LMI-Based Multiobjective Controller Design with Non-Common Lyapunov Variables

Yoshio Ebihara

December, 2001

LMI-Based Multiobjective Controller Design with Non-Common Lyapunov Variables

A Dissertation
Submitted to Kyoto University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Engineering

Yoshio Ebihara

December, 2001

Abstract

This thesis studies linear matrix inequality (LMI) approaches to multiobjective controller design problems. By multiobjective controller design problems, we mean design problems with a mixture of different design specifications such as the H_∞ performance, the H_2 performance, the regional pole placement constraints and so on. Recent studies show that these design specifications are characterized as matrix inequalities that include controller variables and so-called Lyapunov variables in their bilinear forms. When we deal with a single design specification, the matrix inequality corresponding to the design specification can be reduced successfully to an LMI and hence we can obtain a desired controller easily via well-established convex optimization techniques. On the other hand, when we deal with multiple design specifications, this is no longer true. Namely, the coupled matrix inequalities that reflect multiple design specifications are considered to be essentially bilinear matrix inequalities (BMI's). Solving BMI's is a non-convex optimization problem and quite hard from the viewpoint of numerical computation. In order to avoid the difficulties in dealing with such BMI's, a so-called *common* Lyapunov variable has been forced for all design specifications so that the BMI's can be converted into LMI's. However, the restriction to a common Lyapunov variable is quite confining and this approach brings some conservatism into the design. The goal of this thesis is to get around the conservatism, and we tackle the multiobjective controller design problems with *non-common* Lyapunov variables.

This thesis proposes three approaches to the multiobjective controller design problems with non-common Lyapunov variables, where the first and second ones deal with the state-feedback problems, while the third one deals with both state- and output-feedback problems.

In the first approach, we impose some additional constraints on the Lyapunov variables so that we convexify the problem and obtain LMI characterizations while keeping the state-feedback gain directly as one of the variables. Because of the freedom left in the Lyapunov variables under the constraints, our formulation turns out to give a set of LMI characterizations that allow non-common Lyapunov variables.

On the other hand, in the second approach, we perform a standard procedure called change of variables, and represent the resulting variables as a set of affine functions of

yet new variables. These affine functions are chosen to have a crucial characteristic that troublesome non-convex constraints are satisfied regardless of the new variables. With these affine functions, we readily derive a set of LMI characterizations that allow non-common Lyapunov variables. We also show that a simple combination of this second approach with the above first approach leads to an effective iterative algorithm, with which we can get around the conservatism considerably.

The third approach we propose is quite distinct from the above two. In this approach, we derive new dilated matrix inequality characterizations for the design specifications, where the decoupling between the controller variables and the Lyapunov variables has been achieved and hence the bilinear terms between them disappear. This is achieved by the introduction of new auxiliary variables that form product with the controller variables instead of the Lyapunov variables. These new dilated matrix inequalities lead us to a new approach which convexifies the problems with non-common *Lyapunov variables* but with a common *auxiliary variable*. It is shown that we can guarantee this approach to achieve better performance than that with the conventional approach.

Although our main interest in this thesis is the multiobjective controller design problems, it turns out that the new dilated characterizations for the design specifications have another potential in dealing with robust performance analysis and synthesis problems for real polytopic uncertainty. Roughly speaking, the conventional approach to these problems is such that they seek a common Lyapunov variable over the whole uncertainty domain and hence arrives at conservative results. On the other hand, the new dilated characterizations enable us to employ a so-called *parameter-dependent* Lyapunov variable, and hence the conservatism of the conventional approach can be circumvented successfully.

The idea to decouple the Lyapunov variables and the controller variables in the matrix inequality characterizations is quite important in dealing with such involved problems as the multiobjective controller design problems, robust performance analysis and synthesis for real polytopic uncertainty and so on. This thesis offers an intriguing methodology that could cover such involved problems, and ensures improvement of the performance over the conventional approach.

Contents

Notations and Definitions

$\mathbf{R}$	The set of all real numbers.
$\mathbf{R}^n$	The set of all n-dimensional real vectors.
$\mathbf{R}^{n\times m}$	The set of all $n \times m$ real matrices.
$\mathbf{C}$	The set of all complex numbers.
$0_{m,n}$	The $m \times n$ zero matrix. The subscripts m and n are omitted when the size is not relevant or can be determined from the context.
I_n	The $n \times n$ identity matrix. The subscript n is omitted when the size is not relevant or can be determined from the context.
A^T	Transpose of the matrix A.
A^*	Complex conjugate transpose of the matrix A.
A^{-1}	Inverse of the matrix A.
A^{-T}	Shorthand notation for $(A^{-1})^T$.
$\sigma(A)$	The set of the eigenvalues of the matrix A.
$\bar{\sigma}(A)$	The largest singular value of the matrix A.
$\mathrm{trace}(A)$	Trace of the matrix A.
$\mathrm{He}[A]$	Shorthand notation for $A + A^T$.
$A_1 \oplus A_2$	The direct some of the matrices A_1 and A_2.
$A_1 \otimes A_2$	The Kronecker product of the matrices A_1 and A_2.
$A^\perp$	For the matrix $A \in \mathbf{R}^{n\times m}$ where $n > m$, the matrix $A^\perp$ is defined as a matrix satisfying the following three conditions, where r is the rank of A. $A^\perp \in \mathbf{R}^{(n-r)\times n}, \quad A^\perp A = 0, \quad A^\perp A^{\perp T} > 0$

Abbreviations

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
MIMO	Multi-Input Multi-Output
LTI	Linear Time-Invariant

Chapter 1

Introduction

This thesis studies linear matrix inequality (LMI) approaches to the multiobjective controller design problems for continuous-time multi-input multi-output (MIMO) linear time-invariant (LTI) systems. By multiobjective controller design problems, we mean design problems with a mixture of different design specifications such as the H_∞ performance, the H_2 performance, the regional pole placement constraints and so on. Some frequency-domain design objectives are best captured by the H_∞ performance, noise or disturbance insensitivity is naturally expressed by the H_2 performance and the transient behavior is effectively tuned by the constraints on the closed-loop pole locations. Although a design framework to satisfy each of these design specifications is well-established, practical design objectives are rarely covered by a single design specification. Namely, it is important in practice to satisfy these multiple design specifications at the same time. However, once we take these multiple design specifications into account, the design problem turns out to be surprisingly difficult. Thus, the multiobjective controller design problems have their roots in practical controller design, and are also quite attractive from a theoretical point of view. Many researchers have dealt with the multiobjective controller design problems, and some of the contributions are summarized below.

The regional pole placement problem taking account of other design specifications was studied intensively in the 1980's in terms of the linear quadratic (LQ) type regulator theory. Furuta and Kim [13], and Kim and Furuta [24] dealt with the problem to find a static state-feedback controller that minimizes an LQ type cost functional while placing the closed-loop poles in a specified disk. Their approach was such that they seek weighting matrices of the cost functional so that the optimal controller associated with the cost functional places the closed-loop poles appropriately. Namely, the cost functional was not fixed in advance, and their attention was paid mainly on clarifying the relation between the weighting matrices and the closed-loop pole locations. On the other hand, Haddad and Bernstein [19] dealt with the design problem to find state- or output-feedback controllers that minimize the H_2 cost of a closed-loop transfer function subject to the regional pole placement constraints. They showed that a modified Lyapunov equation which reflects the regional pole placement constraints leads directly to an upper bound of the cost functional, and they reduced the problem into a minimization problem of this upper bound subject to the modified Lyapunov equation. This minimization problem was completely solved by Sivashankar et al. [42] via a discrete-time H_2 synthesis technique. The idea to minimize an upper bound of the cost functional is quite important for the tractability of the problem, and this idea led to the multiobjective controller design via LMI optimization.

It has been recognized recently that a wide variety of problems arising in system and control theory can be reduced to optimization problems involving LMI's [4],[43]. Since solving LMI's is a convex optimization problem, the LMI formulations are quite appealing from the viewpoint of numerical computation, and also offer a tractable means for such problems that lack analytical solutions. In addition, because the framework of LMI's enables us to deal with design specifications as constraints on the closed-loop system, many researchers have attacked the multiobjective controller design problems via LMI's [6],[7],[22],[23],[26], [28],[36].

In the framework of LMI's, generally speaking, controller design is carried out in the following two steps.

1. First, the design specifications are characterized as matrix inequalities with respect to the controller variables and some additional variables. Since most practical design specifications inherently require stability of the closed-loop system, the matrix inequalities include the *Lyapunov inequality*, and the additional variable that forms the Lyapunov inequality is called *Lyapunov variable.* Because of this fact, the matrix inequalities that characterize each of the design specifications have bilinear terms between the Lyapunov variables and the controller variables [4],[43]. Namely, the design specifications are characterized as bilinear matrix inequalities (BMI's). It is known that solving BMI's is a non-convex optimization problem and quite hard from the viewpoint of numerical computation [45].

2. Second, some algebraic manipulations are applied to these BMI's so that they can be reduced to LMI's. Representative methods are *elimination of variables* [14],[22], [43] and *change of variables* [4],[6],[17],[23],[26],[28],[36]. In these methods, the controller variables are parametrized as nonlinear functions with respect to the Lyapunov variables and some other variables.

An important fact in the LMI-based controller design is that the design specifications cannot be characterized directly as LMI's with respect to the controller variables, and if we employ the elimination or change of variables techniques, the controller variables are parametrized in such a way that they depend nonlinearly on the Lyapunov variables, as stated above. This fact imposes no limit when we deal with a single design specification, in which situation the above techniques are successfully applied so that we can obtain a desired controller easily. In fact, recent studies show that many controller design problems can be solved efficiently with these two representative methods [36],[43]. However, when we deal with multiple design specifications, this is no longer true. This is because, although the multiple design specifications are naturally characterized with non-common Lyapunov variables for each of the design specifications, the controller parametrization *does not* allow non-common Lyapunov variables. Thus, general multiobjective controller design problems are considered to be the ones essentially characterized as BMI's. In order to avoid the difficulties in dealing with these BMI's, a so-called *common* Lyapunov variable has been forced for all design specifications [6],[7],[26],[36]. As a benefit, these BMI's have been converted into LMI's with the change of variables technique. Specifically, Chilali and Gahinet [6], Masubuchi et al. [26] and Scherer et al. [36] gave a unified framework for general multiobjective controller design problems, based on a common Lyapunov variable. It should be noted, however, that convexity there is recovered essentially by forcing a common Lyapunov variable for all design specifications, and this approach brings some conservatism into the design.

Although the existing LMI approach with a common Lyapunov variable offers a tractable means for the multiobjective controller design problems, the resulting controllers sometimes fail to have a satisfactory performance because of the conservatism of the design. The goal of this thesis is to get around the conservatism, and we tackle the problem with non-common Lyapunov variables.

Several researchers also have tried to solve the problem with non-common Lyapunov variables. We summarize some of the contributions in the following. Shimomura and Fujii have proposed an effective iterative algorithm for the state- and output-feedback multiobjective controller design problems [38]–[40]. They showed that completing the square with respect to the Lyapunov variables and the controller variables converts the bilinear term between them into another set of bilinear terms: quadratic terms with respect to the Lyapunov variables and the controller variables. An advantage of this manipulation is that the resulting bilinear terms can be replaced by their upper bounds, where the upper bounds can be chosen as linear terms with respect to the Lyapunov variables and the controller variables. With a suitable replacement of the parameters in these upper bounds, an effective iterative algorithm has been derived. A similar idea was also proposed by Oliveira et al. [32]. In these approaches, however, we need another effort to determine suitable initial parameters in the upper bounds, and in general, the initial parameters are sought by the conventional approach with a common Lyapunov variable.

In contrast with the iterative approaches, the output-feedback multiobjective controller design problems have been solved with non-common Lyapunov variables via finite-dimensional Q-parametrization [47] by Chen and Wen [5], Hindi et al. [21] and Scherer [37]. In this approach, the controller variables are assembled to a specific part of the Q-parameter so that the bilinear term between the Lyapunov variables and the controller variables disappears. As a benefit, we can employ non-common Lyapunov variables for each design specification without any difficulty. This approach is quite effective in the sense that the conservatism can be made arbitrarily small, but there is inherent inflation of the size of the LMI's and high order controllers tend to be designed.

In the late 1990's, Oliveira et al. showed a new direction for the state- and output-feedback multiobjective controller design problems in the discrete-time setting [29],[30]. They showed that the *dilation* of the matrix inequality characterizations and the introduction of auxiliary variables achieve decoupling between the Lyapunov variables and the controller variables and thus the technical restriction to a common Lyapunov variable can be avoided. They have shown a constructive way to derive dilated characterizations that are equivalent to the original ones. The advantage of working with these dilated characterizations lies in the fact that if we consider a set of dilated matrix inequality characterizations, then it *includes* the corresponding set of the original ones as a special case. More specifically, if one chooses the newly introduced auxiliary variable the same as the Lyapunov variable, the set of dilated characterizations reduces to the original one [29],[30]. Because of this nice property, the dilated characterizations are successfully applied to a wide range of problems including multiobjective control [30] and robust control for real polytopic uncertainty [29] to circumvent the conservatism, with the use of non-common or so-called *parameter-dependent* Lyapunov variables. Unfortunately, however, the study in [29],[30] relies on the features of the matrix inequality characterizations in the discrete-time setting, and hence analogous dilated characterizations in the continuous-time setting do not follow in a parallel fashion. Namely, we need another effort, as is suggested in [1].

In this thesis, we propose three approaches to the multiobjective controller design problems with non-common Lyapunov variables in the continuous-time setting, where the first and second ones deal with the state-feedback problems, while the third one deals with both state- and output-feedback problems.

In the first approach, we impose some additional constraints on the Lyapunov variables so that we convexify the problem and obtain LMI characterizations while keeping the state-feedback gain directly as one of the LMI variables. Because of the freedom left in the

Lyapunov variables under the constraints, our formulation turns out to give a set of LMI's that allow non-common Lyapunov variables. If we choose the additional constraints reasonably, this approach leads to a feedback gain that achieves better performance than the one based on a common Lyapunov variable. Furthermore, an effective iterative algorithm follows immediately from this approach.

In the second approach, we perform the standard procedure called change of variables [4], [17],[23],[28], and represent the resulting variables and the Lyapunov variables as a set of affine functions of yet new variables. The reason why we introduce such new variables is that the variables resulting from the change of variables and the Lyapunov variables are subject to non-convex constraints, since the feedback gain is parametrized only by their nonlinear function. It is to get around the difficulties stemming from such non-convex constraints that the yet new variables are introduced. Indeed, the affine functions are chosen to have a crucial characteristic that the troublesome non-convex constraints are satisfied regardless of the new variables. With these affine functions, we readily derive a set of LMI characterizations that allow non-common Lyapunov variables. If we choose the parameters included in the affine functions reasonably, this approach yields a feedback gain that achieves better performance than that with the conventional approach. Thus, in the first and the second approaches, we arrive at two distinct sets of LMI's for the multiobjective state-feedback controller design problems. It turns out that each set of LMI's is obtained by freezing some different portion of the freedom in the Lyapunov variables. Hence, applying these two approaches by turns, it is expected that we can use the freedom of the frozen portion complementarily. This consideration directly leads to an effective combined iterative algorithm, with which we can get around the conservatism considerably.

The third approach we propose is quite distinct from the above two. Motivated by the study in [29],[30], we propose a general approach to the dilated matrix inequality characterizations for continuous-time controller design. As stated before, the study in [29], [30] fully relies on the features of the matrix inequality characterizations in the discrete-time setting, and hence analogous characterizations in the continuous-time setting do not follow in a parallel fashion. Therefore, making another effort, we reveal that a particular application of the Schur complement technique [4] and the introduction of an auxiliary variable lead to a constructive way to derive dilated characterizations that are suitable for controller synthesis. In addition, it is shown that the set of the new dilated characterizations includes the corresponding set of the original ones as a special case, via a particular choice of the auxiliary variable. These are very nice and interesting properties that are to some extent analogous to the ones already obtained in the discrete-time setting [29], [30]. With these dilated matrix characterizations, we successfully reduce the multiobjective controller design problem to a convex optimization problem with non-common Lyapunov variables.

Although our main interest in this thesis is the multiobjective controller design, the new dilated characterizations have another potential in dealing with the robust performance analysis and synthesis problems for real polytopic uncertainty [4]. Roughly speaking, the conventional approach to these problems is such that they seek a common Lyapunov variable over the whole uncertainty domain [4],[6] and hence arrives at conservative results. On the other hand, the new dilated characterizations enable us to employ a parameter-dependent Lyapunov variable [1],[12],[16],[29],[34],[35], and hence the conservatism of the conventional approach can be circumvented successfully.

The above three approaches have been reported in separate papers [8]–[11]. This thesis assembles these contributions, with a plenty of numerical examples to illustrate the effectiveness of them. The thesis is organized as follows.

Chapter 2 gives a formal description of the multiobjective controller design problems to be dealt with in this thesis. The conventional LMI approach with a common Lyapunov variable [6],[7],[17],[23],[26],[28],[36] is also reviewed, where we point out the conservatism of this approach and clarify the goal of this thesis.

Chapter 3 discusses two LMI approaches to the multiobjective state-feedback controller design problems with non-common Lyapunov variables: we call these approaches a *subspace approach* and an *affine representation approach*, respectively. As stated before, it turns out that two iterative algorithms follow from these approaches. Numerical examples in this chapter demonstrate that the application of the new approaches results in significant improvements over the conventional approach based on a common Lyapunov variable.

Chapter 4 describes a general approach to the dilated matrix inequality characterizations for continuous-time controller design. With these dilated matrix inequality characterizations, we readily reduce the multiobjective controller design problem to a convex optimization problem with non-common *Lyapunov variables* but with a common *auxiliary variable*. A remarkable prominence is that we can guarantee this new approach to achieve a better upper bound than that with the conventional approach. Numerical examples show that the actual cost is also improved, in general, due to the freedom gained by the non-common Lyapunov variables. The dilated characterizations also enable us to develop a new approach to the robust multiobjective synthesis for real polytopic uncertainty [4],[6], where we successfully employ non-common parameter-dependent Lyapunov variables. The effectiveness of this new approach is also illustrated through numerical examples.

Chapter 5 is the conclusion, where we summarize the achievements in this thesis and discuss future topics.

Chapter 2

Multiobjective Controller Design Problems and Conventional LMI Approach with a Common Lyapunov Variable

The purpose of this chapter is to describe formally the multiobjective controller design problems to be dealt with in this thesis. The conventional LMI approach with a common Lyapunov variable [6],[7],[17],[23],[26],[28],[36] is also reviewed, where we point out the conservatism of this approach and clarify the goal of this thesis.

2.1 Multiobjective Controller Design Problem

Throughout this thesis, we consider the continuous-time multi-input multi-output (MIMO) linear time-invariant (LTI) plant described by

$$\left\{\begin{array}{rclclcl} \dot{x} &=& Ax &+& B_w w &+& Bu \\ z &=& C_z x &+& D_{zw} w &+& D_z u \\ y &=& Cx &+& D_w w && \end{array}\right. \tag{2.1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$ are respectively the state, control input and measured output, and w and z are the vectors of exogenous input and output signals related to the performance of the control system.

The controller that we consider is the full-order output-feedback controller K given by

$$\left\{\begin{array}{rclcl} \dot{x}_K &=& A_K x_K &+& B_K y \\ u &=& C_K x_K &+& D_K y \end{array}\right. \tag{2.2}$$

In the state feedback case ($C = I$, $D_w = 0$), we also consider the static state-feedback controller K given by

$$u = Kx \tag{2.3}$$

With the plant (2.1) and the controller given by (2.2) or (2.3), the closed-loop system can be written in the form of

$$\begin{cases} \dot{x}_{\mathrm{cl}} &= \mathcal{A}x_{\mathrm{cl}} + \mathcal{B}w \\ z &= \mathcal{C}x_{\mathrm{cl}} + \mathcal{D}w \end{cases} \tag{2.4}$$

and we denote its transfer matrix from w to z by $T_{zw}(s)$.

In the multiobjective controller design problem, several design specifications are imposed on different channels of the closed-loop system at the same time, which is often necessary to accommodate practical design objectives. In this thesis, we consider the H_∞ performance, the H_2 performance and the regional constraints on the closed-loop pole locations, motivated by the following considerations [36],[43].

- The H_∞ norm of a stable system $T_{zw}(s)$ is defined by

$$||T_{zw}(s)||_\infty := \sup_{\omega \in \mathbf{R}} \bar{\sigma}(T_{zw}(j\omega)) \tag{2.5}$$

  The H_∞ norm can be interpreted in the following two ways. One is a measure for the worst-case disturbance rejection level. The other is a measure for robustness (robust stability or robust performance). For example, from the small gain theorem [47], the closed-loop system remains stable for all perturbations of the plant represented by $w = \Delta z$ with $||\Delta(s)||_\infty \leq \gamma^{-1}$ if and only if $||T_{zw}(s)||_\infty < \gamma$.

- The H_2 norm of a stable and strictly proper system $T_{zw}(s)$ is defined by

$$||T_{zw}(s)||_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\left(T_{zw}(j\omega)T_{zw}^*(j\omega)\right) d\omega\right)^{\frac{1}{2}} \tag{2.6}$$

  The H_2 norm can be interpreted in the following way. In most cases, the disturbance signal is considered to be a Gaussian white noise and an important control objective is to minimize the root-mean-square (RMS) of the output signal of interest. The RMS of z for the Gaussian white noise w is nothing but the H_2 norm $||T_{zw}(s)||_2$.

- The transient behavior of the closed-loop system is closely related to the closed-loop pole locations. By confining close-loop poles within a suitable subregion $\mathbf{D}$ contained in the open left-half plane, some bounds can be put on the time-domain objective such as the rise time, the settling time and so on. This regional pole placement constraints is called $\mathbf{D}$-stability constraints [6],[19],[42].

In dealing with the H_∞ performance and the H_2 performance imposed on different channels of the closed-loop system, the general description of the plant (2.1) amounts to the following specific form.

$$\left\{\begin{array}{lllllll} \dot{x} & = Ax & + B_\infty w_\infty & + B_2 w_2 & + Bu \\ z_\infty & = C_\infty x & + D_\infty w_\infty & & + D_{z\infty} u \\ z_2 & = C_2 x & & + D_2 w_2 & + D_{z2} u \\ y & = Cx & + D_{w\infty} w_\infty & + D_{w2} w_2 & \end{array}\right. \tag{2.7}$$

Here, the pair (w_∞, z_∞) is concerned with the H_∞ performance while the pair (w_2, z_2) is concerned with the H_2 performance. The diagram of the above plant is shown in Fig. 2.1.

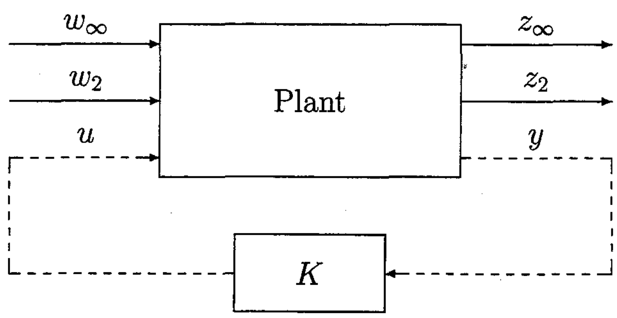

Figure 2.1: Multiobjective Controller Design

Let us denote the closed-loop transfer matrix from w_j to z_j by $T_{z_j w_j}(s)$ $(j = \infty, 2)$. The transfer matrix $T_{z_j w_j}(s)$ has the following state space realization.

$$T_{z_j w_j}(s) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array}\right] \tag{2.8}$$

For the output-feedback controller K, the coefficient matrices in (2.8) are

$$\begin{aligned} &\mathcal{A} = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix}, \quad \mathcal{B}_j = \begin{bmatrix} B_j + BD_K D_{wj} \\ B_K D_{wj} \end{bmatrix}, \\ &\mathcal{C}_j = \begin{bmatrix} C_j + D_{zj} D_K C & D_{zj} C_K \end{bmatrix}, \quad \mathcal{D}_j = D_j + D_{zj} D_K D_{wj} \end{aligned} \tag{2.9}$$

while for the static state-feedback controller K, the coefficient matrices are given by

$$\mathcal{A} = A + BK, \quad \mathcal{B}_j = B_j, \quad \mathcal{C}_j = C_j + D_{zj} K, \quad \mathcal{D}_j = D_j \tag{2.10}$$

For the H_2 norm $||T_{z_2 w_2}(s)||_2$ to be well-defined, it is necessary that $\mathcal{D}_2 = 0$ in (2.8). In order to assure $\mathcal{D}_2 = 0$ in a simple manner, in this thesis, we assume that the plant (2.7) satisfies

$D_2 = 0$. In addition, we force $D_K = 0$ in the output-feedback controller (2.2), which implies that the full-order output-feedback controller K is strictly proper.

Now, we are ready to describe formally the multiobjective controller design problem to be dealt with in this thesis.

Problem Consider the continuous-time multi-input, multi-output (MIMO), linear time-invariant (LTI) plant described by

$$\left\{\begin{array}{lllllll} \dot{x} & = & Ax & + \; B_\infty w_\infty & + \; B_2 w_2 & + & Bu \\ z_\infty & = & C_\infty x & + \; D_\infty w_\infty & & + & D_{z\infty} u \\ z_2 & = & C_2 x & & & + & D_{z2} u \\ y & = & Cx & + \; D_{w\infty} w_\infty & + \; D_{w2} w_2 & & \end{array}\right. \tag{2.11}$$

For the prescribed H_∞ performance $\gamma_\infty > 0$ and the prescribed closed-loop pole placement region $\mathbf{D}$ contained in the open left-half plane, find a controller K, full-order output-feedback (2.2) or static state-feedback (2.3), such that

- the H_∞ performance $||T_{z_\infty w_\infty}(s)||_\infty < \gamma_\infty$ is achieved;
- the closed-loop poles lie in the prescribed region $\mathbf{D}$;
- the H_2 performance $||T_{z_2 w_2}(s)||_2$ is minimized subject to the above two constraints.

The above problem includes the H_2 specification, the H_∞ constraint and the $\mathbf{D}$-stability constraint and hence we call this problem multiobjective $H_2/H_\infty/\mathbf{D}$-stability problem. This problem was addressed by Chilali and Gahinet [6]. In this thesis, we also deal with a special case of the above problem, i.e., multiobjective $H_2/\mathbf{D}$-stability problem without the H_∞ constraint, which was treated by Haddad and Bernstein [19] and Sivashankar et al. [42].

Concerning the pole placement region $\mathbf{D}$, we consider the so-called LMI regions [6] represented by

$$\mathbf{D} \;:=\; \left\{\lambda \in \mathbf{C} : \;\; M + \lambda N + \bar{\lambda} N^T < 0\right\} \tag{2.12}$$

Here, $M = M^T$ and N are constant real matrices that characterize the region $\mathbf{D}$. The LMI region (2.12) includes α-stability regions, circular regions and conic sector regions, which are frequently used in the regional pole placement constraints [6],[19]. The following notations specify these regions contained in the open left-half plane.

$$\begin{array}{lll} \mathcal{H}(\alpha) & := & \{\lambda \in \mathbf{C} : \;\; \mathrm{Re}\,[\lambda] < -\alpha \,\} \quad (\alpha > 0) \\ \mathcal{C}(c, r) & := & \{\lambda \in \mathbf{C} : \;\; |\lambda - c| < r \,\} \;\; (c < -r < 0) \\ \mathcal{S}(k) & := & \{\lambda \in \mathbf{C} : \;\; |\,\mathrm{Im}\,[\lambda]\,| < k|\,\mathrm{Re}\,[\lambda]\,|\,\} \;\; (k > 0) \end{array} \tag{2.13}$$

Furthermore, the intersection of regions is denoted by $\bigcap$. For example, the region shown in Fig. 2.2 is denoted by $\bigcap\{\mathcal{H}(\alpha), \mathcal{C}(c,r), \mathcal{S}(k)\}$. The regions (2.13) are characterized by setting M and N in (2.12) as follows.

$$\begin{aligned} &\text{For } \mathcal{H}(\alpha): \quad M = 2\alpha, && N = 1 \\ &\text{For } \mathcal{C}(c,r): \quad M = \begin{bmatrix} -r & -c \\ -c & -r \end{bmatrix}, && N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &\text{For } \mathcal{S}(k): \quad M = 0, && N = \begin{bmatrix} k & 1 \\ -1 & k \end{bmatrix} \end{aligned} \tag{2.14}$$

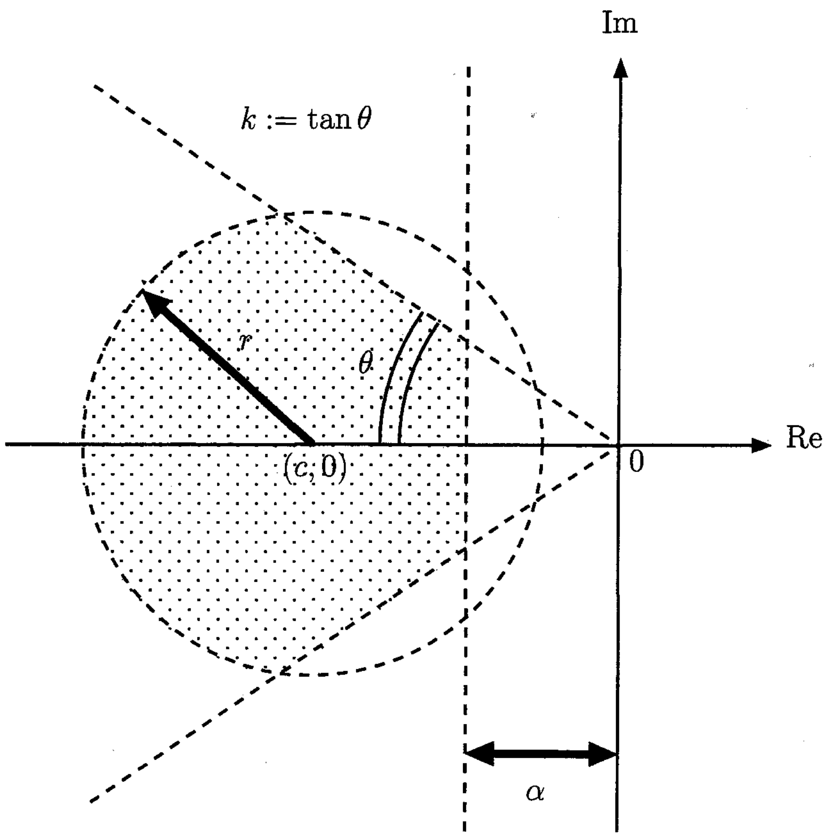

Figure 2.2: The regional pole placement

Some observations concerning the regions $\mathcal{H}(\alpha)$, $\mathcal{C}(c,r)$ and $\mathcal{S}(k)$ are worth noting [6], [19]. For simplicity, let $\lambda = -\zeta\omega_n \pm j\omega_d$ be a pair of closed-loop poles, where $0 < \zeta < 1$ is the damping ratio, $\omega_n = |\lambda|$ is the undamped natural frequency, and $\omega_d := \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Then, if $\lambda \in \mathcal{C}(c,r)$, it follows that

$$\zeta > \sqrt{1 - \left(\frac{r}{c}\right)^2}, \quad \omega_d < r, \quad -c - r < \omega_n < -c + r, \quad -c - r < \zeta\omega_n < -c + r \tag{2.15}$$

In practice, it is important to consider such constraints as $\zeta > \zeta_{\min}$ and $\zeta\omega_n > \xi$. By confining the closed-loop pole λ into $\mathcal{S}(k)$ and $\mathcal{H}(\alpha)$, we have

$$\zeta > \frac{1}{\sqrt{1+k^2}}, \quad \zeta\omega_n > \alpha \tag{2.16}$$

and hence the constraints can be achieved by

$$k = \sqrt{\zeta_{\min}^{-2} - 1}, \quad \alpha = \xi \tag{2.17}$$

Thus, we can ensure an appropriate decay rate, damping ratio, undamped natural frequency and damped natural frequency in a flexible fashion, using the parameters α, c, r and k in (2.13).

Now, we have given some comments on the usefulness of the **D**-stability constraints. Recall that our multiobjective controller design problem includes the H_2 specification and the H_∞ constraint as well as the **D**-stability constraints. In the following, we review some standard results on the characterizations of these design specifications in terms of matrix inequalities [4],[6],[43].

Lemma 2.1 (*The H_∞ Performance*) For the system described by

$$T_{z_\infty w_\infty}(s) := \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_\infty \\ \hline \mathcal{C}_\infty & \mathcal{D}_\infty \end{array}\right], \tag{2.18}$$

the following two conditions are equivalent.

(i) The matrix $\mathcal{A}$ is stable and the H_∞ cost $||T_{z_\infty w_\infty}(s)||_\infty$ is bounded by $\gamma_\infty > 0$. Namely,

$$||T_{z_\infty w_\infty}(s)||_\infty < \gamma_\infty \tag{2.19}$$

(ii) There exists a matrix $X_\infty > 0$ such that

$$\begin{bmatrix} \mathcal{A}X_\infty + X_\infty\mathcal{A}^T & \mathcal{B}_\infty & X_\infty\mathcal{C}_\infty^T \\ \mathcal{B}_\infty^T & -I & \mathcal{D}_\infty^T \\ \mathcal{C}_\infty X_\infty & \mathcal{D}_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{2.20}$$

Lemma 2.2 (*The Regional Pole Placement*) The following two conditions are equivalent.

(i) The matrix $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \subset \mathbf{D}$ for the region

$$\mathbf{D} \; := \; \left\{\lambda \in \mathbf{C} : \;\; M + \lambda N + \bar{\lambda}N^T < 0\right\} \tag{2.21}$$

(ii) There exists a matrix $X_{\mathbf{D}} > 0$ such that

$$M \otimes X_{\mathbf{D}} + N \otimes (\mathcal{A}X_{\mathbf{D}}) + N^T \otimes (X_{\mathbf{D}}\mathcal{A}^T) < 0 \tag{2.22}$$

where $\otimes$ denotes the Kronecker product.

Lemma 2.3 (*The H_2 Performance*) For the system described by

$$T_{z_2w_2}(s) := \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_2 \\ \hline \mathcal{C}_2 & 0 \end{array}\right], \tag{2.23}$$

the following two conditions are equivalent.

(i) The matrix $\mathcal{A}$ is stable and the H_2 cost $||T_{z_2w_2}(s)||_2$ is bounded by $\gamma_2 > 0$. Namely,

$$||T_{z_2w_2}(s)||_2 < \gamma_2 \tag{2.24}$$

(ii) There exist matrices $X_2 > 0$ and $Z_2 > 0$ such that

$$\begin{bmatrix} \mathcal{A}X_2 + X_2\mathcal{A}^T & X_2\mathcal{C}_2^T \\ \mathcal{C}_2X_2 & -\gamma_2^2 I \end{bmatrix} < 0, \quad \begin{bmatrix} Z_2 & \mathcal{B}_2^T \\ \mathcal{B}_2 & X_2 \end{bmatrix} > 0, \quad \mathrm{trace}(Z_2) < 1 \tag{2.25}$$

In Lemmas 2.1–2.3, new variables X_∞, $X_{\mathbf{D}}$, X_2 and Z_2 are introduced to characterize each of the design specifications as matrix inequalities. Since the design specifications inherently require stability of the closed-loop system, we can see that the matrix inequalities (2.20), (2.22) and (2.25) include the *Lyapunov inequality* of the form

$$X_j > 0, \quad \mathcal{A}X_j + X_j\mathcal{A}^T < 0 \quad (j = \infty, \mathbf{D}, 2) \tag{2.26}$$

Because of this fact, the variables X_∞, $X_{\mathbf{D}}$ and X_2 are called *Lyapunov variables* for each design specification.

With (2.20), (2.22) and (2.25), we readily obtain the following formulation of our problem.

Basic Formulation of Our Problem

Minimize γ_2^2 subject to (2.20), (2.22) and (2.25). Here, the variables are X_∞, $X_{\mathbf{D}}$, X_2, Z_2, γ_2^2 and the controller variables included in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

As is easily seen, the characterizations (2.20), (2.22) and (2.25) involve bilinear terms between the Lyapunov variables and the controller variables as in $\mathcal{A}X_j + X_j\mathcal{A}^T$ $(j = \infty, \mathbf{D}, 2)$[†].

[†]Note that the matrix $\mathcal{A}$ contains the controller variables.

Hence, the above formulation is of no use in practice since the resulting optimization problem involves BMI's. Solving BMI's is a non-convex optimization problem and quite hard from the viewpoint of numerical computation [45]. Many researchers have tried to overcome this difficulty, and the change of variables technique has been applied so that the resulting optimization problems involve only LMI's. This conventional approach has been partially successful by forcing a common Lyapunov variable for all design specifications [6],[7],[17],[23],[26],[28],[36], which we will review in the next section.

2.2 Conventional LMI Approach with a Common Lyapunov Variable

In the preceding section, the multiobjective controller design problem has been formulated as an optimization problem involving BMI's. To recover convexity in the optimization problem, a *common* Lyapunov variable

$$X := X_\infty = X_{\mathbf{D}} = X_2 \tag{2.27}$$

has been forced for all design specifications in the previous studies [6],[7],[17],[23],[26],[28],[36]. This conventional approach can be described formally as follows.

LMI Formulation of Our Problem with a Common Lyapunov Variable

Minimize γ_2^2 subject to (2.20), (2.22) and (2.25) with a common Lyapunov variable (2.27). Here, the variables are X, Z_2, γ_2^2, and the controller variables included in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

With the restriction (2.27), the problem actually results in a convex optimization problem represented by LMI's [6],[7],[17],[23],[26],[28],[36]. Clearly, this restriction brings some conservatism into the design and only an upper bound of the cost functional will be minimized, but there is no *further* conservatism [36]. In the rest of this section, we will review an existing method to linearize the BMI's (2.20), (2.22) and (2.25) under the restriction (2.27).

State-Feedback Case

In the state-feedback case, it follows readily from (2.10) that the restriction (2.27) admits a simple change of variables technique [4],[17],[23],[28]

$$Y := KX \tag{2.28}$$

so that the constraints (2.20), (2.22) and (2.25) result in LMI's with respect to X, Y, Z and γ_2^2. Once the variables X and Y have been found, the state-feedback gain K can be determined by

$$K = YX^{-1} \tag{2.29}$$

Thus, we are led to the conclusion that under the restriction (2.27), the linearization is completed without any further conservatism.

Output-Feedback Case

In the output-feedback case, we need a much more involved change of controller variables technique, and we will follow the result proposed by Scherer [36].

Let us partition X and its inverse P as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad P = X^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{2.30}$$

where $X_{11} \in \mathbf{R}^{n \times n}$, $P_{11} \in \mathbf{R}^{n \times n}$ and the other variables have compatible dimensions. We assume that X_{12} and P_{12} are nonsingular without loss of generality [6]. With (2.30) and the controller variables given in (2.2) where $D_K = 0$, we define the following matrices.

$$\Xi_X := \begin{bmatrix} I & X_{11} \\ 0 & X_{12}^T \end{bmatrix}, \quad \Xi_P := \begin{bmatrix} P_{11} & I \\ P_{12}^T & 0 \end{bmatrix} \tag{2.31}$$

$$\bar{B}_K := P_{12} B_K, \; \bar{C}_K := C_K X_{12}^T, \; \bar{A}_K := \begin{bmatrix} P_{11} & \bar{B}_K \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X_{11} \\ \bar{C}_K \end{bmatrix} + P_{12} A_K X_{12}^T \tag{2.32}$$

The matrices Ξ_X and Ξ_P are nonsingular and satisfy the following equality.

$$X \Xi_P = \Xi_X \tag{2.33}$$

Applying appropriate congruence transformations with the matrix Ξ_P given by (2.31) to (2.20), (2.22) and (2.25) under the restriction (2.27), we obtain the following matrix inequalities for the H_∞ constraint, the **D**-stability constraints and the H_2 specification.

$$\begin{bmatrix} \Xi_P^T \mathcal{A} X \Xi_P + \Xi_P^T X \mathcal{A}^T \Xi_P & \Xi_P^T \mathcal{B}_\infty & \Xi_P^T X \mathcal{C}_\infty^T \\ \mathcal{B}_\infty^T \Xi_P & -I & \mathcal{D}_\infty^T \\ \mathcal{C}_\infty X \Xi_P & \mathcal{D}_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{2.34}$$

$$M \otimes (\Xi_P^T X \Xi_P) + N \otimes (\Xi_P^T \mathcal{A} X \Xi_P) + N^T \otimes (\Xi_P^T X \mathcal{A}^T \Xi_P) < 0 \tag{2.35}$$

$$\begin{gathered} \begin{bmatrix} \Xi_P^T \mathcal{A} X \Xi_P + \Xi_P^T X \mathcal{A}^T \Xi_P & \Xi_P^T X \mathcal{C}_2^T \\ \mathcal{C}_2 X \Xi_P & -\gamma_2^2 I \end{bmatrix} < 0, \\ \begin{bmatrix} Z_2 & \mathcal{B}_2^T \Xi_P \\ \Xi_P^T \mathcal{B}_2 & \Xi_P^T X \Xi_P \end{bmatrix} > 0, \quad \mathrm{trace}(Z_2) < 1 \end{gathered} \tag{2.36}$$

We can see that the above matrix inequalities only involve the following terms.

$$\Xi_P^T X \Xi_P = \Xi_P^T \Xi_X = \begin{bmatrix} P_{11} & I \\ I & X_{11} \end{bmatrix} \tag{2.37}$$

$$\begin{aligned} \Xi_P^T \mathcal{B}_j &= \begin{bmatrix} P_{11}B_j + \bar{B}_K D_{wj} \\ B_j \end{bmatrix}, \quad \mathcal{C}_j X \Xi_P = \begin{bmatrix} C_j & C_j X_{11} + D_{zj}\bar{C}_K \end{bmatrix}, \quad (j = \infty, 2) \\ \Xi_P^T \mathcal{A} X \Xi_P &= \Xi_P^T \mathcal{A} \Xi_X = \begin{bmatrix} P_{11}A + \bar{B}_K C & \bar{A}_K \\ A & AX_{11} + B\bar{C}_K \end{bmatrix} \end{aligned} \tag{2.38}$$

The above terms are affine with respect to X_{11}, P_{11}, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$. Accordingly, the matrix inequalities (2.20), (2.22) and (2.25) result in the LMI's (2.34), (2.35) and (2.36) with respect to the variables X_{11}, P_{11}, $\bar{A}_K$, $\bar{B}_K$, $\bar{C}_K$, Z_2 and γ_2^2. Once the variables X_{11}, P_{11}, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$ have been found, the output-feedback controller (2.2) can be determined through (2.32) by

$$\begin{aligned} B_K &= P_{12}^{-1}\bar{B}_K, \quad C_K = \bar{C}_K X_{12}^{-T}, \\ A_K &= P_{12}^{-1}\left\{ \bar{A}_K - \begin{bmatrix} P_{11} & \bar{B}_K \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X_{11} \\ \bar{C}_K \end{bmatrix} \right\} X_{12}^{-T} \end{aligned} \tag{2.39}$$

where X_{12} and P_{12} are nonsingular matrices satisfying

$$X_{12}P_{12}^T = I - X_{11}P_{11} \tag{2.40}$$

Note that above change of variables is based on the congruence transformations, and thus the linearization is completed under (2.27) without any further conservatism.

Now, the existing change of variables techniques have been reviewed. Summing up, our multiobjective controller design problem can be cast as a convex optimization problem involving LMI's only, under the restriction on the Lyapunov variables that they are taken to be common. It follows that the use of a common Lyapunov variable is the core of the change of variables techniques. However, recall that the restriction to a common Lyapunov variable is very confining and hence the conventional LMI approach results in conservative design. This crucial fact motivates us to improve the study on the multiobjective controller design problem. Namely, the purpose of this thesis is to get around the conservatism of the conventional approach arising from seeking a common Lyapunov variable. To this end, we tackle the multiobjective controller design problem with non-common Lyapunov variables.

Chapter 3

LMI-Based Multiobjective State-Feedback Controller Design with Non-Common Lyapunov Variables

In this chapter, we propose two LMI approaches to the multiobjective state-feedback $H_2/H_\infty/$ **D**-stability problems with non-common Lyapunov variables. We call these approaches a *subspace approach* and an *affine representation approach.*

In the subspace approach, we impose some additional constraints to the Lyapunov variables so that we convexify the problem and obtain LMI characterizations while keeping the state-feedback gain directly as one of the LMI variables. Because of the freedom left in the Lyapunov variables under the constraints, our formulation turns out to give a set of LMI characterizations that allow non-common Lyapunov variables. If we choose the additional constraints reasonably, it is shown that this approach leads to a feedback gain that achieves better (no worse) performance than the conventional approach. Furthermore, an effective iterative algorithm follows immediately from this approach.

On the other hand, in the affine representation approach, we perform the change of variables [4],[17],[23],[28] with non-common Lyapunov variables, and represent the resulting variables and the Lyapunov variables as a set of affine functions of yet new variables. Because the feedback gain is parametrized only by a nonlinear function of the Lyapunov variables and those variables resulting from change of variables, they are subject to non-convex constraints. It is to get around this difficulty that we introduce the yet new variables, and the affine functions are chosen to have a crucial characteristic that the troublesome non-convex constraints are satisfied regardless of the new variables. With these affine functions, we readily derive a set of LMI characterizations that allow non-common Lyapunov variables. In addition, a reasonable choice of the parameters included in the affine functions assures that this approach attains better (no worse) perfromance than the conventional approach. Thus, with the above two approaches, we derive two distinct sets of LMI characterizations for the multiobjective state-feedback controller design problems. It turns out that each set of LMI characterizations is derived by freezing some different portion of the freedom in the Lyapunov variables, and hence applying these two approaches by turns, we can use the freedom of the frozen portion complementarily. This consideration directly leads to a combined iterative algorithm, with which we can circumvent the conservatism successfully.

The effectiveness of the two approaches as well as the two iterative algorithms resulting from these approaches are demonstrated by numerical exmaples in this chapter. We also examine the effectiveness of the iterative algorithms in comparison with the algorithm proposed by Shimomura and Fujii [38]–[40]. Numerical examples show that the conventional approach with a common Lyapunov variable is very conservative, and the conservatism is successfully reduced with the use of non-common Lyapunov variables.

3.1 Preliminaries

As in the preceding chapter, we consider the continuous-time MIMO, LTI plant given by

$$\left\{\begin{array}{lllllll} \dot{x} & = & Ax & + & B_\infty w_\infty & + & B_2 w_2 & + & Bu \\ z_\infty & = & C_\infty x & + & D_\infty w_\infty & & & + & D_{z\infty} u \\ z_2 & = & C_2 x & & & & & + & D_{z2} u \end{array}\right. \tag{3.1}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$ and all other signals and matrices have appropriate dimensions.

In the state-feedback problems, the matrix inequalities (2.20), (2.22) and (2.25) for the H_∞ constraint, the $\mathbf{D}$-stability constraint and the H_2 specification, respectively, can be rewritten as follows.

$$X_\infty > 0 \tag{3.2a}$$

$$\begin{bmatrix} \mathrm{He}[(A+BK)X_\infty] & B_\infty & X_\infty(C_\infty + D_{z\infty}K)^T \\ B_\infty^T & -I & D_\infty^T \\ (C_\infty + D_{z\infty}K)X_\infty & D_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.2b}$$

$$X_{\mathbf{D}} > 0 \tag{3.3a}$$

$$M \otimes X_{\mathbf{D}} + \mathrm{He}[N \otimes \{(A+BK)X_{\mathbf{D}}\}] < 0 \tag{3.3b}$$

$$\begin{bmatrix} \mathrm{He}[(A+BK)X_2] & X_2(C_2+D_{z2}K)^T \\ (C_2+D_{z2}K)X_2 & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.4a}$$

$$\begin{bmatrix} Z_2 & B_2^T \\ B_2 & X_2 \end{bmatrix} > 0 \tag{3.4b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.4c}$$

Here, the variables are X_∞, $X_{\mathbf{D}}$, X_2, Z_2, γ_2^2 and the controller variable K. As we have seen in the preceding chapter, the inequalities (3.2)–(3.4) are BMI's. To avoid the difficulties in dealing with these BMI's, a common Lyapunov variable

$$X := X_\infty = X_{\mathbf{D}} = X_2 \tag{3.5}$$

has been forced in [17],[23],[28]. This common Lyapunov variable admits a simple change of variable

$$Y := KX \tag{3.6}$$

so that the constraints (3.2)–(3.4) result in LMI's with respect to X, Y, Z_2 and γ_2^2. Once the variables X and Y have been found, the state-feedback gain based on a common Lyapunov variable can be determined by

$$K_c = YX^{-1} \tag{3.7}$$

Here, assuming that the problem is feasible with a common Lyapunov variable, let us denote by $\tilde{\gamma}_{2c}^2$ the optimal value of γ_2^2 obtained with a common Lyapunov variable. It should be noted that $\tilde{\gamma}_{2c}$ is nothing but an *upper bound* of the H_2 cost achieved by the corresponding feedback gain K_c given by (3.7); the *actual* H_2 cost achieved by K_c can be calculated by minimizing γ_2^2 subject to the following LMI's.

$$\begin{bmatrix} \mathrm{He}[(A+BK_c)Q_2] & Q_2(C_2+D_{z2}K_c)^T \\ (C_2+D_{z2}K_c)Q_2 & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.8a}$$

$$\begin{bmatrix} Z_2 & B_2^T \\ B_2 & Q_2 \end{bmatrix} > 0 \tag{3.8b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.8c}$$

Here the variables are Q_2, Z_2 and γ_2^2. If we denote the optimal value of γ_2^2 obtained by this procedure by γ_{2c}^2, we have

$$\gamma_{2c} \leq \tilde{\gamma}_{2c} \tag{3.9}$$

Remark 3.1 In the following, we assume $n > m$, which means that the number of the states of the plant is larger than that of the control inputs. Note that this is naturally satisfied in practical plants.

3.2 Subspace Approach

In this section, we state the basic idea of the subspace approach, and derive a set of LMI characterizations that allow non-common Lyapunov variables. We also consider an iterative algorithm which follows immediately from this approach.

To begin with, let us define the new variables P_∞, $P_\mathbf{D}$ and P_2 as follows.

$$P_\infty := X_\infty^{-1}, \quad P_\mathbf{D} := X_\mathbf{D}^{-1}, \quad P_2 := X_2^{-1} \tag{3.10}$$

In the following, we also call the variables P_∞, $P_\mathbf{D}$ and P_2 Lyapunov variables. With (3.10), the inequalities (3.2)–(3.4) can be rearranged into

$$P_\infty > 0 \tag{3.11a}$$

$$\begin{bmatrix} \mathrm{He}[(A+BK)^T P_\infty] & P_\infty B_\infty & (C_\infty + D_{z\infty} K)^T \\ B_\infty^T P_\infty & -I & D_\infty^T \\ C_\infty + D_{z\infty} K & D_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.11b}$$

$$P_\mathbf{D} > 0 \tag{3.12a}$$

$$M \otimes P_\mathbf{D} + \mathrm{He}[N \otimes \{(A+BK)^T P_\mathbf{D}\}] < 0 \tag{3.12b}$$

$$\begin{bmatrix} \mathrm{He}[(A+BK)^T P_2] & (C_2 + D_{z2} K)^T \\ C_2 + D_{z2} K & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.13a}$$

$$\begin{bmatrix} Z_2 & B_2^T P_2 \\ P_2 B_2 & P_2 \end{bmatrix} > 0 \tag{3.13b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.13c}$$

where the variables are P_∞, $P_{\mathbf{D}}$, P_2, K, Z_2 and γ_2^2. As is easily seen, these matrix inequalities are not LMI's, either, because of the bilinear terms $P_jBK + K^TB^TP_j$ $(j = \infty, \mathbf{D}, 2)$. However, there is a remarkable difference between the matrix inequalities (3.2)–(3.4) and (3.11)–(3.13); the matrix inequalities (3.2)–(3.4) have bilinear terms of *direct* products between the Lyapunov variables and the controller variable, while the bilinear terms in (3.11)–(3.13) are *indirect* products between the Lyapunov variables P_∞, $P_{\mathbf{D}}$ and P_2 and the controller variable K through the constant matrix $B \in \mathbf{R}^{n\times m}$. Because of this difference, the matrix inequalities (3.11)–(3.13) turn out to enable us to derive a set of LMI characterizations that leave the feedback gain K directly as an LMI variable. This is achieved by freezing only some portion of the Lyapunov variables P_∞, $P_{\mathbf{D}}$ and P_2 and the freedom in the remaining portion enables us to have non-common Lyapunov variables. The rest of this section is devoted to showing the details of such LMI characterizations.

To this end, we impose the following additional constraints on the Lyapunov variables P_∞, $P_{\mathbf{D}}$ and P_2:

$$B^TP_\infty = U_\infty, \quad B^TP_{\mathbf{D}} = U_{\mathbf{D}}, \quad B^TP_2 = U_2 \tag{3.14}$$

Here, $U_\infty \in \mathbf{R}^{m\times n}$, $U_{\mathbf{D}} \in \mathbf{R}^{m\times n}$ and $U_2 \in \mathbf{R}^{m\times n}$ are constant matrices and given in advance in some reasonable way, the details of which will be discussed later in Subsection 3.2.2. Under the constraints (3.14), it is clear that there is still some freedom left in the variables P_∞, $P_{\mathbf{D}}$ and P_2 because of the assumption $n > m$. We begin by analyzing in details the freedom so that we can use it to derive a set of LMI characterizations that allow non-common Lyapunov variables. For the ease of description, we suppress the subscripts ∞, $\mathbf{D}$ and 2 that denote the design specifications for the time being, and consider the constraint

$$B^TP = U \tag{3.15}$$

on the symmetric matrix P, where $U \in \mathbf{R}^{m\times n}$ is some prescribed matrix.

3.2.1 Parametrization of P such that $B^TP = U$

This subsection gives the parametrization of a general solution of P satisfying (3.15). With this parametrization, we represent explicitly the freedom left in the variables P under the constraint (3.15). In the sequel, we assume that the matrix $B \in \mathbf{R}^{n\times m}$ has full column rank for simplicity and let the singular value decomposition of B^T be

$$B^T = W\begin{bmatrix} \bar{\Sigma} & 0_{m,n-m} \end{bmatrix}V, \quad W \in \mathbf{R}^{m\times m}, \quad \bar{\Sigma} \in \mathbf{R}^{m\times m}, \quad V \in \mathbf{R}^{n\times n} \tag{3.16}$$

where $\bar{\Sigma}$ is a positive definite diagonal matrix.

First, we consider if such P exists that satisfies (3.15). The following lemma gives the answer.

Lemma 3.1 [43] Given a matrix $U \in \mathbf{R}^{m\times n}$, there exists some symmetric matrix P satisfying $B^T P = U$ if and only if UB is symmetric.

We can give a general solution of (3.15) based on a constructive proof of the above lemma.

Lemma 3.2 [43] Suppose U is such that UB is symmetric and define

$$[\, R\ S\,] := W^T U V^T, \quad R \in \mathbf{R}^{m\times m}, \quad S \in \mathbf{R}^{m\times(n-m)} \tag{3.17}$$

Then, a general solution of P satisfying (3.15) is given by

$$P = P_0 + V^T \bar{\Pi} V, \quad P_0 := V^T \begin{bmatrix} \bar{\Sigma}^{-1}R & \bar{\Sigma}^{-1}S \\ S^T\bar{\Sigma}^{-1} & 0 \end{bmatrix} V, \quad \bar{\Pi} = 0_{m,m} \oplus \Pi \tag{3.18}$$

where $\Pi \in \mathbf{R}^{(n-m)\times(n-m)}$ is an arbitrary symmetric matrix.

Lemma 3.2 shows that the constraint (3.15) freezes only some portion of the symmetric matrix $P \in \mathbf{R}^{n\times n}$; to put it reverse, we still have freedom which is indicated by $\Pi \in \mathbf{R}^{(n-m)\times(n-m)}$ under the constraint. This observation is quite important to derive a set of LMI characterizations that allow non-common Lyapunov variables, as is described in the following subsection.

3.2.2 New LMI Characterization with the Subspace Approach

Now, we are ready to give our main result in this section, where we give a set of LMI characterizations for the multiobjective state-feedback controller design problem that allow non-common Lyapunov variables. Applying Lemma 3.2 to the constraints (3.14), we define

$$P_{\infty 0} := V^T \begin{bmatrix} \bar{\Sigma}^{-1}R_\infty & \bar{\Sigma}^{-1}S_\infty \\ S_\infty^T\bar{\Sigma}^{-1} & 0 \end{bmatrix} V, \quad [\, R_\infty\ S_\infty\,] := W^T U_\infty V^T \tag{3.19}$$

$$P_{\mathbf{D}0} := V^T \begin{bmatrix} \bar{\Sigma}^{-1}R_{\mathbf{D}} & \bar{\Sigma}^{-1}S_{\mathbf{D}} \\ S_{\mathbf{D}}^T\bar{\Sigma}^{-1} & 0 \end{bmatrix} V, \quad [\, R_{\mathbf{D}}\ S_{\mathbf{D}}\,] := W^T U_{\mathbf{D}} V^T \tag{3.20}$$

$$P_{20} := V^T \begin{bmatrix} \bar{\Sigma}^{-1}R_2 & \bar{\Sigma}^{-1}S_2 \\ S_2^T\bar{\Sigma}^{-1} & 0 \end{bmatrix} V, \quad [\, R_2\ S_2\,] := W^T U_2 V^T \tag{3.21}$$

Then, the general solutions of P_∞, $P_{\mathbf{D}}$ and P_2 satisfying (3.14) are given by

$$P_\infty = P_{\infty 0} + V^T \bar{\Pi}_\infty V, \quad \bar{\Pi}_\infty = 0_{m,m} \oplus \Pi_\infty, \quad \Pi_\infty \in \mathbf{R}^{(n-m)\times(n-m)} \tag{3.22}$$

$$P_{\mathbf{D}} = P_{\mathbf{D}0} + V^T \bar{\Pi}_{\mathbf{D}} V, \quad \bar{\Pi}_{\mathbf{D}} = 0_{m,m} \oplus \Pi_{\mathbf{D}}, \quad \Pi_{\mathbf{D}} \in \mathbf{R}^{(n-m)\times(n-m)} \tag{3.23}$$

$$P_2 = P_{20} + V^T \bar{\Pi}_2 V, \quad \bar{\Pi}_2 = 0_{m,m} \oplus \Pi_2, \quad \Pi_2 \in \mathbf{R}^{(n-m)\times(n-m)} \tag{3.24}$$

Using (3.22)–(3.24) together with (3.14), we can rewrite the BMI's (3.11)–(3.13) into the LMI's (3.25)–(3.27) given below, where (3.25) reflects the H_∞ constraint, (3.26) the **D**-stability constraint and (3.27) the H_2 specification.

$$P_{\infty 0} + V^T \bar{\Pi}_\infty V > 0 \tag{3.25a}$$

$$\begin{bmatrix} \mathrm{He}[A^T(P_{\infty 0} + V^T\bar{\Pi}_\infty V) + K^T U_\infty] & (P_{\infty 0} + V^T \bar{\Pi}_\infty V)B_\infty & (C_\infty + D_{z\infty}K)^T \\ B_\infty^T(P_{\infty 0} + V^T\bar{\Pi}_\infty V) & -I & D_\infty^T \\ C_\infty + D_{z\infty}K & D_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.25b}$$

$$P_{\mathbf{D}0} + V^T \bar{\Pi}_{\mathbf{D}} V > 0 \tag{3.26a}$$

$$M \otimes (P_{\mathbf{D}0} + V^T\bar{\Pi}_{\mathbf{D}}V) + \mathrm{He}[N \otimes \{A^T(P_{\mathbf{D}0} + V^T\bar{\Pi}_{\mathbf{D}}V) + K^T U_{\mathbf{D}}\}] < 0 \tag{3.26b}$$

$$\begin{bmatrix} \mathrm{He}[A^T(P_{20} + V^T\bar{\Pi}_2 V) + K^T U_2] & (C_2 + D_{z2}K)^T \\ C_2 + D_{z2}K & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.27a}$$

$$\begin{bmatrix} Z_2 & B_2^T(P_{20} + V^T\bar{\Pi}_2 V) \\ (P_{20} + V^T\bar{\Pi}_2 V)B_2 & P_{20} + V^T\bar{\Pi}_2 V \end{bmatrix} > 0 \tag{3.27b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.27c}$$

Here, the variables are $\bar{\Pi}_\infty$, $\bar{\Pi}_{\mathbf{D}}$, $\bar{\Pi}_2$, K, Z_2 and γ_2^2 with $\bar{\Pi}_\infty$, $\bar{\Pi}_{\mathbf{D}}$ and $\bar{\Pi}_2$ given by (3.22)–(3.24).

Observe that the above characterizations (3.25)–(3.27) are in fact LMI's that leave the feedback gain K directly as one of the LMI variables. With these LMI characterizations, the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problem can be cast into a *convex* optimization problem as described formally in the following.

Subspace Approach with Non-common Lyapunov Variables

Minimize γ_2^2 subject to the LMI's (3.25)–(3.27). Here, the variables are $\bar{\Pi}_\infty$, $\bar{\Pi}_{\mathbf{D}}$, $\bar{\Pi}_2$, K, Z_2 and γ_2^2 with $\bar{\Pi}_\infty$, $\bar{\Pi}_{\mathbf{D}}$ and $\bar{\Pi}_2$ given by (3.22)–(3.24).

Remark 3.2 In the optimization subject to the LMI's (3.25)–(3.27), the variables $\bar{\Pi}_\infty$, $\bar{\Pi}_{\mathbf{D}}$ and $\bar{\Pi}_2$ can take distinct values and hence the Lyapunov variables P_∞, $P_{\mathbf{D}}$ and P_2 given by (3.22)–(3.24) are not the same, in general. The implication is that we have given a new approach that allows non-common Lyapunov variables.

The approach presented above with non-common Lyapunov variables was obtained by introducing the additional constraints (3.14). Hence, the choice of U_∞, $U_{\mathbf{D}}$ and U_2 in the constraints would have strong influence on the feasibility of the LMI's (3.25)–(3.27) as well as the control performance achieved by the resulting feedback gain. Regarding the choice of U_∞, $U_{\mathbf{D}}$ and U_2, they have to satisfy at least the conditions that $U_\infty B$, $U_{\mathbf{D}}B$ and U_2B are all symmetric (see Lemma 3.1). With this in mind, we propose to determine U_∞, $U_{\mathbf{D}}$ and U_2 by

$$U_\infty = U_{\mathbf{D}} = B^T X^{-1}, \quad U_2 = B^T Q_2^{-1} \tag{3.28}$$

Here, X is the common Lyapunov variable (3.5) obtained by the conventional approach [6] and Q_2 is obtained by minimizing γ_2^2 in (3.8) for the feedback gain K_c resulting from the conventional approach. By this choice, $U_\infty B$, $U_{\mathbf{D}}B$ and U_2B become symmetric. Moreover, it is easy to see that the constraints (3.14) with (3.28) admit special solutions

$$P_\infty = P_{\mathbf{D}} = X^{-1}, \quad P_2 = Q_2^{-1} \tag{3.29}$$

Because the inequalities (3.11)–(3.13) are feasible for $\gamma_2 = \gamma_{2c}$ with the Lyapunov variables (3.29) and the feedback gain $K = K_c$, we are led to the following result.

Theorem 3.1 If we take U_∞, $U_{\mathbf{D}}$ and U_2 given by (3.28), then the LMI's (3.25)–(3.27) are feasible. Moreover, suppose we minimize γ_2^2 subject to the LMI's (3.25)–(3.27) with U_∞, $U_{\mathbf{D}}$ and U_2 given by (3.28), and denote the optimal value of γ_2^2 by $\tilde{\gamma}_{2n}^2$. Then, we have

$$\tilde{\gamma}_{2n} \leq \gamma_{2c} \tag{3.30}$$

Namely, we can obtain a feedback gain that achieves better (no worse) performance than the one based on a common Lyapunov variable.

If we determine U_∞, $U_{\mathbf{D}}$ and U_2 by (3.28), the explicit description of the subspace approach will be as follows.

Step 0. Minimize γ_2^2 subject to (3.2)–(3.4) with a common Lyapunov variable X and a variable Y given by (3.5) and (3.6), respectively [6]. Denote the optimal value of γ_2^2 by $\tilde{\gamma}_{2c}^2$ and the resulting feedback gain by $K_c := YX^{-1}$.

Step 1. Minimize γ_2^2 subject to (3.8). Denote the optimal value of γ_2^2 by γ_{2c}^2.

Step 2. Define U_∞, $U_{\mathbf{D}}$ and U_2 by (3.28), where X and Q_2 are those obtained in Step 0 and Step 1, respectively.

Step 3. Minimize γ_2^2 subject to the LMI's (3.25)–(3.27) to get the feedback gain K.

In Step 3, let us denote the resulting optimal value of γ_2^2 by $\tilde{\gamma}_{2n}^2$. Since $\tilde{\gamma}_{2n}$ is only an upper bound for the H_2 cost achieved by the gain K, minimize γ_2^2 subject to (3.8) with K_c replaced by K to get the exact value of the H_2 cost achieved by the feedback gain K, and denote the resulting optimal value of γ_2^2 by γ_{2n}^2. Then, it is clear that

$$\gamma_{2n} \leq \tilde{\gamma}_{2n} \leq \gamma_{2c} \leq \tilde{\gamma}_{2c} \tag{3.31}$$

This implies that we can arrive at the H_2 cost γ_{2n}, which is better (no worse) than γ_{2c} achieved by the conventional approach.

3.2.3 Iterative Algorithm Based on the Subspace Approach

In order to get around the conservatism of the conventional approach with a common Lyapunov variable as much as possible, it will be effective to apply the subspace approach iteratively. Such an iterative algorithm is immediately available if the constant matrices U_∞, $U_{\mathbf{D}}$ and U_2 in the additional constraints (3.14) are updated, in a reasonable fashion, based on K resulting from this approach.

One primitive way to obtain new U_∞ with given K is such that we determine U_∞ by $U_\infty := B^T\bar{P}_\infty$, where $\bar{P}_\infty$ is the solution minimizing m_∞ under the following inequality constraint.

$$-\bar{P}_\infty \oplus H_\infty(\bar{P}_\infty, K) < m_\infty I \tag{3.32}$$

Here, $H_\infty(\bar{P}_\infty, K)$ denotes the left-hand side of (3.11b) with P_∞ replaced by $\bar{P}_\infty$.

Similarly, new $U_{\mathbf{D}}$ can be determined by $U_{\mathbf{D}} := B^T\bar{P}_{\mathbf{D}}$, where $\bar{P}_{\mathbf{D}}$ is the solution minimizing $m_{\mathbf{D}}$ under the following inequality constraint.

$$-\bar{P}_{\mathbf{D}} \oplus \mathbf{D}(\bar{P}_{\mathbf{D}}, K) < m_{\mathbf{D}} I \tag{3.33}$$

Here, $\mathbf{D}(\bar{P}_{\mathbf{D}}, K)$ denotes the left-hand side of (3.12b) with $P_{\mathbf{D}}$ replaced by $\bar{P}_{\mathbf{D}}$.

Concerning the constant matrix U_2, it would be reasonable to determine it by $U_2 := B^TQ_2^{-1}$, where Q_2 is the solution of (3.8) with K_c replaced by K.

In the above two procedures to determine new U_∞ and $U_{\mathbf{D}}$, it is assured that m_∞ and $m_{\mathbf{D}}$ result in negative numbers, because the feedback gain K actually achieves both of the H_∞ and $\mathbf{D}$-stability constraints. As a consequence, it follows that the LMI's (3.25)–(3.27) are feasible under these new U_∞, $U_{\mathbf{D}}$ and U_2. This is because the constraints (3.14) with these new U_∞, $U_{\mathbf{D}}$ and U_2 admit special solutions

$$P_\infty = \bar{P}_\infty, \quad P_{\mathbf{D}} = \bar{P}_{\mathbf{D}}, \quad P_2 = Q_2^{-1} \tag{3.34}$$

Note that the inequalities (3.11)–(3.13) are satisfied for γ_2 set to the actual cost achieved by K, with the Lyapunov variables (3.34) and with the given state-feedback gain K.

Noting that Step 0 of the subspace approach is just one simple method to obtain U_∞, $U_{\mathbf{D}}$ and U_2 that assure the feasibility of the LMI's (3.25)–(3.27), we can still follow the remaining steps with these new matrices. This idea leads directly to the following iterative algorithm.

Iterative Algorithm Based on the Subspace Approach

Step 0. Minimize γ_2^2 subject to (3.2)–(3.4) with a common Lyapunov variable X and a variable Y given by (3.5) and (3.6), respectively. Denote the resulting feedback gain by $K^{(i)} := YX^{-1}$, where we set $i = 0$.

Step 1. Minimize γ_2^2 subject to (3.8) with K_c replaced by $K^{(i)}$. Denote the optimal value of γ_2^2 by $(\gamma_2^{(i)})^2$, and check the stop criterion with respect to $\gamma_2^{(i)}$.

Step 2. Calculate $\bar{P}_\infty$ and $\bar{P}_{\mathbf{D}}$ minimizing m_∞ and $m_{\mathbf{D}}$ under the constraints (3.32) and (3.33), respectively, with K replaced by $K^{(i)}$. With $\bar{P}_\infty$ and $\bar{P}_{\mathbf{D}}$ together with Q_2 obtained in Step 1, determine U_∞, $U_{\mathbf{D}}$ and U_2 by

$$U_\infty := B^T\bar{P}_\infty, \quad U_{\mathbf{D}} := B^T\bar{P}_{\mathbf{D}}, \quad U_2 := B^TQ_2^{-1} \tag{3.35}$$

Step 3. Set $i := i+1$ and minimize γ_2^2 subject to the LMI's (3.25)–(3.27) to get the feedback gain $K^{(i)}$. Then, go to Step 1.

As mentioned above, we can always assure the feasibility of the LMI's in Step 3. It is also assured that the H_2 cost $\gamma_2^{(i)}$ is monotonically nonincreasing throughout the iterative algorithm. The stop criterion in Step 1 can be specified with the decreasing rate of the H_2 cost $\gamma_2^{(i)}$ or the number of iterations, and so on.

It should be noted that there exists a conventional iterative algorithm in which the Lyapunov variables P_∞, $P_{\mathbf{D}}$, P_2 and the feedback gain K in (3.11)–(3.13) are alternately fixed and the corresponding LMI's are solved by turns with respect to unfixed variables. Indeed, the conventional method given below is one primitive method to obtain better performance than the one based on a common Lyapunov variable, which clearly shows the similarities and differences between these two iterative algorithms.

Conventional Iterative Algorithm

Steps 0 and **1** are the same as in the iterative algorithm based on the subspace approach.

Step 2. Calculate $\bar{P}_\infty$ and $\bar{P}_{\mathbf{D}}$ minimizing m_∞ and $m_{\mathbf{D}}$ under the constraints (3.32) and (3.33), respectively, with K replaced by $K^{(i)}$.

Step 3. Minimize γ_2^2 subject to (3.11)–(3.13) for the variable K by freezing P_∞, $P_{\mathbf{D}}$ to $\bar{P}_\infty$, $\bar{P}_{\mathbf{D}}$ obtained in Step 2 and P_2 to Q_2^{-1} obtained in Step 1, respectively. Set $i := i+1$ and denote the resulting feedback gain by $K^{(i)}$. Then, go to Step 1.

The effectiveness of the new iterative algorithm will be studied in Section 3.5 in comparison with this conventional iterative algorithm.

3.3 Affine Representation Approach

In the preceding section, we introduced the additional constraints (3.14) to the Lyapunov variables so that we can convexify the problem, and obtained LMI characterizations that leave the feedback gain directly as one of the variables. Because of the freedom left in the Lyapunov variables under the additional constraints, this approach turned out to give a set of LMI characterizations (3.25)–(3.27) that allow non-common Lyapunov variables.

This section describes another approach to the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problems: the affine representation approach. In this new approach, we perform a standard procedure called change of variables [4],[23] and represent the resulting variables as a set of affine functions by introducing yet new variables. These affine functions are chosen to have a crucial characteristic that troublesome non-convex constraints are satisfied regardless of the new variables. With these affine functions, we readily derive a set of LMI characterizations that allow non-common Lyapunov variables. Furthermore, a simple combination of the subspace approach and the affine representation approach leads to another effective iterative algorithm, with which we can get around the conservatism successfully.

3.3.1 Change of Variables via Affine Functions

Let us focus on the inequalities (3.2)–(3.4) again. Applying the *change of variables*

$$Y_\infty := KX_\infty, \quad Y_{\mathbf{D}} := KX_{\mathbf{D}}, \quad Y_2 := KX_2 \tag{3.36}$$

to (3.2)–(3.4), we get

$$X_\infty > 0 \tag{3.37a}$$

$$\begin{bmatrix} \mathrm{He}[AX_\infty + BY_\infty] & B_\infty & (C_\infty X_\infty + D_{z\infty}Y_\infty)^T \\ B_\infty^T & -I & D_\infty^T \\ C_\infty X_\infty + D_{z\infty}Y_\infty & D_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.37b}$$

$$X_{\mathbf{D}} > 0 \tag{3.38a}$$

$$M \otimes X_{\mathbf{D}} + \mathrm{He}[N \otimes (AX_{\mathbf{D}} + BY_{\mathbf{D}})] < 0 \tag{3.38b}$$

$$\begin{bmatrix} \mathrm{He}[AX_2 + BY_2] & (C_2X_2 + D_{z2}Y_2)^T \\ C_2X_2 + D_{z2}Y_2 & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.39a}$$

$$\begin{bmatrix} Z_2 & B_2^T \\ B_2 & X_2 \end{bmatrix} > 0 \tag{3.39b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.39c}$$

Here, the variables are X_∞, $X_{\mathbf{D}}$, X_2, Y_∞, $Y_{\mathbf{D}}$, Y_2, Z_2 and γ_2^2. Note that the size of Y_∞, $Y_{\mathbf{D}}$ and Y_2 are the same as that of the feedback gain K and hence $m \times n$.

It follows from the change of variables (3.36) that the variables X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 are subject to the non-convex constraint given below.

$$K := Y_\infty X_\infty^{-1} = Y_{\mathbf{D}} X_{\mathbf{D}}^{-1} = Y_2 X_2^{-1} \tag{3.40}$$

This is inevitable as long as we do change of variables as in (3.36), and the common variables

$$X := X_\infty = X_{\mathbf{D}} = X_2, \quad Y := Y_\infty = Y_{\mathbf{D}} = Y_2 \tag{3.41}$$

correspond to the simplest way to meet the constraint. Here an intriguing interpretation of the common variables is that the original variables X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 are respectively represented by *affine functions* (in fact, identity functions) of the new variables X and Y in such a way that the constraint (3.40) is satisfied regardless of the new variables X and Y. Generalizing this interpretation, we are led to the key observation in this section that, to convexify the problem, it is actually enough for the variables X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 to satisfy the following two conditions.

Condition 1. The variables X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 are represented as some affine functions of other new variables.

Condition 2. The variables X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 satisfy the constraint (3.40) regardless of these new variables.

Taking account of the first condition, let us introduce the new variables Ω_∞, $\Omega_{\mathbf{D}}$, Ω_2 and Γ_∞, $\Gamma_{\mathbf{D}}$, Γ_2, and, for the moment, represent X_∞, $X_{\mathbf{D}}$, X_2 and Y_∞, $Y_{\mathbf{D}}$, Y_2 as

$$X_\infty = \mathcal{X}_\infty(\Omega_\infty), \quad X_{\mathbf{D}} = \mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}}), \quad X_2 = \mathcal{X}_2(\Omega_2) \tag{3.42a}$$

$$Y_\infty = \mathcal{Y}_\infty(\Gamma_\infty), \quad Y_{\mathbf{D}} = \mathcal{Y}_{\mathbf{D}}(\Gamma_{\mathbf{D}}), \quad Y_2 = \mathcal{Y}_2(\Gamma_2) \tag{3.42b}$$

where $\mathcal{X}_\infty(\cdot)$, $\mathcal{X}_\mathbf{D}(\cdot)$, $\mathcal{X}_2(\cdot)$, $\mathcal{Y}_\infty(\cdot)$, $\mathcal{Y}_\mathbf{D}(\cdot)$ and $\mathcal{Y}_2(\cdot)$ are some matrix-valued affine functions to be determined. Substituting (3.42) into (3.37)–(3.39), the variables are regarded to be Ω_∞, $\Omega_\mathbf{D}$, Ω_2, Γ_∞, $\Gamma_\mathbf{D}$, Γ_2, Z_2 and γ_2^2. In the sequel, we will give explicitly a set of affine functions such that the second condition above is also satisfied, or equivalently,

$$\mathcal{Y}_\infty(\Gamma_\infty)\mathcal{X}_\infty(\Omega_\infty)^{-1} = \mathcal{Y}_\mathbf{D}(\Gamma_\mathbf{D})\mathcal{X}_\mathbf{D}(\Omega_\mathbf{D})^{-1} = \mathcal{Y}_2(\Gamma_2)\mathcal{X}_2(\Omega_2)^{-1} \tag{3.43}$$

regardless of Ω_∞, $\Omega_\mathbf{D}$, Ω_2 and Γ_∞, $\Gamma_\mathbf{D}$, Γ_2 (in fact, we will take $\Gamma_\infty = \Gamma_\mathbf{D} = \Gamma_2$), while the conservatism being circumvented as much as possible. To derive such affine functions, we make the following assumption.

Assumption 3.1 A set of matrices X_∞, $X_\mathbf{D}$, X_2, Y_∞, $Y_\mathbf{D}$, Y_2 (and Z_2) that satisfy (3.37)–(3.39) for some $\gamma_2 = \gamma_{20} > 0$ is given. Furthermore, denoting these matrices by $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20} and $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20}, they satisfy

$$Y_{\infty 0}X_{\infty 0}^{-1} = Y_{\mathbf{D}0}X_{\mathbf{D}0}^{-1} = Y_{20}X_{20}^{-1} =: K_0 \tag{3.44}$$

Without loss of generality, we assume that K_0 has full row rank.

The above assumption is necessary in our derivation of the affine functions. A reasonable way to determine the matrices in Assumption 3.1 will be discussed later in Subsection 3.3.2.

Based on the constant matrices $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20} and $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20}, we derive affine functions as in (3.42) with the above-mentioned properties. The outline of the derivation will be as follows.

1. We first construct symmetric-matrix-valued affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_\mathbf{D}(\Omega_\mathbf{D})$ and $\mathcal{X}_2(\Omega_2)$ such that

   $$Y_{\infty 0}\mathcal{X}_\infty(\Omega_\infty)^{-1} = Y_{\mathbf{D}0}\mathcal{X}_\mathbf{D}(\Omega_\mathbf{D})^{-1} = Y_{20}\mathcal{X}_2(\Omega_2)^{-1} \tag{3.45}$$

   regardless of Ω_∞, $\Omega_\mathbf{D}$ and Ω_2.

2. For the affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_\mathbf{D}(\Omega_\mathbf{D})$ and $\mathcal{X}_2(\Omega_2)$ constructed in the first step, we next construct matrix-valued affine functions $\mathcal{Y}_\infty(\Gamma_\infty)$, $\mathcal{Y}_\mathbf{D}(\Gamma_\mathbf{D})$ and $\mathcal{Y}_2(\Gamma_2)$ such that (3.43) holds regardless of the variables Ω_∞, $\Omega_\mathbf{D}$, Ω_2 and Γ_∞, $\Gamma_\mathbf{D}$, Γ_2 (in fact, we will take $\Gamma_\infty = \Gamma_\mathbf{D} = \Gamma_2$).

Once these affine functions are constructed, our new approach to the multiobjective state-feedback controller design problem will be completed simply by substituting them into (3.37)–(3.39). The rest of this subsection is devoted to the details of the above procedure for the derivation of the affine functions.

In the following, let the singular value decompositions of $Y_{\infty 0} \in \mathbf{R}^{m\times n}$, $Y_{\mathbf{D}0} \in \mathbf{R}^{m\times n}$ and $Y_{20} \in \mathbf{R}^{m\times n}$ be respectively

$$Y_{\infty 0} = W_\infty \Sigma_\infty V_\infty, \quad Y_{\mathbf{D}0} = W_{\mathbf{D}} \Sigma_{\mathbf{D}} V_{\mathbf{D}}, \quad Y_{20} = W_2 \Sigma_2 V_2 \tag{3.46}$$

The matrices Σ_∞, $\Sigma_{\mathbf{D}}$ and Σ_2 can be represented respectively in the following forms.

$$\Sigma_\infty =: \left[\bar{\Sigma}_\infty \ 0_{m,n-m} \right], \quad \Sigma_{\mathbf{D}} =: \left[\bar{\Sigma}_{\mathbf{D}} \ 0_{m,n-m} \right], \quad \Sigma_2 =: \left[\bar{\Sigma}_2 \ 0_{m,n-m} \right] \tag{3.47}$$

Here, $Y_{\infty 0}$, $Y_{\mathbf{D}0}$ and Y_{20} have full row rank because K_0 does by Assumption 3.1. Hence, $\bar{\Sigma}_\infty \in \mathbf{R}^{m \times m}$, $\bar{\Sigma}_{\mathbf{D}} \in \mathbf{R}^{m \times m}$ and $\bar{\Sigma}_2 \in \mathbf{R}^{m \times m}$ are positive-definite diagonal matrices.

Construction of $\mathcal{X}_\infty$, $\mathcal{X}_{\mathbf{D}}$ and $\mathcal{X}_2$

Now, we consider the first step: we construct the symmetric-matrix-valued affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$ satisfying (3.45). For that purpose, it is useful to explore symmetric matrices P_∞, $P_{\mathbf{D}}$ and P_2 satisfying

$$Y_{\infty 0} P_\infty = Y_{\mathbf{D}0} P_{\mathbf{D}} = Y_{20} P_2 \tag{3.48}$$

Regarding the existence of such symmetric matrices, the following proposition is a direct consequence from Lemma 3.1.

Proposition 3.1 Symmetric matrices P_∞, $P_{\mathbf{D}}$ and P_2 satisfying (3.48) exist if and only if there exists some $K \in \mathbf{R}^{m \times n}$ such that

$$K Y_{\infty 0}^T = Y_{\infty 0} K^T, \quad K Y_{\mathbf{D}0}^T = Y_{\mathbf{D}0} K^T, \quad K Y_{20}^T = Y_{20} K^T \tag{3.49}$$

In particular, if (3.49) holds, then there exist symmetric matrices P_∞, $P_{\mathbf{D}}$ and P_2 satisfying

$$Y_{\infty 0} P_\infty = Y_{\mathbf{D}0} P_{\mathbf{D}} = Y_{20} P_2 = K \tag{3.50}$$

It is clear from (3.44) that we actually have (3.49) for $K = K_0$, so that there exist symmetric matrices P_∞, $P_{\mathbf{D}}$ and P_2 satisfying (3.48). Hence, based on Proposition 3.1, let us suppose that we are given a matrix K satisfying (3.49) (perhaps $K = K_0$, but this is not necessarily assumed). Then, a crucial problem is to give a general class of P_∞, $P_{\mathbf{D}}$ and P_2 satisfying (3.50). Regarding this problem, we obtain the following proposition immediately from Lemma 3.2.

Proposition 3.2 Suppose that $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20} and K satisfying (3.49) are given, and consider the singular value decompositions given by (3.46) and (3.47). Then the general solutions of P_∞, $P_{\mathbf{D}}$ and P_2 satisfying (3.50) are given by

$$P_\infty = V_\infty^T \begin{bmatrix} \Xi_{\infty 1} & \Xi_{\infty 12} \\ \Xi_{\infty 12}^T & \Xi_\infty \end{bmatrix} V_\infty, \quad [\, \Xi_{\infty 1} \; \Xi_{\infty 12} \,] := \bar{\Sigma}_\infty^{-1} W_\infty^T K V_\infty^T,$$

$$P_{\mathbf{D}} = V_{\mathbf{D}}^T \begin{bmatrix} \Xi_{\mathbf{D}1} & \Xi_{\mathbf{D}12} \\ \Xi_{\mathbf{D}12}^T & \Xi_{\mathbf{D}} \end{bmatrix} V_{\mathbf{D}}, \quad [\, \Xi_{\mathbf{D}1} \; \Xi_{\mathbf{D}12} \,] := \bar{\Sigma}_{\mathbf{D}}^{-1} W_{\mathbf{D}}^T K V_{\mathbf{D}}^T, \tag{3.51}$$

$$P_2 = V_2^T \begin{bmatrix} \Xi_{2,1} & \Xi_{2,12} \\ \Xi_{2,12}^T & \Xi_2 \end{bmatrix} V_2, \quad [\, \Xi_{2,1} \; \Xi_{2,12} \,] := \bar{\Sigma}_2^{-1} W_2^T K V_2^T$$

where $\Xi_{\infty 1} \in \mathbf{R}^{m\times m}$, $\Xi_{\infty 12} \in \mathbf{R}^{m\times(n-m)}$, $\Xi_{\mathbf{D}1} \in \mathbf{R}^{m\times m}$, $\Xi_{\mathbf{D}12} \in \mathbf{R}^{m\times(n-m)}$, $\Xi_{2,1} \in \mathbf{R}^{m\times m}$ and $\Xi_{2,12} \in \mathbf{R}^{m\times(n-m)}$ are constant matrices while $\Xi_\infty \in \mathbf{R}^{(n-m)\times(n-m)}$, $\Xi_{\mathbf{D}} \in \mathbf{R}^{(n-m)\times(n-m)}$ and $\Xi_2 \in \mathbf{R}^{(n-m)\times(n-m)}$ are arbitrary symmetric matrices.

Recall that the general solution (3.51) is derived under the condition that a matrix K satisfying (3.49) is given. For the time being, we consider to fix K to K_0 given by (3.44) in Assumption 3.1, so that the constant matrices $\Xi_{\infty 1}$, $\Xi_{\infty 12}$, and so on, in Proposition 3.2 will be determined by setting $K = K_0$ in (3.51); we will return to the case of $K \neq K_0$ later on.

We are now in a position to give affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$ satisfying (3.45), or to be more precise,

$$Y_{\infty 0}\mathcal{X}_\infty(\Omega_\infty)^{-1} = Y_{\mathbf{D}0}\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})^{-1} = Y_{20}\mathcal{X}_2(\Omega_2)^{-1} = K_0 \tag{3.52}$$

Proposition 3.3 Under the notation of Proposition 3.2 with K set to K_0 given by (3.44), consider the affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$ given respectively by

$$\begin{aligned} \mathcal{X}_\infty(\Omega_\infty) &= \Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty, \\ \mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}}) &= \Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}}, \\ \mathcal{X}_2(\Omega_2) &= \Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2 \end{aligned} \tag{3.53}$$

where Θ_∞, Λ_∞, $\Theta_{\mathbf{D}}$, $\Lambda_{\mathbf{D}}$, Θ_2 and Λ_2 are constant matrices defined by

$$\Theta_\infty := V_\infty^T \begin{bmatrix} \Xi_{\infty 1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} V_\infty, \quad \Lambda_\infty := \begin{bmatrix} -\Xi_{\infty 12}^T \Xi_{\infty 1}^{-1} & I \end{bmatrix} V_\infty$$

$$\Theta_{\mathbf{D}} := V_{\mathbf{D}}^T \begin{bmatrix} \Xi_{\mathbf{D}1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} V_{\mathbf{D}}, \quad \Lambda_{\mathbf{D}} := \begin{bmatrix} -\Xi_{\mathbf{D}12}^T \Xi_{\mathbf{D}1}^{-1} & I \end{bmatrix} V_{\mathbf{D}} \tag{3.54}$$

$$\Theta_2 := V_2^T \begin{bmatrix} \Xi_{2,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} V_2, \quad \Lambda_2 := \begin{bmatrix} -\Xi_{2,12}^T \Xi_{2,1}^{-1} & I \end{bmatrix} V_2$$

and $\Omega_\infty \in \mathbf{R}^{(n-m)\times(n-m)}$, $\Omega_{\mathbf{D}} \in \mathbf{R}^{(n-m)\times(n-m)}$ and $\Omega_2 \in \mathbf{R}^{(n-m)\times(n-m)}$ are symmetric. Then, $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$ are symmetric and satisfy (3.52) for any Ω_∞, $\Omega_{\mathbf{D}}$ and Ω_2.

This proposition is based on the idea presented in [39]. A proof of the proposition is given below to complete our exposition.

Proof. Applying the matrix inversion formula [47] to P_∞ given by (3.51), we have

$$P_\infty^{-1} = \Theta_\infty + \Lambda_\infty^T(\Xi_\infty - \Xi_{\infty 12}^T \Xi_{\infty 1}^{-1} \Xi_{\infty 12})^{-1} \Lambda_\infty \tag{3.55}$$

Hence, defining $\Omega_\infty := (\Xi_\infty - \Xi_{\infty 12}^T \Xi_{\infty 1}^{-1} \Xi_{\infty 12})^{-1}$, we can see that P_∞^{-1} is nothing but $\mathcal{X}_\infty(\Omega_\infty)$ in (3.53). Now, since P_∞ satisfies $K_0 = Y_{\infty 0} P_\infty$ for any symmetric Ξ_∞ (and hence, for any symmetric Ω_∞) by Proposition 3.2, it is obvious that $\mathcal{X}_\infty(\Omega_\infty)$ given by (3.53) satisfies $K_0 = Y_{\infty 0}\mathcal{X}_\infty(\Omega_\infty)^{-1}$ for any Ω_∞. Similarly for $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$. Q.E.D.

Proposition 3.3 gives a candidate for the set of desirable affine functions $\mathcal{X}_\infty(\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\Omega_2)$ that satisfy (3.45). In fact, they satisfy (3.52). However, from the viewpoint of deriving a new feedback gain K from the initial gain K_0 based on the affine functions $X_\infty = \mathcal{X}_\infty(\Omega_\infty)$, $X_{\mathbf{D}} = \mathcal{X}_{\mathbf{D}}(\Omega_{\mathbf{D}})$ and $X_2 = \mathcal{X}_2(\Omega_2)$, it is imperative for these affine functions to satisfy the less restrictive constraint (3.45), rather than (3.52). In other words, we need to regard K in Proposition 3.2 as a variable. From (3.51), this means that we must pay attention to the dependence of $\Xi_{\infty 1}$, $\Xi_{\infty 12}$, $\Xi_{\mathbf{D}1}$, $\Xi_{\mathbf{D}12}$, $\Xi_{2,1}$ and $\Xi_{2,12}$ on K.

To keep the affine nature of (3.53) with respect to the variables even under this viewpoint, it is necessary to restrict somehow the way we regard K as a variable. Indeed, by inspection, it would be reasonable from (3.51), (3.53) and (3.54) to take $K = \eta^{-1}K_0$, where the scalar η is a new additional variable. Then, we readily arrive at the following proposition.

Proposition 3.4 Consider the affine functions $\mathcal{X}_\infty(\eta, \Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\eta, \Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\eta, \Omega_2)$ given respectively by

$$\begin{aligned} \mathcal{X}_\infty(\eta, \Omega_\infty) &= \eta\Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty, \\ \mathcal{X}_{\mathbf{D}}(\eta, \Omega_{\mathbf{D}}) &= \eta\Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}}, \\ \mathcal{X}_2(\eta, \Omega_2) &= \eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2 \end{aligned} \tag{3.56}$$

where $\eta \in \mathbf{R}$, and $\Omega_\infty \in \mathbf{R}^{(n-m)\times(n-m)}$, $\Omega_{\mathbf{D}} \in \mathbf{R}^{(n-m)\times(n-m)}$, $\Omega_2 \in \mathbf{R}^{(n-m)\times(n-m)}$ while Θ_∞, Λ_∞, $\Theta_{\mathbf{D}}$, $\Lambda_{\mathbf{D}}$, Θ_2 and Λ_2 are all the same as in Proposition 3.3. Then, for any η, Ω_∞, $\Omega_{\mathbf{D}}$ and Ω_2, we have (3.45), or to be more precise,

$$Y_{\infty 0}\mathcal{X}_\infty(\eta, \Omega_\infty)^{-1} = Y_{\mathbf{D}0}\mathcal{X}_{\mathbf{D}}(\eta, \Omega_{\mathbf{D}})^{-1} = Y_{20}\mathcal{X}_2(\eta, \Omega_2)^{-1} = \eta^{-1}K_0 \tag{3.57}$$

Construction of $\mathcal{Y}_\infty$, $\mathcal{Y}_{\mathbf{D}}$ and $\mathcal{Y}_2$

Now, we consider the second step: we construct affine functions $\mathcal{Y}_\infty(\Gamma_\infty)$, $\mathcal{Y}_{\mathbf{D}}(\Gamma_{\mathbf{D}})$ and $\mathcal{Y}_2(\Gamma_2)$ satisfying (3.43), or to be more precise,

$$\mathcal{Y}_\infty(\Gamma_\infty)\mathcal{X}_\infty(\eta,\Omega_\infty)^{-1} = \mathcal{Y}_{\mathbf{D}}(\Gamma_{\mathbf{D}})\mathcal{X}_{\mathbf{D}}(\eta,\Omega_{\mathbf{D}})^{-1} = \mathcal{Y}_2(\Gamma_2)\mathcal{X}_2(\eta,\Omega_2)^{-1} \tag{3.58}$$

for the affine functions $\mathcal{X}_\infty(\eta,\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\eta,\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\eta,\Omega_2)$ given by (3.56). Regarding this construction, we simply introduce a new variable $\Gamma := \Gamma_\infty = \Gamma_{\mathbf{D}} = \Gamma_2$ and define

$$\mathcal{Y}_\infty(\Gamma) = \Gamma Y_{\infty 0}, \quad \mathcal{Y}_{\mathbf{D}}(\Gamma) = \Gamma Y_{\mathbf{D}0}, \quad \mathcal{Y}_2(\Gamma) = \Gamma Y_{20} \tag{3.59}$$

We then have the following result.

Proposition 3.5 For the affine functions $\mathcal{X}_\infty(\eta,\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\eta,\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\eta,\Omega_2)$ given by (3.56) and $\mathcal{Y}_\infty(\Gamma)$, $\mathcal{Y}_{\mathbf{D}}(\Gamma)$ and $\mathcal{Y}_2(\Gamma)$ given by (3.59), we have (3.58) for any η, Ω_∞, $\Omega_{\mathbf{D}}$, Ω_2 and Γ. In particular, we have

$$\mathcal{Y}_\infty(\Gamma)\mathcal{X}_\infty(\eta,\Omega_\infty)^{-1} = \mathcal{Y}_{\mathbf{D}}(\Gamma)\mathcal{X}_{\mathbf{D}}(\eta,\Omega_{\mathbf{D}})^{-1} = \mathcal{Y}_2(\Gamma)\mathcal{X}_2(\eta,\Omega_2)^{-1} = \eta^{-1}\Gamma K_0 \tag{3.60}$$

Remark 3.3 It will turn out in the following subsection that the H_2 cost will be minimized over a set of gains K of the form

$$K = \eta^{-1}\Gamma K_0 \tag{3.61}$$

in our new approach, because of the form (3.60). In this sense, the scalar variable η may look redundant in the above construction, but this is not the case. The variable η, together with Ω_∞, $\Omega_{\mathbf{D}}$ and Ω_2, corresponds to the scaling of X_∞, $X_{\mathbf{D}}$ and X_2 in (3.37)–(3.39) (see (3.65)). It is known that the scaling of the Lyapunov variables is an important factor to get around the conservatism [25].

3.3.2 New LMI Characterization with the Affine Representation Approach

Now, we are ready to give our main result in this section. Substituting the affine functions $\mathcal{X}_\infty(\eta,\Omega_\infty)$, $\mathcal{X}_{\mathbf{D}}(\eta,\Omega_{\mathbf{D}})$ and $\mathcal{X}_2(\eta,\Omega_2)$ given in (3.56) and $\mathcal{Y}_\infty(\Gamma)$, $\mathcal{Y}_{\mathbf{D}}(\Gamma)$ and $\mathcal{Y}_2(\Gamma)$ given in (3.59) into (3.37)–(3.39), and recalling (3.60), we arrive at the LMI's (3.62)–(3.64) given below, where (3.62) reflects the H_∞ constraint, (3.63) the **D**-stability constraint and (3.64) the H_2 specification.

$$\eta\Theta_\infty + \Lambda_\infty^T\Omega_\infty\Lambda_\infty > 0 \tag{3.62a}$$

$$\begin{bmatrix} \mathrm{He}[A(\eta\Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty) + B\Gamma Y_{\infty 0}] & B_\infty & (\eta\Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty)C_\infty^T + Y_{\infty 0}^T \Gamma^T D_{z\infty}^T \\ B_\infty^T & -I & D_\infty^T \\ C_\infty(\eta\Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty) + D_{z\infty}\Gamma Y_{\infty 0} & D_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.62b}$$

$$\eta\Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}} > 0 \tag{3.63a}$$

$$M \otimes (\eta\Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}}) + \mathrm{He}\left[N \otimes \left\{A(\eta\Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}}) + B\Gamma Y_{\mathbf{D}0}\right\}\right] < 0 \tag{3.63b}$$

$$\begin{bmatrix} \mathrm{He}[A(\eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2) + B\Gamma Y_{20}] & (\eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2)C_2^T + Y_{20}^T \Gamma^T D_{z2}^T \\ C_2(\eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2) + D_{z2}\Gamma Y_{20} & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.64a}$$

$$\begin{bmatrix} Z_2 & B_2^T \\ B_2 & \eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2 \end{bmatrix} > 0 \tag{3.64b}$$

$$\mathrm{trace}(Z_2) < 1 \tag{3.64c}$$

Here, the variables are η, Ω_∞, $\Omega_{\mathbf{D}}$, Ω_2, Γ, Z_2 and γ_2^2, and the constant matrices Θ_∞, Λ_∞, $\Theta_{\mathbf{D}}$, $\Lambda_{\mathbf{D}}$, Θ_2 and Λ_2 are defined through (3.46), (3.47), (3.51) and (3.54) with K in (3.51) set to K_0 given by (3.44). The optimal feedback gain K is given by (3.61).

With these LMI characterizations, the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problem can be cast into a *convex* optimization problem as is described formally in the following.

Affine Representation Approach with Non-common Lyapunov Variables

Minimize γ_2^2 subject to the LMI's (3.62)–(3.64). Here, the variables are η, Ω_∞, $\Omega_{\mathbf{D}}$, Ω_2, Γ, Z_2 and γ_2^2

Remark 3.4 In the optimization subject to the LMI's (3.62)–(3.64), the variables Ω_∞, $\Omega_{\mathbf{D}}$ and Ω_2 can take distinct values and hence the Lyapunov variables X_∞, $X_{\mathbf{D}}$ and X_2 given by

$$X_\infty = \eta\Theta_\infty + \Lambda_\infty^T \Omega_\infty \Lambda_\infty, \quad X_{\mathbf{D}} = \eta\Theta_{\mathbf{D}} + \Lambda_{\mathbf{D}}^T \Omega_{\mathbf{D}} \Lambda_{\mathbf{D}}, \quad X_2 = \eta\Theta_2 + \Lambda_2^T \Omega_2 \Lambda_2 \tag{3.65}$$

are not the same, in general. The implication is that the affine representation approach allows non-common Lyapunov variables.

It should be noted that the LMI's described by (3.62)–(3.64) that allow non-common Lyapunov variables have been obtained under Assumption 3.1. Hence, before solving these LMI's, we have to determine the constant matrices $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20}, $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20} (and K_0) satisfying (3.44). Regarding this problem, as in the preceding section, we propose to choose these matrices as

$$X_{\infty 0} = X_{\mathbf{D}0} = X, \quad X_{20} = Q_2, \quad Y_{\infty 0} = Y_{\mathbf{D}0} = Y, \quad Y_{20} = K_c Q_2, \tag{3.66}$$

where X and Y are the common variables (3.41) obtained by the conventional approach [6], K_c is given by $K_c = YX^{-1}$, and Q_2 is obtained by minimizing γ_2^2 in (3.8) for the gain K_c. We assume that K_c is of full row rank; if this is not the case, we can add a small perturbation to the common variable Y so that Assumption 3.1 is satisfied.

An advantage of the choice (3.66) is now shown. Because the inequalities (3.37)–(3.39) are feasible for $\gamma_2 = \gamma_{2c}$ with

$$X_\infty = X_{\mathbf{D}} = X, \quad X_2 = Q_2, \quad Y_\infty = Y_{\mathbf{D}} = Y, \quad Y_2 = K_c Q_2, \tag{3.67}$$

it can be claimed that the LMI's (3.62)–(3.64) are feasible for $\gamma_2 = \gamma_{2c}$ if we set the variables as follows.

$$\begin{cases} \eta = 1, \quad \Gamma = I_m, \\ \Omega_\infty = (\Xi_\infty - \Xi_{\infty 12}^T \Xi_{\infty 1}^{-1} \Xi_{\infty 12})^{-1}, \\ \Omega_{\mathbf{D}} = (\Xi_{\mathbf{D}} - \Xi_{\mathbf{D}12}^T \Xi_{\mathbf{D}1}^{-1} \Xi_{\mathbf{D}12})^{-1}, \\ \Omega_2 = (\Xi_2 - \Xi_{2,12}^T \Xi_{2,1}^{-1} \Xi_{2,12})^{-1} \end{cases} \tag{3.68}$$

Here, Ξ_∞, $\Xi_{\infty 1}$, $\Xi_{\infty 12}$, $\Xi_{\mathbf{D}}$, $\Xi_{\mathbf{D}1}$, $\Xi_{\mathbf{D}12}$, Ξ_2, $\Xi_{2,1}$ and $\Xi_{2,12}$ are defined with the matrices $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20} and the singular value decompositions (3.46) by

$$V_\infty X_{\infty 0}^{-1} V_\infty^T =: \begin{bmatrix} \Xi_{\infty 1} & \Xi_{\infty 12} \\ \Xi_{\infty 12}^T & \Xi_\infty \end{bmatrix}, \; V_{\mathbf{D}} X_{\mathbf{D}0}^{-1} V_{\mathbf{D}}^T =: \begin{bmatrix} \Xi_{\mathbf{D}1} & \Xi_{\mathbf{D}12} \\ \Xi_{\mathbf{D}12}^T & \Xi_{\mathbf{D}} \end{bmatrix}, \; V_2 X_{20}^{-1} V_2^T =: \begin{bmatrix} \Xi_{2,1} & \Xi_{2,12} \\ \Xi_{2,12}^T & \Xi_2 \end{bmatrix} \tag{3.69}$$

To see the above claim, it is enough to note that with the special choice of the variables given in (3.68), each of the affine representations in (3.65) and $Y_\infty = \Gamma Y_{\infty 0}$, $Y_{\mathbf{D}} = \Gamma Y_{\mathbf{D}0}$, $Y_2 = \Gamma Y_{20}$ reduces to

$$\begin{aligned} &X_\infty = X_{\infty 0} = X, \quad X_{\mathbf{D}} = X_{\mathbf{D}0} = X, \quad X_2 = X_{20} = Q_2, \\ &Y_\infty = Y_{\infty 0} = Y, \quad Y_{\mathbf{D}} = Y_{\mathbf{D}0} = Y, \quad Y_2 = Y_{20} = K_c Q_2 \end{aligned} \tag{3.70}$$

Thus, we are led to the following result.

Theorem 3.2 If we take $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20}, $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20} given by (3.66), then the LMI's (3.62)–(3.64) are feasible. Moreover, suppose we minimize γ_2^2 subject to the LMI's (3.62)–(3.64) and denote the optimal value of γ_2^2 by $\tilde{\gamma}_{2n}^2$. Then, we have

$$\tilde{\gamma}_{2n} \leq \gamma_{2c} \tag{3.71}$$

where γ_{2c} is the actual cost achieved the feedback gain K_c. Namely, we can obtain a feedback gain that achieves better (no worse) performance than the one based on a common Lyapunov variable.

If we determine the matrices $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20}, $Y_{\infty 0}$, $Y_{\mathbf{D}0}$, Y_{20} by (3.66), the explicit description of the affine representation approach will be as follows.

Step 0. Minimize γ_2^2 subject to (3.2)–(3.4) with a common Lyapunov variable X and a variable Y given by (3.5) and (3.6), respectively. Denote the optimal value of γ_2^2 by $\tilde{\gamma}_{2c}^2$ and the resulting feedback gain by $K_c := YX^{-1}$.

Step 1. Minimize γ_2^2 subject to (3.8). Denote the optimal value of γ_2^2 by γ_{2c}^2.

Step 2. Define the constant matrices Θ_∞, Λ_∞, $\Theta_{\mathbf{D}}$, $\Lambda_{\mathbf{D}}$, Θ_2 and Λ_2 through (3.66), (3.46), (3.47), (3.51) and (3.54), with K in (3.51) replaced by K_c.

Step 3. Minimize γ_2^2 subject to the LMI's (3.62)–(3.64) to get the feedback gain K given by (3.61).

In Step 3, let us denote the resulting optimal value by $\tilde{\gamma}_{2n}^2$. Furthermore, minimize γ_2^2 subject to (3.8) with K_c replaced by K to get the exact value of the H_2 cost achieved by the feedback gain K designed in Step 3, and denote the resulting optimal value by γ_{2n}^2. Then, it is clear that

$$\gamma_{2n} \leq \tilde{\gamma}_{2n} \leq \gamma_{2c} \leq \tilde{\gamma}_{2c} \tag{3.72}$$

This implies that we can arrive at the H_2 cost γ_{2n}, which is better (no worse) than γ_{2c} achieved by the initial gain K_c.

Now, the affine representation approach has been given explicitly. This approach leads to the LMI's (3.62)–(3.64), that are quite different form those (3.25)–(3.27) with the subspace approach. Moreover, it follows that the affine representation approach has a different property from the subspace approach. Before closing this subsection, let us give a few remarks on the comparison between them as well as the one proposed by Shimomura and Fujii [38]–[40], especially from the viewpoint of the structure of the feedback gains obtained by these approaches. In order to implement each of the approaches, we need K_0, the initial feedback gain. As stated before, the feedback gain K obtained by the affine representation approach is always of the form (3.61), which means that the resulting gain K is dependent on the initial gain K_0 in a particular form. On the other hand, if we take the subspace approach or the one given in [38]–[40], the initial gain K_0 does not restrict the resulting gain K in such a structural way. This is because, in these approaches, the feedback gain K is chosen directly as an LMI variable.

Because of this fact, it seems that iterative applications of the affine representation approach do not work effectively, even though such an iterative algorithm can readily be derived in principle. However, by combining the affine representation approach and the subspace approach, we obtain an effective iterative algorithm as described in the following subsection.

3.3.3 Combined Iterative Algorithm

We have proposed two approaches to the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problem: the subspace approach and the affine representation approach. We can see that each approach obtains a distinct set of LMI characterizations by freezing some different portion of the Lyapunov variables. Thus, applying these approaches by turns, it is expected that we can use the freedom of the frozen potion complementarily. This idea leads directly to the following combined iterative algorithm.

Combined Iterative Algorithm

Step 0. Minimize γ_2^2 subject to (3.2)–(3.4) with a common Lyapunov variable X and a variable Y given by (3.5) and (3.6), respectively. Denote the resulting feedback gain by $K^{(i)} := YX^{-1}$, where we set $i = 0$. For the subsequent design steps, define $X_\infty := X$ and $X_\mathbf{D} := X$.

Step 1. Minimize γ_2^2 subject to (3.8) with K_c replaced by $K^{(i)}$. Denote the optimal value of γ_2^2 by $(\gamma_2^{(i)})^2$, and check the stop criterion with respect to $\gamma_2^{(i)}$. In this step, the variable Q_2 is updated, which will be used in the following Step 2.

Step 2. Define U_∞, $U_\mathbf{D}$ and U_2 by

$$U_\infty := B^T X_\infty^{-1}, \quad U_\mathbf{D} := B^T X_\mathbf{D}^{-1}, \quad U_2 := B^T Q_2^{-1} \tag{3.73}$$

where X_∞, $X_\mathbf{D}$ and Q_2 are those obtained in the previous design steps. Set $i := i + 1$ and minimize γ_2^2 subject to the LMI's (3.25)–(3.27) to get the feedback gain $K^{(i)}$. In this step, the variables P_∞ and $P_\mathbf{D}$ given by (3.22) and (3.23) are updated, which will be used in Step 4.

Step 3. Minimize γ_2^2 subject to (3.8) with K_c replaced by $K^{(i)}$. Denote the optimal value of γ_2^2 by $(\gamma_2^{(i)})^2$, and check the stop criterion with respect to $\gamma_2^{(i)}$. In this step, the variable Q_2 is updated, which will be used in the following Step 4.

Step 4. Define $X_{\infty 0}$, $X_{\mathbf{D}0}$, X_{20}, $Y_{\infty 0}$, $Y_{\mathbf{D}0}$ and Y_{20} by

$$\begin{aligned} X_{\infty 0} &= P_\infty^{-1}, & X_{\mathbf{D}0} &= P_{\mathbf{D}}^{-1}, & X_{20} &= Q_2, \\ Y_{\infty 0} &:= K^{(i)} P_\infty^{-1}, & Y_{\mathbf{D}0} &:= K^{(i)} P_{\mathbf{D}}^{-1}, & Y_{20} &:= K^{(i)} Q_2 \end{aligned} \tag{3.74}$$

where P_∞, $P_{\mathbf{D}}$ and Q_2 are those obtained in the previous design steps. With these constant matrices, further define the constant matrices Θ_∞, Λ_∞, $\Theta_{\mathcal{D}}$, $\Lambda_{\mathbf{D}}$, Θ_2 and Λ_2 through (3.46), (3.47), (3.51) and (3.54), with K in (3.51) replaced by $K^{(i)}$. Set $i := i+1$ and minimize γ_2^2 subject to the LMI's (3.62)–(3.64) to get the feedback gain $K^{(i)}$. Then, go to Step 1. In this step, the variables X_∞ and $X_{\mathbf{D}}$ given by (3.65) are updated, which will be used in Step 2.

In this algorithm, Step 0 corresponds to the conventional approach with a common Lyapunov variable to obtain an initial feedback gain $K^{(0)}$. Step 2 corresponds to the subspace approach provided in the preceding section, and Step 4 corresponds to the affine representation approach. Steps 1 and 3 calculate the actual cost achieved by the feedback gain $K^{(i)}$ resulting from each design step, and check the stop criterion with respect to $\gamma_2^{(i)}$. In this combined algorithm, we can always assure the feasibility of the LMI's in Steps 2 and 4. It is also assured that the resulting H_2 cost $\gamma_2^{(i)}$ is monotonically nonincreasing throughout the iterative algorithm.

Remark 3.5 In Step 2 of the combined iterative algorithm, the matrices U_∞, $U_{\mathbf{D}}$ and U_2 for the subspace approach are updated with the matrices obtained by the affine representation approach in Step 4. This clearly shows the difference between the combined iterative algorithm and the iterative algorithm based on the subspace approach given in Subsection 3.2.3. It is possible for the combined iterative algorithm to include such procedure of minimizing m_∞ and $m_{\mathbf{D}}$ under the constraints (3.32) and (3.33) as in the algorithm based on the subspace approach, although these procedures are not employed for simplicity.

With this combined iterative algorithm, it is expected that we can get around the conservatism in the conventional approach. For the same purpose but in a different fashion, Shimomura and Fujii also proposed an iterative algorithm [38]–[40]. The effectiveness of these iterative algorithms are illustrated and compared through numerical examples in Section 3.5.

3.4 Robust Multiobjective State-Feedback Controller Design for Real Polytopic Uncertainty

In the preceding sections, we have proposed two approaches to the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problem. These approaches can be readily extended to the problem for the plant with polytopic uncertainty [4]. In this section, we deal with a robust multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability controller design problem for real polytopic uncertainty.

Let us consider the plant described by (3.1) again. Supposing that there is no uncertainty, it is enough to represent the plant by just one model $\{A, B_\infty, B_2, B, C_\infty, C_2, D_\infty, D_{z\infty}, D_{z2}\}$. However, if the parameters of the plant have uncertainties, we cannot determine such a single model and need to represent the plant with some set. It is sometimes useful to represent the plant as a polytope [4] described by

$$\left\{\begin{array}{lllllllll} \dot{x} &=& A(\psi)x &+& B_\infty(\psi)w_\infty &+& B_2(\psi)w_2 &+& B(\psi)u \\ z_\infty &=& C_\infty(\psi)x &+& D_\infty(\psi)w_\infty & & &+& D_{z\infty}(\psi)u \\ z_2 &=& C_2(\psi)x & & & & &+& D_{z2}(\psi)u \end{array}\right. \tag{3.75}$$

where

$$\begin{gathered} \begin{bmatrix} A(\psi) & B_\infty(\psi) & B_2(\psi) & B(\psi) \\ C_\infty(\psi) & D_\infty(\psi) & 0 & D_{z\infty}(\psi) \\ C_2(\psi) & 0 & 0 & D_{z2}(\psi) \end{bmatrix} = \sum_{i=1}^{p} \psi_i M_i, \\ \begin{bmatrix} A_i & B_{\infty i} & B_{2i} & B_i \\ C_{\infty i} & D_{\infty i} & 0 & D_{z\infty i} \\ C_{2i} & 0 & 0 & D_{z2i} \end{bmatrix} =: M_i \ (i = 1, \cdots, p), \\ \psi = (\psi_1, \cdots, \psi_p)^T, \quad \psi \in \varPsi := \left\{ \psi \middle| \ \psi_i \geq 0 \ (i = 1, \cdots, p), \ \sum_{i=1}^{p} \psi_i = 1 \right\} \end{gathered} \tag{3.76}$$

We assume that the uncertain parameter ψ is time-invariant, and that the matrices $\{A_i, B_{\infty i}, B_{2i}, B_i, C_{\infty i}, C_{2i}, D_{\infty i}, D_{z\infty i}, D_{z2i}\}$ in M_i $(i = 1, \cdots, p)$ are given matrices.

Consider the multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability controller design problem for the plant (3.75) with the polytopic uncertainty (3.76). The problem is to find a state-feedback gain K minimizing the worst case H_2 cost defined by

$$\gamma_{2,\text{w.c.}} := \max_{\psi \in \varPsi} ||T_{z_2 w_2}(s)||_2 \tag{3.77}$$

subject to the H_∞ and $\mathbf{D}$-stability constraints for all possible values of $\psi \in \varPsi$.

The conventional approach [6] to this problem is such that they minimize γ_2^2 subject to

$$\begin{bmatrix} \mathrm{He}[A_iX + B_iY] & B_{\infty i} & (C_{\infty i}X + D_{z\infty i}Y)^T \\ B_{\infty i}^T & -I & D_{\infty i}^T \\ C_{\infty i}X + D_{z\infty i}Y & D_{\infty i} & -\gamma_\infty^2 I \end{bmatrix} < 0 \quad (i = 1, \cdots, p) \tag{3.78}$$

$$M \otimes X + \mathrm{He}[N \otimes (A_iX + B_iY)] < 0 \quad (i = 1, \cdots, p) \tag{3.79}$$

$$\begin{bmatrix} \mathrm{He}[A_iX + B_iY] & (C_{2i}X + D_{z2i}Y)^T \\ C_2X + D_{z2i}Y & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.80a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X \end{bmatrix} > 0 \tag{3.80b}$$

$$\mathrm{trace}(Z_{2i}) < 1 \quad (i = 1, \cdots, p) \tag{3.80c}$$

where the variables are X, Y, Z_{2i} ($i = 1, \cdots, p$) and γ_2^2. Once the variables X and Y have been found, the state-feedback gain can be determined by $K_c = YX^{-1}$. Note that if (3.78)–(3.80) are satisfied for $\gamma_2 = \tilde{\gamma}_{2c}$, we readily obtain

$$\begin{bmatrix} \mathrm{He}[(A(\psi) + B(\psi)K_c)X] & B_\infty(\psi) & X(C_\infty(\psi) + D_{z\infty}(\psi)K_c)^T \\ B_\infty(\psi)^T & -I & D_\infty(\psi)^T \\ (C_\infty(\psi) + D_{z\infty}(\psi)K_c)X & D_\infty(\psi) & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{3.81}$$

$$M \otimes X + \mathrm{He}[N \otimes \{(A(\psi) + B(\psi)K_c)X\}] < 0 \tag{3.82}$$

$$\begin{bmatrix} \mathrm{He}[(A(\psi) + B(\psi)K_c)X] & X(C_2(\psi) + D_{z2}(\psi)K_c)^T \\ (C_2(\psi) + D_{z2}(\psi)K_c)X & -\tilde{\gamma}_{2c}^2 I \end{bmatrix} < 0 \tag{3.83a}$$

$$\begin{bmatrix} Z_2(\psi) & B_2(\psi)^T \\ B_2(\psi) & X \end{bmatrix} > 0 \tag{3.83b}$$

$$\mathrm{trace}(Z_2(\psi)) < 1 \tag{3.83c}$$

regardless of ψ, where $Z_2(\psi) = \sum_{i=1}^{p} \psi_i Z_{2i}$. Consequently, we can assure the achievement of $\gamma_{2,\text{w.c.}} \leq \tilde{\gamma}_{2c}$ under the H_∞ and **D**-stability constraints.

Remark 3.6 As in the uncertainty free case, the value $\tilde{\gamma}_{2c}$ is an upper bound for $\gamma_{2,\text{w.c.}}$ achieved by K_c. A better upper bound can be sought by minimizing γ_2^2 subject to

$$\begin{bmatrix} \mathrm{He}[(A_i + B_iK_c)X_2] & X_2(C_{2i} + D_{z2i}K_c)^T \\ (C_{2i} + D_{z2i}K_c)X_2 & -\gamma_2^2 I \end{bmatrix} < 0 \tag{3.84a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X_2 \end{bmatrix} > 0 \tag{3.84b}$$

$$\mathrm{trace}(Z_{2i}) < 1 \quad (i = 1, \cdots, p) \tag{3.84c}$$

where the variables are X_2 and Z_{2i} $(i = 1, \cdots, p)$. If we denote the optimal value of γ_2^2 obtained by this procedure by γ_{2c}^2, we have

$$\gamma_{2,\mathrm{w.c.}} \leq \gamma_{2c} \leq \tilde{\gamma}_{2c} \tag{3.85}$$

In contrast to the uncertainty free case, it should be noted that the value γ_{2c} is still an upper bound for $\gamma_{2,\mathrm{w.c.}}$ achieved by K_c. In fact, the calculation of $\gamma_{2,\mathrm{w.c.}}$ is known to be quite hard under the polytopic uncertainty setting. Once the feedback gain K_c is determined, the problem to calculate the worst case H_2 cost $\gamma_{2,\mathrm{w.c.}}$ amounts to an *analysis* problem, and this problem will be addressed as a robust H_2 performance analysis problem for real polytopic uncertainty [35] in Chapter 4.

Although the conventional approach given above offers a tractable means for the problem, this approach is quite conservative, and the conservatism arises from the following two causes.

1. A *common* Lyapunov variable X is forced for all *design specifications*, as in the uncertainty free case.

2. A *fixed* Lyapunov variable X is employed to test performance over the whole *uncertainty domain*, for each of the design specifications (see (3.81)–(3.83)). In dealing with time-invariant uncertainties, however, it is well known that the use of a fixed Lyapunov variable tends to be very conservative [16]. Instead, *parameter-dependent* Lyapunov variables are useful to get around this sort of conservatism, and the methodology for parameter-dependent Lyapunov variables has been studied extensively since the late 1990's [1],[2],[12],[16],[18],[20],[27],[29]–[32],[34],[35].

It is clear that the two approaches presented in the preceding sections work fine to get around the conservatism arising from the first cause stated above, because we can employ non-common Lyapunov variables for different design specifications. Note however that to apply the subspace approach under the polytopic setting, we need an additional assumption that the coefficient matrix B of the plant has no uncertainty. In the application of the affine representation approach, we need no additional assumption.

To circumvent the conservatism arising from the second cause, neither of the two approaches is helpful. Overcoming this problem will be deferred to Chapter 4, where we provide a new approach to the multiobjective controller design problem for real polytopic uncertainty that allows not merely non-common Lyapunov variables but *non-common parameter-dependent* Lyapunov variables for multiple design specifications.

3.5 Illustrative Examples

In this section, we give some numerical examples to illustrate our approaches. The goal of this section is described in the following.

1. We show that the application of the new approaches with non-common Lyapunov variables results in significant improvements over the conventional approach with a common Lyapunov variable.

2. We compare the effectiveness of the new approaches and the one proposed by Shimomura and Fujii [38]–[40] in a single step of the iteration.

3. We examine and compare the effectiveness of the iterative algorithms resulting from the new approaches and the one proposed by Shimomura and Fujii [38]–[40] in their limiting performance.

For the ease of description, we name each of the approaches and the iterative algorithms as follows.

Table 3.1: Name for the approaches

Conventional Approach	The conventional approach with a common Lyapunov variable.
Approach I	The subspace approach presented in Subsection 3.2.2.
Approach II	The affine representation approach presented in Subsection 3.3.2.
Approach III	The approach proposed by Shimomura and Fujii [38]–[40] with the iteration carried out only once.

Table 3.2: Name for the iterative algorithms

Conventional Algorithm	The conventional iterative algorithm presented in Subsection 3.2.3.
Iterative Algorithm I	The iterative algorithm based on the subspace approach, presented in Subsection 3.2.3.
Iterative Algorithm II	The combined iterative algorithm, presented in Subsection 3.3.3.
Iterative Algorithm III	The iterative algorithm proposed by Shimomura and Fujii [38]–[40].

Remark 3.7 Concerning the **D**-stability constraints, we have used in Sections 3.2 and 3.3 a single Lyapunov variable $X_{\mathbf{D}}$ irrespective of the regions, for notational simplicity. In the following numerical examples, we employ non-common Lyapunov variables such as $X_{\mathcal{H}}$, $X_{\mathcal{C}}$ and $X_{\mathcal{S}}$ for each of the regions $\mathcal{H}(\alpha)$, $\mathcal{C}(c,r)$ and $\mathcal{S}(k)$ to get around the conservatism as much as possible. This modification is straightforward and hence we suppress the detailed descriptions for it.

Remark 3.8 We have used such notation as K_c that denotes the feedback gain obtained by Conventional Approach. In the following, we also use such notation as K_{I}, that denotes the feedback gain obtained by Approach I. Similarly, $K_{\mathrm{I}}^{(5)}$ denotes the feedback gain obtained by Iterative Algorithm I with five iterations. Further, K_{I}^{*} denotes the feedback gain resulting from Iterative Algorithm I in the limit. Similarly for $K_c^{(30)}$, K_{II}, K_{III}^{*} and so on.

Remark 3.9 On the implementation of the iterative algorithms, we decided to set the stop criterion by $|\gamma_2^{(i)} - \gamma_2^{(i+1)}| < \varepsilon$, where $\gamma_2^{(i)}$ is the H_2 cost after the ith iteration. We arrive at the value $\varepsilon = 10^{-4}$ by trial and errors, aiming at a reasonable compromise between the resulting H_2 cost and the computation time. The value $\varepsilon = 10^{-4}$ will be sufficiently small to evaluate the limiting performance of each algorithm for the problems treated in this section.

In the following, all LMI-related computations were carried out with the *LMI Control Toolbox* [15], on PENTIUM-III 933MHz.

3.5.1 Multiobjective State-Feedback Controller Design Problems

First, we consider the multiobjective state-feedback controller design problems for systems without uncertainty.

Problem 1 (H_2/D-stability Synthesis)

Consider the LTI plant described by

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \\
z_\infty &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x, \\
z_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u
\end{aligned}
\tag{3.86}
$$

where $k = 0.245$ and $f = 0.0219$ [3]. The problem is to find a state-feedback gain K minimizing $||T_{z_2w}||_2$ subject to the **D**-stability constraint $\sigma(\mathcal{A}) \subset \bigcap\{\mathcal{H}(0.5), \mathcal{S}(\tan(3\pi/8))\}$ (see Figs. 3.1 and 3.2).

Applying to Problem 1 the (non-iterative) approaches listed in Table 3.1, we get the H_2 costs shown in Table 3.3. More specifically, this table shows the *actual* H_2 cost resulting from each approach as well as the computation time. We can see that Approaches I–III achieve better H_2 costs than Conventional Approach. In particular, Approach I successfully achieves the best performance with less computation time than Approaches II and III. Approach III takes much more computation time than the other approaches, and one of the possible reasons is that this approach deals with the LMI's enlarged by some algebraic manipulations to allow non-common Lyapunov variables [39],[40].

Table 3.3: The resulting H_2 costs by the approaches listed in Table 3.1

Approach (Corresponding gain)	H_2 cost	CPU time (sec)
Conventional Approach (K_c)	1.5924	0.15
Approach I $(K_{\rm I})$	1.4848	0.39
Approach II $(K_{\rm II})$	1.5046	0.43
Approach III $(K_{\rm III})$	1.5478	0.98

The feedback gains resulting from these approaches are given below for comparison.

$$K_c = \begin{bmatrix} -4.5752 & -0.9647 & -3.0720 & -13.8032 \end{bmatrix} \tag{3.87}$$

$$K_{\rm I} = \begin{bmatrix} -3.3489 & -0.3486 & -2.4225 & -9.4676 \end{bmatrix} \tag{3.88}$$

$$K_{\rm II} = \begin{bmatrix} -3.3765 & -0.7120 & -2.2671 & -10.1868 \end{bmatrix} \tag{3.89}$$

$$K_{\rm III} = \begin{bmatrix} -4.0642 & -0.6690 & -2.8955 & -11.7070 \end{bmatrix} \tag{3.90}$$

For reference, the H_2 optimal feedback gain (without taking account of the **D**-stability constraint) is given in the following.

$$K_{H_2} = \begin{bmatrix} -1.3271 & -0.0871 & -1.6334 & -1.9464 \end{bmatrix} \tag{3.91}$$

This feedback gain achieves $||T_{z_2w}||_2 = 1.2780$.

Because of the nature of Approach II, the feedback gain $K_{\rm II}$ depends on K_c in the following form.

$$K_{\mathrm{II}} = 0.7380 K_c \tag{3.92}$$

On the other hand, the feedback gains K_{I} and K_{III} do not depend on K_c in such a structural way. As is typically shown in (3.92), it turns out that Approaches I–III arrive at lower gains than that of Conventional Approach. From another point of view, it can be said that the feedback gains (3.88)–(3.90) successfully come close to the optimal gain K_{H_2} given by (3.91) so that they attain better H_2 costs than K_c.

It is expected that Approaches I–III achieves better performance because of their less conservative nature. Indeed, the less conservative nature of Approaches I–III can be seen in the closed-loop pole locations. The closed-loop pole locations under the feedback gains K_{H_2}, K_c and K_{I} are shown in Fig. 3.1, where we suppressed those under K_{II} and K_{III} for simplicity.

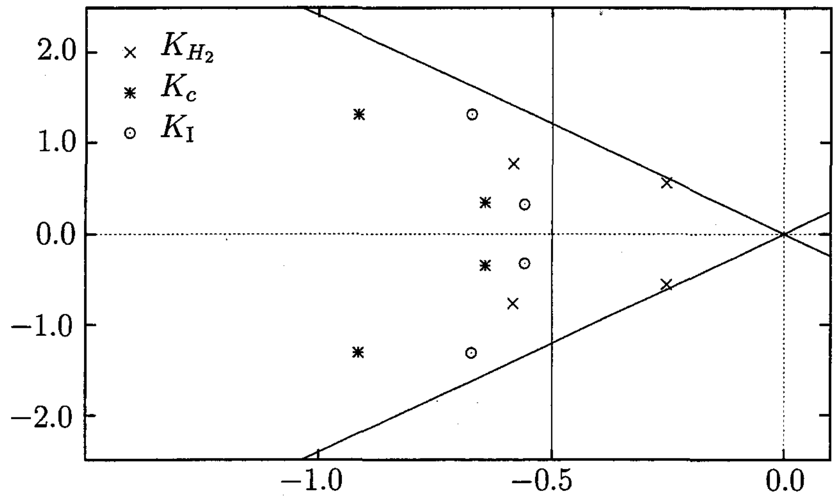

Figure 3.1: Pole locations under K_{H_2}, K_c and K_{I}

The above figure shows that the feedback gain K_{H_2} does not satisfy the **D**-stability constraint, while the feedback gains K_c and K_{I} do satisfy the constraint as required. However, the figure suggests that there is a big margin left for the **D**-stability constraint under K_c. In other words, Conventional Approach yields an excessively high feedback gain so that it achieves the **D**-stability constraint in a conservative fashion. On the other hand, the closed-loop pole locations under K_{I} are close to the boundary and the feedback gain K_{I} indeed achieves considerably better H_2 cost than K_c. These facts lead us to the conclusion that Conventional Approach with a common Lyapunov variable is conservative, and the conservatism has been circumvented by Approach I with the use of non-common Lyapunov variables. Note that similar observations apply also to Approaches II and III.

Next, we investigated the effectiveness of the iterative algorithms. Applying to Problem 1 the iterative algorithms listed in Table 3.2 under the stop criterion $|\gamma_2^{(i)} - \gamma_2^{(i+1)}| < 10^{-4}$, we get the H_2 costs shown in Table 3.4, where we also show the number of iterations and computation time for each algorithm.

Table 3.4: The resulting H_2 costs in the limit by the iterative algorithms listed in Table 3.2 (N: number of iterations)

Algorithm (Corresponding gain)	H_2 cost	N	CPU time (sec)
Conventional Algorithm (K_c^*)	1.3898	1458	286.10
Iterative Algorithm I (K_{I}^*)	1.3887	31	12.78
Iterative Algorithm II (K_{II}^*)	1.3004	26	7.31
Iterative Algorithm III (K_{III}^*)	1.3025	16	14.25

The above table shows that Iterative Algorithms I–III all achieve better performance than Conventional Algorithm with much less computation time. In particular, with Iterative Algorithms II or III, the H_2 cost is considerably improved. In this problem, Iterative Algorithm II successfully achieves the best performance with the least computational effort, which leads to the state-feedback gain

$$K_{\mathrm{II}}^* = K_{\mathrm{II}}^{(26)} = \begin{bmatrix} -1.6287 & 0.2147 & -1.9579 & -2.9135 \end{bmatrix} \tag{3.93}$$

The closed-loop pole locations under K_{II}^* are shown in Fig. 3.2 to see the less conservative nature of Iterative Algorithm II. This figure shows that the feedback gain K_{II}^* achieves the **D**-stability constraint without any margin.

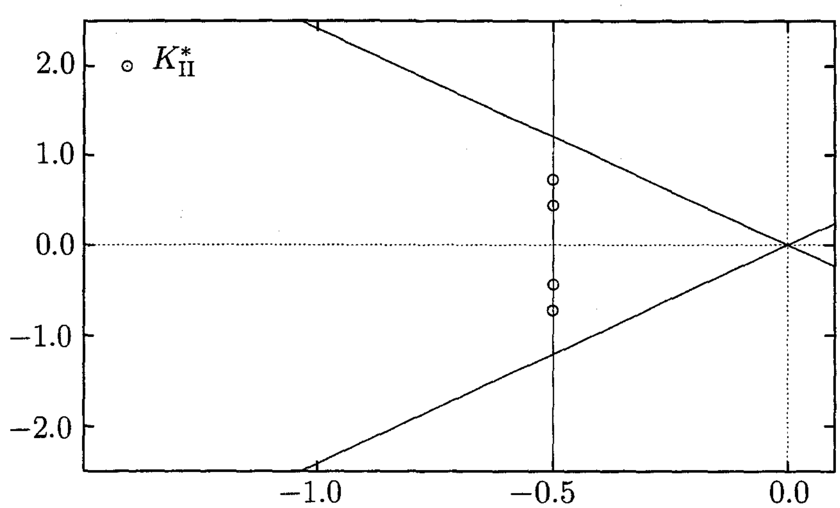

Figure 3.2: Pole locations under K_{II}^*

Although Iterative Algorithm II is such that it uses the feedback gain K_c given by (3.87) as an initial feedback gain, the resulting feedback gain $K_{\rm II}^*$ given by (3.93) is quite different from K_c. It is also a quite interesting fact that the second element of the feedback gain has changed its sign during the iterations from K_c to $K_{\rm II}^*$ in the Iterative Algorithm II.

Here, let us observe the H_∞ costs corresponding to the feedback gains resulting from these iterative algorithms. We take w and z_∞ for the input and output, respectively, to measure the H_∞ costs (see the state space realization (3.86) of the plant). The following Table 3.5 shows the results.

Table 3.5: The resulting H_∞ costs for the iterative algorithms

Algorithm (Corresponding gain)	H_∞ cost ($\|\|T_{z_\infty w}\|\|_\infty$)
Conventional Algorithm (K_c^*)	0.3891
Iterative Algorithm I ($K_{\rm I}^*$)	0.5647
Iterative Algorithm II ($K_{\rm II}^*$)	0.7091
Iterative Algorithm III ($K_{\rm III}^*$)	0.7215

It is clear from the above table that none of the feedback gains achieve the H_∞ costs less than 0.3. With this in mind, we further include the additional constraint $||T_{z_\infty w}||_\infty < 0.3$ to Problem 1. Namely, as the next problem, we consider the following multiobjective state-feedback $H_2/H_\infty/\mathbf{D}$-stability problem.

Problem 2 ($H_2/H_\infty/\mathbf{D}$-stability Synthesis)

Consider again the LTI plant described by (3.86) where $k = 0.245$ and $f = 0.0219$. The problem is to find a state-feedback gain K minimizing $||T_{z_2 w}||_2$ subject to the H_∞ constraint $||T_{z_\infty w}||_\infty < 0.3$ and the $\mathbf{D}$-stability constraint $\sigma(\mathcal{A}) \subset \bigcap\{\mathcal{H}(0.5), \mathcal{S}(\tan(3\pi/8))\}$.

As in the preceding problem, we first show the effectiveness of the (non-iterative) approaches listed in Table 3.1, and we second demonstrate that the performance is further improved by the application of the iterative algorithms listed in Table 3.2.

Applying the non-iterative approaches to Problem 2, we get the H_2 costs shown in Table 3.6, where we also show computation time for each approach. We can see from this table that Approaches I, II and III achieve better H_2 costs than Conventional Approach, and that Approach I achieves the best performance with less computation time than Approaches II and III. It turns out that the H_2 costs shown here are naturally worse than those in Table 3.3 because of the additional H_∞ constraint.

Table 3.6: The resulting H_2 costs by the approaches listed in Table 3.1

Approach (Corresponding gain)	H_2 cost	CPU time (sec)
Conventional Approach (K_c)	1.7111	0.19
Approach I (K_{I})	1.6180	0.51
Approach II (K_{II})	1.6393	0.52
Approach III (K_{III})	1.6696	1.25

The feedback gains obtained by these approaches are given below for comparison.

$$K_c = \begin{bmatrix} -5.9037 & -4.9351 & -3.5089 & -21.8058 \end{bmatrix} \tag{3.94}$$

$$K_{\mathrm{I}} = \begin{bmatrix} -4.7255 & -2.5468 & -2.8125 & -15.7032 \end{bmatrix} \tag{3.95}$$

$$K_{\mathrm{II}} = \begin{bmatrix} -4.7141 & -3.9407 & -2.8019 & -17.4120 \end{bmatrix} = 0.7985K_c \tag{3.96}$$

$$K_{\mathrm{III}} = \begin{bmatrix} -5.3411 & -4.0142 & -3.3389 & -18.9957 \end{bmatrix} \tag{3.97}$$

Similarly to the preceding problem, Approaches I–III yield lower gains than Conventional Approach. For reference, the H_2 optimal feedback gain (without any care for the H_∞ and the **D**-stability constraints) is again given in the following.

$$K_{H_2} = \begin{bmatrix} -1.3271 & -0.0871 & -1.6334 & -1.9464 \end{bmatrix} \tag{3.98}$$

Recall that this H_2 optimal feedback gain achieves $||T_{z_2w}||_2 = 1.2780$.

One of the possible reasons why Approaches I–III arrive at better performance is that they successfully circumvent the conservatism of Conventional Approach. Indeed, we can see the less conservative nature of Approaches I–III than Conventional Approach via the H_∞ costs and the closed-loop pole locations under the resulting feedback gains (3.94)–(3.97). To see this, we show the H_∞ costs achieved by these gains in Table 3.7. The closed-loop pole locations under K_{H_2}, K_c and K_{I} are shown in Fig. 3.3, where those under K_{II} and K_{III} are omitted for simplicity.

Table 3.7: The resulting H_∞ costs under the constraint $||T_{z_\infty w}||_\infty < 0.3$

Approach (Corresponding gain)	H_∞ cost ($||T_{z_\infty w}||_\infty$)
Conventional Approach (K_c)	0.0923
Approach I (K_{I})	0.1375
Approach II (K_{II})	0.1155
Approach III (K_{III})	0.1069
H_2 optimal (K_{H_2})	0.9702

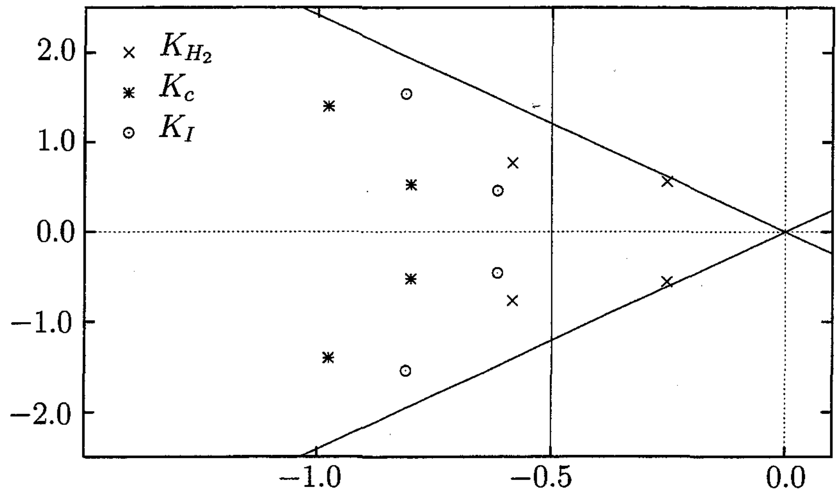

Figure 3.3: Pole locations under K_{H_2}, K_c and K_I

Table 3.7 and Fig. 3.3 show that the feedback gain K_{H_2} satisfies neither the H_∞ constraint nor the **D**-stability constraint, while the feedback gains K_c and K_{I} do satisfy both of the constraints as required. In particular, we can see that the feedback gain K_{I} satisfies both of the constraints in a less conservative fashion: the feedback gain K_{I} leaves less margins for the constraints than K_c. Noting that the feedback gain K_{I} indeed achieves better performance than K_c, we can conclude that Conventional Approach is conservative, and Approach I successfully reduces the conservatism with the use of non-common Lyapunov variables. Similar comments also apply to Approaches II and III.

Next, we investigated how the H_2 cost is further improved by the iterative algorithms listed in Table 3.2. Applying them to Problem 2 under the stop criterion $|\gamma_2^{(i)} - \gamma_2^{(i+1)}| < 10^{-4}$, we get the H_2 costs shown in Table 3.8, where we also show the number of iterations and computation time for each algorithm.

Table 3.8: The resulting H_2 costs in the limit by the iterative algorithms listed in Table 3.2 (N: number of iterations)

Algorithm (Corresponding gain)	H_2 cost	N	CPU time (sec)
Conventional Algorithm (K_c^*)	1.4151	873	233.14
Iterative Algorithm I (K_{I}^*)	1.4152	42	23.36
Iterative Algorithm II (K_{II}^*)	1.4138	16	5.53
Iterative Algorithm III (K_{III}^*)	1.3985	14	14.75

The above table shows that all of the algorithms achieve the H_2 costs nearly 1.4, which is significant improvement over the results of the (non-iterative) approaches where Approach

I achieved the best cost 1.6180. This suggests that these algorithms have worked better to reduce the conservatism. Indeed, the less conservative nature of them can be seen from the way they satisfy the H_∞ and **D**-stability constraints. For example, Iterative Algorithm II arrives at the feedback gain

$$K_{\mathrm{II}}^* = K_{\mathrm{II}}^{(16)} = \begin{bmatrix} -2.5713 & -0.8917 & -2.1339 & -6.9047 \end{bmatrix} \tag{3.99}$$

and this feedback gain achieves H_∞ cost 0.2907 and places the closed-loop poles as shown in Fig. 3.4. Comparing these results with those in Table 3.7 or Fig. 3.3, we can see that the feedback gain K_{II}^* satisfies both constraints in a less conservative fashion.

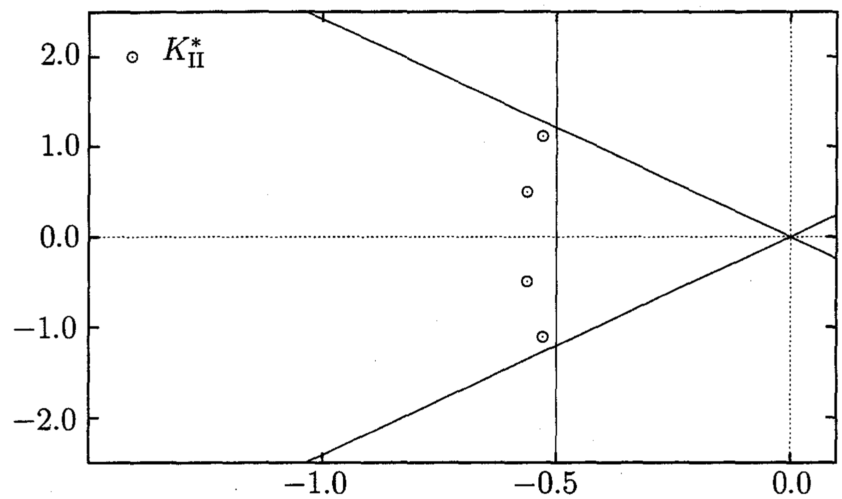

Figure 3.4: Pole locations under K_{II}^*

In this example, as shown in Table 3.8, Iterative Algorithm III arrives at the best performance. Although Conventional Algorithm achieves almost the same performance as Iterative Algorithms I and II, it needs much more computation time than the latter two algorithms. This clearly suggests the advantage of Iterative Algorithms I and II over Conventional Algorithm.

3.5.2 Robust Multiobjective State-Feedback Controller Design Problem for Real Polytopic Uncertainty

In the preceding subsections, we considered multiobjective state-feedback controller design problems for a plant without uncertainty, and demonstrated the effectiveness of the new approaches and the iterative algorithms. The aim of this subsection is to show that they are also effective for the multiobjective state-feedback controller design problems under the polytopic uncertainty setting. To this end, let us consider a simple multiobjective state-feedback controller design problem for real polytopic uncertainty. In dealing with the problem, it is desirable to employ non-common Lyapunov variables for the design specifications where the Lyapunov variables are at the same time parameter-dependent over the uncertainty domain. However, we concentrate our attention on clarifying the advantage of the use of non-common Lyapunov variables for the design specifications, and hence, for the whole uncertainty domain, we evaluate each design specification with a fixed Lyapunov variable. As described in Section 3.4, the two approaches in this chapter enable us to do so.

Problem 3 (H_2/D-stability Synthesis for Real Polytopic Uncertainty)

Consider the LTI plant (3.86) where the parameters k and f have the following ranges of uncertainties.

$$0.09 \le k \le 0.4, \quad 0.0038 \le f \le 0.04 \tag{3.100}$$

The problem is to find a state-feedback gain K minimizing the worst case H_2 cost of the closed-loop system defined by

$$\gamma_{2,\text{w.c.}} := \max_{k,f} ||T_{z_2 w}(s)||_2 \tag{3.101}$$

subject to the $\mathbf{D}$-stability constraint such that the closed-loop poles for all possible values of the parameters k and f lie in $\bigcap\{\mathcal{H}(0.15), \mathcal{S}(\tan(3\pi/8))\}$.

In order to deal with the uncertainties of the parameters k and f, we describe the plant as a polytope with four vertices M_i $(i = 1, \cdots, 4)$ (see Section 3.4). In the sequel, we refer to the model corresponding to each vertex as Model 1, 2, 3 and 4, respectively.

Applying to Problem 3 the non-iterative approaches listed in Table 3.1, we get the H_2 costs shown in Table 3.9, where we also show the computation time for each approach. Recall that the H_2 costs here are obtained by forcing a fixed Lyapunov variable over the whole uncertainty domain and hence nothing but an upper bound for the worst case H_2 cost $\gamma_{2,\text{w.c.}}$ achieved by each approach.

Table 3.9: The resulting H_2 costs by the approaches listed in Table 3.1

Approach (Corresponding gain)	H_2 cost	CPU time (sec)
Conventional Approach (K_c)	2.0199	0.85
Approach I (K_{I})	1.8296	1.82
Approach II (K_{II})	1.9182	2.16
Approach III (K_{III})	1.9856	5.63

The above table shows that Approaches I–III achieve better performance than Conventional Approach. In particular, Approach I achieves considerably better performance than Approaches II and III with less computation time. Note that in the preceding Problems 1 and 2, Approach I also arrived at the best performance. One of the reasons why Approach I (i.e. the subspace approach) works fine is that we can take a large subspace of the Lyapunov variables in these problems because the plant of interest has four states while only one control input (see (3.22)–(3.24)).

These approaches lead to the following feedback gains.

$$K_c = \begin{bmatrix} -10.0449 & 4.5272 & -5.2278 & -30.1554 \end{bmatrix} \tag{3.102}$$

$$K_{\mathrm{I}} = \begin{bmatrix} -6.0300 & 2.9961 & -3.7987 & -16.9341 \end{bmatrix} \tag{3.103}$$

$$K_{\mathrm{II}} = \begin{bmatrix} -7.8782 & 3.5507 & -4.1002 & -23.6509 \end{bmatrix} = 0.7843 K_c \tag{3.104}$$

$$K_{\mathrm{III}} = \begin{bmatrix} -9.3438 & 4.2518 & -4.9842 & -27.6284 \end{bmatrix} \tag{3.105}$$

Although all of the Approaches I–III are such that they use K_c as an initial gain, we can see that the resulting gains K_{I}, K_{II} and K_{III} are quite different from K_c.

It is expected that Approaches I–III successfully arrive at better performance because of their less conservative nature. To see this, the closed-loop pole locations on each vertex under K_{H_2}, K_c and K_{I} are shown in Figs. 3.5, 3.6 and 3.7, respectively. Here, K_{H_2} is a feedback gain that minimizes the upper bound of the worst case H_2 cost without taking account of the **D**-stability constraint and achieves an upper bound 1.7584. These figures show that K_{H_2} slightly violates the **D**-stability constraint, while K_c and K_{I} do satisfy the **D**-stability constraint as required. Specifically, Fig. 3.7 suggests the less conservative nature of Approach I: the feedback gain K_{I} achieves almost the boundary for the **D**-stability constraint.

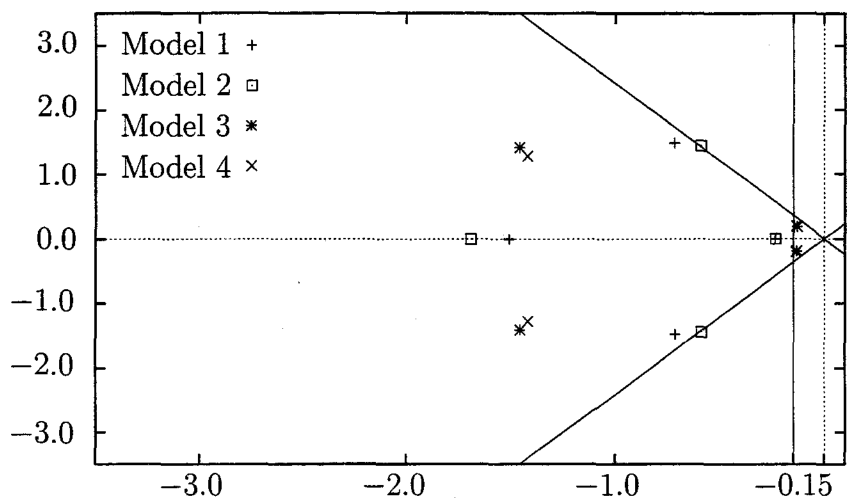

Figure 3.5: Pole locations under K_{H_2}

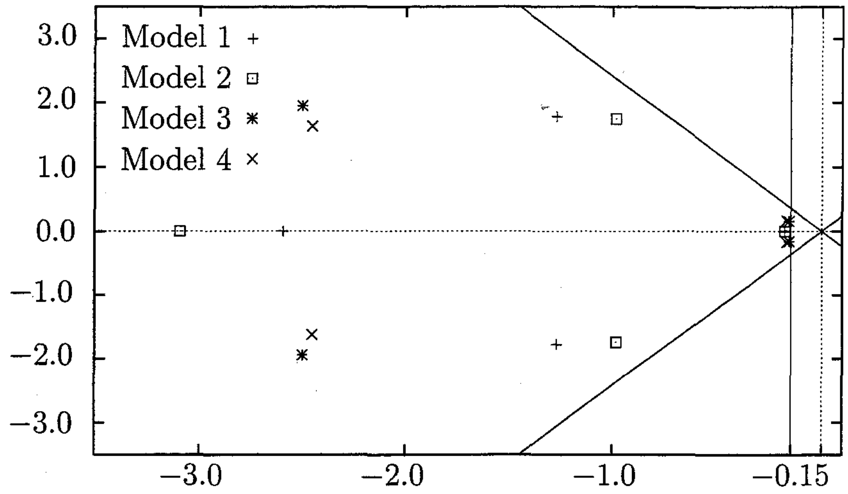

Figure 3.6: Pole locations under K_c

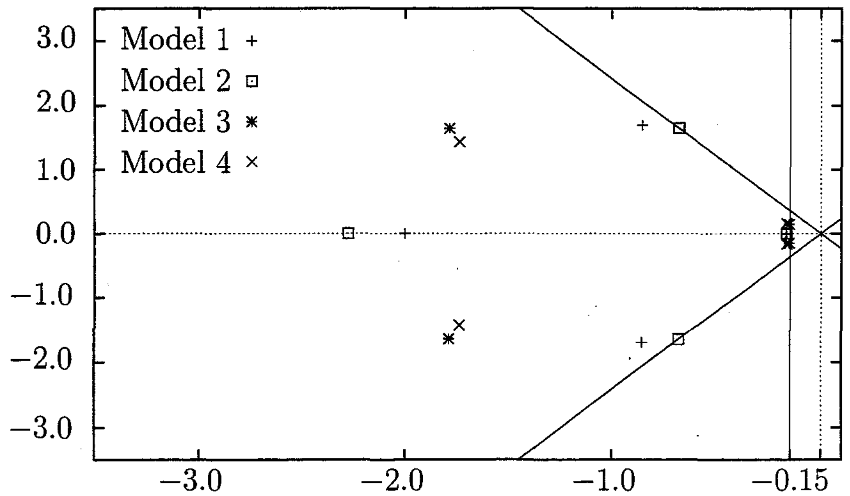

Figure 3.7: Pole locations under K_{I}

Next, we demonstrate the effectiveness of the iterative algorithms under the polytopic setting. Applying to Problem 3 the iterative algorithms listed in Table 3.2 under the stop criterion $|\gamma_2^{(i)} - \gamma_2^{(i+1)}| < 10^{-4}$, we get the H_2 costs shown in Table 3.10, where we also show the number of iterations and computation time for each algorithm.

Table 3.10: The resulting H_2 costs in the limit by the iterative algorithms listed in Table 3.2 (N: number of the iterations)

Algorithm (Corresponding gain)	H_2 cost	N	CPU time (sec)
Conventional Algorithm (K_c^*)	1.9351	75	61.34
Iterative Algorithm I (K_{I}^*)	1.8023	9	14.09
Iterative Algorithm II (K_{II}^*)	1.8044	14	18.25
Iterative Algorithm III (K_{III}^*)	1.7986	44	216.58

As shown in the above table, Iterative Algorithms I–III achieve considerably better performance than Conventional Algorithm. For reference, Iterative Algorithm I arrives at

$$K_{\mathrm{I}}^* = K_{\mathrm{I}}^{(9)} = \begin{bmatrix} -5.2378 & 2.6604 & -3.8654 & -13.8780 \end{bmatrix} \tag{3.106}$$

We can see that the above feedback gain is quite different from the initial feedback gain K_c given by (3.102).

It follows from Table 3.10 that Iterative Algorithms I and II achieve almost the same performance as Iterative Algorithm III with much less computation time. This clearly suggests the effectiveness of Iterative Algorithms I and II.

In this subsection, we clarified the advantage of the use of non-common Lyapunov variables in dealing with the robust multiobjective controller design problems for real polytopic uncertainty. The new approaches indeed enabled us to employ non-common Lyapunov variables, but the Lyapunov variables are fixed over the whole uncertainty domain. To circumvent the conservatism arising from seeking fixed Lyapunov variables, we propose a new approach in Chapter 4 which enables us to employ not merely non-common Lyapunov variables but *non-common parameter-dependent* Lyapunov variables for multiple design specifications.

3.6 Summary

In this chapter, we have proposed two approaches with non-common Lyapunov variables to the multiobjective state-feedback controller design problem. In Section 3.2, we proposed the subspace approach, where we introduced some additional constraints to the Lyapunov variables. This additional constraints successfully enabled us to derive a set of LMI's that leave the feedback gain directly as one of the LMI variables and also allow non-common Lyapunov variables. With a suitable replacement of the parameters included in the additional constraints, we arrived at an iterative algorithm based on the subspace approach. On the other hand, in Section 3.3, we proposed the affine representation approach. In this approach, we performed a standard procedure called change of variables and represented the resulting variables as a set of affine functions. These affine functions are constructed in such a way that troublesome non-convex constraints are avoided. Because of this nice property, we readily derived a set of LMI's that allow non-common Lyapunov variables. In addition, the idea of simply combining the subspace and the affine representation approaches led us to another effective iterative algorithm.

The above approaches and iterative algorithms seem to be effective in reducing the conservatism and attaining better performance as is demonstrated by numerical examples in Section 3.5. Despite the advantages, we have to admit some deficiencies in our study and drawbacks of our approaches. First, we gave no analytical results concerning the effectiveness of our approaches: we introduced no quantitative index for the degree of the conservatism in the conventional approach with a common Lyapunov variable, and hence we could not give any analytical results about how much the conservatism can be circumvented with our approaches. Second, our iterative algorithms as well as the one proposed by Shimomura and Fujii [38]–[40] are not able to guarantee the achievement of global optimality. Concerning the dynamic feedback control problems, some approaches [5],[21],[37] based on the finite-dimensional Q-parametrization achieve global optimality for the multiobjective H_2/H_∞ problems, although there are inherent inflation of the size of the LMI's and thus the order of the controller. These approaches are very effective for the dynamic feedback control problems, but it seems difficult to deal with the static state-feedback control problems in a parallel fashion to [5],[21],[37]. Hence, in spite of the above deficiencies and drawbacks, our approaches will be useful indeed when we need a static state-feedback gain.

Chapter 4

New Dilated LMI Characterizations for Continuous-Time Controller Design and Robust Multiobjective Synthesis

In the preceding chapter, we have proposed two LMI approaches with non-common Lyapunov variables to the multiobjective state-feedback controller design problems. The effectiveness of these approaches was demonstrated through several numerical examples. Although these approaches as well as the one proposed by Shimomura and Fujii [38]–[40] indeed solve the problem with non-common Lyapunov variables, they require some auxiliary steps before being implemented. More specifically, these approaches include some parameters to be determined in advance, and the conventional approach with a common Lyapunov variable seems indispensable for their reasonable and systematic setting. In this context, unfortunately, these approaches cannot be *self-contained* ones that are actually free from the use of a common Lyapunov variable.

As we have seen in the preceding chapters, the reason why we come to employ a common Lyapunov variable is that most matrix inequality characterizations in control theory make use of the Lyapunov variables in such a way that they appear as products with the controller variables [4],[43]. This leads to unnecessary restrictions on the variables for a set of LMI's: a common Lyapunov variable has been forced for all LMI characterizations. This restriction is the most important source of the conservatism not only in the LMI-based multiobjective controller synthesis [6],[26],[36] but also in robust performance analysis and synthesis for real polytopic uncertainty [4],[6].

To overcome the difficulty, Oliveira et al. showed a new direction in the discrete-time setting [29],[30]. They showed that the *dilation* of the matrix inequality characterizations and the introduction of auxiliary variables achieve decoupling between the Lyapunov variables and the controller variables and thus the technical restriction to a common Lyapunov variable can be avoided. They have shown a constructive way to derive dilated characterizations that are equivalent to the original ones. The advantage of working with the dilated characterizations lies in the fact that if we consider a set of dilated matrix inequality characterizations, then it *includes* the corresponding set of the original ones as a special case. More specifically, if one chooses the auxiliary variable the same as the Lyapunov variable, the set of dilated characterizations reduces to the original one [29], [30]. This property is very promising in dealing with a wide range of problems. Indeed, they are successfully applied to multiobjective control [30] and robust control for real polytopic uncertainty [29] to get around the conservatism in the conventional approaches.

Unfortunately, the study in [29],[30] fully relies on the features of matrix inequalities in the discrete-time setting, and hence analogous dilated characterizations in the continuous-time setting do not follow in a parallel fashion. It follows from [29],[30] that in dealing with *synthesis* problems, we have to derive dilated characterizations with a single square auxiliary variable being involved in the products with the controller variables. This restriction can be relaxed when we deal with *analysis* problems, and thus the well-known *Elimination Lemma* [14],[22],[43] works fine to arrive at the dilated characterizations in both the discrete- and continuous-time settings, with the use of several auxiliary variables [33]–[35]. However, the restriction on the number of the auxiliary variables to address synthesis problems cannot be handled with a simple application of the Elimination Lemma. Namely, we need another effort as is suggested in [1],[30].

In this chapter, we propose a general approach to the dilated characterizations in the continuous-time setting. The key idea in this approach is a particular application of the Schur complement technique [4], which leads to a constructive way to derive dilated characterizations that are suitable for controller synthesis. Moreover, it is shown that a set of the new dilated characterizations *includes* the corresponding set of the original ones as a special case, via a particular choice of the auxiliary variables introduced for dilation. These are very nice and interesting features that are to some extent analogous to the ones already obtained in the discrete-time setting. Because of these features, it turns out that the dilated characterizations can be applied to robust multiobjective control for real polytopic uncertainty in a reasonable fashion, with the use of non-common and parameter-dependent Lyapunov variables.

4.1 Useful Results and Relevant Studies

In this chapter, we consider the continuous-time multi-input multi-output (MIMO) linear time-invariant (LTI) system with the state-space representation

$$\begin{cases} \dot{x} = \mathcal{A}x + \mathcal{B}w \\ z = \mathcal{C}x + \mathcal{D}w \end{cases} \tag{4.1}$$

where the state vector $x \in \mathbf{R}^n$ and all other vectors and matrices have appropriate dimensions. We assume that for *analysis* problems, the coefficient matrices $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ are given matrices while for *synthesis* problems, they include controller variables to be determined. The following lemma is useful in characterizing a variety of control performance of the system (4.1) and plays a crucial role in our study.

Lemma 4.1 Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$, scalars $\delta_1 > 0$, $\delta_2 > 0$, a matrix Δ of column dimension n, and a scalar $b = a^{-1} > 0$ be given. Then, the following two conditions are equivalent.

(i) There exists a matrix $\mathcal{X} > 0$ such that

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \delta_1\mathcal{X} + \delta_2\mathcal{A}\mathcal{X}\mathcal{A}^T + \mathcal{X}\Delta^T\Delta\mathcal{X} < 0 \tag{4.2}$$

(ii) There exist matrices $\mathcal{X} > 0$ and G such that

$$\begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix} G \begin{bmatrix} I & -bI & bI & I & b\Delta^T \end{bmatrix} \right\} < 0 \tag{4.3}$$

Moreover, for every solution $\mathcal{X} = X > 0$ of (4.2), $[\ \mathcal{X}\ G\ ] = [\ X \ \ -a(\mathcal{A} - aI)^{-1}X\ ]$ is a solution of (4.3), irrespective of δ_1, δ_2 and Δ. Conversely, every matrix $\mathcal{X} > 0$ such that (4.3) holds for some G also satisfies (4.2), irrespective of δ_1, δ_2 and Δ.

Note that (4.2) in the condition (i) can be regarded as a standard characterization for the analysis and synthesis of continuous-time systems frequently used in the previous studies [4], [43], while (4.3) in the condition (ii) is a new *dilated* characterization of (4.2). Since $\mathcal{X}$ is a Lyapunov variable in (4.2), the above lemma validates us to call $\mathcal{X}$ a Lyapunov variable even in (4.3), although it does not contain such terms as $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T$ as is desired. The matrix G is an *auxiliary* variable introduced for the dilated characterization.

Using the parameters δ_1, δ_2 and Δ in Lemma 4.1, we can represent varieties of control performance. For example, if we simply take $\delta_1 = \delta_2 \to 0$ and $\Delta = 0$, the inequality (4.2) reduces to

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T < 0 \tag{4.4}$$

The above inequality is nothing but the *Lyapunov inequality* that characterizes stability of the matrix $\mathcal{A}$. On the other hand, the inequality (4.3) reduces essentially to

$$\begin{bmatrix} 0 & -\mathcal{X} \\ -\mathcal{X} & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \end{bmatrix} G \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \tag{4.5}$$

which gives our dilated characterization of the Lyapunov inequality.

In the following, we give a proof of Lemma 4.1, in which the relation between the solutions of (4.2) and (4.3) is clear. This nice and interesting relation is quite important in our study, especially in dealing with multiobjective controller synthesis. The following lemma is repeatedly used in the proof.

Lemma 4.2 (*Schur complement*)[4],[43] Let a matrix $\Phi = \Phi^T$ is given with a partition $\Phi =: \left[\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{12}^T & \Phi_{22} \end{array}\right]$. Then, the following three conditions are equivalent.

(i) $\Phi < 0$

(ii) $\Phi_{11} < 0, \quad \Phi_{22} - \Phi_{12}^T \Phi_{11}^{-1} \Phi_{12} < 0$

(iii) $\Phi_{22} < 0, \quad \Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^T < 0$

Proof of Lemma 4.1. First, we show that the condition (i) implies (ii). Applying the Schur complement technique to (4.2) with the given scalar $a > 0$, we have

$$\left[\begin{array}{c|c|ccc} -2a\mathcal{X} & -2a\mathcal{X} & 0 & 0 & 0 \\ \hline -2a\mathcal{X} & (\mathcal{A} - aI)\mathcal{X} + \mathcal{X}(\mathcal{A} - aI)^T & -\mathcal{X} & -\mathcal{A}\mathcal{X} & -\mathcal{X}\Delta^T \\ \hline 0 & -\mathcal{X} & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X}\mathcal{A}^T & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ 0 & -\Delta\mathcal{X} & 0 & 0 & -I \end{array}\right] < 0 \tag{4.6}$$

Here, $(\mathcal{A} - aI)$ is nonsingular because $\mathcal{A}$ is stable by the condition (i). Hence, the above inequality admits a congruence transformation with $I \oplus (\mathcal{A} - aI)^{-1} \oplus I \oplus I \oplus I$ to get

$$
\begin{bmatrix}
-2a\mathcal{X} & -2a\mathcal{X}(\mathcal{A}-aI)^{-T} & 0 & 0 & 0 \\
-2a(\mathcal{A}-aI)^{-1}\mathcal{X} & (\mathcal{A}-aI)^{-1}\mathcal{X}+\mathcal{X}(\mathcal{A}-aI)^{-T} & -(\mathcal{A}-aI)^{-1}\mathcal{X} & -\mathcal{X}-a(\mathcal{A}-aI)^{-1}\mathcal{X} & -(\mathcal{A}-aI)^{-1}\mathcal{X}\varDelta^{T} \\
0 & -\mathcal{X}(\mathcal{A}-aI)^{-T} & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\
0 & -\mathcal{X}-a\mathcal{X}(\mathcal{A}-aI)^{-T} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\
0 & -\varDelta\mathcal{X}(\mathcal{A}-aI)^{-T} & 0 & 0 & -I
\end{bmatrix} < 0 \tag{4.7}
$$

Defining $\widehat{G} := -(\mathcal{A}-aI)^{-1}\mathcal{X}$, we have $\mathcal{X} = -(\mathcal{A}-aI)\widehat{G}$, and thus we readily obtain

$$
\begin{bmatrix}
2a\mathcal{X}+2a\mathrm{He}[(\mathcal{A}-aI)\widehat{G}] & 2a\widehat{G}^{T}-\mathcal{X}-(\mathcal{A}-aI)\widehat{G} & \mathcal{X}+(\mathcal{A}-aI)\widehat{G} & a\mathcal{X}+a(\mathcal{A}-aI)\widehat{G} & \mathcal{X}\varDelta^{T}+(\mathcal{A}-aI)\widehat{G}\varDelta^{T} \\
2a\widehat{G}-\mathcal{X}-\widehat{G}^{T}(\mathcal{A}-aI)^{T} & -\widehat{G}-\widehat{G}^{T} & \widehat{G} & -\mathcal{X}+a\widehat{G} & \widehat{G}\varDelta^{T} \\
\mathcal{X}+\widehat{G}^{T}(\mathcal{A}-aI)^{T} & \widehat{G}^{T} & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\
a\mathcal{X}+a\widehat{G}^{T}(\mathcal{A}-aI)^{T} & -\mathcal{X}+a\widehat{G}^{T} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\
\varDelta\mathcal{X}+\varDelta\widehat{G}^{T}(\mathcal{A}-aI)^{T} & \varDelta\widehat{G}^{T} & 0 & 0 & -I
\end{bmatrix} < 0 \tag{4.8}
$$

The above inequality can be written as follows.

$$
\begin{bmatrix}
2a\mathcal{X} & -\mathcal{X} & \mathcal{X} & a\mathcal{X} & \mathcal{X}\varDelta^{T} \\
-\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\
\mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\
a\mathcal{X} & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\
\varDelta\mathcal{X} & 0 & 0 & 0 & -I
\end{bmatrix}
+\mathrm{He}\left\{
\begin{bmatrix} \mathcal{A}-aI \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}
\widehat{G}\begin{bmatrix} 2aI & -I & I & aI & \varDelta^{T} \end{bmatrix}
\right\} < 0 \tag{4.9}
$$

Performing a congruence transformation with $\begin{bmatrix} I & aI \\ 0 & I \end{bmatrix} \oplus I \oplus I \oplus I$ on (4.9), we arrive at (4.3) in (ii), where $G := a\widehat{G}$.

It remains to show that the condition (ii) implies the condition (i), which is a simple task since

$$
\begin{aligned}
&\begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\perp}
\begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix}
\begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}^{T\perp} \\
&= \begin{bmatrix} I & -\mathcal{A} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}
\begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix}
\begin{bmatrix} I & 0 & 0 & 0 \\ -\mathcal{A}^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&= \left[\begin{array}{c|ccc} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T & \mathcal{X} & \mathcal{A}\mathcal{X} & \mathcal{X}\Delta^T \\ \hline \mathcal{X} & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ \mathcal{X}\mathcal{A}^T & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & -I \end{array}\right] < 0
\end{aligned}
\tag{4.10}
$$

Applying the Schur complement technique to the above inequality, we have (4.2) in the condition (i). Q.E.D.

The advantage of working with the dilated characterization (4.3) instead of (4.2) is that the Lyapunov variable $\mathcal{X}$ appears nowhere as a product with $\mathcal{A}$. This is quite appealing in dealing with a wide range of problems including multiobjective control, robust performance analysis or synthesis for real polytopic uncertainty and so on, because the technical restriction to a common Lyapunov variable (i.e., a common Lyapunov function) in the conventional approach can be avoided. Namely, we can employ non-common and/or parameter-dependent Lyapunov variables. Moreover, the relation between the solutions of (4.2) and (4.3) shown in Lemma 4.1 is quite important to clarify the advantage of the dilated characterization over the conventional one, especially in dealing with the multiobjective controller design problems.

In the following, we repeatedly refer to the relation between the solutions of (4.2) and (4.3). For the convenience in referring to this relation, we introduce the following definition.

Definition 4.1 We say that the dilated characterization (4.3) *recovers* the original one (4.2) via $G = \widetilde{G}(X)$ if the matrix function $\widetilde{G}(X)$ is such that $[\ \mathcal{X}\ G\ ] = [\ X\ \widetilde{G}(X)\ ]$ is a solution of (4.3) whenever $\mathcal{X} = X > 0$ is a solution of (4.2).

With this definition, the relation shown in Lemma 4.1 can be represented simply by saying that the dilated characterization (4.3) recovers (4.2) via $G = \widetilde{G}(X) := -a(\mathcal{A} - aI)^{-1}X$.

In the following, several properties of Lemma 4.1 are studied. First, we give some remarks on a possible independent proof of Lemma 4.1 with the *Elimination Lemma* stated below, which is frequently used to derive dilated characterizations in the previous studies [1],[33]–[35].

Lemma 4.3 (*Elimination Lemma*) [14],[22],[43] Let matrices $E \in \mathbf{R}^{k\times l}$, $F \in \mathbf{R}^{m\times k}$ and $Y = Y^T \in \mathbf{R}^{k\times k}$ be given. Then, the following two conditions are equivalent.

(i) The following two conditions hold.

$$\begin{array}{llll} E^{\perp}Y(E^{\perp})^T < 0 & (k > l) & \text{or} & EE^T > 0 \quad (k \leq l), \\ (F^T)^{\perp}YF^{\perp} < 0 & (k > m) & \text{or} & F^TF > 0 \quad (k \leq m) \end{array} \tag{4.11}$$

(ii) There exists a matrix $Q \in \mathbf{R}^{l\times m}$ such that

$$Y + \mathrm{He}\,[EQF] < 0 \tag{4.12}$$

As is shown in the Appendix section, Section 4.7, it is possible to apply the Elimination Lemma to show the equivalence between (4.2) and (4.3). However, we would like to stress the indispensability of our particular proof stated above, especially in the following two points.

1. The Elimination Lemma originally moves from the condition (ii) to (i) to eliminate the variable Q [14],[22]. Hence, the Elimination Lemma itself gives no constructive way to derive the dilated characterization (4.12) from the original condition (4.11). To see this, note that the original conditions are rarely given in such a form as in (4.11) with the matrices E and F given explicitly, and in general, it is hard to find out these matrices such that the original conditions are equivalent to the conditions in (4.11) (see the proof of Lemma 4.1 with the Elimination Lemma given in the Appendix section). To overcome the difficulty, we certainly need another effort, as is also suggested in [1], [30],[33].

2. We have shown a nice and interesting relation between the solutions of (4.2) and (4.3). Namely, the dilated characterization (4.3) recovers the original one (4.2) via the particular choice of the auxiliary variable

$$G = \tilde{G}(a, \mathcal{A}, X) := -a(\mathcal{A} - aI)^{-1}X \tag{4.13}$$

This relation is successfully obtained by our particular proof and does not follow directly by applying the Elimination Lemmă. This is because the general solution Q to (4.12) given in [14],[22],[43] is a complicated function with respect to Y, E and F. Although the solution G that satisfies (4.3) for a given $\mathcal{X} = X > 0$ is not unique, the above recovery property with a special choice of G is quite important in dealing with multiobjective controller synthesis (see Section 4.3).

In this chapter, we deal with not only the multiobjective controller design problems but also the robust performance analysis and controller synthesis problems for real polytopic uncertainty. It turns out that because of the relation described by (4.13), our new approach based on the dilated characterizations ensures an advantage over the conventional approach in dealing with the multiobjective controller design problems. Unfortunately, however, the choice (4.13) of the auxiliary variable depends on the coefficient matrix $\mathcal{A}$ and hence we need another effort to ensure an advantage in dealing with the synthesis problems *under the polytopic uncertainty setting.* Here, by inspection, we have

$$G = \lim_{a \to \infty} \tilde{G}(a, \mathcal{A}, X) = X \tag{4.14}$$

where the dependence on $\mathcal{A}$ disappears. This suggests that the dilated characterization (4.3) recovers the original condition (4.2) via the particular choice $G = \tilde{G}(X) := X$ for sufficiently large a. Even though the relation (4.14) does not seem strong enough to validate this observation immediately, we can indeed establish the following result.

Lemma 4.4 For every solution $\mathcal{X} = X > 0$ of (4.2), $[\ \mathcal{X}\ G\ ] = [\ X\ X\ ]$ is a solution of (4.3) if $a > a_{\min} > 0$, where $a_{\min} = a_{\min}(X)$ is characterized by the infimum of $a > 0$ satisfying the following two inequalities.

$$\begin{aligned} &\mathcal{A}X + X\mathcal{A}^T + \delta_1 X + \delta_2 \mathcal{A}X\mathcal{A}^T + X\Delta^T\Delta X \\ &\qquad\qquad + \mathcal{R}(X)(-\delta_1 X - X\Delta\Delta^T X + 2aX)^{-1}\mathcal{R}^T(X) < 0 \end{aligned} \tag{4.15}$$

$$-\delta_1 X - X\Delta\Delta^T X + 2aX > 0 \tag{4.16}$$

Here, $\mathcal{R}(X) := \mathcal{A}X + \delta_1 X + X\Delta^T\Delta X$.

Proof. See the Appendix section of this chapter for the proof. Q.E.D.

Remark 4.1 In the above lemma, we have shown that the dilated characterization (4.3) recovers (4.2) via $G = \widetilde{G}(X) := X$ if we take a sufficiently large $a > 0$. If we let $a \to \infty$, however, it is also seen that all admissible auxiliary variables G tend to $\mathcal{X}$ and hence the dilated characterization (4.3) "reduces" to (4.2). To see this, let us consider the inequality (4.5). Note that the feasibility of (4.5) is a necessary condition for the feasibility of (4.3). With this in mind, let us rewrite (4.5) in the following form.

$$\begin{bmatrix} \mathcal{A}G + G^T\mathcal{A}^T & G^T - \mathcal{X} - b\mathcal{A}G \\ G - \mathcal{X} - bG^T\mathcal{A}^T & -b(G + G^T) \end{bmatrix} < 0 \tag{4.17}$$

Applying the Schur complement technique to the above inequality, it follows that (4.5) is equivalent to

$$\begin{aligned} &\mathcal{A}G + G^T\mathcal{A}^T + a(G^T - \mathcal{X})(G + G^T)^{-1}(G - \mathcal{X}) \\ &\qquad -\mathrm{He}[(G^T - \mathcal{X})(G + G^T)^{-1}G^T\mathcal{A}^T] + a^{-1}\mathcal{A}G(G + G^T)^{-1}G^T\mathcal{A}^T < 0 \\ &G + G^T > 0 \end{aligned} \tag{4.18}$$

The above two inequalities imply that if we let $a \to \infty$, $G \neq \mathcal{X}$ is not allowed for the feasibility of (4.5) and hence for the feasibility of (4.3). This together with Lemma 4.4 establishes the assertion.

The relations between the inequalities (4.2) and (4.3) obtained above are summarized in the following.

- The dilated characterization (4.3) recovers the original one (4.2) via the specific choice of the auxiliary variable $G = \widetilde{G}(X) := -a(\mathcal{A} - aI)^{-1}X$.

- The dilated characterization (4.3) recovers the original one (4.2) via the specific choice of the auxiliary variable $G = \widetilde{G}(X) := X$ if we take sufficiently large a. If we let $a \to \infty$, however, all admissible auxiliary variables G tend to $\mathcal{X}$ and hence the dilated characterization (4.3) "reduces" to (4.2). Namely, we lose the advantage of working with the dilated characterization if a is taken excessively large.

Note that the above relations are to some extent analogous to those already obtained in the discrete-time setting [29],[30].

Here, we will give another intriguing interpretation of (4.3) in comparison with the dilated characterizations frequently used in the previous studies [33]–[35]. With Lemma 4.1 and with the idea of the Elimination Lemma, we readily obtain the following result.

Lemma 4.5 The following condition is also equivalent to the conditions (i) and (ii) given in Lemma 4.1.

(iii) There exist matrices $\mathcal{X} > 0$ and $Q = [\ Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ ]$ such that

$$\begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix} Q \right\} < 0 \tag{4.19}$$

Moreover, for every solution $\mathcal{X} = X > 0$ of (4.2), there exists a sufficiently small $\varepsilon > 0$ such that $[\ \mathcal{X}\ Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ ] = [\ X\ X\ \ -\varepsilon X\ 0\ X\ 0\ ]$ is a solution of (4.19), irrespective of δ_1, δ_2 and Δ. Conversely, every matrix $\mathcal{X} > 0$ such that (4.19) holds for some Q_i $(i = 1, \cdots, 5)$ also satisfies (4.2), irrespective of δ_1, δ_2 and Δ.

Proof. See the Appendix section of this chapter for the proof. Q.E.D.

The dilated characterization (4.19) is nothing but a general description of the LMI's used in [33]–[35], where robust performance analysis problems for real polytopic uncertainty are addressed. Note that in dealing with analysis problems, the inequality (4.19) can be regarded as an LMI with respect to $\mathcal{X}$ and Q_i $(i = 1, \cdots, 5)$. However, for synthesis problems, the characterization (4.19) is of little use since it employs several auxiliary variables involved in the products with the controller variables, which prevents us from applying the change of variables technique. In this context, the dilated characterization (4.3) can be interpreted as a special case of (4.19) which successfully reduced the number of auxiliary variables so that controller synthesis problems can be also addressed.

Thus, we have shown that the original characterization (4.2) allows dilated ones (4.3) and (4.19) that are suitable for controller synthesis and performance analysis, respectively. The advantage of these dilated characterizations will be demonstrated in the rest of this chapter. In spite of the advantages, unfortunately, it could be said that the original characterization (4.2) is not thoroughly comprehensive in dealing with practical design specifications. For example, the H_∞ specification is shown to be the one that cannot be characterized in the form of (4.2). However, if we confine our attention to *analysis* problems, we arrive at a more comprehensive matrix inequality characterization than (4.2) that allows a dilated one. This is enabled by the fact that, in dealing with analysis problems, there is no technical restriction on the number of auxiliary variables and hence several difficulties in deriving dilated characterizations are avoided. Before closing this section, we provide such a matrix inequality characterization for the control performance of continuous-time systems.

Lemma 4.6 Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$, matrix functions $\mathcal{M} : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times nl}$, $\mathcal{N} : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times m}$, matrices $\Delta_1 \in \mathbf{R}^{l\times l} > 0$ and $\Delta_2 \in \mathbf{R}^{m\times m} > 0$ be given. Then the following two conditions are equivalent.

(i) There exists a matrix $\mathcal{X} > 0$ such that

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^T + \mathcal{M}(\mathcal{X})(\Delta_1 \otimes \mathcal{X}^{-1})\mathcal{M}^T(\mathcal{X}) + \mathcal{N}(\mathcal{X})\Delta_2\mathcal{N}^T(\mathcal{X}) < 0 \tag{4.20}$$

(ii) There exist matrices $\mathcal{X} > 0$ and $Q \in \mathbf{R}^{n\times\{n(2+l)+m\}}$ such that

$$\begin{bmatrix} 0 & -\mathcal{X} & \mathcal{M}(\mathcal{X}) & \mathcal{N}(\mathcal{X}) \\ -\mathcal{X} & 0 & 0 & 0 \\ \mathcal{M}^T(\mathcal{X}) & 0 & -\Delta_1^{-1} \otimes \mathcal{X} & 0 \\ \mathcal{N}^T(\mathcal{X}) & 0 & 0 & -\Delta_2^{-1} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \end{bmatrix} Q \right\} < 0 \tag{4.21}$$

Moreover, for every solution $\mathcal{X} = X > 0$ of (4.20), there exists a sufficiently small $\varepsilon > 0$ such that the pair $\mathcal{X} = X$, $Q = [\, X \;\; -\varepsilon X \;\; 0_{n,nl+m} \,]$ is a solution of (4.21), irrespective of $\mathcal{M}$, $\mathcal{N}$, Δ_1 and Δ_2. Conversely, every matrix $\mathcal{X} > 0$ such that (4.21) holds for some Q also satisfies (4.20), irrespective of $\mathcal{M}$, $\mathcal{N}$, Δ_1 and Δ_2.

Proof. The assertion follows immediately from the Elimination Lemma and a similar methodology to the one given in the proof of Lemma 4.5. Q.E.D.

Remark 4.2 If the matrix functions $\mathcal{M}$ and $\mathcal{N}$ in Lemma 4.6 are affine functions, then the inequality (4.21) can be regarded as an LMI with respect to $\mathcal{X}$ and Q in dealing with analysis problems.

4.2 New Dilated Characterizations for Continuous-Time Controller Design and Performance Analysis

In this section, we give some new dilated matrix inequality characterizations for practical controller design and performance analysis in the continuous-time setting. These dilated characterizations are readily obtained by the application of Lemmas 4.1, 4.5 and 4.6.

4.2.1 New Dilated Characterizations for Stability

This subsection describes new dilated characterizations for stability. In the following theorem, the equivalence between the conditions (i) and (ii) is well-known [4],[43], and a similar condition to (iv) is also derived in [33],[34].

Theorem 4.1 (*Stability*) Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$ and a scalar $b = a^{-1} > 0$ be given. Then, the following four conditions are equivalent.

(i) The matrix $\mathcal{A}$ is stable in the continuous-time sense.

(ii) (*Lyapunov inequality*) There exists a matrix $X_{\mathcal{L}} > 0$ such that

$$\mathcal{A}X_{\mathcal{L}} + X_{\mathcal{L}}\mathcal{A}^T < 0 \tag{4.22}$$

(iii) There exist matrices $X_{\mathcal{L}} > 0$ and $G_{\mathcal{L}}$ such that

$$\begin{bmatrix} 0 & -X_{\mathcal{L}} \\ -X_{\mathcal{L}} & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \end{bmatrix} G_{\mathcal{L}} \begin{bmatrix} I & -bI \end{bmatrix} \right\} < 0 \tag{4.23}$$

(iv) There exist matrices $X_{\mathcal{L}} > 0$ and $F_{\mathcal{L}} = [\ F_{\mathcal{L}1}\ F_{\mathcal{L}2}\ ]$ such that

$$\begin{bmatrix} 0 & -X_{\mathcal{L}} \\ -X_{\mathcal{L}} & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \end{bmatrix} F_{\mathcal{L}} \right\} < 0 \tag{4.24}$$

Proof. The equivalence between the conditions (i) and (ii) is well-known. The equivalence between the conditions (ii) and (iii) immediately follows by the application of Lemma 4.1 with $\delta_1 = \delta_2 \to 0$, $\varDelta = 0$ and with $\mathcal{X} := X_{\mathcal{L}}$, $G := G_{\mathcal{L}}$. Similarly, the equivalence between the conditions (ii) and (iv) immediately follows by the application of Lemma 4.5 with $\delta_1 = \delta_2 \to 0$, $\varDelta = 0$ and with $\mathcal{X} := X_{\mathcal{L}}$, $Q := F_{\mathcal{L}}$. Q.E.D.

The following two corollaries describe specifically the relation between the solutions of (4.22)–(4.24).

Corollary 4.1 For every solution $X_{\mathcal{L}} = X > 0$ of (4.22), $[\ X_{\mathcal{L}}\ G_{\mathcal{L}}\ ] = [\ X\ -a(\mathcal{A}-aI)^{-1}X\ ]$ is a solution of (4.23). Conversely, every matrix $X_{\mathcal{L}} > 0$ such that (4.23) holds for some $G_{\mathcal{L}}$ also satisfies (4.22).

Corollary 4.2 For every solution $X_{\mathcal{L}} = X > 0$ of (4.22), there exists a sufficiently small $\varepsilon > 0$ such that $[\ X_{\mathcal{L}}\ F_{\mathcal{L}1}\ F_{\mathcal{L}2}\ ] = [\ X\ X\ -\varepsilon X\ ]$ is a solution of (4.24). Conversely, every matrix $X_{\mathcal{L}} > 0$ such that (4.24) holds for some $F_{\mathcal{L}}$ also satisfies (4.22).

4.2.2 New Dilated Characterizations for D-stability

Let us consider the new dilated characterizations for the regional pole placement (**D**-stability) constraints. Concerning the pole placement region **D**, we consider the α-stability region $\mathcal{H}(\alpha)$, the circular region $\mathcal{C}(c,r)$ and the conic sector region $\mathcal{S}(k)$ given by (2.13). The pole placement in each of the LMI regions $\mathcal{H}(\alpha)$, $\mathcal{C}(c,r)$ and $\mathcal{S}(k)$ is given in order, where in each of the theorems, the equivalence between the conditions (i) and (ii) is a well-known result derived in [6]. In the previous studies, Peaucelle et al. investigated some tests for robust **D**-stability analysis problem in [34], and arrived at similar conditions to (iv) given in each of the theorems.

Theorem 4.2 (*α-stability Region*) Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$ and a scalar $b = a^{-1} > 0$ be given. Then, the following four conditions are equivalent.

(i) The matrix $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \subset \mathcal{H}(\alpha)$.

(ii) There exists a matrix $X_{\mathcal{H}} > 0$ such that

$$\mathcal{A}X_{\mathcal{H}} + X_{\mathcal{H}}\mathcal{A}^T + 2\alpha X_{\mathcal{H}} < 0 \tag{4.25}$$

(iii) There exist matrices $X_{\mathcal{H}} > 0$ and $G_{\mathcal{H}}$ such that

$$\begin{bmatrix} 0 & -X_{\mathcal{H}} & X_{\mathcal{H}} \\ -X_{\mathcal{H}} & 0 & 0 \\ X_{\mathcal{H}} & 0 & -\frac{1}{2}\alpha^{-1}X_{\mathcal{H}} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \end{bmatrix} G_{\mathcal{H}} \begin{bmatrix} I & -bI & bI \end{bmatrix} \right\} < 0 \tag{4.26}$$

(iv) There exist matrices $X_{\mathcal{H}} > 0$ and $F_{\mathcal{H}} = [\, F_{\mathcal{H}1} \; F_{\mathcal{H}2} \; F_{\mathcal{H}3} \,]$ such that

$$\begin{bmatrix} X_{\mathcal{H}} & -X_{\mathcal{H}} & X_{\mathcal{H}} \\ -X_{\mathcal{H}} & 0 & 0 \\ X_{\mathcal{H}} & 0 & -\frac{1}{2}\alpha^{-1}X_{\mathcal{H}} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \end{bmatrix} F_{\mathcal{H}} \right\} < 0 \tag{4.27}$$

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [6]. The equivalence between the conditions (ii) and (iii) immediately follows by the application of Lemma 4.1 with $\delta_1 = 2\alpha$, $\delta_2 \to 0$, $\Delta = 0$ and with $\mathcal{X} := X_{\mathcal{H}}$, $G := G_{\mathcal{H}}$. Similarly, the equivalence between the conditions (ii) and (iv) immediately follows by the application of Lemma 4.5 with $\delta_1 = 2\alpha$, $\delta_2 \to 0$, $\Delta = 0$ and with $\mathcal{X} := X_{\mathcal{H}}$, $Q := F_{\mathcal{H}}$. Q.E.D.

Corollary 4.3 For every solution $X_{\mathcal{H}} = X > 0$ of (4.25), $[\, X_{\mathcal{H}} \; G_{\mathcal{H}} \,] = [\, X \; -a(\mathcal{A}-aI)^{-1}X \,]$ is a solution of (4.26). Conversely, every matrix $X_{\mathcal{H}} > 0$ such that (4.26) holds for some $G_{\mathcal{H}}$ also satisfies (4.25).

Corollary 4.4 For every solution $X_{\mathcal{H}} = X > 0$ of (4.25), there exists a sufficiently small $\varepsilon > 0$ such that $[\, X_{\mathcal{H}}\ F_{\mathcal{H}1}\ F_{\mathcal{H}2}\ F_{\mathcal{H}3}\,] = [\, X\ X\ \ -\varepsilon X\ 0\,]$ is a solution of (4.27). Conversely, every matrix $X_{\mathcal{H}} > 0$ such that (4.27) holds for some $F_{\mathcal{H}}$ also satisfies (4.25).

Theorem 4.3 (*Circular Region*) Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$ and a scalar $b = a^{-1} > 0$ be given. Then, the following four conditions are equivalent.

(i) The matrix $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \subset \mathcal{C}(c, r)$.

(ii) There exists a matrix $X_{\mathcal{C}} > 0$ such that

$$\begin{bmatrix} -rX_{\mathcal{C}} & \mathcal{A}X_{\mathcal{C}} - cX_{\mathcal{C}} \\ X_{\mathcal{C}}\mathcal{A}^T - cX_{\mathcal{C}} & -rX_{\mathcal{C}} \end{bmatrix} < 0 \tag{4.28}$$

(iii) There exist matrices $X_{\mathcal{C}} > 0$ and $G_{\mathcal{C}}$ such that

$$\begin{bmatrix} 0 & -X_{\mathcal{C}} & X_{\mathcal{C}} & 0 \\ -X_{\mathcal{C}} & 0 & 0 & -X_{\mathcal{C}} \\ X_{\mathcal{C}} & 0 & \frac{c}{\beta}X_{\mathcal{C}} & 0 \\ 0 & -X_{\mathcal{C}} & 0 & cX_{\mathcal{C}} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \end{bmatrix} G_{\mathcal{C}} \begin{bmatrix} I & -bI & bI & I \end{bmatrix} \right\} < 0 \tag{4.29}$$

where $\beta := c^2 - r^2 (> 0)$.

(iv) There exist matrices $X_{\mathcal{C}} > 0$ and $F_{\mathcal{C}} = [\, F_{\mathcal{C}1}\ F_{\mathcal{C}2}\ F_{\mathcal{C}3}\ F_{\mathcal{C}4}\,]$ such that

$$\begin{bmatrix} 0 & -X_{\mathcal{C}} & X_{\mathcal{C}} & 0 \\ -X_{\mathcal{C}} & 0 & 0 & -X_{\mathcal{C}} \\ X_{\mathcal{C}} & 0 & \frac{c}{\beta}X_{\mathcal{C}} & 0 \\ 0 & -X_{\mathcal{C}} & 0 & cX_{\mathcal{C}} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \end{bmatrix} F_{\mathcal{C}} \right\} < 0 \tag{4.30}$$

where $\beta := c^2 - r^2 (> 0)$.

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [6]. Since (4.28) in the condition (ii) is equivalent to

$$\mathcal{A}X_{\mathcal{C}} + X_{\mathcal{C}}\mathcal{A}^T - \frac{\beta}{c}X_{\mathcal{C}} - \frac{1}{c}\mathcal{A}X_{\mathcal{C}}\mathcal{A}^T < 0, \tag{4.31}$$

the equivalence between the conditions (ii) and (iii) immediately follows by the application of Lemma 4.1 with $\delta_1 = -\frac{\beta}{c}$ (> 0), $\delta_2 = -\frac{1}{c}$ (> 0), $\varDelta = 0$, and with $\mathcal{X} := X_{\mathcal{C}}$, $G := G_{\mathcal{C}}$. Similarly, the equivalence between the conditions (ii) and (iv) immediately follows by the application of Lemma 4.5 with $\delta_1 = -\frac{\beta}{c}$, $\delta_2 = -\frac{1}{c}$, $\varDelta_1 = 0$ and with $\mathcal{X} := X_{\mathcal{C}}$, $Q := F_{\mathcal{C}}$.

Q.E.D.

Corollary 4.5 For every solution $X_{\mathcal{C}} = X > 0$ of (4.28), $[\,X_{\mathcal{C}}\ G_{\mathcal{C}}\,] = [\,X\ \ -a(\mathcal{A}-aI)^{-1}X\,]$ is a solution of (4.29). Conversely, every matrix $X_{\mathcal{C}} > 0$ such that (4.29) holds for some $G_{\mathcal{C}}$ also satisfies (4.28).

Corollary 4.6 For every solution $X_{\mathcal{C}} = X > 0$ of (4.28), there exists a sufficiently small $\varepsilon > 0$ such that $[\,X_{\mathcal{C}}\ F_{\mathcal{C}1}\ F_{\mathcal{C}2}\ F_{\mathcal{C}3}\ F_{\mathcal{C}4}\,] = [\,X\ X\ \ -\varepsilon X\ 0\ X\,]$ is a solution of (4.30). Conversely, every matrix $X_{\mathcal{C}} > 0$ such that (4.30) holds for some $F_{\mathcal{C}}$ also satisfies (4.28).

Theorem 4.4 (*Conic Sector Region*) Let a matrix $\mathcal{A} \in \mathbf{R}^{n\times n}$ and a scalar $b = a^{-1} > 0$ be given. Then, the following four conditions are equivalent.

(i) The matrix $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \subset \mathcal{S}(k)$.

(ii) There exists a matrix $X_{\mathcal{S}} > 0$ such that

$$\begin{bmatrix} k(\mathcal{A}X_{\mathcal{S}} + X_{\mathcal{S}}\mathcal{A}^T) & \mathcal{A}X_{\mathcal{S}} - X_{\mathcal{S}}\mathcal{A}^T \\ X_{\mathcal{S}}\mathcal{A}^T - \mathcal{A}X_{\mathcal{S}} & k(\mathcal{A}X_{\mathcal{S}} + X_{\mathcal{S}}\mathcal{A}^T) \end{bmatrix} < 0 \tag{4.32}$$

(iii) There exist matrices $X_{\mathcal{S}} > 0$ and $G_{\mathcal{S}}$ such that

$$\begin{bmatrix} 0 & -kX_{\mathcal{S}} & X_{\mathcal{S}} & 0 \\ -kX_{\mathcal{S}} & 0 & 0 & -X_{\mathcal{S}} \\ X_{\mathcal{S}} & 0 & 0 & -kX_{\mathcal{S}} \\ 0 & -X_{\mathcal{S}} & -kX_{\mathcal{S}} & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} & 0 \\ I & 0 \\ 0 & I \\ 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} G_{\mathcal{S}} & 0 \\ 0 & G_{\mathcal{S}} \end{bmatrix} \begin{bmatrix} kI & -bkI & bI & I \\ -I & -bI & -bkI & kI \end{bmatrix} \right\} < 0 \tag{4.33}$$

(iv) There exist matrices $X_{\mathcal{S}} > 0$ and $F_{\mathcal{S}i}$ $(i = 1, \cdots, 8)$ such that

$$\begin{bmatrix} 0 & -kX_{\mathcal{S}} & X_{\mathcal{S}} & 0 \\ -kX_{\mathcal{S}} & 0 & 0 & -X_{\mathcal{S}} \\ X_{\mathcal{S}} & 0 & 0 & -kX_{\mathcal{S}} \\ 0 & -X_{\mathcal{S}} & -kX_{\mathcal{S}} & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} & 0 \\ I & 0 \\ 0 & I \\ 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} F_{\mathcal{S}1} & F_{\mathcal{S}2} & F_{\mathcal{S}3} & F_{\mathcal{S}4} \\ F_{\mathcal{S}5} & F_{\mathcal{S}6} & F_{\mathcal{S}7} & F_{\mathcal{S}8} \end{bmatrix} \right\} < 0 \tag{4.34}$$

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [6]. To show the equivalence between the conditions (ii) and (iii) or (ii) and (iv), we cannot apply Lemma 4.1 or Lemma 4.5 directly because of the form (4.32). However, by closely following a similar methodology to that in Lemma 4.1 or Lemma 4.5, we can show that the conditions (ii), (iii) and (iv) are equivalent, the details of which are thoroughly described in the Appendix section of this chapter. Q.E.D.

Because of the proof of this theorem given in the Appendix, we readily obtain the following results.

Corollary 4.7 For every solution $X_{\mathcal{S}} = X > 0$ of (4.32), $[\, X_{\mathcal{S}} \; G_{\mathcal{S}} \,] = [\, X \; -a(\mathcal{A}-aI)^{-1}X \,]$ is a solution of (4.33). Conversely, every matrix $X_{\mathcal{S}} > 0$ such that (4.33) holds for some $G_{\mathcal{S}}$ also satisfies (4.32).

Corollary 4.8 For every solution $X_{\mathcal{S}} = X > 0$ of (4.32), there exists a sufficiently small $\varepsilon > 0$ such that $[\, X_{\mathcal{S}} \; F_{\mathcal{S}1} \; F_{\mathcal{S}2} \; F_{\mathcal{S}3} \; F_{\mathcal{S}4} \; F_{\mathcal{S}5} \; F_{\mathcal{S}6} \; F_{\mathcal{S}7} \; F_{\mathcal{S}8} \,] = [\, X \;\; kX \;\; -\varepsilon X \;\; 0 \;\; X \;\; -X \;\; 0 \;\; -\varepsilon X \;\; kX \,]$ is a solution of (4.34). Conversely, every matrix $X_{\mathcal{S}} > 0$ such that (4.34) holds for some $F_{\mathcal{S}i}$ $(i = 1, \cdots, 8)$ also satisfies (4.32).

Remark 4.3 It should be noted that a simple application of the well-known Elimination Lemma (i.e. Lemma 4.3) does not lead to the equivalence between the conditions (ii) and (iii) in a straightforward fashion, because in (4.33) the solution corresponding to Q in (4.12) has a structure of the form $G_{\mathcal{S}} \oplus G_{\mathcal{S}}$. Since the condition (i) of Theorem 4.4 is equivalent to the stability condition of a complex matrix, it is expected that an extension of the Elimination Lemma to complex-valued matrices [43] might diminish the difficulty. However, it seems that a possible proof in this direction is still involved, so that even if such a direction is successful, the arguments could be even more involved than the methodology given in the Appendix, which is similar to the proof of Lemma 4.1.

4.2.3 New Dilated Characterizations for the H_2 Specification

This subsection describes new dilated characterizations for the H_2 specification. In the following theorem, the equivalence between the conditions (i) and (ii) is a well-known result from [4],[43].

Theorem 4.5 (*The H_2 Performance*) Let us consider the system described by

$$T_{zw}(s) := \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & 0 \end{array}\right] \tag{4.35}$$

For a given scalar $b = a^{-1} > 0$, the following four conditions are equivalent.

(i) The matrix $\mathcal{A}$ is stable and the H_2 cost $||T_{zw}(s)||_2$ is bounded by $\gamma_2 > 0$. Namely,

$$||T_{zw}(s)||_2 < \gamma_2 \tag{4.36}$$

(ii) There exist matrices $X_2 > 0$ and $Z_2 > 0$ such that

$$\begin{aligned}
&\begin{bmatrix} \mathcal{A}X_2 + X_2\mathcal{A}^T & X_2\mathcal{C}^T \\ \mathcal{C}X_2 & -I \end{bmatrix} < 0, \\
&\begin{bmatrix} Z_2 & \mathcal{B}^T \\ \mathcal{B} & X_2 \end{bmatrix} > 0, \quad \text{trace}(Z_2) < \gamma_2^2
\end{aligned} \tag{4.37}$$

(iii) There exist matrices $X_2 > 0$, $Z_2 > 0$ and G_2 such that

$$\begin{aligned}
&\begin{bmatrix} 0 & -X_2 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ \mathcal{C} \end{bmatrix} G_2 \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0, \\
&\begin{bmatrix} Z_2 & \mathcal{B}^T \\ \mathcal{B} & X_2 \end{bmatrix} > 0, \quad \text{trace}(Z_2) < \gamma_2^2
\end{aligned} \tag{4.38}$$

(iv) There exist matrices $X_2 > 0$, $Z_2 > 0$ and $F_2 = \begin{bmatrix} F_{21} & F_{22} & F_{23} \end{bmatrix}$ such that

$$\begin{aligned}
&\begin{bmatrix} 0 & -X_2 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ \mathcal{C} \end{bmatrix} F_2 \right\} < 0, \\
&\begin{bmatrix} Z_2 & \mathcal{B}^T \\ \mathcal{B} & X_2 \end{bmatrix} > 0, \quad \text{trace}(Z_2) < \gamma_2^2
\end{aligned} \tag{4.39}$$

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [4],[43]. The equivalence between the conditions (ii) and (iii) immediately follows by the application of Lemma 4.1 with $\delta_1 = \delta_2 \to 0$, $\Delta = \mathcal{C}$, and with $\mathcal{X} := X_2$, $G := G_2$. More specifically, from Lemma 4.1 together with Lemma 4.2, the first inequality in (4.37) is equivalent to

$$\begin{bmatrix} 0 & -X_2 & X_2\mathcal{C}^T \\ -X_2 & 0 & 0 \\ \mathcal{C}X_2 & 0 & -I \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \end{bmatrix} G_2 \begin{bmatrix} I & -bI & b\mathcal{C}^T \end{bmatrix} \right\} < 0 \tag{4.40}$$

Applying a congruence transformation with $I \oplus \begin{bmatrix} I & 0 \\ \mathcal{C} & I \end{bmatrix}$ on the above inequality, we obtain the first inequality in (4.38).

Similarly, the equivalence between the conditions (ii) and (iv) immediately follows by the application of Lemma 4.5 with $\delta_1 = \delta_2 \to 0$, $\Delta = \mathcal{C}$ and with $\mathcal{X} := X_2$, $Q := F_2$. Namely, from Lemma 4.5 together with Lemma 4.2, the first inequality in (4.37) is equivalent to

$$\begin{bmatrix} 0 & -X_2 & X_2\mathcal{C}^T \\ -X_2 & 0 & 0 \\ \mathcal{C}X_2 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \end{bmatrix} \begin{bmatrix} F_{21} & F_{22} & F_{23} \end{bmatrix} \right\} < 0 \tag{4.41}$$

By the same congruence transformation as stated above, we arrive at

$$\begin{bmatrix} 0 & -X_2 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ \mathcal{C} \end{bmatrix} \begin{bmatrix} F_{21} & F_{22} & F_{22}\mathcal{C}^T + F_{23} \end{bmatrix} \right\} < 0 \tag{4.42}$$

The above inequality is nothing but the first inequality in (4.39), by redefining F_{23} by $F_{22}\mathcal{C}^T + F_{23}$. Q.E.D.

Corollary 4.9 For every solution $[\, X_2 \; Z_2 \,] = [\, X \; Z \,]$ of (4.37), $[\, X_2 \; Z_2 \; G_2 \,] = [\, X \; Z \; -a(\mathcal{A} - aI)^{-1}X \,]$ is a solution of (4.38). Conversely, every pair of the matrices $X_2 > 0$ and $Z_2 > 0$ such that (4.38) holds for some G_2 also satisfies (4.37).

Corollary 4.10 For every solution $[\, X_2 \; Z_2 \,] = [\, X \; Z \,]$ of (4.37), there exists a sufficiently small $\varepsilon > 0$ such that $[\, X_2 \; Z_2 \; F_{21} \; F_{22} \; F_{23} \,] = [\, X \; Z \; X \; -\varepsilon X \; 0 \,]$ is a solution of (4.39). Conversely, every pair of the matrices $X_2 > 0$ and $Z_2 > 0$ such that (4.39) holds for some F_2 also satisfies (4.37).

Remark 4.4 It is a quite important fact that we can rewrite (4.40) as (4.38), and (4.41) as (4.39). In (4.40) and (4.41), the Lyapunov variable X_2 is involved in the product with the matrix $\mathcal{C}$ and hence the decoupling between the Lyapunov variable and the controller variables has not been achieved. On the other hand, in (4.38) and (4.39), the Lyapunov variable X_2 is not involved in any products with the controller variables and hence the decoupling has been achieved completely.

In recent studies, several dilated characterizations similar to (4.38) have been reported [1], [41]. However, these studies achieve the dilation based on the Elimination Lemma and hence they have not reached such result as what we have given in Corollary 4.9. It should be noted that this corollary plays an essential role in dealing with multiobjective controller design problems (see Section 4.3), and that it is our particular proof of Lemma 4.1 that led to this important corollary.

4.2.4 New Dilated Characterization for the H_∞ Specification

This subsection considers a new dilated characterization for the H_∞ specification. In the following theorem, the equivalence between the conditions (i) and (ii) is a well-known result from [4],[43], and a similar condition to (iii) can be found in [33].

Theorem 4.6 (*The H_∞ Performance*) For the system described by,

$$T_{zw}(s) := \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right], \tag{4.43}$$

the following three conditions are equivalent.

(i) The matrix $\mathcal{A}$ is stable and the H_∞ cost $||T_{zw}(s)||_\infty$ is bounded by $\gamma_\infty > 0$. Namely,

$$||T_{zw}(s)||_\infty < \gamma_\infty \tag{4.44}$$

(ii) There exists a matrix $X_\infty > 0$ such that

$$\begin{bmatrix} \mathcal{A}X_\infty + X_\infty \mathcal{A}^T & \mathcal{B} & X_\infty \mathcal{C}^T \\ \mathcal{B}^T & -I & \mathcal{D}^T \\ \mathcal{C}X_\infty & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{4.45}$$

(iii) There exist matrices $X_\infty > 0$ and $F_\infty = [\ F_{\infty 1}\ F_{\infty 2}\ F_{\infty 3}\ F_{\infty 4}\ ]$ such that

$$\begin{bmatrix} 0 & -X_\infty & \mathcal{B} & 0 \\ -X_\infty & 0 & 0 & 0 \\ \mathcal{B}^T & 0 & -I & \mathcal{D}^T \\ 0 & 0 & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ \mathcal{C} \end{bmatrix} F_\infty \right\} < 0 \tag{4.46}$$

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [4], [43]. Since (4.45) in the condition (ii) is equivalent to

$$\mathcal{A}X_\infty + X_\infty \mathcal{A}^T - \begin{bmatrix} \mathcal{B} & X_\infty \mathcal{C}^T \end{bmatrix} \begin{bmatrix} -I & \mathcal{D}^T \\ \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B}^T \\ \mathcal{C}X_\infty \end{bmatrix} < 0, \tag{4.47}$$

the equivalence between the conditions (ii) and (iii) immediately follows by the application of Lemma 4.6 with

$$\mathcal{M} := 0, \quad \mathcal{N}(X_\infty) := \begin{bmatrix} \mathcal{B} & X_\infty \mathcal{C}^T \end{bmatrix}, \quad \Delta_2 := -\begin{bmatrix} -I & \mathcal{D}^T \\ \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix}^{-1} > 0 \tag{4.48}$$

and with $Q := F_\infty$, where $\Delta_1 > 0$ is chosen arbitrarily. More specifically, from Lemma 4.6, the inequality in (4.45) is equivalent to

$$\begin{bmatrix} 0 & -X_\infty & \mathcal{B} & X_\infty\mathcal{C}^T \\ -X_\infty & 0 & 0 & 0 \\ \mathcal{B}^T & 0 & -I & \mathcal{D}^T \\ \mathcal{C}X_\infty & 0 & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} F_{\infty1} & F_{\infty2} & F_{\infty3} & F_{\infty4} \end{bmatrix} \right\} < 0 \tag{4.49}$$

It remains to perform a congruence transformation with $I \oplus \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{C} & 0 & I \end{bmatrix}$ on (4.49) to get

$$\begin{bmatrix} 0 & -X_\infty & \mathcal{B} & 0 \\ -X_\infty & 0 & 0 & 0 \\ \mathcal{B}^T & 0 & -I & \mathcal{D}^T \\ 0 & 0 & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ \mathcal{C} \end{bmatrix} \begin{bmatrix} F_{\infty1} & F_{\infty2} & F_{\infty3} & F_{\infty2}\mathcal{C}^T + F_{\infty4} \end{bmatrix} \right\} < 0 \tag{4.50}$$

The above inequality is nothing but the inequality (4.39), by redefining $F_{\infty4}$ by $F_{\infty2}\mathcal{C}^T + F_{\infty4}$.

Q.E.D.

Corollary 4.11 For every solution $X_\infty = X > 0$ of (4.45), there exists a sufficiently small $\varepsilon > 0$ such that $\begin{bmatrix} X_\infty & F_{\infty1} & F_{\infty2} & F_{\infty3} & F_{\infty4} \end{bmatrix} = \begin{bmatrix} X & X & -\varepsilon X & 0 & 0 \end{bmatrix}$ is a solution of (4.46). Conversely, every matrix $X_\infty > 0$ such that (4.46) holds for some F_∞ also satisfies (4.45).

Remark 4.5 Similarly to Remark 4.4, it is a quite important fact that we can rewrite (4.49) as (4.46). In (4.49), the Lyapunov variable X_∞ is involved in the product with $\mathcal{C}$ and hence the decoupling between the Lyapunov variable and the controller variables has not been achieved. On the other hand, in (4.46), the Lyapunov variable X_∞ is not involved in any products with controller variables and hence the decoupling has been achieved completely.

Remark 4.6 In contrast to the **D**-stability constraints and the H_2 specification, unfortunately, Lemma 4.1 and Lemma 4.5 cannot be applied to the H_∞ specification because we cannot rewrite the characterization (4.45) into the form of (4.2). This is one of the reasons why we introduced Lemma 4.6, with which we can derive the dilated characterization (4.46) for the H_∞ specification.

4.3 Multiobjective H_2/D-stability Synthesis with Non-Common Lyapunov Variables

In the preceding section, we have derived new dilated characterizations for the regional pole placement (**D**-stability) constraints, the H_2 specification and the H_∞ specification. Specifically, for the **D**-stability constraints and the H_2 specification, we have derived dilated characterizations with a single square auxiliary variable involved in the products with the controller variables, which is crucial to addressing synthesis problems. These dilated characterizations are successfully applied in this section to the multiobjective H_2/**D**-stability controller design problems with non-common Lyapunov variables.

Let us consider the continuous-time MIMO, LTI plant described by

$$\begin{cases} \dot{x} = Ax + B_2 w_2 + Bu \\ z_2 = C_2 x + D_{z2} u \\ y = Cx + D_{w2} w_2 \end{cases} \tag{4.51}$$

The controller that we consider is the full-order strictly proper output-feedback controller K given by

$$\begin{cases} \dot{x}_K = A_K x_K + B_K y \\ u = C_K x_K \end{cases} \tag{4.52}$$

In the state feedback case ($C = I$, $D_{w2} = 0$), we also consider the static state-feedback controller

$$u = Kx \tag{4.53}$$

With the plant (4.51) and the controller given by (4.52) or (4.53), the closed-loop system can be written as

$$\begin{cases} \dot{x}_{\text{cl}} = \mathcal{A} x_{\text{cl}} + \mathcal{B} w_2 \\ z_2 = \mathcal{C} x_{\text{cl}} + \mathcal{D} w_2 \end{cases} \tag{4.54}$$

and we denote its transfer function from w_2 to z_2 by $T_{z_2 w_2}(s)$. For the dynamic controller K, the coefficient matrices in (4.54) are given by

$$\mathcal{A} = \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_2 \\ B_K D_{w2} \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C_2 & D_{z2} C_K \end{bmatrix}, \quad \mathcal{D} = 0 \tag{4.55}$$

while for the static state-feedback controller K, the coefficient matrices are given by

$$\mathcal{A} = A + BK, \quad \mathcal{B} = B_2, \quad \mathcal{C} = C_2 + D_{z2} K, \quad \mathcal{D} = 0 \tag{4.56}$$

Now, we consider the multiobjective H_2/**D**-stability controller design problem [6],[19], [26],[36]. Recall that the problem is to find a controller K, full-order output-feedback or static state-feedback, that minimizes $||T_{z_2w_2}(s)||_2$ subject to the regional pole placement constraint $\sigma(\mathcal{A}) \subset \bigcap \{\mathcal{H}(\alpha), \mathcal{C}(c,r), \mathcal{S}(k)\}$.

We can describe two approaches for the problem via the characterizations given in the preceding section.

(i) Conventional Approach [6],[26],[36]

Minimize γ_2^2 subject to the constraints (4.25), (4.28), (4.32) and (4.37).

(ii) New Approach

Minimize γ_2^2 subject to the constraints (4.26), (4.29), (4.33) and (4.38). Here, the scalar $b = a^{-1}$ is arbitrarily chosen in advance.

In the following, we compare the above two approaches in terms of the conservatism of the design, following the arguments given in [36].

As we have seen repeatedly in the preceding sections, the characterizations (4.25), (4.28), (4.32) and (4.37) involve such products between the Lyapunov variables and the controller variables as $\mathcal{A}X_j + X_j\mathcal{A}^T$ $(j = \mathcal{H}, \mathcal{C}, \mathcal{S}, 2)$. Hence, the conventional approach results in a non-convex optimization problem. Convexity can be recovered by forcing those inequalities to have a common Lyapunov variable [6],[26],[36]

$$X := X_{\mathcal{H}} = X_{\mathcal{C}} = X_{\mathcal{S}} = X_2 \tag{4.57}$$

With the restriction (4.57), the conventional approach reduces to a convex optimization problem via the change of controller variables technique [6],[26],[36], as we have seen in Chapter 2. Clearly, this restriction brings conservatism into the design and only an upper bound of the cost functional will be minimized, but there is no further conservatism in this approach as shown in [36].

On the other hand, in the new approach, the characterizations (4.26), (4.29), (4.33) and (4.38) involve no products between the Lyapunov variables and the controller variables and hence it is very promising that we can arrive at the use of non-common Lyapunov variables for different design specifications. Unfortunately, the auxiliary variables G_j $(j = \mathcal{H}, \mathcal{C}, \mathcal{S}, 2)$ form products with the controller variables as in $\mathcal{A}G_j + G_j^T\mathcal{A}^T$, and thus the new approach still results in a non-convex optimization problem. As is clarified below, however, convexity can be recovered by forcing those inequalities to have a common auxiliary variable

$$G := G_{\mathcal{H}} = G_{\mathcal{C}} = G_{\mathcal{S}} = G_2 \tag{4.58}$$

With the restriction (4.58), the new approach reduces to a convex optimization problem involving LMI's only. Clearly, this restriction again brings conservatism into the design, and only an upper bound of the cost functional will be minimized. However there is no further conservatism in this approach, either, the details of which are clarified later on.

Based on the above arguments, we readily arrive at the following theorem, which assures the advantage of the new approach.

Theorem 4.7 For the multiobjective $H_2/\mathbf{D}$-stability controller design problem, suppose that the conventional LMI approach with a common Lyapunov variable (4.57) achieves an upper bound $\gamma_{2c} > 0$ of the cost functional. Then, the new LMI approach with a common *auxiliary variable* (4.58) and a common prescribed scalar $b = a^{-1}$ but with non-common *Lyapunov variables* always achieves a better (no worse) upper bound than γ_{2c}, irrespective of the choice of $a > 0$.

Proof. The assertion follows immediately from Corollaries 4.3, 4.5, 4.7 and 4.9. Indeed, suppose that the conventional LMI approach achieves an upper bound γ_{2c} with the variables

$$X_{\mathcal{H}} = X_{\mathcal{C}} = X_{\mathcal{S}} = X_2 = X, \quad Z_2 = Z \tag{4.59}$$

in the constraints (4.25), (4.28), (4.32) and (4.37). Then, from the above-mentioned corollaries, the new LMI approach ensures the achievement of the same upper bound γ_{2c} with the variables

$$\begin{aligned} &X_{\mathcal{H}} = X_{\mathcal{C}} = X_{\mathcal{S}} = X_2 = X, \quad Z_2 = Z \\ &G_{\mathcal{H}} = G_{\mathcal{C}} = G_{\mathcal{S}} = G_2 = G = \widetilde{G}(a, \mathcal{A}, X) := -a(\mathcal{A} - aI)^{-1}X \end{aligned} \tag{4.60}$$

in the constraints (4.26), (4.29), (4.33) and (4.38). Observe the roles of the above corollaries in ensuring the condition (4.58). Q.E.D.

In the previous studies, the multiobjective $H_2/\mathbf{D}$-stability problems have also been reduced to the convex optimization problems of an upper bound of the cost functional, but the upper bounds there are ensured by forcing a common Lyapunov variable [6],[19],[26], [36]. This could be regarded as a standard tractable approach, but sometimes results in excessively conservative design. The new approach is quite different from those in that non-common Lyapunov variables are employed. Although Theorem 4.7 assures only the improvement of an upper bound of the cost functional, the actual cost will be also improved in general, due to the freedom of non-common Lyapunov variables and the auxiliary variable G (see the illustrative examples in Section 4.5). Moreover, from the above theorem, we can see that the new approach allows a line search with respect to the scalar $a > 0$ in a reasonable fashion, ensuring the achievement of a better (no worse) upper bound than that of the conventional approach.

The rest of this section is devoted to the linearization of the new characterizations (4.26), (4.29), (4.33) and (4.38), via change of controller variables, under the restriction (4.58).

State-Feedback Case

In the state-feedback case, it follows readily from (4.56) that (4.58) admits a simple change of variable

$$W := KG \tag{4.61}$$

so that the constraints (4.26), (4.29), (4.33) and (4.38) result in LMI's with respect to $X_{\mathcal{H}}$, $X_{\mathcal{C}}$, $X_{\mathcal{S}}$, X_2, Z, G, W and γ_2^2. Once the variables G and W have been found, the state-feedback gain K can be determined by

$$K = WG^{-1} \tag{4.62}$$

Note that the nonsingularity of G is assured by the constraint

$$G + G^T > 0 \tag{4.63}$$

included in all of the LMI characterizations (4.26), (4.29), (4.33) and (4.38). Thus, we are led to the conclusion that under the restriction (4.58), the linearization is completed without any further conservatism.

Output-Feedback Case

In the output-feedback case, we need a much more involved change of controller variables technique. The technique given below is based on the result proposed by Scherer [36] and similar to its variant presented in [1],[30].

Let us partition G and its inverse H as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \quad H = G^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{4.64}$$

where $G_{11} \in \mathbf{R}^{n \times n}$, $H_{11} \in \mathbf{R}^{n \times n}$, and other variables have compatible dimensions. We assume that G_{21} and H_{21} are nonsingular without loss of generality [1],[6]. Recall that G is nonsingular by (4.63). With (4.64) and the controller matrices given in (4.52), we define the following matrices.

$$\Xi_G := \begin{bmatrix} I & G_{11} \\ 0 & G_{21} \end{bmatrix}, \quad \Xi_H := \begin{bmatrix} H_{11} & I \\ H_{21} & 0 \end{bmatrix}, \quad \Pi := H_{11}^T G_{11} + H_{21}^T G_{21} \tag{4.65}$$

$$\begin{aligned} &\bar{B}_K := H_{21}^T B_K, \quad \bar{C}_K := C_K G_{21}, \\ &\bar{A}_K := \begin{bmatrix} H_{11}^T & \bar{B}_K \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} G_{11} \\ \bar{C}_K \end{bmatrix} + H_{21}^T A_K G_{21} \end{aligned} \tag{4.66}$$

The matrices Ξ_G and Ξ_H are nonsingular and satisfy the following equality.

$$G\Xi_H = \Xi_G \tag{4.67}$$

Appropriate congruence transformations with the matrix Ξ_H can be applied to (4.26), (4.29), (4.33) and (4.38) so that the resulting constraints only involve the following terms (the detailed manipulations are given in the Appendix, Subsection 4.7.5).

$$\Xi_H^T X_j \Xi_H = \bar{X}_j \quad (j = \mathcal{H}, \mathcal{C}, \mathcal{S}, 2), \quad \Xi_H^T G \Xi_H = \Xi_H^T \Xi_G = \begin{bmatrix} H_{11}^T & \Pi \\ I & G_{11} \end{bmatrix} \tag{4.68}$$

$$\begin{aligned} &\Xi_H^T \mathcal{B} = \begin{bmatrix} H_{11}^T B_2 + \bar{B}_K D_{w2} \\ B_2 \end{bmatrix}, \quad \mathcal{C} G \Xi_H = \mathcal{C} \Xi_G = \begin{bmatrix} C_2 & C_2 G_{11} + D_{z2} \bar{C}_K \end{bmatrix}, \\ &\Xi_H^T \mathcal{A} G \Xi_H = \Xi_H^T \mathcal{A} \Xi_G = \begin{bmatrix} H_{11}^T A + \bar{B}_K C & \bar{A}_K \\ A & A G_{11} + B \bar{C}_K \end{bmatrix} \end{aligned} \tag{4.69}$$

We can see that the above terms are affine with respect to $\bar{X}_j$ $(j = \mathcal{H}, \mathcal{C}, \mathcal{S}, 2)$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$. Accordingly, the matrix inequalities (4.26), (4.33), (4.29) and (4.38) amount to LMI's with respect to the variables $\bar{X}_j$ $(j = \mathcal{H}, \mathcal{C}, \mathcal{S}, 2)$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$, $\bar{C}_K$, Z and γ_2^2. Once the variables G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$ have been found, the output-feedback controller (4.52) can be determined by

$$\begin{aligned} &B_K = H_{21}^{-T} \bar{B}_K, \quad C_K = \bar{C}_K G_{21}^{-1}, \\ &A_K = H_{21}^{-T} \left\{ \bar{A}_K - \begin{bmatrix} H_{11}^T & \bar{B}_K \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} G_{11} \\ \bar{C}_K \end{bmatrix} \right\} G_{21}^{-1} \end{aligned} \tag{4.70}$$

where G_{21} and H_{21} are nonsingular matrices satisfying

$$H_{21}^T G_{21} = \Pi - H_{11}^T G_{11} \tag{4.71}$$

Remark 4.7 It is always possible to find the nonsingular matrices G_{21} and H_{21} satisfying (4.71). This is verified by assuring the nonsingularity of $\Pi - H_{11}^T G_{11}$, which is proved below.

After performing the linearizing congruence transformations and change of variables (4.68) and (4.69), the resulting LMI's include the following constraint corresponding to (4.63).

$$\Xi_H^T(G+G^T)\Xi_H = \Xi_H^T\Xi_G + \Xi_G^T\Xi_H = \begin{bmatrix} H_{11}+H_{11}^T & \Pi + I \\ \Pi^T + I & G_{11}+G_{11}^T \end{bmatrix} > 0 \tag{4.72}$$

The above inequality ensures the nonsingularity of the two matrices $\Xi_H^T\Xi_G$ and G_{11}. With G_{11}, we can define the nonsingular matrix

$$\Phi = \begin{bmatrix} -G_{11} & 0 \\ I & I \end{bmatrix} \tag{4.73}$$

The product of the two matrices $\Xi_H^T\Xi_G$ and Φ leads

$$\Xi_H^T\Xi_G\Phi = \begin{bmatrix} H_{11}^T & \Pi \\ I & G_{11} \end{bmatrix} \begin{bmatrix} -G_{11} & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} \Pi - H_{11}^T G_{11} & \Pi \\ 0 & G_{11} \end{bmatrix} \tag{4.74}$$

which assures the nonsingularity of $\Pi - H_{11}^T G_{11}$.

Remark 4.8 In the change of variables (4.68) and (4.69) and the derivation of the controller matrices (4.70), the variables G_{11}, G_{21}, H_{11} and H_{21} are involved. Recalling that the original variable is G, and H is its inverse as in (4.64), we must ensure the existence of the consistent eliminated matrices G_{12}, G_{22}, H_{12} and H_{22}. However, simple algebraic manipulations show that

$$\begin{aligned} G_{12} &= (I - G_{11}H_{11})H_{21}^{-1}, \quad G_{22} = -G_{21}H_{11}H_{21}^{-1}, \\ H_{12} &= (I - H_{11}G_{11})G_{21}^{-1}, \quad H_{22} = -H_{21}G_{11}G_{21}^{-1} \end{aligned} \tag{4.75}$$

actually satisfy $GH = I$.

Remark 4.9 The above change of variables is based on congruence transformations, and thus the linearization is completed under (4.58) without any further conservatism.

In summary, we gave a linearization technique of the multiobjective $H_2/\mathbf{D}$-stability problem based on the characterizations (4.26), (4.29), (4.33) and (4.38). This, together with (4.58), led to non-common Lyapunov variables for different specifications and convexity is essentially recovered without any further conservatism. This ensures the achievement of a better upper bound than the one based on the conventional approach with a common Lyapunov variable of the form (4.57) (see Theorem 4.7).

4.4 Robust Performance Analysis and Synthesis for Real Polytopic Uncertainty

In the preceding section, the dilated characterizations were successfully applied to the multiobjective controller design problems with the use of non-common Lyapunov variables, where the plant was assumed to be free from uncertainties. In this section, we show that the dilated characterizations are also useful in dealing with the robust performance analysis and robust multiobjective synthesis problems for real polytopic uncertainty.

4.4.1 Robust Performance Analysis for Real Polytopic Uncertainty

In this subsection, let us consider the continuous-time LTI system with polytopic uncertainty [4] described by

$$\begin{cases} \dot{x} = \mathcal{A}(\psi)x + \mathcal{B}(\psi)w \\ z = \mathcal{C}(\psi)x \; + \mathcal{D}(\psi)w \end{cases} \tag{4.76}$$

where

$$\begin{gathered} \begin{bmatrix} \mathcal{A}(\psi) & \mathcal{B}(\psi) \\ \mathcal{C}(\psi) & \mathcal{D}(\psi) \end{bmatrix} = \sum_{i=1}^{p} \psi_i M_{ci}, \quad \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix} =: M_{ci} \; (i = 1, \cdots, p), \\ \psi = (\psi_1, \cdots, \psi_p)^T, \quad \psi \in \Psi := \left\{ \psi | \; \psi_i \geq 0 \; (i = 1, \cdots, p), \; \sum_{i=1}^{p} \psi_i = 1 \right\} \end{gathered} \tag{4.77}$$

We assume that the uncertain parameter ψ is time-invariant, and that the matrices $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i\}$ corresponding to the vertices M_{ci} $(i = 1, \cdots, p)$ are given matrices.

For the uncertain system (4.76) with (4.77), several robust performance analysis problems have been addressed [1],[18],[20],[33]–[35]. In this subsection, we confine ourselves to the robust H_2 performance analysis problem [35] given below, assuming that $\mathcal{A}(\psi)$ is stable and $\mathcal{D}(\psi) = 0$ for all $\psi \in \Psi$.

Robust H_2 Performance Analysis Problem

For the uncertain system (4.76) with (4.77) such that $\mathcal{A}(\psi)$ is stable and $\mathcal{D}(\psi) = 0$ for all $\psi \in \Psi$, find the worst case H_2 cost $\gamma_{2,\text{w.c.}}$ defined by

$$\gamma_{2,\text{w.c.}} := \max_{\psi \in \Psi} ||T_{zw}(s)||_2 \tag{4.78}$$

As is shown in [35], the worst case H_2 cost $\gamma_{2,\text{w.c.}}$ can be characterized as the infimum of $\gamma_2 > 0$ such that

$$
\begin{aligned}
&\begin{bmatrix} \mathcal{A}(\psi)X_2(\psi) + X_2(\psi)\mathcal{A}(\psi)^T & X_2(\psi)\mathcal{C}(\psi)^T \\ \mathcal{C}(\psi)X_2(\psi) & -I \end{bmatrix} < 0, \\
&\begin{bmatrix} Z_2(\psi) & \mathcal{B}(\psi)^T \\ \mathcal{B}(\psi) & X_2(\psi) \end{bmatrix} > 0, \\
&\mathrm{trace}(Z_2(\psi)) < \gamma_2^2 \quad (\forall \psi \in \Psi)
\end{aligned} \tag{4.79}
$$

for some $X_2(\psi) > 0$ and $Z_2(\psi) > 0$. However, the optimization problem of γ_2^2 subject to (4.79) is not easily tractable in this form, first because (4.79) includes an infinite number of inequalities and second because there is no general and systematic way to formally determine $X_2(\psi)$ and $Z_2(\psi)$ as functions of the uncertain parameter vector ψ [29]. As numerically tractable methods, the three approaches described below follow from the LMI characterizations given in Subsection 4.2.3.

(i) Conventional Approach with a common Lyapunov variable.

Minimize γ_2^2 subject to

$$
\begin{bmatrix} \mathcal{A}_i X_2 + X_2 \mathcal{A}_i^T & X_2 \mathcal{C}_i^T \\ \mathcal{C}_i X_2 & -I \end{bmatrix} < 0 \tag{4.80a}
$$

$$
\begin{bmatrix} Z_{2i} & \mathcal{B}_i^T \\ \mathcal{B}_i & X_2 \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.80b}
$$

Here, the variables are X_2, Z_{2i} $(i = 1, \cdots, p)$ and γ_2^2.

(ii) New Approach with a common auxiliary variable G.

Minimize γ_2^2 subject to

$$
\begin{bmatrix} 0 & -X_{2i} & 0 \\ -X_{2i} & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_i \\ I \\ \mathcal{C}_i \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \tag{4.81a}
$$

$$
\begin{bmatrix} Z_{2i} & \mathcal{B}_i^T \\ \mathcal{B}_i & X_{2i} \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.81b}
$$

Here, the variables are X_{2i}, Z_{2i} $(i = 1, \cdots, p)$, G and γ_2^2. In this approach, we have to determine b $(= a^{-1}) > 0$ in advance.

(iii) New Approach with common auxiliary variables F_k $(k = 1, 2, 3)$.

Minimize γ_2^2 subject to

$$\begin{bmatrix} 0 & -X_{2i} & 0 \\ -X_{2i} & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_i \\ I \\ \mathcal{C}_i \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \right\} < 0 \tag{4.82a}$$

$$\begin{bmatrix} Z_{2i} & \mathcal{B}_i^T \\ \mathcal{B}_i & X_{2i} \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.82b}$$

Here, the variables are X_{2i}, Z_{2i} $(i = 1, \cdots, p)$, F_k $(k = 1, 2, 3)$ and γ_2^2.

Observe that a common Lyapunov variable X_2 is forced for all vertices of the polytope Ψ in the approach (i), while a common auxiliary variable G is forced in the approach (ii) and common auxiliary variables F_k $(k = 1, 2, 3)$ are forced in the approach (iii). With these restrictions, all of these approaches result in convex optimization problems subject to a finite number of LMI's, although they ensure the robust H_2 performance in different ways. Namely, the approach (i) ensures the robust H_2 performance via the following inequalities resulting from (4.80).

$$\begin{aligned} &\begin{bmatrix} \mathcal{A}(\psi)X_2 + X_2\mathcal{A}(\psi)^T & X_2\mathcal{C}(\psi)^T \\ \mathcal{C}(\psi)X_2 & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} Z_2(\psi) & \mathcal{B}(\psi)^T \\ \mathcal{B}(\psi) & X_2 \end{bmatrix} > 0, \quad \mathrm{trace}(Z_2(\psi)) < \gamma_2^2, \\ &Z_2(\psi) := \sum_{i=1}^{p} \psi_i Z_{2i} > 0 \end{aligned} \tag{4.83}$$

These inequalities imply that a *fixed* Lyapunov variable X_2 is forced to ensure the robust H_2 performance over the whole uncertainty domain. On the other hand, the approach (ii) performs differently and ensures the robust H_2 performance via

$$\begin{aligned} &\begin{bmatrix} 0 & -X_2(\psi) & 0 \\ -X_2(\psi) & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}(\psi) \\ I \\ \mathcal{C}(\psi) \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0, \\ &\begin{bmatrix} Z_2(\psi) & \mathcal{B}(\psi)^T \\ \mathcal{B}(\psi) & X_2(\psi) \end{bmatrix} > 0, \quad \mathrm{trace}(Z_2(\psi)) < \gamma_2^2, \\ &X_2(\psi) := \sum_{i=1}^{p} \psi_i X_{2i} > 0, \quad Z_2(\psi) := \sum_{i=1}^{p} \psi_i Z_{2i} > 0 \end{aligned} \tag{4.84}$$

The above inequalities are readily obtained from (4.81), and clearly shows an interesting fact that the approach (ii) ensures the robust H_2 performance with the use of the *parameter-dependent* Lyapunov variable

$$X_2(\psi) = \sum_{i=1}^{p} \psi_i X_{2i} > 0 \tag{4.85}$$

Namely, the restriction to a fixed Lyapunov variable in the approach (i) has been avoided in the approach (ii). Similar comments also apply to the approach (iii).

As we have seen, the new approaches (ii) and (iii) have very promising properties. In the following, several results on the comparison between the above three approaches are given. First, the advantage of the approach (iii) over the approach (i) is clarified.

Proposition 4.1 For the robust H_2 performance analysis problem, suppose that the LMI's (4.80) with the approach (i) are feasible. Then the LMI's (4.82) with the approach (iii) are feasible. Moreover, if we denote the optimal values of γ_2 achieved by the approaches (i) and (iii) by $\gamma_{2c} > 0$ and $\gamma_{2F} > 0$, respectively, then we have

$$\gamma_{2,\text{w.c.}} \leq \gamma_{2F} \leq \gamma_{2c} \tag{4.86}$$

Namely, the approach (iii) achieves a better (no worse) upper bound for $\gamma_{2,\text{w.c.}}$ than that with the approach (i).

Proof. The assertion follows immediately from Corollary 4.10. Indeed, suppose that the conventional approach (i) achieves an upper bound γ_{2c} with the variables

$$X_2 = X, \quad Z_{2i} = Z_i \ (i = 1, \cdots, p) \tag{4.87}$$

in the inequalities (4.80). Then, from Corollary 4.10, there exists a sufficiently small $\varepsilon > 0$ such that the inequalities (4.82) hold for

$$X_{2i} = X, \quad Z_{2i} = Z_i \ (i = 1, \cdots, p), \quad F_1 = X, \quad F_2 = -\varepsilon X, \quad F_3 = 0, \quad \gamma_2 = \gamma_c \tag{4.88}$$

This implies that the new approach (iii) ensures the achievement of the same upper bound γ_{2c}, which completes the proof. Q.E.D.

Next, we show the advantage of the approach (iii) over the approach (ii).

Proposition 4.2 For the robust H_2 performance analysis problem, suppose that the LMI's (4.81) with the approach (ii) are feasible. Then the LMI's (4.82) with the approach (iii) are feasible. Moreover, if we denote the optimal values of γ_2 achieved by the approaches (ii) and (iii) by $\gamma_{2G} > 0$ and $\gamma_{2F} > 0$, respectively, then we have

$$\gamma_{2,\text{w.c.}} \leq \gamma_{2F} \leq \gamma_{2G} \tag{4.89}$$

Namely, the approach (iii) achieves a better (no worse) upper bound for $\gamma_{2,\text{w.c.}}$ than that with the approach (ii).

Proof. The assertion follows immediately because (4.81) is a special case of (4.82) with $[\, F_1 \; F_2 \; F_3 \,] = G [\, I \; -bI \; 0 \,]$. Q.E.D.

Finally, we compare the approaches (i) and (ii). Unfortunately, for an arbitrarily chosen $a > 0$, we cannot conclude that the approach (ii) achieves a better upper bound than that with the approach (i), in spite of the very promising use of a parameter-dependent Lyapunov variable in the approach (ii). This is in sharp contrast to the case without plant uncertainties studied in the preceding section. The difficulties in the comparison here arise from the unfortunate fact that in the case of (polytopic) uncertainties, we cannot ensure the existence of a common auxiliary variable G satisfying (4.81) even if there exists a common Lyapunov variable X_2 satisfying (4.80). Observe that the crucial choice of the auxiliary variable in the uncertainty free case represented by

$$G = \widetilde{G}(X_2) := -a(\mathcal{A} - aI)X_2 \tag{4.90}$$

does not generate a common G under the polytopic uncertainty setting, because the coefficient matrix $\mathcal{A}$ in (4.90) should be replaced by $\mathcal{A}_i (i = 1, \cdots, p)$ for each vertex. However, Lemma 4.4 in Section 4.1 plays an important role in this situation to arrive at the following result, which clarifies the advantage of the approach (ii) over the conventional approach (i) under an appropriate condition.

Proposition 4.3 For the robust H_2 performance analysis problem, suppose that the LMI's (4.80) are feasible and let us denote the optimal value of γ_2 achieved by the approach (i) by $\gamma_{2c} > 0$. Then, there exists $a_{\min} > 0$ such that whenever $a > a_{\min}$, the LMI's (4.81) are feasible and the new approach (ii) ensures the achievement of a better (no worse) upper bound than γ_{2c}. Namely, if we denote the optimal value of γ_2 achieved by the approach (ii) by $\gamma_{2G} > 0$, we have

$$\gamma_{2,\text{w.c.}} \leq \gamma_{2G} \leq \gamma_{2c} \quad (a > a_{\min}) \tag{4.91}$$

Proof. It follows from Lemma 4.4 with $\delta_1 = \delta_2 \to 0$, $\Delta = \mathcal{C}_i$ and with $\mathcal{A} := \mathcal{A}_i$, $\mathcal{X} := X_{2i}$ that the inequality (4.81a) for the new approach (ii) corresponding to the ith vertex M_i recovers

$$\begin{bmatrix} \mathcal{A}_i X_{2i} + X_{2i}\mathcal{A}_i^T & X_{2i}\mathcal{C}_i^T \\ \mathcal{C}_i X_{2i} & -I \end{bmatrix} < 0 \tag{4.92}$$

via $G = \widetilde{G}(X_{2i}) := X_{2i}$ whenever $a > a_{\min,i}(X_{2i})$, where $a_{\min,i}(X_{2i})$ depends not only on X_{2i} but also on M_i. However, the approach (i) corresponds to the common Lyapunov variable $X_{2i} = X_2$ $(i = 1, \cdots, p)$. Hence, confining ourselves to this common Lyapunov variable, we see that $a_{\min,i}(X_{2i}) = a_{\min,i}(X_2)$ depends only on the vertex M_i, and (4.92) reduces to (4.80a). Since there are only a finite number of vertices, $a_{\min} := \max_{i=1,\cdots,p} a_{\min,i}(X_2)$ is well-defined. Summarizing the above arguments, if $a > a_{\min}$ and if $X_2 = X > 0$ satisfies the inequality (4.80a), then $X_{2i} = X (i = 1, \cdots, p)$ and $G = X$ satisfy the inequality (4.81a) for $i = 1, \cdots, p$. This completes the proof. Q.E.D.

Remark 4.10 It should be noted that the specific choice of $a > a_{\min}$ is only a sufficient condition for the approach (ii) to achieve a better (no worse) upper bound than γ_{2c}. Because the restriction to a common Lyapunov variable in the approach (i) is considerably relaxed in the approach (ii), the latter results in improvements in most problems, even without special care on a. For example, a simple thoughtless choice such as $a = 1.0$ would even give a satisfactory result (see illustrative examples in Section 4.5).

Although we have confined ourselves to the robust H_2 performance analysis problem in this subsection, the robust **D**-stability analysis problems [34] or the robust H_∞ performance analysis problems can be addressed in a similar fashion, based on the dilated characterizations given in Subsections 4.2.2 and 4.2.4. Similar results to Propositions 4.1, 4.2 and 4.3 also follow in the context of such problems.

4.4.2 Robust Controller Synthesis for Real Polytopic Uncertainty

As we have seen in the preceding subsection, the dilated characterizations exhibit their potentials under the robust performance analysis problems for real polytopic uncertainty. It is shown in this subsection that the dilated characterizations are also useful in dealing with robust controller synthesis for real polytopic uncertainty.

Let us consider the case where the plant (4.51) has the polytopic uncertainty described by

$$\begin{cases} \dot{x} = A(\psi)x + B_2(\psi)w_2 + B(\psi)u \\ z_2 = C_2(\psi)x + D_{z2}(\psi)u \\ y = C(\psi)x + D_{w2}(\psi)w_2 \end{cases} \tag{4.93}$$

where

$$\begin{bmatrix} A(\psi) & B_2(\psi) & B(\psi) \\ C_2(\psi) & 0 & D_{z2}(\psi) \\ C(\psi) & D_{w2}(\psi) & 0 \end{bmatrix} = \sum_{i=1}^{p} \psi_i M_i, \quad \begin{bmatrix} A_i & B_{2i} & B_i \\ C_{2i} & 0 & D_{z2i} \\ C_i & D_{w2i} & 0 \end{bmatrix} =: M_i \ (i = 1, \cdots, p),$$
$$\psi = (\psi_1, \cdots, \psi_p)^T, \quad \psi \in \Psi := \left\{ \psi | \ \psi_i \geq 0 \ (i = 1, \cdots, p), \ \sum_{i=1}^{p} \psi_i = 1 \right\} \tag{4.94}$$

As in the previous problem, we assume that the uncertain parameter ψ is time-invariant. Note that $\{A_i, B_{2i}, B_i, C_{2i}, C_i, D_{z2i}, D_{w2i}\}$ $(i = 1, \cdots, p)$ are given matrices.

The problem considered here is the robust state-feedback H_2 synthesis problem for real polytopic uncertainty described in the following.

Robust H_2 Synthesis Problem

For the uncertain system (4.93) with (4.94), find a state-feedback controller K minimizing the worst case H_2 cost $\gamma_{2,\text{w.c.}}$ defined by

$$\gamma_{2,\text{w.c.}} := \max_{\psi \in \Psi} ||T_{z_2 w_2}(s)||_2 \tag{4.95}$$

In the case where the plant has the polytopic uncertainty, the closed-loop system also has the polytopic uncertainty of the form

$$\begin{cases} \dot{x} = \mathcal{A}(\psi)x + \mathcal{B}(\psi)w_2 \\ z_2 = \mathcal{C}(\psi)x + \mathcal{D}(\psi)w_2 \end{cases},$$
$$\begin{bmatrix} \mathcal{A}(\psi) & \mathcal{B}(\psi) \\ \mathcal{C}(\psi) & \mathcal{D}(\psi) \end{bmatrix} = \sum_{i=1}^{p} \psi_i M_{ci}, \quad \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix} =: M_{ci} \ (i = 1, \cdots, p) \tag{4.96}$$

It turns out that the matrices $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i\}$ in the vertex M_{ci} are

$$\mathcal{A}_i = A_i + B_i K, \quad \mathcal{B}_i = B_{2i}, \quad \mathcal{C}_i = C_{2i} + D_{z2i}K, \quad \mathcal{D}_i = 0 \qquad (i = 1, \cdots, p) \tag{4.97}$$

Note that (4.97) denotes the coefficient matrices of the closed-loop system corresponding to the vertex M_i of the polytope (4.94).

For this problem, we can provide two approaches by the LMI characterizations (4.37) and (4.38) given in Subsection 4.2.3.

(i) Conventional Approach with a common Lyapunov variable.

Minimize γ_2^2 subject to

$$\begin{bmatrix} (A_i + B_i K)X_2 + X_2(A_i + B_i K)^T & X_2(C_{2i} + D_{z2i}K)^T \\ (C_{2i} + D_{z2i}K)X_2 & -I \end{bmatrix} < 0, \tag{4.98a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X_2 \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.98b}$$

Here, the variables are X_2, Z_{2i} $(i = 1, \cdots, p)$, γ_2^2 and the controller variable K.

(ii) New Approach with a common auxiliary variable G.

Minimize γ_2^2 subject to

$$\begin{bmatrix} 0 & -X_{2i} & 0 \\ -X_{2i} & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_i + B_i K \\ I \\ C_{2i} + D_{z2i} K \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \tag{4.99a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X_{2i} \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.99b}$$

Here, the scalar $b = a^{-1}$ is arbitrarily chosen in advance and the variables are X_{2i}, Z_{2i}, $(i = 1, \cdots, p)$, G, γ_2^2 and the controller variable K.

It can be seen that the approach (i) forces a common Lyapunov variable X_2 for all vertices of the polytope, while the approach (ii) forces only a common auxiliary variable G. With these restrictions, each approach results in LMI's with a simple change of controller variables techniques represented by (2.28) and (4.61), respectively. Note that the dilated characterization (4.39) cannot be applied to synthesis problems because of the multiple products between the controller variable and the auxiliary variables F_{2k} $(k = 1, 2, 3)$.

Similarly to the preceding analysis problem, the approach (ii) performs differently from the approach (i) and the former attains robust H_2 performance via a parameter-dependent Lyapunov variable of the form (4.85). Namely, the restriction to a fixed Lyapunov variable in the approach (i) has been avoided successfully in the approach (ii).

In spite of this attractive feature, however, we cannot conclude that the approach (ii) achieves a better upper bound than the approach (i) for an arbitrarily chosen $a > 0$. The reason is the same as what has been described for the analysis problem in the preceding subsection. However, Lemma 4.4 again plays a crucial role to lead us to the following result, which clarifies the advantage of the new approach (ii) over the conventional approach (i).

Proposition 4.4 For the robust H_2 synthesis problem, suppose that the LMI's (4.98) are feasible and let us denote the optimal value of γ_2 achieved by the approach (i) by $\gamma_{2c} > 0$. Then, there exists $a_{\min} > 0$ such that whenever $a > a_{\min}$, the LMI's (4.99) are feasible and the new approach (ii) ensures the achievement of a better (no worse) upper bound than γ_{2c}. Namely, if we denote the optimal value of γ_2 achieved by the approach (ii) by $\gamma_{2G} > 0$, we have

$$\gamma_{2,\text{w.c.}} \le \gamma_{2G} \le \gamma_{2c} \quad (a > a_{\min}) \tag{4.100}$$

Proof. The proof is similar to that of Proposition 4.3 and hence omitted. Q.E.D.

Remark 4.11 As in the preceding analysis problem, the specific choice of $a > a_{\min}$ is only a sufficient condition for the approach (ii) to achieve a better (no worse) upper bound than γ_{2c} and similar comments to Remark 4.10 apply.

Although we have confined ourselves to the robust H_2 synthesis problem in this subsection, the robust **D**-stability synthesis problems can be addressed in a similar fashion, based on the dilated characterizations given in Subsection 4.2.2. Similar results to Proposition 4.4 follow also in that context, which show the advantage of the new approach over the conventional approach.

4.4.3 Robust Multiobjective H_2/D-stability Synthesis for Real Polytopic Uncertainty

In the preceding subsection, we have proposed a new approach to the robust controller synthesis for real polytopic uncertainty. The idea is readily extended to the robust *multiobjective* controller synthesis for real polytopic uncertainty in this subsection.

Let us consider again the case where the plant has the polytopic uncertainty described by (4.93) with (4.94). The problem considered here is the robust multiobjective H_2/**D**-stability synthesis problem for real polytopic uncertainty. For the ease of description, let us consider the following example of robust multiobjective H_2/**D**-stability synthesis problems.

Robust Multiobjective H_2/D-stability Synthesis Problem

For the uncertain system (4.93) with (4.94), find a state-feedback controller K minimizing the worst case H_2 cost $\gamma_{2,\text{w.c.}}$ defined by (4.95), subject to the **D**-stability constraint $\sigma(\mathcal{A}(\psi)) \subset \mathcal{H}(\alpha)$ $(\forall \psi \in \Psi)$.

As in the preceding problem, we can describe two approaches for this problem.

(i) Conventional Approach with a common Lyapunov variable.

Minimize γ_2^2 subject to

$$\begin{bmatrix} (A_i + B_i K)X + X(A_i + B_i K)^T & X(C_{2i} + D_{z2i}K)^T \\ (C_{2i} + D_{z2i}K)X & -I \end{bmatrix} < 0 \tag{4.101a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X \end{bmatrix} > 0, \quad \text{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.101b}$$

and

$$(A_i + B_iK)X + X(A_i + B_iK)^T + 2\alpha X < 0 \quad (i = 1, \cdots, p) \tag{4.102}$$

Here, the variables are X, Z_{2i} $(i = 1, \cdots, p)$, γ_2^2 and the controller variable K.

(ii) New Approach with a common auxiliary variable G.
Minimize γ_2^2 subject to

$$\begin{bmatrix} 0 & -X_{2i} & 0 \\ -X_{2i} & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_i + B_iK \\ I \\ C_{2i} + D_{z2i}K \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} < 0 \tag{4.103a}$$

$$\begin{bmatrix} Z_{2i} & B_{2i}^T \\ B_{2i} & X_{2i} \end{bmatrix} > 0, \quad \mathrm{trace}(Z_{2i}) < \gamma_2^2 \quad (i = 1, \cdots, p) \tag{4.103b}$$

and

$$\begin{bmatrix} 0 & -X_{\mathcal{H}i} & X_{\mathcal{H}i} \\ -X_{\mathcal{H}i} & 0 & 0 \\ X_{\mathcal{H}i} & 0 & -\frac{1}{2}\alpha^{-1}X_{\mathcal{H}i} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_i + B_iK \\ I \\ 0 \end{bmatrix} G \begin{bmatrix} I & -bI & bI \end{bmatrix} \right\} < 0 \ (i = 1, \cdots, p) \tag{4.104}$$

Here, the scalar $b = a^{-1}$ is arbitrarily chosen in advance and the variables are X_{2i}, $X_{\mathcal{H}i}$, Z_{2i} $(i = 1, \cdots, p)$, G, γ_2^2 and the controller variable K.

Observe that the conventional approach (i) forces a common Lyapunov variable X for the two design specifications as well as for all vertices of the polytope. On the other hand, the new approach (ii) forces only a common auxiliary variable G and a common scalar $b = a^{-1} > 0$. Here, the latter common scalar is enforced only to simplify the exposition, and it is indeed possible to use distinct scalars for each design specification.

Because of the restriction on the Lyapunov variables or auxiliary variables, the change of controller variables techniques represented by (2.28) and (4.61) are successfully applied so that each of the approaches results in LMI's. In particular, the dilated characterizations in the approach (ii) enables us to employ Lyapunov variables

$$X_2(\psi) = \sum_{i=1}^{p} \psi_i X_{2i}, \quad X_{\mathcal{H}}(\psi) = \sum_{i=1}^{p} \psi_i X_{\mathcal{H}i} \tag{4.105}$$

to attain robust H_2 performance under the $\mathbf{D}$-stability constraint, where these Lyapunov variables are *parameter-dependent* and at the same time *non-common* for the two design specifications.

Yet again, however, for an arbitrarily chosen $a > 0$, we cannot conclude that the approach (ii) achieves a better upper bound than the approach (i), in spite of the very promising use of non-common and parameter-dependent Lyapunov variables in the approach (ii). The reason is the same as what we have seen for the analysis and synthesis problems in the preceding subsections. As before, however, we are led to the following result, by which the advantage of the approach (ii) over the conventional approach (i) is ensured.

Proposition 4.5 For the robust multiobjective $H_2/\mathbf{D}$-stability synthesis problem, suppose that the set of LMI's (4.101) and (4.102) is feasible and let us denote the optimal value of γ_2 achieved by the approach (i) by $\gamma_{2c} > 0$. Then, there exists $a_{\min} > 0$ such that whenever $a > a_{\min}$, the set of LMI's (4.103) and (4.104) is feasible and the new approach (ii) ensures the achievement of a better (no worse) upper bound than γ_{2c}. Namely, if we denote the optimal value of γ_2 achieved by the approach (ii) by $\gamma_{2G} > 0$, we have

$$\gamma_{2,\text{w.c.}} \leq \gamma_{2G} \leq \gamma_{2c} \quad (a > a_{\min}) \tag{4.106}$$

Proof. It is a direct consequence from Proposition 4.3 that there exists $a_{\min,2} > 0$ with the following property.

- If $X > 0$ and a feedback gain K satisfy the LMI's (4.101) and if $a > a_{\min,2}$, then the matrices $X_{2i} = X$ $(i = 1, \cdots, p)$ and $G = X$ satisfy the LMI's (4.103) for the same K.

Similarly, it follows that there exists $a_{\min,\mathcal{H}} > 0$ with the following property.

- If $X > 0$ and a feedback gain K satisfy the LMI's (4.102) and if $a > a_{\min,\mathcal{H}}$, then the matrices $X_{\mathcal{H}i} = X$ $(i = 1, \cdots, p)$ and $G = X$ satisfy the LMI's (4.104) for the same K.

These facts clearly show that if $X > 0$ and a feedback gain K satisfy the LMI's (4.101) and (4.102) and if $a > a_{\min} := \max\{a_{\min,2}, a_{\min,\mathcal{H}}\}$, then $X_{2i} = X_{\mathcal{H}i} = X$ $(i = 1, \cdots, p)$ and $G = X$ satisfy the LMI's in (4.103) and (4.104) for the same K, which completes the proof. Q.E.D.

Similarly to Propositions 4.3 and 4.4, it should be noted that the specific choice $a > a_{\min}$ in Proposition 4.5 is only a sufficient condition to ensure the advantage of the new approach (ii). This can be viewed from the proof given above that evaluates the performance of the new approach (ii) with only a common Lyapunov variable $X_{2i} = X_{\mathcal{H}i} = X$ $(i = 1, \cdots, p)$.

However, recall that the new approach (ii) essentially allows non-common and parameter-dependent Lyapunov variables, and this fact is quite promising in reducing the conservatism of the conventional approach (i). Because of this nice property, we can say that the new approach (ii) is more effective than the conventional approach (i) in general, even without special care on the scalar a. Indeed, a numerical example in Section 4.5 demonstrates that the application of the new approach (ii) results in significant improvements over the conventional approach (i), with a simple thoughtless choice of the scalar a such as $a = 1.0$.

4.5 Illustrative Examples

This section illustrates the effectiveness of the new dilated LMI approaches provided in the preceding section through numerical examples. In the following, all LMI related computations were carried out with the *LMI Control Toolbox* [15], on PENTIUM-III 933MHz.

4.5.1 Multiobjective H_2/D-stability Controller Design

First of all, let us demonstrate the effectiveness of the new approach to the multiojbective H_2/**D**-stability controller design problems presented in Section 4.3. Here, the plant is assumed to be free from uncertainty.

State-Feedback Problem

Consider the LTI plant described by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u \\ z &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \end{aligned} \tag{4.107}$$

where $k = 0.245$ and $f = 0.0219$ [3]. The problem is to find a state-feedback gain K minimizing $||T_{zw}(s)||_2$ subject to the **D**-stability constraint $\sigma(\mathcal{A}) \subset \bigcap\{\mathcal{H}(0.5), \mathcal{S}(\tan(3\pi/8))\}$ (see Fig. 4.1). This problem is nothing but Problem 1 studied in the preceding chapter, Section 3.5.

Applying to this problem Conventional Approach and New Approach provided in Section 4.3, we get the H_2 costs given in Table 4.1, where we show both upper bounds and the actual costs resulting from these approaches and computation time. Recall that New Approach successfully employs non-common Lyapunov variables to circumvent the conservatism of Conventional Approach. Indeed, this table shows that the upper bound of the cost functional is considerably improved by New Approach (as is expected from Theorem 4.7). Note that actual cost is also improved. It is an interesting fact that even the upper bound ensured by New Approach is considerably better than the actual cost achieved by Conventional Approach. Unfortunately, however, it is inevitable for New Approach to increase computation time because of the the dilation of the matrix inequalities and the introduction of the auxiliary variable.

Table 4.1: The resulting H_2 costs

Approach (Corresponding gain)	upper bound	actual cost	CPU time (sec)
Conventional Approach (K_c)	1.7545	1.5924	0.15
New Approach with $a = 1.0$ (K)	1.4878	1.4197	1.58

As shown in the above table, New Approach performs considerably better than Conventional Approach. Indeed, New Approach with $a = 1.0$ arrives at the feedback gain

$$K = \begin{bmatrix} -2.7551 & -0.1113 & -2.3042 & -7.0435 \end{bmatrix} \tag{4.108}$$

which is quite different from K_c given by

$$K_c = \begin{bmatrix} -4.5752 & -0.9647 & -3.0720 & -13.8032 \end{bmatrix} \tag{4.109}$$

For reference, the H_2 optimal feedback gain (without taking account of the **D**-stability constraint) is given in the following.

$$K_{H_2} = \begin{bmatrix} -1.3271 & -0.0871 & -1.6334 & -1.9464 \end{bmatrix} \tag{4.110}$$

This feedback gain achieves $||T_{zw}(s)||_2 = 1.2780$.

It is expected that the less conservative nature of New Approach leads to the improvement of the cost functional over Conventional Approach. To see this, Fig. 4.1 shows the closed-loop pole locations under K_{H_2}, K_c and K. It follows from this figure that the feedback gain K_{H_2} does not satisfy the **D**-stability constraint. The feedback gains K_c and K do satisfy the constraint as required, and in particular, the feedback gain K achieves the constraint in a less conservative fashion than K_c.

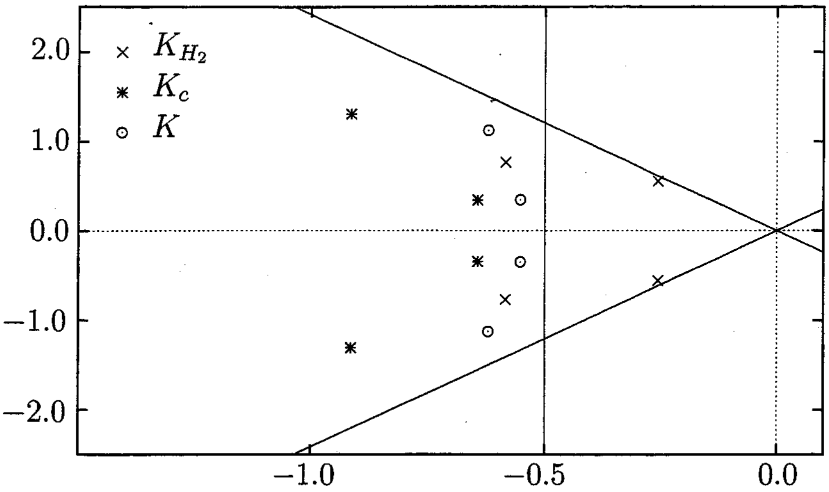

Figure 4.1: Pole locations under K_{H_2}, K_c and K

It is worth mentioning that New Approach has another advantage over Conventional Approach. Namely, as we have seen in Section 4.3, New Approach allows a line search with respect to the scalar $a > 0$ in a reasonable fashion, ensuring the achievement of a better (no worse) upper bound than that with Conventional Approach. In order to demonstrate this nice property, the line search with respect to $a > 0$ is performed to get the result shown in Fig. 4.2. From this figure, we can ascertain that New Approach achieves better upper bounds than 1.7545 for the cost functional (which is achieved by Conventional Approach) irrespective of $a > 0$. This figure also suggests that New Approach leads to satisfactory results even without special care on the scalar a.

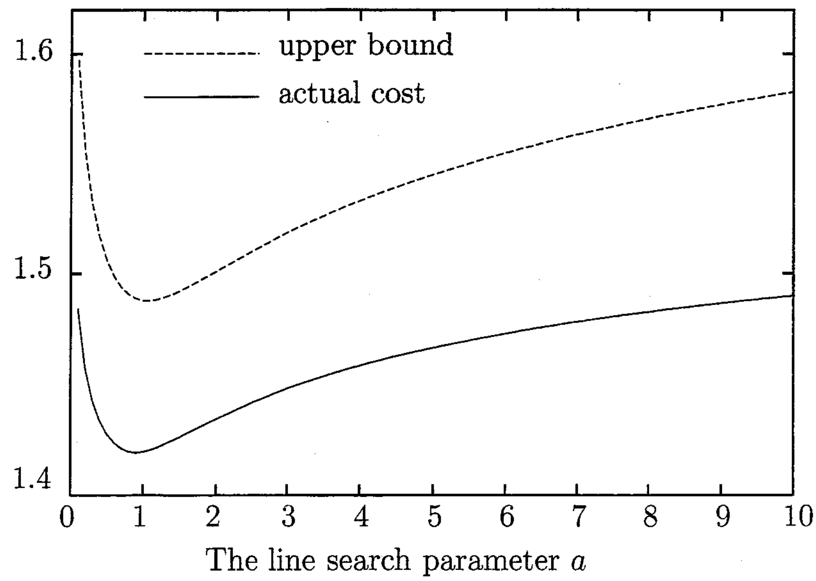

Figure 4.2: The H_2 cost achieved by New Approach

As stated before, this problem was also dealt with in Section 3.5. For reference, the results there are summarized in the following.

- Approach I (i.e. the subspace approach provided in Subsection 3.2.2) achieved the best H_2 cost 1.4848 among the non-iterative approaches as shown in Table 3.3. The computation time was 0.39 (sec).
- Iterative Algorithm II (i.e. the combined iterative algorithm provided in Subsection 3.3.3) achieved the best H_2 cost 1.3004 among the iterative algorithms as shown in Table 3.4. The computation time was 7.31 (sec).

Recall that New Approach presented in this chapter arrives at the H_2 cost 1.4197 with the computation time is 1.58 (sec). Although New Approach takes more computation time than Approach I, the former achieves a considerably better H_2 cost. On the other hand, Iterative algorithm II indeed achieves a better H_2 cost than New Approach, but those iterative algorithms naturally lead to drastic increase in computation time. Here, recalling that the iterative algorithms provided in the preceding chapter need an initial feedback gain for their implementation, it is clear that New Approach has another usefulness. Namely, New Approach will be also helpful to those algorithms in providing better initial gains so that further better performance and quick convergence can be obtained.

In the next problem, we deal with the *output-feedback* multiobjective $H_2/\mathbf{D}$-stability controller design problem. Output-feedback multiobjective synthesis with non-common Lyapunov variables is an important achievement in this chapter. Note that the approaches and the algorithms in the preceding chapter only deal with state-feedback problems.

Output-Feedback Problem

Consider the LTI plant described by

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\
z &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x + 2\, w
\end{aligned}
\tag{4.111}
$$

The problem is to find a full-order dynamic controller K minimizing $||T_{zw}(s)||_2$ subject to the $\mathbf{D}$-stability constraint $\sigma(\mathcal{A}) \subset \mathcal{C}(-20, 19)$ (see Fig. 4.3).

Applying to this problem Conventional Approach and New Approach provided in Section 4.3, we get the H_2 costs shown in Table 4.2, where we show both upper bounds and the actual costs resulting from these approaches and computation time. Similarly to the preceding state-feedback problem, this table shows the advantage of New Approach. Namely, New Approach achieves a considerably better upper bound of the cost functional as is expected from Theorem 4.7, and indeed arrives at a better actual cost. Unfortunately, however, New Approach takes much more computation time than Conventional Approach.

Table 4.2: The resulting H_2 costs

Method (Corresponding controller)	upper bound	actual cost	CPU time (sec)
Conventional Approach (K_c)	71.3675	36.2192	0.31
New Approach with $a = 1.0$ (K)	42.6631	30.8958	19.75

The two approaches arrive at the following (full-order) controllers.

$$K_c(s) = \frac{-36.3896(s+5.0979)(s+1.7193)}{(s+8.4616)(s+4.9328)(s+2.7321)} \tag{4.112}$$

$$K(s) = \frac{-28.9937(s+5.0907)(s+0.9733)}{(s+7.2741)(s+4.9554)(s+2.1750)} \tag{4.113}$$

These controllers place the closed-loop poles as shown in Fig. 4.3 where we also show the closed-loop pole locations under the H_2 optimal controller (without any care for the **D**-stability constraint) given by

$$K_{H_2}(s) = \frac{-5.0701(s+5.0951)(s-0.2754)}{(s+5.0863)(s^2+3.3732s+9.9288)} \tag{4.114}$$

This controller achieves $||T_{zw}(s)||_2 = 13.7335$. From this figure, we can see that the controller K_{H_2} does not satisfy the **D**-stability constraint. Although both of the controllers K_c and K do satisfy the **D**-stability constraint as required, the controller K leaves a less margin for the constraint, which suggests the less conservative nature of New Approach.

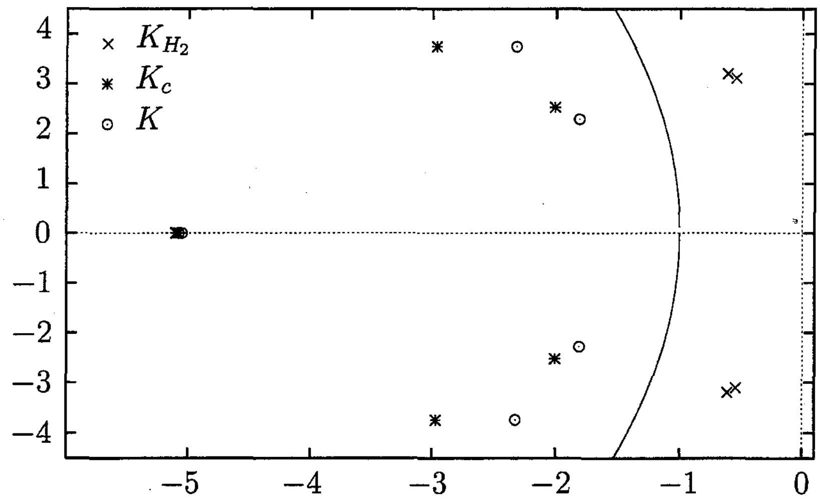

Figure 4.3: Pole locations under K_{H_2}, K_c and K

4.5.2 Robust Performance Analysis for Real Polytopic Uncertainty

As we have seen in Subsection 4.4.1, the dilated LMI characterizations have very promising nature that they allow the use of parameter-dependent Lyapunov variables in dealing with the robust performance analysis problems for real polytopic uncertainty. This subsection demonstrates the effectiveness of the dilated LMI's in such problems through simple numerical experiments.

Robust Stability Analysis Problem [18]

Let us consider a simple robust stability analysis problem. The problem is to determine the maximum value of $\bar{\theta}$ such that a set of matrices $A(\theta) := A + \theta gh$ remains stable for all $|\theta| < \bar{\theta}$, where A, g and h are given in the following.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -12 & -25 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad h = \begin{bmatrix} 3 & 3 & 0 & 0 \end{bmatrix} \qquad (4.115)$$

It follows that the uncertain matrix $A(\theta)$ ($|\theta| < \bar{\theta}$) can be described as a polytope with two vertices $A(\bar{\theta})_1 := A + \bar{\theta}gh$ and $A(\bar{\theta})_2 := A - \bar{\theta}gh$. We applied the following three approaches to determine the maximum value of $\bar{\theta}$.

(i) Conventional Approach with a common Lyapunov variable.

Maximize $\bar{\theta}$ subject to (4.22) with $\{\mathcal{A}, X_{\mathcal{L}}\}$ replaced by $\left\{A(\bar{\theta})_i, X\right\}$ $(i = 1, 2)$. Here, the variables are X and $\bar{\theta}$.

(ii) New Approach with a common auxiliary variable G.

Maximize $\bar{\theta}$ subject to (4.23) with $\{\mathcal{A}, X_{\mathcal{L}}, G_{\mathcal{L}}\}$ replaced by $\left\{A(\bar{\theta})_i, X_i, G\right\}$ $(i = 1, 2)$. Here, the variables are X_i $(i = 1, 2)$, G and $\bar{\theta}$. On the choice of the scalar $b = a^{-1} > 0$, we test $a = 1$, 10 and 100.

(iii) New Approach with common auxiliary variables F_k $(k = 1, 2)$.

Maximize $\bar{\theta}$ subject to (4.24) with $\{\mathcal{A}, X_{\mathcal{L}}, F_{\mathcal{L}1}, F_{\mathcal{L}2}\}$ replaced by $\left\{A(\bar{\theta})_i, X_i, F_1, F_2\right\}$ $(i = 1, 2)$. Here, the variables are X_i $(i = 1, 2)$, F_1, F_2 and $\bar{\theta}$.

The above approaches correspond to the three approaches provided in Subsection 4.4.1, respectively. Recall that the approaches (ii) and (iii) ensure robust stability with the use of parameter-dependent Lyapunov variables.

Applying the above three approaches to this problem, we get the maximum values of $\bar{\theta}$ shown in Table 4.3. In this problem, we can verify easily that the exact maximum value of $\bar{\theta}$ is 4.0. As is expected from Propositions 4.1 and 4.2, the approach (iii) achieves the best value among these approaches, and it successfully attains the exact maximum value. Although the approach (iii) provides only a sufficient condition, this approach turns out to be not conservative in this example. On the other hand, the approaches (i) and (ii) lead to conservative results. However, the approach (ii) achieves better results than the approach (i) irrespective of the scalar $a > 0$, which suggests the advantage of the approach (ii).

These results indicate that the robust stability analysis with a common Lyapunov variable is quite conservative, and the conservatism has been circumvented successfully with the use of parameter-dependent Lyapunov variables.

Table 4.3: Maximum value of $\bar{\theta}$

Method	$\bar{\theta}$
Approach (i)	2.15
Approach (ii) with $a = 1$	3.42
Approach (ii) with $a = 10$	2.74
Approach (ii) with $a = 100$	2.36
Approach (iii)	4.00
The exact maximum value	4.00

Robust D-stability Analysis Problem

The goal here is to clarify the advantage of the dilated LMI's in dealing with the robust **D**-stability analysis problems. For that purpose, let us consider the robust **D**-stability analysis problem for a polytope matrix $A(\psi)$ given by

$$A(\psi) = \psi A_1 + (1-\psi)A_2, \quad A_1 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -3 & 0 \\ -1 & -2.2 \end{bmatrix}, \; 0 \le \psi \le 1 \quad (4.116)$$

In the following, we examine whether the three approaches given below work fine to ensure $\sigma(A(\psi)) \in \mathbf{D}$, where $\mathbf{D}$ is a prescribed region such that $\sigma(A(\psi)) \in \mathbf{D}$ is indeed the case whenever $0 \le \psi \le 1$. Note that a similar experiment was also carried out in [34].

(i) Conventional Approach with a common Lyapunov variable X based on the LMI's (4.25), (4.28), and (4.32).

(ii) New Approach with a common auxiliary variable G based on the dilated LMI's (4.26), (4.29), and (4.33), which admits the use of a parameter-dependent Lyapunov variable to ensure the robust **D**-stability. On the choice of the scalar $b = a^{-1} > 0$, we test $a = 1$, 10 and 100.

(iii) New Approach with common auxiliary variables F_k based on the dilated LMI's (4.27), (4.30), and (4.34), which admits the use of a parameter-dependent Lyapunov variable to ensure the robust **D**-stability.

The regions considered here are $\mathcal{H}(1.9)$, $\mathcal{C}(-2, 1.4)$, $\mathcal{C}(-2, 1.3)$, $\mathcal{C}(-2, 1.2)$, $\mathcal{C}(-2, 1.1)$ and the intersection of these regions. Fig. 4.4 shows these regions, as well as the variation of the eigenvalues of $A(\psi)$ when the parameter ψ moves from 0 to 1. We can see that the eigenvalues of $A(\psi)$ are actually contained in the regions considered here regardless of $0 \le \psi \le 1$.

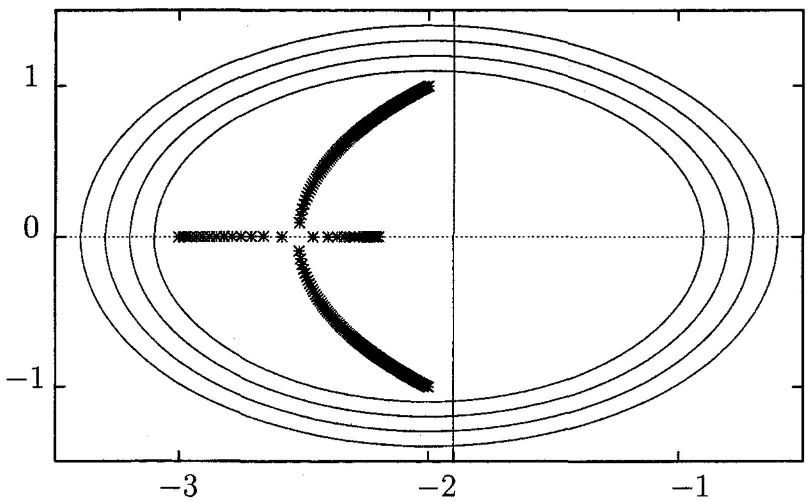

Figure 4.4: Pole placement regions **D** and pole locations

Applying the three approaches to each region, we get the results shown in Table 4.4, where the symbol $\circ$ denotes the success in ensuring robust **D**-stability while $\times$ denotes the failure. As is expected from Propositions 4.1 and 4.2, the approach (iii) works better than the other two approaches. In particular, the approach (iii) are successful in all regions considered here, which suggests the effectiveness of this approach. Although the approaches (i) and (ii) arrive at conservative results, the approach (ii) achieves better results than the approach (i) irrespective of $a > 0$.

Table 4.4: Pole placement regions and results

Pole placement regions	Approach (i)	Approach (ii)			Approach (iii)
		$a = 1$	$a = 10$	$a = 100$	
$\mathcal{H}(1.9)$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\mathcal{C}(-2, 1.4)$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\mathcal{C}(-2, 1.3)$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\mathcal{C}(-2, 1.2)$	$\times$	$\circ$	$\circ$	$\circ$	$\circ$
$\mathcal{C}(-2, 1.1)$	$\times$	$\times$	$\times$	$\times$	$\circ$
$\bigcap\{\mathcal{H}(1.9), \mathcal{C}(-2, 1.4)\}$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\bigcap\{\mathcal{H}(1.9), \mathcal{C}(-2, 1.3)\}$	$\times$	$\times$	$\circ$	$\circ$	$\circ$
$\bigcap\{\mathcal{H}(1.9), \mathcal{C}(-2, 1.2)\}$	$\times$	$\times$	$\times$	$\times$	$\circ$
$\bigcap\{\mathcal{H}(1.9), \mathcal{C}(-2, 1.1)\}$	$\times$	$\times$	$\times$	$\times$	$\circ$

Robust H_∞ Analysis Problem

As for the H_∞ specification, we have arrived at a dilated characterization (4.46). In dealing with robust H_∞ analysis problems for real polytopic uncertainty, this dilated characterization becomes a powerful tool as illustrated below.

Consider the LTI plant described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -k & -f \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_\infty \\ z_\infty &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{4.117}$$

where k and f have the following ranges of uncertainties.

$$0.6 \le k \le 0.8, \quad 0.2 \le f \le 0.8 \tag{4.118}$$

The problem here is to find the worst case H_∞ cost $\gamma_{\infty,\mathrm{w.c.}}$ defined by

$$\gamma_{\infty,\text{w.c.}} := \max_{k,f} ||T_{z_\infty w_\infty}(s)||_\infty \tag{4.119}$$

for all possible values of the parameters k and f.

Since k and f have the uncertainties (4.118), we describe the plant as a polytope with four vertices and applied the following two approaches to this problem.

(i) Conventional Approach with a common Lyapunov variable X based on the LMI (4.45).

(ii) New Approach with common auxiliary variables F_k based on the dilated LMI (4.46), which allows the use of a parameter-dependent Lyapunov variable to ensure the robust H_∞ performance.

Solving the problem with these two approaches, we get the results shown in Table 4.5. We can see the effectiveness of the new approach (ii) over the conventional approach (i).

Table 4.5: The upper bounds for the worst case H_∞ cost

Approach	upper bound
Approach (i)	15.1212
Approach (ii)	6.5094

In this problem, it can be shown analytically that the exact value of $\gamma_{\infty,\text{w.c.}}$ is nothing but the H_∞ cost on the vertex corresponding to $k = 0.6$ and $f = 0.2$, whose value is 6.5094. Hence, we can conclude that the approach (ii) successfully achieves the exact maximum value.

4.5.3 Robust H_2 Synthesis for Real Polytopic Uncertainty

In this subsection, we deal with an example of the robust H_2 synthesis problem for real polytopic uncertainty, based on the arguments provided in Subsection 4.4.2.

Consider the LTI plant described by (4.107), where the parameters k and f have the following ranges of uncertainties [15].

$$0.09 \le k \le 0.4, \quad 0.0038 \le f \le 0.04 \tag{4.120}$$

The problem is to find a state-feedback gain K minimizing the worst case H_2 cost defined by

$$\gamma_{2,\text{w.c.}} := \max_{k,f} ||T_{zw}(s)||_2 \tag{4.121}$$

for all possible values of the parameters k and f.

Since k and f have the uncertainties (4.120), the plant can be described as a polytope with four vertices. In the following, we refer to the model corresponding to each vertex as Model 1, 2, 3 and 4, respectively.

Solving this problem by Conventional Approach and New Approach provided in Subsection 4.4.2, we get the upper bounds of the worst case H_2 cost shown in Table 4.6, where we also show the computation time for each approach.

Table 4.6: The upper bounds for the worst case H_2 cost

Approach (Corresponding gain)	upper bound	CPU time (sec)
Conventional Approach (K_c)	1.7584	0.23
New Approach with $a = 1.0$ (K)	1.3989	0.96

These approaches arrive at the state-feedback gains given below.

$$K_c = \begin{bmatrix} -4.2380 & 1.8329 & -3.0918 & -9.6993 \end{bmatrix} \tag{4.122}$$

$$K = \begin{bmatrix} -1.6224 & 0.1055 & -2.1086 & -3.6594 \end{bmatrix} \tag{4.123}$$

It follows that New Approach leads to the above gain K that is quite different from K_c, yielding a considerably better upper bound of the worst case H_2 cost. Recall that New Approach designs the state-feedback gain K through a parameter-dependent Lyapunov variable so that the conservatism of Conventional Approach can be reduced.

Although the upper bound is indeed improved by New Approach, this result is not strong enough to conclude that the feedback gain K achieves better performance than K_c. To see this more carefully, the following Table 4.7 shows the H_2 costs on each vertex achieved by these gains. This table shows that the H_2 costs achieved by K_c on each vertex are larger than the upper bound 1.3989 of the worst case H_2 cost achieved by K. Hence, we can conclude that the feedback gain K resulting from New Approach indeed achieves better performance than K_c.

Table 4.7: The H_2 costs on each vertex achieved by K_c and K

	Model 1	Model 2	Model 3	Model 4
K_c	1.5763	1.5751	1.5320	1.5258
K	1.3251	1.3442	1.2904	1.3012

4.5.4 Robust Multiobjective H_2/D-stability Synthesis for Real Polytopic Uncertainty

As we have seen in Subsection 4.4.3, the dilated characterizations enabled us to propose a new approach to the robust multiobjective H_2/**D**-stability synthesis problem for real polytopic uncertainty, with the use of non-common parameter-dependent Lyapunov variables. This subsection demonstrates the effectiveness of this new approach through a simple numerical example.

Consider again the LTI uncertain plant described by (4.107) and (4.120). The problem here is to find a state-feedback gain K minimizing the worst case H_2 cost defined by (4.121) subject to the **D**-stability constraint such that the closed-loop poles for all possible values of the parameters k and f lie in $\bigcap\{\mathcal{H}(0.15), \mathcal{S}(\tan(3\pi/8))\}$. This problem is nothing but the Problem 3 studied in Section 3.5.

We can see that the two approaches provided in Subsection 4.4.3 are ready to be applied to this problem, by including the corresponding LMI's for the sector region. Applying them to this problem, we get the upper bounds of the worst case H_2 cost shown in Table 4.8, where we also show the computation time for each approach. Although New Approach takes much more computation time, it achieves significant improvement of the upper bound over Conventional Approach. Note that these upper bounds are naturally worse than those of the preceding problem because of the additional **D**-stability constraint.

Table 4.8: The upper bounds for the worst case H_2 cost

Approach (Corresponding gain)	upper bound	CPU time (sec)
Conventional Approach (K_c)	2.1816	0.88
New Approach with $a = 1.0$ (K)	1.7801	11.51

The state-feedback gains resulting from these approaches are given in the following for comparison.

$$K_c = \begin{bmatrix} -10.0449 & 4.5272 & -5.2278 & -30.1554 \end{bmatrix} \tag{4.124}$$

$$K = \begin{bmatrix} -5.1862 & 2.4977 & -4.0846 & -13.1273 \end{bmatrix} \tag{4.125}$$

In order to examine the performance of these feedback gains, we calculate the H_2 costs achieved by them on each vertex and obtain the results shown in Table 4.9. This table shows that the H_2 costs achieved by K_c on each vertex are larger than the upper bound 1.7801 of the worst case H_2 cost achieved by K, which leads us to the conclusion that the feedback gain K indeed achieves better performance than K_c.

Table 4.9: The H_2 costs on each vertex achieved by K_c and K

	Model 1	Model 2	Model 3	Model 4
K_c	1.9714	1.9559	1.9265	1.9075
K	1.6954	1.6901	1.6648	1.6568

One of the possible reasons why New Approach arrives at better performance than Conventional Approach is that it circumvent the conservatism of Conventional Approach successfully. Indeed, we can see the less conservative nature of New Approach via the closed-loop pole locations on each vertex under the feedback gains K_c and K shown in Figs 4.5 and 4.6. With a comparison between these two figures, we can see that New Approach is less conservative: the feedback gain K achieves almost the boundary for the **D**-stability constraint.

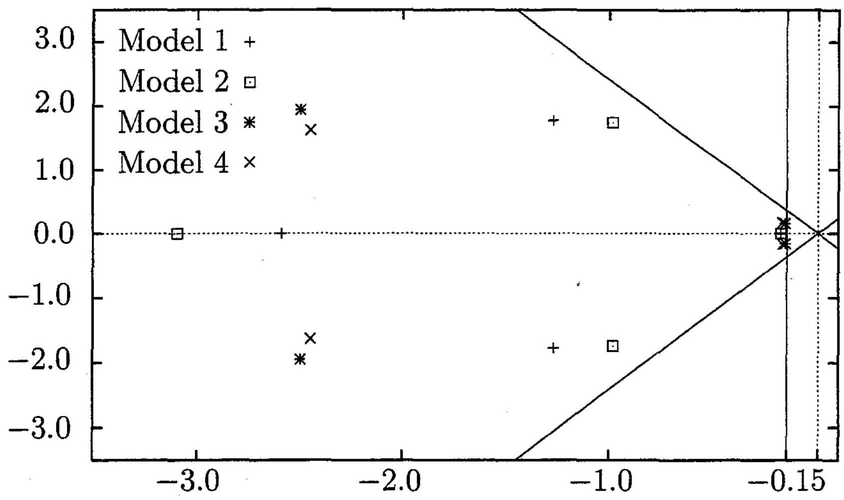

Figure 4.5: Pole locations under K_c

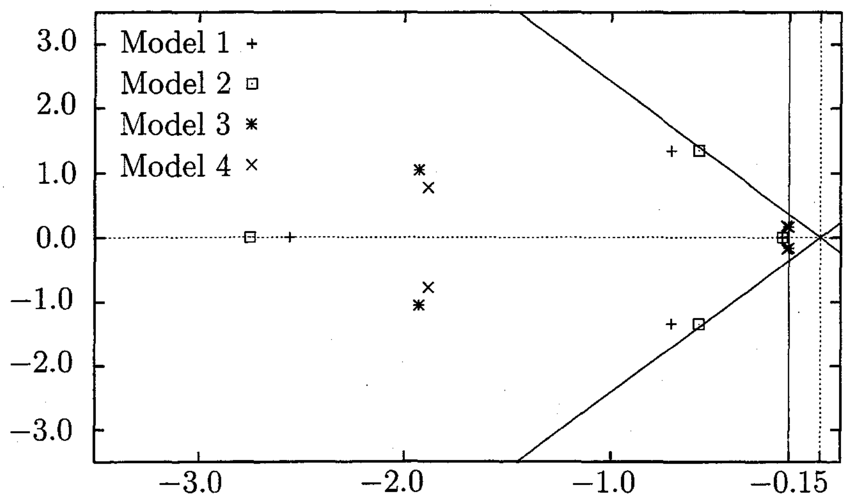

Figure 4.6: Pole locations under K

As stated before, this problem was also dealt with in Section 3.5, where we applied some approaches that only allow non-common Lyapunov variables for the design specifications. Namely, in contrast with New Approach, the Lyapunov variables there were fixed over the whole uncertainty domain. The results there are quickly reviewed in the following.

- Approach I (i.e. the subspace approach) achieved the best upper bound 1.8296 of the worst case H_2 cost among the non-iterative approaches as shown in Table 3.9. The corresponding feedback gain K_{I} is given by (3.103).

- Iterative Algorithm I–III achieved almost the same performance, and Iterative Algorithm I (i.e. the iterative algorithm based on the subspace approach) achieved an upper bound 1.8023 of the worst case H_2 cost with the least computation time 14.09 (sec) as shown in Table 3.10. The corresponding feedback gain K_{I}^* is given by (3.106).

Summing up these results and that of New Approach, we obtain the following table, which shows that the feedback gain K resulting from New Approach achieves the best upper bound.

Table 4.10: The upper bounds for the worst case H_2 cost

Approach (Corresponding gain)	upper bound	CPU time (sec)
Approach I (K_{I})	1.8296	1.82
Iterative Algorithm I (K_{I}^*)	1.8023	14.09
New Approach with $a = 1.0$ (K)	1.7801	11.51

Although the feedback gain K achieves the best upper bound, we cannot conclude that the gain K achieves the best performance. In order to evaluate the performance achieved by K_{I}, K_{I}^* and K more carefully, we calculate the H_2 costs on each vertex achieved by these gains and obtain Table 4.11. Here, aiming at the exact evaluation, we further solve the robust H_2 performance analysis problems for given gains K_{I}, K_{I}^* and K by the approach (iii) provided in Subsection 4.4.1. This enables us to have better upper bounds of the worst case H_2 cost than those in Table 4.10, and the resulting upper bounds are denoted by γ_{2F} in Table 4.11.

Table 4.11: The H_2 costs on each vertex achieved by K_I, K_I^* and K

	Model 1	Model 2	Model 3	Model 4	γ_{2F}
K_{I}	1.7333	1.7256	1.6793	1.6658	1.7362
K_{I}^*	1.6878	1.6821	1.6486	1.6390	1.6906
K	1.6954	1.6901	1.6648	1.6568	1.6980

The above table readily leads us to the following conclusions.

- The feedback gain K indeed achieves better performance than K_{I}, which follows immediately from the fact that the H_2 costs achieved by K_{I} on Models 1 and 2 are larger than the upper bound $\gamma_{2F} = 1.6980$ of the worst case H_2 cost ensured by K.

- The feedback K_{I}^* indeed achieves better performance than K, which follows immediately from the fact that the H_2 cost achieved by K on Model 1 is larger than the upper bound $\gamma_{2F} = 1.6906$ of the worst case H_2 cost ensured by K_{I}^*.

Although New Approach fails to attain better performance than the Iterative Algorithm I, we can see that New Approach achieves comparable performance. It should be noted that New Approach achieves the performance with less computational effort than Iterative Algorithm I as shown in Table 4.10.

In this subsection, we have dealt with an example of the robust multiobjective $H_2/\mathbf{D}$-stability synthesis problem for real polytopic uncertainty. The problem treated here fortunately allows a feasible common Lyapunov variable, which is indispensable in applying the approaches and the algorithms provided in the preceding chapter. To put it reverse, they are of little use for such problems that lack a feasible common Lyapunov variable. In dealing with such problems, however, New Approach in this chapter will be still helpful so that parameter-dependent Lyapunov variables can be sought.

4.6 Summary

In this chapter, we have derived new dilated matrix inequality characterizations for continuous-time controller design and performance analysis. We have shown that a particular application of the Schur complement technique leads to a constructive way to derive dilated characterizations, exhibiting some analogous properties to the ones already obtained in the discrete-time setting [29],[30].

The results obtained about the dilated characterizations could be summarized as follows.

1. For the $\mathbf{D}$-stability constraints and the H_2 specification, we derived new dilated characterizations that are suitable for controller synthesis. These dilated characterizations enabled us to propose a new approach to the multiobjective $H_2/\mathbf{D}$-stability controller design problems with non-common Lyapunov variables. It was shown that the new approach leads to a better (no worse) upper bound for the cost functional than that with the conventional approach [6],[26],[36]. Numerical examples showed that the actual cost is also improved, and the application of the new approach resulted in significant improvements over the conventional approach.

2. With the dilated characterizations, we proposed a new approach to the robust multiobjective $H_2/\mathbf{D}$-stability synthesis for real polytopic uncertainty, where we successfully employed non-common parameter-dependent Lyapunov variables. We showed that a specific choice of the scalar included in the dilated characterizations ensures the new

approach to achieve a better (no worse) upper bound for the cost functional than that with the conventional approach. This choice is only a sufficient condition to ensure the advantage of the new approach, and a numerical example showed that the new approach achieves considerably better performance than the conventional approach even without a special care on the choice of that scalar.

3. For the **D**-stability constraints, the H_2 performance and the H_∞ performance, we have derived dilated matrix inequality characterizations that are suitable for robust performance analysis for real polytopic uncertainty [33]–[35]. The effectiveness of these new characterizations is demonstrated through numerical examples.

The above three are the most important achievements in this chapter, gained by the dilated matrix inequality characterizations. Specifically, it is a remarkable contribution that we have reduced the multiobjective $H_2/\mathbf{D}$-stability problem into a convex optimization problem with *non-common* Lyapunov variables in a reasonable fashion.

In spite of the above achievements, we have the following future topics.

1. For the H_∞ performance, we have *not* derived a dilated characterization with a single square auxiliary variable being involved in the product with the controller variables. Such a characterization is indispensable to address the multiobjective H_2/H_∞ problem [23] or the multiobjective $H_2/H_\infty/\mathbf{D}$-stability problem [6],[26],[36] with non-common Lyapunov variables in a straightforward fashion.

2. We expect that the new dilated characterizations presented in this chapter have another potential to show new directions in such problems as the fixed order dynamic output-feedback control problem [44], the decentralized control problem [46] and so on. It is known that these problems are quite hard to solve with the conventional matrix inequality characterizations because of the product between the Lyapunov variables and the controller variables.

We would like to stress, however, that the first topic given above has been *partially* achieved by the dilated characterization presented in this chapter. To see this, let us focus on the following dilated characterization for the H_∞ performance, which is derived in Theorem 4.6.

$$\begin{bmatrix} 0 & -X_\infty & \mathcal{B} & 0 \\ -X_\infty & 0 & 0 & 0 \\ \mathcal{B}^T & 0 & -I & \mathcal{D}^T \\ 0 & 0 & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ \mathcal{C} \end{bmatrix} F_\infty \right\} < 0 \tag{4.126}$$

Here, the variables are X_∞ and $F_\infty = [\ F_{\infty 1}\ F_{\infty 2}\ F_{\infty 3}\ F_{\infty 4}\ ]$. This inequality characterizes a necessary and sufficient condition for $||T_{zw}(s)||_\infty < \gamma_\infty$ where $T_{zw}(s) = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right]$. Moreover, we have shown that this inequality successfully recovers the original one (4.45) via $F_\infty = X_\infty [\ I\ \ -\varepsilon I\ 0\ 0\ ]$ if the scaler $\varepsilon > 0$ is taken sufficiently small (see Corollary 4.11).

We have dealt with the dilated characterization (4.126) as a tool for robust performance analysis problems in this chapter. However, it should be noted that this characterization can be readily converted into a suitable form for controller synthesis if we impose some restriction on the variable F_∞. To keep the above mentioned nice recovery property of (4.126) even under such restriction, it is reasonable to consider $F_\infty = G [\ I\ \ -bI\ 0\ 0\ ]$, where G is a new square variable and $b = a^{-1}$ is a positive scaler. Then, we obtain the following matrix inequality with respect to X_∞ and G, which characterizes a *sufficient* condition for $||T_{zw}(s)||_\infty < \gamma_\infty$. Note that a similar characterization can be found in [41].

$$\begin{bmatrix} 0 & -X_\infty & \mathcal{B} & 0 \\ -X_\infty & 0 & 0 & 0 \\ \mathcal{B}^T & 0 & -I & \mathcal{D}^T \\ 0 & 0 & \mathcal{D} & -\gamma_\infty^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ \mathcal{C} \end{bmatrix} G \begin{bmatrix} I & -bI & 0 & 0 \end{bmatrix} \right\} \tag{4.127}$$

An advantage of working with this inequality is that it allows controller synthesis. Moreover, we can see that this inequality has the following interesting properties.

- There exists a sufficiently large $a\ (= b^{-1}) > 0$ such that the dilated characterization (4.127) recovers the original one (4.45) via $G = \widetilde{G}(X_\infty) := X_\infty$, which is a direct consequence from Corollary 4.11.
- It is also true that if we let $a \to \infty$, all admissible auxiliary variables G in (4.127) tend to X_∞ and hence the dilated characterization (4.127) "reduces" to (4.45).

These facts have strong similarities to what we have given in Section 4.1 on the dilated characterization (4.3). However, these similarities arise only when a sufficiently large a is considered. Namely, for arbitrarily chosen $a > 0$, the dilated characterization (4.127) performs differently from (4.3) and the former reduces to only a sufficient condition for the original one. It should be noted that the dilated characterization (4.3) is equivalent to the original one (4.2) irrespective of $a > 0$, and this property has played an essential role in dealing with the multiobjective controller synthesis for plants without uncertainties.

Based on the above arguments, we give the following remarks on the use of the dilated characterization (4.127).

Remark 4.12 It is possible, in principle, to address such problems as the multiobjective H_2/H_∞ problem [23] or the multiobjective $H_2/H_\infty/\mathbf{D}$-stability problem [6],[26],[36] with non-common Lyapunov variables, using the LMI's (4.26), (4.29), (4.33), (4.38) and (4.127). However, the inclusion of the H_∞ specification weakens the corresponding assertion to Theorem 4.7. Namely, all we can assure for the new approach reduces to that there exists $a_{\min} > 0$ such that whenever $a > a_{\min}$, the new approach ensures the achievement of a better (no worse) upper bound of the cost functional than that with the conventional approach, which clearly shows the difference between the assertion of Theorem 4.7 for the multiobjective $H_2/\mathbf{D}$-stability problem.

Remark 4.13 It is possible, in principle, to address such problems as the robust multiobjective $H_2/H_\infty/\mathbf{D}$-stability problem for real polytopic uncertainty [6] with non-common parameter-dependent Lyapunov variables, using the LMI's (4.26), (4.29), (4.33), (4.38) and (4.127). All we can assure for the new approach is that there exists $a_{\min} > 0$ such that whenever $a > a_{\min}$, the new approach ensures the achievement of a better (no worse) upper bound of the cost functional than that with the conventional approach, which is the same consequence, on the surface, as we have given in Proposition 4.5. However, it should be noted that the inclusion of the H_∞ specification brings another sort of conservatism into the new approach arising from the fact that the inequality (4.127) itself is merely a sufficient condition for the H_∞ specification.

4.7 Appendix

4.7.1 Proof of Lemma 4.1 with the Elimination Lemma

Proof. Let us define the following matrices.

$$Y := \begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix}, \quad E := \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Q := G, \quad F := \begin{bmatrix} I \\ -bI \\ I \\ bI \\ b\Delta \end{bmatrix}^T \tag{4.128}$$

Then, the inequality (4.3) in the condition (ii) in Lemma 4.1 can be described as (4.12). Hence, to establish the equivalence between the conditions (i) and (ii) in Lemma 4.1, it is enough to show that the two conditions $E^\perp Y E^{\perp T} < 0$ and $(F^T)^\perp Y F^\perp < 0$ with the matrices given by (4.128) are equivalent to the original condition (4.2).

As we have seen in (4.10), the condition $E^{\perp}YE^{\perp^T} < 0$ is equivalent to the original condition (4.2). On the other hand, it turns out that the condition $(F^T)^{\perp}YF^{\perp} < 0$ is equivalent to an *implicit* condition hidden in (4.2). To see this, let us consider (4.2) again. Completing the square with respect to $\mathcal{A}$ in (4.2) yields

$$(\mathcal{A}+\delta_2^{-1}I)\delta_2\mathcal{X}(\mathcal{A}+\delta_2^{-1}I)^T + (\delta_1-\delta_2^{-1})\mathcal{X} + \mathcal{X}\Delta^T\Delta\mathcal{X} < 0 \tag{4.129}$$

Hence, we can see that

$$(\delta_1-\delta_2^{-1})\mathcal{X} + \mathcal{X}\Delta^T\Delta\mathcal{X} < 0 \tag{4.130}$$

is a necessary condition for the feasibility of (4.2). Applying the Schur complement technique to (4.130) with the given scalar $a > 0$, we have

$$\left[\begin{array}{cc|c|c} -2a\mathcal{X} & 0 & -2a\mathcal{X} & 0 \\ 0 & -\delta_1^{-1}\mathcal{X} & -\mathcal{X} & 0 \\ \hline -2a\mathcal{X} & -\mathcal{X} & -2a\mathcal{X}-\delta_2^{-1}\mathcal{X} & -\mathcal{X}\Delta^T \\ \hline 0 & 0 & -\Delta\mathcal{X} & -I \end{array}\right] < 0 \tag{4.131}$$

From the above inequality, we readily obtain

$$\begin{aligned}
&(F^T)^{\perp}YF^{\perp} \\
&= \begin{bmatrix} I \\ -bI \\ bI \\ I \\ b\Delta \end{bmatrix}^{\perp} \begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} I \\ -bI \\ bI \\ I \\ b\Delta \end{bmatrix}^{T\perp} \\
&= \begin{bmatrix} I & aI & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 \\ 0 & aI & 0 & I & 0 \\ 0 & \Delta & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ aI & I & aI & \Delta^T \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \qquad (4.132) \\
&= \begin{bmatrix} -2a\mathcal{X} & 0 & -2a\mathcal{X} & 0 \\ 0 & -\delta_1^{-1}X & -\mathcal{X} & 0 \\ -2a\mathcal{X} & -\mathcal{X} & -2a\mathcal{X}-\delta_2^{-1}X & -\mathcal{X}\Delta^T \\ 0 & 0 & -\Delta\mathcal{X} & -I \end{bmatrix} < 0
\end{aligned}$$

which completes the proof. Q.E.D.

4.7.2 Proof of Lemma 4.4

Proof. Note that $a_{\min}$ is well-defined since (4.2) holds with $\mathcal{X} = X$ by assumption and since $X > 0$. It is obvious that the inequalities (4.15) and (4.16) hold for any $a > a_{\min}$. With this in mind, let us apply the Schur complement technique to (4.15) under the condition (4.16) to get

$$\begin{bmatrix} \mathcal{A}X + X\mathcal{A}^T + \delta_1 X + \delta_2 \mathcal{A}X\mathcal{A}^T + X\Delta^T\Delta X & \mathcal{A}X + \delta_1 X + X\Delta^T\Delta X \\ X\mathcal{A}^T + \delta_1 X + X\Delta^T\Delta X & \delta_1 X + X\Delta\Delta^T X - 2aX \end{bmatrix} < 0 \tag{4.133}$$

Applying again the Schur complement technique to (4.133), we obtain

$$\left[\begin{array}{cc:ccc} \mathcal{A}X + X\mathcal{A}^T & \mathcal{A}X & X & \mathcal{A}X & X\Delta^T \\ X\mathcal{A}^T & -2aX & X & 0 & X\Delta^T \\ \hdashline X & X & -\delta_1^{-1}X & 0 & 0 \\ X\mathcal{A}^T & 0 & 0 & -\delta_2^{-1}X & 0 \\ \Delta X & \Delta X & 0 & 0 & -I \end{array}\right] < 0 \tag{4.134}$$

It remains to perform a congruence transformation with $\begin{bmatrix} I & b\mathcal{A} \\ 0 & bI \end{bmatrix} I \oplus I \oplus I$ on the above inequality to get

$$\begin{bmatrix} \mathcal{A}X + X\mathcal{A}^T & -b\mathcal{A}X & b\mathcal{A}X + X & \mathcal{A}X & b\mathcal{A}X\Delta^T + X\Delta^T \\ -bX\mathcal{A}^T & -2bX & bX & 0 & bX\Delta^T \\ bX\mathcal{A}^T + X & bX & -\delta_1^{-1}X & 0 & 0 \\ X\mathcal{A}^T & 0 & 0 & -\delta_2^{-1}X & 0 \\ b\Delta X\mathcal{A}^T + \Delta X & b\Delta X & 0 & 0 & -I \end{bmatrix} < 0 \tag{4.135}$$

This inequality is nothing but (4.3) with $\mathcal{X} = G = X$. This completes the proof. Q.E.D.

4.7.3 Proof of Lemma 4.5

Proof. It is easy to show the equivalence between (4.2) and (4.19) by the Elimination Lemma, with

$$
Y := \begin{bmatrix} 0 & -\mathcal{X} & \mathcal{X} & 0 & \mathcal{X}\Delta^T \\ -\mathcal{X} & 0 & 0 & -\mathcal{X} & 0 \\ \mathcal{X} & 0 & -\delta_1^{-1}\mathcal{X} & 0 & 0 \\ 0 & -\mathcal{X} & 0 & -\delta_2^{-1}\mathcal{X} & 0 \\ \Delta\mathcal{X} & 0 & 0 & 0 & -I \end{bmatrix}, \quad E := \begin{bmatrix} \mathcal{A} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.136}
$$

$$
Q := \begin{bmatrix} Q_1 \, Q_2 \, Q_3 \, Q_4 \, Q_5 \end{bmatrix}, \quad F := I
$$

and with (4.10). In the following, however, we give another proof of the fact that (4.2) implies (4.19), in which the relation between the solutions of (4.2) and (4.19) is clear.

For every solution $\mathcal{X} = X > 0$, there exists $\varepsilon > 0$ such that

$$
\mathcal{A}X + X\mathcal{A}^T + \delta_1 X + \delta_2 \mathcal{A}X\mathcal{A}^T + X\Delta^T\Delta X + \frac{1}{2}\varepsilon \mathcal{A}X\mathcal{A}^T < 0 \tag{4.137}
$$

Applying the Schur complement technique to the above inequality, we obtain

$$
\left[\begin{array}{c|cccc} \mathcal{A}X + X\mathcal{A}^T & -\varepsilon \mathcal{A}X & X & \mathcal{A}X & X\Delta^T \\ \hline -\varepsilon X\mathcal{A}^T & -2\varepsilon X & 0 & 0 & 0 \\ X & 0 & -\delta_1^{-1}X & 0 & 0 \\ X\mathcal{A}^T & 0 & 0 & -\delta_2^{-1}X & 0 \\ \Delta X & 0 & 0 & 0 & -I \end{array}\right] < 0 \tag{4.138}
$$

The above inequality is nothing but the LMI condition (4.19) with $[\, \mathcal{X} \; Q_1 \; Q_2 \; Q_3 \; Q_4 \; Q_5 \,] = [\, X \; X \; -\varepsilon X \; 0 \; X \; 0 \,]$. Q.E.D.

4.7.4 Proof of Theorem 4.4

Proof. The equivalence between the conditions (i) and (ii) is a well-known result [6]. The condition (iii) and (iv) imply the condition (ii) since

$$
\begin{aligned}
&\begin{bmatrix} \mathcal{A} & 0 \\ I & 0 \\ 0 & I \\ 0 & \mathcal{A} \end{bmatrix}^{\perp} \begin{bmatrix} 0 & -kX_S & X_S & 0 \\ -kX_S & 0 & 0 & -X_S \\ X_S & 0 & 0 & -kX_S \\ 0 & -X_S & -kX_S & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A} & 0 \\ I & 0 \\ 0 & I \\ 0 & \mathcal{A} \end{bmatrix}^{T\perp} \\
&= \begin{bmatrix} I & -\mathcal{A} & 0 & 0 \\ 0 & 0 & -\mathcal{A} & I \end{bmatrix} \begin{bmatrix} 0 & -kX_S & X_S & 0 \\ -kX_S & 0 & 0 & -X_S \\ X_S & 0 & 0 & -kX_S \\ 0 & -X_S & -kX_S & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -\mathcal{A}^T & 0 \\ 0 & -\mathcal{A}^T \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} k(\mathcal{A}X_S + X_S\mathcal{A}^T) & \mathcal{A}X_S - X_S\mathcal{A}^T \\ X_S\mathcal{A}^T - \mathcal{A}X_S & k(\mathcal{A}X_S + X_S\mathcal{A}^T) \end{bmatrix} < 0
\end{aligned} \tag{4.139}
$$

Because the condition (iii) implies the condition (iv), it remains to show that the condition (ii) implies (iii). In the following, we give a proof of the fact that the condition (ii) implies (iii), in which the relation between the solutions of (4.32) and (4.33) is clear.

Applying the Schur complement technique to (4.32) with the given scalar a and with a simple manipulation $\mathcal{A}X_S - X_S\mathcal{A}^T = (\mathcal{A}-aI)X_S - X_S(\mathcal{A}-aI)^T$, we obtain

$$\left[\begin{array}{c:cc:c} -2akX_S & -2akX_S & 0 & 0 \\ \hdashline -2akX_S & k\mathrm{He}\left[(\mathcal{A}-aI)X_S\right] & (\mathcal{A}-aI)X_S - X_S(\mathcal{A}-aI)^T & 0 \\ 0 & X_S(\mathcal{A}-aI)^T - (\mathcal{A}-aI)X_S & k\mathrm{He}\left[(\mathcal{A}-aI)X_S\right] & -2akX_S \\ \hdashline 0 & 0 & -2akX_S & -2akX_S \end{array}\right] < 0 \quad (4.140)$$

Here, $(\mathcal{A}-aI)$ is nonsingular because $\mathcal{A}$ is stable by the condition (ii). Hence, the above inequality admits a congruence transformation with $I \oplus (\mathcal{A}-aI)^{-1} \oplus (\mathcal{A}-aI)^{-1} \oplus I$ to get

$$\begin{bmatrix} -2akX_S & -2akX_S(\mathcal{A}-aI)^{-T} & 0 & 0 \\ -2ak(\mathcal{A}-aI)^{-1}X_S & k\mathrm{He}\left[X_S(\mathcal{A}-aI)^{-T}\right] & X_S(\mathcal{A}-aI)^{-T} - (\mathcal{A}-aI)^{-1}X_S & 0 \\ 0 & (\mathcal{A}-aI)^{-1}X_S - X_S(\mathcal{A}-aI)^{-T} & k\mathrm{He}\left[X_S(\mathcal{A}-aI)^{-T}\right] & -2ak(\mathcal{A}-aI)^{-1}X_S \\ 0 & 0 & -2akX_S(\mathcal{A}-aI)^{-T} & -2akX_S \end{bmatrix} < 0 \quad (4.141)$$

Defining $\widehat{G}_S := -(\mathcal{A}-aI)^{-1}X_S$, we have $X_S = -(\mathcal{A}-aI)\widehat{G}_S$, and thus we readily obtain

$$\begin{bmatrix} 2akX_S + 2ak\mathrm{He}\left[(\mathcal{A}-aI)\widehat{G}_S\right] & 2ak\widehat{G}_S^T - kX_S - k(\mathcal{A}-aI)\widehat{G}_S & X_S + (\mathcal{A}-aI)\widehat{G}_S & 0 \\ 2ak\widehat{G}_S - kX_S - k\widehat{G}_S^T(\mathcal{A}-aI)^T & -k(\widehat{G}_S + \widehat{G}_S^T) & \widehat{G}_S - \widehat{G}_S^T & -X_S - \widehat{G}_S^T(\mathcal{A}-aI)^T \\ X_S + \widehat{G}_S^T(\mathcal{A}-aI)^T & \widehat{G}_S^T - \widehat{G}_S & -k(\widehat{G}_S + \widehat{G}_S^T) & 2ak\widehat{G}_S - kX_S - k(\mathcal{A}-aI)\widehat{G}_S \\ 0 & -X_S - (\mathcal{A}-aI)\widehat{G}_S & 2ak\widehat{G}_S^T - kX_S - k\widehat{G}_S^T(\mathcal{A}-aI)^T & 2akX_S + 2ak\mathrm{He}\left[(\mathcal{A}-aI)\widehat{G}_S\right] \end{bmatrix} < 0 \quad (4.142)$$

The above inequality can be written as follows.

$$\begin{bmatrix} 2akX_S & -kX_S & X_S & 0 \\ -kX_S & 0 & 0 & -X_S \\ X_S & 0 & 0 & -kX_S \\ 0 & -X_S & -kX_S & 2akX_S \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}-aI & 0 \\ I & 0 \\ 0 & I \\ 0 & \mathcal{A}-aI \end{bmatrix} \begin{bmatrix} \widehat{G}_S & 0 \\ 0 & \widehat{G}_S \end{bmatrix} \begin{bmatrix} 2akI & -kI & I & 0 \\ 0 & -I & -kI & 2akI \end{bmatrix} \right\} < 0 \tag{4.143}$$

Performing a congruence transformation with $\begin{bmatrix} I & aI \\ 0 & I \end{bmatrix} \oplus \begin{bmatrix} I & 0 \\ aI & I \end{bmatrix}$ on (4.143), we have (4.33) in (iii), where $G_S := a\widehat{G}_S$ and $b := a^{-1}$.

It is easy to give an alternative proof of the fact that the condition (ii) implies (iv), where the relation between the solutions of (4.32) and (4.34) is clear. If (4.32) holds, there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} k(\mathcal{A}X_S + X_S\mathcal{A}^T) + \frac{1}{2}\varepsilon\mathcal{A}X_S\mathcal{A}^T & \mathcal{A}X_S - X_S\mathcal{A}^T \\ X_S\mathcal{A}^T - \mathcal{A}X_S & k(\mathcal{A}X_S + X_S\mathcal{A}^T) + \frac{1}{2}\varepsilon\mathcal{A}X_S\mathcal{A}^T \end{bmatrix} < 0 \tag{4.144}$$

Applying the Schur complement technique to the above inequality, we have

$$\left[\begin{array}{cc:cc} k(\mathcal{A}X_S + X_S\mathcal{A}^T) & X_S\mathcal{A}^T - \mathcal{A}X_S & -\varepsilon\mathcal{A}X_S & 0 \\ X_S\mathcal{A} - \mathcal{A}^TX_S & k(\mathcal{A}X_S + X_S\mathcal{A}^T) & 0 & -\varepsilon\mathcal{A}X_S \\ \hdashline -\varepsilon X_S\mathcal{A}^T & 0 & -2\varepsilon X_S & 0 \\ 0 & -\varepsilon X_S\mathcal{A}^T & 0 & -2\varepsilon X_S \end{array}\right] < 0 \tag{4.145}$$

This inequality admits a congruence transformation with $\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix}$ to get

$$\begin{bmatrix} k(\mathcal{A}X_S + X_S\mathcal{A}^T) & -\varepsilon\mathcal{A}X_S & 0 & \mathcal{A}X_S - X_S\mathcal{A}^T \\ -\varepsilon X_S\mathcal{A}^T & -2\varepsilon X_S & 0 & 0 \\ 0 & 0 & -2\varepsilon X_S & -\varepsilon X_S\mathcal{A}^T \\ X_S\mathcal{A}^T - \mathcal{A}X_S & 0 & -\varepsilon\mathcal{A}X_S & k(\mathcal{A}X_S + X_S\mathcal{A}^T) \end{bmatrix} < 0 \tag{4.146}$$

It is easy to see that the above inequality is nothing but the LMI condition (4.34) in (iv) with $[\ F_{S1}\ F_{S2}\ F_{S3}\ F_{S4}\ F_{S5}\ F_{S6}\ F_{S7}\ F_{S8}\ ] = [\ kX_S\ \ -\varepsilon X_S\ \ 0\ \ X_S\ \ -X_S\ \ 0\ \ -\varepsilon X_S\ \ kX_S\ ]$.

Q.E.D.

4.7.5 LMI's for the Output-Feedback Controller Synthesis

In this subsection, we show that the constraints (4.26), (4.29), (4.33) and (4.38) can be reduced to LMI's by congruence transformations with Ξ_H given by (4.65).

α-stability region (4.26)

Performing a congruence transformation with $\Xi_H^T \oplus \Xi_H^T \oplus \Xi_H^T$ on (4.26) under the constraint (4.58), we obtain

$$\begin{bmatrix} 0 & -\Xi_H^T X_{\mathcal{H}} \Xi_H & \Xi_H^T X_{\mathcal{H}} \Xi_H \\ -\Xi_H^T X_{\mathcal{H}} \Xi_H & 0 & 0 \\ \Xi_H^T X_{\mathcal{H}} \Xi_H & 0 & -\frac{1}{2}\alpha^{-1}\Xi_H^T X_{\mathcal{H}} \Xi_H \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} \Xi_H^T \mathcal{A} G \Xi_H & -b\Xi_H^T \mathcal{A} G \Xi_H & b\Xi_H^T \mathcal{A} G \Xi_H \\ \Xi_H^T G \Xi_H & -b\Xi_H^T G \Xi_H & b\Xi_H^T G \Xi_H \\ 0 & 0 & 0 \end{bmatrix}\right\} < 0 \tag{4.147}$$

The above constraint only involves the terms $\Xi_H^T X_{\mathcal{H}} \Xi_H$, $\Xi_H^T G \Xi_H$ and $\Xi_H^T \mathcal{A} G \Xi_H$. Since these matrices can be represented as in (4.68) and (4.69), it is an LMI with respect to $\bar{X}_{\mathcal{H}}$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$.

Circular region (4.29)

Performing a congruence transformation with $\Xi_H^T \oplus \Xi_H^T \oplus \Xi_H^T \oplus \Xi_H^T$ on (4.29) under the constraint (4.58), we obtain

$$\begin{bmatrix} 0 & -\Xi_H^T X_{\mathcal{C}} \Xi_H & \Xi_H^T X_{\mathcal{C}} \Xi_H & 0 \\ -\Xi_H^T X_{\mathcal{C}} \Xi_H & 0 & 0 & -\Xi_H^T X_{\mathcal{C}} \Xi_H \\ \Xi_H^T X_{\mathcal{C}} \Xi_H & 0 & \frac{c}{\beta}\Xi_H^T X_{\mathcal{C}} \Xi_H & 0 \\ 0 & -\Xi_H^T X_{\mathcal{C}} \Xi_H & 0 & c\Xi_H^T X_{\mathcal{C}} \Xi_H \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} \Xi_H^T \mathcal{A} G \Xi_H & -b\Xi_H^T \mathcal{A} G \Xi_H & b\Xi_H^T \mathcal{A} G \Xi_H & \Xi_H^T \mathcal{A} G \Xi_H \\ \Xi_H^T G \Xi_H & -b\Xi_H^T G \Xi_H & b\Xi_H^T G \Xi_H & \Xi_H^T G \Xi_H \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} < 0 \tag{4.148}$$

The above constraint only involves the terms $\Xi_H^T X_{\mathcal{C}} \Xi_H$, $\Xi_H^T G \Xi_H$ and $\Xi_H^T \mathcal{A} G \Xi_H$, which enables us to see that it is an LMI with respect to $\bar{X}_{\mathcal{C}}$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$.

Sector region (4.33)

Performing a congruence transformation with $\Xi_H^T \oplus \Xi_H^T \oplus \Xi_H^T \oplus \Xi_H^T$ on (4.33) under the constraint (4.58), we arrive at

$$\begin{aligned}&\begin{bmatrix} 0 & -k\Xi_H^T X_S \Xi_H & \Xi_H^T X_S \Xi_H & 0 \\ -k\Xi_H^T X_S \Xi_H & 0 & 0 & -\Xi_H^T X_S \Xi_H \\ \Xi_H^T X_S \Xi_H & 0 & 0 & -k\Xi_H^T X_S \Xi_H \\ 0 & -\Xi_H^T X_S \Xi_H & -k\Xi_H^T X_S \Xi_H & 0 \end{bmatrix} \\ &\quad + \mathrm{He}\left\{ \begin{bmatrix} k\Xi_H^T \mathcal{A} G \Xi_H & -bk\Xi_H^T \mathcal{A} G \Xi_H & b\Xi_H^T \mathcal{A} G \Xi_H & \Xi_H^T \mathcal{A} G \Xi_H \\ k\Xi_H^T G \Xi_H & -bk\Xi_H^T G \Xi_H & b\Xi_H^T G \Xi_H & \Xi_H^T G \Xi_H \\ -\Xi_H^T G \Xi_H & -b\Xi_H^T G \Xi_H & -bk\Xi_H^T G \Xi_H & k\Xi_H^T G \Xi_H \\ -\Xi_H^T \mathcal{A} G \Xi_H & -b\Xi_H^T \mathcal{A} G \Xi_H & -bk\Xi_H^T \mathcal{A} G \Xi_H & k\Xi_H^T \mathcal{A} G \Xi_H \end{bmatrix} \right\} < 0 \end{aligned} \tag{4.149}$$

Similarly to (4.147) and (4.148), the above constraint only involves the terms $\Xi_H^T X_S \Xi_H$, $\Xi_H^T G \Xi_H$ and $\Xi_H^T \mathcal{A} G \Xi_H$ and hence (4.149) is an LMI with respect to $\bar{X}_S$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$.

The H_2 Specification (4.38)

Performing a congruence transformation with $\Xi_H^T \oplus \Xi_H^T \oplus I$ on the first inequality in (4.38) under the constraint (4.58), we get

$$\begin{bmatrix} 0 & -\Xi_H^T X_2 \Xi_H & 0 \\ -\Xi_H^T X_2 \Xi_H & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \Xi_H^T \mathcal{A} G \Xi_H & -b\Xi_H^T \mathcal{A} G \Xi_H & 0 \\ \Xi_H^T G \Xi_H & -b\Xi_H^T G \Xi_H & 0 \\ \mathcal{C} G \Xi_H & -b\mathcal{C} G \Xi_H & 0 \end{bmatrix} \right\} < 0 \tag{4.150}$$

On the other hand, a congruence transformation with $I \oplus \Xi_H^T$ on the second inequality in (4.38) leads to

$$\begin{bmatrix} Z_2 & \mathcal{B}^T \Xi_H \\ \Xi_H^T \mathcal{B} & \Xi_H^T X_2 \Xi_H \end{bmatrix} > 0 \tag{4.151}$$

It turns out that the constraints (4.150) and (4.151) involve only the terms given in (4.68) and (4.69) and hence we can conclude that they are LMI's with respect to $\bar{X}_2$, G_{11}, H_{11}, Π, $\bar{A}_K$, $\bar{B}_K$ and $\bar{C}_K$.

Chapter 5

Conclusion

In this thesis, we proposed several LMI approaches to the multiobjective controller design problems with non-common Lyapunov variables. The robust performance analysis and synthesis problems for real polytopic uncertainty are also studied, based on the dilated LMI characterizations derived for the multiobjective controller design. As a concluding chapter, we now summarize the achievements in this thesis and discuss future topics.

In Chapter 2, we gave a formal description of the multiobjective controller design problems that we dealt with in this thesis. The conventional LMI approach was also reviewed, where we clarified that the conventional approach is conservative because of the use of a common Lyapunov variable.

In order to get around the conservatism, in Chapter 3, we provided two LMI approaches to the multiojbective state-feedback controller design problems with non-common Lyapunov variables. First, we proposed the subspace approach. The key to this approach was an introduction of the additional constraints on the Lyapunov variables, which led us to a set of LMI's that leave the state-feedback gain directly as one of the LMI variables. This was achieved by freezing only some portion of the freedom in the Lyapunov variables. Hence, using the unconstrained portion of the freedom, the set of LMI's turned out to allow non-common Lyapunov variables. In particular, it was shown that this approach yields a feedback gain that achieves better (no worse) performance than that with the conventional approach if we choose the parameters included in the constraints reasonably. An iterative algorithm was also derived with a suitable replacement of these parameters.

Second, we proposed the affine representation approach. In this approach, we performed a standard procedure called change of variables and represented the resulting variables as a set of affine functions by introducing yet new variables. These affine functions were chosen to have a crucial characteristic that troublesome non-convex constraints are satisfied regardless of the new variables. With these affine functions, we readily derived a set of LMI characterizations that allow non-common Lyapunov variables. In addition, we showed that a reasonable choice of the parameters included in the affine functions assures an advantage of this approach over the conventional approach. The affine representation approach also enabled us to have another effective iterative algorithm by simply combining it with the subspace approach.

The effectiveness of these approaches as well as the iterative algorithms was demonstrated through numerical examples in this chapter. Applying them to several problems, we obtained considerably better performance than that with the conventional approach. These results suggest that the conventional approach with a common Lyapunov variable is conservative, and the conservatism is circumvented successfully with the use of non-common Lyapunov variables.

Although the approaches provided in Chapter 3 achieved satisfactory results in numerical examples, they have some drawbacks. From a theoretical point of view, the most crucial drawback lies in the fact that they cannot be self-contained ones that are actually free from the use of a common Lyapunov variable, which is also the case with the algorithms presented in the previous studies [32],[38]–[40]. To overcome this drawback, we derived in Chapter 4 new dilated matrix inequality characterizations for continuous-time controller design. We showed that a particular application of the Schur complement technique leads to a constructive way to derive dilated characterizations, exhibiting some nice *recovery* properties. The advantage of working with these dilated characterizations is that the technical restriction to a common Lyapunov variable can be avoided. Indeed, they enabled us to propose some very promising approaches to several problems including the multiobjective controller design problems, as summarized in the following.

- We proposed a new LMI approach to the multiobjective $H_2/\mathbf{D}$-stability controller design problems with *non-common* Lyapunov variables. This approach was readily obtained by the dilated characterizations for the $\mathbf{D}$-stability constraints and the H_2 specification. Specifically, because of the nice recovery property of these dilated characterizations, we showed that the new approach achieves a better (no worse) upper bound for the cost functional than the conventional approach. Numerical examples illustrated that the actual cost is also improved, and the application of the new approach resulted in significant improvements over the conventional approach.

- A new approach was proposed to the robust multiobjective $H_2/\mathbf{D}$-stability controller design problems for real polytopic uncertainty, where we successfully employed non-common parameter-dependent Lyapunov variables. The dilated characterizations led us directly to this new approach. In addition, we proved that a specific choice of the scalar included in the dilated characterizations ensures the new approach to achieve a better (no worse) upper bound for the cost functional than that with the conventional approach. This choice is only a sufficient condition to ensure the advantage of the new approach, and numerical examples demonstrated that the new approach without a special care on the choice of the scalar even achieves considerably better performance than the conventional approach.

- We proposed new LMI approaches to the robust performance analysis problems for real polytopic uncertainty with the use of parameter-dependent Lyapunov variables. This was readily achieved by the dilated matrix inequality characterizations for the **D**-stability constraints, the H_2 specification and the H_∞ specification we derived. The effectiveness of the new approaches was illustrated through several numerical examples.

The above three are the most important achievements in Chapter 4, gained by the dilated matrix inequality characterizations. Specifically, it is a remarkable contribution that we directly reduced the multiobjective $H_2/\mathbf{D}$-stability problem into a convex optimization problem represented by LMI's with *non-common* Lyapunov variables in a reasonable fashion.

In closing, we describe some future topics and possible extensions of the results on the dilated characterizations obtained in this thesis. As for the H_∞ specification, although we arrived at a new dilated characterization that is suitable for the robust H_∞ performance *analysis* problems, it is desirable to derive another dilated characterization that enables us to address the H_∞ *synthesis* problems in a straightforward fashion. On the other hand, in contrast with the H_∞ specification, we actually derived new dilated characterizations for the **D**-stability constraints and the H_2 specification that are suitable for controller synthesis. These dilated characterizations are indeed useful in dealing with the multiobjective controller design problems and robust controller synthesis for real polytopic uncertainty. Furthermore, we expect that they have another potential to show new directions in such problems as fixed order dynamic output-feedback controller design, decentralized controller design and so on. In dealing with these problems, it seems indispensable to attain the decoupling between the Lyapunov variables and controller variables in the matrix inequalities, which is achieved in the dilated characterizations provided in this thesis.

Acknowledgements

The author wishes to express his sincere gratitude to Professor Tomomichi Hagiwara of the Department of Electrical Engineering, Kyoto University. His enthusiastic guidance and constant encouragement enabled the author to complete this work.

The author is also grateful to Professor Mituhiko Araki, Professor of the Department of Electrical Engineering, Kyoto University, for his kindness, advice and especially for his providing an opportunity to study control theory.

Gratitude is also due to the members of Professor Hagiwara's and Professor Araki's research groups for their assistance.

Bibliography

[1] P. Apkarian, H. D. Tuan and J. Bernussou, "Analysis, Eigenstructure Assignment and H_2 Multi-Channel Synthesis with Enhanced LMI Characterization," Proc. Conference on Decision and Control, pp. 1489–1494 (2000).

[2] P. Apkarian, P. C. Pellanda and H. D. Tuan, "Mixed H_2/H_∞ Multi-Channel Linear Parameter-Varying Control in Discrete Time," *Systems & Control Letters*, Vol. 41, pp. 333–346 (2000).

[3] R. M. Biernacki, H. Hwang and S. P. Bhattacharyya, "Robust Stability with Structured Real Parameter Perturbations," *IEEE Trans. Automatic Control*, Vol. 32, No. 6, pp. 495–506 (1987).

[4] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia (1998).

[5] X. Chen and J. T. Wen, "A Linear Matrix Inequality Approach to the General Mixed H_2/H_∞ Control Problem," Proc. American Control Conference, pp. 1443–1447 (1995).

[6] M. Chilali and P. Gahinet, "H_∞ Design with Pole Placement Constraints: An LMI Approach," *IEEE Trans. Automatic Control*, Vol. 41, No. 3, pp. 358–363 (1996).

[7] M. Chilali, P. Gahinet and P. Apkarian, "Robust Pole Placement in LMI Regions," *IEEE Trans. Automatic Control*, Vol. 44, No. 12, pp. 2257–2270 (1999).

[8] Y. Ebihara and T. Hagiwara, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: A Subspace Approach," Proc. of the 9th IFAC Symposium on Large Scale Systems: Theory and Applications, pp. 290–295 (2001).

[9] Y. Ebihara, T. Hagiwara and T. Shimomura, "Multiobjective State-Feedback Control Design with Non-common LMI Solutions: Change of Variables via Affine Functions," Proc. American Control Conference, pp. 848–853 (2001).

[10] Y. Ebihara and T. Hagiwara, "Mixed $H_2/\mathbf{D}$-stability Synthesis with New Dilated LMI Characterizations," Proc. SICE 1st Annual Conference on Control Systems, pp. 619–624 (2001).

[11] Y. Ebihara and T. Hagiwara, "Application of Dilated LMI's to $H_2/\mathbf{D}$-stability Synthesis for Real Polytopic Uncertainty," Proc. of the 30th SICE Symposium on Control Theory, pp. 263–268 (2001).

[12] E. Feron, P. Apkarian, and P. Gahinet, "Analysis and Synthesis of Robust Control Systems via Parameter-Dependent Lyapunov Functions," *IEEE Trans. Automatic Control*, Vol. 41, No. 7, pp. 1041–1046 (1996).

[13] K. Furuta and S. B. Kim, "Pole Assignment in a Specified Disk," *IEEE Trans. Automatic Control*, Vol. 32, No. 5, pp. 423–426 (1987).

[14] P. Gahinet and P. Apkarian, "A Linear Matrix Inequality Approach to H_∞ Control," *Int. J. Robust Nonlinear Control*, Vol. 4, pp. 421–448 (1994).

[15] P. Gahinet, A. Nemirovskii, A. J. Laub and M. Chilali, *LMI Control Toolbox*, The Mathworks Inc. (1995).

[16] P. Gahinet, P. Apkarian, and M. Chilali, "Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty," *IEEE Trans. Automatic Control*, Vol. 41, No. 3, pp. 436–442 (1996).

[17] J. C. Geromel, P. L. D. Peres and S. R. Souza, "H_2 Guaranteed Cost Control for Uncertain Discrete-Time Linear Systems," *Int. J. Control*, Vol. 47, No. 1, pp. 143–160 (1988).

[18] J. C. Geromel, M. C. Oliveira and L. Hsu, "LMI Characterization of Structural and Robust Stability," *Linear Algebra and its Applications*, Vol. 285, pp. 69–80 (1998).

[19] W. M. Haddad and D. S. Bernstein, "Controller Design with Regional Pole Constraints," *IEEE Trans. Automatic Control*, Vol. 37, No. 1, pp. 54–69 (1992).

[20] D. Henrion, D. Arzelier, D. Peaucelle and M. Sebek, "An LMI Condition for Robust Stability of Polynomial Matrix Polytopes," *Automatica*, Vol. 37, No. 3, pp. 461–468 (2001).

[21] H. A. Hindi, B. Hassibi and S. P. Boyd, "Multiobjective H_2/H_∞-Optimal Control via Finite Dimensional Q-parametrization and Linear Matrix Inequalities," Proc. American Control Conference, pp. 3244–3249 (1998).

[22] T. Iwasaki and R. E. Skelton, "All Controllers for the General H_∞ Control Problem: LMI Existence Conditions and State Space Formulas," *Automatica*, Vol. 30, No. 8, pp. 1307–1317 (1994).

[23] P. P. Khargonekar and M. A. Rotea, "Mixed H_2/H_∞ Control: A Convex Optimization Approach," *IEEE Trans. Automatic Control*, Vol. 36, No. 7, pp. 824–837 (1991).

[24] S. B. Kim and K. Furuta, "Regulator Design," *Int. J. Control*, Vol. 47, No. 1, pp. 143–160 (1988).

[25] K. S. Kim and F. Jabbari, "Using Scales in the Multiobjective Approach," *IEEE Trans. Automatic Control*, Vol. 45, No. 5, pp. 973–977 (2000).

[26] I. Masubuchi, A. Ohara and N. Suda, "LMI-based Controller Synthesis: A Unified Formulation and Solution," *Int. J. Robust Nonlinear Control*, Vol. 8, No. 8, pp. 669–686 (1998).

[27] T. Mori and H. Kokame, "A Parameter-Dependent Lyapunov Function for a Polytope of Matrices," *IEEE Trans. Automatic Control*, Vol. 45, No. 8, pp. 1516–1519 (2000).

[28] A. Ohara, I. Masubuchi and N. Suda, "Quadratic Stabilization and Robust Disturbance Attenuation Using Parametrization of Stabilizing State Feedback Gains: Convex Optimization Approach," Proc. American Control Conference, pp. 2816–2820 (1993).

[29] M. C. Oliveira, J. Bernussou, and J. C. Geromel, "A New Discrete-Time Robust Stability Condition," *Systems & Control Letters*, Vol. 37, pp. 261–265 (1999).

[30] M. C. Oliveira, J. C. Geromel and J. Bernussou, "An LMI Optimization Approach to Multiobjective Controller Design for Discrete-Time Systems," Proc. Conference on Decision and Control, pp. 3611–3616 (1999).

[31] M. C. Oliveira, J. C. Geromel and L. Hsu, "LMI Characterization of Structural and Robust Stability: the Discrete-time Case," *Linear Algebra and its Applications*, Vol. 296, pp. 27–38 (1999).

[32] M. C. Oliveira, J. F. Camino and R. E. Skelton, "A Convexifying Algorithm for the Design of Structured Linear Controller," Proc. Conference on Decision and Control, pp. 2781–2786 (2000).

[33] M. C. Oliveira and R. E. Skelton, "Stability Tests for Constrained Linear Systems," *Perspectives in Robust Control*, Lecture Notes in Control and Information Sciences 268 edited by S. O. Reza Moheimani, Springer (2001).

[34] D. Peaucelle, D. Arzelier, O. Bachelier and J. Bernussou, "A New Robust $\mathcal{D}$-stability Condition for Real Convex Polytopic Uncertainty," *Systems & Control Letters*, Vol. 40, pp. 21–30 (2000).

[35] D. Peaucelle and D. Arzelier, "New LMI-Based Conditions for Robust H_2 Performance Analysis," Proc. American Control Conference, pp. 317–321 (2000).

[36] C. Scherer, P. Gahinet and M. Chilali, "Multiobjective Output-Feedback Control via LMI Optimization," *IEEE Trans. Automatic Control*, Vol. 42, No. 7, pp. 896–911 (1997).

[37] C. Scherer, "An Efficient Solution to Multi-Objective Control Problems with LMI Objectives," *Systems & Control Letters*, Vol. 40, pp. 43–57 (2000).

[38] T. Shimomura and T. Fujii, "Subspace Controller Design to the Mixed H_2/H_∞ Synthesis with Uncommon LMI Solutions," Proc. American Control Conference, pp. 547–549 (1998).

[39] T. Shimomura and T. Fujii, "An Iterative Method for Mixed H_2/H_∞ Control Design with Uncommon LMI Solutions," Proc. American Control Conference, pp. 3292–3296 (1999).

[40] T. Shimomura and T. Fujii, "Multiobjective Control Design via Successive Over-Bounding of Quadratic Terms," Proc. Conference on Decision and Control, pp. 2763–2768 (2000).

[41] T. Shimomura and T. Fujii, "Extended-Space Control Design with Parameter Dependent Lyapunov Functions," Proc. SICE 1st Annual Conference on Control Systems, pp. 613–618 (2001).

[42] N. Sivashankar, I. Kaminer and P. P. Khargonekar, "Optimal Controller Synthesis with $\mathcal{D}$ Stability," *Automatica*, Vol. 30, No. 6, pp. 1003–1008 (1994).

[43] R. E. Skelton, T. Iwasaki and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis (1997).

[44] H. Tanaka and T. Sugie, "New Characterization of Fixed-Order Controllers based on LMI," *Int. J. Control*, Vol. 72, No. 1, pp. 58–74 (1999).

[45] O. Toker and H. Ozbay, "On the NP-Hardness of solving Bilinear Matrix Inequalities and Simultaneous Stabilizations with Static Output Feedback," Proc. American Control Conference, pp. 2525–2526 (1995).

[46] G. Zhai, M. Ikeda and Y. Fujisaki, "Decentralized H_∞ Controller Design: A Matrix Inequality Approach ," *Automatica*, Vol. 37, No. 4, pp. 565–572 (2001).

[47] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice Hall (1998).

List of Publications by the Author

Journal Papers

1. Y. Ebihara, T. Hagiwara and M. Araki, "Sequential Tuning Methods of LQ/LQI Controllers for Multivariable Systems and Their Application to Hot Strip Mills," *Int. J. Control*, Vol. 73, No. 15, pp. 1392–1404 (2000).

2. K. Kyoizumi, Y. Fujita and Y. Ebihara, "Simple Adaptive Control Method with Automatic Tuning of PFC and Its Application to Positioning Control of a Pneumatic Servo System" (in Japanese), *Trans. Institute of Systems, Control and Information Engineers*, Vol. 14, No. 3, pp. 102–109 (2001).

3. Y. Ebihara, T. Hagiwara and T. Shimomura, "Multiobjective State-Feedback Control Design with Non-common LMI Solutions: Change of Variables via Affine Functions," submitted for possible publication.

4. Y. Ebihara and T. Hagiwara, "Multiobjective State-Feedback Control Design with Non-common LMI Solutions: A Subspace Approach," submitted for possible publication.

Conference Papers (International)

1. Y. Ebihara, T. Hagiwara and M. Araki, "Sequential Tuning Methods of LQ/LQI Controllers for Multivariable Systems and Their Application to Hot Strip Mills," Proc. Conference on Decision and Control, pp. 767–772 (1999).

2. Y. Ebihara, T. Hagiwara and T. Shimomura, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: Change of Variables via Affine Functions," Proc. American Control Conference, pp. 848–853 (2001).

3. Y. Ebihara and T. Hagiwara, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: A Subspace Approach," Proc. of the 9th IFAC Symposium on Large Scale Systems: Theory and Applications, pp. 290–295 (2001).

4. Y. Ebihara and T. Hagiwara, "New Dilated LMI Characterizations for Continuous-Time Control Design and Robust Multiobjective Control," to appear in Proc. American Control Conference (2002).

Conference Papers (Domestic)

1. Y. Ebihara, S. Tanaka and M. Araki, "A Study on the Digital Simulation of Automatic Voltage Regulators" (in Japanese), Proc. of the 41st Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 113–114 (1997).

2. Y. Ebihara, T. Hagiwara and M. Araki, "A Study on the Sequential Design Procedure of Multi-Input Optimal Regulators" (in Japanese), Proc. of the 42nd Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 177–178 (1998).

3. Y. Ebihara, T. Hagiwara and M. Araki, "A Sequential Design Procedure for Tuning of Optimal Control Systems under Operation" (in Japanese), Proc. of SICE Annual Conference in Kansai, pp. 20–23 (1998).

4. Y. Ebihara, T. Hagiwara and M. Araki, "A Sequential Tuning Method for Multivariable Optimal Control Systems" (in Japanese), in *Basic Studies for On-Line Tuning of Multivariable Control Systems*, pp. 57–72, ISIJ (1998).

5. Y. Ebihara, T. Hagiwara and M. Araki, "Sequential Tuning Methods for Optimal Control Systems and their Application to Hot Strip Mills" (in Japanese), Proc. of poster sessions in the 137th ISIJ Meeting, p. 13 (1999).

6. Y. Ebihara, T. Hagiwara and M. Araki, "A Continuous Tuning Method of Optimal Regulator Gains" (in Japanese), Proc. of the 43rd Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 557–558 (1999).

7. Y. Ebihara, T. Hagiwara and M. Araki, "A Sequential Tuning Method of Multivariable LQI Servo Systems and Its Application to Hot Strip Mills" (in Japanese), Proc. of the 43rd Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 581–582 (1999).

8. Y. Ebihara and T. Hagiwara, "A Sequential Tuning Method of Controllers based on H_2 performance" (in Japanese), Proc. of poster sessions in the 138th ISIJ Meeting, p. 14 (1999).

9. Y. Ebihara and T. Hagiwara, "A Solution of Multiobjective Control Design Problems with Non-Common LMI Solutions" (in Japanese), Proc. of the 44th Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 281–282 (2000).

10. K. Kyoizumi, Y. Fujita and Y. Ebihara, "Simple Adaptive Control Method with Automatic Tuning of PFC and Its Application to Positioning Control of a Pneumatic Servo System" (in Japanese), Proc. of the 44th Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 7–8 (2000).

11. Y. Ebihara and T. Hagiwara, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: An Approach without Change of Variables," Proc. of the 29th SICE Symposium on Control Theory, pp. 271–276 (2000).

12. Y. Ebihara, T. Hagiwara and T. Shimomura, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: An Approach with Change of Variables via Affine Functions," Proc. of the 23rd SICE Dynamical System Theory Symposium, pp. 9–14 (2000).

13. Y. Ebihara and T. Hagiwara, "An Equivalent LMI Condition to the Lyapunov Inequality and Its Application" (in Japanese), Proc. of the 33rd SICE Annual Conference in Hokkaido, pp. 103–106 (2001).

14. T. Hosokawa, Y. Ebihara and T. Hagiwara, "Model Reduction Preserving the Steady-State Gain and Its Application to Electric Power Systems" (in Japanese), Proc. of the 45th Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 175–176 (2001).

15. K. Tokuyama, Y. Ebihara and T. Hagiwara, "Multiobjective State-Feedback Control Design with Non-Common LMI Solutions: An Iterative Algorithm" (in Japanese), Proc. of the 45th Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 415–416 (2001).

16. Y. Ebihara and T. Hagiwara, "An Equivalent LMI Characterization to the Lyapunov Inequality and Multiobjective Control Design Problems" (in Japanese), Proc. of the 45th Annual Conference of the Institute of Systems, Control and Information Engineers, pp. 417–418 (2001).

17. Y. Ebihara and T. Hagiwara, "Mixed $H_2/\mathbf{D}$-stability Synthesis with New Dilated LMI Characterizations," Proc. of the 1st SICE Annual Conference on Control Systems, pp. 619–624 (2001).

18. Y. Ebihara and T. Hagiwara, "Application of Dilated LMI's to $H_2/\mathbf{D}$-stability Synthesis for Real Polytopic Uncertainty," Proc. of the 30th SICE Symposium on Control Theory, pp. 263–268 (2001).