does not depend on α, for any $\varepsilon>0$ and any ρ $(0<\rho<1)$, we have

(8.3.24) $E\{\|u(t_f)\|_H^2 \,|\, u(t_0)=u_0\} \leq \varepsilon(1-\rho)$.

The proof has been completed.

The conditions stated in Theorem-8.3.1 can not be easily checked without the aid of computer. If the explicit forms of evolution operators Φ and $\tilde{\Phi}$ are obtained, the conditions stated in Theorems-8.3.2 and -8.3.3 can be examined. Then, setting some restrictions on the operators $B(t)$ and $A^D(t)$, we obtain the following theorem, giving easily checked sufficient conditions for the stochastic controllability.

[Theorem-8.3.4](Stochastic controllability)†: We assume that $B(t)= b(x)\cdot I$ (identity operator). The initial state u_0 of the system Σ_3 is stochastically controllable within the time interval $\overline{T}$, if the following conditions are satisfied:

(Condition-8.3.6) There exists a sequence $\{\lambda_i, \phi_i;\ i=1,2,\cdots\}$ of the eigenvalues and eigenfunctions such that

(i) $(A^D(t)\phi_i,\ \psi)_H = (\tilde{\lambda}_i\phi_i,\ \psi)$ for $\forall\psi\in H$.

(ii) $0<\tilde{\lambda}_1<\tilde{\lambda}_2<\cdots<\tilde{\lambda}_i<\cdots<\lim_{i\to\infty}\tilde{\lambda}_i=\infty$.

(iii) $\{\phi_i;\ i=1,2,\cdots\}$ is complete orthonormal in H.

(Condition-8.3.7) $B_{ij}\triangleq(\phi_i,\ BB^*\phi_j)_H$ can be partitioned in a form of

(8.3.25) $B_{ij} = \tilde{B}_i\cdot\tilde{B}_j$

which satisfies $\tilde{B}_i \neq 0$ for all i.

(Condition-8.3.8) The initial condition satisfies

(8.3.26a) $\tilde{B}_i(u_0,\ \phi_i)_H \geq 0$ for all i

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† As was mentioned in Part 1, i.e.the stochastic controllability theorem contains the stochastic ε-controllability theorem, we consider only problem of stochastic controllability.

or

(8.3.26b) $\tilde{B}_i(u_0, \phi_i)_H \le 0$ for all i .

Proof: Since the proof is similar to that of Theorem-8.3.3, from Condition-8.3.7, we may show only that there exists a constant $\tilde{M}$ satisfying Condition-8.3.4. From Condition-8.3.6 the fundamental solution $\tilde{\Phi}$ relative to $A^D(t)$ is expressed by

(8.3.27) $$\tilde{\Phi}(t_f,s) \overset{\Delta}{=} \sum_{i=1}^{\infty} e^{-\tilde{\lambda}_i(t_f-s)}\phi_i(\phi_i,\cdot)_H .$$

Using Eq.(8.3.27), for $\psi\in R(\tilde{\Phi}(t_f,t))$† we have

(8.3.28) $$\begin{aligned}(\psi,\ W(t_f,t_0)\psi)_H &\overset{\Delta}{=} (\psi,\ \tilde{\Phi}^{-1}(t_f,t_0)\int_{t_0}^{t_f}\tilde{\Phi}(t_f,s)BB^*\tilde{\Phi}^*(t_f,s)ds \\ &\qquad \times\tilde{\Phi}^{*-1}(t_f,t_0)\psi)_H \\ &= \int_{t_0}^{t_f}[\sum_{i=1}^{\infty} e^{\tilde{\lambda}_i(s-t_0)}\tilde{B}_i(\psi,\ \phi_i)_H]^2 ds \\ &\ge \frac{1}{t_f - t_0}\{\int_{t_0}^{t_f}\sum_{i=1}^{\infty} e^{\tilde{\lambda}_i(s-t_0)}\tilde{B}_i(\psi,\ \phi_i)_H ds\}^2 \\ &= \frac{1}{t_f - t_0}\{\sum_{i=1}^{\infty}\frac{e^{\tilde{\lambda}_i(t_f - t_0)} - 1}{\tilde{\lambda}_i}\tilde{B}_i(\psi,\ \phi_i)\}^2 .\end{aligned}$$

If ψ satisfies Condition-8.3.7, we have

(8.3.29) $$\begin{aligned}(\psi,\ W(t_f,t_0)\psi)_H &\ge \frac{1}{t_f - t_0}\sum_{i=1}^{\infty}\tilde{B}_i^2\{\frac{e^{\tilde{\lambda}_i(t_f-t_0)} - 1}{\tilde{\lambda}_i}\}^2(\psi,\ \phi_i)_H^2 \\ &= \frac{1}{t_f - t_0}\min_i[\tilde{B}_i^2\{\frac{e^{\tilde{\lambda}_i(t_f-t_0)} - 1}{\tilde{\lambda}_i}\}^2]\|\psi\|_H^2 .\end{aligned}$$

Since from Eq.(8.3.28) it may be observed that the operator $W(t_f,t_0)$

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† $R(\cdot)$ denotes the range space of the operator $(\cdot)$.

is positive definite, it follows that

$$(8.3.30)\quad (u_0, W^{-1}(t_f,t_0)u_0)_H = (u_0, \tilde{\Phi}(t_f,t_0)[\int_{t_0}^{t_f}\tilde{\Phi}(t_f,s)BB^*\tilde{\Phi}^*(t_f,s)ds]^{-1} \times\tilde{\Phi}(t_f,t_0)u_0)_H$$

$$\le \frac{t_f - t_0}{\min_i \tilde{B}_i^2(\frac{e^{\tilde{\lambda}_i(t_f - t_0)} - 1}{\tilde{\lambda}_i})^2}\| u_0 \|^2$$

$$\le \tilde{M} .$$

The proof has been completed.

8.4. Examples of Stochastic Controllability

8.4.1. Evamples of Stochastic Controllability with Distributed Input.

[Example-8.4.1] A somewhat artificial but typical example is considered. The mathematical model is given by the one-dimensional parabolic partial differential equation with white Gaussian noise coefficient,

$$(8.4.1a)\quad \frac{\partial u(t,x)}{\partial t} + a_4\frac{\partial^4 u(t,x)}{\partial x^4} + c\eta(t,\omega)\frac{\partial^2 u(t,x)}{\partial x^2} = \int_G \delta(x-y)f(t,y)dy ,$$

$$\text{for } (t,x)\in T\times G=]0,1[$$

with the initial condition,

$$(8.4.1b)\quad u(t_0,x) = u_0(x) \in L^\infty(G) \quad \text{a.s.}$$

and the boundary conditions,

$$(8.4.1c)\quad u(t,0)=u(t,1)= 0 , \text{ for } t\in T$$

and

$$(8.4.1d)\quad \frac{\partial^2 u(t,x)}{\partial x^2}\bigg|_{x=1} = \frac{\partial^2 u(t,x)}{\partial y^2}\bigg|_{x=0} = 0 , \text{ for } t\in T ,$$

where a_4 and c are constants which satisfy $a_4 - \frac{1}{2}c^2 > 0$ and $\eta(t,\omega)$ is a white Gaussian noise with zero mean and unit variance.

The partial differential operaors A^D and $\hat{A}^s$ are respectively expressed by

$$(8.4.2)\quad A^D = a_4\frac{\partial^4}{\partial x^4}$$

and

$$(8.4.3)\quad \hat{A}^s = c\frac{\partial^2}{\partial x^2}\quad .$$

Noting that $B(t) = \int_G \delta(x-y)(\ )dy$, from Eq.(8.3.1) we have

$$(8.4.4)\quad K(t) = \frac{1}{2}c^2\frac{\partial^4}{\partial x^4}\quad .$$

From Eq.(8.3.3), the evolution operator $\Phi(t_f,t_0)$ becomes

$$(8.4.5)\quad \Phi(t_f,t_0) = \sum_{i=1}^{\infty} \exp\{-(i\pi)^4(a_4 - \tfrac{1}{2}c^2)(t_f - t_0)\}\phi_i(\phi_i,\cdot)_H$$

where $\phi_i = \sqrt{2}\sin(i\pi x)$.

From Eqs.(8.3.11) and (8.4.5), it follows that

$$(8.4.6)\quad \Phi^*(t_f,t_0)\Big[\int_{t_0}^{t_f}\Phi(t_f,t)\Phi^*(t_f,t)dt\Big]^{-1}\Phi(t_f,t_0)$$

$$= \sum_{i=1}^{\infty}\frac{2(i\pi)^4(a_4 - \frac{1}{2})}{\exp\{2(i\pi)^4(a_4 - \frac{1}{2}c^2)(t_f - t_0)\} - 1}\phi_i(\phi_i,\cdot)_H.$$

From the fact that $u_0 \in L^\infty(G)$, Eq.(8.4.6) yields

$$(8.4.7)\quad (\Phi(t_f,t_0)u_0,\ \Big[\int_{t_0}^{t_f}\Phi(t_f,t)BB^*\Phi^*(t_f,t)dt\Big]^{-1}\Phi(t_f,t_0)u_0)_H \le M\ .$$

Therefore, since Conditions-8.3.1 and -8.3.3 are satisfied, the initial condition u_0 of the system Eq.(8.5.1) is stochastically controllable in the time interval $\overline{T}$.

[Example-8.4.2] Consider the following heat equation

(8.4.8a) $$\frac{\partial u(t,x)}{\partial t} + a_2\frac{\partial^2 u(t,x)}{\partial x^2} + c\eta(t,\omega)\frac{\partial u(t,x)}{\partial x} = \delta(x - \nu)f(t,x)$$

for $(t,x)\in T\times G =]0,1[$

together with the initial and boundary conditions,

(8.4.8b) $u(t_0,x) = 2\sin(2\pi x)\cos(\pi x)$, for $x\in]0,1[$

and

(8.4.8c) $u(t,0) = u(t,1) = 0$, for $t\in T$,

where $\eta(t,\omega)$ is the white Gaussian noise with zero mean and unit variance, which denotes the random velocity of a fluid medium in the heat conduction.

From Coercivity condition-2.2.1 in Sec.2.2 of Chap.2, it is assumed that

(8.4.9) $a_2 + \frac{1}{2}c^2 < 0$.

Since the partial differential operator A^D and $\hat{A}^s$ are respectively given by

(8.4.10) $$A^D = a_2\frac{\partial^2}{\partial x^2}$$

and

(8.4.11) $$\hat{A}^s = c\frac{\partial}{\partial x} ,$$

the eigenfunction satisfied in Condition-8.3.6 becomes

(8.4.12) $\phi_i(x) = \sqrt{2}\sin(i\pi x)$, for $i=1,2,\cdots$

In this example, noting that $B = \delta(x - \nu)$, then, from Condition-8.3.7 B_{ij} becomes

(8.4.13) $B_{ij} = (\phi_i, B^2\phi_j)_H = \sqrt{2}\sin(i\nu\pi)\sin(j\nu\pi)$.

From Eq.(8.4.13), if ν is an irrational number in G, then Condition-8.3.7 is satisfied. From the fact that the initial condition $u(t_0,x)$ can be expanded by

$$(8.4.14)\quad u(t_0,x) = \sin(\pi x) + \sin(3\pi x) \quad ,$$

if the irrational number ν satisfies

$$(8.4.15)\quad 0<\nu<\frac{1}{3} \ , \ \frac{2}{3}<\nu<1 \ ,$$

then, Condition-8.3.8 is satisfied, i.e., the initial condition u_0 given by Eq.(8.4.8b) is stchastically controllable.

8.4.2. <u>An Example of Stochastic Controllability with Boundary Input</u>

In the following example, the results obtained in Sec.8.3 are extended to the boundary controllability.

[Example-8.4.3] Consider once again the heat equation given by the previous example-8.4.2,

$$(8.4.16a)\quad \frac{\partial u(t,x)}{\partial t} + a_2\frac{\partial^2 u(t,x)}{\partial x^2} + c\eta(t,\omega)\frac{\partial u(t,x)}{\partial x} = 0 \ ,$$

$$\text{for } (t,x)\in T\times G =]0,1[\ ,$$

with the initial condition,

$$(8.4.16b)\quad u(t_0,x) = u_0 \ , \text{ for } x\in]0,1[,$$

and the boundary condition,

$$(8.4.16c)\quad \frac{\partial u(t,1)}{\partial x} = f(t) \ , \text{ and } \frac{\partial u(t,0)}{\partial x} = 0, \text{ for } t\in T \ ,$$

where $f(t)$ is a boundary control and parameters in Eq.(8.4.16) are assumed to be same as in Example-8.4.2.

In this example, denoting

$$(8.4.17)\quad A^D = a_2\frac{\partial^2}{\partial x^2}$$

and

(8.4.18) $$\hat{A}^s = c^2\frac{\partial}{\partial x},$$

we can construct an evolution operator $\tilde{\Phi}(t,s)$ such that

(8.4.19) $$\tilde{\Phi}(t,s) = \sum_{i=1}^{\infty} \exp\{-(i\pi)^2 a_2(t - s)\}\phi_i(\phi_i,\cdot)_H,$$

with the homogeneous Neumann condition, where $\phi_i(x) = \sqrt{2}\cos(i\pi x)$.

In order to examine the stochastic controllability, using the same procedure as in Theorem-8.3.4, for $\psi \in R(\tilde{\Phi}(t_f,t_0))$, we have

(8.4.20) $$(\psi, W(t_f,t_0)\psi)_H = (\psi, \tilde{\Phi}^{-1}(t_f,t_0)[\int_{t_0}^{t_f} B[\tilde{\Phi}(t_f,s)]B[\tilde{\Phi}(t_f,s)]ds] \times\tilde{\Phi}^{-1}(t_f,t_0)\psi)_H,$$

where

(8.4.21) $$B[\tilde{\Phi}(t_f,s)] = \tilde{\Phi}(t_f,s)\Big|_{x=1}$$

$$= \sum_{i=1}^{\infty} \exp\{-(i\pi)^2 a_2(t_f - s)\}\sqrt{2}\cos(i\pi)(\phi_i,\cdot)_H .$$

Equation (8.4.20) becomes

(8.4.22) $$(\psi, W(t_f,t_0)\psi)_H = \int_{t_0}^{t_f} [\sum_{i=1}^{\infty} e^{(i\pi)^2 a_2(s - t_0)}\sqrt{2}\cos(i\pi) \times(\phi_i, \psi)_H]^2 ds$$

$$\geq \frac{1}{t_f - t_0}[\sum_{i=1}^{\infty} \frac{\sqrt{2}}{(i\pi)^2}\{e^{(i\pi)^2 a_2(t_f - t_0)} - 1\} \times\cos(i\pi)(\phi_i, \psi)_H]^2 .$$

If ψ satisfies $(\cos(i\pi))(\psi, \phi_i)_H \leq 0$ for all i (or $(\cos(i\pi))(\psi, \phi_i)_H \geq 0$ for all i), then it follows that

$$(8.4.23)\quad (\psi,\ W(t_f,t_0)\psi)_H \geq \frac{1}{t_f - t_0}\frac{2}{\pi^4}\{\exp(\pi^2 a_2(t_f - t_0)) - 1\}\|\psi\|_H^2.$$

Consequentry, if u_0 satisfies $(\cos(i\pi))(u_0,\phi_i)_H \leq 0$ for all i (or $(\cos(i\pi))(u_0,\phi_i)_H \geq 0$ for all i), we have

$$(8.4.24)\quad (u_0,\ W^{-1}(t_f,t_0)u_0)_H \leq \frac{t_f - t_0}{\frac{\pi^4}{2}\{\exp(\pi^2 a_2(t_f - t_0)) - 1\}^2}\|u_0\|_H^2 .$$

Hence, the initial condition u_0 becomes stochastically controllable.

8.5. Discussions and Summary

In this chapter, according to the definitions of both the stochastic ε-controllability and the stochastic controllability, four theorems have been developed, which give sufficient conditions. As mentioned in Part 1, one of significant defferences between the deterministic thory of controllability and stochastic one is the fact that the control signal must be obtained in a concrete form, which transfer the given initial state to the desired target domain, because of the randomness of the system state caused by the stochastic coefficients. From this fact, in order to show explicit conditions of the stochastic controllability for distributed parameter systems without the computer implementation, the controllable initial state accompanies with somewhat severely restricted conditions. However, if the graphical procedure of hitting problem mentioned in Chap.5 of Part 1 is introduced, a wide class of stochastically controllable initial states derived by milder conditions stated in Theorem-8.3.3 can be examined.

For stochastic distributed parameter systems with the additive noise disturbances, Condition-8.3.1 is always satisfied. Then, from the results of Part 1, checking Conditions-8.3.2 or -8.3.3, the stochastic controllability problem can be examined (See ref.[S16]).

The controllability problem for Hyperbolic systems with white Gaussian coefficients is easily investigated from the same method as described in this chapter.

9. Conclusions

9.1. Concluding Remarks

In Part 2, chracterizing the randomness of coefficients by white Gaussian process and Markov chain process, the mathematical models of distributed parameter systems with stochastic coefficients have been formulated on the well-known Sobolev spaces. Furthermore, two types of stochastic partial differential equations governed by Parabolic and Hyperbolic types were considered in order to establish the common precedure to find the optimal control of stochastic systems which will hopefully be of a generalized mathematical model of physically existing systems.

By using the Dynamic programming and Maximum principle approches, the optimal distributed and boundary control signals were derived. It should be emphasized that the boundary control signal depends on the randomness of coefficients and then the optimal boundary control becomes complicated.

For the system with white Gaussian noise coefficients, the optimal control problem can be solved, because of Markov property of the state space. However, in the case of Markov chain coefficients, we can only obtain the sub-optimal control signal from the information of the state variable with the aid of stochastic eigenvalue problem. The proposed methods in Part 2 will contribute to obtain a feasible solution to the practical design of feedback control for stochastic distributed parameter systems.

9.2. Discussions

The optimal control problem for systems with the mixed coefficients given by Definitions-2.3.5 and -2.3.6 in Sec.2.3 of Chap.2

was not discussed in this thesis. From the results given in Chap.3, it is easily found that the optimal control signal for the mixed coefficients type can not be attained without the information about the Markov chain coefficients. However, the stochastic eigenvalue problem proposed in Chap.3 suggests us to construct the approximation method of generating the sub-optimal control signal with the aid of the estimation theory for Poisson process,[S6],[S17].

Furthermore, there are many cases where coefficients in a system operator have the random property in the spatial variable x. How to characterize the randomness for the spatial reagion and how to formulate the mathematical model are further problems.

APPENDIX A. (Proof of Theorem-2.2.1)

We assume the Sobolev space V is separable. Let $e_1, e_2, \cdots e_m, \cdots$ be an orthonormal basis of H. Let $V_m = [e_1, e_2, \cdots, e_m]$ and define the approximated operators by

$$\text{(A-1)} \quad A_m^M(t,\omega) = \sum_{i=1}^{m} \langle A^M(t,\omega)(\cdot),\ e_i \rangle e_i$$

and

$$\text{(A-2)} \quad dA_m^s(t,\omega)[\cdot] = \sum_{i=1}^{m} (dA^s(t,\omega)[\cdot],\ e_i)_H e_i \ .$$

From Coercivity condition-2.2.1, we may find

$$\text{(A-3)} \quad A_m^M(t,\omega) \in \mathcal{L}(V_m; V_m{}') \qquad \text{w.p.1} \ .$$

Using $A_m^M(t,\omega)$ and $dA_m^s(t,\omega)$, we can approximate Eq.(2.2.1) by

$$\text{(A-4)} \quad u_m(t) + \int_{t_0}^{t_f} A_m^M(s,\omega) u_m(s) ds + \int_{t_0}^{t_f} dA_m^s(s,\omega)[u_m(s)] = u_{0m} \ ,$$

where

$$\text{(A-5)} \quad u_{0m} = \sum_{i=1}^{m} (u_0,\ e_i)_H e_i \ , \quad \sum_{i=1}^{m} (u_0,\ e_i)_H e_i \to u_0 \text{ in } H \text{ as } m\to\infty \ .$$

From Proposition 2.2.2, it follows that

$$\text{(A-6)} \quad E\{(\int_{t_0}^{t} dA_m^s(s,\omega)[u_m(s)],\ \int_{t_0}^{t} dA_m^s(s,\omega)[u_m(s)])_H\}$$

$$= \sum_{i=1}^{m} E\{[\sum_{|p|\le n/2} \int_{t_0}^{t} (a_{|p|}^s(t,x) D_x^p u_m(s),\ e_i)_H dw_{|p|}(s)]^2\}$$

$$= E\{\int_{t_0}^{t} (\hat{A}_m^s(s) u_m(s),\ \hat{A}_m^s(s) u_m(s))_H ds\}$$

$$\le E\{\int_{t_0}^{t} (\hat{A}^s(s) u_m(s),\ \hat{A}^s(s) u_m(s))_H ds\}$$

$$= E\{\int_{t_0}^{t} \langle \hat{A}^{s^2}(s)u_m(s),\ u_m(s)\rangle ds\}\ .$$

Noting that Eq.(A-4) is an m-dimensional ordinary Itô-stochastic equation and using the well-known stochastic calculas, we have

$$\text{(A-7)}\quad \tfrac{1}{2}E\{\| u_m(t)\|_H^2\} + E\{\int_{t_0}^{t} \langle A^M(s,\omega)u_m(s),\ u_m(s)\rangle ds\}$$

$$- \tfrac{1}{2}E\{\int_{t_0}^{t} (\hat{A}_m^s(s)u_m(s),\ \hat{A}_m^s(s)u_m(s))_H ds\} = \tfrac{1}{2}E\{\| u_{0m}\|_H^2\}.$$

From the inequality (A-6), Eq.(A-7) becomes

$$\text{(A-8)}\quad \tfrac{1}{2}E\{\| u_m(t)\|_H^2\} + E\{\int_{t_0}^{t} [\langle A^M(s,\omega)u_m(s),\ u_m(s)\rangle$$

$$- \tfrac{1}{2}\langle \hat{A}^{s^2}(s)u_m(s),\ u_m(s)\rangle] ds\} \le \tfrac{1}{2}E\{\| u_{0m}\|_H^2\}.$$

Using Coercivity condition-2.2.1, we have

$$\text{(A-9)}\quad E\{\| u_m(t)\|_H^2\} + \alpha E\{\int_{t_0}^{t} \| u_m(s)\|_V^2 ds\} \le E\{\| u_{0m}\|_m^2\}\ .$$

From Eq.(A-9), there exist some constants C_1 and C_2 independent of m such that

$$\text{(A-10)}\quad E\{\| u_m(t)\|_H^2\} \le C_1\ ,\ \text{for } \forall t\in T$$

and

$$\text{(A-11)}\quad E\{\int_{t_0}^{t_f} \| u_m(s)\|_V^2 ds\} \le C_2\ .$$

From Eqs.(A-10) and (A-11), it is easily shown that we can extract a subsequence $u_m \to u$ in $L^2(\Omega,P;L^2(T;V))$. Then, let e_i be an arbitrary but fixed element of the basis. For $m \ge i$, we have

$$(A-12)\quad (u_m(t), e_i)_H - (u_{om}, e_i)_H + \int_{t_0}^{t} \langle A^M(s,\omega)u_m(s), e_i\rangle ds + \int_{t_0}^{t} (dA^S(s,\omega)[u_m(s)], e_i)_H = 0 ,$$

and passing to the limit, we have

$$(A-13)\quad (u(t), e_i)_H - (u_0, e_i)_H + \int_{t_0}^{t} \langle A^M(s,\omega)u(s), e_i\rangle ds + \int_{t_0}^{t} (dA^S(s,\omega)[u(s)], e_i)_H = 0 ,$$

from which it follows that there exists a solution which belongs to the class

$$(A-14)\quad L^2(\Omega,P;L^2(T;V))\cap L^\infty(T;L^2(\Omega,P;H)) \quad .$$

Furthermore, from the weak convergence of $u_m \to u$, we obtain the energy inequality

$$(A-15)\quad E\{\| u(t)\|_H^2\} + \alpha E\{\int_{t_0}^{t} \| u(s)\|_V^2 ds\} \le E\{\| u_0\|_H^2\} .$$

On the other hand, from Eq.(A-4), we have

$$(A-16)\quad \frac{1}{2}\| u_m(t)\|_H^2 + \int_{t_0}^{t} \langle A^M(s,\omega)u_m(s), u_m(s)\rangle ds - \frac{1}{2}\int_{t_0}^{t} (\hat{A}_m^s(s)u_m(s), \hat{A}_m^s(s)u_m(s))_H ds = \frac{1}{2}\| u_{0m}\|_H^2 + \int_{t_0}^{t} (dA_m^s(s,\omega)[u_m(s)],u_m(s))_H .$$

Using Coercivity condition-2.2.1, it follows that

$$(A-17)\quad \frac{1}{2}E\{\sup_{t\varepsilon T} \| u_m(t)\|_H^2\} + \alpha E\{\int_{t_0}^{t_f} \| u_m(s)\|_V^2 ds\} \le E\{\| u_{0m}\|_H^2\} + E\{\sup_{t\varepsilon T}|\int_{t_0}^{t} (dA_m^s(s,\omega)[u_m(s)], u_m(s))_H|\} .$$

Applying the martingale inequality[P1], we have for some $C_3>0$

$$\text{(A-18)}\quad E\{\sup_{t\varepsilon T}|\int_{t_0}^{t}(dA_m^s(s,\omega)[u_m(s)],\ u_m(s))_H|\}$$
$$\leq \frac{1}{6}E\{\sup_{t\varepsilon T}\|u_m(t)\|_H^2\} + C_3E\{\int_{t_0}^{t_f}\|u_m(s)\|_V^2 ds\} .$$

From Eqs.(A-11) and (A-18), (A-17) becomes

$$\text{(A-19)}\quad E\{\sup_{t\varepsilon T}\|u_m(t)\|_H^2\} \leq C_4 \quad \text{for some } C_4>0 .$$

The inequality (A-19) implies that

$$\text{(A-20)}\quad u_m \ \varepsilon\ L^2(\Omega,P;C(\overline{T};H)).$$

Consequently, we can show that

$$\text{(A-21)}\quad u \ \varepsilon\ L^2(\Omega,P;C(\overline{T};H)).$$

Now suppose that there exist two solutions $u^1(t)$ and $u^2(t)$ which satisfy Eq.(2.2.1) with the same initial condition u_0. Defining $\tilde{u}(t) \triangleq u^1(t) - u^2(t)$, it is easy to show that $\tilde{u}(t)$ also satisfies Eq.(2.2.1) with zero initial condition. Then, from Eq.(A-15) we have $E\{\|\tilde{u}(t)\|_H^2\} = 0$. This shows that Eq.(2.2.1) has a unique solution. Thus proof has been completed.

APPENDIX B. (Proof of Theorem 2.2.2)

As in Appendix A, let $e_1, e_2, \cdots, e_m, \cdots$ be a basis of V. We consider

$$\text{(B-1)} \quad \dot{v}_{0m} = \sum_{i=1}^{m} (\dot{v}_0, e_i)_H e_i \quad v_{0m} \to v_0 \text{ in H as } m \to \infty$$

and

$$\text{(B-2)} \quad v_{0m} = \sum_{i=1}^{m} (v_0, e_i)_H e_i \quad v_{0m} \to v_0 \text{ in V as } m \to \infty$$

By using the same approximation procedure as in Appendix A, we can define an approximation solution $z_m(t)$ of Eq.(2.2.27) by

$$\text{(B-3)} \quad z_m(t) + \int_{t_0}^{t} \tilde{A}_m^D(s) z_m(s) ds + \int_{t_0}^{t} d\tilde{A}_m^S(s,\omega)[z_m(s)] = z_{0m},$$

where

$$\text{(B-4)} \quad z_{0m} = [v_{0m}, \dot{v}_{0m}]'$$

$$\text{(B-5)} \quad \tilde{A}_m^D(t) = \begin{bmatrix} 0 & -I \\ A_m^D(t) & 0 \end{bmatrix}$$

and

$$\text{(B-6)} \quad d\tilde{A}_m^S(t,\omega)[z_m(t)] = [\,0,\ dA_m^S(t,\omega)[v_m(t)]]'.$$

Noting that $z_m(t)$ is a solution to an ordinary Itô stochastic equation and using the stochastic caluculus, we obtain

$$\text{(B-7)} \quad E\{[z_m(t), z_m(t)]_H\} + 2E\{\int_{t_0}^{t} [\tilde{A}_m(s) z_m(s), z_m(s)]_H ds\}$$

$$= E\{[z_{0m}, z_{0m}]_H\} + E\{\int_{t_0}^{t} [\hat{G}_m^S(s) z_m(s), \hat{G}_m^S(s) z_m(s)]_H ds\}.^{\dagger}$$

† See Eq.(6.2.9) for definition of the operator $\hat{G}_s(t)$

Noting that the definition of inner product $[\cdot,\cdot]_H$, Eq.(B-7) becomes

$$(B\text{-}8)\quad E\{\| v_m(t)\|_H^2\} + E\{\| \dot{v}_m(t)\|_H^2\} = E\{\| v_{0m}\|_H^2\} + E\{\| \dot{v}_{0m}\|_H^2\}$$
$$+ 2E\{\int_{t_0}^{t} (\dot{v}_m(s), v_m(s))_H ds\} - 2E\{\int_{t_0}^{t} (A_m^D(s)v_m(s), \dot{v}_m(s))_H ds\}$$
$$+ 2E\{\int_{t_0}^{t} (\hat{A}_m^s(s)v_m(s), \hat{A}_m^s(s)v_m(s))_H ds\} .$$

From Eqs.(B-1) and (B-6), it follows that

$$(B\text{-}9)\quad E\{\| v_m(t)\|_H^2\} + E\{\| \dot{v}_m(t)\|_H^2\} \le E\{\| v_{0m}\|_H^2\} + E\{\| \dot{v}_{0m}\|_H^2\}$$
$$+ 2E\{\int_{t_0}^{t} (\dot{v}_m(s), v_m(s))_H ds\} - 2E\{\int_{t_0}^{t} \langle A^D(s)v_m(s), \dot{v}_m(s)\rangle ds\}$$
$$+ E\{\int_{t_0}^{t} \langle \hat{A}^{s^2}(s)v_m(s), v_m(s)\rangle ds\}.$$

From the relations

$$(B\ 10)\quad 2\int_{t_0}^{t} (\dot{v}_m(s), v_m(s))_H ds = \| v_m(t)\|_H^2 - \| v_{0m}\|_H^2 ,$$

and

$$(B\text{-}11)\quad - 2\int_{t_0}^{t} \langle A^D(s)v_m(s), \dot{v}_m(s)\rangle ds = \langle A^D(t_0)v_{0m}, v_{0m}\rangle$$
$$- \langle A^D(t)v_m(t), v_m(t)\rangle + \int_{t_0}^{t} \langle \dot{A}^D(s)v_m(s), v_m(s)\rangle ds,$$

the inequality (B-9) becomes

$$(B\text{-}12)\quad E\{\| \dot{v}_m(t)\|_H^2\} + E\{\langle A^D(t)v_m(t), v_m(t)\rangle\} \le E\{\| \dot{v}_{0m}\|_H^2\}$$
$$+ E\{\langle A^D(t_0)v_{0m}, v_{0m}\rangle\} + E\{\int_{t_0}^{t} \langle A^{s^2}(s)v_m(s), v_m(s)\rangle ds\}$$
$$+ E\{\int_{t_0}^{t} \langle \dot{A}^D(s)v_m(s), v_m(s)\rangle ds\} .$$

From Coercivity conditions -2.2.2 and -2.2.3, we have†

$$(\text{B-13})\quad E\{\|\dot{v}_m(t)\|_H^2\} + \alpha_1 E\{\|v_m(t)\|_V^2\} \le E\{\|\dot{v}_{0m}\|_H^2\} + \tilde{\gamma}_1 E\{\|v_{0m}\|_V^2\} + E\{\int_{t_0}^t \gamma_1\|v_m(s)\|_V^2 ds\} + E\{\int_{t_0}^t \gamma_2\|v_m(s)\|_V^2 ds\}\ .$$

For some positive constants C_1 and C_2, the inequality (B-13) becomes

$$(\text{B-14})\quad E\{\|\dot{v}_m(t)\|_H^2\} + E\{\|v_m(t)\|_V^2\} \le C_1[E\{\|\dot{v}_{0m}\|_H^2\} + E\{\|v_{0m}\|_V^2\}] + C_2\int_{t_0}^t [E\{\|\dot{v}_m(s)\|_H^2\} + E\{\|v_m(s)\|_V^2\}]ds.$$

Setting as

$$(\text{B-15})\quad Y_m(t) \triangleq E\{\|\dot{v}_m(t)\|_H^2\} + E\{\|v_m(t)\|_V^2\}\ ,$$

it follows that

$$(\text{B-16})\quad Y_m(t) \le C_1 Y_m(t_0) + C_2\int_{t_0}^t Y_m(s)ds\ .$$

From Gronwall's inequality, we have

$$(\text{B-17})\quad Y_m(t) \le M = \text{constant independedt of } m, \text{ for } \forall t \in T\ .$$

The inequality (B-16) yields that there exist subsequences $v_m \to v$ in $L^2(T;L^2(\Omega,P;V))$ and $\dot{v}_m \to \dot{v}$ in $L^2(T;L^2(\Omega,P;H))$.

Furthermore, from Eq.(B-3), we easily obtain

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† From Eq.(2.2.25a), we can easily find that there exists $\tilde{\gamma}>0$ satisfying Eq.(B-13).

$$\text{(B-18)} \quad E\{\sup_{t\varepsilon T}\|\dot{v}_m(t)\|_H^2\} + E\{\sup_{t\varepsilon T}\|v_m(t)\|_H^2\}$$
$$\leq C_1[E\{\|\dot{v}_{0,m}\|_H^2\} + E\{\|v_{0,m}\|_V^2\}] + C_2E\{\int_{t_0}^{t_f}[\|\dot{v}_m(t)\|_H^2$$
$$+ \|v_m(t)\|_V^2]dt\} + E\{\sup_{t\varepsilon T}|\int_{t_0}^{t}[dA_m^s(s,\omega)[z_m(s)],z_m(s)]_H|\} .$$

Using the martingale inequality, it follows that

$$\text{(B-19)} \quad E\{\sup_{t\varepsilon T}|\int_{t_0}^{t}[dA_m^s(s,\omega)[z_m(s)],\ z_m(s)]_H|\}$$
$$\leq \frac{1}{6}E\{\sup_{t\varepsilon T}\|\dot{v}_m(s)\|_H^2\} + C_3E\{\int_{t_0}^{t_f}\|v_m(s)\|_V^2 ds\} \quad .$$

From the inequalities (B-17) and (B-19), (B-18) becomes

$$\text{(B-20)} \quad \frac{5}{6}E\{\sup_{t\varepsilon T}\|\dot{v}_m(t)\|_H^2\} + E\{\sup_{t\varepsilon T}\|v_m(t)\|_V^2\} \leq M_2$$

where M_2 is independent of m.

Consequently, we also have $z \;\varepsilon\; L^2(\Omega,P;C(\overline{T};V))\times L^2(\Omega,P;C(\overline{T};H))$.

The remainder part of this proof is the same as in Appendix-A. Thus proof has been completed.

APPENDIX C. (Weak Solution of System $\sum_4$)

1) Stable Boundary Condition Case

Let $e_1, e_2, \cdots, e_m, e_{m+1}, \cdots$ be an orthonormal basis of H made up with elements of $\mathcal{D}(A^{D*}) = \{\psi |\ \Psi \varepsilon H^n(G)$ and $\mathcal{B}_j^* \psi = 0$ on ∂G for $j=1,2,\cdots, n/2\}$. We take $\psi(x,t) = \phi(t)e_i$ where $\phi \varepsilon C^1(\overline{T})$ and $\phi(t_f)=0$. From Eq.(2.3.10b), we introduce the following finite dimensional approximation system

(C-1) $$u^m(t) \triangleq \sum_{i=1}^{m} y_i(t)e_i$$

where

(C-2a) $$y_i(t) + \sum_{k=1}^{m}\int_{t_0}^{t} (e_k,\ A^{D*}(s)e_i)_H y_k(s)ds + \sum_{k=1}^{m}\sum_{j=1}^{n/2}\int_{t_0}^{t} (e_k,\ \hat{A}_j^{s*}(s)e_i)_H y_k(s)dw_j(s) = y_i(t_0) + \sum_{j=1}^{n/2}\int_{t_0}^{t} (g_j(s),\ \mathcal{T}_j(s)e_i)_{L^2(\partial G)} ds$$

and

(C-2b) $$\hat{A}_j^s(s) = \sum_{|p|\le j} a_j^s(s,x)D_x^p \ .$$

From Eq.(C-1), we have the following system in the sense of Eq.(2.3.10b):

(C-3) $$u^m(t) + \int_{t_0}^{t} A_m^D(s)u^m(s)ds + \sum_{j=1}^{n/2}\int_{t_0}^{t} \hat{A}_{jm}^s(s)u^m(s)dw_j(s) = u_{0m} + \sum_{j=1}^{n/2}\int_{t_0}^{t} g_j^m(s)ds,$$

where

$$\text{(C-4)}\quad A_m^D(s)(\cdot) = \sum_{k=1}^{m} (\cdot,\ A^{D*}(s)e_k)_H e_k$$

$$\text{(C-5)}\quad \hat{A}_{jm}^s(\cdot) = \sum_{k=1}^{m} (\cdot,\ \hat{A}_j^{s*}(s)e_k)_H e_k$$

and

$$\text{(C-6)}\quad g_j^m(s) = \sum_{k=1}^{m} (g_j(s),\ \mathcal{T}_j^*(s)e_k)_{L^2(\partial G)} e_k\ .$$

For the system (C-3), by using the Itô's formula defined by Pardoux[P1], it follows that

$$\text{(C-7)}\quad E\{\| u^m(t)\|_H^2\} + E\{\int_{t_0}^{t} [2\langle A^D(s)u^m(s),\ u^m(s)\rangle - \sum_{j=1}^{n/2} (\hat{A}_j^s(s)u^m(s),\ \hat{A}_j^s(s)u^m(s))_H]ds\}$$
$$\le E\{\| u_{0n}\|_H^2\} + 2\sum_{j=1}^{n/2} E\{\int_{t_0}^{t} (g_j^m(s),\ u^m(s))ds\}\ .$$

On the other hand, we introduce the following system:

$$\text{(C-8)}\quad \tilde{z}^m(t) + \int_{t_0}^{t} [A^D(s) - \frac{1}{2}\sum_{j=1}^{n/2} \hat{A}_j^{s*}(s)\hat{A}_j^s(s)]\tilde{z}^m(s)ds$$
$$= u_{0m} + \sum_{j=1}^{n/2}\int_{t_0}^{t} g_j^m(s)ds$$

with the boundary condition $\mathcal{B}_j(t)\tilde{z}^m(t)=0$ on $T\times\partial G$ for $j=1,2,\cdots,\frac{n}{2}$.

From Eq.(C-8), it is easy to show that

$$\text{(C-9)}\quad E\{\| \tilde{z}^m(t)\|_H^2\} + E\{\int_{t_0}^{t} [2\langle A^D(s)\tilde{z}^m(s),\ \tilde{z}^m(s)\rangle - \sum_{j=1}^{n/2} (\hat{A}_j^s(s)\tilde{z}^m(s),\ \hat{A}_j^s(s)\tilde{z}^m(s))_H]ds\}$$
$$= E\{\| u_{0m}\|_H^2\} + 2\sum_{j=1}^{n/2} E\{\int_{t_0}^{t} (g_j(s),\ \tilde{z}^m(s))_H ds\}\ .$$

Consequently, with the aid of comparison theorem, we have

(C-10) $E\{\|u^m(t)\|_H^2\} \leq E\{\|\tilde{z}^m(t)\|_H^2\}$,

Then, in the remainder half, we must show that

(C-11) $E\{\int_{t_0}^{t_f}\|\tilde{z}^m(t)\|_H^2 dt\} \leq M$ (M is independent of m) .

From (C-10) and (C-11), we get

(C-12) $E\{\int_{t_0}^{t_f}\|u^m(t)\|_H^2 dt\} \leq E\{\int_{t_0}^{t_f}\|z^m(t)\|_H^2 dt\} \leq M$.

From the fact that there exists an analytic semigroup $\Phi(t,s)$ with the infinitesimal generator $A^D(t) - \frac{1}{2}\sum_{j=1}^{n/2}\hat{A}_j^{s*}(t)\hat{A}_j^s(t)$ whose domain is $\mathcal{D}(A^D)$. Then, the solution to Eq.(C-8) becomes

(C-13) $\tilde{z}^m(t) = \Phi(t,t_0)u_{0m} + \sum_{j=1}^{n/2}\int_{t_0}^{t}\Phi(t,s)g_j^m(s)ds$.

By using the well-known semigroup property and reversing the order of integration, it is easy to show that, $K>0$,

(C-14) $$E\{\int_{t_0}^{t_f}\|\sum_{j=1}^{n/2}\int_{t_0}^{t}\Phi(t,s)g_j^m(s)ds\|_H^2 dt\}$$

$$\leq K\sum_{j=1}^{n/2}E\{\int_{t_0}^{t_f}\|g_j(t)\|_{\partial G}^2 dt\} .$$

Then, noting that $g_j \in L^2(\Omega,P;L^2(T;L^2(\partial G)))$, we have

(C-15) $E\{\int_{t_0}^{t_f}\|u^m(t)\|_H^2 dt\} \leq E\{\int_{t_0}^{t_f}\|\tilde{z}^m(t)\|_H^2 dt\} \leq M$.

(C-15) implies that we can extract a subsequence $u^m \to u$ in $L^2(\Omega,P;L^2(T;H))$ weakly. By using the same procedure mentioned in Appendix-A, we may show that $u \in L^2(\Omega,P;L^2(T;H))$.

2) Mixed Boundary Condition Case

In this appendix, we restrict ourselves to the following simple but interesting mixed boundary condition

$$\text{(C-16)}\quad \{\mathcal{B}_j(t)\}_{j=1}^{\ell} = \{(\partial/\partial\nu)^{\frac{n}{2}+j-\ell-1}\}_{j=1}^{\ell}$$

and

$$\text{(C-17)}\quad \{\mathcal{B}_j(t)\}_{j=\ell+1}^{n/2} = \{(\partial/\partial\nu)^{\frac{n}{2}+j-1}\}_{j=\ell+1}^{n/2},$$

where ν is the exterior normal derivative on ∂G and we further assume

$$\text{(C-18)}\quad \ell \leq \frac{n}{4}.$$

By using Green's formula, it is easy to show that the adjoint boundary condition of $\{\mathcal{B}_j(t)\}_{j=1}^{\ell}$ becomes $\{C_j(t)\}_{j=\frac{n}{2}-\ell-1}^{n/2}$ $= \{(\partial/\partial\nu)^{n-\ell-j}\}_{j=\frac{n}{2}-\ell+1}^{n/2}$. Then, for any $\psi_1 \varepsilon H^{n/2}(G)$, $\psi_2 \varepsilon H^n(G)$ and $C_j(t)\psi_2$ $= 0$ on $T\times\partial G$ for $j=1,2,\cdots,n/2$, there exists a boundary system $\{\hat{T}_{j+\frac{n}{2}-\ell}(t)\}_{j=2\ell-\frac{n}{2}+1}^{\ell}$ such that

$$\text{(C-19)}\quad \sum_{j=1}^{n/2}(\hat{A}_j^s(t)\psi_1,\ \psi_2)_H - (\psi_1, \sum_{j=1}^{n/2}\hat{A}_j^{s*}(t)\psi_2)_H$$
$$= \sum_{j=2\ell-\frac{n}{2}+1}^{\ell}(\mathcal{B}_j(t)\psi_1,\ T_{j+\frac{n}{2}-\ell}(t)\psi_2)_{L^2(\partial G)}.$$

[Theorem-C]: For any $\psi\varepsilon\mathcal{D}(A^{D*})$, $\psi\varepsilon C(\overline{T};H)$ and $\psi(t_f)=0$, we define the following weak sense solution:

$$\text{(C-20)}\quad \int_{t_0}^{t_f}(u(t),\ -\frac{\partial\psi}{\partial t} + A^{D*}(t)\psi)_H dt + \sum_{j=1}^{n/2}\int_{t_0}^{t_f}(u(t),\ \hat{A}_j^{s*}(t)\psi)_H dw_j(t)$$
$$+ \sum_{j=2\ell-\frac{n}{2}+1}^{\ell}\int_{t_0}^{t_f}(g_j(t),\ \hat{T}_{j+\frac{n}{2}-\ell}(t)\psi)_{L^2(\partial G)} dw_j(t)$$

$$= (u_0, \psi(t_0))_H + \sum_{j=1}^{n/2} \int_{t_0}^{t} (g_j(t), \tilde{T}_j(t)\psi)_{L^2(\partial G)} dt .$$

If, in addition to the conditions of Theorem-2.2.1, Eq.(C-18) is satisfied, there exists a unique weak solution $u \in L^2(\Omega, P; L^2(\overline{T}; H))$ to Eq.(C-20).

Proof: By using the same approach as mentioned in stable condition case, the approximated system to Eq.(C-20) is given by

$$\text{(C-21)}\quad u^m(t) + \int_{t_0}^{t} A_m^D(s) u^m(s) ds + \sum_{j=1}^{n/2-\ell} \int_{t_0}^{t} \hat{A}_{jm}^s(s) u^m(s) dw_j(s)$$
$$+ \sum_{j=n/2-\ell+1}^{n/2} \int_{t_0}^{t} [\hat{A}_{jm}^s(s) u^m(s) + \tilde{g}_j^m(s)] dw_j(s)$$
$$= u_{0m} + \sum_{j=1}^{n/2} \int_{t_0}^{t} g_j^m(s) ds ,$$

where

$$\text{(C-22)}\quad \tilde{g}_j^m(s) = \sum_{k=1}^{m} (g_j(s), \hat{T}_{j+\frac{n}{2}-\ell}(s) e_k)_{L^2(\partial G)} e_k$$

$$\text{(C-23)}\quad g_j^m(s) = \sum_{k=1}^{m} (g_j(s), \tilde{T}_j(s) e_k)_{L^2(\partial G)} e_k$$

and $A_m^D(s)$ and $\hat{A}_{jm}^s(s)$ are defined by Eqs.(C-4) and (C-5), respectively. Then, by using the Itô's formula, we have

$$\text{(C-24)}\quad E\{\| u^m(t) \|_H^2\} + E\{\int_{t_0}^{t} [2\langle A^D(s) u^m(s), u^m(s)\rangle$$
$$- \sum_{j=1}^{n/2-\ell} (\hat{A}_j^s(s) u^m(s), \hat{A}_j^s(s) u^m(s))_H$$
$$- \sum_{j=n/2-\ell+1}^{n/2} \{(\hat{A}_j^s(s) u^m(s) + \tilde{g}_j^m(s), \hat{A}_j^s(s) u^m(s) + \tilde{g}_j^m(s))_H\}] ds\}$$

$$\leq E\{\| u_{0m}\|_H^2\} + 2\sum_{j=1}^{n/2} E\{\int_{t_0}^{t}(g_j^m(s),\ u^m(s))_H ds\}\ .$$

Furhtermore, it follows that, for any $\varepsilon>0$ and some $C(\varepsilon)>o$,

$$\text{(C-25)}\quad 2|(\hat{A}_j^s(s)u^m(s),\ g_j^m(s))_H|$$
$$\leq \varepsilon\|\hat{A}_j^s(s)u^m(s)\|_H^2 + C(\varepsilon)\|\tilde{g}_j^m(s)\|_H^2\ .$$

From Eq.(C-25), we have

$$\text{(C-26)}\quad E\{\| u^m(t)\|_H^2\} + E\{\int_{t_0}^{t}[2\langle A^D(s)u^m(s),\ u^m(s)\rangle$$
$$-(\sum_{j=1}^{n/2-\ell} + (1+\varepsilon)\sum_{j=n/2-\ell+1}^{n/2})(\hat{A}_j^s(s)u^m(s),\ \hat{A}_j^s(s)u^m(s))_H]ds\}$$
$$-\sum_{j=n/2-\ell+1}^{n/2} C(\varepsilon)E\{\int_{t_0}^{t}(\tilde{g}_j^m(s),\ \tilde{g}_j^m(s))_H ds\}$$
$$\leq E\{\| u_{0m}\|_H^2\} + 2\sum_{j=1}^{n/2} E\{\int_{t_0}^{t}(g_j^m(s),\ u^m(s))_H ds\}$$

On the other hand, we introduce the following system:

$$\text{(C-27)}\quad \tilde{Z}^m(t) + \int_{t_0}^{t}[A^D(s) - \frac{1}{2}(\sum_{j=1}^{n/2-\ell} + (1+\varepsilon)\sum_{j=n/2-\ell+1}^{n/2})\hat{A}_j^{s*}(s)\hat{A}_j^s(s)]\tilde{z}^m(s)ds$$
$$+\sum_{j=n/2-\ell+1}^{n/2}\sqrt{C(\varepsilon)}\int_{t_0}^{t}\tilde{g}_j^m(s)dw_j(s)$$
$$= u_{0m} + \sum_{j=1}^{n/2}\int_{t_0}^{t} g_j^m(s)ds\ .$$

with the boundary condition $\mathcal{B}_j(t)\tilde{z}^m(t)=0$ on $T\times\partial G$ for $j=1,2,\cdots,n/2$. As is mentioned in the stable condition case, from the well-known comparison theorem, we have

$$\text{(C-28)}\quad E\{\|u^m(t)\|_H^2\} \le E\{\|\tilde{z}^m(t)\|_H^2\} \text{ ,}$$

Choosing ε as a sufficiently small constant such that there exists an analytic semigroup $\Phi_\varepsilon(t,s)$ with the infinitesimal generator $A^D(t)-\frac{1}{2}\left(\sum_{j=1}^{n/2-\ell} + (1+\varepsilon)\sum_{j=n/2-\ell+1}^{n/2}\right)\hat{A}_j^{s*}(t)\hat{A}_j^s(t)$ whose domain is $\mathcal{D}(A^D)$, Eq.(C-27) becomes

$$\text{(C-29)}\quad \tilde{z}^m(t) = \Phi_\varepsilon(t,t_0)u_{0m} + \sum_{j=1}^{n/2}\int_{t_0}^{t}\Phi_\varepsilon(t,s)g_j^m(s)ds$$

$$-\sum_{j=n/2-\ell+1}^{n/2}\sqrt{C(\varepsilon)}\int_{t_0}^{t}\Phi_\varepsilon(t,s)\tilde{g}_j^m(s)dw_j(s) \text{ .}$$

By using the semigroup property, it follows that, for $K_1>0$,

$$\text{(C-30)}\quad E\{\int_{t_0}^{t_f}\|\int_{t_0}^{t}\Phi_\varepsilon(t,s)g_j^m(s)dw_j(s)\|_H^2 dt\}$$

$$= E\{\int_{t_0}^{t_f}\int_{t_0}^{t}\|\Phi_\varepsilon(t,s)\tilde{g}_j^m(s)\|_H^2 dsdt\}$$

$$\le K_1 E\{\int_{t_0}^{t_f}\int_{t_0}^{t}|t-s|^{-2\frac{\text{order}\{\hat{T}_{j+n/2-\ell}\}+1}{n}}\|g_j(s)\|_{L^2(\partial G)}^2 ds\} \text{ .}$$

Noting that $\max_{n/2-\ell+1\le j\le n/2}\text{order}\{\hat{T}_{j+n/2-\ell}\}\le\ell-1$, we have from Eq.(C-18)

$$\text{(C-31)}\quad \frac{\text{order}\{\hat{T}_{j+n/2-\ell}\}+1}{n} \le \frac{\ell}{n} \le \frac{1}{4} \text{ .}$$

Consequently, it follows that for, $K_2>0$,

$$\text{(C-32)}\quad E\{\int_{t_0}^{t_f}\|\int_{t_0}^{t}\Phi_\varepsilon(t,s)\tilde{g}_j^m(s)dw_j(s)\|_H^2 dt\} < K_2 E\{\int_{t_0}^{t_f}\|g_j(t)\|_{L^2(\partial G)}^2 dt\} \text{ .}$$

Then, we can easily show that $u^m \to u \in L^2(\Omega, P; L^2(T;H))$ weakly, by using the same approach as in Appendix-A.

The proof has been completed.

The reference with an asterisk is the work of the author.

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