Bull. Inst. Chem. Res., Kyoto Univ., Vol. 57, No. 1, 1979

Three-Particle Relativistic Kinematics

Shigeru KAKIGI*, Kiyoji FUKUNAGA*,
Takao OHSAWA**, and Seiji TANAKA**

Received October 11, 1978

For reactions leading to three particles in the final state, five independent variables are assigned to the polar and azimuthal angles of two particles and the kinetic energy of one of them. Formulas are given for calculations of other variables from these five. Formulas are also given for transformations of polar and azimuthal angles from the laboratory system to the rest frames of two-particle systems. A FORTRAN programme using these formulas is included in an appendix.

KEY WORDS Three-particle kinematics / Relativistic formulas / FORTRAN programme /

I. INTRODUCTION

For three-particle reactions of the form $a+b\rightarrow 1+2+3$, five independent variables are required to define the final state completely. In experiments of two-particle correlation measurements, two detectors are set at angles (θ_1, ϕ_1) and (θ_2, ϕ_2) respectively and kinetic energies T_1 and T_2 of particles 1 and 2 are measured in coincidence. Therefore, quantities T_1, θ_1, ϕ_1, θ_2, and ϕ_2 are taken as five independent variables and differential cross sections $d^5\sigma/dT_1d\Omega_1d\Omega_2$ are measured as functions of these variables. Kinetic energy T_2, an extra variable, can be used to identify true events of the reaction.

Coincident energy spectra are characterized by relative kinetic energies between two final-state particles. These energies are calculated from the five independent variables. For the purpose of doing it, nonrelativistic formulas are summarized in Ref. 1. Since two-particle correlation experiments are performed frequently at intermediate incident energies, relativistic calculations are done in this report. Moreover the following matters are examined. First, quasi-free scattering proceeds in two ways. One is that the target b consists of two particles and one of them collides with the projectile a with leaving the rest particle as a spectator. The other is that the projectile a consists of two particles and one of them collides with the target b. In the former, quasi-free scattering occurs near the energy corresponding to the spectator kinetic energy of zero in the target system and in the latter, of zero in the beam system. These two processes of quasi-free scattering are expected to occur especially when particles a and b are identical.[2] Then, kinetic energies T_i of particles i are calculated in both the target and

* 柿木 茂, 福永清二: Nuclear Science Research Facility, Institute for Chemical Research, Kyoto University, Kyoto.

** 大沢孝夫, 田中誠二: Laboratory of Nuclear Reaction, Institute for Chemical Research, Kyoto University, Kyoto.

beam systems. Secondly, in angular-correlation experiments, differential cross sections $d^5\sigma/dT_1d\Omega_1d\Omega_2$ are measured as functions of angles (θ_2, ϕ_2) of particle 2 if one assigns particle 1 to the particle emitted from the production process of the (23) system and particle 2 to the succeeding decaying one. Angular correlation functions are obtained through the transformation of differential cross sections to the ones in the rest frame of (23) system. Several times, according to physical or experimental requirements, detector 2 is placed off the reaction plane which is defined by the direction of detector 1 and that of incident beam. Then transformation equations for the azimuthal angle ϕ_2 as well as for the polar angle θ_2 are required.

II. CALCULATION FORMULAS

1. Reference Systems

Four types of reference system[3] are used to describe kinetic motions of particles, that is, laboratory system (LS), overall center-of-momentum system (CMS), beam system (BS) and rest frames of two-particle system (R12, R23, and R31). Target b is assumed to rest in the LS and therefore the target system is identical with the LS. The CMS quantities are denoted by an asterisk and the Rij quantities by an index Rij. An index L for the LS quantities is omitted for simplicity.

The z-axis of the LS is defined as the momentum of projectile and the z-axis in the CMS is defined as the direction parallel to the LS one. The zx-plane ($\phi=0$) can be defined arbitrary.

2. Kinetic Energies

Quantities to describe the motion of particle i are as follows: the total energy E_i, the momentum $\mathbf{P}_i$, the rest mass m_i, the kinetic energy T_i, and the velocity $\mathbf{v}_i$. The momentum vector is expressed in terms of its absolute value P_i, the polar angle θ_i, and the azimuthal angle ϕ_i in the polar coordinate system. Among these quantities exist the following relations.

$$E_i^2=P_i^2+m_i^2, \tag{1}$$
$$E_i=T_i+m_i, \tag{2}$$
$$P_i^2=T_i^2+2m_iT_i, \tag{3}$$
$$\mathbf{v}_i=\mathbf{P}_i/E_i. \tag{4}$$

The similar relations hold for a two-particle system consisting of particles i and j with definition of its energy E_{ij}, momentum $\mathbf{P}_{ij}$ and invariant mass M_{ij} as

$$E_{ij}=E_i+E_j, \tag{5}$$
$$\mathbf{P}_{ij}=\mathbf{P}_i+\mathbf{P}_j, \tag{6}$$
$$M^2_{ij}=(E_i+E_j)^2-(\mathbf{P}_i+\mathbf{P}_j)^2. \tag{7}$$

For the total system its energy E_0, momentum $\mathbf{P}_0$ and invariant mass M_0 are calculated from the quantities of the initial state with assigning the particle a to the projectile and the particle b to the target. They are written as

$$E_0=E_a+E_b=T_a+m_a+m_b, \tag{8}$$
$$\mathbf{P}_0=\mathbf{P}_a+\mathbf{P}_b=\mathbf{P}_a, \tag{9}$$

$$P_0=P_a=(T_a^2+2m_aT_a)^{1/2}, \tag{10}$$
$$M_0^2=E_0^2-P_0^2=(m_a+m_b)^2+2m_bT_a. \tag{11}$$

If, for the particle 1, the kinetic energy T_1 and the angles (θ_1, ϕ_1) are known in the LS, the (23) system consisting of particles 2 and 3 is uniquely determined owing to the energy-momentum conservation equations

$$E_0-E_1=E_2+E_3, \tag{12}$$
$$\mathbf{P}_0-\mathbf{P}_1=\mathbf{P}_2+\mathbf{P}_3. \tag{13}$$

Then kinematics for two-particle reactions[4] can be applied to solve P_2 as a function of the angles (θ_2, ϕ_2). The calculacions are performed as follows. First, in the R23, the momentum and the energy of particle 2 are independent of the angles and are expressed in terms of the invariant mass M_{23}:

$$P_2^{R23}=\lambda^{1/2}(M_{23}^2,\ m_2^2,\ m_3^2)/(2M_{23}), \tag{14}$$
$$E_2^{R23}=(M_{23}^2+m_2^2-m_3^2)/(2M_{23}), \tag{15}$$

where

$$\begin{aligned} M_{23}^2&=(E_2+E_3)^2-(\mathbf{P}_2+\mathbf{P}_3)^2\\ &=(E_0-E_1)^2-(\mathbf{P}_0-\mathbf{P}_1)^2\\ &=M_0^2+m_1^2-2E_0E_1+2P_0P_1\cos\theta_1 \end{aligned} \tag{16}$$

and the function $\lambda(x, y, z)$ is defined as

$$\lambda(x,\ y,\ z)=x^2+y^2+z^2-2xy-2yz-2zx. \tag{17}$$

Secondly, the momentum of particle 2 in the LS is calculated from the momentum and energy in the R23 through the Lorentz transformation between these systems. The transformation is done along the z′-axis which is defined as $\mathbf{P}_{23}$ in the LS. The velocity $\mathbf{v}_{23}$ of the R23 in the LS and the associated Lorentz factor γ_{23} are given by

$$\mathbf{v}_{23}=\mathbf{P}_{23}/E_{23}, \tag{18}$$
$$\gamma_{23}=E_{23}/M_{23}, \tag{19}$$
$$\gamma_{23}v_{23}=P_{23}/M_{23}, \tag{20}$$

where

$$E_{23}=E_0-E_1, \tag{21}$$
$$P_{23}=|\mathbf{P}_0-\mathbf{P}_1|=(P_0^2+P_1^2-2P_0P_1\cos\theta_1)^{1/2}. \tag{22}$$

The angle θ_{23} of (23) system with respect to the z-axis is given through the momentum conservation, Eq. (13), as follows:

$$\sin\theta_{23}=P_1\sin\theta_1/P_{23}, \tag{23}$$
$$\cos\theta_{23}=(P_0-P_1\cos\theta_1)/P_{23}. \tag{24}$$

Now, the Lorentz transformation equation solved for P_2 is written as

$$\begin{aligned} E_2^{R23}&=-\gamma_{23}v_{23}P_2\cos\theta'_2+\gamma_{23}E_2\\ &=-\gamma_{23}v_{23}P_2\cos\theta'_2+\gamma_{23}(P_2^2+m_2^2)^{1/2}. \end{aligned} \tag{25}$$

The angle θ'_2 is measured with respect to the z′-axis and $\cos\theta'_2$ is expressed by

$$\begin{aligned} \cos\theta'_2&=\cos\theta_2\cos\theta_{23}-\sin\theta_2\cos(\phi_2-\phi_1)\sin\theta_{23},\\ &=(P_0\cos\theta_2-P_1\cos\theta_{1-2})/P_{23}, \end{aligned} \tag{26}$$

where

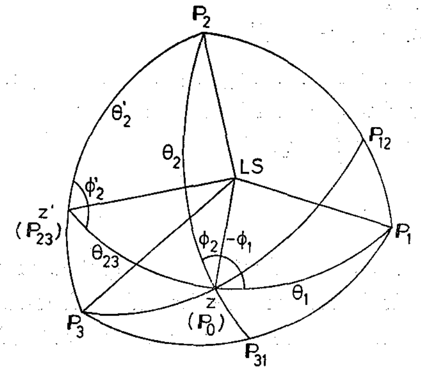

Fig. 1. Angle coordinate in the laboratory system.

$$\cos\theta_{1-2} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1). \tag{27}$$

The z'x'-plane ($\phi'=0$) is defined as the plane including the z-axis (Fig. 1). Finally, the solution P_2 is given by

$$P_2^{\pm} = P_2^{R23}(B\pm\sqrt{D})/A, \tag{28}$$

with

$$D = \gamma_{23}^2(1-g^2) + g^2(\gamma_{23}/\gamma_2^{R23})^2\cos^2\theta'_2, \tag{29}$$

$$A = \gamma_{23}(1-v_{23}^2\cos^2\theta'_2), \tag{30}$$

$$B = g\cos\theta'_2, \tag{31}$$

$$g = v_{23}/v_2^{R23}, \tag{32}$$

where v_2^{R23} is the velocity of particle 2 in the R23 and and γ_2^{R23} is the associated Lorentz factor and they are given by

$$v_2^{R23} = P_2^{R23}/E_2^{R23}, \tag{33}$$

$$\gamma_2^{R23} = E_2^{R23}/m_2. \tag{34}$$

Concerning the existence of solutions $P_2^{\pm}$, one has two cases depending on the relative magnitudes of v_{23} and v_2^{R23}. (1) If $1>g$, always $D>0$ as found from Eq. (29) and, however, $D^{1/2}>|B|$ and consequently $P_2^{-}<0$. The latter relation is found from the equation

$$B^2 - D = (g^2-1)\gamma_{23}^2(1-v_{23}^2\cos^2\theta'_2). \tag{35}$$

Then, one has only the solution P_2^{+}. (2) If $1\leqq g$, the signs of D and $\cos\theta'_2$ should be examined. (i) If $D\geqq 0$ and $\cos\theta'_2\geqq 0$, one has two solutions $P_2^{\pm}$. (ii) If $D\geqq 0$ but $\cos\theta'_2 < 0$, one has no physical solution because $P_2^{\pm}$ are always found to be negative as seen from Eqs. (31) and (35). (iii) If $D<0$, regardless of the sign of $\cos\theta'_2$, one has no solution. The energy E_2 is given by

$$E_2 = (P_2^2 + m_2^2)^{1/2}. \tag{36}$$

For the particle 3, the energy E_3 and momentum P_3 are calculated from the quantities for the particles 1 and 2 through the energy and momentum conservation equations ;

$$E_3 = E_0 - E_1 - E_2, \tag{37}$$

$$\mathbf{P}_3=\mathbf{P}_0-\mathbf{P}_1-\mathbf{P}_2. \tag{38}$$

From the latter equations, P_3 is derived as follows.

$$P_3=(P_0{}^2+P_1{}^2+P_2{}^2-2P_0P_1\cos\theta_1-2P_0P_2\cos\theta_2+2P_1P_2\cos\theta_{1\text{-}2})^{1/2}. \tag{39}$$

The kinetic energy T_i of particle i is given by

$$T_i=E_i-m_i. \tag{40}$$

The kinetic energies of the relative motions between the particles j and k and between the particle i and the (jk) system are given by

$$T_{j\text{-}k}=M_{jk}-m_j-m_k \tag{41}$$

and

$$T_{i\text{-}jk}=M_0-m_i-M_{jk} \tag{42}$$

respectively. M_{jk} are obtained by cyclic permutations of indexes in Eq. (16).

Once the energy E_i is obtained in the LS, the energy $E_i{}^B$ in the BS can be calculated through the Lorentz transformation between these systems with $\gamma_a=E_a/m_a$ and $\gamma_a v_a=P_a/m_a$ and is given by

$$E_i{}^B=(E_aE_i-P_aP_i\cos\theta_i)/m_a. \tag{43}$$

3. Polar and Azimuthal Angles

In order to evaluate an azimuthal angle ϕ, both values of $\sin\phi$ and $\cos\phi$ are needed and on the contrary a polar angle θ is uniquely evaluated from a value of $\cos\theta$. However, in practical calculations with a computer, it is frequently convenient to use both values $\sin\theta$ and $\cos\theta$ for evaluating θ. In the following both formulas for $\sin\theta$ and $\cos\theta$ are given.

The LS angles (θ_3, ϕ_3) of particle 3 are calculated from the quantities for the particles 1 and 2 through the momentum conservation, Eq. (38), and are given by

$$\sin\theta_3=(P_1{}^2\sin^2\theta_1+P_2{}^2\sin^2\theta_2+2P_1P_2\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1))^{1/2}/P_3, \tag{44}$$

$$\cos\theta_3=(P_0-P_1\cos\theta_1-P_2\cos\theta_2)/P_3, \tag{45}$$

$$\sin\phi_3=(-P_1\sin\theta_1\sin\phi_1-P_2\sin\theta_2\sin\phi_2)/(P_3\sin\theta_3), \tag{46}$$

$$\cos\phi_3=(-P_1\sin\theta_1\cos\phi_1-P_2\sin\theta_2\cos\phi_2)/(P_3\sin\theta_3). \tag{47}$$

Practically the factor $P_3\sin\theta_3$ in Eqs. (46) and (47) is not needed because ϕ_3 is evaluated through the ratio $\tan\phi_3=\sin\phi_3/\cos\phi_3$ associated with an examination of the sign of $\sin\phi_3$ and $\cos\phi_3$ ($\sin\theta_3\geqq 0$). For calculation of θ_3 through Eq. (45) only, the value of P_3 is given by Eq. (39).

The CMS angles $(\theta_i{}^*, \phi_i{}^*)$ are calculated from the LS ones through the Lorentz transformation equations with $\gamma_0=E_0/M_0$ and $\gamma_0v_0=P_0/M_0$ which are evaluated with Eqs. (8), (10) and (11), and are given by

$$\sin\theta_i{}^*=P_i\sin\theta_i/P_i{}^*, \tag{48}$$

$$\cos\theta_i{}^*=(E_0P_i\cos\theta_i-P_0E_i)/(M_0P_i{}^*), \tag{49}$$

$$\phi_i{}^*=\phi_i. \tag{50}$$

If one calculates $\theta_i{}^*$ through Eq. (49) only, the value of $P_i{}^*$ is needed and is given by

$$P_i{}^*=\lambda^{1/2}(M^2{}_{jk}, M_0{}^2, m_i{}^2)/(2M_0). \tag{51}$$

In the Rjk, the z-axis is defined as the direction opposite to the momentum of particle i, that is, $-\mathbf{P}_i^{Rjk}$. Then the polar angle θ_j^{Rjk} equals to $\pi-\theta_{i\text{-}j}^{Rjk}$ and is expressed in terms of the invariant mass M_{ij} and the energies and momenta of particles i and j in the Rjk as follows:

$$\cos\theta_j^{Rij}=(M^2_{ij}-m_i^2-m_j^2-2E_i^{Rjk}E_j^{Rjk})/(2P_i^{Rjk}P_j^{Rjk}), \qquad (52)$$

where

$$E_i^{Rjk}=(M_0^2-M_{jk}^2-m_i^2)/(2M_{jk}), \qquad (53)$$
$$P_i^{Rjk}=\lambda^{1/2}(M_0^2,\ M_{jk}^2,\ m_i^2)\,(2M_{jk}) \qquad (54)$$

and E_j^{Rjk} and P_j^{Rjk} are obtained by cyclic permutations of indexes in Eqs. (14) and (15). The Rjk azimuthal angle ϕ_j^{Rjk} of particle j is calculated from the LS angles through three steps of transformations. The first is the Lorentz transformation between the LS and the CMS along the z-axis. The azimuthal angle is invariant, that is, $\phi_j=\phi_j^*$. The second is a rotation of coordinate system in the CMS which transform the z-axis to the z″-axis. The latter axis is defined as $\mathbf{P}_{jk}^*=-\mathbf{P}_i^*$ and the rotation angle equals to the angle $\theta_{jk}^*=\pi-\theta_i^*$. By using formulas of spherical trigonometry, ϕ_j^* is transformed to ϕ_j'' through the following equations.

$$\sin\theta_j''\sin\phi_j''=\sin\theta_j^*\sin(\phi_j^*-\phi_i^*) \qquad (55)$$
$$\sin\theta_j''\cos\phi_j''=\cos\theta_j^*\sin\theta_i^*-\sin\theta_j^*\cos\theta_i^*\cos(\phi_j^*-\phi_i^*). \qquad (56)$$

The z″x″-plane ($\phi''=0$) is defined as the plane including the z-axis (Fig. 2). The last is a Lorentz transformation between the CMS and the Rjk along the z″-axis and the equations are as follows:

$$P_j^{Rjk}\sin\theta_j^{Rjk}=P_j^*\sin\theta_j'', \qquad (57)$$
$$\phi_j^{Rjk}=\phi_j''. \qquad (58)$$

By substitution of Eqs. (48), (49), (50), (57) and (58) into Eqs. (55) and (56), the following equations are obtained:

$$\sin\phi_j^{Rjk}=P_j\sin\theta_j\sin(\phi_j-\phi_i)/(P_j^{Rjk}\sin\theta_j^{Rjk}), \qquad (59)$$

$$\cos\phi_j^{Rjk}=[(E_0P_j\cos\theta_j-P_0E_j)P_i\sin\theta_i-(E_0P_i\cos\theta_i-P_0E_i)P_j\sin\theta_j\cos(\phi_j-\phi_i)]$$
$$\times(M_0P_i^*)^{-1}(P_j^{Rjk}\sin\theta_j^{Rjk})^{-1}. \qquad (60)$$

Concerning to the factor $(P_j^{Rjk}\sin\theta_j^{Rjk})$, is noted the same fact as mentioned about Eqs. (46) and (47).

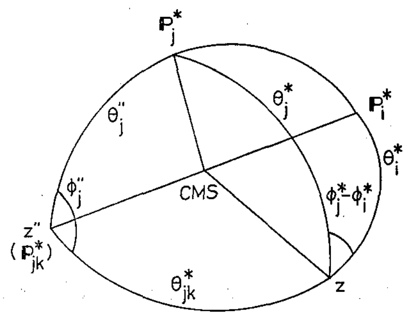

Fig. 2. Angle coordinate in the center-of-momentum system.

4. Phase Space Factor

If the differential cross section is written in a form as follows[1]:

$$d^5\sigma/dT_1d\Omega_1d\Omega_2=(2\pi/\hbar v_a)\rho_1(E_1)|M|^2, \tag{61}$$

then the phase space factor $\rho_1(E_1)$ is given by

$$\rho_1(E_1)=(2\pi\hbar)^{-6}P_1P_2E_1E_2E_3|E_2+E_3-E_2(\mathbf{P}_0-\mathbf{P}_1)\cdot\mathbf{P}_2/P_2^2|^{-1}. \tag{62}$$

This equation corresponds to the following definition:

$$\rho_1(E_1)dT_1d\Omega_1d\Omega_2=(2\pi\hbar)^{-6}\int d^3\mathbf{P}_1d^3\mathbf{P}_2d^3\mathbf{P}_3 \times\delta^3(\mathbf{P}_0-\mathbf{P}_1-\mathbf{P}_2-\mathbf{P}_3)\delta(E_0-E_1-E_2-E_3) \tag{63}$$

with the integrations over the variables P_3, θ_3, ϕ_3, and P_2. This expression is a non-invariant form and differs from a Lorentz-invariant one by inclusion of a factor of $8E_1E_2E_3$. In the R23, the phase space factor is calculated in the form:

$$\rho_1(E_1^{R23})=(2\pi\hbar)^{-6}P_1^{R23}P_2^{R23}E_1^{R23}E_2^{R23}E_3^{R23}/M_{23}. \tag{64}$$

Then the Jacobian $\partial(T_1, \Omega_1, \Omega_2)/\partial(T_1^{R23}, \Omega_1^{R23}, \Omega_2^{R23})$ for the transformation of the differential cross section $d^5\sigma/dT_1d\Omega_1d\Omega_2$ from the LS to the R23 is given by

$$\partial(T_1, \Omega_1, \Omega_2)/\partial(T_1^{R23}, \Omega_1^{R23}, \Omega_2^{R23})=\rho_1(E_1^{R23})/\rho_1(E_1). \tag{65}$$

III. PROGRAMME

A FORTRAN programme is given in an appendix. The input quantities are the masses of five particles participating in the reaction: m_a, m_b, m_1, m_2, and m_3 (in AMU), the kinetic energy of the projectile in the laboratory system: T_a (in MeV) and the polar and azimuthal angles of the particles 1 and 2 in the laboratory system: θ_1, ϕ_1, θ_2, and ϕ_2 (in degrees). In calculations the kinetic energies of particle 1: T_1 (in MeV) are given successively by a starting value plus a increment value multiplied by integers. These values are also given in the input data. In the programme is put a restriction that a given value of T_1 should satisfy a limit $M_{23}-m_2-m_3\geq 0$, before the examinations mentioned in the preceding section. The output quantities are represented in a matrix form as used in Ref. 1.

ACKNOWLEDGMENTS

The authors would like to express their thanks to Dr. N. Fujiwara and Mr. A. Okihana for their useful discussions.

The calculations were made with FACOM 230-48 computer at the Institute for Chemical Research of Kyoto University.

REFERENCES

(1) G. G. Ohlsen, *Nucl. Instr. and Meth.*, **37**, 240 (1965).
(2) K. Fukunaga, N. Fujiwara, S. Kakigi, T. Ohsawa, H. Nakamura-Yokota, S. Tanaka, A. Okihana, T. Sekioka, T. Higo, and T. Miyanaga, *J. Phys. Soc. Japan*, **45**, 1783 (1978).
(3) E. Byckling and K. Kajantie, "Particle Kinematics", John Wiley & Sons, London, 1972, p.19.
(4) E. Byckling and K. Kajantie, *ibid.*, p. 36.

APPENDIX

```
SOURCE    LIST

      C     THREE BODY RELATIVISTIC KINEMATICS
      C
      C     P=PROJECTILE
      C     T=TARGET
      C     1=DETECTED AT (TH1L,PH1L)
      C     2=DETECTED AT (TH2L,PH2L)
      C     3=UNDETECTED
      C
      C     L   =LAB.
      C     C   =CM OF TOTAL SYSTEM
      C     RIJ=CM OF (IJ)SYSTEM
      C     B   =BEAM
      C
      C     XM  =REST MASS (AMU)
      C     T   =KINETIC ENERGY (MEV)
      C     TH  =POLAR ANGLE (DEG.)
      C     PH  =AZIMUTHAL ANGLE (DEG.)
      C     TI-J=RELATIVE ENERGY OF PARTICLES I-J
      C     RHO1=PHASE SPACE (T1L)
      C     PHO2=PHASE SPACE (T1L,T2L)
      C     GR23=JACOBIAN FOR TRANSFORMATION L-R23
      C
            COMMON/DIN1/XMP,XMT,XM1,XM2,XM3,TPL,TH1L,TH2L,PH1L,PH2L
            COMMON/DIN2/START,STEP
            DOUBLE PRECISION XMP,XMT,XM1,XM2,XM3
            READ(5,1000) XMP
            READ(5,1000) XMT
            READ(5,1010) TPL
            READ(5,1020) START,STEP
         10 READ(5,1000) XM1
            READ(5,1000) XM2
            READ(5,1000) XM3
            IF(XM1.EQ.0.0) GO TO 900
         20 READ(5,1020) TH1L,PH1L
            IF(TH1L.EQ.360.0) GO TO 10
         30 READ(5,1020) TH2L,PH2L
            IF(TH2L.EQ.360.0) GO TO 20
            CALL KINEMA
            CALL PRINT
            GO TO 30
        900 CONTINUE
            STOP
       1000 FORMAT(D14.8)
       1010 FORMAT(F6.2)
       1020 FORMAT(2F7.2)
            END
```

```
            SUBROUTINE KINEMA
            COMMON/DIN1/XMP,XMT,XM1,XM2,XM3,TPL,TH1L,TH2L,PH1L,PH2L
            COMMON/DIN2/START,STEP
            COMMON/DOUT1/T1L(300,2),T2L(300,2),T3L(300,2)
           X,T23(300),T31(300,2),T12(300,2),TH3L(300,2),PH3L(300,2)
           X,TH1C(300),TH2C(300,2),TH3C(300,2),RHO1(300,2),RHO2(300,2)
           X,TH2R23(300,2),TH3R31(300,2),TH1R12(300,2)
           X,PH2R23(300,2),PH3R31(300,2),PH1R12(300,2)
           X,GRCM(300,2)
           X,T1B(300),T2B(300,2),T3B(300,2)
            COMMON/DOUT2/RQV,E0,P0,RS0
            COMMON/LNXICD/LMIN,LMAX,ICODE,LCODE
            DOUBLE PRECISION XMP,XMT,XM1,XM2,XM3,YMP,YMT,YM1,YM2,YM3,Z1,Z2,Z3
           X,ZP,ZT
           X,AUMV,ARG,PAI
           X,TH1,TH2,TH3,PH1,PH2,PH3,TH1CM,TH2CM,TH3CM
           X,E0,P0,S0,RS0,RQV
           X,P1,P2,P3,E1,E2,E3,E23,P23,V23,G23,COTH12,COTH2P
           X,E1R23,E2R23,E3R23,P1R23,P2R23,V2R23,G2R23
           X,S23,S31,S12,TA1,TA2,TA3
           X,A,B,D,VR,DVR,RT23
           X,THCN,THRCM,PHRCM,ARCTAN,RAMDA,X,Y,Z
           X,RH1,RH1RCM,FACT,EFACT1,EFACT2
           X,SI,CO,BNS,BNB
            RAMDA(X,Y,Z)=X**2+Y**2+Z**2-2.0*X*Y-2.0*Y*Z-2.0*Z*X
            AUMV=931.504
            ARG=-1.0
            PAI=DARCOS(ARG)
            YMP=XMP*AUMV
            YMT=XMT*AUMV
            YM1=XM1*AUMV
            YM2=XM2*AUMV
            YM3=XM3*AUMV
            ZP=YMP**2
            ZT=YMT**2
            Z1=YM1**2
            Z2=YM2**2
            Z3=YM3**2
            TH1=TH1L*PAI/180.0
            TH2=TH2L*PAI/180.0
            PH1=PH1L*PAI/180.0
            PH2=PH2L*PAI/180.0
            E0=YMP+YMT+TPL
            P0=DSQRT(TPL**2+2.0*YMP*TPL)
            S0=E0**2-P0**2
            RS0=DSQRT(S0)
            EP=YMP+TPL
            RQV=YMP+YMT-YM1-YM2-YM3
            ICODE=0
            LCODE=0
            LMIN=1
            LMAX=300
            DO 505 L=1,300
            XL=L-1
            T1=START+STEP*XL
            E1=T1+YM1
         10 P1=DSQRT(T1**2+2.0*YM1*T1)
            S23=S0+Z1-2.0*E0*E1+2.0*P0*P1*DCOS(TH1)
            RT23=DSQRT(S23)-YM2-YM3
            IF(RT23) 60,15,15
         15 CONTINUE
            E23=E0-E1
            P23=DSQRT(P0**2+P1**2-2.0*P0*P1*DCOS(TH1))
            V23=P23/E23
            G23=E23/DSQRT(S23)
            P1R23=DSQRT(RAMDA(S0,S23,Z1))/(2.0*DSQRT(S23))
            P2R23=DSQRT(RAMDA(S23,Z2,Z3))/(2.0*DSQRT(S23))
            E1R23=(S0-S23-Z1)/(2.0*DSQRT(S23))
            E2R23=(S23+Z2-Z3)/(2.0*DSQRT(S23))
            E3R23=(S23+Z3-Z2)/(2.0*DSQRT(S23))
            V2R23=P2R23/E2R23
            G2R23=E2R23/YM2
            VR=V23/V2R23
            COTH12=DCOS(TH1)*DCOS(TH2)+DSIN(TH1)*DSIN(TH2)*DCOS(PH2-PH1)
            COTH2P=(P0*DCOS(TH2)-P1*COTH12)/P23
            D=(G23**2)*(1.0-VR**2)+(VR**2)*(COTH2P**2)+(G23/G2R23)**2
            A=G23*(1.0-(V23**2)*(COTH2P**2))
            B=VR*COTH2P
            DVR=1.0-VR
            IF(DVR) 25,40,40
         25 IF(COTH2P) 60,30,30
         40 LOWCD=0
            T1L(L,1)=T1
            T1L(L,2)=0.0
            GO TO 90
         30 IF(D) 60,50,50
         50 LOWCD=1
            LCODE=1
            T1L(L,1)=T1
            T1L(L,2)=T1
```

```
      GO TO 90
   60 CONTINUE
      IF(ICODE.EQ.0) GO TO 500
      LMAX=L-1
      GO TO 600
   90 IF(ICODE.EQ.1) GO TO 80
      LMIN=L
      ICODE=1
   80 K=1
      P2=P2R23*(B+DSQRT(D))/A
      GO TO 100
   95 K=2
      P2=P2R23*(B-DSQRT(D))/A
  100 CONTINUE
      E2=DSQRT(P2**2+Z2)
      P3=DSQRT(P0**2+P1**2+P2**2-2.0*P0*P1*DCOS(TH1)
     X-2.0*P0*P2*DCOS(TH2)+2.0*P1*P2*COTH12)
      E3=DSQRT(P3**2+Z3)
      S31=S0+Z2-2.0*E0*E2+2.0*P0*P2*DCOS(TH2)
      S12=Z1+Z2+2.0*E1*E2-2.0*P1*P2*COTH12
      T2L(L,K)=E2-YM2
      T3L(L,K)=E3-YM3
      T23(L)   =DSQRT(S23)-YM2-YM3
      T31(L,K)=DSQRT(S31)-YM3-YM1
      T12(L,K)=DSQRT(S12)-YM1-YM2
      TA1=ZP+Z1-2.0*EP*E1+2.0*P0*P1*DCOS(TH1)
      TA2=ZP+Z2-2.0*EP*E2+2.0*P0*P2*DCOS(TH2)
      TA3=Z1+Z2+ZT+2.0*(E1*E2-YMT*(E1+E2)-P1*P2*COTH12)
      T1R(L)   =((YMP-YM1)**2-TA1)/(2.0*YMP)
      T2R(L,K)=((YMP-YM2)**2-TA2)/(2.0*YMP)
      T3R(L,K)=((YMP-YM3)**2-TA3)/(2.0*YMP)
      SI=DSQRT(P1**2*(DSIN(TH1))**2+P2**2*(DSIN(TH2))**2
     X+2.0*P1*P2*DSIN(TH1)*DSIN(TH2)*DCOS(PH2-PH1))
      CO=P0-P1*DCOS(TH1)-P2*DCOS(TH2)
      TH3=ARCTAN(SI,CO)
      TH3L(L,K)=TH3*180.0/PAI
      SI=-P1*DSIN(TH1)*DSIN(PH1)-P2*DSIN(TH2)*DSIN(PH2)
      CO=-P1*DSIN(TH1)*DCOS(PH1)-P2*DSIN(TH2)*DCOS(PH2)
      PH3=ARCTAN(SI,CO)
      PH3L(L,K)=PH3*180.0/PAI
      TH1CM=THCM(TH1,S0,E0,P0,E1,P1)
      TH2CM=THCM(TH2,S0,E0,P0,E2,P2)
      TH3CM=THCM(TH3,S0,E0,P0,E3,P3)
      TH1C(L)   =TH1CM*180.0/PAI
      TH2C(L,K)=TH2CM*180.0/PAI
      TH3C(L,K)=TH3CM*180.0/PAI
      TH2R23(L,K)=THRCM(S0,S12,S23,Z1,Z2,Z3)
      TH3R31(L,K)=THRCM(S0,S23,S31,Z2,Z3,Z1)
      TH1R12(L,K)=THRCM(S0,S31,S12,Z3,Z1,Z2)
      PH2R23(L,K)=PHRCM(TH1CM,TH2CM,PH1,PH2)
      PH3R31(L,K)=PHRCM(TH2CM,TH3CM,PH2,PH3)
      PH1R12(L,K)=PHRCM(TH3CM,TH1CM,PH3,PH1)
      FACT=1.0
      EFACT1=E1*E2*E3
      BNS=P1*P2*EFACT1
      BNB=DABS(E0-E1-E2*P23*COTH2P/P2)*FACT
      RH1=BNS/BNB
      RHO1(L,K)=RH1
      EFACT2=E1R23*E2R23*E3R23
      BNS=P1R23*P2R23*EFACT2
      BNB=DSQRT(S23)*FACT
      RH1RCM=BNS/BNB
      GRCM(L,K)=RH1RCM/RH1
      IF(LOWCD.EQ.0) GO TO 500
      IF(K.EQ.2) GO TO 500
      GO TO 95
  500 CONTINUE
  505 CONTINUE
  600 RETURN
      END
```

```
      FUNCTION ARCTAN(Y,X)
      DOUBLE PRECISION ARCTAN,X,Y,PAI
      DOUBLE PRECISION ARG
      ARG=-1.0
      PAI=DARCOS(ARG)
      IF(X) 300,301,302
  300 IF(Y) 303,303,303
  303 ARCTAN=DATAN(Y/X)+PAI
      GO TO 390
  301 IF(Y) 306,307,308
  306 ARCTAN=3.0*PAI/2.0
      GO TO 390
  307 ARCTAN=9999*PAI/180.00
      GO TO 390
  308 ARCTAN=PAI/2.0
      GO TO 390
  302 IF(Y) 309,310,310
  309 ARCTAN=DATAN(Y/X)+2.0*PAI
      GO TO 390
  310 ARCTAN=DATAN(Y/X)
  390 CONTINUE
      RETURN
      END
```

```
      FUNCTION THCM(TH1,S0,E0,P0,E1,P1)
      DOUBLE PRECISION TH1,S0,E0,P0,E1,P1,BNS,BNB,THCM
      DOUBLE PRECISION ARCTAN
      BNS=DSQRT(S0)*DSIN(TH1)
      BNB=E0*DCOS(TH1)-P0*(E1/P1)
      THCM=ARCTAN(BNS,BNB)
      RETURN
      END
```

```
      FUNCTION THRCM(S0,S12,S23,Z1,Z2,Z3)
      DOUBLE PRECISION THRCM,S0,S12,S23,Z1,Z2,Z3,RAMDA,U,V,W
     X,R1,R2,BNS,BNB,X,PAI
      DOUBLE PRECISION ARG
      RAMDA(U,V,W)=U**2+V**2+W**2-2.0*U*V-2.0*V*W-2.0*W*U
      ARG=-1.0
      PAI=DARCOS(ARG)
      R1=RAMDA(S0,S23,Z1)
      R2=RAMDA(S23,Z2,Z3)
      BNS=(S0-S23-Z1)*(S23+Z2-Z3)+2.0*S23*(Z1+Z2-S12)
      BNB=DSQRT(R1)*DSQRT(R2)
      X=-BNS/BNB
      THRCM=DARCOS(X)*180.0/PAI
      RETURN
      END
```

```
      FUNCTION PHRCM(TH1CM,TH2CM,PH1,PH2)
      DOUBLE PRECISION PHRCM,TH1CM,TH2CM,PH1,PH2,PH12,SI,CO,PAI,ARG
     X,ARCTAN
      ARG=-1.0
      PAI=DARCOS(ARG)
      PH12=PH2-PH1
      SI=DSIN(TH2CM)*DSIN(PH12)
      CO=DCOS(TH2CM)*DSIN(TH1CM)-DCOS(TH1CM)*DSIN(TH2CM)*DCOS(PH12)
      PHRCM=ARCTAN(SI,CO)
      PHRCM=PHRCM*180.0/PAI
      RETURN
      END
```