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Note

Some Comments on Radioactive Decay and Gowth

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I. INTRODUCTION

Now at present the electronic computer gives good approximated solutions of the complicated differential equations on radioactive decay and growth. However, the exact solutions are yet useful in the simple cases. In this paper some conclusive solutions, which were not found in the text books and literatures, are commented. The processes for solving the equations are omitted.

II. LINEAR DECAY SERIES

1. General Solution[1)]

The differential equation

$$\frac{dN_n}{dt}=\lambda_{n-1}N_{n-1}-\lambda_n N_n$$

for the linear decay series

$$1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \cdots n \xrightarrow{\lambda_n}$$

was solved by Laplace's transformation. Here, N is number of nuclides, λ is decay constant, and t is time. The solution is

$$N_n=\frac{1}{\lambda_n}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{\lambda_j\lambda_{j+1}\cdots\lambda_n}{(\lambda_j-\lambda_i)(\lambda_{j+1}-\lambda_i)\cdots(\lambda_n-\lambda_i)}N_j^0 e^{-\lambda_i t}, \qquad (1)$$

where the symbol (^{0}) represents "at $t=0$". When a factor $(\lambda-\lambda_i)$ in the denominator in the summation becomes zero, it must be converted to unity by the promise.

2. Applications

a) *The 4th member* When $n=4$, the general solution leads to the solution

$$\begin{aligned}N_4=&\left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}N_1^0\right]e^{-\lambda_1 t}\\ &+\left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}N_1^0+\frac{\lambda_2\lambda_3}{(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}N_2^0\right]e^{-\lambda_2 t}\\ &+\left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}N_1^0\right.\\ &\qquad\left.+\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}N_2^0+\frac{\lambda_3}{\lambda_4-\lambda_3}N_3^0\right]e^{-\lambda_3 t}\end{aligned}$$

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$$+\left[\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}N_1^0\right.$$
$$\left.+\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}N_2^0+\frac{\lambda_3}{\lambda_3-\lambda_4}N_3^0+N_4^0\right]e^{-\lambda_4 t}. \qquad (2)$$

This is the typical solution, which allows to extend formally over other n values smaller or larger than 4.

b) Initial absence of daughters When $N_2^0=N_3^0=\ldots N_n^0=0$, j is valid only for $j=1$. The general equation leads to the expression

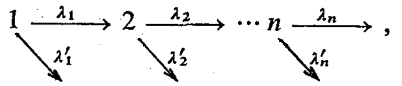
$$N_n=\frac{1}{\lambda_n}\sum_{i=1}^{n}\frac{\lambda_1\lambda_2\cdots\lambda_n}{(\lambda_1-\lambda_i)(\lambda_2-\lambda_i)\cdots(\lambda_n-\lambda_i)}N_1^0 e^{-\lambda_i t}$$
$$=\lambda_1\lambda_2\cdots\lambda_{n-1}N_1^0\sum_{i=1}^{n}\frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)(\lambda_2-\lambda_i)\cdots(\lambda_n-\lambda_i)}. \qquad (3)$$

This is the well-known Bateman's solution. If $\lambda_1\ll\lambda_2, \lambda_3,\cdots\lambda_n$, so that $e^{-\lambda_1 t}\gg e^{-\lambda_2 t}$, $e^{-\lambda_3 t},\cdots e^{-\lambda_n t}$, i is valid only for $i=1$. The above expression leads to the formula

$$\lambda_n N_n=\lambda_1 N_1^0 e^{-\lambda_1 t}=\lambda_1 N_1,$$

which means that every daughter in the decay series is in equilibrium to the parent after the sufficiently long time.

c) Branching When the members have branchings such as

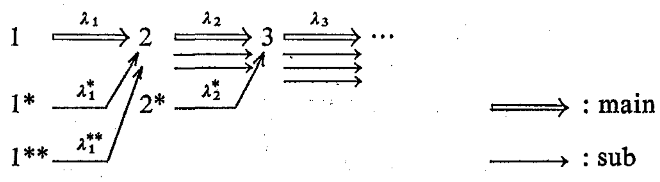


the general equation is converted to

$$N_n=\frac{1}{\lambda_n}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{\lambda_j\lambda_{j+1}\cdots\lambda_n}{(\Lambda_j-\Lambda_i)(\Lambda_{j+1}-\Lambda_i)\cdots(\Lambda_n-\Lambda_i)}N_j^0 e^{-\Lambda_i t}, \qquad (4)$$

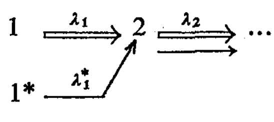
where $\Lambda=\lambda+\lambda'$.

d) Unification When the unifications such as



are included, i) at first a main chain is selected, and the general equation is applied, ii) sub-chains are constructed, and the general equation is applied putting $N^0=0$ for the member at the unification point and the members following it, and then iii) the derived equations are summed simply.

For the most simple example



the solution is as follows:

$$N_2 = \frac{\lambda_1}{\lambda_2-\lambda_1} N_1^0 e^{-\lambda_1 t} + \frac{\lambda_1^*}{\lambda_2-\lambda_1^*} N_1^{*0} e^{-\lambda_1^* t}$$
$$+ \left[\frac{\lambda_1}{\lambda_2-\lambda_1} N_1^0 + \frac{\lambda_1^*}{\lambda_1^*-\lambda_2} N_1^{*0} + N_2^0 \right] e^{-\lambda_2 t}. \tag{5}$$

For a complicated case such as

$$\begin{tikzcd}
 & 2 \arrow[dr, "\lambda_2"] \arrow[dd, Rightarrow, "\lambda_2'"] & & \\
1 \arrow[ur, Rightarrow, "\lambda_1"] \arrow[dr, "\lambda_1'"'] & & 3 \arrow[r, Rightarrow, "\lambda_3"] & \cdots, \\
 & 2^* \arrow[ur, Rightarrow, "\lambda_2^*"'] & &
\end{tikzcd}$$

for example, the solution for N_3 is as follows:

$$N_3 = \left[\frac{\lambda_1\lambda_2'\lambda_2^*}{\{(\lambda_2+\lambda_2')-(\lambda_1+\lambda_1')\}\{\lambda_2^*-(\lambda_1+\lambda_1')\}\{\lambda_3-(\lambda_1+\lambda_1')\}} \right.$$
$$+ \frac{\lambda_1'\lambda_2^*}{\{\lambda_2^*-(\lambda_1+\lambda_1')\}\{\lambda_3-(\lambda_1+\lambda_1')\}}$$
$$\left. + \frac{\lambda_1\lambda_2}{\{(\lambda_2+\lambda_2')-(\lambda_1+\lambda_1')\}\{\lambda_3-(\lambda_1+\lambda_1')\}} \right] N_1^0 e^{-(\lambda_1+\lambda_1')t}$$
$$+ \left[\left\{ \frac{\lambda_1\lambda_2'\lambda_2^*}{\{(\lambda_1+\lambda_1')-(\lambda_2+\lambda_2')\}\{\lambda_2^*-(\lambda_2+\lambda_2')\}\{\lambda_3-(\lambda_2+\lambda_2')\}} \right. \right.$$
$$\left. + \frac{\lambda_1\lambda_2}{\{(\lambda_1+\lambda_1')-(\lambda_2+\lambda_2')\}\{\lambda_3-(\lambda_2+\lambda_2')\}} \right\} N_1^0$$
$$\left. + \left\{ \frac{\lambda_2'\lambda_2^*}{\{\lambda_2^*-(\lambda_2+\lambda_2')\}\{\lambda_3-(\lambda_2+\lambda_2')\}} + \frac{\lambda_2}{\lambda_3-(\lambda_2+\lambda_2')} \right\} N_2^0 \right] e^{-(\lambda_2+\lambda_2')t}$$
$$+ \left[\left\{ \frac{\lambda_i\lambda_2'\lambda_2^*}{\{(\lambda_1+\lambda_1')-\lambda_2^*\}\{(\lambda_2+\lambda_2')-\lambda_2^*\}\{\lambda_3-\lambda_2^*\}} \right. \right.$$
$$\left. + \frac{\lambda_1'\lambda_2^*}{\{(\lambda_1+\lambda_1')-\lambda_2^*\}\{\lambda_3-\lambda_2^*\}} \right\} N_1^0$$
$$\left. + \frac{\lambda_2'\lambda_2^*}{\{(\lambda_2+\lambda_2')-\lambda_2^*\}\{\lambda_3-\lambda_2^*\}} N_2^0 + \frac{\lambda_2^*}{\lambda_3-\lambda_2^*} N_2^{*0} \right] e^{-\lambda_2^* t}$$
$$+ \left[\left\{ \frac{\lambda_1\lambda_2'\lambda_2^*}{\{(\lambda_1+\lambda_1')-\lambda_3\}\{(\lambda_2+\lambda_2')-\lambda_3\}\{\lambda_2^*-\lambda_3\}} \right. \right.$$
$$\left. + \frac{\lambda_1'\lambda_2^*}{\{(\lambda_1+\lambda_1')-\lambda_3\}\{\lambda_2^*-\lambda_3\}} + \frac{\lambda_1\lambda_2}{\{(\lambda_1+\lambda_1')-\lambda_3\}\{(\lambda_2+\lambda_2')-\lambda_3\}} \right\} N_1^0$$
$$+ \left\{ \frac{\lambda_2'\lambda_2^*}{\{(\lambda_2+\lambda_2')-\lambda_3\}\{\lambda_2^*-\lambda_3\}} + \frac{\lambda_2}{(\lambda_2+\lambda_2')-\lambda_3} \right\} N_2^0 + \frac{\lambda_2^*}{\lambda_2^*-\lambda_3} N_2^{*0}$$
$$\left. + N_3^0 \right] e^{-\lambda_3 t}. \tag{6}$$

III. OVERLAPPING OF NUCLEAR REACTIONS[2)]

The rate of nuclear reaction is expressed as $-(\mathrm{d}N/\mathrm{d}t)=rN$, which is the same form as for the decay, $-(\mathrm{d}N/\mathrm{d}t)=\lambda N$. Here, r is rate constant. If the direction of reaction is also the same as for the decay, r is added on λ simply in the decay and growth calculations. However, if the direction is opposite as in the case

$$1 \underset{\beta^-}{\overset{(n,p)}{\rightleftarrows}} 2\,,$$

for example, the solution of the differential equation is difficult, and the general solution is not given. Only typical and simple cases are described below. For simplicity r is put as λ.

1. First Member Reproduced

The differential equations for the case

$$1 \underset{\lambda_2^0}{\overset{\lambda_1}{\rightleftarrows}} 2 \xrightarrow{\lambda_2}$$

are

$$\frac{dN_1}{dt}=\lambda_2^0 N_2-\lambda_1 N_1$$

$$\frac{dN_2}{dt}=\lambda_1 N_1-(\lambda_2+\lambda_2^0)N_2\,.$$

The solution is as follows:

$$N_1=\frac{\lambda_2^0}{\mu_1-\mu_2}\left[\left\{\frac{1}{\lambda_1+\mu_1}N_1^0+N_2^0\right\}e^{\mu_1 t}-\left\{\frac{\lambda_1}{\lambda_1+\mu_2}N_1^0+N_2^0\right\}e^{\mu_2 t}\right] \tag{7}$$

$$N_2=\frac{1}{\mu_1-\mu_2}\left[(\lambda_1+\mu_1)\left\{\frac{\lambda_1}{\lambda_1+\mu_1}N_1^0+N_2^0\right\}e^{\mu_1 t}-(\lambda_1+\mu_2)\left\{\frac{\lambda_1}{\lambda_1+\mu_2}N_1^0+N_2^0\right\}e^{\mu_2 t}\right], \tag{8}$$

where μ_1 and μ_2 are two roots of the equation

$$\mu^2+(\lambda_1+\lambda_2+\lambda_2^0)\mu+\lambda_1\lambda_2=0. \tag{9}$$

2. Second Member Reproduced

The solution for the case

$$1 \xrightarrow{\lambda_1} 2 \underset{\lambda_2^0}{\overset{\lambda_2}{\rightleftarrows}} 3 \xrightarrow{\lambda_3}$$

is as follows:

$$\left.\begin{aligned} N_1&=N_1^0 e^{-\lambda_1 t}\\ N_2&=\frac{\lambda_1}{\lambda_2-\lambda_1}N_1^0 e^{-\lambda_1 t}+\lambda_2^0\left\{\frac{A}{\lambda_2+\mu_3}e^{\mu_3 t}+\frac{B}{\lambda_2+\mu_4}e^{\mu_4 t}+\frac{C}{\lambda_2-\lambda_1}e^{-\lambda_1 t}\right\}\\ N_3&=Ae^{\mu_3 t}+Be^{\mu_4 t}+Ce^{-\lambda_1 t}\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} A&=\alpha+\frac{\lambda_1\lambda_2 N_1^0}{(\lambda_1+\mu_3)(\mu_3-\mu_4)} \\ B&=\beta+\frac{\lambda_1\lambda_2 N_1^0}{(\lambda_1+\mu_4)(\mu_3-\mu_4)} \\ C&=\frac{\lambda_1\lambda_2 N_1^0}{(\lambda_1+\mu_3)(\lambda_1+\mu_4)} \end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned} \alpha&=\frac{1}{\mu_3-\mu_4}\{\lambda_2 N_2^0+(\lambda_2+\mu_3)N_3^0\} \\ \beta&=\frac{1}{\mu_4-\mu_3}\{\lambda_2 N_2^0+(\lambda_2+\mu_4)N_3^0\} \end{aligned}\right\} \quad (12)$$

$$\mu_3,\ \mu_4\text{: Two roots of } \mu^2+(\lambda_2+\lambda_2^0+\lambda_3)\mu+\lambda_2\lambda_3=0 \quad (13)$$

REFERENCES

(1) K. Otozai and M. Matsuoka, unpublished. (Under the guide of Prof. K. Tamada, Faculty of Engineering, Kyoto University)

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