Harmonic Analysis of Linear Continuous-Time Periodic Systems

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Abstract

In this thesis, plants that are described by finite-dimensional linear continuous-time periodic (FDLCP) differential equations are examined via the harmonic analysis. That is, the control system concerned has a linear continuous-time state-space realization with a finite-dimensional state vector and the system matrices are periodic with respect to the time variable. The well-developed Fourier series analysis technique and its relevant theorems and lemmas are the main tools and working bases of this study but these results are utilized mainly from an operator-theoretic point of view. The discussion focus lies on analysis of a class of general FDLCP systems. Topics include: asymptotic stability and the harmonic Lyapunov equation; the frequency response operator and its properties; and the H_2 and H_∞ norms and their individual equivalence between the time-domain and frequency-domain definitions. Numerical implementations for stability criteria and norm computations, with various convergence problems taken into account, are tackled rigorously.

Previous efforts on analysis and synthesis about FDLCP systems are briefly reviewed in Chapter 1, centering around asymptotic stability, frequency-domain analysis and the H_2 and H_∞ norms. In particular, the frequency response definitions through the lifting technique, fast sampling/fast hold approximation, parametric transfer function and input/output steady-state analysis are sketched, and their individual advantages and drawbacks are pointed out and compared, while for the H_2 and H_∞ norm computations it is shown that the solutions of periodic Lyapunov and/or Riccati differential equations are also useful approaches. The basic properties of FDLCP systems such as the Floquet theorem, and several convergence lemmas about the Fourier series pertinent to our arguments are quickly summarized in Chapter 2. As further preparations, mathematical notations and preliminaries such as the Toeplitz transformation are also included in Chapter 2.

In Chapter 3, at first from the Floquet theorem and the Toeplitz transformation, the Floquet transformation on state vectors is shown to be equivalent to what we call the similarity transformation relations stated on some infinite-dimensional linear spaces (l_2 and l_1, respectively under suitable conditions) in terms of the state transition matrix knowledge of FDLCP systems. Next, by means of the similarity transformation relations, the harmonic Lyapunov equation densely defined on the Hilbert space l_2 is established for the asymptotic stability analysis of FDLCP systems for the first time. The harmonic Lyapunov equation is also useful and necessary in establishing the *exact* trace formula for the H_2 norm in FDLCP

systems, which is parallel to the trace formula that we have in linear time-invariant (LTI) continuous-time systems but in terms of infinite-dimensional input or output matrices and the solution of a corresponding harmonic Lyapunov equation. Also through the similarity transformation relations, the Gerschgorin theorem is extended to operators defined on the Hilbert space l_2, which leads to a sufficient disc-group stability condition for asymptotic stability of FDLCP systems. Again, from the similarity transformation relations, the frequency response operator is established for FDLCP systems via the input/output steady-state analysis. It is shown that the frequency response operator thus introduced is guaranteed to be densely defined on the Hilbert space l_2 and be well-defined on the whole Banach space l_1 under suitably strengthened conditions. The equivalences of the H_2 norm as well as the H_∞ norm between the time-domain and frequency-domain definitions are verified on the frequency response operator thus defined.

In contrast to the operator-theoretic arguments of Chapter 3 about the basic properties of FDLCP systems, Chapter 4 is devoted to the numerical implementation problems of FDLCP systems analysis. First, for asymptotic stability testing of FDLCP systems, an approximate modeling approach is suggested, which gives a necessary and sufficient condition if an approximate model is constructed in a dense subset and the transition matrix of the approximate model can be determined explicitly. Corollaries giving necessary and/or sufficient conditions are derived thereupon, which have lower computational loads. Second, for the H_2 and H_∞ norm computations, the skew truncation and its modification, the staircase truncation, are introduced on the frequency response operators such that these two norms can be asymptotically computed by means of finite-dimensional LTI *continuous-time* systems, while the well-known lifting technique converts the problems to those of finite-dimensional linear shift-invariant (LSI) *discrete-time* systems. Although the H_2 and H_∞ norm computations can only be asymptotically carried out, uniform convergence is ensured under mild assumptions in most practical systems. Under these mild assumptions, upper bounds for the norm computation errors can be given explicitly, which leads to size assessments inequalities for the truncations. Furthermore, the limit of the trace formula for the H_2 norm computation developed via the skew truncation on the frequency response operator goes to the *exact* trace formula developed in Chapter 3 in terms of the harmonic Lyapunov equation. On the other hand, the staircase truncation analysis makes it possible to extend the Hamiltonian test for the H_∞ norm to the FDLCP setting and thus a modified bisection algorithm is developed for the H_∞ norm computation. Finally, the H_2 and H_∞ norm computations via approximate models are also considered. There are examples to illustrate the computation efficacy for the above problems.

In the final chapter, Chapter 5, we first summarize the main contributions of this work in which an operator-theoretic harmonic analysis approach is adopted in the analysis of FDLCP systems, and then we suggest some subsequent research directions and possible extensions, and sketch difficulties in the suggested research directions.

The major contributions of this thesis contain: firstly, asymptotic stability of FDLCP

systems is connected to the harmonic Lyapunov equation and a Gerschgorin-like stability criterion is established; secondly, the existence conditions and properties of frequency response operators defined through the input/output steady-state analysis are completely clarified; thirdly, the well-definedness of the H_2 and H_∞ norms of the frequency response operators and their time-domain/frequency-domain equivalences are fully investigated and manifested. Finally, the numerical implementations of the above theoretical analysis results form another group of achievements of this study.

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Notations and Glossary

a.e.	almost everywhere
CAC	continuous and absolutely convergent
CPCD	continuous and piecewise continuously differentiable
EMP	exponentially modulated periodic
FR	frequency response
FSFH	fast sampling/fast hold
LSI	linear shift-invariant
LTI	linear time-invariant
FDLCP	finite-dimensional linear continuous-time periodic
FDLDP	finite-dimensional linear discrete-time periodic
pc	piecewise constant
PCC	piecewise continuous and convergent
PCD	piecewise continuous and differentiable a.e.

$\mathcal{C}$	the field of complex numbers
$\mathcal{F}$	the Fourier series expansion operator from $L_2[0,h]$ to l_2
$\mathcal{I}_0$	the frequency interval $[-\frac{\omega_h}{2}, \frac{\omega_h}{2})$ where $\omega_h := 2\pi/h$ (h is the period)
$\mathcal{R}$	the field of real numbers
$\mathcal{S}^+$	the set of all strictly positive-definite self-adjoint bounded operators on the Hilbert space l_2, i.e., if $S \in \mathcal{S}^+$, then $\langle S\underline{x}, \underline{x}\rangle > 0$, $\forall 0 \neq \underline{x} \in l_2$
$\mathcal{Z}$	the ring of integers
$\lambda(\cdot)$	the set of all the eigenvalues of a matrix $(\cdot)$
$\|\|\cdot\|\|$	the Euclidean norm of a finite-dimensional vector $(\cdot)$ and the matrix norm induced by this vector norm
$\underline{x}$	an infinite-dimensional vector with finite-dimensional vector entries x_k, $\underline{x} := [\cdots, x_{-1}^T, x_0^T, x_1^T, \cdots]^T$
$[\underline{x}]_k$	the k-th entry of the infinite-dimensional vector $\underline{x}$, i.e., $[\underline{x}]_k = x_k$
$[\,\cdot\,]_{(i,k)}$	the (i,k)-th entry of an infinite-dimensional matrix $(\cdot)$

C_0^1	the set of continuously differentiable functions with compact support
l_∞	the set of infinite-dimensional vectors $\underline{x}$ satisfying

$$||\underline{x}||_{l_\infty} := \sup_k ||x_k|| < \infty$$

l_p — the set of infinite-dimensional vectors $\underline{x}$ satisfying

$$||\underline{x}||_{l_p} := [\sum_{k=-\infty}^{+\infty} ||x_k||^p]^{1/p} < \infty$$

where $1 \le p < \infty$.

L_p — the set of all vector functions $x(t)$ defined on $[0,\infty)$ satisfying

$$||x(\cdot)||_{L_p} := [\int_0^\infty ||x(t)||^p dt]^{1/p} < \infty,$$

where $1 \le p < \infty$.

L_∞ — the set of all vector functions $x(t)$ defined on $[0,\infty)$ satisfying

$$||x(\cdot)||_\infty := \sup_{t\in[0,\infty)} ||x(t)|| < \infty$$

where the sup should be understood as the essential supremum.

$L_2[0,h]$ — the set of all vector functions $x(t)$ defined on $[0,h]$ satisfying

$$||x(\cdot)||_{L_2[0,h]} := [\int_0^h ||x(t)||^2 dt]^{1/2} < \infty$$

$L_\infty[0,h]$ — the set of all vector functions $x(t)$ defined on $[0,h]$ satisfying

$$||x(\cdot)||_\infty := \sup_{t\in[0,h]} ||x(t)|| < \infty$$

where the sup should be understood as the essential supremum.

$L_{\rm PCD}[0,h]$ — the set of all functions f defined on $[0,h]$ that are piecewise continuous and differentiable at a.e. $t \in [0,h]$

$L_{\rm PCC}[0,h]$ — the set of all functions f defined on $[0,h]$ that are piecewise continuous and whose Fourier series expansions are convergent to $f(t_0)$ for a.e. $t_0 \in [0,h]$

$L_{\rm CAC}[0,h]$ — the set of all continuous functions f defined on $[0,h]$ whose Fourier series expansions are absolutely convergent

$L_{\rm CPCD}[0,h]$ — the set of all continuous functions f defined on $[0,h]$ whose first-order derivatives are piecewise continuous in $[0,h]$

$L_{\rm pc}[0,h]$ — the set of all piecewise constant functions f defined on $[0,h]$

$F(t) \in L_2[0,h]$ — means that F is a matrix function, each element of which is h-periodic and belongs to $L_2[0,h]$ when its domain is restricted to $[0,h]$. Similarly for other function sets defined over $[0,h]$.

$||\cdot||_X$ — the endowed norm on the linear normed space X

$||\cdot||_{X/Y}$ — the induced norm of the operator $(\cdot)$ from Y to X

$||\cdot||_{X/Y(Z)}$ — the induced norm of the operator $(\cdot)$ from the subset Y of Z to X, where the norm $||\cdot||_Z$ endowed on the linear normed space Z is used to the subset Y, that is,

$$||\cdot||_{X/Y(Z)} := \sup_{0\ne y\in Y} \frac{||(\cdot)y||_X}{||y||_Z}$$

Chapter 1

Introduction

Quite a large class of practical control plants are such systems that can be described by periodically time-varying continuous-time models. Typical representatives are machines whose dynamic state motions have rotating characteristics such as steam turbines, alternating current electric generators and propellers of helicopters. Periodic models also come from the nature system itself; for instance, the sun rises up in the east and falls down in the west everyday; four seasons are running consecutively one round once a year; tide surges in and ebbs out monthly; and so on and so forth. Such examples are actually too numerous to mention here. In summary, periodic vibration phenomena exist ubiquitously both in artificial engineering systems and in the evolution of the nature itself. Thus, it is a quite natural and even primitive desire but actually an unavoidable task of the human being to try to understand and then apply control to systems which consist of periodically time-varying components internally or are driven by some external periodic forces. This prevalence can best explain the reason that efforts in this direction have such a long and unbroken history both in pure and applied mathematics and in the relatively young control theory as well. To reflect this historic fact, we mention several prestigious mathematicians who had made outstanding contributions to the periodic world: Parseval (1755), Fourier (1768), Faraday (1831), Mathieu (1835), Floquet (1883), Raileigh (1883), and Hill (1886). Obviously, it is impossible to retrieve all the motivations behind the long-lasting enthusiasm among researchers, and we can only mention several points from our own research experience facing the universality of cycling vibration examples around us.

In this thesis, plants whose dynamic behaviors are described by finite-dimensional linear continuous-time periodic (FDLCP) differential equations are examined via the harmonic analysis approach from an operator-theoretic viewpoint. That is, the plant concerned has a linear continuous-time state-space realization with a finite-dimensional state vector and the system matrices are all periodic with respect to the time variable t. Namely, we will consider the FDLCP system given by the state space equation

$$\begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases} \tag{1.1}$$

where the system matrices $A(t), B(t), C(t)$ and $D(t)$ are h-periodic, and x, u, y are finite-dimensional state vector, input vector and output vector, respectively. For this sort of FDLCP systems, stability analysis [25], [38], [51], [71], controllability/observability [9], [56], [80], frequency-domain properties and evaluation [39], [69], [70], [84], [85], controller synthesis [21] are frequently attacked research targets. Before any concrete scrutiny for some of these problems, we must clarify the motivations and significance of these efforts first.

- Periodic process analysis itself is quite intuitive and has simple engineering interpretations. In particular, there are inseparable connections between the time-domain properties of periodic precesses and those of the complex-domain counterparts. In fact, the Fourier analysis about periodic functions is the prelude and kernel of various frequency-domain techniques in signal, systems and control theory [10], [31], [42] by virtue of the various well-established theorems and lemmas about uniqueness, convergence, completeness and so on in the Fourier analysis [17], [28], [41], [47]. Therefore, works in periodic systems are significant in developing frequency-domain techniques for various control problems in FDLCP and linear time-invariant (LTI) systems.

- Confining only to the system control field, periodic control can bring some excellent control effects that are otherwise difficult or impossible to be realized through conventional LTI control designs; these include gain margin improvement [43] and zero assignment [50]. In particular, in the latest decade, together with the swift development of computer technology, the popular research on the sampled-data systems has aroused great interests [1], [2], [4], [5], [16], [24], [27], [34], [63], [72], [77], [78], [79], in which a continuous-time plant is closed with discrete-time controller feedback. In this sort of hybrid systems where continuous-time signals and discrete-time ones co-exist, if the input/output relations are analyzed from the continuous-time viewpoint, the sampled-data system is *periodically time-varying* because of the installation of samplers and holds. Due to this specific structure property of the sampled-data systems, it is urgently needed to find ways to deal with analysis and synthesis problems in control systems when discrete-time control algorithms are involved. These necessities mentioned above force researchers and engineers to re-evaluate the periodic control problems in a hybrid system configuration background.

- Analysis and synthesis of FDLCP control systems can be a feasible intermediate bridge between the well-established control theory for LTI and/or linear shift-invariant (LSI) systems and its possible extensions to general time-varying control systems. Indeed, discussions about periodic systems, both continuous-time and discrete-time ones, are frequently included as a chapter in most textbooks of differential and/or difference equations analysis, such as [6], [19], [25], [38], [51], [54], [59]. It is expected reasonably that if we succeed in dealing with periodic systems at a certain control problem, then we would apply the same technique to general time-varying systems via the period interval extension in some appropriate sense.

It must be pointed out that although researchers have devoted lots of their attentions and efforts to the periodic world, some quite important problems are still remaining open because it is difficult to give a closed-form and exact motion description for a general FDLCP system. This situation is best manifested by the stability problem of FDLCP systems. There are many ramifications or branch cases in stability analysis of FDLCP systems [3], [25], [53], and only those with specific state matrix structures have been understood fully [51], [54], [71]; otherwise, one has to make do with only primitive results, such as the Floquet theorem, or resort to approximate analysis. This is also the case for other problems in FDLCP systems such as controllability/observability problem [9], [56], [80] and the H_2 and H_∞ control [20], [21]. Bearing these in mind, we are in a position to survey the previous works about the topics related to FDLCP systems which will fall into the scope of this thesis. To be concise, different topics are surveyed in different sections.

1.1 Asymptotic Stability of FDLCP Systems

Compared with the stability analysis of LTI continuous-time systems, it is much harder to deal with asymptotic stability of a general FDLCP system. The difficulty comes from the fact that it is impossible to determine the state transition matrix for an FDLCP system exactly in a handy form, though there are lasting efforts [68] in this direction. The celebrated Floquet theorem reveals that such a transition matrix can be expressed in the so-called Floquet factorization form and asymptotic stability is completely determined by the eigenvalues of the corresponding monodromy matrix. Generally speaking, however, except in some special cases, for example, the system state matrix $A(t)$ is scalar and continuous with respect to the time variable [54], or $A(t)$ is piecewise constant [25], [71] or $A(t)$ is commutative [51], the monodromy matrix cannot be determined explicitly in a closed form. Facing this difficulty, many researchers turn their eyes to methods that test stability of the original FDLCP system via that of some approximate models if stability of the approximate models can be determined easily. One typical way was suggested in [38] which relies on an LTI continuous-time approximate model. This result is proved by the variation-of-constants formula about the solutions of differential equations and the well-known Gronwall's Lemma [25], [38]. The proof frame is an asymptotic analysis process of differential equations, which is a prevailing technique in tackling the stability problem in FDLCP systems.

There are also some studies which try to solve the stability problem in FDLCP systems by a frequency-domain approach. For example, in [39], [70], the Nyquist criterion is extended to closed-loop FDLCP systems via the frequency response operators. The generalized Nyquist criterion is claimed through determinant relations defined on the frequency response operators. However, since the frequency response operators are infinite-dimensional, the implementation of the extended Nyquist criterion remains open. One possible way to solve this problem is by truncating the determinant defined on the infinite-dimensional frequency response operators as suggested in [39], [70], but the convergence induced has not been warranted. As a side note of this Nyquist criterion, it should be pointed out that the well-definedness of the determinant on the frequency response operators should be scrutinized further for general FDLCP systems. The thesis of [70] did provide some arguments for this definition validity based on absolute convergence results of infinite-dimensional determinants in [53, pp. 36-38], but it is validated only when the FDLCP system concerned can be described by the canonical form of the Mathieu differential equation and thus the infinite-dimensional determinant is actually a pre-Hill determinant [70, p. 50]. Our observation about the same problem reveals that it is generally not trivial to show that the infinite-dimensional determinant thus defined belongs to the class of so-called trace-class operators [29, vol. I, pp. 104-119],[83] (for a trace-class operator G, the determinant of the operator $I + G$, i.e., $\det(I + G)$, is well-defined).

For FDLCP systems that have the Mathieu or Hill differential equations as their dynamic behavior descriptions, the stability problem has been attacked more deeply and some better understandings exist [3], [53], [59], [81]. These classes of FDLCP systems have attracted much attention in the study of the vibrations of stretched elliptical membranes, gravitationally stabilized earth pointing satellites, and the rolling motion of ships. Other important examples include the control of helicopter vibrations and wind turbines.

1.2 Frequency Responses of FDLCP Systems

There are several ways to define frequency response relations in finite-dimensional LTI continuous-time systems, for example, the steady-state analysis [64], the Fourier transform [42] and the Laplace transform [18] (via impulse convolution relations). It is well-known that these definitions are equivalent to each other when the convergence region [42] of the Laplace transform of the impulse response $g(t)$ of the LTI system concerned contains the imaginary axis, in which case the Fourier transform of $g(t)$ is meaningful in the sense that the Fourier transform is well-defined. This can be guaranteed if the LTI system is asymptotically stable with a proper rational transfer function. However, in general FDLCP systems, definitions are much more difficult because it is hard to compute the impulse response and establish the Fourier transform and/or Laplace transform relations about the input and output signals. The lifting technique [4], [5], fast sampling/fast hold approximation [44], parametric transfer function [48], [57], [82] and input/output steady-state analysis [70] are the frequently adopted approaches in the literature for the frequency response definitions in FDLCP systems. Generally speaking, each of these methods has its own advantages and drawbacks in theoretical analysis and numerical computations. The main points of these approaches are summarized as follows.

In defining frequency response relations of FDLCP systems, the *continuous-time* lifting technique is a powerful tool. By the lifting treatment, a continuous-time signal $f(\cdot) \in L_p$, $1 \leq p \leq \infty$ is first segmented into (infinitely many) sub-signals $f_k(\cdot)$. For each k, $f_k(\cdot)$ belongs to $L_p[0, h]$ and takes the value of $f(\cdot)$ during the time interval $[kh, (k + 1)h)$. Now create an (infinite-dimensional) vector $\underline{f}(\cdot) := [f_0(\cdot)^T, f_1(\cdot)^T, \cdots,]^T$, which is called the lifting of $f(\cdot)$ and denoted by $\underline{f}(\cdot) = \mathcal{W}_h f(\cdot)$. Then, it is shown [4] that the lifting operator $\mathcal{W}_h : L_p \rightarrow l^p_{L_p[0,h]}$ is invertible and $\mathcal{W}_h$ and $\mathcal{W}_h^{-1}$ are isometrically isomorphic, where $l^p_{L_p[0,h]}$ is the set of all $\underline{f}(\cdot)$ defined in the above. If the lifting is applied to the input and output signals simultaneously of the h-periodic FDLCP system (1.1), and thus we get $\underline{u}(\cdot) = \mathcal{W}_h u(\cdot)$ and $\underline{y}(\cdot) = \mathcal{W}_h y(\cdot)$, then the lifted system operator $\hat{G} : \underline{u}(\cdot) \mapsto \underline{y}(\cdot)$ commutes with the standard shift operator on $l^p_{L_p[0,h]}$. This feature can be taken as the indication of the time invariance of the operator $\hat{G}$. From this, an integral operator-valued sequence $\{\hat{G}_k\}$ can be explicitly determined to express the operator $\hat{G}$. Noting that the sequence $\{\hat{G}_k\}$ can connect the segmented input signal and the segmented output signal in a *discrete-time* convolution manner, it is natural to introduce a Z-transform to represent this relation. Based on this consideration and some results of [65], it is shown [4] that in the case of $p = 2$, $G(z) := \sum_{k=0}^{\infty} \hat{G}_k z^k$ is a well-defined operator-valued bounded analytic function (on the unit disk) and has all the standard properties of the Z-transform. From $G(z)$, the frequency response relation is defined by letting $z = e^{j\varphi h}$, which is an operator acting on $L_2[0,h]$. It is shown [74] that this frequency response relation can be interpreted in a steady-state sense. The frequency response relation defined in the above has brought fruitful applications in the sampled-data control problems [4], [5], [72], [73], [74], [75], [76]. It should be pointed out that the frequency response relation defined by the *continuous-time* lifting involves an operator-valued complex function. This brings us difficulties in such numerical computations as frequency response gains, and we will give the reasons for this difficulty in the forthcoming H_2 and H_∞ norms section of this chapter.

Another physically intuitive way to define the frequency response relations in FDLCP systems is through the input/output steady-state analysis. The general idea is completely the same as what we do in LTI continuous-time systems, but there is an essential difference. In LTI continuous-time systems, corresponding to a sinusoid wave input, the steady-state output response is also a sinusoid of the same angular frequency with a (probably) different amplitude and phase if the system is asymptotically stable. However, in stable FDLCP systems this is not the case, which can be shown by simple input/output computations [39], [70]. In fact, in stable FDLCP systems, for each sinusoid input there are infinitely many sinusoid waves of the angular frequencies that are higher or lower than that of the input sinusoid wave by integer multiples of the angular frequency corresponding to the system period. From this observation, the input signal is switched to a summation of infinitely many sinusoids called an EMP signal [70], where EMP stands for exponentially modulated periodic. To this sort of inputs, the steady-state output response is expected to be also EMP, and then the frequency response operator is introduced to connect these two EMP signals in a harmonic balance fashion. This idea has been throughly examined in establishing the so-called FR-operator (FR is abbreviated from frequency response) in sampled-data systems [2], [34], [35], [36], [37], [73]. However, in FDLCP systems, the existence conditions and properties of the frequency response operator thus defined have not been well understood because of the various convergence problems induced by the Fourier series analysis and an unboundedness property of operators related to differential operations. Our study reveals that the central problem here is how to interpret the similarity transformation relations suggested in [70].

The works of [44], [76] suggest that the frequency response relations of FDLCP systems can also be defined 'approximately' via the *discrete-time* lifting after proper 'discretization approximation' about the input and output signals. The *discrete-time* lifting has been applied for defining frequency responses in discrete-time periodic systems [30], [85], i.e., the systems described by finite-dimensional linear periodic difference equations; for brevity this class of systems will be termed FDLDP systems. The survey paper [7] provides a thorough investigation about the frequency response relations in FDLDP systems defined via the *discrete-time* lifting, cyclic, frequency lifting and Fourier analysis besides a novel notion of generalized frequency response suggested therein. However, before applying *discrete-time* lifting to define an approximate frequency response relation in an FDLCP system, the FDLCP system should be first approximated by an FDLDP model. It is the studies of [44], [76] that suggest to get such a 'discretization approximation' model via the fast sampling/fast hold (FSFH) technique. The general idea of FSFH is: by subdividing the period h into N subintervals, inputs are approximated in each subinterval by step functions. This operation is denoted by $u_a(\cdot) = \mathcal{H}_{h/N}\mathcal{S}_{h/N}u(\cdot)$ if applied on the input signal of the system (1.1), where $\mathcal{S}_{h/N}$ and $\mathcal{H}_{h/N}$ are the operators corresponding to sampling and (zero-order) hold. Outputs are likewise approximated by taking sampled values from these subintervals and denoted by $y_a(\cdot) = \mathcal{H}_{h/N}\mathcal{S}_{h/N}y(\cdot)$. Then an approximate input/output relation of the system (1.1) can be given by $\mathcal{H}_{h/N}\mathcal{S}_{h/N}\mathcal{G}\mathcal{H}_{h/N}\mathcal{S}_{h/N} : u(\cdot) \mapsto y_a(\cdot)$, where $\mathcal{G}$ denotes the mapping from $u(\cdot)$ to $y(\cdot)$ in the system (1.1). It is clear by the Floquet theorem and simple input/output analysis that the approximated system (more precisely, the discretized system $\mathcal{S}_{h/N}\mathcal{G}\mathcal{H}_{h/N} : u_d(\cdot) \mapsto y_d(\cdot)$, where $u_d(\cdot)$ and $y_d(\cdot)$ are the sampled-data counterparts of $u(t)$ and $y(t)$) then turns out to be a finite-dimensional discrete-time periodic (FDLDP) system in the approximated input/output sense, and then the frequency response of this FDLDP approximate model can be defined via the *discrete-time* lifting or other approaches as suggested in [7] and be represented by a finite-dimensional matrix. Different from the definitions by the *continuous-time* lifting and the input/output steady-state analysis, the frequency response defined via the FSFH approximation and *discrete-time* lifting is only an approximation of that of the original FDLCP system at most, although the matrix expression of the latter frequency response definition is explicit. It is expected intuitively that as $N \to \infty$, the frequency response goes to that of the FDLCP system. However, the installation of sampling and holding processes inevitably imposes some constraints both on the structure of the original FDLCP system and the admissible input signals since the operator $\mathcal{H}_{h/N}\mathcal{S}_{h/N}$ is guaranteed to work well on signals that are relatively smooth [14]. In other words, to ensure the desired convergence in some specific sense, some extra conditions on the system structure and input signals are needed. Unfortunately, these problems have not been fully understood in the literature. Further explanations about these problems are given in the H_2 and H_∞ norm section of this chapter.

Parametric transfer function is another worthwhile method to define frequency relations for time-varying systems [82], in particular for FDLCP systems and sampled-data systems [48]. The parametric transfer function $w(t, s)$ for a time-varying system is defined by a τ-variable Laplace transform on the impulse response $g(t, \tau)$ of the system, i.e., $w(t, s) := \int_0^\infty g(t, \tau)e^{-s\tau}d\tau$ for each fixed parameter t under appropriate convergence conditions. It is shown that the parametric transfer function $w(t, s)$ possesses general properties similar to those of the standard transfer function for LTI continuous-time systems [57], [58]. Thus, hopefully, by letting $s = j\omega$ in $w(t, s)$ a frequency-domain relation $w(t, j\omega)$ is established between input and output. This idea does work in FDLCP systems, at least in theory, since in this case the time-domain input/output relation is a Volterra integral operator (see, e.g., [55] for the definition of Volterra integral operators) by the Floquet theorem [61], so that the single-variable Laplace transform introduced in the above will become well-defined if the convergence conditions for the Laplace transform and the conditions for integral-order interchanges involved are satisfied. Unfortunately, however, the definition of the parametric transfer function (and thus its corresponding frequency-domain relation) also relies on the transition matrix knowledge, which is not easy to calculate by a handy and closed-form formula in general FDLCP systems.

1.3 H_2 and H_∞ Norms of FDLCP Systems

The H_2 and H_∞ norms are used to quantify system performances and as objectives for control system synthesis [32], [84], [85], [91]. Their computations in LTI systems have been solved respectively by the trace formula involving the solution of algebraic Lyapunov equations, and by the solution of algebraic Riccati equations according to the well-known bounded real lemma or the Hamiltonian test [32]. However, in FDLCP systems, the computations are much more difficult. The well-known lifting technique, differential equation solutions, fast sampling/fast hold approximation, parametric transfer function approach and truncations on frequency response operators defined via steady-state analysis are the frequently adopted approaches in the literature.

By the lifting technique [4], [5], the H_2 and H_∞ norms of periodic systems can be computed with some corresponding 'equivalent' LSI discrete-time systems. As one of the most successful applications of this technique, in sampled-data systems which are also periodic [24], explicit formulas for the H_2 and H_∞ norm computations are given in terms of corresponding 'equivalent' LSI discrete-time systems [4], [5], [13], [15], [72]. However, no readily and numerically implementable algorithms are available if the systems are FDLCP. To be more precise, by the *continuous-time* lifting, an FDLCP system can be represented by an *operator-valued* shift-invariant discrete-time system (with a finite-dimensional state space while the input and output spaces are infinite-dimensional) equivalently in the H_2 and H_∞ norm sense (see also Section 1.2 in the above). It is based on this *operator-valued* shift-invariant discrete-time system that the equivalences between the frequency-domain H_2 and H_∞ norms and their time-domain counterparts are verified. Unfortunately, however, this *operator-valued* shift-invariant discrete-time model is by no means numerically implementable, though the finite-dimensional state space structure guarantees that this *operator-valued* shift-invariant discrete-time system can have an equivalent finite-dimensional LSI discrete-time state space realizations through operator composition computations. However, the operator compositions involved are neither explicit nor trivial in a general FDLCP system if one notes that the monodromy matrices of some augmented h-periodic state matrices are needed in the operator composition computations [4], [5].

Another available method for the H_2 and H_∞ norm computations of FDLCP systems is through solutions of differential equations. For example, for the H_2 norm computation, it can be done by solving a periodic Lyapunov equation and doing integration of a certain trace function about the solution [20]. The existence of the solution of the periodically time-varying Lyapunov equations can be guaranteed under some standard assumptions [9]. In general, the solutions can be determined only numerically. As for the H_∞ norm, the well-known bounded real lemma leads us to the necessary and sufficient Hamiltonian test [20] for the H_∞ norm of the FDLCP system (1.1) to be less than or equal to a prescribed positive scalar γ. This Hamiltonian test is stated via an associated h-periodic Hamiltonian matrix $H(t,\gamma)$. Hence the H_∞ norm can be computed to any degree of accuracy via a bisection algorithm by checking if $H(t,\gamma)$ has characteristic multiplier (see Remark 2.1 for its definition) on the unit circle. In general, this method also needs repeated numerical computations of the monodromy matrices corresponding to $H(t,\gamma)$ because of the iterative steps with respect to the prescribed scalar γ. Another celebrated contribution of the differential equation approach is that the parameterization of state-feedback H_2 and H_∞ controllers in FDLCP systems is solved [21].

There are also efforts to compute the H_2 norm by the parametric transfer functions of FDLCP systems [48], which lead to a closed-form formula for the H_2 norm. This formula is stated by defining a so-called correlation function of the parametric transfer function $w(t,s)$, which is given by the integral process $B_0(s) := (1/h)\int_0^h w(t,-s)w^*(t,s)dt$. Therefore, the H_2 norm formula thus derived is actually a multiple integral about the complex function $w(t,s)$ so that its numerical implementation is not so simple, besides the computation problem of an infinite summation defined on the above correlation function $B_0(s)$. In addition, how to compute the H_∞ norm via the parametric transfer function has not yet been discussed, and the equivalence of the H_2 norm defined on the parametric transfer function with the usual time-domain counterpart remains to be an open problem.

As for the H_2 and H_∞ norm computations of FDLCP systems by the frequency response operator defined via input/output steady-state analysis [39], [69], [70], the numerical implementation is also not trivial since the frequency response operators are infinite-dimensional. To solve this problem, the square truncation is proposed in [69]. However, its convergence has not been verified, which is nontrivial especially when the operator involved is non-compact. There have been no discussions to clarify the relations between the original FDLCP frequency response operator and the square truncated one, either. The possible reasons may be attributed to the fact that the square truncation neglects the 'symmetrical' mathematical structure of the frequency response operator, which makes such discussions hard.

One can also consider the norm computations via the fast sampling/fast hold (FSFH) approximation of the frequency response of FDLCP systems. The FSFH approach is first proposed in [44] and recently is applied by [76] to the frequency response approximation in sampled-data systems. As we have seen in Section 1.2, it is naturally expected that the frequency response relation defined via the FSFH treatment in an FDLCP system approaches that of the original FDLCP system as $N \to \infty$, and thus so do the H_2 and H_∞ norms. To show these convergences is seemingly trivial. However, the works of [14], [76] show that this is actually not very easy. One of the difficulties we would encounter is how to ensure that in the FDLCP system (1.1), as $N \to \infty$, both the imposed input $u(t)$ and the corresponding output $y(t)$ can be suitably approximated by the FSFH approximation. This becomes a serious question because of the introduction of the operator $\mathcal{H}_{h/N}\mathcal{S}_{h/N}$, which is unbounded on L_p for any $1 \le p < \infty$ [14]. For example, in the H_∞ norm case, the worst-case input/output is relevant. A natural question is if these 'worst' input and output can be properly represented by their FSFH counterparts. Unfortunately, however, no one can know the 'worst' input and output in advance. It is well-known that a proper representation of a signal can be ensured if the signal is relatively smooth [14], [16]. Bearing this in mind, it follows naturally that to satisfy this smoothness requirement (and therefore the convergence desired) in a general FDLCP system, the input should be confined to an admissible signal set to satisfy the well-behavedness of the operator $\mathcal{H}_{h/N}\mathcal{S}_{h/N}$; at the same time the system concerned should be of certain structure such that even the worst-case output signal (for the H_∞ norm) can be properly approximated. In less rigorous words, the FSFH approach would work well if the frequency response of the original system is low-pass and the input signal is chosen from a set of signals that are relatively smooth. Some similar problems also appear in the H_2 norm computation. It is evident that the FSFH operator $\mathcal{H}_{h/N}\mathcal{S}_{h/N}$ does not behave well on the δ-function no matter what N is taken since the δ-function is neither smooth nor band-limited. Hence, the approximation error between the actual impulse response of the original FDLCP system and the FSFH approximation is hard to be assessed in the time-domain. Therefore, a time-domain proof for the H_2 norm convergence seems to be nontrivial. We believe that a frequency-domain proof for the desired convergence also needs much more involved discussions about the relations among the frequency responses defined via the FSFH treatment, continuous-time lifting and input/output steady-state analysis, and these discussions are of independent significance from the experience of this author. However, we will not probe into these topics in the main context of this thesis.

Finally, it must be pointed out that there exist no available formulas or algorithms in the literature established via an FSFH approach for the H_2 and H_∞ norms computations in the FDLCP setting. Thus, the real purpose to include the above paragraph here is to show some considerations about why the FSFH approach has not been successful so far, instead of a survey about the FSFH approach and its application.

1.4 Scope of the Thesis

Having given a survey on the existing studies on FDLCP systems in the preceding sections, we will concentrate our attention in the forthcoming chapters only on the analysis of FDLCP systems through the harmonic analysis approach both theoretically and numerically. Before our formal discussions, the contents of the rest of the thesis are sketched as follows.

The basic properties of FDLCP systems such as the well-known Floquet theorem and the principal results of the Fourier series analysis closely related to our arguments are quickly summarized in Chapter 2. As further preparations, some other mathematical notations and preliminaries such as the Toeplitz transformation are also included in Chapter 2.

In Chapter 3, at first from the Floquet theorem and the Toeplitz transformation, it is shown that the Floquet transformation about the state vector can be equivalently expressed as the similarity transformation relations stated on some infinite-dimensional linear spaces (l_2 and l_1, respectively under suitable conditions) in terms of the transition matrix of an FDLCP system. Next, by means of the similarity transformation relations, the harmonic Lyapunov equation densely defined on the linear space l_2 is established for the asymptotic stability analysis of FDLCP systems for the first time. The proof arguments are given only through simple matrix algebra so that the existence problem of steady-state periodic solutions of a periodically time-varying Lyapunov differential equation [9] is circumvented completely. The harmonic Lyapunov equation is helpful in proving a stability criterion for FDLCP systems based on approximate modeling in Chapter 4, and this equation is also useful and necessary in establishing the *exact* trace formula for the H_2 norm in FDLCP systems, which is parallel to the trace formula expression that we have in LTI continuous-time systems but in terms of infinite-dimensional input and/or output matrices and the solution of a corresponding harmonic Lyapunov equation. Also through the similarity transformation relations, the Gerschgorin theorem is extended to operators defined on the Hilbert space l_2, which leads to a sufficient disc-group stability condition for FDLCP systems. Next, again from the similarity transformation relations, the frequency response operators are established for FDLCP systems via the input/output steady-state analysis. It is shown that the frequency response operator thus introduced is guaranteed to be densely defined on the Hilbert space l_2 and be well-defined on the whole Banach space l_1 under suitably strengthened conditions. The respective equivalences about the H_2 and H_∞ norm between the time-domain and frequency-domain definitions are verified when the frequency response operator thus defined is used in their frequency-domain definitions.

In contrast to the operator-theoretic arguments of Chapter 3, Chapter 4 is devoted to the numerical implementations of the results in Chapter 3. First, for asymptotic stability of

FDLCP systems, an approximate modeling approach is suggested, which yields a necessary and sufficient condition if the transition matrix of an approximate model can be determined explicitly. Here, the sufficiency proof is via the harmonic Lyapunov equation, while the necessity one follows from the Gronwall's Lemma and the variation-of-constants formula [38]. Several corollaries giving necessary and sufficient conditions are derived thereupon, which relax the requirements on the transition matrices of approximate models. Second, for the H_2 and H_∞ norm computations, the skew truncation and its modification, the staircase truncation, are introduced on the frequency response operator such that these two norms can be asymptotically computed by means of finite-dimensional LTI *continuous-time* systems, while the lifting technique converts the problems to those of finite-dimensional LSI *discrete-time* systems. Although the computations are only asymptotically carried out, uniform convergence can be easily ensured under mild assumptions in most practical systems. Upper bounds of computation errors can be given under these mild conditions so that in most practical FDLCP systems, it is possible to assess the truncation size in advance. Moreover, the limit of the asymptotic trace formula for the H_2 norm computation developed via the skew truncation on the frequency response operator is shown to go to the *exact* trace formula developed in Chapter 3 through the infinite-dimensional harmonic Lyapunov equation. On the other hand, the staircase truncation analysis also makes it possible to extend the Hamiltonian test for the H_∞ norm to the FDLCP setting and thus a modified bisection algorithm is developed for the H_∞ norm computation. Finally, the H_2 and H_∞ norm computations via approximate models are also considered. There are numerical examples to illustrate the efficacy of the numerical implementation algorithms.

In Chapter 5, we summarize the main results of this thesis and point out the problems that have not been solved up to the present stage, and sketch the difficulties we have encountered in solving them. Finally we move on to suggest some possible subsequent research topics.

Chapter 2

Preliminaries to FDLCP Systems

The purpose of this chapter is to lay the mathematical foundations for the subsequent discussions. In Section 2.1, the state-space description of finite-dimensional linear continuous-time periodic (FDLCP) systems is presented first and the Floquet theorem is reviewed. Remarks about the Floquet theorem are given, which play a key role in understanding the transition matrix and asymptotic stability of a general FDLCP system. Next in Section 2.2, several convergence lemmas about the Fourier series expansions of periodic functions are quoted from textbooks. Based on these preparations, the Toeplitz transformation is re-defined rigorously and several important lemmas and propositions stated via the Toeplitz transformation for FDLCP systems are proved thereafter. These lemmas and propositions are the main tools in establishing the similarity transformation formulas of FDLCP systems in Section 2.3 and useful in discussing the eigenvalue structure of FDLCP systems in Section 2.4. These lemmas and propositions will also be used to assure various convergence and validity in theoretical analysis and numerical computations in the coming chapters.

2.1 FDLCP Systems and the Floquet Theorem

Consider the FDLCP system

$$G : \begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases} \tag{2.1}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m$ and $y \in \mathcal{R}^l$ are the state vector, input vector and output vector, respectively. Accordingly, $A(t), B(t), C(t)$ and $D(t)$ are the $n \times n$ state matrix, $n \times m$ input matrix, $l \times n$ output matrix and $l \times m$ feedthrough matrix, which are h-periodic time-varying matrices. The transition matrix of the system (2.1) is denoted by $\Phi(t, t_0)$ when the initial time is t_0. The system (2.1) is said to be strictly proper if $D(t) \equiv 0, \forall t \in [0, h]$. In the following, all the dimensionality subscripts will be suppressed if no confusion is caused.

Theorem 2.1 (Floquet Theorem [38], [51], [61]) *Assume in the system (2.1) that* $A(t) \in L_1[0, h]$. *Then the transition matrix* $\Phi(t, t_0)$ *is continuous with respect to* t *and can be expressed as* $\Phi(t, t_0) = P(t, t_0)e^{Q(t-t_0)}$ *where* $P(t, t_0)$ *is absolutely continuous in* t*, nonsingular and* h*-periodic both in* t *and* t_0*, and* Q *is a constant-matrix. Moreover, the system is asymptotically stable if and only if the eigenvalues of the monodromy matrix,* $\Phi(h + t_0, t_0)$*, are in the open unit disk, or equivalently, the eigenvalues of* Q *lie in the open left-half plane.*

Remark 2.1 *In [19], [38], [51], [56], [59], the eigenvalues of the monodromy matrix* $\Phi(h + t_0, t_0)$ *are also called characteristic multipliers of* $A(t)$ *while the eigenvalues of the matrix* Q *are named characteristic exponents of* $A(t)$*. The characteristic exponents are unique in the sense of* modulo $(2\pi j/h)$*, and the characteristic multipliers are actually independent of the initial time* t_0*. Hence it would lose no generality to let* $t_0 = 0$ *in the discussions so that we will take a zero initial time in general. Noting that absolute continuity implies continuity [55], [60], it follows that* $P(t, t_0)$ *is continuous with respect to* t*. Generally speaking,* Q *is complex (derived through a matrix logarithm [19]) and this may bring difficulties in certain practical design problems. This difficulty can be overcome by resorting to a real Floquet factorization, that is,* Q *can be given by an appropriate real matrix, as stated in Corollary 8.1.4 of [51]. In [81], Theorem* 2.1 *is named the Floquet-Lyapunov theorem, in which it is also asserted that for any arbitrary constant matrix* Q *and* h*-periodic matrix* $P(t, 0)$ *that is nonsingular for all* t*, continuous, and has a piecewise-continuous derivative, there is some* h*-periodic system whose transition matrix is* $P(t, 0)e^{Qt}$*.*

Remark 2.2 *In the literature there are two ways to express the Floquet factorization (or decomposition) of the transition matrix of an* FDLCP *system. One way is as we have stated in Theorem* 2.1*. In some references [56], [61], one can also see the Floquet factorization in the form of* $\Phi(t, t_0) = P(t)e^{Q(t-t_0)}P^{-1}(t_0)$*. It is worth mentioning that these two forms are actually equivalent. This can be proved as follow.*

From Theorem 2.1*, it is clear that* $\Phi(t, 0) = P(t, 0)e^{Qt}$ *and* $\Phi(t_0, 0) = P(t_0, 0)e^{Qt_0}$*. Then, we obtain from the basic properties of a transition matrix that*

$$\begin{aligned}\Phi(t, t_0) &= \Phi(t, 0)\Phi(0, t_0) = \Phi(t, 0)\Phi^{-1}(t_0, 0) \\ &= P(t, 0)e^{Qt}\big(P(t_0, 0)e^{Qt_0}\big)^{-1} = P(t, 0)e^{Q(t-t_0)}P^{-1}(t_0, 0)\end{aligned}$$

If we rewrite $P(t) = P(t, 0)$*, then the equivalent expression follows immediately.*

Therefore, the Floquet factorization of the transition matrix $\Phi(t, t_0)$ *of the* FDLCP *system (2.1) always means that* $\Phi(t, t_0) = P(t, t_0)e^{Q(t-t_0)}$ *in the discussions of this thesis.*

The Floquet theorem is merely an existence theorem and as such is useful in theoretical work. However, the computation for the Floquet factorization pair $(P(t, t_0), Q)$ of the state transition matrix of a general FDLCP system is usually difficult to the best knowledge of the author except in the cases when the state matrix $A(t)$ has special structures; for example, $A(t)$ is a scalar and continuous (Theorem 2.4.1 in [54]), $A(t)$ is piecewise constant ([25], [71]) or $A(t)$ is commutative ([51]). It is worth mentioning that in the last case, only the computation of Q is reduced to the 'DC' matrix computation about $A(t)$ on $[0, h]$, and the computation for the periodic portion $P(t, t_0)$ is still difficult in general.

Combining the Floquet theorem with Theorem 6.3.2 of [51], simple deductions yield

$$\begin{cases} P(t,0) = \Phi(t,0)e^{-Qt}, & \frac{d}{dt}P(t,0) = [A(t)\Phi(t,0) - \Phi(t,0)Q]e^{-Qt} \quad \text{(a.e.)} \\ P^{-1}(t,0) = e^{Qt}\Phi(0,t), & \frac{d}{dt}P^{-1}(t,0) = e^{Qt}[Q\Phi(0,t) - \Phi(0,t)A(t)] \quad \text{(a.e.)} \end{cases} \tag{2.2}$$

The equations of (2.2) play a key role in analyzing the convergence properties of the Fourier series expansions of the periodic functions $P(t, 0)$ and $P^{-1}(t, 0)$.

Based on the Floquet theorem, introduce the state transformation $\hat{x} = P^{-1}(t, 0)x$ to the FDLCP system (2.1). Then it follows readily after simple derivations that

$$\hat{G} : \begin{cases} \dot{\hat{x}} = Q\hat{x} + \hat{B}(t)u \\ y = \hat{C}(t)\hat{x} + D(t)u \end{cases} \tag{2.3}$$

with the matrices $\hat{B}(t)$ and $\hat{C}(t)$ given by

$$\hat{B}(t) := P^{-1}(t,0)B(t), \quad \hat{C}(t) = C(t)P(t,0) \tag{2.4}$$

It is clear that the FDLCP system (2.3) is equivalent to that of (2.1) in the Lyapunov sense. That is, the system (2.1) is asymptotically stable if and only if the system (2.3) is. Another important structure feature of the system (2.3) is that the state matrix is a constant matrix. In the literature, the state transformation $\hat{x} = P^{-1}(t, 0)x$ is called the Floquet state transformation, which brings some mathematical convenience in the discussions.

2.2 Fourier Series and the Toeplitz Transformation

In this section, the Toeplitz transformation is introduced and its validity is considered. Some results from the Fourier series analysis that pertain to the subsequent arguments are summarized simply as mathematical preparations. To this end, first let us assume that $X(t)$ is an h-periodic time-varying matrix function belonging to $L_2[0, h]$. Now define

$$\omega_h = 2\pi/h$$

and expand $X(t)$ to its Fourier series expansion, i.e., $X(t) = \sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_h t}$, which is well-defined in the sense that

$$||X(\cdot) - \sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_h(\cdot)}||_{L_2[0,h]} = 0$$

The Toeplitz transformation on $X(t)$ [70], denoted by $\mathcal{T}\{X(t)\}$, maps the matrix function $X(t) \in L_2[0,h]$ into a doubly infinite-dimensional block Toeplitz operator [70] (or to be more precise, block Laurent operator [29, Vol. II, p. 564]) of the form

$$\mathcal{T}\{X(t)\} := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & X_0 & X_{-1} & X_{-2} & \cdots \\ \cdots & X_1 & X_0 & X_{-1} & \cdots \\ \cdots & X_2 & X_1 & X_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} =: \underline{X} \tag{2.5}$$

It is straightforward to show that

$$\mathcal{T}\{X(t)+Y(t)\} = \mathcal{T}\{X(t)\} + \mathcal{T}\{Y(t)\}$$

when $X(t)$ and $Y(t)$ are h-periodic and belong to $L_2[0,h]$. However, the situation is different for the Toeplitz transformation of the product of two h-periodic matrix functions. To clarify the conditions under which the Toeplitz transformation can be interchanged with matrices multiplication computations, the Mertens theorem is stated below.

Lemma 2.1 (Mertens Theorem [12], [45]) *Let $\sum_{m=0}^{\infty} a_m$ and $\sum_{m=0}^{\infty} b_m$ be two convergent infinite series and let $c_m = \sum_{p+q=m} a_p b_q$. The series $\sum_{m=0}^{\infty} c_m$ is called the Cauchy product of $\sum_{m=0}^{\infty} a_m$ and $\sum_{m=0}^{\infty} b_m$. Provided that one of the infinite series $\sum_{m=0}^{\infty} a_m$ and $\sum_{m=0}^{\infty} b_m$ is absolutely convergent, then $\sum_{m=0}^{\infty} c_m$ is convergent and satisfies*

$$\sum_{m=0}^{\infty} c_m = \sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} b_n$$

It is worth mentioning that by the proof of the Mertens theorem provided in [12], [45], it is nontrivial to extend this theorem to two-sided infinite series like $\sum_{m=-\infty}^{\infty} a_m$. Because in this latter case, the Cauchy product involves terms of infinite summations so that an iterative relation of partial summations fails to hold (this relation is an essential point of that proof). This is why we give a lengthy and seemingly redundant proof for the following lemma in which the Mertens theorem is applied. By Lemma 2.1, the interchangeability problem we mentioned in the above can be solved. Now we consider two compatible h-periodic matrix functions $X(t)$ and $Y(t)$ with the Fourier series expansions $X(t) = \sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_h t}$ and $Y(t) = \sum_{m=-\infty}^{+\infty} Y_m e^{jm\omega_h t}$, respectively. Then we have the following lemma, in which it is implicitly assumed that both $X(t)Y(t)$ and $Y(t)X(t)$ make sense.

Lemma 2.2 *Suppose that the Fourier series expansion of $X(t) \in L_2[0,h]$ converges to $X(t_0)$ for almost every (a.e.) $t_0 \in [0,h]$. Also suppose that $Y(t) \in L_2[0,h]$ is continuous and the Fourier series expansion of $Y(t)$ is absolutely convergent. Then, the Fourier series expansion of $X(t)Y(t)$ (or that of $Y(t)X(t)$, respectively) converges to $X(t_0)Y(t_0)$ ($Y(t_0)X(t_0)$, respectively) for a.e. $t_0 \in [0,h]$, and*

$$\mathcal{T}\{X(t)Y(t)\} = \mathcal{T}\{X(t)\}\mathcal{T}\{Y(t)\}, \quad \mathcal{T}\{Y(t)X(t)\} = \mathcal{T}\{Y(t)\}\mathcal{T}\{X(t)\}$$

Proof By the absolute convergence of the Fourier series expansion of $Y(t)$, it follows that it is uniformly convergent with respect to t over $[0, h]$. Thus, the Fourier series expansion of $Y(t)$ defines a continuous function over $[0, h]$. By the property of the Fourier series expansion as noted above, together with the continuity of $Y(t)$, it follows that this continuous function is nothing but $Y(t)$. In other words, for every $t_0 \in [0, h]$, the Fourier series expansion of $Y(t)$ converges to $Y(t_0)$. Now rewrite the Fourier series expansions of $X(t)$ and $Y(t)$ as

$$X(t) = \sum_{m=0}^{+\infty} X_m(t), \quad Y(t) = \sum_{m=0}^{+\infty} Y_m(t)$$

with $X_0(t) = X_0$ and $X_m(t) = X_m e^{jm\omega_h t} + X_{-m}e^{-jm\omega_h t}, m = 1, 2, \cdots$, and $Y_m(t)$, $m = 0, 1, 2, \cdots$ are defined similarly. Hence, by the assumption on $X(t)$ and the Mertens theorem, for each $t_0 \in [0, h]$ at which the Fourier series expansion of $X(t)$ converges to $X(t_0)$, we have

$$X(t_0)Y(t_0) = \Big(\sum_{m=0}^{+\infty} X_m(t_0)\Big)\Big(\sum_{m=0}^{+\infty} Y_m(t_0)\Big) = \sum_{m=0}^{+\infty} Z_m(t_0) \tag{2.6}$$

where $Z_m(t_0) = \sum_{u+v=m} X_u(t_0)Y_v(t_0)$. That is, the right-hand side of (2.6) is the Cauchy product of $\sum_{m=0}^{+\infty} X_m(t_0)$ and $\sum_{m=0}^{+\infty} Y_m(t_0)$. Simple computations lead us to

$$\begin{aligned}
Z_0(t_0) &= X_0Y_0 \\
Z_1(t_0) &= (X_0Y_1 + X_1Y_0)e^{j\omega_h t_0} + (X_0Y_{-1} + X_{-1}Y_0)e^{-j\omega_h t_0} \\
Z_2(t_0) &= (X_0Y_2 + X_1Y_1 + X_0Y_2)e^{j2\omega_h t_0} + (X_1Y_{-1} + X_{-1}Y_1) \\
& \quad +(X_0Y_{-2} + X_{-1}Y_{-1} + X_{-2}Y_0)e^{-j2\omega_h t_0} \\
\cdots & \quad \cdots\cdots\cdots
\end{aligned}$$

Now we construct the following array from the above computation results.

$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	$\cdots$
0	0	$(X_0Y_{-2} + X_{-1}Y_{-1} + X_{-2}Y_0)e^{-j2\omega_h t_0}$	$\cdots$
0	$(X_0Y_{-1} + X_{-1}Y_0)e^{-j\omega_h t_0}$	0	$\cdots$
X_0Y_0	0	$(X_1Y_{-1} + X_{-1}Y_1)$	$\cdots$
0	$(X_0Y_1 + X_1Y_0)e^{j\omega_h t_0}$	0	$\cdots$
0	0	$(X_0Y_2 + X_1Y_1 + X_0Y_2)e^{j2\omega_h t_0}$	$\cdots$
0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For simplicity, the (p, q)-th entry in this array is simply denoted by a_{pq}. Here, $\{a_{pq}\}$ with $p \in \mathcal{Z}$ and $q = 0, 1, 2, \cdots$ forms a double sequence [12]. It is obvious that the summation of all the entries in the q-th column is just $Z_q(t_0), q = 0, 1, 2, \cdots$ and $\sum_{q=0}^{\infty} Z_q(t_0)$ is nothing but the sum of the repeated-series-by-columns of the double sequence $\{a_{pq}\}$ according to the terminology of [12], [52]. In other words, we obtain

$$\sum_{q=0}^{+\infty}\Big(\sum_{p=-\infty}^{+\infty} a_{pq}\Big)=\sum_{q=0}^{\infty} Z_q(t_0)$$

Similarly, $\sum_{p=-\infty}^{+\infty}\left(\sum_{q=0}^{+\infty} a_{pq}\right)$ is called the sum of the repeated-series-by-rows of the given double sequence. To our current purpose, we further consider the so-called rectangular partial summation defined on the double sequence $\{a_{pq}\}$, which is given by

$$S_{lr} := \sum_{p=-l}^{l}\sum_{q=0}^{r} a_{pq}$$

According to [52], the convergence of the sum of the repeated-series-by-rows as well as that of the repeated-series-by-columns can be ensured based on the convergence of the rectangular partial summation. To exploit this fact, we first aim at establishing the convergence of the rectangular partial summation, which can be guaranteed by the convergence of the sum of the repeated-series-by-columns by the specific structure of the double sequence $\{a_{pq}\}$. Indeed, it is easy to see that if $l \geq r$, then $S_{lr} = \sum_{q=0}^{r} Z_q(t_0)$. Hence, the convergence of $\sum_{q=0}^{+\infty} Z_q(t_0)$ (by the Mertens Theorem under the given assumptions) implies that the double sequence $\{S_{lr}\}$ itself is convergent to $\sum_{q=0}^{+\infty} Z_q(t_0)$, which in turn is called the sum of the double sequence $\{a_{pq}\}$ by definition and can be expressed as

$$S := \sum_{q=0}^{\infty} Z_q(t_0)$$

On the other hand, for any fixed $p \in \mathcal{Z}$, the p-th row of the array $\{a_{pq}\}$ satisfies

$$\sum_{q=0}^{+\infty} a_{pq} = \sum_{k=-\infty}^{+\infty} X_{p-k}Y_k e^{jp\omega_h t_0}$$

Then it is clear by the Cauchy-Schwarz inequality and the Parseval theorem that

$$||\sum_{q=0}^{+\infty} a_{pq}|| \leq \sum_{k=-\infty}^{+\infty} ||X_{p-k}||\cdot||Y_k|| \leq [\sum_{k=-\infty}^{+\infty} ||X_{p-k}||^2]^{1/2}[\sum_{k=-\infty}^{+\infty} ||Y_k||]^{1/2} < \infty$$

which says that the array $\{a_{pq}\}$ is convergent for each row. Summarizing these arguments, the conclusion of [52] tells us that the sum of the repeated-series-by-rows of $\{a_{pq}\}$, i.e., $\sum_{p=-\infty}^{+\infty}\left(\sum_{q=0}^{\infty} a_{pq}\right)$, is also convergent and the sum of the repeated-series-by-rows is equal to S. That is

$$\sum_{p=-\infty}^{+\infty}\Big(\sum_{q=0}^{\infty} a_{pq}\Big) = \sum_{p=-\infty}^{+\infty}\Big(\sum_{k=-\infty}^{+\infty} X_{p-k}Y_k\Big)e^{jp\omega_h t_0} = S = \sum_{q=0}^{\infty} Z_q(t_0)$$

Using this relation in (2.6), we eventually obtain

$$X(t_0)Y(t_0) = \sum_{p=-\infty}^{+\infty}\Big(\sum_{k=-\infty}^{+\infty} X_{p-k}Y_k\Big)e^{jp\omega_h t_0} \tag{2.7}$$

which holds for almost all $t_0 \in [0,h]$ by the assumption on $X(t)$. Noting that it is in the form of the Fourier series expansion, it immediately follows that $\{\sum_{k=-\infty}^{+\infty} X_{p-k}Y_k\}_{m=-\infty}^{+\infty}$ is indeed the Fourier coefficients sequence of $X(t)Y(t)$, because we readily have

$$\Big\|X(\cdot)Y(\cdot) - \sum_{p=-\infty}^{+\infty}\Big(\sum_{k=-\infty}^{+\infty} X_{p-k}Y_k\Big)e^{jp\omega_h(\cdot)}\Big\|_{L_2[0,h]} = 0$$

(since (2.7) holds for a.e. $t_0 \in [0,h]$) and the Fourier series expansion is unique. This gives $\mathcal{T}\{X(t)Y(t)\} = \mathcal{T}\{X(t)\}\mathcal{T}\{Y(t)\}$. Similarly for $\mathcal{T}\{Y(t)X(t)\} = \mathcal{T}\{Y(t)\}\mathcal{T}\{X(t)\}$. This completes the proof. **Q.E.D.**

The following lemma [17, p. 104, Theorem 2] gives some sufficient conditions under which the Fourier series expansion of a given h-periodic matrix function is absolutely convergent.

Lemma 2.3 *Let $X(t)$ be h-periodic and continuous, and suppose that its first-order derivative is piecewise continuous. Then the convergence of the Fourier series expansion of $X(t)$ is absolute and thus uniform with respect to $t \in [0,h]$.*

If the conditions in Lemma 2.3 are relaxed [28, p. 173, Theorem 10′], we get Lemma 2.4.

Lemma 2.4 *Let $X(t)$ be h-periodic, piecewise continuous and differentiable at a.e. $t \in [0,h]$. Then, the Fourier series expansion of $X(t)$ converges to $X(t_0)$ for a.e. $t_0 \in [0,h]$.*

To validate a useful result about the Toeplitz transformation on the derivative of an h-periodic time-varying matrix function (i.e., the equation (2.15) given below), we need the following lemma [17, p. 106, Theorem 3].

Lemma 2.5 *Let $X(t)$ be h-periodic and continuous, and suppose that the first-order derivative of $X(t)$ is piecewise continuous. Then, at $t \in [0,h]$ where the second-order derivative of $X(t)$ exists, $\dot{X}(t) = \sum_{m=-\infty}^{+\infty} jm\omega_h X_m e^{jm\omega_h t}$. Namely, the termwise differentiation is valid.*

Remark 2.3 *A function $X(t)$ defined on the interval $[0,h]$ is said to be piecewise continuous if $[0,h]$ can be divided into finitely many sub-intervals, on each of which $X(t)$ is continuous and the unilateral limits of $X(t)$ at the ends from the interior of the sub-interval exist [55]. $X(t)$ is said to be piecewise smooth on $[0,h]$ if $X(t)$ is continuous on the whole interval $[0,h]$ and continuously differentiable except at finitely many points of $[0,h]$, at each of which the left and right derivatives exist. Following the proofs of Lemmas 2.3 and 2.5 [17], or following the arguments in [28] regarding the results corresponding to these lemmas, we can readily see that the conditions on $X(t)$ in these two lemmas can actually be replaced by the weaker condition that $X(t)$ is h-periodic and piecewise smooth. Given this fact, but with a slight abuse of terminology, we neglect the slight difference in these two conditions in this thesis. Namely, when we say that a function is continuous and its first-order derivative is piecewise continuous, the exact meaning should be interpreted to be that the function is piecewise smooth. We follow this convention, since the former wording seems more intuitive and provides some ease in descriptions. However, this does not cause any loss of rigorousness.*

Now we define some sets of h-periodic functions as follows.

$$L_{\mathrm{PCD}}[0,h] := \left\{ f(t) : \begin{array}{l} f(t) \text{ is piecewise continuous and} \\ \text{differentiable at a.e. } t \in [0,h] \end{array} \right\}$$

$$L_{\mathrm{PCC}}[0,h] := \left\{ f(t) : \begin{array}{l} f(t) \text{ is piecewise continuous and its Fourier series} \\ \text{expansion is convergent to } f(t_0) \text{ for a.e. } t_0 \in [0,h] \end{array} \right\}$$

$$L_{\mathrm{CAC}}[0,h] := \left\{ f(t) : \begin{array}{l} f(t) \text{ is continuous and the Fourier series} \\ \text{expansion of } f(t) \text{ is absolutely convergent} \end{array} \right\} \subset L_{\mathrm{PCC}}[0,h]$$

$$L_{\mathrm{CPCD}}[0,h] := \left\{ f(t) : \begin{array}{l} f(t) \text{ is continuous and the derivative of} \\ f(t) \text{ is piecewise continuous in } [0,h] \end{array} \right\} \subset L_{\mathrm{PCD}}[0,h]$$

where PCD stands for piecewise continuous and differentiable and PCC is short for piecewise continuous and convergent while CAC and CPCD are abbreviated from continuous and absolute convergent, and continuous and piecewise continuously differentiable, respectively. By Lemma 2.3 and Lemma 2.4, respectively, it is clear that

$$L_{\mathrm{CPCD}}[0,h] \subset L_{\mathrm{CAC}}[0,h], \quad L_{\mathrm{CPCD}}[0,h] \subset L_{\mathrm{PCD}}[0,h] \subset L_{\mathrm{PCC}}[0,h]$$

which are helpful in interpreting relevant results stated on different classes of FDLCP systems. The following results are helpful in our subsequent arguments.

It is not hard to show the following lemma [46].

Lemma 2.6 *If the matrix functions $X(t), Y(t) \in L_{\mathrm{CAC}}[0,h]$ have compatible dimensions, then $X(t)Y(t) \in L_{\mathrm{CAC}}[0,h]$.*

Proof Expand $X(t)$ and $Y(t)$ into their Fourier series expansions as we have done just before Lemma 2.2. Then under the assumptions about $X(t)$ and $Y(t)$, it is clear that

$$\sum_{m=-\infty}^{+\infty} ||X_m|| < \infty, \quad \sum_{m=-\infty}^{+\infty} ||Y_m|| < \infty$$

On the other hand, by the proof of Lemma 2.2, it follows that for any $t \in [0,h]$

$$X(t)Y(t) = \sum_{p=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} X_{p-k} Y_k e^{jp\omega_h t}$$

and $\sum_{k=-\infty}^{+\infty} X_{p-k} Y_k$ is the p-th Fourier coefficient of the h-periodic matrix function $X(t)Y(t)$. To complete the proof, it remains only to show that $\sum_{p=-\infty}^{+\infty} || \sum_{k=-\infty}^{+\infty} X_{p-k} Y_k|| < \infty$. To this end, we observe that

$$\sum_{p=-\infty}^{+\infty} || \sum_{k=-\infty}^{+\infty} X_{p-k} Y_k|| \leq \sum_{p=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} ||X_{p-k}|| \cdot ||Y_k||$$
$$= \left[\sum_{k=-\infty}^{+\infty} ||Y_k|| \right] \left[\sum_{p=-\infty}^{+\infty} ||X_{p-k}|| \right] < \infty$$

where we used the fact that $\{||X_{p-k}|| \cdot ||Y_k||\}$ is a double-side infinite sequence with non-negative terms, and thus the convergence of the summation $\sum_{p=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}||X_{p-k}|| \cdot ||Y_k||$ does not change no matter how the terms are rearranged. **Q.E.D.**

Lemma 2.7 *$L_{\mathrm{CAC}}[0,h]$ is dense in $L_2[0,h]$.*

Proof Take an arbitrary $f(t) \in L_2[0,h]$ and expand it into the Fourier series expansion $f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{jn\omega_h t}$ with $\underline{f} := [\cdots, f_{-1}^T, f_0^T, f_1^T, \cdots]^T \in l_2$ (by the Parseval theorem). Noting that for any $\epsilon > 0$, we can find $\underline{d} \in l_1$ such that $||\underline{f} - \underline{d}||_{l_2} < \epsilon$ since l_1 is dense in l_2. Now construct $d(t) := \sum_{n=-\infty}^{+\infty} d_n e^{jn\omega_h t}$. It is easy to see that $d(t) \in L_{\mathrm{CAC}}[0,h]$ and

$$||f(\cdot) - d(\cdot)||_{L_2[0,h]} = ||\sum_{n=-\infty}^{+\infty}(f_n - d_n)e^{jn\omega_h(\cdot)}||_{L_2[0,h]} = ||\underline{f} - \underline{d}||_{l_2} < \epsilon$$

which follows again by the Parseval theorem. This completes the proof. **Q.E.D.**

Lemma 2.8 *Let $X(t) \in L_{\mathrm{PCC}}[0,h]$ and $\underline{X}$ be the Toeplitz transformation of $X(t)$ defined in (2.5), i.e., $\underline{X} = \mathcal{T}\{X(t)\}$. Then, $||\underline{X}||_{l_2/l_2} = \sup_{t\in[0,h]}||X(t)||$ and $\underline{X}$ is bounded on l_2.*

Proof Taking $f(t) \in L_{\mathrm{CAC}}[0,h]$ and expanding it into the Fourier series expansion $f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{jn\omega_h t}$, it follows readily that $\underline{f} := [\cdots, f_{-1}^T, f_0^T, f_1^T, \cdots]^T \in l_1$. It is evident that the converse is also true. On the other hand, by the assumption on $X(t)$, it follows from Lemma 2.2 that $\underline{X}\,\underline{f} = \underline{y}$, where $\underline{y}$ is defined similarly to $\underline{f}$ but in terms of the Fourier coefficients of $X(t)f(t)$. Thus it follows that

$$||\underline{X}||_{l_2/l_1(l_2)} := \sup_{0\neq \underline{f}\in l_1}\left\{\frac{||\underline{X}\,\underline{f}||_{l_2}}{||\underline{f}||_{l_2}}\right\} = \sup_{0\neq f(t)\in L_{\mathrm{CAC}}[0,h]}\left\{\frac{||X(\cdot)f(\cdot)||_{L_2[0,h]}}{||f(\cdot)||_{L_2[0,h]}}\right\} =: ||X(\cdot)||_*$$

Obviously, $||\underline{X}||_{l_2/l_1(l_2)} = ||\underline{X}||_{l_2/l_2}$ since l_1 is dense in l_2. Similarly, from Lemma 2.7, it follows that $||X(\cdot)||_* = ||X(\cdot)||_{L_2[0,h]/L_2[0,h]}$. Hence, we obtain

$$||\underline{X}||_{l_2/l_2} = ||X(\cdot)||_{L_2[0,h]/L_2[0,h]} = \sup_{t\in[0,h]}||X(t)||$$

Since $X(t) \in L_{\mathrm{PCC}}[0,h]$ by the assumption, the boundedness assertion follows. **Q.E.D.**

Remark 2.4 *By the terminology of [29, p. 565], the h-periodic matrix function $X(t)$ of Lemma* 2.8 *is the defining function of the (block Laurent) operator $\underline{X}$. Then, it is easy to see that Lemma* 2.8 *is only a special case of Corollary 2.2 of [29, p. 567]. However, an independent proof is provided here for Lemma* 2.8 *through the harmonic analysis approach.*

The following proposition describes the basic properties of the Fourier series expansions of the h-periodic matrices $P(t,0)$ and $P^{-1}(t,0)$, which are the periodic portion and the corresponding inverse of the transition matrix $\Phi(t,0)$ of the FDLCP system (2.1), together with the characteristics of their Toeplitz operator expressions.

Proposition 2.1 *Assume in the system (2.1) that the state matrix $A(t)$ is piecewise continuous in $[0,h]$ and let $\mathcal{T}\{P(t,0)\} =: \underline{P}$. Then, the Fourier series expansions of $P(t,0)$ and $P^{-1}(t,0)$ are absolutely convergent, or equivalently, $P(t,0)$ and $P^{-1}(t,0)$ belong to $L_{\rm CAC}[0,h]$. Moreover, $\mathcal{T}\{P^{-1}(t,0)\} = \underline{P}^{-1}$.*

Proof From (2.2), the assumption on $A(t)$ clearly says that $P(t,0)$ and $P^{-1}(t,0)$ are continuous and their first-order derivatives are piecewise continuous. Hence, by Lemma 2.3, the Fourier series expansions of $P(t,0)$ and $P^{-1}(t,0)$ are absolutely convergent. On the other hand, $P(t,0)P^{-1}(t,0) = I, \forall t \in [0,h]$. Hence, from Lemma 2.2, applying the Toeplitz transformation on the above equation gives

$$\underline{I} = \mathcal{T}\{P(t,0)P^{-1}(t,0)\} = \mathcal{T}\{P(t,0)\}\mathcal{T}\{P^{-1}(t,0)\}$$

where $\underline{I} := \mathcal{T}\{I\}$ is the identity operator on l_2. Similarly, $\underline{I} = \mathcal{T}\{P^{-1}(t,0)\}\mathcal{T}\{P(t,0)\}$. Hence, we have $\mathcal{T}\{P^{-1}(t,0)\} = \underline{P}^{-1}$ by the uniqueness of the inverse operator [22]. **Q.E.D.**

To show various convergences involved in the subsequent chapters, we derive the following proposition as a further preparation.

Proposition 2.2 *In the system (2.1), assume that $A(t) \in L_{\rm PCD}[0,h]$ and $B(t), C(t) \in L_{\rm CAC}[0,h]$. Then $\hat{B}(t)$ and $\hat{C}(t)$ belong to $L_{\rm CAC}[0,h]$, where $\hat{B}(t)$ and $\hat{C}(t)$ are given in (2.4). Furthermore, if letting $\underline{\hat{B}} := \mathcal{T}\{\hat{B}(t)\}$ and $\underline{\hat{C}} := \mathcal{T}\{\hat{C}(t)\}$, then it holds that*

$$||\underline{\hat{B}}||_{l_2/l_2} \leq ||B(\cdot)||e^{(||A(\cdot)||+||Q||)h}, \quad ||\underline{\hat{C}}||_{l_2/l_2} \leq ||C(\cdot)||e^{(||A(\cdot)||+||Q||)h}$$

where $||A(\cdot)|| := \sup_{t\in[0,h]} ||A(t)||$, and $||B(\cdot)||$ and $||C(\cdot)||$ are defined similarly but in terms of $B(t)$ and $C(t)$, respectively.

Proof From the assumption on $A(t)$, Proposition 2.1 shows that $P^{-1}(t,0)$ and $P(t,0)$ belong to $L_{\rm CAC}[0,h]$. Since $B(t), C(t) \in L_{\rm CAC}[0,h]$, it follows readily from Lemma 2.6 that $\hat{B}(t) = P^{-1}(t,0)B(t)$, $\hat{C}(t) = C(t)P(t,0)$ belong to $L_{\rm CAC}[0,h]$. This gives the first assertion. Thus, it makes sense to do the following arguments by Lemma 2.8.

$$\begin{aligned} ||\underline{\hat{B}}||_{l_2/l_2} &= ||P^{-1}(\cdot,0)B(\cdot)||_{L_2[0,h]/L_2[0,h]} \leq \sup_{t\in[0,h]} ||P^{-1}(t,0)|| \sup_{t\in[0,h]} ||B(t)|| \\ &\leq ||B(\cdot)|| \sup_{t\in[0,h]} ||e^{Qt}|| \sup_{t\in[0,h]} ||\Phi(0,t)|| \leq ||B(\cdot)||e^{(||A(\cdot)||+||Q||)h} \end{aligned}$$

The last inequality follows from Lemma 4.2 in Chapter 4 (i.e., Lemma 6.3.1 of [51]) and leads to the inequality for $||\underline{\hat{B}}||_{l_2/l_2}$. The inequality for $||\underline{\hat{C}}||_{l_2/l_2}$ follows similarly. **Q.E.D.**

2.3 Similarity Transformation Formulas

Equipped with the Floquet theorem and the lemmas in Section 2.2, we are ready to establish the (Floquet) similarity transformation relations in FDLCP systems. These relations play a key role in verifying such features as validity of various definitions, convergence and stability related to FDLCP systems. The rigorous proof and interpretation for the similarity transformation relations will be established separately on the linear spaces l_2 and l_1 in this section, which form one of the main contributions of this thesis, but it should be pointed out that the original ideas partially come from [69], [70].

2.3.1 Similarity Transformation Formula on l_2

To state the similarity transformation formula, we define the infinite-dimensional matrix

$$\underline{E}(j\varphi) = \text{diag}[\cdots, j\varphi_{-2}I, j\varphi_{-1}I, j\varphi_0 I, j\varphi_1 I, j\varphi_2 I, \cdots]$$

Here

$$\varphi_k := \varphi + k\omega_h, \quad \varphi \in [-\frac{\omega_h}{2}, \frac{\omega_h}{2}) =: \mathcal{I}_0 \tag{2.8}$$

and the $j\varphi_0 I$-block is at the center of $\underline{E}(j\varphi)$. The infinite-dimensional matrix $\underline{E}(j\varphi)$ will also be used for the frequency-response operator definition via the input/output steady-state analysis in Chapter 3. Now we further define the subset l_E of l_2 by

$$l_E := \{\underline{x} \in l_2 : \underline{E}(j0)\underline{x} \in l_2\} \tag{2.9}$$

where $\underline{E}(j0) = \underline{E}(j\varphi)|_{\varphi=0}$. Now we prove the following lemma which describes some basic properties of the subset l_E of l_2.

Lemma 2.9 *l_E is dense in the Hilbert space l_2. Also, l_E is a proper and dense subset of l_1 in the l_2-norm sense.*

Proof Let $\underline{x} \in l_2$. For any $\epsilon > 0$, there exists $\underline{x}' \in l_2$ with only finite nonzero entries such that $||\underline{x} - \underline{x}'||_{l_2} < \epsilon$. Obviously, $\underline{E}(j0)\underline{x}' \in l_2$ and thus $\underline{x}' \in l_E$, which implies that the subset l_E of l_2 is dense in l_2.

Furthermore, it is clear that $\underline{x} \in l_E$ if and only if $\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} m^2\omega_h^2||[\underline{x}]_m||^2 < \infty$. It follows from the Cauchy-Schwarz inequality that if $\underline{x} \in l_E$, then

$$\begin{aligned}\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} ||[\underline{x}]_m|| &= \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \cdot m||[\underline{x}]_m|| \\ &\leq \Big(\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m^2}\Big)^{\frac{1}{2}} \Big(\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} m^2||[\underline{x}]_m||^2\Big)^{\frac{1}{2}} \leq M < \infty\end{aligned}$$

for some $M > 0$. Thus, if $\underline{x} \in l_E$, then $\underline{x} \in l_1$. The fact that l_E is dense in l_1 can be shown exactly the same as in the proof for the first assertion (i.e., via truncation). Now take

$$[\underline{x}]_m = \begin{cases} \frac{1}{|m|^2}[1, 0, \cdots, 0]^T & (m \neq 0) \\ 0 & (m = 0) \end{cases}$$

Clearly, $\underline{x} \in l_1$ but $\underline{E}(j0)\underline{x} \notin l_2$. Namely, l_E is a proper subset of l_1. **Q.E.D.**

Now we are in the right position to state the so-called (Floquet) similarity transformation relations in a general class of FDLCP systems.

Theorem 2.2 *In the system (2.1), let $A(t) \in L_{\mathrm{PCD}}[0,h]$ and $B(t), C(t) \in L_{\mathrm{PCC}}[0,h]$. Then, l_E is $\underline{P}$-, $\underline{P}^{-1}$-, $\underline{P}^*$- and $\underline{P}^{-*}$-invariant, where $\underline{P}^{-*} := [\underline{P}^{-1}]^*$. $\underline{P}$ is invertible on l_E and the unique inverse of $\underline{P}$ on l_E is $\underline{P}^{-1}$ restricted to l_E. It holds on $l_E \subset l_2$ that*

$$\underline{P}(\underline{E}(j0) - \underline{Q})\underline{P}^{-1} = \underline{E}(j0) - \underline{A} \tag{2.10}$$

where $\underline{Q} = \mathcal{T}\{Q\}$. Moreover, letting $\underline{\hat{B}} := \mathcal{T}\{P^{-1}(t,0)B(t)\}$ and $\underline{\hat{C}} := \mathcal{T}\{C(t)P(t,0)\}$, it holds on the whole l_2 that

$$\underline{\hat{B}} = \underline{P}^{-1}\underline{B}, \quad \underline{\hat{C}} = \underline{C}\,\underline{P} \tag{2.11}$$

Proof By the equations in (2.2), we obtain

$$P(t,0)Q = A(t)P(t,0) - \dot{P}(t,0) \quad \text{(a.e.)} \tag{2.12}$$

By Proposition 2.1, the Fourier series expansion of $P(t,0)$ is absolutely convergent. Note also that by Lemma 2.4, the Fourier series expansion of $A(t)$ converges to $A(t_0)$ for a.e. $t_0 \in [0,h]$ from the assumption. Hence, by Lemma 2.2, we have

$$\mathcal{T}\{A(t)P(t,0)\} = \mathcal{T}\{A(t)\}\mathcal{T}\{P(t,0)\} \tag{2.13}$$

Again by (2.2), under the assumption about $A(t)$, the first-order derivative of $P(t,0)$ is piecewise continuous and the second-order derivative of $P(t,0)$ exists a.e. in $[0,h]$. Thus, by Lemma 2.5 it holds that

$$\dot{P}(t,0) = \sum_{m=-\infty}^{+\infty} jm\omega_h P_m e^{jm\omega_h t} \quad \text{(a.e.)} \tag{2.14}$$

through the termwise differentiation, where $\{P_m\}_{m=-\infty}^{+\infty}$ is the Fourier coefficients sequence of $P(t,0)$. In other words, $\{jm\omega_h P_m\}_{m=-\infty}^{+\infty}$ is the Fourier coefficients sequence of $\dot{P}(t,0)$, so that by some simple algebra [70], we are led to

$$\mathcal{T}\{\dot{P}(t,0)\} = \underline{E}(j0)\underline{P} - \underline{P}\,\underline{E}(j0) \tag{2.15}$$

Note that $\mathcal{T}\{\dot{P}(t,0)\}$ is bounded on l_2 (which follows from Lemma 2.8 since $\dot{P}(t,0)$ belongs to $L_{\mathrm{PCC}}[0,h]$ by the assumption on $A(t)$ and Lemma 2.4), but that the two operators on the right-hand side of the above equation are unbounded on l_2 since $\underline{E}(j0)$ is. This means that we are not allowed to consider the operators $\underline{E}(j0)\underline{P}$ and $\underline{P}\,\underline{E}(j0)$ separately if the underlying space is l_2. To get around the problem, we have to restrict the domain of these operators to $l_E \subset l_2$. Now take $\underline{x} \in l_E \subset l_2$. Then $\mathcal{T}\{\dot{P}(t,0)\}\underline{x} \in l_2$ by Lemma 2.4 and Lemma 2.8 if we note the properties of $\dot{P}(t,0)$ mentioned above. Also, $\underline{P}\,\underline{E}(j0)\underline{x} \in l_2$ since $\underline{E}(j0)\underline{x} \in l_2$ and $\underline{P}$ is bounded on l_2 (which follows again from Lemma 2.8 by the fact that the Fourier series expansion of $P(t,0)$ is absolutely convergent as stated in Proposition 2.1). It follows from (2.15) that $\underline{E}(j0)\underline{P}\,\underline{x} \in l_2$, which clearly says that l_E is $\underline{P}$-invariant.

Similarly, by repeating the arguments about $\dot{P}(t,0)$ on $\dot{P}^{-1}(t,0)$, it readily follows that l_E is also $\underline{P}^{-1}$-invariant. Hence, $\underline{P}$ and $\underline{P}^{-1}$ are actually mappings on l_E. From this, it can be asserted that $\underline{P}$ is invertible on l_E and the unique inverse of $\underline{P}$ on l_E is nothing but $\underline{P}^{-1}$ restricted to $l_E \subset l_2$ since $\underline{P}^{-1}\underline{P}\,\underline{x} = \underline{P}\,\underline{P}^{-1}\underline{x} = \underline{x}, \forall \underline{x} \in l_E$.

On the other hand, the equations (2.13) and (2.15) actually say that the Toeplitz transformation applies to each term of (2.12) under the given assumptions, so that we obtain

$$\underline{P}\,\underline{Q} = \underline{A}\,\underline{P} - \underline{E}(j0)\underline{P} + \underline{P}\,\underline{E}(j0)$$

Therefore, if we work on l_E instead of l_2, the operators involved are well-defined from l_E to l_2, i.e., the above equation can be rewritten as

$$\underline{P}(\underline{E}(j0) - \underline{Q}) = (\underline{E}(j0) - \underline{A})\underline{P} \tag{2.16}$$

which, together with the fact that $\underline{P}$ is invertible on l_E, gives (2.10).

To see that l_E is $\underline{P}^*$-invariant, we note that

$$\underline{P}^* = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & P_0^* & P_1^* & P_2^* & \cdots \\ \cdots & P_{-1}^* & P_0^* & P_1^* & \cdots \\ \cdots & P_{-2}^* & P_{-1}^* & P_0^* & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} =: \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \tilde{P}_0 & \tilde{P}_{-1} & \tilde{P}_{-2} & \cdots \\ \cdots & \tilde{P}_1 & \tilde{P}_0 & \tilde{P}_{-1} & \cdots \\ \cdots & \tilde{P}_2 & \tilde{P}_1 & \tilde{P}_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which implies that

$$\tilde{P}_m = P_{-m}^* = \left\{\frac{1}{h}\int_0^h P(t,0)e^{jm\omega_h t}dt\right\}^* = \frac{1}{h}\int_0^h P^*(t,0)e^{-jm\omega_h t}dt$$

From this, it follows readily that $\underline{P}^* = \mathcal{T}\{P^*(t,0)\}$. It is evident from the assumption about $A(t)$ that the first-order derivative of $P^*(t,0)$ is piecewise continuous in $[0,h]$ and the second-order derivative of $P^*(t,0)$ exists a.e. in $[0,h]$. Then from Lemma 2.5, it is true that

$$\mathcal{T}\{\dot{P}^*(t,0)\} = \underline{E}(j0)\underline{P}^* - \underline{P}^*\underline{E}(j0)$$

From this the assertion follows immediately. Similarly, one can show that l_E is $\underline{P}^{-*}$-invariant.

Note that the Fourier series expansion of $P^{-1}(t,0)$ is absolutely convergent by Proposition 2.1. This, together with the assumption on $B(t)$ and Lemma 2.2, implies that

$$\mathcal{T}\{P^{-1}(t,0)B(t)\} = \mathcal{T}\{P^{-1}(t,0)\}\mathcal{T}\{B(t)\}$$

Combining this with the second assertion of Proposition 2.1, the first equation of (2.11) follows. Similarly for the second relation of (2.11). This completes the proof. **Q.E.D.**

Now we give remarks about the (Floquet) similarity transformation formulas.

Remark 2.5 *The infinite-dimensional matrix equations (2.10) and (2.11) form the so-called similarity transformation relations in terms of the Toeplitz operators on the Hilbert space l_2, which are first given partially in [69], [70], but the proofs given there have not paid full attention to the convergence problems related to the Fourier analysis. This is why we include an alternative proof here. In mathematical form, (2.10) and (2.11) are similar to what we have when a state vector transformation is introduced to* LTI *continuous-time systems. That is why the equations (2.10) and (2.11) are also called similarity transformation formulas. Since (2.10) and (2.11) are only stated under the assumption that the Floquet factorization of the state transition matrix $\Phi(t,0)$ of the* FDLCP *system (2.1) is available, it seems better to include the term 'Floquet' to clarify this. However, for the sake of simple statements, the term 'Floquet' will be dropped in this thesis. Also, it is important to notice that (2.10) is stated only on the dense subset l_E of l_2 (by Lemma 2.9) while (2.11) has no such constraint. In other words, (2.10) is guaranteed to be only densely defined on the Hilbert space l_2 [29], [55, p. 486], while (2.11) is well-defined on the whole l_2. Generally speaking, densely defined operators are related to derivative operations [22], as seen in (2.15).*

It is straightforward to see that the relation (2.16) is true under the assumption that $A(t) \in L_{\text{PCD}}[0,h]$ when we view it as an aggregated expression of infinitely many simultaneous finite-dimensional matrix equations. However, when we try to establish the inverse of $\underline{E}(j0) - \underline{A}$ through (2.16) (this is needed in the frequency response operator definition), we quickly come to the fact that both $\underline{E}(j0) - \underline{Q}$ and $\underline{E}(j0) - \underline{A}$ are unbounded in the l_2-induced norm (though $\underline{A}$ and $\underline{Q}$ are bounded on l_2), so that the ranges of $\underline{E}(j0) - \underline{Q}$ and $\underline{E}(j0) - \underline{A}$ are not in l_2. This causes us a difficulty to define the inverses of these two operators. It is also for us to get around this obstacle that the set l_E is introduced and, thereupon, (2.10) is claimed only on this set l_E. From our discussions in the proof of Theorem 2.2, one can conclude that by restricting the domains of $\underline{E}(j0) - \underline{Q}$ and $\underline{E}(j0) - \underline{A}$ to l_E, these two operator can be treated as operators from l_E to l_2.

Now we answer the question that under what conditions the operator $\underline{E}(j0) - \underline{A}$ is invertible. This problem remains unsolved in [69], [70], on which this thesis is particularly based, and thus we also tackle basic properties about the inverse operator of $\underline{E}(j0) - \underline{A}$. It is evident by (2.10) that $\underline{E}(j0) - \underline{A}$ is invertible if and only if $\underline{E}(j0) - \underline{Q}$ is and that if such an inverse exists, the inverse operator is a mapping from l_2 to l_E. Theorem 2.3 gives the answer to this question in a slightly more general form.

Theorem 2.3 *Assume in the system (2.1) that $A(t) \in L_{\text{PCD}}[0,h]$ and that the system is asymptotically stable. Then, $\underline{E}(j\varphi) - \underline{A} : l_E \to l_2$ is invertible for all $\varphi \in \mathcal{I}_0$ and*

$$(\underline{E}(j\varphi) - \underline{A})^{-1} = \underline{P}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{P}^{-1} \tag{2.17}$$

where $\underline{E}(j\varphi) = \underline{E}(j0) + j\varphi\underline{I}$ and

$$(\underline{E}(j\varphi) - \underline{Q})^{-1} = \text{diag}[\cdots, (j\varphi_{-1}I - Q)^{-1}, (j\varphi_0 I - Q)^{-1}, (j\varphi_1 I - Q)^{-1}, \cdots] \tag{2.18}$$

with $\varphi_m := \varphi + m\omega_h, m \in \mathcal{Z}$. *Moreover,* $(\underline{E}(j\varphi) - \underline{A})^{-1} : l_2 \to l_E$ *is compact and uniformly bounded over* $\varphi \in \mathcal{I}_0$.

Proof From the assumption on $A(t)$, we have (2.10) so that for any $\varphi \in \mathcal{I}_0$

$$\underline{P}(\underline{E}(j\varphi) - \underline{Q})\underline{P}^{-1} = \underline{E}(j\varphi) - \underline{A}$$

Also, by the stability assumption, all the eigenvalues of $Q - j\varphi_m I, \forall m \in \mathcal{Z}$ have negative real parts. Thus, the operator on the right-hand side of (2.18), denoted by $\underline{D}(Q, \varphi)$, is well-defined and bounded on l_2. To see the latter, we note that there exists $K > 0$ such that

$$||(j\varphi_m I - Q)^{-1}|| \leq K f(m) \quad (m \in \mathcal{Z}, \forall \varphi \in \mathcal{I}_0) \tag{2.19}$$

where f is defined in Appendix A.1. Noting that $\underline{D}(Q, \varphi)$ is block-diagonal, it follows that

$$||\underline{D}(Q, \varphi)||_{l_2/l_2} = \sup_{m \in \mathcal{Z}} ||(j\varphi_m I - Q)^{-1}|| \leq K \quad (\forall \varphi \in \mathcal{I}_0) \tag{2.20}$$

Simple computations show that

$$\underline{D}(Q, \varphi)(\underline{E}(j\varphi) - \underline{Q}) = \underline{I}, \quad (\underline{E}(j\varphi) - \underline{Q})\underline{D}(Q, \varphi) = \underline{I}$$

which, together with the fact that $\underline{P}$ and $\underline{P}^{-1}$ are invertible on l_2 and l_E, respectively, establishes (2.17). Noting that $(\underline{E}(j\varphi) - \underline{Q})^{-1}$ is uniformly bounded from l_2 to l_E over $\varphi \in \mathcal{I}_0$ by (2.20) and that $\underline{P}$ and $\underline{P}^{-1}$ are bound on l_E and l_2, respectively, then the uniform boundedness of $(\underline{E}(j\varphi) - \underline{A})^{-1}$ from l_2 to l_E over $\varphi \in \mathcal{I}_0$ follows from (2.17).

To see the compactness of $(\underline{E}(j\varphi) - \underline{Q})^{-1}$, we define

$$[(\underline{E}(j\varphi) - \underline{Q})^{-1}]_N = \text{diag}[\cdots, 0, (j\varphi_{-N} I - Q)^{-1}, \cdots, (j\varphi_N I - Q)^{-1}, 0, \cdots]$$

It is clear that for any fixed N, the operator $[(\underline{E}(j\varphi) - \underline{Q})^{-1}]_N$ is bounded on l_2 by (2.19) and its range has finite rank so that $[(\underline{E}(j\varphi) - \underline{Q})^{-1}]_N$ is a compact operator. Furthermore, it is easy to see from (2.19) that for any $\varphi \in \mathcal{I}_0$,

$$\lim_{N \to \infty} [(\underline{E}(j\varphi) - \underline{Q})^{-1}]_N = (\underline{E}(j\varphi) - \underline{Q})^{-1}$$

in the l_2-induced norm sense. These facts tell us from Theorem 2 of [22, p. 112] that $(\underline{E}(j\varphi) - \underline{Q})^{-1}$ is a compact mapping on l_2. Noting that $\underline{P}$ and $\underline{P}^{-1}$ are bounded on l_E and l_2, respectively, it follows from (2.17) that $(\underline{E}(j\varphi) - \underline{A})^{-1}$ is also compact. **Q.E.D.**

2.3.2 Similarity Transformation Formula on l_1

To state the similarity transformation formulas on the linear space l_1, we define the set

$$l_e = \{\underline{x} \in l_1 : \underline{E}(j0)\underline{x} \in l_1\} \tag{2.21}$$

and state the following lemma, which can be shown in a similar way to Lemma 2.9.

Lemma 2.10 . *l_e is a proper dense subset of l_1 and $l_e \subset l_E$.*

Proof We only show that $l_e \subset l_E$. Let $\underline{x} \in l_e$. By definition

$$\begin{aligned} \|\underline{E}(j0)\underline{x}\|_{l_2}^2 &= \sum_{m=-\infty}^{+\infty} |m|^2\omega_h^2 \|[\underline{x}]_m\|^2 \\ &\le \max_m\{|m|\omega_h\|[\underline{x}]_m\|\} \sum_{m=-\infty}^{+\infty} |m|\omega_h\|[\underline{x}]_m\| \\ &= \max_m\{|m|\omega_h\|[\underline{x}]_m\|\}\, \|\underline{E}(j0)\underline{x}\|_{l_1} \end{aligned}$$

Since $\underline{E}(j0)\underline{x} \in l_1$ by the definition of l_e, $\max_m\{|m|\omega_h\|[\underline{x}]_m\|\}$ is well-defined. Hence, it can be asserted that $\underline{E}(j0)\underline{x} \in l_2$, which implies that $\underline{x} \in l_E$. **Q.E.D.**

Lemma 2.10 says that the role of the subset l_e of l_1 is similar to that of the subset l_E of l_2 so that $(\underline{E}(j0) - \underline{A})^{-1}$ can be derived as a mapping from l_1 to l_e from (2.16). For brevity, the exact assertions are stated in the following theorem, which is helpful in establishing the frequency response relation of the FDLCP system in terms of a mapping on l_1.

Theorem 2.4 *In the system (2.1), let $A(t) \in L_{\mathrm{CPCD}}[0,h]$ and $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$. Then, $\underline{P}$ and $\underline{P}^{-1}$ are bounded on l_1. l_e is $\underline{P}$- and $\underline{P}^{-1}$-invariant, and hence $\underline{P}$ is invertible on l_e. The unique inverse of $\underline{P}$ on l_e is $\underline{P}^{-1}$ restricted to l_e. It is true that on $l_e \subset l_1$*

$$\underline{P}(\underline{E}(j0) - \underline{Q})\underline{P}^{-1} = \underline{E}(j0) - \underline{A} \tag{2.22}$$

Moreover, it holds on the whole l_1 that

$$\underline{\hat{B}} = \underline{P}^{-1}\underline{B}, \quad \underline{\hat{C}} = \underline{C}\,\underline{P} \tag{2.23}$$

Furthermore, if the system (2.1) is asymptotically stable in the Floquet theorem sense, then $\underline{E}(j\varphi) - \underline{A} : l_e \to l_1$ is invertible for all $\varphi \in \mathcal{I}_0$, and

$$\underline{P}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{P}^{-1} = (\underline{E}(j\varphi) - \underline{A})^{-1} \tag{2.24}$$

which is a mapping from l_1 to l_e. Also, $(\underline{E}(j\varphi) - \underline{A})^{-1} : l_1 \to l_e$ is compact and uniformly bounded over $\varphi \in \mathcal{I}_0$.

Proof A complete proof can be given by similar steps to those in the proofs of Theorems 2.2 and 2.3. Here, it remains only to show the nontrivial points that the operators $\mathcal{T}\{\dot{P}(t,0)\}$, $\underline{P}, \underline{P}^{-1}, \underline{B}$ and $\underline{C}$ are bounded on l_1, and that $(\underline{E}(j\varphi) - \underline{Q})^{-1}$ is uniformly bounded from l_1 to l_e over $\varphi \in \mathcal{I}_0$. By (2.2) and the assumption on $A(t)$, it follows that $\dot{P}(t,0)$ is continuous and the first-order derivative of $\dot{P}(t,0)$ is piecewise continuous in $[0,h]$. Hence, by Lemma 2.3, the Fourier series expansion of $\dot{P}(t,0)$ is absolutely convergent. Now we denote the Fourier coefficients sequence of $\dot{P}(t,0)$ by $\{\hat{P}_m\}_{m=-\infty}^{+\infty}$. It is easy to see that if $\underline{x} \in l_1$, then

$$\begin{aligned} \|\mathcal{T}\{\dot{P}(t,0)\}\underline{x}\|_{l_1} &= \sum_{m=-\infty}^{+\infty} \Big\| \sum_{k=-\infty}^{+\infty} \hat{P}_{m-k} x_k \Big\| \\ &\le \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \|\hat{P}_{m-k}\| \cdot \|x_k\| = \Big(\sum_{m=-\infty}^{+\infty} \|\hat{P}_m\| \Big) \|\underline{x}\|_{l_1} \end{aligned}$$

where $\sum_{m=-\infty}^{+\infty} ||\hat{P}_m|| < \infty$ by the absolute convergence. From this, it follows readily that the operator $\mathcal{T}\{\dot{P}(t,0)\}$ is bounded on l_1. Similarly, since the Fourier series expansions of $P(t,0)$ and $P^{-1}(t,0)$ are absolutely convergent, $\underline{P}$ and $\underline{P}^{-1}$ are bounded on l_1. The boundedness of $\underline{B}$ and $\underline{C}$ on l_1 follows directly from the assumption that $B(t)$ and $C(t)$ belong to $L_{\mathrm{CAC}}[0,h]$. The last assertion follows from the above discussions, (2.19) and (2.24). **Q.E.D.**

Remark 2.6 *The relations (2.22) and (2.23) form the similarity transformation relations on the Banach space l_1 in terms of the Toeplitz transformation (see also Remark 2.5). In addition, the compactness of $(\underline{E}(j\varphi) - \underline{A})^{-1}$ should be interpreted on the Banach space l_1 in Theorem 2.4, which is in contrast to the Hilbert space l_2 in Theorem 2.3.*

2.4 Eigenvalue Properties of FDLCP systems

In Section 2.3, it is clarified that (2.16) holds under the assumption that $A(t) \in L_{\mathrm{PCD}}[0,h]$ when we see this equation as an aggregated expression of infinitely many simultaneous finite-dimensional matrix equations. However, when one tries to establish the eigenvalue-eigenvector relation on and between $\underline{A} - \underline{E}(j0)$ and $\underline{Q} - \underline{E}(j0)$, the difficulty that $\underline{A} - \underline{E}(j0)$ and $\underline{Q} - \underline{E}(j0)$ are unbounded (even though $\underline{A}$ and $\underline{Q}$ are bounded on l_2) emerges again. To recover the desired eigenvalue-eigenvector relation, the introduction of the set l_E is also helpful. In fact, from the above section, if the domains of $\underline{A} - \underline{E}(j0)$ and $\underline{Q} - \underline{E}(j0)$ are restricted to $l_E \subset l_2$, then the ranges of these operators will fall into l_2. Therefore, the eigenvalue and eigenvector concepts are recovered by following Definition 6.5.1 of [55]. To be clear, the definition is restated as follows.

Definition 2.1 *Let T be a linear transformation with its domain l_E and range l_2. A scalar λ such that there does exist an $x \in l_E, x \neq 0$, satisfying $Tx = \lambda x$, is said to be an eigenvalue of $T : l_E \to l_2$. Here, x is said to be an eigenvector of T corresponding to the eigenvalue λ.*

Then, the next task is to determine the set of the eigenvalues of the system operator $\underline{A} - \underline{E}(j0)$ and to make sure that for each eigenvalue of $\underline{A} - \underline{E}(j0)$ there is a corresponding eigenvector $\underline{x}$ belonging to l_E. To this end, let us define

$$\Lambda := \{\lambda \in \mathcal{C} : (\underline{Q} - \underline{E}(j0))\underline{x} = \lambda \underline{x}, 0 \neq \exists\, \underline{x} \in l_E\}$$

That is, Λ is the set of all eigenvalues of the operator $\underline{Q} - \underline{E}(j0)$. Now let λ be an eigenvalue of Q with an associated eigenvector x. Clearly, x is also an eigenvector of $Q + jm\omega_h I$, $\forall m \in \mathcal{Z}$, corresponding to the eigenvalue $\lambda + jm\omega_h$. For each specific $m \in \mathcal{Z}$, taking $\underline{x} = [\cdots, 0^T, x^T, 0^T, \cdots]^T (\in l_2)$ in which x is located at the m-th position of $\underline{x}$, we observe that $(\underline{Q} - \underline{E}(j0))\underline{x} = (\lambda + jm\omega_h)\underline{x}$, which implies immediately that

$$\Lambda = \{\lambda(Q) + jm\omega_h : m \in \mathcal{Z}\} \tag{2.25}$$

To see this, it is enough to show that there is no $\alpha \notin \{\lambda(Q)+jm\omega_h : m \in \mathcal{Z}\}$ and $0 \neq \underline{x} \in l_E$ such that $(\underline{Q} - \underline{E}(j0))\underline{x} = \alpha\underline{x}$. Suppose the contrary. Then by the block-diagonal structure of $\underline{Q} - \underline{E}(j0)$, one must conclude that $(Q - jm\omega_h)[\underline{x}]_m = \alpha[\underline{x}]_m$, where $[\underline{x}]_m$ means the m-th position entry of $\underline{x}$. This implies that α is the eigenvalue of $Q - jm\omega_h I$ if $[\underline{x}]_m \neq 0$. This is contradictory to the assumption. Now we show the following result.

Theorem 2.5 *Suppose in the system (2.1) that $A(t) \in L_{\mathrm{PCD}}[0, h]$. Then the system is asymptotically stable if and only if the set Λ of the eigenvalues of $\underline{Q} - \underline{E}(j0)$ lies in the open left-half plane. Moreover, $\Lambda = \Lambda_A$ where Λ_A is the set of the eigenvalues of $\underline{A} - \underline{E}(j0)$.*

Proof From the Floquet theorem and (2.25), the first assertion follows immediately. For each $\lambda \in \Lambda$, there exists a nonzero $\underline{x} \in l_E$ such that $(\underline{Q} - \underline{E}(j0))\underline{x} = \lambda\underline{x}$ by definition. Noting that $\underline{P}\,\underline{x} \in l_E$ since l_E is $\underline{P}$-invariant, it follows from Theorem 2.2 that

$$(\underline{A} - \underline{E}(j0))\underline{P}\,\underline{x} = \underline{P}(\underline{Q} - \underline{E}(j0))\underline{x} = \lambda\underline{P}\,\underline{x}$$

This implies that $\Lambda \subset \Lambda_A$ since $\underline{P}$ is invertible on l_E. Similarly we obtain $\Lambda_A \subset \Lambda$. **Q.E.D.**

Now we give the definition of the eigenvalues of an FDLCP system.

Definition 2.2 *The operator $\underline{A} - \underline{E}(j0) : l_E \rightarrow l_2$ is called the system operator of the* FDLCP *system (2.1). By the eigenvalues of this* FDLCP *system, we mean the eigenvalues of its system operator.*

It is easy to see from Theorem 2.5 that the set of all the eigenvalues of an FDLCP system is countably infinite. This can be interpreted as another reflection of the uniqueness modulo $j2\pi/h = j\omega_h$ of the characteristic exponents of the system matrix $A(t)$ (see also Remark 2.1). One can also see that if $A(t)$ is $n \times n$, the eigenvalues of the corresponding FDLCP system are distributed on n lines parallel to the imaginary axis, on each of which eigenvalues are located equitably with the distance $2\pi/h = \omega_h$ for any adjacent two. This geometrical interpretation is useful in extending the Gerschgorin theorem to the system operator $\underline{A} - \underline{E}(j0)$ that is infinite-dimensional as discussed in Chapter 3.

Chapter 3

Theoretical Harmonic Analysis of FDLCP Systems

The objective of this chapter is to develop an operator-theoretic explanation about properties of a class of FDLCP systems in a similar way to what has been done in LTI continuous-time systems. The main mathematical tools, as we have reviewed in Chapter 2, are the Fourier series analysis and the Toeplitz transformation. Due to the widespread utilization of the Fourier analysis in the discussions, the analysis approach adopted in this thesis is named 'harmonic analysis' by following the conventional terminology. The targets in this chapter include: asymptotic stability analysis via the operator-valued harmonic Lyapunov equation involving the system operator $\underline{A} - \underline{E}(j0)$ in Section 3.1, an extended Gerschgorin criterion in Section 3.2 stated for the system operator, the frequency response operator derived through the input/output steady-state analysis and its properties in Section 3.3, the H_2 and H_∞ norms and their respective equivalences between the time-domain and frequency-domain definitions in Section 3.4; in particular, in Subsection 3.4.3, the trace formula is recovered for the H_2 norm of FDLCP systems based on the harmonic Lyapunov equation.

3.1 Stability and Harmonic Lyapunov Equation

Asymptotic stability analysis is one of the central topics about FDLCP systems, which is much harder to deal with than that of LTI systems, and only some primitive results are available [25], [38]. Roughly speaking, the well-known Floquet theorem seems to be the best result at hand when we deal with the stability problem of general FDLCP systems [39], [51]. In this section, we establish a stability test in terms of an infinite-dimensional Lyapunov equation for an FDLCP system, which can be seen to be the counterpart to the Lyapunov equation for a finite-dimensional LTI continuous-time system. Although methods to solve this equation and the positive definiteness test of the solution remain to be problems, the harmonic Lyapunov equation really reveals that an FDLCP system is essentially linear time-invariant as long as asymptotic stability is concerned. The harmonic Lyapunov equation also gives help in establishing a trace formula for the H_2 norm of FDLCP systems and proving applicable stability criteria from an operator-theoretic viewpoint. The main results in this section have been presented in [89].

To begin with, we need some discussions on the adjoint operator of the unbounded operator $\underline{A} - \underline{E}(j0)$ viewed on l_E, which we denote by $(\underline{A} - \underline{E}(j0))^*$. By Lemma 2.9, l_E is dense in l_2, and thus it is said that $\underline{A} - \underline{E}(j0)$ is densely defined on l_2 [55, p. 486]. However, the functional $F_{\underline{y}}$ defined on l_E by $F_{\underline{y}} := \langle(\underline{A} - \underline{E}(j0))\underline{x}, \underline{y}\rangle$ with $\underline{y} \in l_2$ may be unbounded and thus the Riesz representation theorem (Theorem 5.21.1 of [55]) does not apply. In other words, the adjoint of the operator $\underline{A} - \underline{E}(j0)$ may not exist in the usual definition for a linear bounded operator. Hence we must modify the definition of $(\underline{A} - \underline{E}(j0))^*$.

From [29, vol. I, p. 290], the domain of the operator $(\underline{A} - \underline{E}(j0))^*$ is given by

$$\mathcal{D}\{(\underline{A} - \underline{E}(j0))^*\} = \left\{\underline{y} \in l_2 : \sup_{0 \neq \underline{x} \in l_E} \frac{|\langle(\underline{A} - \underline{E}(j0))\underline{x}, \underline{y}\rangle|}{||\underline{x}||_{l_2}} < \infty\right\}$$

However, since $\underline{A}$ is bounded on l_2 (by the assumption that $A(t) \in L_{\mathrm{PCD}}[0,h] \subset L_{\mathrm{PCC}}[0,h]$ and Lemma 2.8), it follows that

$$\mathcal{D}\{(\underline{A} - \underline{E}(j0))^*\} = \left\{\underline{y} \in l_2 : \sup_{0 \neq \underline{x} \in l_E} \frac{|\langle\underline{E}(j0)\underline{x}, \underline{y}\rangle|}{||\underline{x}||_{l_2}} < \infty\right\} = \mathcal{D}\{\underline{E}(j0)^*\}$$

Now take $\underline{y} \in \mathcal{D}\{\underline{E}(j0)^*\}$. Noting that l_E is dense in l_2, it follows from [22] that the closure $\bar{l}_E$ of l_E is nothing but l_2, i.e., $\bar{l}_E = l_2$. Hence, the functional $F_{\underline{y}}$ has a unique bounded linear extension $\bar{F}_{\underline{y}}$ to the whole l_2. Therefore, the Riesz representation theorem ensures the existence of a unique $\underline{z} \in l_2$ such that $\bar{F}_{\underline{z}}(\underline{x}) = \langle\underline{x}, \underline{z}\rangle, \forall \underline{x} \in l_E$. If we define $(\underline{A} - \underline{E}(j0))^*$ by $(\underline{A} - \underline{E}(j0))^*\underline{y} = \underline{z}$, then

$$\langle(\underline{A} - \underline{E}(j0))\underline{x},\, \underline{y}\rangle = \langle\underline{x},\, (\underline{A} - \underline{E}(j0))^*\underline{y}\rangle, \quad (\underline{x} \in l_E, \underline{y} \in \mathcal{D}\{\underline{E}(j0)^*\}) \tag{3.1}$$

The above arguments show that $(\underline{A} - \underline{E}(j0))^*$ is well-defined if its domain is restricted to $\mathcal{D}\{\underline{E}(j0)^*\}$. To complete our understanding about $(\underline{A} - \underline{E}(j0))^*$, we must know the structure of $\mathcal{D}\{\underline{E}(j0)^*\}$ and the matrix representation of $(\underline{A} - \underline{E}(j0))^*$. The following lemma gives these desired answers, which can be proved by following Example 7.10.1 of [55, p. 528] (a complete proof is given in Appendix A.2). Here $\mathcal{D}\{\underline{E}(j0)\} = l_E$ by definition.

Lemma 3.1 $\mathcal{D}\{\underline{E}(j0)^*\} = \mathcal{D}\{\underline{E}(j0)\} = l_E$. *Moreover, the matrix representation of* $\underline{E}(j0)^*$ *coincides with the usual complex conjugate transpose of that of* $\underline{E}(j0)$.

From Lemma 3.1, we have actually verified that $(\underline{A} - \underline{E}(j0))^*$ is also defined on l_E and that the matrix representation of $(\underline{A} - \underline{E}(j0))^*$ coincides with the usual complex conjugate transpose of that of $\underline{A} - \underline{E}(j0)$. Hence, from (3.1), we have

$$\langle(\underline{A} - \underline{E}(j0))\underline{x},\, \underline{y}\rangle = \langle\underline{x},\, (\underline{A} - \underline{E}(j0))^*\underline{y}\rangle \quad (\underline{x}, \underline{y} \in l_E) \tag{3.2}$$

The above results indicate that the operator $(\underline{A} - \underline{E}(j0))^*$ can be interpreted as both an infinite-dimensional matrix and an operator densely defined on l_2 (i.e., a mapping from l_E to l_2). Equipped with these facts, we are now in a position to give the theorem about the harmonic Lyapunov equation. In this theorem $\mathcal{S}^+$ is the set of all strictly positive-definite self-adjoint bounded operators on the Hilbert space l_2, i.e., if the operator S belongs to $\mathcal{S}^+$, then $\langle S\underline{x}, \underline{x}\rangle > 0$, for all $0 \neq \underline{x} \in l_2$.

Theorem 3.1 *Suppose in the system (2.1) that $A(t) \in L_{\mathrm{PCD}}[0,h]$. Then the system (2.1) is asymptotically stable if and only if for any $\underline{W} \in \mathcal{S}^+$, there exists a unique $\underline{V} \in \mathcal{S}^+$ satisfying*

$$(\underline{A} - \underline{E}(j0))^*\underline{V} + \underline{V}(\underline{A} - \underline{E}(j0)) = -\underline{W} \tag{3.3}$$

which is called the (infinite-dimensional) harmonic Lyapunov equation of the system (2.1) densely defined on l_2 (or more precisely, defined on the dense subset l_E of l_2).

Before giving the proof, we make a few remarks about the harmonic Lyapunov equation. First the harmonic Lyapunov equation (3.3) should be viewed as an operator-valued Lyapunov equation densely defined on l_2 (i.e., the domains of the operators involved in the equation are restricted to l_E, which is dense in l_2 by Lemma 2.9). The precise implication of this is that when we post-multiply $\underline{x} \in l_2$ on (3.3), $\underline{x}$ should belong to l_E to guarantee that it makes sense to deal with $(\underline{A} - \underline{E}(j0))^*\underline{V}\,\underline{x}$ and $\underline{V}(\underline{A} - \underline{E}(j0))\underline{x}$ separately and that the inner product is validated in the sense of (3.2). Now we show that this is indeed the case.

This is equivalent to showing that for the solution $\underline{V}$ of (3.3), $\underline{V}\,\underline{x} \in l_E$ for any $\underline{x} \in l_E$. To this end, take $\underline{x} \in l_E$ and post-multiply it on (3.3). Then, since $\underline{W}$, $\underline{V}$ and $\underline{A}$ are bounded on l_2, it follows that $\underline{V}(\underline{A}-\underline{E}(j0))\underline{x} \in l_2$ and $\underline{W}\,\underline{x} \in l_2$. Now we are led to $(\underline{A}-\underline{E}(j0))^*\underline{V}\,\underline{x} \in l_2$ which, in particular, implies that $-\underline{E}(j0)^*\underline{V}\,\underline{x} \in l_2$. However, since $-\underline{E}(j0)^* = \underline{E}(j0)$, we can conclude that $\underline{V}\,\underline{x} \in l_E$ as we claimed. The meaning of this first remark is that it makes sense to consider the inner product $\langle(\underline{A} - \underline{E}(j0))^*\underline{V}\,\underline{x},\, \underline{y}\rangle = \langle\underline{V}\,\underline{x},\, (\underline{A} - \underline{E}(j0))\underline{y}\rangle$ for any $\underline{x}, \underline{y} \in l_E$ since $\underline{V}\,\underline{x} \in l_E = \mathcal{D}((\underline{A} - \underline{E}(j0))^*)$. Simply speaking, in the following arguments, the inner product is validated by this sort of reasoning.

Second, it is easy to see that (3.3) could be derived by applying the Toeplitz transformation to the h-periodic time-varying Lyapunov equation $-\frac{d}{dt}V(t) = A(t)^TV(t) + V(t)A(t) + W(t)$ provided that $W(t) \in L_2[0,h]$ and that there exists a steady-state h-periodic solution to this matrix equation. However, to accomplish the proof in this direction, one has to impose some conditions to ensure that such an h-periodic solution has an absolutely convergent Fourier series expansion before applying the Toeplitz transformation. In contrast, our approach will involve only simple matrix algebra analysis and will not rely on the assumption about the existence of a periodic solution. Therefore, in Theorem 3.1, $\underline{W}$ is not confined to an infinite-dimensional block Toeplitz operator. For the solution of a class of general h-periodic Lyapunov equations, some results are given in [9].

The final point is that the harmonic Lyapunov equation (3.3) is not a special case of the operator Lyapunov equation in Theorem I.6.1 of [29] since the operator $\underline{A} - \underline{E}(j0)$ is unbounded. It should be noted that Theorem I.6.1 of [29] is stated and established under the assumption that the operator involved in the Lyapunov equation is bounded. Therefore, this study shows that the Lyapunov equation established by the harmonic analysis about $A(t)$ is nontrivial. Moreover, the solution can be expressed in a closed form with explicitly defined components without directly defining the exponential operator $\exp[(\underline{A} - \underline{E}(j0))t]$.

Proof of Theorem 3.1 From Theorem 2.5, the system (2.1) is asymptotically stable if and only if all the eigenvalues of $\underline{A} - \underline{E}(j0)$ lie in the open left-half plane. The proof will follow some similar steps to what we do in the LTI case. However, since we are dealing with infinite-dimensional matrices, the validity of such deductions must be justified.

(Sufficiency) Suppose the equation (3.3) holds for some $\underline{W}, \underline{V} \in \mathcal{S}^+$. Let λ be an eigenvalue of $\underline{A} - \underline{E}(j0)$ with an associated eigenvector $\underline{x} \in l_E \subset l_2$. Then, post-multiplying (3.3) by $\underline{x}$ and taking the inner product with $\underline{x}$, we obtain

$$\lambda + \bar{\lambda} = -\frac{\langle \underline{W}\,\underline{x}, \underline{x}\rangle}{\langle \underline{V}\,\underline{x}, \underline{x}\rangle} < 0$$

where the inner product is validated from the above discussions. This inequality implies that $\mathrm{Re}(\lambda) < 0$. Thus, the stability assertion follows.

(Necessity) Assume that the FDLCP system (2.1) is asymptotically stable. It must be shown that for any $\underline{W} \in \mathcal{S}^+$, there exists a unique operator $\underline{V} \in \mathcal{S}^+$ such that (3.3) holds on l_E. To this end, we define the infinite-dimensional exponential matrix function

$$\underline{e}(Q,t) := \mathrm{diag}[\cdots, e^{(Q+j\omega_h I)t}, e^{Qt}, e^{(Q-j\omega_h I)t}, \cdots] \tag{3.4}$$

By the definition, it follows that $\underline{e}(Q,t)$ is well-defined and uniformly bounded on l_2 over $t \geq 0$ under the stability assumption. To see this, noting that all the eigenvalues of Q have negative real parts, then, it follows from [25, p. 20] that there exist a pair of numbers $\hat{K} > 0$ and $\alpha > 0$ such that

$$\mathrm{Re}\{\lambda(Q)\} < -\alpha, \quad ||e^{Qt}|| \leq \hat{K}e^{-\alpha t} \quad (\forall t \geq 0) \tag{3.5}$$

On the other hand, observing that $\underline{e}(Q,t)$ is block-diagonal, we have

$$||\underline{e}(Q,t)||_{l_2/l_2} = \sup_{m\in\mathcal{Z}} ||e^{(Q+jm\omega_h I)t}|| = \sup_{m\in\mathcal{Z}} ||e^{Qt}|| \leq \hat{K}e^{-\alpha t} \leq \hat{K} \quad (\forall t \geq 0) \tag{3.6}$$

Next, we construct the operator

$$\hat{\underline{V}} := \int_0^\infty \underline{e}(Q,\tau)^* \hat{\underline{W}}\, \underline{e}(Q,\tau) d\tau \tag{3.7}$$

where $\hat{\underline{W}} = \underline{P}^* \underline{W}\, \underline{P} \in \mathcal{S}^+$. In the sequel, we first show that $\hat{\underline{V}} \in \mathcal{S}^+$ and then show that it is a unique solution of

$$(\underline{Q} - \underline{E}(j0))^* \hat{\underline{V}} + \hat{\underline{V}}(\underline{Q} - \underline{E}(j0)) = -\hat{\underline{W}} \tag{3.8}$$

in the elementwise sense (that is, here we regard (3.8) as infinitely many simultaneous equations of finite-dimensional matrices with infinitely many finite-dimensional matrix variables). We complete this in three steps.

Step 1. It is shown that $\underline{\hat{V}} \in \mathcal{S}^+$. Since $\underline{e}(Q,\tau)$ is invertible for all τ, it follows that $\underline{\hat{V}}$ is strictly positive definite. Furthermore, from (3.6), we have

$$\begin{aligned}
&||\underline{\hat{V}}||_{l_2/l_2} \leq \sup_{\underline{x}\in l_2}\Big\{\int_0^\infty ||\underline{e}(Q,\tau)^* \underline{\hat{W}}\, \underline{e}(Q,\tau)\underline{x}||_{l_2/l_2} d\tau : ||\underline{x}||_{l_2} = 1\Big\} \\
\leq\ & \sup_{\underline{x}\in l_2}\Big\{\int_0^\infty ||\underline{e}(Q,\tau)^* \underline{\hat{W}}\, \underline{e}(Q,\tau)||_{l_2/l_2} ||\underline{x}||_{l_2} d\tau : ||\underline{x}||_{l_2} = 1\Big\} \\
\leq\ & \int_0^\infty ||\underline{e}(Q,\tau)||^2_{l_2/l_2} ||\underline{\hat{W}}||_{l_2/l_2} d\tau \\
\leq\ & \int_0^\infty \hat{K}^2 e^{-2\alpha\tau} d\tau ||\underline{\hat{W}}||_{l_2/l_2} = \frac{\hat{K}^2}{2\alpha} ||\underline{\hat{W}}||_{l_2/l_2}
\end{aligned} \tag{3.9}$$

Finally, it is evident from the definition that $\underline{\hat{V}}^* = \underline{\hat{V}}$. Hence, we have $\underline{\hat{V}} \in \mathcal{S}^+$.

Step 2. It is shown that $\underline{\hat{V}}$ of (3.7) is a solution of (3.8). By the definition of $\underline{e}(Q,t)$, it is straightforward to see that

$$\underline{e}(Q,t)|_{t=0} = \underline{I}, \qquad \underline{e}(Q,t)|_{t=\infty} = \underline{0} \tag{3.10}$$

$$\frac{d}{dt}\underline{e}(Q,t) = (\underline{Q} - \underline{E}(j0))\underline{e}(Q,t) = \underline{e}(Q,t)(\underline{Q} - \underline{E}(j0)) \tag{3.11}$$

Using (3.10) and (3.11) in (3.8), we obtain

$$\begin{aligned}
&(\underline{Q} - \underline{E}(j0))^* \underline{\hat{V}} + \underline{\hat{V}}(\underline{Q} - \underline{E}(j0)) \\
=\ & \int_0^\infty \big(d\underline{e}(Q,\tau)\big)^* \underline{\hat{W}}\, \underline{e}(Q,\tau) + \int_0^\infty \underline{e}(Q,\tau)^* \underline{\hat{W}} \big(d\underline{e}(Q,\tau)\big) \\
=\ & \underline{e}(Q,\tau)^* \underline{\hat{W}}\, \underline{e}(Q,\tau)\Big|_0^\infty = -\underline{\hat{W}}
\end{aligned} \tag{3.12}$$

In the above deductions, there are frequently order interchanges of infinite-dimensional matrix $\underline{Q} - \underline{E}(j0)$ with infinite integral and derivative defined on infinite-dimensional matrix. Noting that both $\underline{Q} - \underline{E}(j0)$ and $\underline{e}(Q,\tau)$ are block-diagonal, the just-mentioned order interchanges are actually ones acting in the elementwise sense and thus are validated.

Step 3. It is shown that (3.8) has a unique solution. To see this, denote the (m,n)-th block entry of $\underline{\hat{V}}$ and $\underline{\hat{W}}$ by $[\underline{\hat{V}}]_{(m,n)}$ and $[\underline{\hat{W}}]_{(m,n)}$, respectively. Then, by comparing both sides of (3.8), it follows that

$$(Q + jm\omega_h I)^* [\underline{\hat{V}}]_{(m,n)} + [\underline{\hat{V}}]_{(m,n)}(Q + jn\omega_h I) = -[\underline{\hat{W}}]_{(m,n)} \tag{3.13}$$

By the stability assumption, all eigenvalues of Q have negative real parts, so that

$$\lambda[(Q + jm\omega_h I)^*] + \lambda[(Q + jn\omega_h I)] \neq 0 \quad \forall m,n \in \mathcal{Z}$$

where $\lambda[\cdot]$ denotes the eigenvalues of the matrix $[\cdot]$. Hence, by Theorem 4.4.6 of [40], the equation (3.13) has a unique solution $[\underline{\hat{V}}]_{(m,n)}$ for any $[\underline{\hat{W}}]_{(m,n)}$. Since $m, n \in \mathcal{Z}$ are arbitrary, it follows that (3.8) has a unique solution.

The above arguments indicate that the assertion we made about the equation (3.8) is true in the elementwise sense. However, repeating the arguments about the adjoint operator of $\underline{A} - \underline{E}(j0)$ on $\underline{Q} - \underline{E}(j0)$, it follows readily that (3.8) can also be viewed as an operator-valued (but with an infinite-dimensional matrix representation) Lyapunov equation defined on $l_E \subset l_2$ and that l_E is $\underline{\hat{V}}$-invariant.

Noting that l_E is $\underline{P}^{-1}$-invariant by Theorem 2.2, it follows that on $l_E \subset l_2$

$$\underline{P}^{-*}(\underline{Q} - \underline{E}(j0))^* \underline{\hat{V}}\,\underline{P}^{-1} + \underline{P}^{-*}\underline{\hat{V}}(\underline{Q} - \underline{E}(j0))\underline{P}^{-1} = -\underline{P}^{-*}\underline{\hat{W}}\,\underline{P}^{-1}$$

by pre-multiplying $\underline{P}^{-*}(= [\underline{P}^{-1}]^*)$ and post-multiplying $\underline{P}^{-1}$ on (3.8). On the other hand, since l_E is $\underline{P}^{-1}$-, $\underline{\hat{V}}$- and $\underline{P}^{-*}$-invariant, it is clear that $\underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1}\underline{x} \in l_E$ if $\underline{x} \in l_E$. Therefore, it can be claimed that on $l_E \subset l_2$

$$\underline{P}^{-*}(\underline{Q} - \underline{E}(j0))^* \underline{P}^* \underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1} + \underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1}\underline{P}(\underline{Q} - \underline{E}(j0))\underline{P}^{-1} = -\underline{P}^{-*}\underline{\hat{W}}\,\underline{P}^{-1}$$

where $\underline{P}^*\underline{P}^{-*} = \underline{I}$ and $\underline{P}^{-1}\underline{P} = \underline{I}$ correspond to the identity operators on l_E and l_2, respectively. Thus, it follows that on $l_E \subset l_2$

$$(\underline{A} - \underline{E}(j0))^* \underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1} + \underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1}(\underline{A} - \underline{E}(j0)) = -\underline{P}^{-*}\underline{\hat{W}}\,\underline{P}^{-1}$$

because $\underline{P}(\underline{Q} - \underline{E}(j0))\underline{P}^{-1} = \underline{A} - \underline{E}(j0)$ on l_E by (2.10) of Theorem 2.2 and because $\underline{P}^{-*}(\underline{Q} - \underline{E}(j0))^*\underline{P}^* = (\underline{A} - \underline{E}(j0))^*$ from (2.10) and the fact that the matrix representation of $(\underline{Q} - \underline{E}(j0))^*$ is the complex conjugate transpose of that of $\underline{Q} - \underline{E}(j0)$. Finally, since $\underline{P}^{-*}\underline{\hat{W}}\,\underline{P}^{-1} = \underline{W}$, it follows that $\underline{V} := \underline{P}^{-*}\underline{\hat{V}}\,\underline{P}^{-1} \in \mathcal{S}^+$ is a unique solution of (3.3). **Q.E.D.**

Remark 3.1 *Theorem* 3.1 *clearly says that* FDLCP *systems are essentially linear time-invariant as long as the asymptotic stability property is concerned and the conclusion is necessary and sufficient. Moreover, the solution of the harmonic Lyapunov equation has been expressed in a closed form as follows.*

$$\underline{V} := \underline{P}^{-*} \int_0^\infty \underline{e}(Q,\tau)^* \underline{P}^* \underline{W}\,\underline{P}\,\underline{e}(Q,\tau) d\tau \underline{P}^{-1} \tag{3.14}$$

However, it still involves the knowledge of the state transition matrix, which is hard to determine in general FDLCP *systems. In view of this, the value of Theorem* 3.1 *is limited to theoretical analysis. However, it plays a key role in establishing a trace formula for the* H_2 *norm in* FDLCP *systems. Our discussions in Chapter* 4 *show that Theorem* 3.1 *can also provide us with some very useful stability tests which are practically applicable.*

3.2 Extended Gerschgorin Criterion

As explained in the last paragraph of Section 2.4, the eigenvalues of the system operator $\underline{A} - \underline{E}(j0)$ have a strip distribution pattern along the imaginary axis. This inspires us with the idea to extend the Gerschgorin criterion to exploit this fact. In this section, we discuss a sufficient stability condition by extending the Gerschgorin theorem to linear operators on the linear space l_2, which utilizes the fact that the eigenvalues of $\underline{A} - \underline{E}(j0)$ can be constrained by countably infinitely many discs.

Theorem 3.2 *Assume that the $n \times n$ matrix $A(t)$ belongs to $L_{\text{PCD}}[0,h] \cap L_{\text{CAC}}[0,h]$ and $\{A_m\}_{m=-\infty}^{+\infty}$ is the Fourier coefficients sequence of $A(t)$. Then the system (2.1) is asymptotically stable if the disc-group $\mathcal{D}_0 := \bigcup_{k=1}^{n} \mathcal{D}_{0k}$ lies in the open left-half plane. Here*

$$\mathcal{D}_{0k} := \{z \in \mathcal{C} : |z - a_{0kk}| \le \Delta_k\} \quad k = 1, 2, \cdots, n$$

with $\Delta_k = \sum_{i=1}^{n} \sum_{m=-\infty}^{+\infty} |a_{mki}| - |a_{0kk}|$ where a_{mki} is the (k,i)-th entry of the matrix A_m.

Furthermore, if there are $m(< n)$ discs $\mathcal{D}_{0i_1}, \mathcal{D}_{0i_2}, \cdots, \mathcal{D}_{0i_m}$ such that $\mathcal{D}'_0 + jl\omega_h$ and $\mathcal{D}''_0$, with $\mathcal{D}'_0 := \bigcup_{k=i_1,\cdots,i_m} \mathcal{D}_{0k}$ and $\mathcal{D}''_0 := \bigcup_{k \ne i_1,\cdots,i_m} \mathcal{D}_{0k}$, are disjoint for all $l \in \mathcal{Z}$. Then, the system (2.1) is unstable if either $\mathcal{D}'_0$ or $\mathcal{D}''_0$ lies in the closed right-half plane.

First we give some remarks. The assumption $A(t) \in L_{\text{CAC}}[0,h]$ guarantees that the disc-group is meaningful in the sense that $\Delta_k < \infty, \forall k = 1, 2, \cdots, n$. By Lemma 2.3, $L_{\text{CPCD}}[0,h] \subset L_{\text{PCD}}[0,h] \cap L_{\text{CAC}}[0,h]$. Hence, Theorem 3.2 applies if the state matrix $A(t)$ belongs to $L_{\text{CPCD}}[0,h]$. The assumption that $A(t) \in L_{\text{CPCD}}[0,h]$ is satisfied by a large class of practical FDLCP systems and can be tested simply.

Proof of Theorem 3.2 Since $A(t) \in L_{\text{PCD}}[0,h]$, Theorem 2.5 applies. Now let λ be an eigenvalue of $\underline{A} - \underline{E}(j0)$ with $\underline{x} \in l_E \subset l_2$ being an associated eigenvector. Then $(\underline{A} - \underline{E}(j0))\underline{x} = \lambda \underline{x}$. Now denote $\underline{x} = [\cdots, x_{-1}, x_0, x_1, \cdots]^T$ where x_i is a scalar, and let $|x_s| = \max_{m \in \mathcal{Z}} |x_m| > 0$ which can be attained at a finite s since $\underline{x} \in l_2$. Using arguments similar to [49], one can show that the eigenvalues of $\underline{A} - \underline{E}(j0)$ lie in $\bigcup_{k=1}^{n} \bigcup_{l=-\infty}^{+\infty} \mathcal{D}_{lk}$ where

$$\mathcal{D}_{lk} = \{z \in \mathcal{C} : |z - a_{0kk} + jl\omega_h| \le \Delta_k\}, \quad l \in \mathcal{Z}, \quad k \in \{1, 2, \cdots, n\}.$$

However, by the definition of $\mathcal{D}_{lk}$, it follows that for each k, $\mathcal{D}_{lk} = \mathcal{D}_{mk} + j(l-m)\omega_h$ in the pointwise sense $(\forall l, m \in \mathcal{Z})$. Hence, if for some $l \in \mathcal{Z}$, the disc-group $\bigcup_{k=1}^{n} \mathcal{D}_{lk}$ lie in the open left-half plane, then so do all the other disc-groups. This gives the first assertion.

To see the second part, we define $A(\rho, t) = D + \rho(A(t) - D)$ with ρ being a constant in $[0,1]$ and $D = \text{diag}[a_{011}, a_{022}, \cdots, a_{0nn}]$. It is clear that for each $\rho \in [0,1]$, $A(\rho, t) \in L_{\text{PCD}}[0,h] \cap L_{\text{CAC}}[0,h]$. Hence, by means of Theorem 2.5 and Gronwall's Lemma [25], [38] (i.e., Lemma 4.1), it can be shown that the eigenvalues of $\underline{A}(\rho) - \underline{E}(j0)$, with $\underline{A}(\rho) := \mathcal{T}\{A(\rho, t)\}$, are continuous with respect to ρ (i.e., Proposition 4.1 in Chapter 4).

Now let us define the discs

$$\mathcal{D}_{lk}(\rho) = \{z \in \mathcal{C} : |z - a_{0kk} + jl\omega_h| \le \rho\Delta_k\}, \quad l \in \mathcal{Z}, \quad k \in \{1, 2, \cdots, n\}$$

Noting that $\mathcal{D}_{lk}(\rho) \subset \mathcal{D}_{lk}, \forall \rho \in [0,1]$, $l \in \mathcal{Z}$ and $k = 1, 2, \cdots, n$, it turns out that $\mathcal{D}'(\rho) := \bigcup_{l=-\infty}^{+\infty} \bigcup_{k=i_1,\cdots,i_m} \mathcal{D}_{lk}(\rho)$ and $\mathcal{D}''(\rho) := \bigcup_{l=-\infty}^{+\infty} \bigcup_{k\neq i_1,\cdots,i_m} \mathcal{D}_{lk}(\rho)$ lie in $\bigcup_{l=-\infty}^{+\infty} \bigcup_{k=i_1,\cdots,i_m} \mathcal{D}_{lk}$ and $\bigcup_{l=-\infty}^{+\infty} \bigcup_{k\neq i_1,\cdots,i_m} \mathcal{D}_{lk}$, respectively.

On the other hand, note that $\mathcal{D}_{lk}(\rho)$ is defined from $A(\rho, t)$ in the same geometric meaning as $\mathcal{D}_{lk}$ is defined from $A(t)$. Therefore, the first assertion says that for any $\rho \in [0,1]$, the eigenvalues of $\underline{A}(\rho) - \underline{E}(j0)$ lie in $\mathcal{D}(\rho) := \bigcup_{k=1}^{n} \bigcup_{l=-\infty}^{+\infty} \mathcal{D}_{lk}(\rho) = \mathcal{D}'(\rho) \bigcup \mathcal{D}''(\rho)$. Then the second assertion follows from Theorem 2.5 and the assumptions about $\mathcal{D}'_0$ and $\mathcal{D}''_0$, which are the discs defined by letting $\rho = 1$, if we show that for any $\rho \in [0,1]$, both $\mathcal{D}'(\rho)$ and $\mathcal{D}''(\rho)$ contain at least one eigenvalue of $\underline{A}(\rho) - \underline{E}(j0)$.

To see this, let $\rho = 0$ and note that the eigenvalues of $\underline{A}(0) - \underline{E}(j0)$ are $a_{011} + jl\omega_h, a_{022} + jl\omega_h, \cdots, a_{0nn} + jl\omega_h, l \in \mathcal{Z}$, which are the centers of the discs $\mathcal{D}_{lk}(\rho), k = 1, \cdots, n, l \in \mathcal{Z}, \rho \in [0,1]$. By the continuity of the eigenvalues of $\underline{A}(\rho) - \underline{E}(j0)$ in ρ and the fact that $\mathcal{D}'(\rho)$ and $\mathcal{D}''(\rho)$ are disjoint since $\mathcal{D}'_0$ and $\mathcal{D}''_0$ are disjoint, the desired result follows. **Q.E.D.**

Generally speaking, the Gerschgorin criterion is conservative and cannot be used directly on the Fourier series expansion of $A(t)$ when the average matrix A_0 of $A(t)$ lacks some diagonal predominance. One way to get around this problem is to introduce the constant similarity transformation R on $A(t)$ such that $R^{-1}A_0R$ is diagonal or at least diagonal predominant. However, this means that the original state matrix is changed to $R^{-1}A(t)R$. Since the transform matrix R may be complex, it should be ensured that after such transformation, the results above are still valid because the Floquet theorem is stated for real systems. Now we show that this is the case.

In the original (real) system, we have known that

$$\underline{P}(\underline{Q} - \underline{E}(j0))\underline{P}^{-1} = \underline{A} - \underline{E}(j0)$$

on $l_E \subset l_2$ under the assumption that $A(t) \in L_{\mathrm{PCD}}[0, h]$. Now pre-multiplying the above equation with $\mathcal{T}\{R^{-1}\} = \underline{R}^{-1}$ and post-multiplying with $\mathcal{T}\{R\} = \underline{R}$ and noting that l_E is $\underline{R}$-invariant by the block-diagonal form of $\underline{R}$, it follows that

$$\tilde{\underline{P}}(\underline{R}^{-1}\underline{Q}\,\underline{R} - \underline{E}(j0))\tilde{\underline{P}}^{-1} = \underline{R}^{-1}\underline{A}\,\underline{R} - \underline{E}(j0) \tag{3.15}$$

on $l_E \subset l_2$ with $\tilde{\underline{P}} := \underline{R}^{-1}\underline{P}\,\underline{R}$. Here, it is easy to show that l_E is $\tilde{\underline{P}}$-invariant and $\tilde{\underline{P}}^{-1}$-invariant. This, together with (3.15), implies that the set of the eigenvalues of $\underline{R}^{-1}\underline{A}\,\underline{R} - \underline{E}(j0)$ is equal to that of $\underline{R}^{-1}\underline{Q}\,\underline{R} - \underline{E}(j0)$, which in turn is clearly equal to that of $\underline{Q} - \underline{E}(j0)$. However, $\underline{R}^{-1}\underline{A}\,\underline{R}$ is just the Toeplitz operator of $R^{-1}A(t)R$. Therefore, Theorem 3.2 is valid even if a complex similarity transformation is applied to $A(t)$.

Example 3.1 *We consider the stability problem of the lossy Mathieu differential equation [3], [59], [70] by the Gerschgorin criterion. The* FDLCP *system is given by*

$$\ddot{x}(t) + 2\xi\dot{x}(t) = [1 - 2\beta\cos\omega_h t]u(t), \quad \omega_h = 2 \quad (i.e., h = \pi) \tag{3.16}$$

which leads to the state-space model below:

$$A_o(t) = \begin{bmatrix} 0 & 1 \\ 0 & -2\xi \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 - 2\beta\cos\omega_h t \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$$

By the output feedback $u(t) = ky(t)$ where k is a scalar, the closed-loop state matrix is

$$A(t) = \begin{bmatrix} 0 & 1 \\ k(1 - 2\beta\cos\omega_h t) & -2\xi \end{bmatrix}$$

which is h-periodic and each element of $A(t)$ is continuous and differentiable in $[0, h]$.

It is clear that the closed-loop state matrix satisfies the condition that $A(t) \in L_{\mathrm{CPCD}}[0, h]$. However, since the structure of $A(t)$ prevents us from applying the criterion effectively, it is necessary to introduce a similarity transformation on $A(t)$ so that the 'DC part' becomes diagonal. This does not affect the stability as we described just before this example.

The results are given in Figure 3.1. In these figures, areas marked by circles correspond to coefficients ranges at which stability of the corresponding FDLCP systems is uncertain (that is, stability cannot be tested by the Gerschgorin criterion). The areas left empty are the coefficients range in which the FDLCP systems are asymptotically stable. □

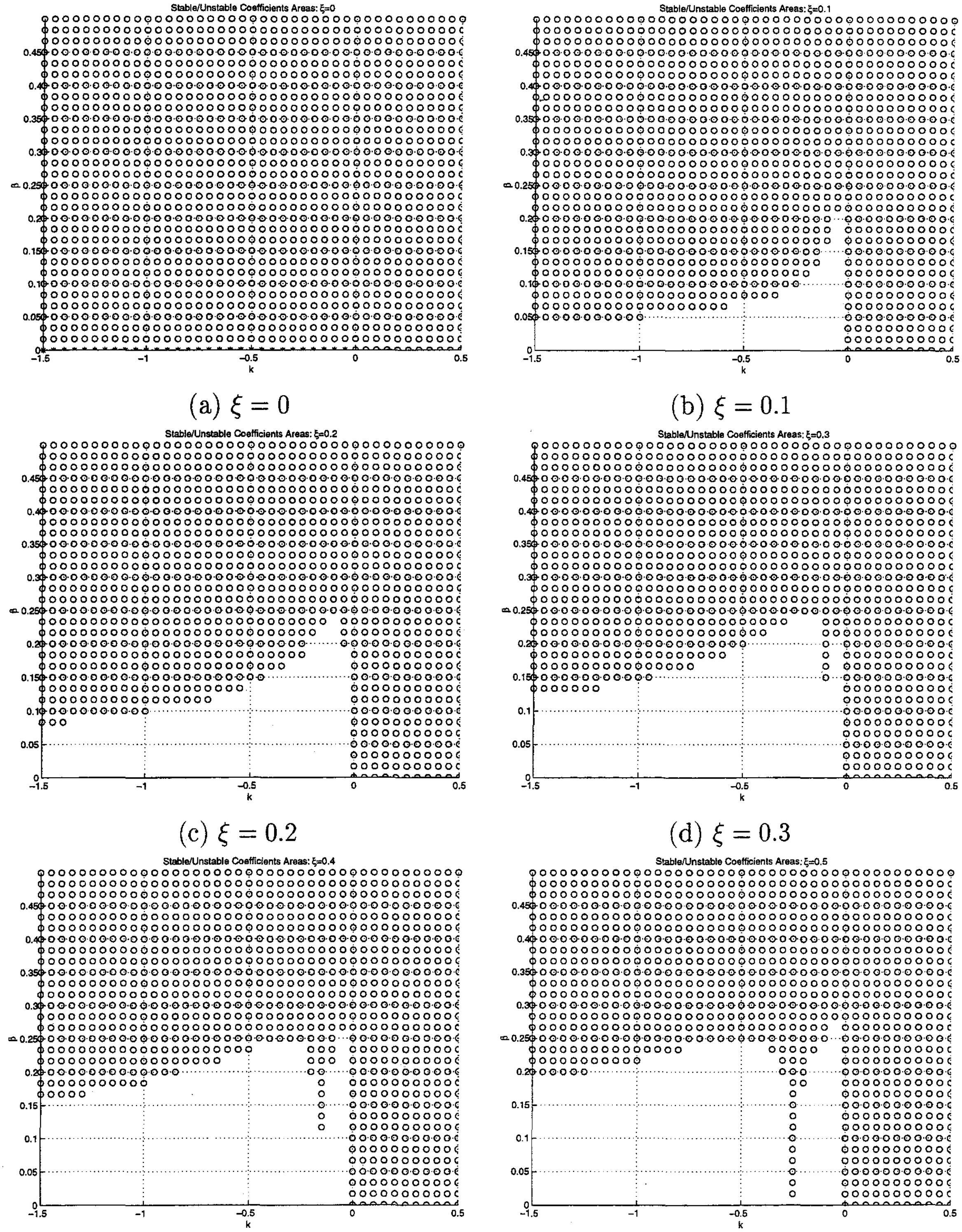
3.3 Frequency Response of FDLCP Systems

Another important aspect of FDLCP systems analysis is how to establish their frequency response relations, which may give us an alternative tool to deal with problems in periodically time-varying systems. Several ways to define frequency response relations have been surveyed in the introduction chapter, so that here we concentrate our attention only on the definition through the input/output steady-state analysis. This is first proposed in [69], [70], and the basic idea can be described as follows. First, impose an l_p-EMP signal u with $1 \leq p < \infty$ (where EMP stands for exponentially modulated periodic) to the system (2.1). By definition, such u is given by

$$u(t) = \sum_{m=-\infty}^{+\infty} u_m e^{j(\varphi + m\omega_h)t} = \sum_{m=-\infty}^{+\infty} u_m e^{j\varphi_m t} \quad (t \geq 0, \varphi \in \mathcal{I}_0, \varphi_m = \varphi + m\omega_h)$$

where the infinite-dimensional vector $\underline{u} := [\cdots, u_{-1}^T, u_0^T, u_1^T, \cdots]^T$ belongs to l_p. Second, measure the steady-state output y of the system, which is (assumed to be) also an l_p-EMP signal under the asymptotic stability assumption of the FDLCP system and represent the signal y by the infinite-dimensional vector $\underline{y} := [\cdots, y_{-1}^T, y_0^T, y_1^T, \cdots]^T \in l_p$ according to the definition of l_p-EMP signals. Finally, the input-output response relation observed in the above is expressed as a mapping $\underline{G}(j\varphi) : \underline{u} \mapsto \underline{y} : l_p \to l_p$.

In the above arguments the Fourier series expansions of $A(t), B(t), C(t)$ and $D(t)$ as well as the Toeplitz operators expressions of these h-periodic matrix functions are used



(a) $\xi = 0$

(b) $\xi = 0.1$

(c) $\xi = 0.2$

(d) $\xi = 0.3$

(e) $\xi = 0.4$

(f) $\xi = 0.5$

Figure 3.1: Stable coefficients areas (blank: G stable, circle: stability of G unknown by the Gerschgorin criterion, i.e., Theorem 3.2)

repeatedly. It should be pointed out that the validity of such use has not been verified rigorously in [69], [70]. In the following, we will reconsider the above-mentioned arguments and scrutinize the convergence problems of the Fourier series expansions and the Toeplitz transformations involved [87]. Only the cases of $p = 2$ and $p = 1$ are considered.

3.3.1 Frequency Response Operator Viewed on l_2

First we investigate the frequency response relation when the input is an l_2-EMP signal. To this purpose, we note from the Floquet theorem and Remark 2.2 that

$$\begin{aligned} y(t) &= C(t)P(t,0)e^{Q(t-t_0)}P^{-1}(t_0,0)x_0 \\ &\quad + C(t)P(t,0)\int_{t_0}^{t} e^{Q(t-\tau)}P^{-1}(\tau,0)B(\tau)u(\tau)d\tau + D(t)u(t) \\ &= \hat{C}(t)[e^{Q(t-t_0)}q_0 + \int_{t_0}^{t} e^{Q(t-\tau)}\hat{B}(\tau)u(\tau)d\tau] + D(t)u(t) \end{aligned} \tag{3.17}$$

with $\hat{B}(t) := P^{-1}(t,0)B(t), \hat{C}(t) := C(t)P(t,0)$ and $q_0 := P^{-1}(t_0,0)x_0$. The second relation of (3.17) says that if we introduce the initial value transformation $q_0 = P^{-1}(t_0,0)x_0$, the system (2.1) can be represented equivalently in the input-output sense by the system configuration shown in Figure 3.2.

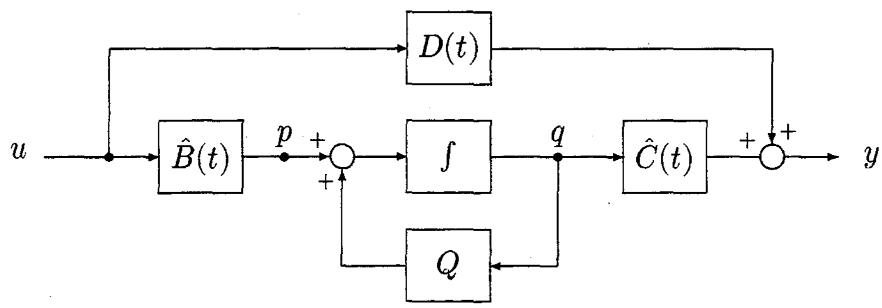


Figure 3.2: Equivalent system configuration

Now we are in a position to establish the frequency response relation in the system of Figure 3.2 by imposing an l_2-EMP sinusoid input u to the system and measuring the steady-state output y. From Figure 3.2, this can be completed by showing that under certain assumptions given below, the steady-state responses of p, q and y are also l_2-EMP signals so that the input-output response relation $u \mapsto y$ can be written as a mapping of $\underline{u} \mapsto \underline{y} : l_2 \to l_2$. We complete this in three steps.

Step 1. Take an h-periodic continuous signal $\hat{u}(t) \in L_2[0,h]$ such that $\mathcal{F}\{\hat{u}(\cdot)\} =: \underline{\hat{u}} \in l_1 \subset l_2$. Then, the Fourier series expansion of $\hat{u}(t)$ is absolutely convergent. Constructing the input l_2-EMP signal as $u(t) = \hat{u}(t)e^{j\varphi t}, \varphi \in \mathcal{I}_0$ (where $\mathcal{I}_0$ is defined in (2.8)), it follows that the corresponding output of $\hat{B}(t)$ to this l_2-EMP signal is

$$p(t) = \hat{p}(t)e^{j\varphi t} \quad (t \geq 0, \varphi \in \mathcal{I}_0)$$

where $\hat{p}(t) = \hat{B}(t)\hat{u}(t) = P^{-1}(t,0)B(t)\hat{u}(t)$.

Now assume that $B(t) \in L_{\rm PCC}[0,h]$. Then, based on the choice of $\hat{u}$ and Lemma 2.2 (the assertion is expressed in terms of the operator $\mathcal{F}$ rather than the Toeplitz transformation, by taking the central column), we obtain

$$\mathcal{F}\{B(\cdot)\hat{u}(\cdot)\} = \underline{B}\,\mathcal{F}\{\hat{u}(\cdot)\} = \underline{B}\,\underline{\hat{u}}$$

where $\underline{\hat{u}} := \mathcal{F}\{\hat{u}(\cdot)\}$ and the Fourier series expansion of $B(t)\hat{u}(t)$ is convergent to $B(t_0)\hat{u}(t_0)$ for a.e. $t_0 \in [0,h]$.

Furthermore, let us assume that $A(t) \in L_{\rm PCD}[0,h]$. Then from Proposition 2.1, $P^{-1}(t,0)$ is continuous and the Fourier series expansion of $P^{-1}(t,0)$ is absolutely convergent. Again by Lemma 2.2 and Proposition 2.1, we obtain

$$\mathcal{F}\{P^{-1}(\cdot,0)B(\cdot)\hat{u}(\cdot)\} = \mathcal{T}\{P^{-1}(\cdot,0)\}\mathcal{F}\{B(\cdot)\hat{u}(\cdot)\} = \underline{P}^{-1}\underline{B}\,\underline{\hat{u}}$$

which can be interpreted as

$$\mathcal{F}\{\hat{p}(\cdot)\} =: \underline{\hat{p}} = \underline{P}^{-1}\underline{B}\,\underline{\hat{u}} \in l_2 \tag{3.18}$$

The assertion that $\underline{\hat{p}} \in l_2$ follows from the facts that $\underline{P}^{-1}$ and $\underline{B}$ are bounded on l_2 under the assumptions on $A(t)$ and $B(t)$. From these arguments, it follows that $\hat{p}(t)e^{j\varphi t} = p(t)$ is also l_2-EMP. In other words, one can conclude that $p(t) = \sum_{m=-\infty}^{+\infty} p_m e^{j(\varphi+m\omega_h)t}$ with $p_m := [\underline{\hat{p}}]_m$.

Remark 3.2 *The reason why we constrain the domain of $\hat{u}$ is that if we work on a general $\hat{u}(t) \in L_2[0,h]$, we may not arrive at the above conclusions for some $\underline{\hat{u}} \in l_2$ because of the convergence problems in the Fourier series expansions and the Toeplitz transformation.*

Step 2. Now impose the signal p to the LTI subsystem of Figure 3.2. We suppose that this subsystem is asymptotically stable (i.e., all the eigenvalues of Q have negative real parts). Then, by the superposition principle of linear systems (Theorem 5.6.2 [55, p. 237]), the output q of the LTI subsystem to p is

$$\begin{aligned} q(t) &= e^{Qt}q_0 + \int_0^t e^{Q(t-\tau)} \sum_{m=-\infty}^{+\infty} p_m e^{j(\varphi+m\omega_h)\tau} d\tau \\ &= e^{Qt}\Big(q_0 + \sum_{m=-\infty}^{+\infty} \int_0^t e^{(j\varphi_m I - Q)\tau} d\tau p_m\Big) \\ &= e^{Qt}\Big(q_0 + \sum_{m=-\infty}^{+\infty} (Q - j\varphi_m I)^{-1} p_m\Big) + \sum_{m=-\infty}^{+\infty} (j\varphi_m I - Q)^{-1} p_m e^{j\varphi_m t} \end{aligned} \tag{3.19}$$

On the other hand, by the stability assumption of Q, (2.19) is true for all $\varphi \in \mathcal{I}_0$. Therefore, we observe by the Cauchy-Schwarz inequality and Lemma A in Appendix $A.1$ that

$$\sum_{m=-\infty}^{+\infty} ||(Q - j\varphi_m I)^{-1} p_m|| \le \sum_{m=-\infty}^{+\infty} ||(Q - j\varphi_m I)^{-1}|| \cdot ||p_m||$$

$$\leq \Big(\sum_{m=-\infty}^{+\infty} \|(Q - j\varphi_m I)^{-1}\|^2\Big)^{\frac{1}{2}} \Big(\sum_{m=-\infty}^{+\infty} \|p_m\|^2\Big)^{\frac{1}{2}}$$

$$\leq K\Big(\sum_{m=-\infty}^{+\infty} f(m)^2\Big)^{\frac{1}{2}} \|\underline{p}\|_{l_2} \leq \sqrt{5}K\,\|\underline{p}\|_{l_2} \tag{3.20}$$

where

$$\underline{p} := [\cdots, p_{-1}^T, p_0^T, p_1^T, \cdots] = \underline{\hat{p}} \in l_2 \tag{3.21}$$

The inequality (3.20) implies that the summation $\sum_{m=-\infty}^{+\infty}(Q - j\varphi_m I)^{-1}p_m$ is absolutely convergent for any $\varphi \in \mathcal{I}_0$. Combining this fact with (3.19), it follows that as $t \to \infty$, the steady-state response of q is

$$\Big(\sum_{m=-\infty}^{+\infty} (j\varphi_m I - Q)^{-1} p_m e^{jm\omega_h t}\Big) e^{j\varphi t} \quad (t \geq 0)$$

since $e^{Qt} \to 0$. This steady-state output q of the LTI subsystem can be expressed as

$$q(t) := \hat{q}(t)e^{j\varphi t} \quad (t \geq 0)$$

where $\hat{q}(t) = \sum_{m=-\infty}^{+\infty} \hat{q}_m e^{jm\omega_h t}$ with $\hat{q}_m := (j\varphi_m I - Q)^{-1}p_m$. The inequality (3.20) indicates that $\mathcal{F}\{\hat{q}(\cdot)\} =: \underline{\hat{q}} \in l_1 \subset l_2$. Consequently, $\hat{q}(t) \in L_2[0, h]$ and the Fourier series expansion of $\hat{q}(t)$ is absolutely convergent. Obviously, $q(t)$ is an l_2-EMP signal.

Here, the fact that $\underline{\hat{q}} \in l_1 \subset l_2$ can be shown in another way. By the definition of $\underline{\hat{q}}$,

$$\underline{\hat{q}} = (\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{p} \tag{3.22}$$

where $(\underline{E}(j\varphi) - \underline{Q})^{-1} := \mathrm{diag}[\cdots, (j\varphi_{-1}I - Q)^{-1}, (j\varphi_0 I - Q)^{-1}, (j\varphi_1 I - Q)^{-1}, \cdots]$. Then the assertion that $\underline{\hat{q}} \in l_E \subset l_1$ follows readily since $(\underline{E}(j\varphi) - \underline{Q})^{-1}$ is a mapping from l_2 to $l_E \subset l_1$ by Theorem 2.3 and Lemma 2.9.

Step 3. Since the Fourier series expansion of $P(t, 0)$ is absolutely convergent, the assertion in the last paragraph of Step 2 actually says that the Fourier series expansion of $P(t, 0)\hat{q}(t)$ is also absolutely convergent by Lemma 2.6. Now repeating the arguments in Step 1 on the matrix function $C(t)P(t, 0)\hat{q}(t)$, the relation

$$\mathcal{F}\{\hat{y}(\cdot)\} =: \underline{\hat{y}} = \underline{C}\,\underline{P}\,\underline{\hat{q}} \tag{3.23}$$

can be asserted if $C(t) \in L_{\mathrm{PCC}}[0, h]$, where $\hat{y}$ is the output of $\hat{C}(t) = C(t)P(t, 0)$ to $\hat{q}(t)$. It is clear that $\underline{\hat{y}} \in l_2$, and thus the output y of $\hat{C}(t)$ to the input $q(t) = \hat{q}(t)e^{j\varphi t}, \varphi \in \mathcal{I}_0$ is an l_2-EMP signal.

Finally, from (3.18), (3.21), (3.22) and (3.23) and by Theorem 2.3, we obtain

$$\underline{y} = \underline{C}\,\underline{P}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{P}^{-1}\underline{B}\,\underline{u} = \underline{C}(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}\,\underline{u}$$

by setting $\underline{u} := \underline{\hat{u}}$ and $\underline{y} := \underline{\hat{y}}$. Summarizing the above discussions together with $D(t)$ taken into consideration, we can state the following main result of this subsection.

Theorem 3.3 *Assume in the system (2.1) that $A(t)$ belongs to $L_{\rm PCD}[0,h]$, $B(t), C(t)$ and $D(t)$ belong to $L_{\rm PCC}[0,h]$ and that the system is asymptotically stable. Then the steady-state response of the* FDLCP *system (2.1) to the l_2-*EMP *input $u(t) = \sum_{m=-\infty}^{+\infty} u_m e^{j\varphi_m t}$ with $\underline{u} = [\cdots, u_{-1}^T, u_0^T, u_1^T, \cdots]^T \in l_1 \subset l_2$ is also an l_2-*EMP *signal $y(t) = \sum_{m=-\infty}^{+\infty} y_m e^{j\varphi_m t}$ with $\underline{y} = [\cdots, y_{-1}^T, y_0^T, y_1^T, \cdots]^T = \underline{G}(j\varphi)\underline{u} \in l_2$, where*

$$\underline{G}(j\varphi) := \underline{C}(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B} + \underline{D} \tag{3.24}$$

which is a densely defined mapping on l_2 for each $\varphi \in \mathcal{I}_0$. Also, $\underline{G}(j\varphi)$ is uniformly bounded over $\varphi \in \mathcal{I}_0$ in the sense that $||\underline{G}(j\varphi)||_{l_2/l_1(l_2)} \leq K < \infty, \forall \varphi \in \mathcal{I}_0$ for some $K > 0$, where

$$||\underline{G}(j\varphi)||_{l_2/l_1(l_2)} := \sup_{0 \neq \underline{x} \in l_1} \left\{ \frac{||\underline{G}(j\varphi)\underline{x}||_{l_2}}{||\underline{x}||_{l_2}} \right\}$$

Proof By the assumption on $D(t)$, it is easy to see that $\mathcal{F}\{D(\cdot)u(\cdot)\} = \underline{D}\,\underline{u}$. From this fact, together with the preceding arguments, the assertion about (3.24) follows. It is also clear from the above arguments that $\underline{G}(j\varphi)$ is a mapping from l_1 into l_2. However, l_1 is dense in l_2 so that the frequency response operator established via the input/output steady-state analysis is a densely defined operator on l_2 [55, p. 486]. To see the uniform boundedness of $\underline{G}(j\varphi)$ over the interval $\mathcal{I}_0$, we note that $\underline{B}, \underline{C}$ and $\underline{D}$ are bounded on l_2 by the assumptions on $B(t), C(t)$ and $D(t)$ from Lemma 2.8. Then, the uniform boundedness assertion of $\underline{G}(j\varphi)$ follows from Theorem 2.3. **Q.E.D.**

Remark 3.3 *Note in Theorem* 3.3 *that we have used the l_2-norm on the linear space l_1. Accordingly, $||\underline{G}(j\varphi)||_{l_2/l_1(l_2)}$ is the l_2-induced norm of $\underline{G}(j\varphi)$ on the dense subset l_1 of l_2. It is also clear from the mathematical expression of the frequency response operator of the* FDLCP *system (2.1) that this operator can have two interpretations. The first one is to view it as a mapping from l_1 into l_2, which has a clear steady-state analysis interpretation as we discussed in the above; the second is to treat it as a mapping on l_2. The second viewpoint makes sense because it can be seen as a mapping with the extended domain l_2 instead of the original domain l_1 and this mapping itself is bounded on l_2 (since all the operators in $\underline{G}(j\varphi)$ are bounded on l_2, and this fact is used in the uniform boundedness proof of $\underline{G}(j\varphi)$).*

Here, to distinguish these two interpretations in the above remark about the frequency response operator, the frequency response operator in the first interpretation is denoted by $\underline{\tilde{G}}(j\varphi)$ while the second is by $\underline{G}(j\varphi)$. In other words, $\underline{\tilde{G}}(j\varphi)$ and $\underline{G}(j\varphi)$ have the same matrix expression but are defined on different domains. Compared with $\underline{G}(j\varphi)$, the frequency response operator $\underline{\tilde{G}}(j\varphi)$ defined via the input/output steady-state analysis is 'deficient' in the sense that the domain of $\underline{\tilde{G}}(j\varphi)$ is a dense subset of l_2. However, the following corollary shows that the l_2-induced norm of $\underline{\tilde{G}}(j\varphi)$ from l_1 to l_2 coincides with the l_2-induced norm of $\underline{G}(j\varphi)$ on l_2. This validates the existing studies on the definition (Section 3.4) and computation (Chapter 4) of the H_∞ norm of the FDLCP system (2.1) based on the frequency response operator $\underline{G}(j\varphi)$.

Corollary 3.1 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\rm PCD}[0,h]$, $B(t), C(t)$ and $D(t)$ belong to $L_{\rm PCC}[0,h]$ and that the system is asymptotically stable. Then, $\underline{G}(j\varphi)$ is bounded on l_2 uniformly over $\varphi \in \mathcal{I}_0$ and for all $\varphi \in \mathcal{I}_0$, it holds*

$$||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)} = ||\underline{G}(j\varphi)||_{l_2/l_2}$$

Proof By Theorem 3.3 and Remark 3.3, the first assertion that $\underline{G}(j\varphi)$ is bounded on l_2 uniformly over $\varphi \in \mathcal{I}_0$ follows readily. To see the second assertion, we note that $l_1 \subset l_2$. Then it is obvious that $||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)} \le ||\underline{G}(j\varphi)||_{l_2/l_2}$. Hence, the proof becomes complete if we show that for all $\varphi \in \mathcal{I}_0$,

$$||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)} \ge ||\underline{G}(j\varphi)||_{l_2/l_2} \tag{3.25}$$

By definition, for any $\mu > 0$, there exists $\underline{x} \in l_2$ with $||\underline{x}||_{l_2} = 1$ such that

$$||\underline{G}(j\varphi)||_{l_2/l_2} \le ||\underline{G}(j\varphi)\underline{x}||_{l_2} + \mu$$

Since l_1 is dense in l_2, for any $\epsilon > 0$, there exists $\underline{x}' \in l_1$ such that $||\underline{x} - \underline{x}'||_{l_2} < \epsilon$. Therefore, from the fact that $\underline{\tilde{G}}(j\varphi)$ and $\underline{G}(j\varphi)$ have the same matrix expression on l_1, we observe

$$\begin{aligned}
||\underline{G}(j\varphi)||_{l_2/l_2} &\le ||\underline{G}(j\varphi)\underline{x}'||_{l_2} + ||\underline{G}(j\varphi)(\underline{x} - \underline{x}')||_{l_2} + \mu \\
&\le ||\underline{\tilde{G}}(j\varphi)\underline{x}'||_{l_2} + ||\underline{G}(j\varphi)||_{l_2/l_2}||\underline{x} - \underline{x}'||_{l_2} + \mu \\
&< ||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)}||\underline{x}'||_{l_2} + \epsilon\, ||\underline{G}(j\varphi)||_{l_2/l_2} + \mu \\
&< ||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)}(1+\epsilon) + \epsilon\, ||\underline{G}(j\varphi)||_{l_2/l_2} + \mu
\end{aligned}$$

If ϵ is small enough, the above inequality can be rewritten as

$$||\underline{G}(j\varphi)||_{l_2/l_2} < \frac{1+\epsilon}{1-\epsilon}||\underline{\tilde{G}}(j\varphi)||_{l_2/l_1(l_2)} + \frac{\mu}{1-\epsilon}$$

which implies the desired assertion of (3.25) since μ can be arbitrarily small and for sufficiently small $\epsilon > 0$, it is true that $\frac{1+\epsilon}{1-\epsilon} > 1$ and $\lim_{\epsilon \to 0} \frac{1+\epsilon}{1-\epsilon} = 1$. **Q.E.D.**

3.3.2 Frequency Response Operator Viewed on l_1

Under certain conditions, the frequency response operator can be established via the steady-state analysis as a mapping on l_1 (i.e., from l_1 to l_1) based on the similarity transformation formulas on l_1 stated in Subsection 2.3.2. Now imposing an l_1-EMP signal, $u(t) = \sum_{m=-\infty}^{+\infty} u_m e^{j\varphi_m t}, \varphi \in \mathcal{I}_0$ with $\underline{u} := [\cdots, u_{-1}^T, u_0^T, u_1^T, \cdots]^T \in l_1$, to the system of Figure 3.2, the steady-state output $y(t)$ is measured, from which the frequency response operator is defined similarly as in Subsection 3.3.1. Theorem 3.4 summarizes such discussions.

Theorem 3.4 *Assume that $A(t) \in L_{\rm CPCD}[0,h]$, $B(t)$, $C(t)$ and $D(t)$ belong to $L_{\rm CAC}[0,h]$ and that the system (2.1) is asymptotically stable. Then the steady-state response of the system (2.1) to the l_1-EMP input $u(t) = \sum_{m=-\infty}^{+\infty} u_m e^{j\varphi_m t}$ with $\underline{u} = [\cdots, u_{-1}^T, u_0^T, u_1^T, \cdots]^T \in l_1$*

is also an l_1-EMP *signal* $y(t) = \sum_{m=-\infty}^{+\infty} y_m e^{j\varphi_m t}$ *with* $\underline{y} = [\cdots, y_{-1}^T, y_0^T, y_1^T, \cdots]^T = \underline{G}(j\varphi)\underline{u} \in l_1$, *where* $\underline{G}(j\varphi)$ *is given in (*3.24*). Hence the frequency response operator* $\underline{G}(j\varphi)$ *is well-defined on* l_1 *for each* $\varphi \in \mathcal{I}_0$. *Also, it is uniformly bounded over* $\varphi \in \mathcal{I}_0$ *in the sense that* $||\underline{G}(j\varphi)||_{l_1/l_1} \leq K < \infty, \forall \varphi \in \mathcal{I}_0$ *for some* $K > 0$.

Remark 3.4 *It is worth mentioning that Theorem* 3.4 *is not a special case of Theorem* 3.3. *To see this, the following facts are mentioned. (i). In Theorem* 3.3, *the* EMP *signal* $u(t)$ *is viewed as an* l_2-EMP *signal even though* $u(t)$ *itself is* l_1-EMP; *(ii). Theorem* 3.3 *and Theorem* 3.4 *are proved by using Theorem* 2.3 *and Theorem* 2.4, *respectively, which hold on different linear spaces (see also Remark* 2.6*); (iii). The uniform boundedness of the frequency response operator* $\underline{G}(j\varphi)$ *in Theorem* 3.3 *is stated in the* l_2*-induced norm sense from the dense subset* l_1 *of* l_2 *to* l_2 *while that of Theorem* 3.4 *is in the* l_1*-induced norm sense.*

3.4 H_2 and H_∞ Norms of FDLCP Systems

As the first task of this section, it is shown that the H_2 and H_∞ norms are well-defined on the frequency response operator of the FDLCP system (2.1) that is defined through the input/output steady-state analysis as discussed in Section 3.3. The definition validity of the H_2 and H_∞ norms of the frequency response operator is in question if the definitions are given simply in some direct extended forms from what we have in LTI continuous-time systems, noting the facts that the frequency response operator is infinite-dimensional and 'deficient' in the sense that the frequency response operator is densely defined on l_2 instead of on the whole Hilbert space l_2. It is also from these facts about the frequency response operator that the respective equivalences of the H_2 and H_∞ norms between the time-domain definitions and their frequency-domain counterparts need to be re-examined carefully before any applications. The re-examination of the equivalences is the second task of this section. It must be pointed out that the respective equivalences of the H_2 and H_∞ norms between their time-domain and frequency-domain definitions in general FDLCP systems have been verified only through the lifting approach [4], [5] so far in the literature (in [20], the same conclusions are drawn again by solution of differentiable equations but still the lifting technique is utilized), in which the frequency-domain H_2 and H_∞ norms are defined on the frequency response relations derived from the lifted system operator [4], [5], [73]. In sampled-data systems, these equivalence questions have been solved both via the lifting approach and the so-called FR-operator approach [2], [34], [35], [36], [37].

3.4.1 Time-Domain Definitions and Computation Formulas

First we give the time-domain definition for the H_2 norm and the definition of the L_2-induced norm in general FDLCP systems. The latter L_2-induced norm is conventionally called the time-domain counterpart of the H_∞ norm defined on the frequency response operator because of their equivalence between the time and frequency domains we will show shortly. Thus, with a bit of abuse of terminology, the L_2-induced norm is also called the time-domain H_∞ norm of the given FDLCP system. To complete our tasks here, let us first define the modal (or formal) frequency response operator of the system in Figure 3.2 by

$$\underline{\hat{G}}(j\varphi) = \underline{\hat{C}}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\hat{B}} + \underline{D}$$

where $\underline{\hat{B}} = \mathcal{T}\{\hat{B}(t)\}$, $\underline{\hat{C}} = \mathcal{T}\{\hat{C}(t)\}$ and $\underline{D} = \mathcal{T}\{D(t)\}$. For $\underline{\hat{G}}(j\varphi)$ to make sense, we assume that $A(t) \in L_2[0,h]$ and the system is asymptotically stable so that $\underline{E}(j\varphi) - \underline{Q}$ is invertible for all $\varphi \in \mathcal{I}_0$. We also assume that $\underline{\hat{B}}, \underline{\hat{C}}$ and $\underline{D}$ are bounded on l_2. Thus, the assumptions on the system matrices $\{A(t), B(t), C(t), D(t)\}$ are 'seemingly' relaxed. However, one must bear in mind that $\underline{\hat{G}}(j\varphi)$ is defined without any connections to the input/output steady-state analysis. It is from this that comes the name of the 'modal' frequency response operator.

The purpose to introduce this 'modal' frequency response operator is that the time-domain H_2 and L_2-induced norm definitions of an FDLCP system are independent of the steady-state analysis so that some mismatch appears in the arguments if we use $\underline{G}(j\varphi)$ directly, which is derived by the steady-state analysis. It will be seen shortly that these time-domain norms are more naturally connected with $\underline{\hat{G}}(j\varphi)$ rather than $\underline{G}(j\varphi)$ although under some strengthened assumptions on the system matrices given below, the matrix representations of these two frequency response operators can be shown eventually to coincide with each other. This coincidence will help to recover the equivalences between these two norms and their counterparts in terms of the frequency response operator.

Now we link the time-domain H_2 norm with the modal frequency response operators of FDLCP systems. In the sequel, we assume that the FDLCP system (2.1) is strictly proper whenever the H_2 norm problem is considered. The definition given below is widely used [5], [13], [16], [32], [70], [84] and a typical proof for its validity (in the sense that the H_2 norm is finite) is given in [32] for general time-varying continuous-time systems.

Definition 3.1 *The time-domain H_2 norm of the* FDLCP *system (2.1) is the quantity*

$$||\mathcal{G}||_{T,2} = \Big\{\frac{1}{h}\int_0^h \int_{-\infty}^{+\infty} \mathrm{trace}\big(g(t,\tau)^* g(t,\tau)\big) dt d\tau\Big\}^{\frac{1}{2}}$$

where $g(\cdot,\cdot)$ is the impulse response of the system (2.1).

The proposition below links the time-domain H_2 norm with the modal frequency response operator introduced in the above.

Proposition 3.1 *Suppose in the system (2.1) that $A(t) \in L_2[0,h]$, the system is asymptotically stable. Also, assume that $\hat{B}(t)$ and $\hat{C}(t)$ belong to $L_{\mathrm{CAC}}[0,h]$. Then, it holds that*

$$||\mathcal{G}||_{T,2} = \Big\{\frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{trace}\big(\underline{\hat{G}}(j\varphi)^* \underline{\hat{G}}(j\varphi)\big) d\varphi\Big\}^{\frac{1}{2}}$$

Proof By the Floquet theorem, i.e, Theorem 2.1, the state transition matrix of the FDLCP system (2.1) can be written as $\Phi(t,0) = P(t,0)e^{Qt}$ when the initial time $t_0 = 0$. Thus, the impulse response of the system to the input $e_i\delta(t-\tau)$ ($\delta(t-\tau)$ is the delta function imposed at $t = \tau \geq 0$, e_i is the i-th natural basis of $\mathcal{R}^m$) is given by

$$\Big(Ge_i\delta_\tau\Big)(t) = \begin{cases} C(t)P(t,0)e^{Q(t-\tau)}P^{-1}(\tau,0)B(\tau)e_i & (t \geq \tau) \\ 0 & (t < \tau) \end{cases} \tag{3.26}$$

Here, we further define $\tilde{B}^T := [\cdots, \hat{B}_{-1}^T, \hat{B}_0^T, \hat{B}_1^T, \cdots]^T$, $\tilde{C} := [\cdots, \hat{C}_1, \hat{C}_0, \hat{C}_{-1}, \cdots]$ and $\Lambda(t) := [\cdots, e^{j\omega_h t}I, I, e^{-j\omega_h t}I, \cdots]^T$ with $\{\hat{B}\}_{m=-\infty}^{+\infty}$ and $\{\hat{C}\}_{m=-\infty}^{+\infty}$ being the Fourier coefficients sequence of $\hat{B}(t)$ and $\hat{C}(t)$, respectively. Then, we obtain

$$C(t)P(t,0)e^{Q(t-\tau)}P^{-1}(\tau,0)B(\tau)e_i = \tilde{C}\Lambda(t)e^{Q(t-\tau)}\hat{B}(\tau)e_i$$

since by the assumptions on $\hat{B}(t)$ and $\hat{C}(t)$, $\hat{B}(t) = \Lambda(\tau)^*\tilde{B}$ and $\hat{C}(t) = \tilde{C}\Lambda(t)$ hold. Therefore, taking the Fourier transformation on (3.26) about t, we obtain

$$\begin{aligned} F\Big[\Big(Ge_i\delta_\tau\Big)(t)\Big](j\omega) &= \int_\tau^{+\infty} \tilde{C}\Lambda(t)e^{Q(t-\tau)}\hat{B}(\tau)e_ie^{-j\omega t}dt \\ &= \int_\tau^{+\infty} \tilde{C}\Lambda(t)e^{Q(t-\tau)}e^{-j\omega t}dt\hat{B}(\tau)e_i \\ &= \tilde{C}\int_\tau^{+\infty} \Lambda(t)e^{Q(t-\tau)}e^{-j\omega t}dt\Lambda(\tau)^*\tilde{B}e_i \\ &= \tilde{C}(\underline{E}(j\omega) - \underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B}e_ie^{-j\omega\tau} \end{aligned} \tag{3.27}$$

In (3.27), the order of the integral and the infinite summation caused by $\hat{C}(t)\Lambda(t)$ is interchanged. Now we show that this is valid under the given assumptions. To this end, we note from [25] that the inequality (3.5) holds by the stability assumption of the system. Therefore, we obtain

$$\begin{aligned} &\sum_{m=-\infty}^{+\infty} \int_\tau^{+\infty} ||\hat{C}_m e^{jm\omega_h t}e^{Q(t-\tau)}e^{-j\omega t}||dt \\ \leq\ &\sum_{m=-\infty}^{+\infty} ||\hat{C}_m|| \int_\tau^{+\infty} ||e^{Q(t-\tau)}||dt \\ \leq\ &\sum_{m=-\infty}^{+\infty} ||\hat{C}_m|| \int_\tau^{+\infty} \hat{K}e^{-\alpha(t-\tau)}dt \leq \frac{\hat{K}}{\alpha}\sum_{m=-\infty}^{+\infty} ||\hat{C}_m|| < \infty \end{aligned} \tag{3.28}$$

since $\hat{C}(t) \in L_{\mathrm{CAC}}[0,h]$. This, together with the Levi theorem [55, p. 577], tells us that the order interchange mentioned above is valid.

Hence, by the time-domain definition of the H_2 norm, we have

$$\begin{aligned} ||\mathcal{G}||_{T,2}^2 &= \frac{1}{h}\int_0^h \int_0^{+\infty} \Big(\sum_{i=1}^m (Ge_i\delta_\tau)^*(t)(Ge_i\delta_\tau)(t)\Big)dt\,d\tau \\ &= \frac{1}{h}\int_0^h \int_0^{+\infty} \mathrm{trace}\Big(\sum_{i=1}^m (Ge_i\delta_\tau)(t)(Ge_i\delta_\tau)^*(t)\Big)dt\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\int_0^h \sum_{i=1}^m \mathrm{trace}\Big(\int_0^{+\infty} (Ge_i\delta_\tau)(t)(Ge_i\delta_\tau)^*(t)\Big)dt\,d\tau \\
&= \frac{1}{2\pi h}\int_0^h \sum_{i=1}^m \mathrm{trace}\Big(\int_{-\infty}^{+\infty} F\big[(Ge_i\delta_\tau)(t)\big](j\omega)F\big[(Ge_i\delta_\tau)(t)\big]^*(j\omega)d\omega\Big)d\tau \\
&= \frac{1}{2\pi h}\int_0^h \sum_{i=1}^m \mathrm{trace}\Big(\int_{-\infty}^{+\infty} \tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B}e_i \\
&\qquad\qquad \cdot e_i^*\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*d\omega\Big)d\tau \\
&= \frac{1}{2\pi h}\int_0^h \mathrm{trace}\Big(\int_{-\infty}^{+\infty} \tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B} \\
&\qquad\qquad \cdot\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*d\omega\Big)d\tau \\
&= \frac{1}{2\pi h}\int_{-\infty}^{+\infty} \mathrm{trace}\Big(\int_0^h \tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B} \\
&\qquad\qquad \cdot\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*d\tau\Big)d\omega
\end{aligned}
\tag{3.29}
$$

by the Parseval Theorem and (3.27). In the last equation of (3.29), the order of the double integrals are interchanged. This can be validated by the fact that

$$
\mathrm{trace}\Big(\tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B}\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*\Big) \geq 0
$$

and the Fubini theorem [55, p. 598]. The fact that the trace computations here are actually only finite summations is also used repeatedly.

Next, it is shown that (3.29) can be rewritten as

$$
\begin{aligned}
||\mathcal{G}||_{T,2}^2 &= \frac{1}{2\pi h}\int_{-\infty}^{+\infty} \mathrm{trace}\Big(\tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1} \\
&\qquad \cdot\Big[\int_0^h \Lambda(\tau)\Lambda(\tau)^*\tilde{B}\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*d\tau\Big](\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*\Big)d\omega
\end{aligned}
\tag{3.30}
$$

To this purpose, define the infinite-dimensional vector function

$$
[\cdots, s_{-1}(\tau)^T, s_0(\tau)^T, s_1(\tau)^T, \cdots]^T := \Lambda(\tau)\Lambda(\tau)^*\tilde{B}\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*
$$

Then some direct computations give

$$
s_m(\tau) = e^{jm\omega_h\tau}\hat{B}(\tau)\hat{B}(\tau)^*\Lambda(\tau)^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^* \quad (m \in \mathcal{Z})
$$

By (2.19) and the assumptions on $\hat{B}(t)$ and $\hat{C}(t)$, there exists $\tilde{K} > 0$ such that

$$
||s_m(\tau)|| \leq \tilde{K} \quad (\forall m \in \mathcal{Z}, \forall \tau \in [0,h], \forall \omega \in (-\infty,+\infty))
\tag{3.31}
$$

Again, by (2.19) and the assumption of $\hat{C}(t) \in L_{\mathrm{CAC}}[0,h]$, we can conclude that the infinite series $\sum_{m=-\infty}^{+\infty} ||\hat{C}_m(j(\omega+m\omega_h)I-Q)^{-1}\}||$ is absolutely convergent over $\omega \in (-\infty,+\infty)$. These facts lead to

$$
\begin{aligned}
&\sum_{m=-\infty}^{+\infty}\int_0^h ||\hat{C}_m(j(\omega+m\omega_h)I-Q)^{-1}s_m(\tau)||d\tau \\
\leq\ & h\tilde{K}\sum_{m=-\infty}^{+\infty} ||\hat{C}_m(j(\omega+m\omega_h)I-Q)^{-1}|| < \infty
\end{aligned}
$$

which implies from the Levi theorem that

$$\begin{aligned}
&\int_0^h \tilde{C}(\underline{E}(j\omega) - \underline{Q})^{-1}\Lambda(\tau)\Lambda(\tau)^*\tilde{B}\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega) - \underline{Q})^{-*}\tilde{C}^* d\tau \\
= \; & \tilde{C}(\underline{E}(j\omega) - \underline{Q})^{-1}\Big[\int_0^h \Lambda(\tau)\Lambda(\tau)^*\tilde{B}\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^*(\underline{E}(j\omega) - \underline{Q})^{-*}\tilde{C}^* d\tau\Big]
\end{aligned} \tag{3.32}$$

Repeating the above arguments on the integral term of the right-hand side of (3.32), the relation (3.30) follows. The repeated arguments are developed on an infinite-dimensional vector term by term. However, this brings in no essential difficulty in the discussions.

Furthermore, it is easy to see that

$$\begin{aligned}
&\Lambda(\tau)\Lambda(\tau)^*\tilde{B}\,\tilde{B}^*\Lambda(\tau)\Lambda(\tau)^* \\
= \; & \Lambda(\tau)\hat{B}(\tau)\,\hat{B}(\tau)^*\Lambda(\tau)^* \\
= \; & \begin{bmatrix} \vdots \\ e^{j\omega_h\tau}\hat{B}(\tau) \\ \hat{B}(\tau) \\ e^{-j\omega_h\tau}\hat{B}(\tau) \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{j\omega_h\tau}\hat{B}(\tau) \\ \hat{B}(\tau) \\ e^{-j\omega_h\tau}\hat{B}(\tau) \\ \vdots \end{bmatrix}^* \\
= \; & R(\tau)\Lambda(\tau)\Lambda(\tau)^*R(\tau)^*
\end{aligned} \tag{3.33}$$

where $R(\tau) := \text{diag}[\cdots, \hat{B}(\tau), \hat{B}(\tau), \hat{B}(\tau), \cdots]$. Then, from this definition, the (n, k)-th entry of the matrix $R(\tau)\Lambda(\tau)\Lambda(\tau)^*R(\tau)^*$ satisfies

$$[R(\tau)\Lambda(\tau)\Lambda(\tau)^*R(\tau)^*]_{(n,k)} = \hat{B}(\tau)\hat{B}(\tau)^*e^{j\omega_h(k-n)\tau}$$

By the assumption, $\hat{B}(\tau)$ can be extended into an absolutely convergent Fourier series expansion $\hat{B}(\tau) = \sum_{q=-\infty}^{+\infty} \hat{B}_q e^{jq\omega_h\tau}$. Hence, it follows from the proof of Lemma 2.2 that $\hat{B}(\tau)\hat{B}(\tau)^*$ has the absolutely convergent Fourier series expansion and

$$\hat{B}(\tau)\hat{B}(\tau)^* = \sum_{m=-\infty}^{+\infty}\Big(\sum_{n=-\infty}^{+\infty} \hat{B}_{m-n}\hat{B}_n^*\Big)e^{jm\omega_h\tau}$$

Hence, it follows readily that

$$\frac{1}{h}\int_0^h \hat{B}(\tau)\hat{B}(\tau)^*e^{j\omega_h(k-n)\tau}d\tau = \sum_{q=-\infty}^{+\infty} \hat{B}_{q+n}\hat{B}_{q+k}^*$$

which implies that

$$\frac{1}{h}\int_0^h R(\tau)\Lambda(\tau)\Lambda(\tau)^*R(\tau)^* d\tau = \underline{\hat{B}}\,\underline{\hat{B}}^* \tag{3.34}$$

with $\hat{\underline{B}} := \mathcal{T}\{\hat{B}(t)\}$. Finally, using (3.34) in (3.30) yields

$$
\begin{aligned}
||\mathcal{G}||_{T,2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{trace}\Big(\tilde{C}(\underline{E}(j\omega)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\omega)-\underline{Q})^{-*}\tilde{C}^*\Big)d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{trace}\Big(\tilde{C}(\underline{E}(j\varphi_m)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\varphi_m)-\underline{Q})^{-*}\tilde{C}^*\Big)d\varphi \\
&= \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}}\sum_{m=-\infty}^{+\infty} \mathrm{trace}\Big(\tilde{C}(\underline{E}(j\varphi_m)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\varphi_m)-\underline{Q})^{-*}\tilde{C}^*\Big)d\varphi \\
&= \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{trace}\Big(\hat{\underline{C}}(\underline{E}(j\varphi)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\varphi)-\underline{Q})^{-*}\hat{\underline{C}}^*\Big)d\varphi \\
&= \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)\hat{\underline{G}}(j\varphi)^*\Big)d\varphi
\end{aligned}
$$

In the above, we have interchanged the order of the integral and the summation. To validate this, it suffices to show that the convergence of

$$
\begin{aligned}
&\sum_{|m|\le M} \mathrm{trace}\Big(\tilde{C}(\underline{E}(j\varphi_m)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\varphi_m)-\underline{Q})^{-*}\tilde{C}^*\Big) \\
&= \mathrm{trace}\Big(\underline{I}_M\tilde{C}(\underline{E}(j\varphi)-\underline{Q})^{-1}\hat{\underline{B}}\,\hat{\underline{B}}^*(\underline{E}(j\varphi)-\underline{Q})^{-*}\tilde{C}^*\underline{I}_M^*\Big) \\
&= \mathrm{trace}\Big(\underline{I}_M\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\underline{I}_M^*\Big) \to \mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\Big)
\end{aligned}
$$

is uniform over $\varphi \in \mathcal{I}_0$, where

$$
\underline{I}_M := \mathrm{diag}[\cdots,0,\underbrace{I,\cdots,I}_{2M+1},0,\cdots] \tag{3.35}
$$

To see this, we note from [55] that if $\{\underline{u}_k\}_{k=-\infty}^{+\infty}$ is an orthonormal basis of l_2 with $\underline{u}_n = [\cdots,0^T,u_n^T,0^T,\cdots]^T$, then it holds

$$
\begin{aligned}
&\mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\Big) - \mathrm{trace}\Big(\underline{I}_M\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\underline{I}_M^*\Big) \\
&= \sum_{k=-\infty}^{+\infty}\Big(||\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}^2 - ||\underline{I}_M\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}^2\Big) \\
&\le \sum_{k=-\infty}^{+\infty}\Big(||\underline{I}_M\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2} + ||\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}\Big)\Big(||(\underline{I}-\underline{I}_M)\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}\Big) \\
&\le \Big[\sum_{k=-\infty}^{+\infty}\Big(||\underline{I}_M\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2} + ||\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}\Big)^2\Big]^{\frac{1}{2}}\Big[\sum_{k=-\infty}^{+\infty}||(\underline{I}-\underline{I}_M)\hat{\underline{G}}(j\varphi)\underline{u}_k||_{l_2}^2\Big]^{\frac{1}{2}} \\
&\le \sqrt{2}\Big[\mathrm{trace}\Big(\underline{I}_M\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\underline{I}_M^*\Big) + \mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\Big)\Big]^{\frac{1}{2}} \\
&\quad\cdot\Big[\sum_{k=-\infty}^{+\infty}||(\underline{I}-\underline{I}_M)\hat{\underline{C}}(\underline{E}(j\varphi)-\underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2\Big]^{\frac{1}{2}}
\end{aligned}
$$

Noting also that $\mathrm{trace}\Big(\underline{I}_M\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\underline{I}_M^*\Big) \le \mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\Big)$ for any $\varphi \in \mathcal{I}_0$ and that $\mathrm{trace}\Big(\hat{\underline{G}}(j\varphi)^*\hat{\underline{G}}(j\varphi)\Big)$ is uniformly bounded over $\varphi \in \mathcal{I}_0$ [86], it is easy to see that the proof will become complete if it is shown that $\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2$ goes to zero uniformly over $\varphi \in \mathcal{I}_0$ as $M \to \infty$. To see this, we note that

$$
\begin{aligned}
&\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \\
\le\ & 2\Big[\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}\,\underline{I}_N(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \\
&+ \sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}(\underline{I} - \underline{I}_N)(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2\Big] \\
\le\ & 2\Big[||(\underline{I} - \underline{I}_M)\hat{\underline{C}}\,\underline{I}_N||_{l_2/l_2}^2 \sum_{k=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \\
&+ ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}||_{l_2/l_2}^2 \sum_{m=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_N)(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2\Big] \\
\le\ & 2\Big[(\sum_{|n|>N} ||\hat{C}_n||)^2 \sum_{k=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \\
&+ ||\hat{\underline{C}}||_{l_2/l_2}^2 \sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_N)(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2\Big]
\end{aligned}
\tag{3.36}
$$

where we have assumed that $M \ge 2N$. This assumption ensures that

$$||(\underline{I} - \underline{I}_M)\hat{\underline{C}}\,\underline{I}_N||_{l_2/l_2} \le \sum_{|n|>N} ||\hat{C}_n||$$

which can be shown by noting the structure of $(\underline{I} - \underline{I}_M)\hat{\underline{C}}\,\underline{I}_N$, and has been used in (3.36).

Furthermore, noting that $\sum_{k=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2$ is uniformly bounded over $\varphi \in \mathcal{I}_0$ by (2.19), it follows from the assumption of $\hat{C}(t) \in L_{\mathrm{CAC}}[0,h]$ that

$$(\sum_{|n|>N} ||\hat{C}_n||)^2 \sum_{k=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \to 0 \quad (\forall \varphi \in \mathcal{I}_0) \tag{3.37}$$

as $N \to \infty$. Also, it is easy to see by the choice of the orthonormal basis $\{\underline{u}_k\}_{k=-\infty}^{+\infty}$ that

$$
\begin{aligned}
&\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_N)(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \\
\le\ & \sum_{k=-\infty}^{+\infty} \sum_{|n|>N} ||(j\varphi_n I - Q)^{-1}||^2 ||\hat{B}_{n-k}||^2 \le \sum_{|n|>N} ||(j\varphi_n I - Q)^{-1}||^2 \sum_{k=-\infty}^{+\infty} ||\hat{B}_k||^2
\end{aligned}
$$

which, together with (2.19), implies that

$$\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_N)(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 \to 0 \quad (\forall \varphi \in \mathcal{I}_0) \tag{3.38}$$

as $N \to \infty$. Using (3.37) and (3.38) in (3.36), one can claim that for any $\epsilon > 0$, there exists an integer $N(\epsilon) > 0$ such that

$$\sum_{k=-\infty}^{+\infty} ||(\underline{I} - \underline{I}_M)\hat{\underline{C}}(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_k||_{l_2}^2 < \epsilon \quad (\forall M \geq 2N(\epsilon), \forall \varphi \in \mathcal{I}_0)$$

which implies the desired result. **Q.E.D.**

Next we define the L_2-induced norm of FDLCP systems and discuss its computation through what we call the modal frequency response operator. This will lead to a proposition which is useful for verifying the equivalence of the H_∞ norm in FDLCP systems between the time-domain definition and the frequency-domain one.

Definition 3.2 *The L_2-induced norm of the* FDLCP *system (2.1) is*

$$||\mathcal{G}||_{L_2/L_2} = \sup_{0 \neq u \in L_2} \frac{||y(\cdot)||_{L_2}}{||u(\cdot)||_{L_2}}$$

To state the following proposition, we introduce the so-called SD-Fourier transform [2]. For $x \in L_2$, its SD-Fourier transform is defined as

$$\underline{X}_{\mathrm{SD}}(j\varphi) := [\cdots, X(j\varphi_{-1})^T, X(j\varphi_0)^T, X(j\varphi_1)^T, \cdots]^T \tag{3.39}$$

where $X(j\omega)$ is the Fourier transform of $x \in L_2$ and $X(j\varphi_n) = X(j(\varphi + n\omega_h)), n \in \mathcal{Z}, \varphi \in \mathcal{I}_0$. It can also be said that $\underline{X}_{\mathrm{SD}}(j\varphi)$ is the lifted version of $X(j\omega)$ in the frequency domain. This kind of frequency-domain lifting technique has been used in sampled-data system sensitivity analysis [11] and signal processing [62].

Proposition 3.2 *Suppose in the system (2.1) that $A(t)$ belongs to $L_2[0, h]$, $\hat{B}(t), \hat{C}(t)$ and $D(t)$ belong to $L_{\mathrm{CAC}}[0, h]$ and that the system is asymptotically stable. Then*

1). $\underline{Y}_{\mathrm{SD}}(j\varphi) = \hat{\underline{G}}(j\varphi)\underline{U}_{\mathrm{SD}}(j\varphi), \forall \varphi \in \mathcal{I}_0$ *for any $u(t) \in C_0^1$, where C_0^1 denotes the space of continuously differentiable functions with compact support;*

2). $||y(\cdot)||_{L_2}^2 = \frac{1}{2\pi}\int_{\mathcal{I}_0} \underline{U}_{\mathrm{SD}}^*(j\varphi)\hat{\underline{G}}^*(j\varphi)\hat{\underline{G}}(j\varphi)\underline{U}_{\mathrm{SD}}(j\varphi)d\varphi$ *for any $u(t) \in C_0^1$;*

where $\underline{U}_{\mathrm{SD}}(j\varphi)$ and $\underline{Y}_{\mathrm{SD}}(j\varphi)$ are the SD*-Fourier transforms of $u(t)$ and $y(t)$, respectively.*

Proof By the asymptotic stability assumption, together with the Floquet theorem, the L_2-stability assertion is obvious [67]. From this, for any $u(t) \in L_2$, the corresponding output $y(t)$ belongs to L_2. Also, C_0^1 is a dense subset of L_2 (Exercise D.13.3, [55, p. 593]). Therefore, it makes sense to define the Fourier transforms $U(j\omega)$ and $Y(j\omega)$ for the input $u(t) \in C_0^1$ and the corresponding output $y(t)$. Now we compute $Y(j\omega)$ in four steps.

Step 1. The Fourier transform of the signal p (see Figure 3.2) is given by

$$P(j\omega) = \int_{-\infty}^{+\infty}\Big(\sum_{m=-\infty}^{+\infty} \hat{B}_m e^{jm\omega_h t}\Big)u(t)e^{-j\omega t}dt = \sum_{m=-\infty}^{+\infty} \hat{B}_m U(j(\omega - m\omega_h)) \tag{3.40}$$

which is well-defined since $\hat{B}(t)$ is L_2-stable (by the boundedness of $\hat{B}(t)$ on $[0,h]$). Here, the order of infinite integral ($\int_{-\infty}^{+\infty}$) and infinite series ($\sum_{m=-\infty}^{+\infty}$) is interchanged. This is valid by Levi Theorem [55] because of the absolute convergence of the Fourier series expansion of $\hat{B}(t)$ and the fact that $u(t)$ has compact support.

Step 2. Imposing p to the LTI subsystem of Figure 3.2, the Fourier transform of q is

$$Q(j\omega) = (j\omega I - Q)^{-1} \sum_{m=-\infty}^{+\infty} \hat{B}_m U(j(\omega - m\omega_h)) \tag{3.41}$$

Since $u(t) \in C_0^1$, it is clear that $\hat{B}(t)u(t) \in L_1$. Also, by the stability assumption, the LTI subsystem of Figure 3.2 is L_1-stable (Theorem 6.30 of [67]). Hence, $q(t) \in L_1$. Now truncate $q(t)$ as follows. It is easy to see that $q_T(t) \in L_1$.

$$q_T(t) = \begin{cases} q(t) & (0 \le t \le T) \\ 0 & (t > T) \end{cases}$$

Based on the fact that $q(t)$ and $q_T(t)$ belong to L_1, $\forall T > 0$, we have

$$\lim_{T\to\infty} Q_T(j\omega) = Q(j\omega) \tag{3.42}$$

uniformly over $\omega \in (-\infty, +\infty)$ for the Fourier transform $Q_T(j\omega)$ of $q_T(t)$ since

$$\begin{aligned} ||Q_T(j\omega) - Q(j\omega)|| &= ||\int_0^{\infty} (q_T(t) - q(t))e^{-j\omega t}dt|| \\ &\le \int_0^{\infty} ||q_T(t) - q(t)||dt \to 0 \quad (T \to \infty) \end{aligned}$$

Step 3. Let $\hat{y}(t)$ be the output of $\hat{C}(t)$ to the input $q(t)$, and let $\hat{y}_T(t)$ be the output of $\hat{C}(t)$ corresponding to the truncated signal $q_T(t)$, which has compact support. Then we clearly have $\hat{y}_T(t) = \hat{C}(t)q_T(t)$, so that by repeating the arguments about (3.40) on $\hat{C}(t)$, the Fourier transform of $y_T(t)$ is given by

$$\hat{Y}_T(j\omega) = \sum_{n=-\infty}^{+\infty} \hat{C}_n Q_T(j(\omega - n\omega_h)) \tag{3.43}$$

It is obvious that $\hat{y}(t)$ and $\hat{y}_T(t)$ belong to L_1 since $\hat{C}(t)$ is bounded on $t \ge 0$. Based on this fact, repeating the arguments about $q(t)$ and $q_T(t)$ on $y(t)$ and $y_T(t)$, it follows that $\lim_{T\to\infty} Y_T(j\omega) = Y(j\omega)$ uniformly over $\omega \in (-\infty, +\infty)$. This further gives the relation

$$\hat{Y}(j\omega) = \sum_{n=-\infty}^{+\infty} \hat{C}_n Q(j(\omega - n\omega_h)) \tag{3.44}$$

since it is evident that

$$\begin{aligned} & ||\sum_{n=-\infty}^{+\infty} \hat{C}_n Q_T(j(\omega - n\omega_h)) - \sum_{n=-\infty}^{+\infty} \hat{C}_n Q(j(\omega - n\omega_h))|| \\ \le & \sum_{n=-\infty}^{+\infty} ||\hat{C}_n|| \cdot ||Q_T(j(\omega - n\omega_h)) - Q(j(\omega - n\omega_h))|| \to 0 \quad (T \to \infty) \end{aligned}$$

uniformly over $\omega \in (-\infty, +\infty)$ by (3.42) and $\sum_{n=-\infty}^{+\infty} ||\hat{C}_n|| < \infty$, which follows from the absolute convergence assumption of the Fourier series expansion of $\hat{C}(t)$.

Step 4. Taking the term $D(t)$ into consideration and lifting the Fourier transform $Y(j\omega)$ of the whole output to its SD-Fourier transform $\underline{Y}_{\rm SD}(j\varphi)$ leads to the assertion 1).

To show the assertion 2), by the well-known Parseval theorem, we note that

$$\begin{aligned} ||y(\cdot)||_{L_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)^*Y(j\omega)d\omega = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{\mathcal{I}_m} Y(j\omega)^*Y(j\omega)d\omega \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{\mathcal{I}_0} Y(j\varphi_m)^*Y(j\varphi_m)d\varphi = \frac{1}{2\pi}\int_{\mathcal{I}_0} \underline{Y}_{\rm SD}(j\varphi)^*\underline{Y}_{\rm SD}(j\varphi)d\varphi \\ &= \frac{1}{2\pi}\int_{\mathcal{I}_0} \underline{U}_{\rm SD}(j\varphi)^*\underline{\hat{G}}(j\varphi)^*\underline{\hat{G}}(j\varphi)\underline{U}_{\rm SD}(j\varphi)d\varphi \end{aligned}$$

where $\mathcal{I}_m := [-\omega_h/2 + jm\omega_h, \omega_h/2 + jm\omega_h), m \in \mathcal{Z}$. To complete the proof, it remains to show that the order interchange of the integral and summation is valid. To this end, it is enough to show that the convergence of $\sum_{m=-M}^{M} Y(j\varphi_m)^*Y(j\varphi_m) \to \underline{Y}_{\rm SD}(j\varphi)^*\underline{Y}_{\rm SD}(j\varphi)$ as $M \to \infty$ is uniform over $\varphi \in \mathcal{I}_0$. We accomplish this in three steps.

Step 4.1. Since $u(t) \in C_0^1$, there exist numbers $K_u > 0$ and $\Omega_u > 0$ such that

$$||U(j\omega)|| \le \begin{cases} K_u & (|\omega| \le \Omega_u) \\ K_u\Omega_u/|\omega| & (|\omega| > \Omega_u) \end{cases} \tag{3.45}$$

which can be shown similarly to Lemma A of [2], and implies that $\underline{U}_{\rm SD}(j\varphi) \in l_2, \forall\varphi \in \mathcal{I}_0$.

Furthermore, from (3.41) and (3.45), we have

$$\begin{aligned} ||Q(j\omega)|| &\le ||(j\omega I - Q)^{-1}|| \sum_{m=-\infty}^{+\infty} ||\hat{B}_m|| \cdot ||U(j(\omega - n\omega_h))|| \\ &\le K_u||(j\omega I - Q)^{-1}|| \sum_{m=-\infty}^{+\infty} ||\hat{B}_m|| \end{aligned} \tag{3.46}$$

which implies that there exist $\tilde{K}_Q > 0$ and $\Omega_Q > 0$ satisfying

$$||Q(j\omega)|| \le \begin{cases} \tilde{K}_Q & (|\omega| \le \Omega_Q) \\ \tilde{K}_Q\Omega_Q/|\omega| & (|\omega| > \Omega_Q) \end{cases} \tag{3.47}$$

Step 4.2. Let us define the infinite-dimensional vector $\underline{Y}_M(j\varphi)$ by

$$\underline{Y}_M(j\varphi) := [\cdots, 0^T, Y(j\varphi_{-M})^T, \cdots, Y(j\varphi_0)^T, \cdots, Y(j\varphi_M)^T, 0^T, \cdots]^T$$

Apparently, $\underline{Y}_M(j\varphi) = \underline{I}_M\underline{Y}_{\rm SD}(j\varphi)$ where $\underline{I}_M$ is given by (3.35). We now show that $\underline{Y}_M(j\varphi)$ converges to $\underline{Y}_{\rm SD}(j\varphi)$ uniformly over $\varphi \in \mathcal{I}_0$ as $M \to \infty$. To this end, we first note that

$$||\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi)||_{l_2} \le ||\underline{\hat{I}}_M\underline{\hat{C}}\,\underline{Q}_{\rm SD}(j\varphi)||_{l_2} + ||\underline{\hat{I}}_M\underline{\hat{D}}\,\underline{U}_{\rm SD}(j\varphi)||_{l_2} \tag{3.48}$$

with $\underline{\hat{I}}_M := \underline{I} - \underline{I}_M$. Now observe that the first term on the right-hand side of (3.48) satisfies

$$||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_{\rm SD}(j\varphi)||_{l_2} \le ||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_N(j\varphi)||_{l_2} + ||\hat{\underline{I}}_M\hat{\underline{C}}\,(\underline{Q}_{\rm SD}(j\varphi) - \underline{Q}_N(j\varphi))||_{l_2} \tag{3.49}$$

where $\underline{Q}_N(j\varphi)$ is defined similarly to $\underline{Y}_M(j\varphi)$ but in terms of $Q(j\varphi_m), m = 0, \pm 1, \cdots, \pm N$. We also assume that $M \ge 2N$. Noting that only the $(2N+1)$ block-columns of $\hat{\underline{I}}_M\hat{\underline{C}}$ at the center are involved in the computation of $||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_N(j\varphi)||_{l_2}$, it follows readily that

$$\begin{aligned} ||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_N(j\varphi)||_{l_2} &= ||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{I}_N\underline{Q}_{\rm SD}(j\varphi)||_{l_2} \le ||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{I}_N||_{l_2/l_2}||\underline{Q}_{\rm SD}(j\varphi)||_{l_2} \\ &\le \sum_{|n|>N} ||\hat{C}_n|| \cdot ||\underline{Q}_{\rm SD}(j\varphi)||_{l_2} \end{aligned} \tag{3.50}$$

On the other hand, since $\hat{\underline{I}}_M\hat{\underline{C}}$ is only a sub-matrix of $\hat{\underline{C}}$, it follows immediately that

$$||\hat{\underline{I}}_M\hat{\underline{C}}\,(\underline{Q}_{\rm SD}(j\varphi) - \underline{Q}_N(j\varphi))||_{l_2} \le ||\hat{\underline{C}}||_{l_2/l_2}||\underline{Q}_{\rm SD}(j\varphi) - \underline{Q}_N(j\varphi)||_{l_2} \tag{3.51}$$

Combining (3.51) with (3.47), one can claim that for any $\epsilon > 0$, there is an integer $N(\epsilon) > 0$ sufficiently large such that

$$||\hat{\underline{I}}_M\hat{\underline{C}}\,(\underline{Q}_{\rm SD}(j\varphi) - \underline{Q}_N(j\varphi))||_{l_2} < \frac{\epsilon}{4} \quad (\forall N \ge N(\epsilon), \forall\varphi \in \mathcal{I}_0)$$

At the same time, from (3.50) together with (3.47), it is guaranteed that for this $N(\epsilon)$, another integer $M(N(\epsilon), \epsilon) \ge 2N(\epsilon)$ can be taken such that

$$||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_N(j\varphi)||_{l_2} \le \frac{\epsilon}{4} \quad (\forall M \ge M(N(\epsilon), \epsilon), \forall\varphi \in \mathcal{I}_0)$$

since $\sum_{|n|>N} ||\hat{C}_n|| \to 0$ as $N \to \infty$ by the assumption on $\hat{C}(t)$.

Summarizing the above discussions, the inequality (3.49) actually tells us that for any $\epsilon > 0$, there exists some integer $M(\epsilon) > 0$ ensuring that

$$||\hat{\underline{I}}_M\hat{\underline{C}}\,\underline{Q}_{\rm SD}(j\varphi)||_{l_2} < \frac{\epsilon}{2} \quad (\forall M > M(\epsilon), \forall\varphi \in \mathcal{I}_0)$$

Noting that $\underline{D}$ has the same structure as $\hat{\underline{C}}$ and that the inequality (3.45) is similar to (3.47), we can repeat the above arguments to the second term on the right-hand side of (3.48). Hence it follows immediately that $\underline{Y}_M(j\varphi)$ converges to $\underline{Y}_{\rm SD}(j\varphi)$ uniformly over $\varphi \in \mathcal{I}_0$ as $M \to \infty$.

Step 4.3. We show that $\sum_{m=-M}^{M} Y(j\varphi_m)^*Y(j\varphi_m)$ converges to $\underline{Y}_{\rm SD}(j\varphi)^*\underline{Y}_{\rm SD}(j\varphi)$ uniformly over $\varphi \in \mathcal{I}_0$ as $M \to \infty$. We note by the Cauchy-Schwarz inequality that

$$\begin{aligned} & ||\underline{Y}_M(j\varphi)^*\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi)^*\underline{Y}_{\rm SD}(j\varphi)|| \\ \le\ & ||\underline{Y}_M(j\varphi)^*(\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi))|| + ||(\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi))^*\underline{Y}_{\rm SD}(j\varphi)|| \\ \le\ & 2||\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi)||_{l_2}||\underline{Y}_{\rm SD}(j\varphi)||_{l_2} \\ \le\ & 2||\underline{Y}_M(j\varphi) - \underline{Y}_{\rm SD}(j\varphi)||_{l_2}||\hat{\underline{G}}(j\varphi)||_{l_2/l_2}||\underline{U}_{\rm SD}(j\varphi)||_{l_2} \end{aligned}$$

Here the fact that $||\underline{Y}_M(j\varphi)||_{l_2} \le ||\underline{Y}_{\rm SD}(j\varphi)||_{l_2}$ is used. Thus, the assertion follows readily by the uniform convergence of $\underline{Y}_M(j\varphi)$, since $||\underline{U}_{\rm SD}(j\varphi)||_{l_2}$ is uniformly bounded by (3.45) and $||\hat{\underline{G}}(j\varphi)||_{l_2/l_2}$ is uniformly bounded over $\varphi \in \mathcal{I}_0$ (which can be seen in a similar fashion to the arguments for $\underline{G}(j\varphi)$). This completes the proof. **Q.E.D.**

3.4.2 Time-Domain/Frequency-Domain Equivalences

Proposition 3.1 and Proposition 3.2 clearly relate the time-domain H_2 and L_2-induced norms of FDLCP systems with the modal frequency response operators. It is quite intuitive to draw the respective equivalences of the H_2 and H_∞ norms between the time- and frequency-domain definitions if the equivalence of the modal frequency response operator $\hat{\underline{G}}(j\varphi)$ and the frequency response operator $\underline{G}(j\varphi)$ is established. Before we state the final results about the desired equivalences, we first define the H_2 and H_∞ norms on the frequency response operator $\underline{G}(j\varphi)$ and examine their well-definedness.

First we consider the H_2 norm of FDLCP systems. Here it is our standing assumption that the FDLCP system is strictly proper whenever the H_2 norm is concerned.

Definition 3.3 ([70], [84]) *The frequency-domain H_2 norm of the* FDLCP *system (2.1) is the quantity*

$$||\mathcal{G}||_{F,2} = \Big\{\frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \text{trace}\Big(\underline{G}(j\varphi)^*\underline{G}(j\varphi)\Big)d\varphi\Big\}^{\frac{1}{2}}$$

Since in this H_2 norm definition a trace operation is involved on an infinite-dimensional operator, it is necessary to clarify the validity of such definition before any further discussions. The following lemma gives an answer to this question. We must stress that the frequency response operator $\underline{G}(j\varphi)$ is defined on the whole l_2, which is formed by the domain extension described in Remark 3.3. The proof idea for this lemma will also be used frequently in the subsequent convergence arguments with respect to truncations.

Lemma 3.2 *Suppose in the system (2.1) that $A(t) \in L_{\text{PCD}}[0,h]$, $B(t), C(t) \in L_{\text{PCC}}[0,h]$ and that the system is asymptotically stable. Then, the frequency-domain H_2 norm of the system (2.1) is well-defined in the sense that $||\mathcal{G}||_{F,2} < \infty$.*

Proof Note under the given assumptions that $\underline{G}(j\varphi)$ is uniformly bounded and compact for any $\varphi \in \mathcal{I}_0$ by Theorem 3.3 and Theorem 2.3. Thus, by [55, p. 392], we have

$$\text{trace}\,\{\underline{G}(j\varphi)^*\underline{G}(j\varphi)\} = \sum_{i=1}^{k}\sum_{n=-\infty}^{\infty} ||\underline{G}(j\varphi)\underline{u}_{ni}||_{l_2}^2 \tag{3.52}$$

where $\{\underline{u}_{ni} : i = 1, 2, \cdots, k\}_{n=-\infty}^{\infty}$ is any orthonormal basis of the linear space l_2^k. For simplicity, we assume that $k = 1$ and this will result in no loss of generality. For our purpose, define $\underline{u}_n := [\cdots, 0, u_n, 0, \cdots]^T$ with $||u_n|| = 1, \forall n \in \mathcal{Z}$. Then

$$\begin{aligned}
\text{trace}\,\{\underline{G}(j\varphi)^*\underline{G}(j\varphi)\} &\leq ||\hat{\underline{C}}||_{l_2/l_2}^2 \sum_{n=-\infty}^{\infty} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_n||_{l_2}^2 \\
&= ||\hat{\underline{C}}||_{l_2/l_2}^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} ||(j\varphi_m I - Q)^{-1}\hat{B}_{m-n}u_n||^2 \\
&\leq ||\hat{\underline{C}}||_{l_2/l_2}^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} ||(j\varphi_m I - Q)^{-1}||^2||\hat{B}_{m-n}||^2||u_n||^2
\end{aligned}$$

$$\leq ||\underline{\hat{C}}||^2_{l_2/l_2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K^2 f(m)^2 ||\hat{B}_{m-n}||^2$$

$$= ||\underline{\hat{C}}||^2_{l_2/l_2} \sum_{m=-\infty}^{\infty} K^2 f(m)^2 \sum_{n=-\infty}^{\infty} ||\hat{B}_{m-n}||^2 \tag{3.53}$$

In (3.53), we have used the facts that the norm of the m-th entry of the infinite-dimensional vector $(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\hat{B}}\,\underline{u}_n$ can be bounded from above by $||(j\varphi_m I - Q)^{-1}|| \cdot ||\hat{B}_{m-n}||$ and that there exists a number $K > 0$ such that

$$||(j\varphi_m I - Q)^{-1}|| \leq K f(m) \tag{3.54}$$

where the function $f(\cdot)$ is defined in the Appendix A.1. Indeed, the inequality (3.54) is a re-statement of (2.19) for reading convenience. Since Q is stable by the stability assumption, K can be chosen to be independent of $\varphi \in \mathcal{I}_0$. Noting that $C(t) \in L_{\text{PCC}}[0, h]$ by the assumption, it follows that $\underline{\hat{C}}$ is bounded on l_2 by Lemma 2.2, Proposition 2.1 and Lemma 2.8. Furthermore, since $B(t) \in L_{\text{PCC}}[0, h]$, it follows that $\sum_{m=-\infty}^{\infty} ||\hat{B}_m||^2 < \infty$. Hence, the assertion follows from (3.53) and Appendix A.1. **Q.E.D.**

Remark 3.5 *The proof of Lemma* 3.2 *actually shows that under the given assumptions about the* FDLCP *system (*2.1*), the frequency response operator $\underline{G}(j\varphi)$ is a Hilbert-Schmidt operator [55, p. 387] on l_2 for each $\varphi \in \mathcal{I}_0$. Since $\underline{G}(j\varphi)$ is compact, it is suggested that we can assess the H_2 norm by truncating $\underline{G}(j\varphi)$, which is left as a topic in the next chapter.*

Now we state the equivalence of the H_2 norms between the time and frequency domains.

Theorem 3.5 *Suppose in the system (*2.1*) that $A(t)$ belongs to $L_{\text{PCD}}[0, h]$, $B(t)$ and $C(t)$ belong to $L_{\text{CAC}}[0, h]$ and that the system is asymptotically stable. Then, $||\mathcal{G}||_{T,2} = ||\mathcal{G}||_{F,2}$.*

Proof Under the given conditions, it is clear from Theorems 2.2 and 2.3 that $\underline{\hat{G}}(j\varphi) = \underline{G}(j\varphi)$. Hence, by the result in Proposition 3.1, it remains to show that the Fourier series expansions of $\hat{B}(t) = P^{-1}(t, 0)B(t)$ and $\hat{C}(t) = C(t)P(t, 0)$ are absolutely convergent. To see this, it is enough to note that the Fourier series expansions of $P^{-1}(t, 0)$ and $P(t, 0)$ are absolutely convergent from Proposition 2.1. Then, by Lemma 2.6 and the assumptions on $B(t)$ and $C(t)$ and the stability assumption on $A(t)$, we have the desired results. **Q.E.D.**

Because of the equivalence stated in Theorem 3.5, we will not distinguish in which domain the H_2 norm is defined and simply denote it by $||\mathcal{G}||_2$ in the following.

Next we consider the frequency-domain H_∞ norm of an FDLCP system.

Definition 3.4 ([70]) *The frequency-domain H_∞ norm of the* FDLCP *system (*2.1*) is*

$$||\mathcal{G}||_\infty := \max_{\varphi \in \mathcal{I}_0} ||\underline{G}(j\varphi)||_{l_2/l_2} \tag{3.55}$$

Lemma 3.3 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\rm PCD}[0,h]$, $B(t), C(t)$ and $D(t)$ belong to $L_{\rm PCC}[0,h]$ and that the system is asymptotically stable. Then, the frequency-domain H_∞ norm of the system is well-defined.*

Proof By Corollary 3.1, $||\underline{G}(j\varphi)||_{l_2/l_2}$ is well-defined for each $\varphi \in \mathcal{I}_0$ under the given conditions. Furthermore, by Theorem 2.3 and Theorem 3.3, it is straightforward to show that $||\underline{G}(j\varphi)||_{l_2/l_2}$ is continuous with respect to $\varphi \in \mathcal{I}_0$ in the l_2-induced norm sense. Hence the maximum value is attainable, and this implies that $||\mathcal{G}||_\infty$ is well-defined. **Q.E.D.**

Based on Proposition 3.2, we establish the equivalence between the L_2-induced norm (which is called the time-domain H_∞ norm) of the system (2.1) and the maximum (3.55) of the l_2-induced norm of $\underline{G}(j\varphi)$ over $\varphi \in \mathcal{I}_0$.

Theorem 3.6 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\rm PCD}[0,h]$, $B(t), C(t)$ and $D(t)$ belong to $L_{\rm CAC}[0,h]$ and that the system is asymptotically stable. Then*

$$||\mathcal{G}||_{L_2/L_2} = ||\mathcal{G}||_\infty$$

Proof By the assumptions on $A(t), B(t)$ and $C(t)$, together with Proposition 2.1 and Lemma 2.6, it follows that the Fourier series expansions of $\hat{B}(t)$ and $\hat{C}(t)$ are also absolutely convergent. This implies that Proposition 3.2 applies to the FDLCP system (2.1) under the given assumptions. In view of this, it follows that

$$||\mathcal{G}||_{L_2/C_0^1(L_2)} := \sup_{0\neq u\in C_0^1} \frac{||y(\cdot)||_{L_2}}{||u(\cdot)||_{L_2}} = \max_{\varphi\in\mathcal{I}_0} ||\underline{\hat{G}}(j\varphi)||_{l_2/l_2} \tag{3.56}$$

which can be established by similar arguments to those in the proof of Theorem 5 of [2]. Furthermore, under the given assumptions, it is obvious that $\underline{\hat{G}}(j\varphi) = \underline{G}(j\varphi)$ by Theorems 2.2 and 2.3. On the other hand, since C_0^1 is dense in L_2 and the system is L_2-stable (i.e., $\mathcal{G}$ is bounded on L_2), it follows by similar arguments to those in the proof of Corollary 3.1 that $||\mathcal{G}||_{L_2/C_0^1(L_2)} = ||\mathcal{G}||_{L_2/L_2}$. This, together with (3.56), completes the proof. **Q.E.D.**

3.4.3 Trace Formula Based on the Harmonic Lyapunov Equation

In this subsection, the relation is discussed between the H_2 norm of the frequency response operator of the FDLCP system (2.1) and the harmonic Lyapunov equation (3.3) in Theorem 3.1. The purpose of this study [90] is to express the H_2 norm by a trace formula via the solution of the harmonic Lyapunov equation so that the well-known trace formula is recovered in an LTI continuous-time fashion but with an infinite-dimensional matrix expression. In some less rigorous sense, the H_2 norm of an FDLCP system can be 'computed' just as we do in finite-dimensional LTI continuous-time systems. Again in this subsection, the FDLCP system is assumed to be strictly proper, i.e., $D(t) = 0, \forall t \in [0,h]$.

Theorem 3.7 *Suppose in the system (2.1) that $A(t) \in L_{\rm PCD}[0,h]$, $B(t), C(t) \in L_{\rm CAC}[0,h]$ and that the system is asymptotically stable. Then it holds that*

$$||\mathcal{G}||_2^2 = \text{trace}(\underline{b}^* \underline{V}\, \underline{b}) = \text{trace}(\underline{c}\, \underline{W}\, \underline{c}^*) \tag{3.57}$$

where $\underline{b} := [\cdots, B_{-1}^T, B_0^T, B_1^T, \cdots]^T$, $\underline{c} := [\cdots, C_1, C_0, C_{-1}, \cdots]$ *with* $\{B_m\}_{-\infty}^{+\infty}$ *and* $\{C_m\}_{-\infty}^{+\infty}$ *being the Fourier coefficients of* $B(t)$ *and* $C(t)$*, respectively. The (infinite-dimensional) matrices* $\underline{V}$ *and* $\underline{W}$ *are, respectively, the solutions of the harmonic Lyapunov equations*

$$(\underline{A} - \underline{E}(j0))^* \underline{V} + \underline{V}(\underline{A} - \underline{E}(j0)) = -\underline{C}^* \underline{C} \tag{3.58}$$

$$(\underline{A} - \underline{E}(j0)) \underline{W} + \underline{W}(\underline{A} - \underline{E}(j0))^* = -\underline{B}\, \underline{B}^* \tag{3.59}$$

Before giving a proof to this theorem, we make a few remarks about the harmonic Lyapunov equations involved and the basic idea of the proof. In Section 3.1, it is shown that the harmonic Lyapunov equation should be viewed as operator-valued equation densely defined on l_2, or more precisely on l_E (which is dense in l_2 by Lemma 2.9). It is also clarified that the adjoint operator, denoted by $(\underline{A} - \underline{E}(j0))^*$, of the unbounded operator $\underline{A} - \underline{E}(j0)$ defined on l_E is also defined on the whole l_E and that the matrix expression of $(\underline{A} - \underline{E}(j0))^*$ is just the complex conjugate transpose of the matrix expression of $\underline{A} - \underline{E}(j0)$. Therefore, the relation between the second equality of (3.57) and the harmonic Lyapunov equation (3.59) can be proved exactly in the same way as in showing the relation between the first equality of (3.57) and (3.58) by introducing a complex conjugate transpose dual system. In view of this, only the proof for the latter relation is given. In addition, the proof will follow some idea similar to what we do in LTI continuous-time systems. Because of the infinite-dimensional structure of the frequency response operator $\underline{G}(j\varphi)$, however, there are frequent order interchanges between infinite integrals and infinite summations so that one must pay attention to the validity of such order interchanges.

Proof of Theorem 3.7 Let $\hat{\underline{b}} := [\cdots, \hat{B}_{-1}^T, \hat{B}_0^T, \hat{B}_1^T, \cdots]^T$ where $\{\hat{B}_m\}_{m=-\infty}^{+\infty}$ is the Fourier coefficients sequence of $\hat{B}(t) = P^{-1}(t,0)B(t)$. Apparently, under the given assumptions, it holds that $\hat{\underline{b}} = \underline{P}^{-1}\underline{b}$. Hence by Theorems 2.2 and 2.3 and the structure of $\hat{\underline{B}}$, we obtain

$$\begin{aligned}
||\mathcal{G}||_2^2 &= \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{+\frac{\omega_h}{2}} \text{trace}\Big(\hat{\underline{B}}^* (\underline{E}(j\varphi) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\varphi) - \underline{Q})^{-1} \hat{\underline{B}}\Big) d\varphi \\
&= \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{+\frac{\omega_h}{2}} \sum_{m=-\infty}^{+\infty} \text{trace}\Big(\hat{\underline{b}}^* (\underline{E}(j\varphi_m) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\varphi_m) - \underline{Q})^{-1} \hat{\underline{b}}\Big) d\varphi \\
&= \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \int_{-\frac{\omega_h}{2}}^{+\frac{\omega_h}{2}} \text{trace}\Big(\hat{\underline{b}}^* (\underline{E}(j\varphi_m) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\varphi_m) - \underline{Q})^{-1} \hat{\underline{b}}\Big) d\varphi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}\Big(\hat{\underline{b}}^* (\underline{E}(j\omega) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\omega) - \underline{Q})^{-1} \hat{\underline{b}}\Big) d\omega
\end{aligned} \tag{3.60}$$

In (3.60), the order of the integral and the infinite summation is interchanged. To see the validity of this interchange, it suffices to show that the convergence of

$$\begin{aligned}
&\sum_{|m|\le M} \text{trace}\Big(\hat{\underline{b}}^* (\underline{E}(j\varphi_m) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\varphi_m) - \underline{Q})^{-1} \hat{\underline{b}}\Big) \\
\to\ &\text{trace}\Big(\hat{\underline{B}}^* (\underline{E}(j\varphi) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\varphi) - \underline{Q})^{-1} \hat{\underline{B}}\Big)
\end{aligned} \tag{3.61}$$

is uniform over $\varphi \in \mathcal{I}_0$ as $M \to \infty$. We also note that the last term of (3.61), i.e., $\text{trace}\big(\underline{\hat{B}}^*(\underline{Q}-\underline{E}(j\varphi))^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{Q}-\underline{E}(j\varphi))^{-1}\underline{\hat{B}}\big)$, is bounded with an upper bound independent of φ (see the proof of Lemma 3.2). Under the given conditions, some similar arguments to those in the proof of Proposition 3.1 will lead to the above assertion (3.61).

Now we truncate the infinite-dimensional vector $\underline{\hat{b}}$ to $\underline{\hat{b}}_N$, which is defined by

$$\underline{\hat{b}}_N := [\cdots, 0, \hat{B}_{-N}^T, \cdots, \hat{B}_0^T, \cdots, \hat{B}_N{}^T, 0, \cdots]^T$$

Noting that we are dealing with the trace of a finite-dimensional matrix, it is clear that

$$\begin{aligned}
||\mathcal{G}||_2^2 &= \frac{1}{2\pi}\text{trace}\Big(\int_{-\infty}^{+\infty}(\lim_{N\to\infty}\underline{\hat{b}}_N^*)(\underline{E}(j\omega)-\underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega)-\underline{Q})^{-1}\underline{\hat{b}}\,d\omega\Big) \\
&= \lim_{N\to\infty}\frac{1}{2\pi}\text{trace}\Big(\int_{-\infty}^{+\infty}\underline{\hat{b}}_N^*(\underline{E}(j\omega)-\underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega)-\underline{Q})^{-1}\underline{\hat{b}}\,d\omega\Big) \\
&= \lim_{N\to\infty}\frac{1}{2\pi}\text{trace}\Big(\underline{\hat{b}}_N^*\int_{-\infty}^{+\infty}(\underline{E}(j\omega)-\underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega)-\underline{Q})^{-1}\underline{\hat{b}}\,d\omega\Big) \qquad (3.62)
\end{aligned}$$

by changing first the order of the infinite integral ($\int_{-\infty}^{+\infty}$) and the limit ($\lim_{N\to\infty}$), and then the order of the infinite integral ($\int_{-\infty}^{+\infty}$) and the summation caused by the multiplication with $\underline{\hat{b}}_N$. The latter order interchange is validated by the fact that for any fixed N, the corresponding summation in fact is only a finite one. It should be pointed out that only $\underline{\hat{b}}^*$ is truncated, but $\underline{\hat{b}}$ is not.

To see the validity of the first order interchanged we just mentioned, we need some extra work which is given in Appendix $A.3$ to keep our mainstream proof clear.

Next we further show that (3.62) can be rewritten as

$$||\mathcal{G}||_2^2 = \lim_{N\to\infty}\frac{1}{2\pi}\text{trace}\Big(\underline{\hat{b}}_N^*\int_{-\infty}^{+\infty}(\underline{E}(j\omega)-\underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega)-\underline{Q})^{-1}d\omega\,\underline{\hat{b}}\Big) \qquad (3.63)$$

by altering the order of the infinite integral ($\int_{-\infty}^{+\infty}$) and the infinite summation caused by the infinite-dimensional vector $\underline{\hat{b}}$, the validity proof of which is also given in Appendix $A.3$.

Furthermore, since $(\underline{E}(j\omega)-\underline{Q})^{-1}$ is block-diagonal, the integral ($\int_{-\infty}^{+\infty}$) can apply to each entry of $\underline{\hat{C}}^*\underline{\hat{C}}$. Denoting the (i,k)-th block entry of a matrix by $[\cdot]_{(i,k)}$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\Big[\int_{-\infty}^{+\infty}(\underline{E}(j\omega)-\underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega)-\underline{Q})^{-1}d\omega\Big]_{(i,k)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j(\omega+i\omega_h)I-Q)^{-*}[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}(j(\omega+k\omega_h)I-Q)^{-1}d\omega \\
&= \int_0^{+\infty}(e^{Q\tau}e^{-ji\omega_h\tau})^*[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}\,e^{Q\tau}e^{-jk\omega_h\tau}d\tau \\
&= \Big[\int_0^{+\infty}\underline{e}(Q,\tau)^*\underline{\hat{C}}^*\underline{\hat{C}}\,\underline{e}(Q,\tau)d\tau\Big]_{(i,k)} \qquad (3.64)
\end{aligned}$$

by the well-known Parseval theorem. In the last equation of (3.64), the fact that $\underline{e}(Q,\tau)$ is block-diagonal is used again. Now substituting (3.64) into (3.63), and using the fact that the operator $\int_0^\infty \underline{e}(Q,\tau)^* \underline{\hat{C}}^* \underline{\hat{C}}\, \underline{e}(Q,\tau) d\tau$ is bounded on l_2, it follows that

$$\begin{aligned}
||\mathcal{G}||_2^2 &= \lim_{N\to\infty} \mathrm{trace}\Big(\underline{\hat{b}}_N^* \int_0^\infty \underline{e}(Q,\tau)^* \underline{\hat{C}}^* \underline{\hat{C}}\, \underline{e}(Q,\tau) d\tau\, \underline{\hat{b}}\Big) \\
&= \mathrm{trace}\Big(\big(\lim_{N\to\infty} \underline{\hat{b}}_N^*\big) \int_0^\infty \underline{e}(Q,\tau)^* \underline{\hat{C}}^* \underline{\hat{C}}\, \underline{e}(Q,\tau) d\tau\, \underline{\hat{b}}\Big) \\
&= \mathrm{trace}\Big(\underline{\hat{b}}^* \underline{\hat{V}}\, \underline{\hat{b}}\Big) = \mathrm{trace}\Big(\underline{b}^* \underline{P}^{-*} \underline{\hat{V}}\, \underline{P}^{-1} \underline{b}\Big) = \mathrm{trace}\Big(\underline{b}^* \underline{V}\, \underline{b}\Big)
\end{aligned}$$

where $\underline{\hat{V}} := \int_0^{+\infty} \underline{e}(Q,\tau)^* \underline{\hat{C}}^* \underline{\hat{C}}\, \underline{e}(Q,\tau) d\tau$ and $\underline{V} := \underline{P}^{-*} \underline{\hat{V}}\, \underline{P}^{-1}$, which, by Theorem 3.1 (see Remark 3.1), is the unique solution of the harmonic Lyapunov equation (3.58). **Q.E.D.**

In general, it is hard to find the solutions of (3.58) and (3.59). Corollary 3.2 below states the results in terms of $\underline{Q} - \underline{E}(j0)$, which gives convenience in analysis and computations.

Corollary 3.2 *Under the same assumptions as in Theorem* 3.7*, it holds that* $||\mathcal{G}||_2^2 = \mathrm{trace}(\underline{\hat{b}}^* \underline{\hat{V}}\, \underline{\hat{b}}) = \mathrm{trace}(\underline{\hat{c}}\, \underline{\hat{W}}\, \underline{\hat{c}}^*)$. *Here the infinite-dimensional matrices* $\underline{\hat{V}}$ *and* $\underline{\hat{W}}$ *are, respectively, the solutions of the harmonic Lyapunov equations*

$$(\underline{Q} - \underline{E}(j0))^* \underline{\hat{V}} + \underline{\hat{V}}(\underline{Q} - \underline{E}(j0)) = -\underline{\hat{C}}^* \underline{\hat{C}} \tag{3.65}$$

$$(\underline{Q} - \underline{E}(j0)) \underline{\hat{W}} + \underline{\hat{W}}(\underline{Q} - \underline{E}(j0))^* = -\underline{\hat{B}}\, \underline{\hat{B}}^* \tag{3.66}$$

where $\underline{\hat{b}}$ *and* $\underline{\hat{c}}$ *are defined similarly to* $\underline{b}$ *and* $\underline{c}$ *but in terms of* $\hat{B}(t)$ *and* $\hat{C}(t)$*, respectively.*

Remark 3.6 *If the* FDLCP *system (*2.1*) is* LTI *continuous-time, the harmonic Lyapunov equation can be seen as the 'lifted' version of the usual algebraic Lyapunov equation. Hence the trace formula of Theorem* 3.7 *reduces to that in the* LTI *continuous-time case [32], [91]. Unfortunately, however, the trace formula for general* FDLCP *systems involves the infinite-dimensional matrices* $\underline{b}$ *and* $\underline{V}$*. This difficulty more or less confines the value of Theorem* 3.7 *to the theoretical analysis. In Chapter* 4*, we derive some modified trace formulas for the* H_2 *norm via the approximate modeling approach, in which the trace formulas of Theorem* 3.7 *and Corollary* 3.2 *play a central role.*

3.4.4 Upper Bound Formula for Frequency Response Gains

In Chapter 4, a bisection algorithm will be developed for the H_∞ norm computation in FDLCP systems. As is well-known in [91], in using this kind of algorithms, the knowledge about upper bounds of the H_∞ norm of the corresponding frequency response operator is necessary. In this subsection, we derive an upper bound for the l_2-induced norm of the frequency response operator $\underline{G}(j\varphi)$ for each $\varphi \in \mathcal{I}_0$, that is, the frequency response gain of the FDLCP system at the frequency $\varphi \in \mathcal{I}_0$. Although the upper bound is claimed for the frequency response gains on each fixed frequency φ in the frequency interval $\mathcal{I}_0$, it is straightforward to see that it actually also gives a way to determine an upper bound for the H_∞ norm that is needed for the bisection algorithm iterative computations.

Theorem 3.8 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\rm PCD}[0,h]$ while $B(t), C(t)$ and $D(t)$ belong to $L_{\rm CAC}[0,h]$, and that the system is asymptotically stable. Then*

$$||\underline{G}(j\varphi)||_{l_2/l_2} \le \gamma_B\gamma_C \sup_{m\in\mathcal{Z}}\{||(j\varphi_m I - Q)^{-1}||\} + \gamma_D$$

for every $\varphi \in \mathcal{I}_0$. Here,

$$\gamma_B := \max_{t\in[0,h]} ||P^{-1}(t,0)B(t)||, \quad \gamma_C := \max_{t\in[0,h]} ||C(t)P(t,0)||, \quad \gamma_D := \max_{t\in[0,h]} ||D(t)||$$

Proof By the definition of the frequency response operator $\underline{G}(j\varphi)$, it follows that

$$||\underline{G}(j\varphi)||_{l_2/l_2} \le ||\underline{C}\,\underline{P}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2}||\underline{P}^{-1}\underline{B}||_{l_2/l_2} + ||\underline{D}||_{l_2/l_2}$$

From the block-diagonal structure of the operator $(\underline{E}(j\varphi) - \underline{Q})^{-1}$, it is easy to see that

$$||(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2} = \sup_{m\in\mathcal{Z}}\{||(j\varphi_m I - Q)^{-1}||\}$$

Furthermore, under the given assumptions and by Lemma 2.8, we obtain that

$$||\underline{P}^{-1}\underline{B}||_{l_2/l_2} = \sup_{t\in[0,h]} ||P^{-1}(t,0)B(t)|| = \max_{t\in[0,h]} ||P^{-1}(t,0)B(t)||(= \gamma_B)$$

The last equality comes from the fact that $P^{-1}(t,0)B(t)$ belongs to $L_{\rm CAC}[0,h]$ by Proposition 2.1 and thus $P^{-1}(t,0)B(t)$ is continuous with respect to $t \in [0,h]$. Similarly for the coefficients γ_C and γ_D. **Q.E.D.**

Remark 3.7 *γ_B, γ_C and γ_D are the maximum singular values of the finite-dimensional matrices defined on a finite time interval. Hence the algorithms can be implemented. In addition, γ_B, γ_C and γ_D are time-domain factors while* $\sup_{m\in\mathcal{Z}}\{||(j\varphi_m I - Q)^{-1}||\}$ *is given in the frequency domain. Therefore, the upper bound is a mixed-type estimation for the l_2-induced norm. If only the H_∞ norm is concerned, an obvious upper bound is given by*

$$||\mathcal{G}||_\infty \le \gamma_B\gamma_C||\mathcal{Q}||_\infty + \gamma_D$$

where $||\mathcal{Q}||_\infty := \sup_{\omega\in(-\infty,\infty)} ||(j\omega I - Q)^{-1}||$ *is the H_∞ norm of the equivalent* LTI *continuous-time subsystem (Q, I, I). In addition, by Proposition* 2.2, *some upper bound formulas for the frequency response gains and H_∞ norm can even be given without the knowledge of the periodic portion $P(t,0)$ and $P^{-1}(t,0)$ of the transition matrix of a given* FDLCP *system but at the price of higher conservativeness.*

Chapter 4

Numerical Harmonic Analysis of FDLCP Systems

Chapter 3 consists of the theoretical results about FDLCP systems derived through the Fourier analysis but at an operator-theoretic level. These results clarify the basic properties of FDLCP systems and lay the foundations for further analysis and synthesis discussions from an operator-theoretic viewpoint. However, simple observations reveal that the matrix expressions for these results are usually infinite-dimensional. This is a hurdle for the numerical computations based thereupon. The main purpose of this chapter is to implement these equations and formulas numerically and prove the resulting convergences when problems are reduced to finite-dimensional ones. The problems in this chapter include: a necessary and sufficient stability theorem based on approximate modeling (derived from the harmonic Lyapunov equation and Gronwall's lemma) and its corollary in Section 4.1; asymptotic trace formulas for the H_2 norm computation and an asymptotic Hamiltonian test for the H_∞ norm computation developed via skew and staircase truncations on the frequency response operator in Section 4.2 [86]. In addition, since it is hard to get the closed-form knowledge of the transition matrix of a general FDLCP system, the H_2 and H_∞ norm computations via approximate modeling are also considered in Section 4.3. The implementation problem of the trace formulas of Theorem 3.7 and Corollary 3.2 is discussed in Section 4.4.

4.1 Stability Criteria via Approximate Modeling

In general, the difficulty in applying the Floquet theorem is that we have to determine the transition matrix that is usually much harder to find, compared with the cases in linear discrete-time periodic systems [8] and sampled-data systems [26]. One may consider to compute $\Phi(h + t_0, t_0)$ by a numerical solution of the corresponding differential equation. In this case, however, an approximate modeling error will be inevitable. To put it another way, this approach amounts to testing merely stability of some approximate model of the given

FDLCP system unless the modeling error is taken into account. An obstacle in using the harmonic Lyapunov equation (Theorem 3.1) is that the solution is infinite-dimensional so that there is no way to test the positive definiteness of this solution besides the difficulty in determining the solution itself. To surmount these difficulties, we revisit the approximate modeling method [25], [38]. The basic idea is that if we construct an approximate model to the original FDLCP systems in some sense such that the transition matrix of this approximate model can be determined explicitly in a closed form (so that this transition matrix knowledge can be used in stability testing of the approximate model), then we are confronted with such a question: under what condition, can one guarantee the stability of the original FDLCP system by that of the approximate model? The main difficulties in such a stability analysis method include: how to measure the modeling error and how to assess its effect on the stability of the actual system.

4.1.1 Stability Criteria Derived via Different Approaches

There are several ways to deal with the modeling error and investigate its effect on asymptotic stability of the actual systems. In this subsection, two approaches will be considered: the harmonic analysis of an *approximate* system operator, $\underline{A}_a - \underline{E}(j0)$, and the asymptotic analysis of an *approximate* differential equation, $\dot{x}(t) = A_a(t)x(t)$.

To express the approximate modeling idea, we decompose the state matrix of (2.1) as

$$A(t) = A_a(t) + A_\Delta(t) \tag{4.1}$$

where $A_a(t)$ is an approximate state matrix and $A_\Delta(t)$ is the error matrix. Here, we assume that $A_a(t)$ and $A_\Delta(t)$ are h-periodic. Now construct the approximate FDLCP model

$$G_a : \dot{\tilde{x}} = A_a(t)\tilde{x} \tag{4.2}$$

which has the (explicit) transition matrix $\Phi_a(t,0) = P_a(t,0)e^{Q_a t}$. By (4.1), $\underline{A} = \underline{A}_a + \underline{A}_\Delta$ with $\underline{A}_a := \mathcal{T}\{A_a(t)\}$ and $\underline{A}_\Delta := \mathcal{T}\{A_\Delta(t)\}$. Again, by using the Fourier series expansion from $L_2[0,h]$ to l_2, it follows from Lemma 2.8 that

$$||\underline{A}_\Delta||_{l_2/l_2} = \sup_{t\in[0,h]} ||A_\Delta(t)|| =: ||A_\Delta(\cdot)||$$

if $A_\Delta(t) \in L_{\text{PCD}}[0,h] \subset L_{\text{PCC}}[0,h]$. Based on these preparations, the following theorem gives an answer to the question we posed.

Theorem 4.1 *Suppose $A(t) \in L_{\text{PCD}}[0,h]$ and let $L_a[0,h]$ be a dense subset of $L_{\text{PCD}}[0,h]$ in the $L_\infty[0,h]$-norm sense (and hence $L_a[0,h]$ is dense in $L_{\text{PCD}}[0,h]$ also in the $L_2[0,h]$-norm sense). Then the system (2.1) is asymptotically stable if and only if there exists an approximate h-periodic system G_a as defined in (4.2) such that*

1). $A_a(t) \in L_a[0,h]$;

2). *G_a has the transition matrix $\Phi_a(t,0) = P_a(t,0)e^{Q_a t}$ and all the eigenvalues of Q_a have negative real parts;*

3). *for $A_\Delta(t) = A(t) - A_a(t)$, there exist numbers $K_a > 0$ and $\alpha > 0$ satisfying*

$$||e^{Q_a t}|| \le K_a e^{-\alpha t} \quad (\forall t \ge 0), \quad \sup_{t\in[0,h]} ||P_a^{-1}(t,0)A_\Delta(t)P_a(t,0)|| < \alpha/K_a^2$$

From the condition 3) of Theorem 4.1, it can be said that the asymptotic stability of FDLCP systems is essentially robust in the sense that a stable FDLCP system can be approximated by an h-periodic model which remains stable under some weak perturbations. The necessity and sufficiency proofs for Theorem 4.1 will be given separately.

Sufficiency Proof of Theorem 4.1 Assume that the conditions 1) through 3) hold. Since $L_a[0,h] \subset L_{\mathrm{PCD}}[0,h]$, Theorem 3.1 applies to the approximate model G_a. Therefore, the assumption that the approximate system G_a is asymptotically stable implies that, for any $\underline{W}_a \in \mathcal{S}^+$, the harmonic Lyapunov equation

$$(\underline{A}_a - \underline{E}(j0))^*\underline{V}_a + \underline{V}_a(\underline{A}_a - \underline{E}(j0)) = -\underline{W}_a \tag{4.3}$$

has a unique solution $\underline{V}_a \in \mathcal{S}^+$. In particular, let $\underline{P}_a^*\underline{W}_a\underline{P}_a = \underline{I} \in \mathcal{S}^+$ by taking $\underline{W}_a = \underline{P}_a^{-*}\underline{P}_a^{-1} \in \mathcal{S}^+$. Then, we have

$$\underline{V}_a = \underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1} \tag{4.4}$$

On the other hand, from (4.3), we obtain

$$\begin{aligned}
&(\underline{A}_a + \underline{A}_\Delta - \underline{E}(j0))^*\underline{V}_a + \underline{V}_a(\underline{A}_a + \underline{A}_\Delta - \underline{E}(j0)) \\
= \;& -\underline{W}_a + \underline{A}_\Delta^*\underline{V}_a + \underline{V}_a\underline{A}_\Delta = -(\underline{P}_a^{-*}\underline{P}_a^{-1} - \underline{A}_\Delta^*\underline{V}_a - \underline{V}_a\underline{A}_\Delta)
\end{aligned} \tag{4.5}$$

Now take $0 \ne \underline{x} \in l_E \subset l_2$. Then

$$\begin{aligned}
&\langle(\underline{P}_a^{-*}\underline{P}_a^{-1} - \underline{A}_\Delta^*\underline{V}_a - \underline{V}_a\underline{A}_\Delta)\underline{x},\, \underline{x}\rangle \\
= \;& \langle\underline{P}_a^{-1}\underline{x},\, \underline{P}_a^{-1}\underline{x}\rangle - \langle\underline{A}_\Delta^*\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{x},\, \underline{x}\rangle \\
& -\langle\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{A}_\Delta\underline{x},\, \underline{x}\rangle
\end{aligned} \tag{4.6}$$

By the well-known Cauchy-Schwarz inequality [22], we obtain

$$\begin{aligned}
&|\langle\underline{A}_\Delta^*\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{x},\, \underline{x}\rangle| \\
= \;& |\langle\underline{P}_a^*\underline{A}_\Delta^*\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{x},\, \underline{P}_a^{-1}\underline{x}\rangle| \\
\le \;& \Big[\langle\underline{P}_a^{-1}\underline{x},\, \underline{P}_a^{-1}\underline{x}\rangle\, \langle\underline{P}_a^*\underline{A}_\Delta^*\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{x}, \\
& \qquad \underline{P}_a^*\underline{A}_\Delta^*\underline{P}_a^{-*}\Big\{\int_0^\infty \underline{e}(Q_a,\tau)^*\underline{e}(Q_a,\tau)d\tau\Big\}\underline{P}_a^{-1}\underline{x}\rangle\Big]^{\frac{1}{2}}
\end{aligned}$$

$$\leq \; ||\underline{P}_a^{-1}\underline{x}||_{l_2}^2 \; ||\underline{P}_a^* \underline{A}_\Delta^* \underline{P}_a^{-*}||_{l_2/l_2} \int_0^\infty ||\underline{e}(Q_a,\tau)^* \underline{e}(Q_a,\tau)||_{l_2/l_2} d\tau$$

$$\leq \; ||\underline{P}_a^{-1}\underline{x}||_{l_2}^2 \; ||\underline{P}_a^{-1} \underline{A}_\Delta \underline{P}_a||_{l_2/l_2} \int_0^\infty K_a^2 e^{-2\alpha\tau} d\tau$$

$$= \; ||\underline{P}_a^{-1}\underline{x}||_{l_2}^2 \sup_{t\in[0,h]} ||P_a^{-1}(t,0)A_\Delta(t)P_a(t,0)|| \frac{K_a^2}{2\alpha} < \frac{1}{2}||\underline{P}_a^{-1}\underline{x}||_{l_2}^2 \tag{4.7}$$

where we used the assumption 3) and followed a similar derivation as in the proof of Lemma 2.8 on the operator $\underline{P}_a^{-1}\underline{A}_\Delta\underline{P}_a$ since $P_a^{-1}(t,0)A_\Delta(t)P_a(t,0) \in L_{\text{PCD}}[0,h]$. It is clear that the arguments in (4.7) can be repeated on the third term of the right-hand side of (4.6). Also, for any $0 \neq \underline{x} \in l_E \subset l_2$, $\underline{P}_a^{-1}\underline{x} \neq 0$. Summarizing the above arguments, it can be concluded that for any $0 \neq \underline{x} \in l_E \subset l_2$

$$\langle(\underline{P}_a^{-*}\underline{P}_a^{-1} - \underline{A}_\Delta^*\underline{V}_a - \underline{V}_a\underline{A}_\Delta)\underline{x},\, \underline{x}\rangle > 0 \tag{4.8}$$

Finally, we confine $\underline{x}$ to be an eigenvector of $\underline{A}_a + \underline{A}_\Delta - \underline{E}(j0)(= \underline{A} - \underline{E}(j0))$ corresponding to an eigenvalue λ. Post-multiplying $\underline{x}$ on (4.5) and taking the inner product with $\underline{x}$, it follows from (4.8) that $2\text{Re}(\lambda)\langle\underline{V}_a\underline{x},\, \underline{x}\rangle < 0$. Noting that $\underline{V}_a \in \mathcal{S}^+$, this inequality actually says that all the eigenvalues of $\underline{A} - \underline{E}(j0)$ have negative real parts. This ensures by Theorem 2.5 that the original FDLCP system G is asymptotically stable. **Q.E.D.**

It is worth mentioning that the sufficiency proof does not rely on the assumption that $L_a[0,h]$ is dense in $L_{\text{PCD}}[0,h]$. This implies that $A_a(t)$ can be any approximate model as long as $A_a(t) \in L_{\text{PCD}}[0,h]$ and its corresponding transition matrix can be determined explicitly by some approach. Indeed, [38] gave a similar stability test by using constant state matrix approximation, which is derived by the well-known Gronwall's Lemma [25], [38], [61]. In the following, we state this lemma for the necessity proof of Theorem 4.1. A complete proof for this lemma is given in Appendix A.4.

Lemma 4.1 (Gronwall's Lemma) *Let u and f be continuous functions defined on the interval $[t_1, t_2]$, $f(t) \geq 0, \forall t \in [t_1, t_2]$ and K is a constant. If $u(t) \leq K + \int_{t_1}^t f(\tau)u(\tau)d\tau$ for $t \in [t_1, t_2]$, then $u(t) \leq K\exp(\int_{t_1}^t f(\tau)d\tau)$.*

In the necessity proof of Theorem 4.1, we also need the following lemma (Lemma 6.3.1 of [51]) about the norm inequality of the transition matrix of a general linear time-varying system. This lemma also plays a key role in simplifying the stability conditions of Theorem 4.1 to get Corollary 4.1 in the following discussions.

Lemma 4.2 *Let $A(t)$ be the state matrix of a time-varying state-space system. Assume that $A(t)$ is locally integrable on the time interval $J \subset [0, \infty)$ in the sense that $\int_J ||A(t)||dt \leq K < \infty$. Then for all $(t,\tau) \in J \times J$, the corresponding transition matrix $\Phi(t,\tau)$ satisfies*

$$||\Phi(t,\tau)|| \leq e^{|\int_\tau^t ||A(\sigma)||d\sigma|}$$

Necessity Proof of Theorem 4.1 Now assuming the FDLCP system (2.1) is asymptotically stable, it is shown that there is an approximate FDLCP system defined as in (4.2) such that the conditions of Theorem 4.1 are satisfied. We accomplish the proof in four steps.

Step 1. It is shown that for any $t \in [0, h]$

$$\lim_{||A_\Delta(\cdot)|| \to 0} ||\Phi(t,0) - \Phi_a(t,0)|| = 0, \qquad \lim_{||A_\Delta(\cdot)|| \to 0} ||e^{Q_a t} - e^{Qt}|| = 0 \tag{4.9}$$

where the convergence is uniform with respect to t. From (4.2), we observe

$$\dot{\Phi}_a(t,0) = A(t)\Phi_a(t,0) - A_\Delta(t)\Phi_a(t,0)$$

According to the variation-of-constants formula [38], it follows that

$$\begin{aligned} & \Phi_a(t,0) - \Phi(t,0) \\ = & -\int_0^t \Phi(t,\tau)A_\Delta(\tau)\Phi_a(\tau,0)d\tau \\ = & -\int_0^t \Phi(t,\tau)A_\Delta(\tau)[\Phi_a(\tau,0) - \Phi(\tau,0)]d\tau - \int_0^t \Phi(t,\tau)A_\Delta(\tau)\Phi(\tau,0)d\tau \end{aligned}$$

which implies that for any $t \in [0, h]$

$$\begin{aligned} & ||\Phi_a(t,0) - \Phi(t,0)|| \\ \leq & \int_0^t ||\Phi(t,\tau)|| \cdot ||A_\Delta(\tau)|| \cdot ||\Phi_a(\tau,0) - \Phi(\tau,0)||d\tau \\ + & \int_0^t ||\Phi(t,\tau)|| \cdot ||A_\Delta(\tau)|| \cdot ||\Phi(\tau,0)||d\tau \end{aligned}$$

It is obvious that there exists a number $\check{K} > 0$ such that

$$\sup_{\tau \in [0,h]} ||\Phi(\tau,0)|| \leq \sup_{\tau,t \in [0,h]} ||\Phi(t,\tau)|| = \check{K} < \infty$$

since $\Phi(t,\tau)$ is continuous on $[0,h] \times [0,h]$. Then it follows that

$$\begin{aligned} & ||\Phi_a(t,0) - \Phi(t,0)|| \\ \leq & \check{K}^2 h \sup_{t \in [0,h]} ||A_\Delta(t)|| + \check{K} \sup_{t \in [0,h]} ||A_\Delta(t)|| \int_0^t ||\Phi_a(\tau,0) - \Phi(\tau,0)||d\tau \end{aligned} \tag{4.10}$$

Since $L_{\text{PCD}}[0,h]$ consists only of piecewise continuous functions, $||A_\Delta(\cdot)|| = \sup_{t \in [0,h]} ||A_\Delta(t)||$ is well-defined regardless of the choice of $L_a[0,h]$ and $A_a(t)$. Furthermore, by the assumption, $||A_\Delta(\cdot)||$ can be made arbitrarily small by a suitable choice of $A_a(t)$. Noting also that $\Phi(t,0)$ and $\Phi_a(t,0)$ are continuous, it follows from Lemma 4.1 that

$$||\Phi_a(t,0) - \Phi(t,0)|| \leq \check{K}^2 h ||A_\Delta(\cdot)|| \exp(\check{K}h||A_\Delta(\cdot)||) \tag{4.11}$$

which says that as $||A_\Delta(\cdot)|| \to 0$, $\Phi_a(t,0) \to \Phi(t,0)$ uniformly with respect to $t \in [0,h]$.

To show the second relation of (4.9), we define the set

$$\mathcal{A}_\delta := \left\{A_a(t) \in L_a[0,h] : \sup_{t\in[0,h]} ||A_\Delta(t)|| \le \delta\right\} \tag{4.12}$$

with δ being a constant. Now we further denote the closure of $\mathcal{A}_\delta$ by $\overline{\mathcal{A}}_\delta$, which is well-defined by Exercise 3.1.4(c,e) of [22, p. 107] since $L_a[0,h]$ is dense in $L_{\text{PCD}}[0,h]$. Clearly, $\overline{\mathcal{A}}_\delta$ is bounded and closed for any fixed δ. From (4.11), we observe that

$$\lim_{||A_\Delta(\cdot)||\to 0} \Phi_a(h,0) = \Phi(h,0) \tag{4.13}$$

which implies that the eigenvalues of $\Phi_a(h,0)$ tend to those of $\Phi(h,0)$ as $||A_\Delta(\cdot)|| \to 0$. Since $\Phi_a(h,0)$ and $\Phi(h,0)$ are nonsingular, it can be asserted that as $||A_\Delta(\cdot)|| \to 0$, each of the eigenvalues of $\Phi_a(h,0)$ can be situated in an arbitrarily small ϵ-neighborhood, which does not contain the origin of the complex plane, of the corresponding one of those of $\Phi(h,0)$. This, together with the fact that $\Phi_a(h,0)$ and $\Phi(h,0)$ have only finitely many eigenvalues, shows that it is always possible to find a real number $R > 1$ and a real number $\theta \in [0, 2\pi)$ together with sufficiently small $\delta > 0$ such that all the eigenvalues of $\Phi_a(h,0)$ and $\Phi(h,0)$ lie in the simply connected region $D_{R,\theta}$ on the complex plane for all $A_a(t) \in \overline{\mathcal{A}}_\delta$. Here $D_{R,\theta}$ denotes the region in the complex plane between the circle $|z| = R$ and $|z| = 1/R$, excluding the ray segment $\{z = re^{j\theta} : 1/R \le r \le R\}$. Let Γ denote the boundary of $D_{R,\theta}$, traversed in the positive sense, and for each $A_a(t) \in \overline{\mathcal{A}}_\delta$, define

$$\operatorname{Log}\Phi_a(h,0) := \frac{1}{2\pi j}\oint_\Gamma (\log z)(zI - \Phi_a(h,0))^{-1}dz \tag{4.14}$$

where the principal branch of the scalar logarithm is used. Then, by Theorem 6.4.20 of [40], we obtain that $\exp(\operatorname{Log}\Phi_a(h,0)) = \Phi_a(h,0)$ and $\operatorname{Log}(\cdot)$ is continuous over the set of $\Phi_a(h,0)$ corresponding to $A_a(t) \in \overline{\mathcal{A}}_\delta$. Hence, it follows by (4.13) and Theorem 3.7.1 of [55] that

$$\begin{aligned}\lim_{||A_\Delta(\cdot)||\to 0} Q_a &= \frac{1}{h}\lim_{||A_\Delta(\cdot)||\to 0}\operatorname{Log}\Phi_a(h,0)\\ &= \frac{1}{h}\operatorname{Log}\lim_{||A_\Delta(\cdot)||\to 0}\Phi_a(h,0) = \frac{1}{h}\operatorname{Log}\Phi(h,0) = Q\end{aligned} \tag{4.15}$$

which says that the condition 2) is satisfied.

Now considering two LTI state space differential equations $\dot\mu = Q_a\mu$ and $\dot v = Qv$ and repeating the arguments around (4.10) and (4.11) on these two equations and applying Lemma 4.1, we obtain that for any $t \in [0,h]$

$$||e^{Q_at} - e^{Qt}|| \le \hat{K}^2h||Q_a - Q||\exp(\hat{K}h||Q_a - Q||) \tag{4.16}$$

where $\hat{K}$ is equal to that in (3.5) since the system G is assumed to be stable. The equalities (4.15) and the inequality (4.16) complete the proof of the second relation of (4.9).

Step 2. It is shown that for any $t \in [0,h]$

$$\lim_{||A_\Delta(\cdot)||\to 0} ||\Phi_a^{-1}(t,0) - \Phi^{-1}(t,0)|| = 0, \qquad \lim_{||A_\Delta(\cdot)||\to 0} ||e^{-Q_at} - e^{-Qt}|| = 0 \tag{4.17}$$

where the convergence is uniform with respect to t. The second relation of (4.17) can be verified by considering two state space differential equations $\dot{\mu} = -Q_a\mu$ and $\dot{v} = -Qv$ as we did for the second relation in (4.9).

To show the first relation of (4.17) is equal to verifying that for any $t \in [0, h]$

$$\lim_{||A_\Delta(\cdot)|| \to 0} ||\Phi_a(0,t) - \Phi^{-1}(t,0)|| = 0$$

By Theorem 6.3.2 of [51], it is true that

$$\frac{d}{dt}\Phi_a(0,t) = -\Phi_a(0,t)A_a(t)$$

which can be equivalently rearranged as

$$\frac{d}{dt}\Phi_a^T(0,t) = -A_a^T(t)\Phi_a^T(0,t) = -A^T(t)\Phi_a^T(0,t) + A_\Delta^T(t)\Phi_a^T(0,t)$$

To apply the variation-of-constants formula, we denote the transition matrix of the state-space differential equation $\dot{q} = -A^T(t)q$ by $\Phi_-(t,\tau)$. Then, we have

$$\begin{aligned} & \Phi_a^T(0,t) - \Phi_-(t,0) \\ = & \int_0^t \Phi_-(t,\tau)A_\Delta^T(\tau)\Phi_a^T(0,\tau)d\tau \\ = & \int_0^t \Phi_-(t,\tau)A_\Delta^T(\tau)[\Phi_a^T(0,\tau) - \Phi_-(\tau,0)]d\tau + \int_0^t \Phi_-(t,\tau)A_\Delta^T(\tau)\Phi_-(\tau,0)d\tau \end{aligned}$$

which, together with Lemma 4.1, yields

$$||\Phi_a^T(0,t) - \Phi_-(t,0)|| \le \tilde{K}^2 h ||A_\Delta(\cdot)|| \exp(\tilde{K}h||A_\Delta(\cdot)||) \quad (4.18)$$

where $\tilde{K} := \sup_{t,\tau\in[0,h]} ||\Phi_-(t,\tau)||$ is well-defined.

On the other hand, by the definition of $\Phi_-(t,0)$, it follows that

$$\frac{d}{dt}\Phi_-(t,0) = -A^T(t)\Phi_-(t,0)$$

or equivalently

$$\frac{d}{dt}\Phi_-^T(t,0) = -\Phi_-^T(t,0)A(t)$$

Again by Theorem 6.3.2 of [51] and the uniqueness of the transition matrix, it follows that

$$\Phi_-^T(t,0) = \Phi(0,t) = \Phi^{-1}(t,0)$$

Using this in (4.18), the desired result follows.

Step 3. Recall the sets $\mathcal{A}_\delta$ and $\overline{\mathcal{A}}_\delta$ introduced in Step 1. Since the first inequality in (3.5) is strict, it follows from (4.15) that there exists a small enough $\delta > 0$ such that for some $\beta > \alpha$, every $A_a(t) \in \overline{\mathcal{A}}_\delta$ will be β-stable in the sense that every eigenvalue of Q_a corresponding to $A_a(t)$ has the real part less than $-\beta$. In the sequel, we take one such small enough δ. Then, for each $A_a(t) \in \overline{\mathcal{A}}_\delta$ there exists a finite number $K_a(Q_a) > 0$ such that

$$||e^{Q_a t}|| \leq K_a(Q_a)e^{-\beta t} \quad (\forall t \geq 0) \tag{4.19}$$

Now we are in a position to show that $\sup_{t\in[0,h]} ||P_a^{-1}(t,0)||$ and $\sup_{t\in[0,h]} ||P_a(t,0)||$ have uniform upper bounds over the set $\overline{\mathcal{A}}_\delta$. To see this, note that

$$\sup_{t\in[0,h]} ||P_a(t,0)|| = \sup_{t\in[0,h]} ||\Phi_a(t,0)e^{-Q_a t}|| \leq \sup_{t\in[0,h]} ||\Phi_a(t,0)|| \sup_{t\in[0,h]} ||e^{-Q_a t}|| \tag{4.20}$$

Here, by the definition of $\overline{\mathcal{A}}_\delta$ and the second relation of (4.17), there exists $M > 0$ such that

$$\sup_{t\in[0,h]} ||e^{-Q_a t}|| \leq M + \sup_{t\in[0,h]} ||e^{-Qt}|| \quad (\forall A_a(t) \in \overline{\mathcal{A}}_\delta) \tag{4.21}$$

Similarly, from the first relation of (4.9), there exists $N > 0$ such that

$$\sup_{t\in[0,h]} ||\Phi_a(t,0)|| \leq N + \sup_{t\in[0,h]} ||\Phi(t,0)|| \quad (\forall A_a(t) \in \overline{\mathcal{A}}_\delta) \tag{4.22}$$

Hence, we are led to the uniform boundedness of $||P_a(t,0)||$. The above arguments can be repeated on $\sup_{t\in[0,h]} ||P_a^{-1}(t,0)||$ by using the second relation of (4.9) and the first relation of (4.17).

Step 4. It is shown that the condition 3) holds. To this end, observe

$$\begin{aligned} \mu &:= \sup_{t\in[0,h]} ||P_a^{-1}(t,0)A_\Delta(t)P_a(t,0)|| \\ &\leq \sup_{t\in[0,h]} ||P_a^{-1}(t,0)|| \sup_{t\in[0,h]} ||A_\Delta(t)|| \sup_{t\in[0,h]} ||P_a(t,0)|| \end{aligned} \tag{4.23}$$

Therefore, by the uniform boundedness of the first and third factors in the right-hand side of the inequality (4.23) and the fact that $L_a[0,h]$ is dense in $L_{\text{PCD}}[0,h]$ in the $L_\infty[0,h]$-norm sense, μ can be made arbitrarily small by taking appropriate $A_a(t) \in \mathcal{A}_\delta \subset L_a[0,h]$. Therefore, the proof becomes complete if we show that the first requirement in the condition 3) can be satisfied for all $A_a(t) \in \mathcal{A}_\delta$ with a fixed $K_a > 0$ and a fixed $\alpha > 0$ independent of $A_a(t) \in \mathcal{A}_\delta$. In the following, we show this is indeed the case. More specifically, we show that there exists $K_a > 0$ such that

$$||e^{Q_a t}|| \leq K_a e^{-\alpha t} \quad (\forall t \geq 0, \forall A_a(t) \in \mathcal{A}_\delta) \tag{4.24}$$

where α is given in (3.5). This can be completed by showing that

$$||e^{Q_a t}|| \leq K_a e^{-\alpha t} \quad (\forall t \geq 0, \forall A_a(t) \in \overline{\mathcal{A}}_\delta) \tag{4.25}$$

since $\mathcal{A}_\delta$ is a subset of $\overline{\mathcal{A}}_\delta$.

To show (4.25), we first fix an $A_{a1}(t) \in \overline{\mathcal{A}}_\delta$ with the associated Q_{a1} satisfying (4.19) with $K_a(Q_{a1})$. Then, for another $A_{a2}(t) \in \overline{\mathcal{A}}_\delta$ with the associated Q_{a2}, we denote $\Delta Q_a := Q_{a2} - Q_{a1}$ and consider the matrix differential equation

$$\dot{X}(t) = Q_{a1}X(t) + \Delta Q_a X(t) \tag{4.26}$$

It is clear that the solution is just $X(t) = e^{(Q_{a1}+\Delta Q_a)t} = e^{Q_{a2}t}$. On the other hand, by using the variation-of-constants formula in (4.26), in a similar way to Step 1, we have

$$X(t) = e^{Q_{a1}t} + \int_0^t e^{Q_{a1}(t-\tau)} \Delta Q_a X(\tau) d\tau$$

which leads to

$$\begin{aligned} ||X(t)|| &\le ||e^{Q_{a1}t}|| + \int_0^t ||e^{Q_{a1}(t-\tau)}|| \cdot ||\Delta Q_a|| \cdot ||X(\tau)|| d\tau \\ &\le K_a(Q_{a1}) e^{-\beta t} + \int_0^t K_a(Q_{a1}) e^{-\beta(t-\tau)} ||\Delta Q_a|| \cdot ||X(\tau)|| d\tau \end{aligned}$$

or equivalently,

$$||X(t)e^{\beta t}|| \le K_a(Q_{a1}) + \int_0^t K_a(Q_{a1}) ||\Delta Q_a|| \cdot ||X(\tau)e^{\beta\tau}|| d\tau$$

Hence, by Gronwall's Lemma (Lemma 4.1), we obtain

$$||X(t)|| \le K_a(Q_{a1}) e^{(K_a(Q_{a1})||\Delta Q_a|| - \beta)t} \quad (\forall t \ge 0) \quad .$$

which, together with the fact that $X(t) = e^{(Q_{a1}+\Delta Q_a)t}$, clearly says that for each Q_{a1}, there exists a neighborhood $\mathcal{N}(Q_{a1})$ of Q_{a1} and a constant number $M_1 > 0$ dependent only on the matrix Q_{a1} such that

$$||e^{Q_a t}|| \le M_1 e^{-\alpha t} \quad (\forall t \ge 0, \forall Q_a \in \mathcal{N}(Q_{a1})) \tag{4.27}$$

The above arguments indicate that if we show that the set of all Q_a associated with $A_a(t) \in \overline{\mathcal{A}}_\delta$, which is denoted by $\mathcal{Q}_\delta$, is also bounded and closed, then the inequality (4.27) and the Heine-Borel finite-covering theorem [60, p. 36] will lead to the existence of $K_a > 0$ such that (4.25) holds. Here the fact that a closed and bounded set of finite-dimensional matrices is compact is used. The existence of such a $K_a > 0$ in turn gives (4.24) as claimed.

Therefore, to complete the proof, it suffices to show that $\mathcal{Q}_\delta$ is bounded and closed. To this end, we further denote the set of all $\Phi_a(h, 0)$ associated with $A_a(t) \in \overline{\mathcal{A}}_\delta$ by Φ_δ. This proof is nontrivial if we note that the mappings $T_1 : A_a(t) \mapsto \Phi_a(h,0) : \overline{\mathcal{A}}_\delta \to \Phi_\delta$ and $T_2 : \Phi_a(h, 0) \mapsto Q_a : \Phi_\delta \to \mathcal{Q}_\delta$ are nonlinear by Definition 4.3.1 of [55, p. 165].

It is clear by (4.11) that the mapping $T_1 : A_a(t) \mapsto \Phi_a(h,0) : \overline{\mathcal{A}}_\delta \to \Phi_\delta$ is continuous. It is also evident from Lemma 4.2 that for any fixed $\delta > 0$

$$\begin{aligned} ||\Phi_a(h,0)|| &\le \exp\left[h \sup_{t\in[0,h]} ||A_a(t)||\right] = \exp\left[h \sup_{t\in[0,h]} ||A(t) - A_\Delta(t)||\right] \\ &\le \exp\left[h(\sup_{t\in[0,h]} ||A(t)|| + \delta)\right] < \infty \end{aligned}$$

which clearly says that T_1 is bounded on $\overline{\mathcal{A}}_\delta$. Hence, by taking the continuity of T_1 into account, it follows that Φ_δ is bounded and closed.

On the other hand, it is well-known that $\log z$ is continuous with respect to z if the principal branch is considered. Obviously, the matrix $(zI - \Phi_a(h,0))^{-1}$ is continuous with regard to z and $\Phi_a(h, 0)$. Hence the maximum of $|\log z| \cdot ||(zI - \Phi_a(h, 0))^{-1}||$ over the closed sets Γ and Φ_δ is attainable and can be denoted by

$$\max_{z\in\Gamma}\max_{\Phi_a(h,0)\in\Phi_\delta}\{|\log z|\cdot||(zI-\Phi_a(h,0))^{-1}||\} =: M_\delta$$

Then by the complex-integral inequality [66, p. 47], the equation (4.14) tells us that

$$\begin{aligned} ||Q_a|| &= ||\frac{1}{2\pi hj}\oint_\Gamma(\log z)(zI-\Phi_a(h,0))^{-1}dz|| \\ &\le \frac{1}{2\pi h}\oint_\Gamma|\log z|\cdot||(zI-\Phi_a(h,0))^{-1}||\cdot|dz| \\ &\le \frac{1}{2\pi h}M_\delta L_\Gamma < \infty \end{aligned} \tag{4.28}$$

where L_Γ is the length of the integral contour Γ given by

$$L_\Gamma := \oint_\Gamma |dz| = 2\pi(R+1/R)+2(R-1/R)$$

by the definition that Γ is the boundary of the simply connected region $D_{R,\theta}$. The inequality (4.28) implies that the mapping $T_2 : \Phi_a(h,0)\mapsto Q_a : \Phi_\delta \to \mathcal{Q}_\delta$ is bounded on Φ_δ. This, together with the continuity of T_2 as claimed in (4.14), shows that $\mathcal{Q}_\delta$ is bounded and closed. This completes the necessity proof. **Q.E.D.**

Most methods that use approximate models to analyze stability of FDLCP systems have a common point, i.e., the transition matrix of the approximate model approaches that of the original FDLCP system. This is also the case in Theorem 4.1 and can have an eigenvalue approaching explanation by (4.15). In other words, $\Lambda_a \to \Lambda$ as $||A_\Delta(\cdot)|| \to 0$ in the elementwise sense, where Λ_a is the set of the eigenvalues of the approximate model G_a defined similarly to Λ (the definition of Λ is given in Section 2.4).

Theorem 4.1 shows that computing merely the eigenvalues of the corresponding monodromy matrix of an approximate model is not sufficient, theoretically speaking, to check whether or not an FDLCP system is stable however high the approximation accuracy may be since there exist modeling errors in the approximation treatments. Bearing this in mind, it can be inferred that any direct but approximate computation of the monodromy matrix $\Phi(t_0+h,t_0)$ is equally insufficient for testing stability of a general FDLCP system unless the modeling error bounds are taken into account. The importance of Theorem 4.1 lies in the fact that it can ensure stability provided that the approximate model is stable enough in the sense that the condition 3) is satisfied.

In spite of a large freedom in choosing $A_a(t)$, however, trial-and-error is needed in choosing $A_a(t)$ to show stability of an FDLCP system with Theorem 4.1. In such a case, it is sensible to consider the dense subset $L_a[0,h]$ from which $A_a(t)$ is taken, and a reasonable candidate for $L_a[0,h]$ is the set of all piecewise constant functions, which is denoted by $L_{\rm pc}[0,h]$. It is well-known [25], [71] that for any $A_a(t)\in L_{\rm pc}[0,h]$, the transition matrix $\Phi_a(t,0)$ can be computed explicitly, so that the condition 3) of Theorem 4.1 is easy to check. The necessity of Theorem 4.1 ensures that it is always possible to find an approximate model in $L_{\rm pc}[0,h]$ to satisfy the conditions by letting $||A_\Delta(\cdot)|| \to 0$ when the system is stable. However, due to the finite-word-length problem in the numerical monodromy matrix computations, to construct approximate models from $L_{\text{pc}}[0,h]$ may not work well. This is because if we let the size of the subintervals tend to zero to get better approximation, then the exponential function computed over each subinterval will tend to the identity matrix. This implies that the monodromy matrix computation of a piecewise-constant approximate matrix $A_a(t)$, which is theoretically quite simple, may actually become ill-conditioned.

Also it should be pointed out that the sufficiency part can be verified by using the variation-of-constants formula and Gronwall's Lemma after some modifications on the condition 3). Indeed, this is just what Theorem 4.2 claims as given later, which is also a conclusion about stability analysis of FDLCP systems via approximate modeling. However, the proof through the harmonic Lyapunov equation is a new direction, which explains the asymptotic stability of a class of general FDLCP systems from an operator-theoretic viewpoint instead of asymptotic analysis of differential equation solutions.

Observations about the condition 3) of Theorem 4.1 indicate that, generally speaking, it is not easy to check whether or not this condition is satisfied since the interval $[0,h]$ is involved. There are two problems. Firstly, K_a and α can only be estimated in a sufficient fashion; secondly, the second inequality of the condition 3) of Theorem 4.1 can only be checked 'discretely' on $[0,h]$ and thus approximately, if a numerical procedure is utilized. To get around the second problem, however, we can use upper bounds about the norms of $P_a(t,0)$ and $P_a^{-1}(t,0)$. Indeed, from Lemma 4.2, Theorem 4.1 can be reduced to some simpler form, which will be summarized in the following corollary. Before we state and prove this corollary, we stress that in the following discussions only the case that approximate models are constructed by piecewise constant approximation on $A(t)$, i.e., $A_a(t) \in L_a[0,h] = L_{\text{pc}}[0,h]$, is considered, though the idea applies to more general cases about $L_a[0,h]$. To our purpose, let $[t_i, t_{i+1}]$ be the i-th sub-interval on $[0,h]$ defined according to the piecewise constant approximation of $A(t)$. Noting that if $A(t) \in L_{\text{PCD}}[0,h]$, then by the definition of piecewise continuous functions, $A_a(t)$ will be well-defined provided that $A_a(t)$ is given by

$$A_a(t) = \begin{cases} A(t_i) & \forall t \in [t_i, t_{i+1}) \ \text{(if } A(\cdot) \text{ is continuous at } t_i) \\ \lim\limits_{t \to t_i + 0} A(t) & \forall t \in [t_i, t_{i+1}) \ \text{(if } A(\cdot) \text{ is discontinuous at } t_i) \end{cases} \tag{4.29}$$

where $\lim_{t \to t_i+0}$ denotes the right limit. Clearly, $\forall t \in [0,h], ||A_a(t)|| \leq \sup_{t\in[0,h]} ||A(t)||$ in such an approximation treatment, and it makes sense to define

$$\tilde{K} := \sup_{t\in[0,h]} ||A(t)|| \geq \sup_{t\in[0,h]} ||A_a(t)|| \tag{4.30}$$

Here it is evident that $\tilde{K}$ is independent of $A_a(t)$.

Corollary 4.1 *Suppose that $A(t) \in L_{\text{PCD}}[0,h]$ and $L_a[0,h] = L_{\text{pc}}[0,h]$. Let $A_a(t)$ be defined as in (4.29) and $\tilde{K} > 0$ be given in (4.30). Then, the system (2.1) is asymptotically stable if and only if there exists an approximate* FDLCP *model G_a as defined in (4.2) such that*

1). $A_a(t) \in L_a[0,h]$;

2). *the constant portion of the transition matrix of* $A_a(t)$ *is* Q_a *and all the eigenvalues of* Q_a *have negative real parts;*

3). *for* $A_\Delta(t) = A(t) - A_a(t)$, *there exist numbers* $K_a > 0$ *and* $\alpha > 0$ *satisfying*

$$||e^{Q_a t}|| \leq K_a e^{-\alpha t} \quad (\forall t \geq 0), \qquad \sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha}{K_a^3} e^{-(2\tilde{K}+||Q_a||)h}$$

Proof (Sufficiency) Noting in the approximate model (4.2) that $P_a(t,0) = \Phi_a(t,0)e^{-Q_a t}$, it holds for any $t \in [0,h]$ that

$$||P_a(t,0)|| \leq ||\Phi_a(t,0)|| \cdot ||e^{-Q_a t}|| \leq e^{\int_0^t ||A_a(t)||dt} e^{||Q_a||t} \leq e^{(\tilde{K}+||Q_a||)h} \tag{4.31}$$

where Lemma 4.2 and (4.30) have been used. Similarly, we have

$$||P_a^{-1}(t,0)|| \leq ||\Phi_a(0,t)|| \cdot ||e^{Q_a t}|| \leq e^{\int_0^t ||A_a(t)||dt} K_a e^{-\alpha t} \leq K_a e^{\tilde{K}h}$$

Then it follows readily from the above two inequalities that

$$\sup_{t\in[0,h]} ||P_a^{-1}(t,0)A_\Delta(t)P_a(t,0)|| \leq K_a e^{(2\tilde{K}+||Q_a||)h} \sup_{t\in[0,h]} ||A_\Delta(t)||$$

which says that if the condition 3) here is satisfied, so is the condition 3) of Theorem 4.1.

(Necessity) Now assume that the FDLCP system (2.1) is asymptotically stable. To show the assertion, we recall the claim in the necessity proof of Theorem 4.1 that under the stability assumption, for sufficiently small $\delta > 0$, all $A_a'(t) \in \mathcal{A}_\delta$ are stable, and there are uniform upper bounds of K_a' and $||Q_a'||$ for all $A_a'(t) \in \mathcal{A}_\delta$, which are denoted by K_δ $(\geq K_a')$ and $\tilde{K}_\delta$ $(\geq ||Q_a'||)$, respectively. Noting also that $L_a[0,h] = L_{\rm pc}[0,h]$ is dense in $L_{\rm PCD}[0,h]$, it follows that there is always an approximate model $A_a(t)$ in $L_a[0,h] \cap \mathcal{A}_\delta$ such that

$$\sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha}{K_\delta^3} e^{-(2\tilde{K}+\tilde{K}_\delta)h}$$

since the right-hand side depends only on $A(t)$ and δ. Recalling that $K_\delta \geq K_a$ and $\tilde{K}_\delta \geq ||Q_a||$ with K_a and Q_a associated with $A_a(t)$, this yields the desired result. **Q.E.D.**

Comparing Theorem 4.1 and Corollary 4.1, it is clear the stability condition 3) of Corollary 4.1 does not involve the periodic portion of the Floquet factorization of the transition matrix of the corresponding approximate model G_a. This means that Corollary 4.1 applies whenever the constant portion Q_a of the transition matrix of G_a can be computed explicitly. Thus, we have much more freedom in choosing approximate models and at the same time the computation loads are reduced. Because of these simplifications in the condition 3) of Corollary 4.1, it becomes possible to test stability of an FDLCP system in a sufficient fashion by only using upper bounds of the modeling error, $\sup_{t\in[0,h]} ||A_\Delta(t)||$, and this may give stability assertions that completely get rid of the difficulty in the supremum computation in the condition 3) of Theorem 4.1 or that of Corollary 4.1.

It might sound strange if we talk about the conservativeness of the stability conditions in this section, since these results here give *necessary and sufficient* conditions for asymptotic stability. However, since we will develop yet another necessary and sufficient condition for FDLCP systems and do comparisons among the necessary and sufficiency conditions derived, it would be convenient to talk about conservativeness. For instance, to see what we mean by conservativeness here, recall that the sufficiency part of Corollary 4.1 is guaranteed by the existence of some approximate model satisfying certain conditions. Thus, Corollary 4.1 can conclude stability of the original system only when such an approximate model $A_a(t)$ can indeed be found. However, if the original system is asymptotically stable, it would be quite often the case that almost all approximate models in $\mathcal{A}_\delta$ are actually asymptotically stable with δ much larger than the modeling error corresponding to a specific approximate model that is found. We mean this fact by the conservativeness of the condition 3) of Corollary 4.1 *when we regard it only as a sufficient condition.*

Now return to Corollary 4.1. It is clear from the proof of Corollary 4.1 that the estimation giving the number $\frac{\alpha}{K_a^3}\exp(-(2\tilde{K}+||Q_a||)h)$ is conservative in the sense that the approximate model should be of quite high accuracy to satisfy the the second inequality of the condition 3) of Corollary 4.1 (i.e., $A_a(t)$ should be taken from $\mathcal{A}_\delta$ with fairly small δ). This in turn may result in unacceptable computation time. Simple observations will reveal that the conservativeness is caused mainly by the exponential function $\exp(-(2\tilde{K}+||Q_a||)t)$ if $\tilde{K}$ and $||Q_a||$ are numerically too large and t is taken to be $t = h$.

Corollary 4.1 is derived from Theorem 4.1, whose sufficiency is established via the harmonic analysis. Therefore, it is meaningful to say that Corollary 4.1 also follows from the harmonic analysis. Now we take a short break from our main framework of the harmonic analysis to show that another necessary and sufficient stability condition for FDLCP systems can also be established through the well-known asymptotic analysis approach. Nevertheless, before stating the criterion, it should be pointed out that the stability condition 3) in Theorem 4.1 must be modified to accommodate this variation of technique in the proof.

Theorem 4.2 *Suppose in the* FDLCP *system (*2.1*) that* $A(t) \in L_{\mathrm{PCD}}[0,h]$ *and* $L_a[0,h] = L_{\mathrm{pc}}[0,h]$. *Let* $A_a(t)$ *be defined as in (*4.29*) and* $\tilde{K} > 0$ *be given in (*4.30*). Then, the system (*2.1*) is asymptotically stable if and only if there exists an approximate* FDLCP *model* G_a *as defined in (*4.2*) such that*

1). $A_a(t) \in L_a[0,h]$;

2). *the constant portion of the transition matrix of* $A_a(t)$ *is* Q_a *and all the eigenvalues of* Q_a *have negative real parts;*

3). *for* $A_\Delta(t) := A(t) - A_a(t)$, *there exist numbers* $K_a > 0$ *and* $\alpha > 0$ *satisfying*

$$||e^{Q_a t}|| \le K_a e^{-\alpha t} \quad (\forall t \ge 0), \qquad \sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha}{K_a} e^{-(\check{K}+||Q_a||)h}$$

Proof The sufficiency proof is given by slightly modifying the arguments of Theorem 1.11 of [38]. Rearrange the equation $\dot{\Phi}(t,0) = A(t)\Phi(t,0)$ as

$$\dot{\Phi}(t,0) = A_a(t)\Phi(t,0) + A_\Delta(t)\Phi(t,0)$$

Then the variation-of-constants formula yields

$$\Phi(t,0) = \Phi_a(t,0) + \int_0^t \Phi_a(t,\tau)A_\Delta(\tau)\Phi(\tau,0)d\tau$$

which leads to the inequality

$$||\Phi(t,0)|| \le ||P_a(t,0)|| \cdot ||e^{Q_a t}|| + \int_0^t ||P_a(t,\tau)||\, ||e^{Q_a(t-\tau)}||\, ||A_\Delta(\tau)||\, ||\Phi(\tau,0)||d\tau \quad (4.32)$$

Since $P_a(t,\tau)$ is h-periodic for both t and τ, the arguments around (4.31) can be applied to $P_a(t,\tau)$ similarly. To be more specific, it is actually true for all $t,\tau \in [0,\infty)$ that

$$||P_a(t,\tau)|| \le e^{(\check{K}+||Q_a||)h} =: \check{K}$$

Hence, by substituting the above inequality and the first inequality of the condition 3) of Theorem 4.2 into the inequality (4.32), it follows that

$$||\Phi(t,0)|| \le K_a\check{K}e^{-\alpha t} + K_a\check{K}\int_0^t e^{-\alpha(t-\tau)}||A_\Delta(\tau)||\, ||\Phi(\tau,0)||d\tau \quad (4.33)$$

Note that $||A_\Delta(t)||$ is not continuous in general due to some assumptions about $L_a[0,h]$ that are required in the approximation. Hence, Gronwall's Lemma can not be applied directly to (4.33). To surmount this difficulty, the inequality (4.33) is changed to the following inequality since $A_\Delta(t)$ is h-periodic.

$$||\Phi(t,0)e^{\alpha t}|| \le K_a\check{K} + K_a\check{K}\int_0^t \sup_{t\in[0,h]} ||A_\Delta(t)||\, ||\Phi(\tau,0)e^{\alpha\tau}||d\tau$$

This, together with Gronwall's Lemma, implies that

$$||\Phi(t,0)e^{\alpha t}|| \le K_a\check{K}\exp\Big[K_a\check{K}\int_0^t \sup_{t\in[0,h]} ||A_\Delta(t)||d\tau\Big]$$

Or, equivalently, we obtain

$$||\Phi(t,0)|| \le K_a\check{K}\exp\Big[-\alpha t + K_a\check{K}t \sup_{t\in[0,h]} ||A_\Delta(t)||\Big]$$

In particular, when $t = nh$ (n being a positive integer)

$$||\Phi(nh,0)|| \le K_a\check{K}\exp\Big[-\alpha nh + K_a\check{K}nh \sup_{t\in[0,h]} ||A_\Delta(t)||\Big]$$

Simple deductions tell us immediately that if

$$nh \sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha nh}{K_a\check{K}} + \frac{1}{K_a\check{K}} \ln \frac{q}{K_a\check{K}} \quad (q < 1) \tag{4.34}$$

then it holds that $||\Phi(nh,0)|| < q < 1$. This says that the eigenvalues of the matrix $\Phi(nh,0)$ are located in the open unit disc under the condition (4.34) .

To complete the sufficiency proof, we reduce the inequality (4.34) to

$$\sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha}{K_a\check{K}} + \frac{1}{nhK_a\check{K}} \ln \frac{q}{K_a\check{K}} \quad (q < 1)$$

It follows that if $\sup_{t\in[0,h]} ||A_\Delta(t)|| < \frac{\alpha}{K_a\check{K}} = \frac{\alpha}{K_a} e^{-(\check{K}+||Q_a||)h}$, then $||\Phi(nh,0)|| < q < 1$ will be assured for sufficiently large n. Finally, noting that $A(t)$ is also nh-periodic, one can assert that $\Phi(nh,0)$ is nothing but the monodromy matrix of such an nh-periodic matrix $A(t)$. Then, the asymptotic stability follows immediately from the Floquet theorem.

For the necessity proof, we notice that $\sup_{t\in[0,h]} ||A_\Delta(t)||$ is well-defined since $L_a[0,h]$ is dense in $L_{\mathrm{PCD}}[0,h]$ in the $L_\infty[0,h]$-norm sense. Hence, $\sup_{t\in[0,h]} ||A_\Delta(t)||$ can be made as small as desired by properly choosing $A_a(t)$. Then arguments similar to those in the necessity proof of Corollary 4.1 will lead to the desired assertion. **Q.E.D.**

Before closing this subsection, we indicate that as a by-product of the arguments in the necessity proof for Theorem 4.1, the continuity property of eigenvalues of an FDLCP system is actually derived. The result is summarized in the following proposition, which is proved by the arguments around (4.15). The continuity characteristic is extremely important in ensuring convergence of algorithms for H_2 and H_∞ norm computations established on approximate modeling, which are the major topics of Sections 4.3.

Proposition 4.1 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\mathrm{PCD}}[0,h]$. Then the eigenvalues of the operator $\underline{A} - \underline{E}(j0)$ are continuous with respect to the elements of $A(t)$.*

4.1.2 Numerical Examples

To illustrate the stability criteria developed in Subsection 4.1.1, we again consider to test stability of the lossy Mathieu differential equation of Example 3.1 by numerically implementing Theorem 4.1 and Theorem 4.2, respectively.

Example 4.1 *The (closed-loop) state matrix is given by*

$$A(t) = \begin{bmatrix} 0 & 1 \\ k(1-2\beta\cos\omega_h t) & -2\xi \end{bmatrix}, \quad \omega_h = 2 \quad (\text{i.e.}, h = \pi)$$

where k, β and ξ are parameters. Here we first consider to test the asymptotic stability of $A(t)$ by Theorem 4.1 numerically.

Here the piecewise constant approximation given in (4.29) is adopted. To construct an approximate model for each set of parameters, we divide the period h into $N_a = 120$ subintervals of the same length of h/N_a, during each of which $A(t)$ is approximated by a constant matrix as in (4.29). Then, we can easily compute the monodromy matrix $\Phi_a(h,0)$ by matrix exponentiations, as well as Q_a by taking a matrix logarithm. To this constant matrix Q_a, two numbers $K_a > 0$ and $\alpha > 0$ can be found by working on the Jordan canonical form of $Q_a := T_a J_a T_a^{-1}$ and the transition matrix $e^{Q_a t} = T_a e^{J_a t} T_a^{-1}$ such that the first inequality of the condition 3) of Theorem 4.1 is satisfied. We further take $N_c = 30$ points equitably distributed on each subinterval, and compute the periodic portion $P_a(t,0)$ on each of these $N_a N_c$ points, which is again carried out by matrix exponentiations since $A_a(t)$ is piecewise constant and Q_a is already known. Then the second inequality of the condition 3) is tested point-by-point on all the $N_a N_c$ points. Figure 4.1 is the computation results for different ξ's (i.e., $\xi = 0$, 0.1, 0.2, 0.3, 0.4, and 0.5), in which the blank areas correspond to the parameters for k and β where the approximate model is stable and satisfies the condition 3) of Theorem 4.1, the asterisks ($*$'s) indicate the parameters area corresponding to an unstable approximate model, while in the areas marked by crosses ($+$'s) the approximate model is stable but the condition 3) of Theorem 4.1 is not satisfied for the above K_a and α. □

Example 4.2 *For the same state matrix $A(t)$ of Example 4.1, we can also consider to test asymptotic stability of $A(t)$ by means of Theorem* 4.2.

The computation results are given in Figure 4.2. Here, the matrix Q_a is computed under the piecewise constant approximation on $A(t)$ as in the application of Theorem 4.1, but with $N_a = 25000$, and to check the second inequality in the condition 3) of Theorem 4.2, we took $N_c = 10$. The symbols in Figure 4.2 have the same meaning as what we have described for Figure 4.1. Note that in this second approach, the knowledge on the periodic portion of the transition matrices of the corresponding approximate models is not required. Also in this latter case, the approximation parameter N_a is fairly large, compared with the case of Theorem 4.1. This is caused by the unfortunate fact that in applying Theorem 4.2, the number $(\alpha/K_a)e^{-(\tilde{K}+\|Q_a\|)h}$ is relatively small, so that we need to construct approximate models with high accuracy and this in turn forces us to choose large N_a. □

Theoretically speaking, Corollary 4.1 can also be used in stability testing for the above FDLCP system. Unfortunately, the condition 3) of Corollary 4.1 is overly conservative in this example (compared with that of Theorem 4.2) so that we will be forced to construct approximate models with extremely high accuracy. Therefore, the computation loads for the given numerical examples are far beyond the capacity of the computers at hand, and thus the application of Corollary 4.1 is abandoned here.

One might suggest to repeat the stability testing of the given FDLCP system by applying Theorem 4.1 again but with $N_a = 25000$ and $N_c = 10$ so that we can compare the results of Figures 4.1 and 4.2 under the same approximation treatments. Unfortunately, however, this suggestion also results in unacceptable computation-time consumption if the computers

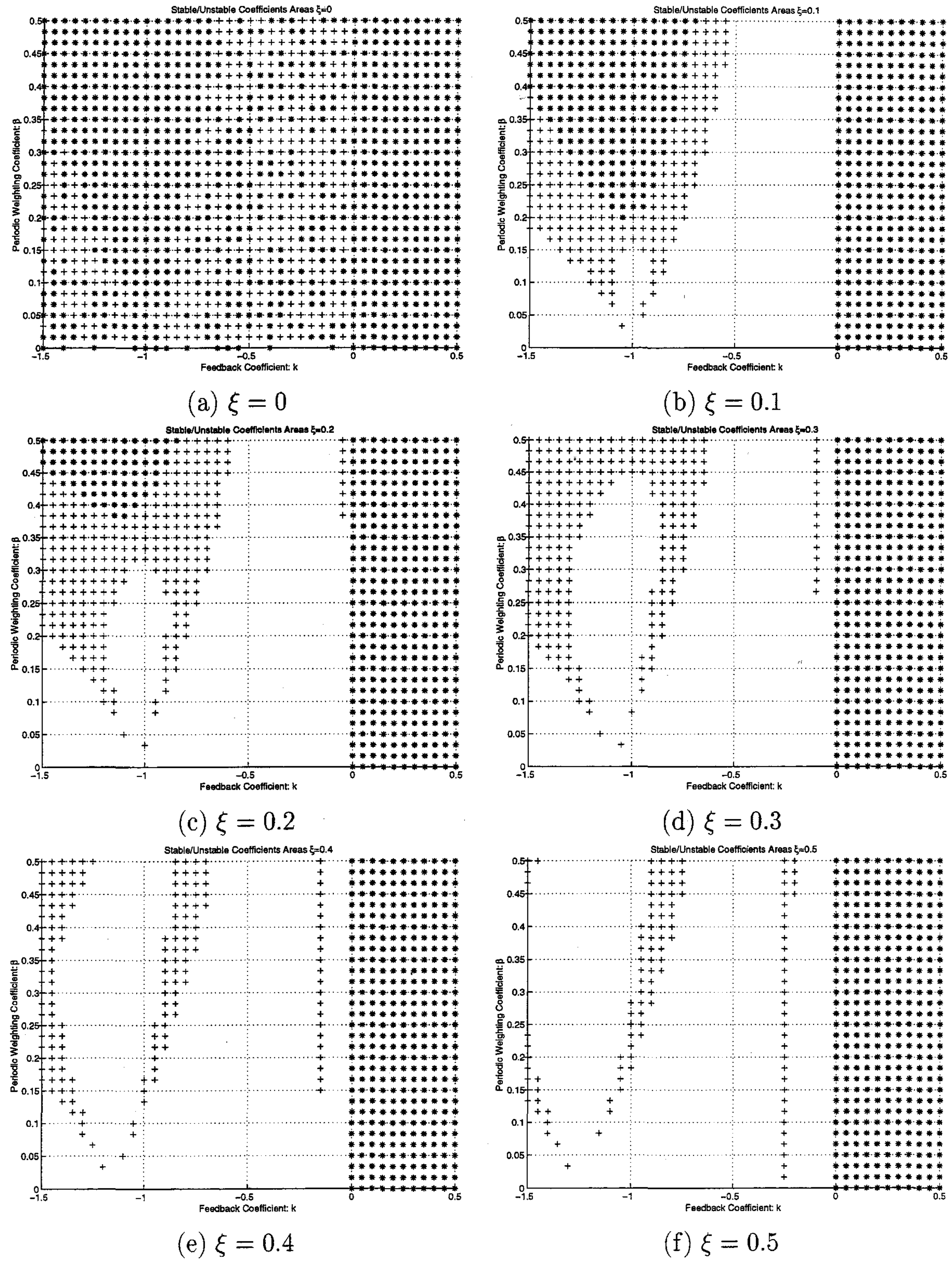
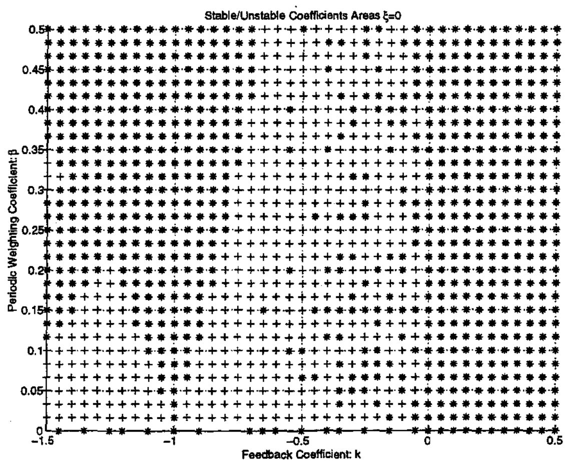
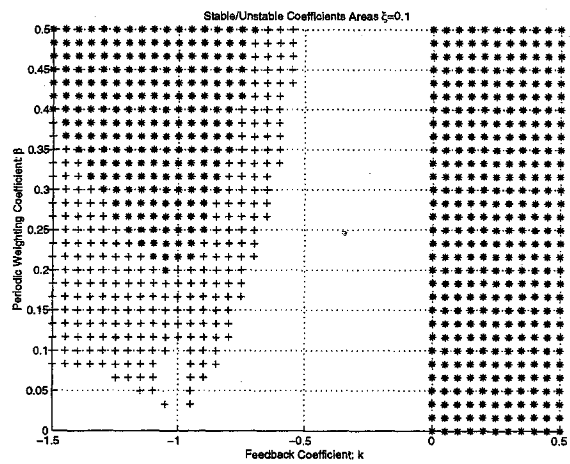


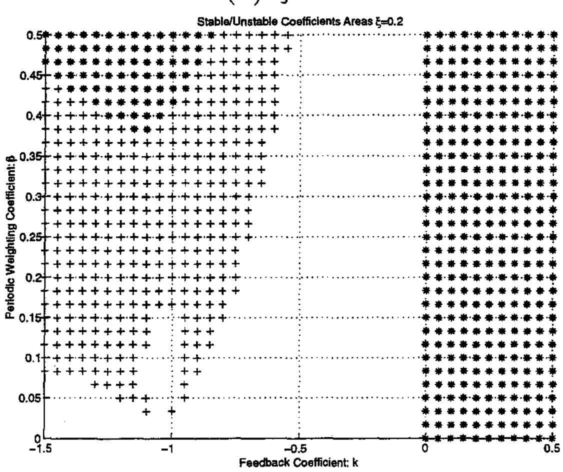
Figure 4.1: Stable coefficients areas of the lossy Mathieu equation via Theorem 4.1 (blank: G stable; asterisk: G_a unstable; cross: G_a stable but stability of G unknown)



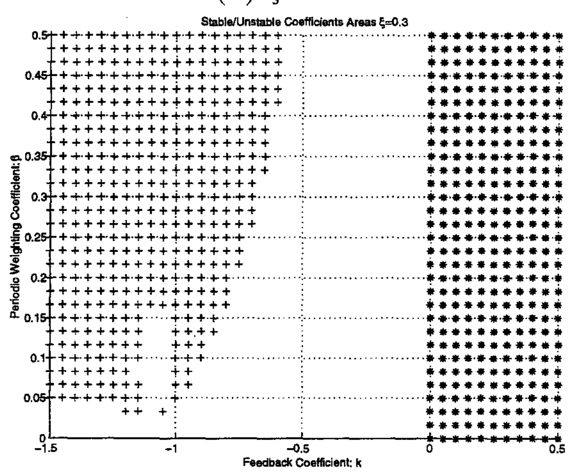
(a) $\xi = 0$



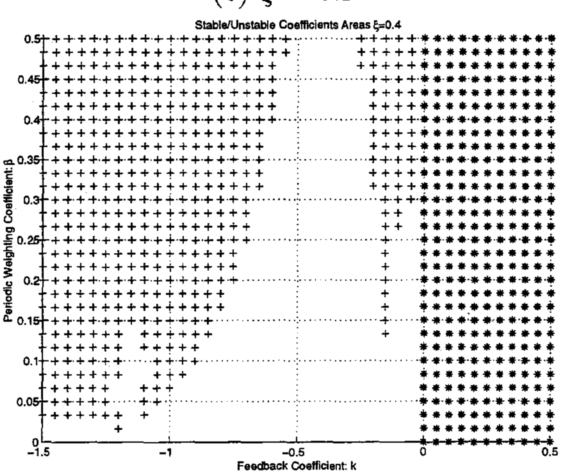
(b) $\xi = 0.1$



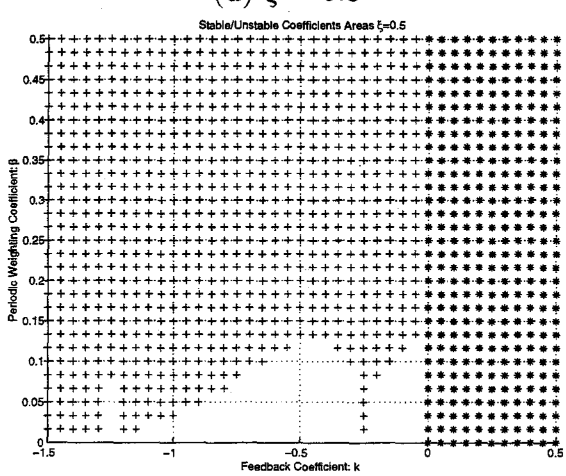
(c) $\xi = 0.2$



(d) $\xi = 0.3$



(e) $\xi = 0.4$



(f) $\xi = 0.5$

Figure 4.2: Stable coefficients areas of the lossy Mathieu equation via Theorem 4.2 (blank: G stable; asterisk: G_a unstable; cross: G_a stable but stability of G unknown)

at hand are used. In view of these problems, we are obliged to alert the readers to the fact that the results given in Figures 4.1 and 4.2 are not comparable directly, although they are given for the same FDLCP system. What could be suggested by the comparison of these two figures would be that Theorem 4.2 gives much more conservative sufficient conditions, given that N_a is taken much larger in Figure 4.2 than in Figure 4.1, while the areas marked with crosses (+'s) are much larger in Figure 4.2.

As side notes about the above numerical example, it should be pointed out that there are two problems remaining unclear from direct observations. The first problem is that if an approximate model can be found (not only exists), in a finite number of steps, to determine whether or not the original system is stable. It is easy to see that this testing process needs at least another instability criterion to figure out instability cases based on approximate modeling. Unfortunately, however, this kind of criteria still remain as open problems. The second problem is some numerical computation errors associated when the stability conditions therein are implemented/investigated through certain numerical analysis tools. Apparently, this kind of numerical errors are essentially different from the modeling errors that we have discussed in Theorem 4.1, Corollary 4.1 and Theorem 4.2 and should be treated as a separate problem. The problem of numerical computation errors will not be considered in this thesis.

4.2 H_2 and H_∞ Norm Computations via Truncations

As for the H_2 and H_∞ norm computations of FDLCP systems by the frequency response operator defined through the input/output steady-state analysis, the square truncation techniques were proposed in [2], [69], [70]. However, the convergence of such algorithms has not been verified, which is nontrivial especially when the operator involved is non-compact. There have been no discussions to clarify the relations between the original FDLCP frequency response operator and the square truncated one, either. The possible reasons may be attributed to the fact that this truncation neglects the 'symmetrical' structure of the frequency response operator, which makes such discussions hard. To surmount these difficulties, the skew and staircase truncations are proposed for the H_2 and H_∞ norm computations in this thesis with rigorous proofs for convergence. The implication of the work is twofold: on one hand, these truncations do give ways to compute the H_2 and H_∞ norms, by extending the trace formula and the Hamiltonian test to FDLCP systems in an LTI continuous-time fashion (as opposed to an LSI discrete-time fashion via the lifting approach); on the other hand, these truncation treatments bridge a gap between the theory and practice on the harmonic analysis of FDLCP systems. In theory, harmonic analysis leads to the notion of the (infinite-dimensional) harmonic Lyapunov equation for FDLCP systems, from whose solution an 'exact' trace formula was obtained for the H_2 norm of FDLCP systems in Subsection 3.4.3. It should be remarked that the limit of the 'asymptotic' trace formula provided in this section as the truncation parameter tends to infinity coincides with this 'exact' trace formula as shown in Section 4.3. This coincidence reveals the effectiveness of the skew truncation methods for dealing with the infinite-dimensionality of the frequency response operators of FDLCP systems. The skew truncation is first introduced in [88] as a basic tool for the frequency response gains computation in FDLCP systems, but in this section we further elaborate on it to derive a Lyapunov-equation-based method for the H_2 norm computation. The staircase truncation is introduced to compute the H_∞ norm as well as to extend the Hamiltonian test into FDLCP systems, and is also useful for the frequency response gain computation of FDLCP systems, as an alternative to the skew-rectangular truncation proposed in [88].

4.2.1 Skew Truncation on the Frequency Response Operator: Asymptotic Trace Formula

To overcome the infinite-dimensionality of the frequency response operator, the skew truncation is suggested in this subsection, from which an asymptotic trace formula with desired convergence is developed via the solution of a finite-dimensional Lyapunov equation. It is shown that the truncation errors can be assessed in most practical FDLCP systems.

Now let us describe the skew truncation on the frequency response operator $\underline{G}(j\varphi)$ of the system (2.1) when it is strictly proper. Let us take $N \geq 1$ and approximate $\underline{G}(j\varphi)$ by

$$\underline{G}_{[N]}(j\varphi) = \underline{\hat{C}}_{[N]}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\hat{B}}_{[N]} \tag{4.35}$$

where $\underline{\hat{B}}_{[N]}$ is formed by skew truncating $\underline{\hat{B}}$ as follows:

$$\underline{\hat{B}}_{[N]} := \begin{bmatrix} \ddots & \cdots & \ddots & \cdots & \ddots & & & & 0 \\ & \hat{B}_N & \cdots & \hat{B}_0 & \cdots & \hat{B}_{-N} & & & \\ & & \hat{B}_N & \cdots & \hat{B}_0 & \cdots & \hat{B}_{-N} & & \\ & & & \hat{B}_N & \cdots & \hat{B}_0 & \cdots & \hat{B}_{-N} & \\ 0 & & & & \ddots & \cdots & \ddots & \cdots & \ddots \end{bmatrix}$$

It is clear that $\underline{\hat{B}}_{[N]} := \mathcal{T}\{\hat{B}_N(t)\}$, where $\hat{B}_N(t) := \sum_{n=-N}^{N} \hat{B}_n e^{jn\omega_h t}$ with $\{\hat{B}_m\}$ being the Fourier coefficients sequence of $\hat{B}(t)$. The operator $\underline{\hat{C}}_{[N]}$ is constructed similarly in terms of $\underline{\hat{C}}$, that is, $\underline{\hat{C}}_{[N]} := \mathcal{T}\{\hat{C}_N(t)\}$ with $\hat{C}_N(t) := \sum_{n=-N}^{N} \hat{C}_n e^{jn\omega_h t}$. The expression of (4.35) is also called the skew truncation of $\underline{G}(j\varphi)$. This truncation of $\underline{G}(j\varphi)$ can provide mathematical convenience in trace computation. Hence it will be used in the H_2 norm computation.

The reason to introduce the skew truncation is that otherwise there is no way to do the trace computation in the H_2 norm (Definition 3.3) due to the infinite-dimensionality of the operator involved. Now we consider the quantity

$$||\mathcal{G}_N||_2^2 = \frac{1}{2\pi}\int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \text{trace}\Big(\underline{G}_{[N]}(j\varphi)^* \underline{G}_{[N]}(j\varphi)\Big) d\varphi \tag{4.36}$$

which can be seen as the H_2 norm of the strictly proper FDLCP system $(Q, \hat{B}_N(t), \hat{C}_N(t))$ given below that is associated with the above skew truncation of size N.

$$\begin{cases} \dot{\tilde{x}} = Q\tilde{x} + \hat{B}_N(t)u \\ y = \hat{C}_N(t)\tilde{x} \end{cases} \tag{4.37}$$

Obviously, $\underline{G}_{[N]}(j\varphi)$ is the frequency response operator of the FDLCP system (4.37). It is easy to see that $\underline{G}_{[N]}(j\varphi)$ has a skew-strip structure, and it will be shown that this structure of $\underline{G}_{[N]}(j\varphi)$ makes it possible to do the trace computation as required in (4.36), though $\underline{G}_{[N]}(j\varphi)$ is still infinite-dimensional. However, before we attack the actual computation problem, we have to face such a convergence problem: does $||\mathcal{G}_N||_2$ tend to $||\mathcal{G}||_2$ as $N \to \infty$?

Now we are in a position to claim the convergence for the skew truncation in the asymptotic H_2 norm computation through (4.36).

Lemma 4.3 *Suppose in the system of (2.1) that the system is asymptotically stable and strictly proper and that $A(t) \in L_{\mathrm{PCD}}[0,h]$, $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$. Then for any $\epsilon > 0$, there exists an integer N_0 such that $\Big| ||\mathcal{G}||_2 - ||\mathcal{G}_N||_2 \Big| < \epsilon \quad (\forall N \geq N_0)$.*

Proof Since $||\mathcal{G}||_2$ and $||\mathcal{G}_N||_2$ are defined through integrations on a finite interval, it suffices to show that for any $\epsilon > 0$, there exists an integer N_0 such that for any $\varphi \in \mathcal{I}_0$ and $\forall N \geq N_0$

$$\Big|\mathrm{trace}\Big(\underline{G}(j\varphi)^*\underline{G}(j\varphi)\Big) - \mathrm{trace}\Big(\underline{G}_{[N]}(j\varphi)^*\underline{G}_{[N]}(j\varphi)\Big)\Big| < \epsilon \tag{4.38}$$

By (3.52), for the orthonormal basis $\{\underline{u}_n\}_{n=-\infty}^{\infty}$ of l_2 defined in Lemma 3.2, we have

$$\begin{aligned}
&\Big|\mathrm{trace}\Big(\underline{G}(j\varphi)^*\underline{G}(j\varphi)\Big) - \mathrm{trace}\Big(\underline{G}_{[N]}(j\varphi)^*\underline{G}_{[N]}(j\varphi)\Big)\Big| \\
= &\Big| \sum_{n=-\infty}^{\infty} \Big(||\underline{G}(j\varphi)\underline{u}_n||_{l_2} + ||\underline{G}_{[N]}(j\varphi)\underline{u}_n||_{l_2}\Big)\Big(||\underline{G}(j\varphi)x_n||_{l_2} - ||\underline{G}_{[N]}(j\varphi)\underline{u}_n||_{l_2}\Big)\Big| \\
\leq &\sum_{n=-\infty}^{\infty} \Big(||\underline{G}(j\varphi)\underline{u}_n||_{l_2} + ||\underline{G}_{[N]}(j\varphi)\underline{u}_n||_{l_2}\Big)\Big(||(\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi))\underline{u}_n||_{l_2}\Big) \\
\leq &\Big[\sum_{n=-\infty}^{\infty} \Big(||\underline{G}(j\varphi)\underline{u}_n||_{l_2} + ||\underline{G}_{[N]}(j\varphi)\underline{u}_n||_{l_2}\Big)^2\Big]^{\frac{1}{2}}\Big[\sum_{n=-\infty}^{\infty} ||\Big(\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi)\Big)\underline{u}_n||_{l_2}^2\Big]^{\frac{1}{2}} \\
\leq &\sqrt{2}\Big[\mathrm{trace}\Big(\underline{G}(j\varphi)^*\underline{G}(j\varphi)\Big) + \mathrm{trace}\Big(\underline{G}_{[N]}(j\varphi)^*\underline{G}_{[N]}(j\varphi)\Big)\Big]^{\frac{1}{2}} \\
&\cdot\Big[\sum_{n=-\infty}^{\infty} ||(\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi))\underline{u}_n||_{l_2}^2\Big]^{\frac{1}{2}}
\end{aligned} \tag{4.39}$$

In (4.39), we used (3.52) and the fact that $\underline{G}(j\varphi)$ and $\underline{G}_{[N]}(j\varphi)$ are uniformly bounded on l_2 over $\varphi \in \mathcal{I}_0$. Following a similar procedure as in (3.53) yields

$$\begin{aligned}
\mathrm{trace}\Big(\underline{G}_{[N]}(j\varphi)^*\underline{G}_{[N]}(j\varphi)\Big) &< 5K^2||\underline{\hat{C}}_{[N]}||_{l_2/l_2}^2 \sum_{|m|\leq N} ||\hat{B}_m||^2 \\
&\leq 5K^2||\underline{\hat{C}}_{[N]}||_{l_2/l_2}^2 \sum_{m=-\infty}^{\infty} ||\hat{B}_m||^2 = K_\alpha < \infty
\end{aligned} \tag{4.40}$$

for some $K_\alpha > 0$, which can be taken independent of N and φ by the fact that $||\hat{\underline{C}}_{[N]}||_{l_2/l_2} \leq \sum_{|m|\leq N} ||\hat{C}_m|| \leq \sum_{m=-\infty}^{+\infty} ||\hat{C}_m|| < \infty$ since $\hat{C}(t) \in L_{\mathrm{CAC}}[0,h]$ (by the assumption that $C(t) \in L_{\mathrm{CAC}}[0,h]$ and Proposition 2.1). This, in particular, implies that (4.36) is well-defined. Also by (4.39), to show (4.38), it is enough to show that for any $\epsilon' > 0$, there exists an integer $N_0 > 0$ such that for any $\varphi \in \mathcal{I}_0$ and $N \geq N_0$

$$\sum_{n=-\infty}^{\infty} ||(\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi))\underline{u}_n||_{l_2}^2 < \epsilon' \tag{4.41}$$

Here, we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} ||(\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi))\underline{u}_n||_{l_2}^2 \\
= &\sum_{n=-\infty}^{\infty} ||\Big([\hat{\underline{C}} - \hat{\underline{C}}_{[N]}](\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}} + \hat{\underline{C}}_{[N]}(\underline{E}(j\varphi) - \underline{Q})^{-1}[\hat{\underline{B}} - \hat{\underline{B}}_{[N]}]\Big)\underline{u}_n||_{l_2}^2 \\
\leq\ &2\sum_{n=-\infty}^{\infty} \Big(||[\hat{\underline{C}} - \hat{\underline{C}}_{[N]}](\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_n||_{l_2}^2 \\
&+||\hat{\underline{C}}_{[N]}(\underline{E}(j\varphi) - \underline{Q})^{-1}[\hat{\underline{B}} - \hat{\underline{B}}_{[N]}]\underline{u}_n||_{l_2}^2\Big)
\end{aligned} \tag{4.42}$$

Also, by a similar derivation to the proof of Lemma 3.2, it can be shown that

$$\sum_{n=-\infty}^{\infty} ||[\hat{\underline{C}} - \hat{\underline{C}}_{[N]}](\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}\,\underline{u}_n||_{l_2}^2 \leq 5K^2||\hat{\underline{B}}||_{l_2/l_2}^2 \sum_{|m|>N} ||\hat{C}_m||^2 \tag{4.43}$$

$$\sum_{n=-\infty}^{\infty} ||\hat{\underline{C}}_{[N]}(\underline{E}(j\varphi) - \underline{Q})^{-1}[\hat{\underline{B}} - \hat{\underline{B}}_{[N]}]\underline{u}_n||_{l_2}^2 \leq 5K^2||\hat{\underline{C}}_{[N]}||_{l_2/l_2}^2 \sum_{|m|>N} ||\hat{B}_m||^2 \tag{4.44}$$

Hence, applying Proposition 2.2 to (4.43) and the uniform boundedness of $||\hat{\underline{C}}_{[N]}||_{l_2/l_2}$ over N to (4.44) (as in (4.40)), we obtain (4.41) from (4.42) and hence (4.38) holds. **Q.E.D.**

Remark 4.1 *It should be noted that the assumption of $B(t)$, $C(t) \in L_{\mathrm{CAC}}[0,h]$ guarantees that $\underline{G}_{[N]}(j\varphi)$ is uniformly bound on l_2 over $\varphi \in \mathcal{I}_0$ and for all integers N; otherwise, the trace computation as in (3.52) may not be applicable in (4.39), though (4.40), (4.43) and (4.44) only involve the uniform convergence of the Fourier series expansion of $\hat{C}(t)$.*

Lemma 4.3 implies that the H_2 norm $||\mathcal{G}||_2$ can be approximated to any degree of accuracy by $||\mathcal{G}_N||_2$. Based on this observation, we define the following LTI continuous-time system with complex state-space matrices given by

$$G_N(s) := \left[\begin{array}{c|c} \mathcal{Q}_N & \mathcal{B}_N \\ \hline \mathcal{C}_N & 0 \end{array}\right] \tag{4.45}$$

where

$$\mathcal{B}_N := [\hat{B}_{-N}^T, \cdots, \hat{B}_{-1}^T, \hat{B}_0^T, \hat{B}_1^T, \cdots, \hat{B}_N^T]^T$$

$$\mathcal{C}_N := \begin{bmatrix} \hat{C}_{-N} & & & & 0 \\ \vdots & \ddots & & & \\ \hat{C}_0 & & \hat{C}_{-N} & & \\ \vdots & \ddots & \vdots & \ddots & \\ \hat{C}_N & \cdots & \hat{C}_0 & \cdots & \hat{C}_{-N} \\ & \ddots & \vdots & \ddots & \vdots \\ & & \hat{C}_N & & \hat{C}_0 \\ & & & \ddots & \vdots \\ 0 & & & & \hat{C}_N \end{bmatrix}$$

$$\begin{aligned} \mathcal{Q}_N &:= \text{diag}[Q + jN\omega_h I, \cdots, Q + j\omega_h I, Q, Q - j\omega_h I, \cdots, Q - jN\omega_h I] \\ &=: Q_N - E_{N0}(j0) \end{aligned} \tag{4.46}$$

with $Q_N := \text{diag}\underbrace{[Q, Q, \cdot, Q]}_{2N+1}$ and $E_{N0}(j\varphi) := \text{diag}[j\varphi_N I, \cdots, j\varphi_1 I, j\varphi_0 I, j\varphi_{-1} I, \cdots, j\varphi_{-N} I]$. The matrix $E_{N0}(j\varphi)$ will also be used in the staircase truncation in the next subsection.

Since in (4.35), $(\underline{E}(j\varphi) - \underline{Q})^{-1}$ is block-diagonal and $\underline{\hat{B}}_{[N]}$ and $\underline{\hat{C}}_{[N]}$ have $2N$ sub-diagonal strips along the main diagonal, the m-th entry of $\underline{G}_{[N]}(j\varphi)^* \underline{G}_{[N]}(j\varphi)$ on the main diagonal is

$$\mathcal{B}_N^*(E_{N0}(j\varphi) + jm\omega_h I_N - Q_N)^{-*} \mathcal{C}_N^* \mathcal{C}_N (E_{N0}(j\varphi) + jm\omega_h I_N - Q_N)^{-1} \mathcal{B}_N$$

where I_N is defined similarly to Q_N but in term of the identity matrix I. By the definition, it is clear that $E_{N0}(j\varphi) = E_{N0}(j0) + j\varphi I_N$. Hence

$$\begin{aligned} ||\mathcal{G}_N||_2^2 &= \frac{1}{2\pi} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \sum_{m=-\infty}^{+\infty} \text{trace}\Big(\mathcal{B}_N^*(j(\varphi + m\omega_h)I_N - \mathcal{Q}_N)^{-*} \mathcal{C}_N^* \\ &\qquad \cdot \mathcal{C}_N (j(\varphi + m\omega_h)I_N - \mathcal{Q}_N)^{-1} \mathcal{B}_N\Big) d\varphi \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \text{trace}\Big(G_N(j\varphi_m)^* G_N(j\varphi_m)\Big) d\varphi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\Big(G_N(j\omega)^* G_N(j\omega)\Big) d\omega = ||G_N||_2^2 \end{aligned} \tag{4.47}$$

where $||G_N||_2$ is the H_2 norm of the LTI continuous-time system $G_N(s)$ of (4.45). In the derivation of (4.47), the infinite summation and the integral are interchanged. The validity of this order interchange is guaranteed by the Levy theorem [55] and the fact that $\sum_{m=-k}^{k} \text{trace}(G_N(j\varphi_m)^* G_N(j\varphi_m))$ is absolutely convergent over k (this fact is shown in the proof of Proposition 3.1). Finally we obtain the following theorem.

Theorem 4.3 *Suppose that the system (2.1) is asymptotically stable and strictly proper and that $A(t) \in L_{\text{PCD}}[0, h]$, $B(t), C(t) \in L_{\text{CAC}}[0, h]$. Then*

$$\lim_{N\to\infty} \mathrm{trace}\big(\mathcal{B}_N^* \mathcal{V}_N \mathcal{B}_N\big) = \lim_{N\to\infty} \mathrm{trace}\big(\mathcal{C}_N \mathcal{W}_N \mathcal{C}_N^*\big) = ||\mathcal{G}||_2^2$$

Here $\mathcal{V}_N$ and $\mathcal{W}_N$ are respectively the observability and controllability Gramians that can be obtained by solving the finite-dimensional Lyapunov equations

$$\mathcal{Q}_N^* \mathcal{V}_N + \mathcal{V}_N \mathcal{Q}_N + \mathcal{C}_N^* \mathcal{C}_N = 0, \quad \mathcal{Q}_N \mathcal{W}_N + \mathcal{W}_N \mathcal{Q}_N^* + \mathcal{B}_N \mathcal{B}_N^* = 0$$

Proof By the stability assumption, all the eigenvalues of Q have negative real parts, and so do all the eigenvalues of $\mathcal{Q}_N$ by the definition. Therefore the system of (4.45) is stable. Then, following the proof of Lemma 4.4 in [91], it still follows in the situation of complex state-space matrices that

$$||G_N||_2^2 = \mathrm{trace}\big(\mathcal{B}_N^* \mathcal{V}_N \mathcal{B}_N\big) = \mathrm{trace}\big(\mathcal{C}_N \mathcal{W}_N \mathcal{C}_N^*\big)$$

This, together with (4.47) and Lemma 4.3, completes the proof. **Q.E.D.**

4.2.2 Staircase Truncation on the Frequency Response Operator: Asymptotic Hamiltonian Test

To compute the H_∞ norm of an FDLCP system, only the skew truncation is not enough to convert the problem to the maximum singular value computation of a finite-dimensional matrix. Now we introduce what we call the staircase truncation to give a solution for the H_∞ norm computation problem. The staircase truncation can be viewed as a modified skew truncation. For the purposes of this subsection, it is assumed that the feedthrough matrix $D(t)$ is constant. The staircase truncation on $\underline{G}(j\varphi)$ is defined as

$$\underline{G}_{[N,M]}(j\varphi) = \underline{\hat{C}}_{[N,M]}(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\hat{B}}_{[N,M]} + \underline{D} \tag{4.48}$$

Here, letting $\{\hat{B}_m\}$ and $\{\hat{C}_m\}$ be the Fourier coefficients sequences of $\hat{B}(t)$ and $\hat{C}(t)$, respectively, the infinite-dimensional matrices $\underline{\hat{B}}_{[N,M]}$ and $\underline{\hat{C}}_{[N,M]}$ are given by

$$\underline{\hat{B}}_{[N,M]} := \mathrm{diag}[\cdots, \hat{B}_{NM}, \hat{B}_{NM}, \hat{B}_{NM}, \cdots], \quad \underline{\hat{C}}_{[N,M]} := \mathrm{diag}[\cdots, \hat{C}_{NM}, \hat{C}_{NM}, \hat{C}_{NM}, \cdots]$$

with the finite-dimensional matrices

$$\hat{B}_{NM} = \underbrace{\begin{bmatrix} \hat{B}_0 & \cdots & \hat{B}_{-N} & & 0 \\ \vdots & \ddots & & \ddots & \\ \hat{B}_N & & \ddots & & \hat{B}_{-N} \\ & \ddots & & \ddots & \vdots \\ 0 & & \hat{B}_N & \cdots & \hat{B}_0 \end{bmatrix}}_{(2M+1)-\text{blocks}}, \quad \hat{C}_{NM} = \underbrace{\begin{bmatrix} \hat{C}_0 & \cdots & \hat{C}_{-N} & & 0 \\ \vdots & \ddots & & \ddots & \\ \hat{C}_N & & \ddots & & \hat{C}_{-N} \\ & \ddots & & \ddots & \vdots \\ 0 & & \hat{C}_N & \cdots & \hat{C}_0 \end{bmatrix}}_{(2M+1)-\text{blocks}} \tag{4.49}$$

where we assume $M \geq N+1$. No truncation is done on $\underline{D}$ since it already has the skew truncated form. Furthermore, to conform to the block-diagonal form of $\hat{\underline{B}}_{[N,M]}$ and $\hat{\underline{C}}_{[N,M]}$, the infinite-dimensional but block-diagonal operators $\underline{E}(j\varphi)$, $\underline{Q}$ and $\underline{D}$ are also partitioned into diagonal blocks accordingly so that (4.48) can be rewritten as

$$\underline{G}_{[N,M]}(j\varphi) = \hat{\underline{C}}_{[N,M]}(\underline{E}_M(j\varphi) - \underline{Q}_M)^{-1}\hat{\underline{B}}_{[N,M]} + \underline{D}_M \tag{4.50}$$

where

$$\begin{aligned} \underline{Q}_M &:= \operatorname{diag}[\cdots, Q_M, Q_M, Q_M, \cdots](= \underline{Q}) \\ \underline{D}_M &:= \operatorname{diag}[\cdots, D_M, D_M, D_M, \cdots](= \underline{D}) \\ \underline{E}_M(j\varphi) &:= \operatorname{diag}[\cdots, E_{M,-1}(j\varphi), E_{M0}(j\varphi), E_{M1}(j\varphi), \cdots](= \underline{E}(j\varphi)) \end{aligned}$$

with $Q_M = \operatorname{diag}\underbrace{[Q, Q, \cdots, Q]}_{(2M+1)}$ and

$$\begin{aligned} E_{Mm}(j\varphi) = \operatorname{diag}\Big[& j\big(\varphi + (m(2M+1) - M)\omega_h\big)I, \cdots, j\big(\varphi + m(2M+1)\omega_h\big)I, \\ & \cdots, j\big(\varphi + (m(2M+1) + M)\omega_h\big)I\Big] \end{aligned}$$

for $m \in \mathcal{Z}$. The block-diagonal matrix D_M is defined similar to Q_M but in terms of D. Our task in this subsection is to establish a computation formula for the H_∞ norm through the staircase truncation treatment. To state the final result, we need to establish some convergence lemmas associated with the staircase truncation on the frequency response operator $\underline{G}(j\varphi)$. The following two lemmas ensure such convergence.

Lemma 4.4 *Assume that the system (2.1) is asymptotically stable and $A(t) \in L_{\mathrm{PCD}}[0,h]$, $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$ and $D(t)$ is a constant matrix. Then for any $\epsilon > 0$, there exists an integer $N_0 > 0$ such that $||\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi)||_{l_2/l_2} < \epsilon \quad (\forall N \geq N_0, \forall \varphi \in \mathcal{I}_0)$.*

Proof By the assumptions, together with Proposition 2.1 and Lemma 2.6, $\hat{B}(t), \hat{C}(t) \in L_{\mathrm{CAC}}[0,h]$. By the structure of $\hat{\underline{B}} - \hat{\underline{B}}_{[N]}$, clearly $||\hat{\underline{B}} - \hat{\underline{B}}_{[N]}||_{l_2/l_2} \leq \sum_{|m|>N} ||\hat{B}_m||$. Similarly for $||\hat{\underline{C}} - \hat{\underline{C}}_{[N]}||_{l_2/l_2}$. Therefore, for any $\mu > 0$, there exists an integer $N_0 > 0$ such that

$$||\hat{\underline{B}} - \hat{\underline{B}}_{[N]}||_{l_2/l_2} < \mu, \quad ||\hat{\underline{C}} - \hat{\underline{C}}_{[N]}||_{l_2/l_2} < \mu \quad (\forall N \geq N_0) \tag{4.51}$$

Furthermore, there exist $K_C > 0$ independent of N such that $||\hat{\underline{C}}_{[N]}||_{l_2/l_2} \leq K_C$, since it holds that $||\hat{\underline{C}}_{[N]}||_{l_2/l_2} \leq \sum_{|n|\leq N} ||\hat{C}_n|| \leq \sum_{n=-\infty}^{+\infty} ||\hat{C}_n||$. On the other hand, we have

$$\begin{aligned} ||\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi)||_{l_2/l_2} \leq\ & ||\hat{\underline{C}} - \hat{\underline{C}}_{[N]}||_{l_2/l_2} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}||_{l_2/l_2} \\ +\ & ||\hat{\underline{C}}_{[N]}||_{l_2/l_2} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2} ||\hat{\underline{B}} - \hat{\underline{B}}_{[N]}||_{l_2/l_2} \end{aligned} \tag{4.52}$$

Hence, by taking μ to be

$$\mu = \frac{\epsilon}{2}\Big[\max\big\{\max_{\varphi\in\mathcal{I}_0} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}||_{l_2/l_2}, K_C \max_{\varphi\in\mathcal{I}_0} ||(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2}\big\}\Big]^{-1}$$

the result follows by (4.51) and (4.52). **Q.E.D.**

By Lemma 4.4, to show that $\underline{G}_{[N,M]}(j\varphi)$ converges to $\underline{G}(j\varphi)$ uniformly over $\varphi \in \mathcal{I}_0$ in the l_2-induced norm sense as $N, M \to \infty$, it suffices to show that $\underline{G}_{[N,M]}(j\varphi)$ converges to $\underline{G}_{[N]}(j\varphi)$ as $M \to \infty$ in the same sense. This is established by the following lemma.

Lemma 4.5 *Assume that the system (2.1) is asymptotically stable and $A(t) \in L_{\rm PCD}[0,h]$, $B(t), C(t) \in L_{\rm CAC}[0,h]$ and $D(t)$ is a constant matrix. Then, for each fixed N and for any $\epsilon > 0$, there exists an integer $M_0(N,\epsilon) > 0$ such that $||\underline{G}_{[N]}(j\varphi) - \underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} < \epsilon \quad (\forall M \geq M_0(N,\epsilon), \forall \varphi \in \mathcal{I}_0)$.*

Proof To prove this lemma, we focus on the inequality

$$\begin{aligned} ||\underline{G}_{[N]}(j\varphi) - \underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} &\leq ||\underline{\check{C}}_{[N,M]}(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2}||\underline{\hat{B}}_{[N,M]}||_{l_2/l_2} \\ &+ ||\underline{\hat{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\check{B}}_{[N,M]}||_{l_2/l_2} \end{aligned} \quad (4.53)$$

where $\underline{\check{B}}_{[N,M]} := \underline{\hat{B}}_{[N]} - \underline{\hat{B}}_{[N,M]}$. More explicitly, it is given by the infinite-dimensional matrix

$$\underline{\check{B}}_{[N,M]} := \begin{bmatrix} \ddots & \ddots & \ddots & & & & \mathbf{0} \\ & \check{B}_{NMl} & 0 & \check{B}_{NMu} & & & \\ & & \check{B}_{NMl} & 0 & \check{B}_{NMu} & & \\ & & & \check{B}_{NMl} & 0 & \check{B}_{NMu} & \\ \mathbf{0} & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

with the entry matrices given by

$$\check{B}_{NMl} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \hat{B}_N & \cdots & \hat{B}_1 \\ & \ddots & & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \hat{B}_N \\ & & & \ddots & & 0 \\ & & & & \ddots & \vdots \\ \mathbf{0} & & & & & 0 \end{bmatrix}}_{(2M+1)}, \quad \check{B}_{NMu} = \underbrace{\begin{bmatrix} 0 & & & & & \mathbf{0} \\ \vdots & \ddots & & & & \\ 0 & & \ddots & & & \\ \hat{B}_{-N} & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & & \ddots & \\ \hat{B}_{-1} & \cdots & \hat{B}_{-N} & 0 & \cdots & 0 \end{bmatrix}}_{(2M+1)}$$

The matrix $\underline{\check{C}}_{[N,M]}$ is defined similarly but in terms of $\{\hat{C}_m\}_{m=-N}^{N}$.

Here, noticing the block-diagonal structure of $\underline{\hat{B}}_{[N,M]}$, it is not hard to show that

$$||\underline{\hat{B}}_{[N,M]}||_{l_2/l_2} = ||\hat{B}_{NM}|| \quad (4.54)$$

Now, since $\hat{B}_{NM}$ is contained in $\underline{\hat{B}}_{[N]}$ as a sub-matrix, it readily follows that $||\underline{\hat{B}}_{[N]}||_{l_2/l_2} \geq ||\hat{B}_{NM}||$. Obviously, there is an upper bound for $||\underline{\hat{B}}_{[N]}||_{l_2/l_2}$ independent of N, which follows readily from $\hat{B}(t) \in L_{\rm CAC}[0,h]$ and the arguments in the proof of Lemma 4.4 about the fact that there is an upper bound for $||\underline{\hat{C}}_{[N]}||_{l_2/l_2}$ independent of N. In other words, it follows that $||\underline{\hat{B}}_{[N,M]}||_{l_2/l_2}$ and $||\underline{\hat{C}}_{[N]}||_{l_2/l_2}$ are uniformly bounded with respect to N and M.

We are now in a position to prove the main result of Lemma 4.5. It is clear that

$$\begin{aligned} & ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}}_{[N,M]}||_{l_2/l_2} \\ \leq \; & ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}_l}_{[N,M]}||_{l_2/l_2} + ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}_u}_{[N,M]}||_{l_2/l_2} \end{aligned} \tag{4.55}$$

where $\underline{\breve{B}_l}_{[N,M]}$ and $\underline{\breve{B}_u}_{[N,M]}$ are the lower and upper triangle portions of $\underline{\breve{B}}_{[N,M]}$, respectively. Hence, by noting the structure of $\underline{E}_M(j\varphi)$ and $\underline{Q}_M$, and the skew structure of $\underline{\breve{B}_l}_{[N,M]}$ as well as the fact that the entries of $\breve{B}_{NMl}$ are zero except its right-upper blocks, we have

$$\begin{aligned} & ||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}_l}_{[N,M]}||_{l_2/l_2} \\ = \; & ||(\underline{E}_M(j\varphi) - \underline{Q}_M)^{-1}\underline{\breve{B}_l}_{[N,M]}||_{l_2/l_2} \\ = \; & \sup_{m\in\mathcal{Z}}\left\{||(E_{Mm}(j\varphi) - Q_M)^{-1}\breve{B}_{NMl}||\right\} \\ \leq \; & \sup_{m\in\mathcal{Z}}\left\{||\partial_N\big((E_{Mm}(j\varphi) - Q_M)^{-1}\big)||\right\} \cdot ||\breve{B}_{NMl}|| \end{aligned} \tag{4.56}$$

where $\partial_N(\cdot)$ means taking out the first N block columns from a matrix. Moreover, by a similar argument to the above, it readily follows that $||\breve{B}_{NMl}||$ has an upper bound independent of M and N (note that $\breve{B}_{NMl}$ is essentially a sub-matrix of $\hat{B}_{NM}$).

Furthermore, by the stability assumption, the inequality (2.19) is true. That is, there is $K > 0$ such that $||(j\varphi_m I - Q)^{-1}|| \leq Kf(m)$, where f is defined in Appendix A.1. Then, it is easy to see that (under our standing assumption $M \geq N+1$)

$$\begin{aligned} & \sup_{m\in\mathcal{Z}}\left\{||\partial_N\big((E_{Mm}(j\varphi) - Q_M)^{-1}\big)||\right\} \\ = \; & \sup_{m\in\mathcal{Z}}\left\{\max_{k\in\{0,1,2,\cdots,N-1\}} ||(j\varphi_{m(2M+1)-M+k}I - Q)^{-1}||\right\} \\ \leq \; & \sup_{m\in\mathcal{Z}}\left\{\max_{k\in\{0,1,2,\cdots,N-1\}} Kf(m(2M+1) - M + k)\right\} \\ \leq \; & K\sup_{m\in\mathcal{Z}} f(\min_{k\in\{0,1,2,\cdots,N-1\}} |m(2M+1) - M + k|) \\ \leq \; & Kf(M-N+1) \end{aligned} \tag{4.57}$$

Combining (4.56) and (4.57), one can conclude that for each fixed N and for any $\epsilon > 0$, there exists an integer $M_0'(N,\epsilon) > 0$ such that

$$||\underline{\hat{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}_l}_{[N,M]}||_{l_2/l_2} < \frac{\epsilon}{4} \quad (\forall M \geq M_0'(N,\epsilon), \forall\varphi \in \mathcal{I}_0) \tag{4.58}$$

because $f(n)$ is monotonically decreasing to 0 for $n \geq 1$ (see Appendix A.1). The above arguments can be repeated on the second term of the right-hand side of (4.55). Hence, for the same $\epsilon > 0$ and $M_0'(N,\epsilon)$, it is easy to see that

$$||\underline{\hat{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}\underline{\breve{B}_u}_{[N,M]}||_{l_2/l_2} < \frac{\epsilon}{4} \quad (\forall M \geq M_0'(N,\epsilon), \forall\varphi \in \mathcal{I}_0) \tag{4.59}$$

where we used the fact that $||\breve{B}_{NMl}||$ and $||\breve{B}_{NMu}||$ have the same upper bound. From (4.58) and (4.59), it follows that

$$||\hat{\underline{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi)-\underline{Q})^{-1}\check{\underline{B}}_{[N,M]}||_{l_2/l_2} < \frac{\epsilon}{2} \quad (\forall M \geq M_0'(N,\epsilon), \forall\varphi \in \mathcal{I}_0) \tag{4.60}$$

In a similar way, one can conclude that for each fixed N and any $\epsilon > 0$, there exists an integer $M_0''(N,\epsilon) > 0$ such that

$$||\hat{\underline{B}}_{[N,M]}||_{l/2l_2}||\check{\underline{C}}_{[N,M]}(\underline{E}(j\varphi)-\underline{Q})^{-1}||_{l_2/l_2} < \frac{\epsilon}{2} \quad (\forall M \geq M_0''(N,\epsilon), \forall\varphi \in \mathcal{I}_0) \tag{4.61}$$

Then, the desired convergence assertion follows from (4.53), (4.60) and (4.61) by taking $M_0(N,\epsilon) = \max\{M_0'(N,\epsilon), M_0''(N,\epsilon)\}$. **Q.E.D.**

Now, let us further define the LTI continuous-time system

$$G_{NM}(s) := \left[\begin{array}{c|c} \mathcal{Q}_M & \mathcal{B}_{NM} \\ \hline \mathcal{C}_{NM} & \mathcal{D}_M \end{array}\right] \tag{4.62}$$

where $\mathcal{B}_{NM} := \hat{B}_{NM}$, $\mathcal{C}_{NM} := \hat{C}_{NM}$ and $\mathcal{D}_M := D_M$, while $\mathcal{Q}_M$ is given by (4.46). The following theorem is helpful in establishing a Hamiltonian test for the H_∞ norm computation.

Theorem 4.4 *Assume that the system (2.1) is asymptotically stable and $A(t) \in L_{\mathrm{PCD}}[0,h]$, $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$ and $D(t)$ is a constant matrix. Then*

$$||\mathcal{G}||_\infty = \lim_{N\to\infty}\lim_{M\to\infty}||G_{NM}||_{\infty,\mathcal{I}^M}$$

where $||G_{NM}||_{\infty,\mathcal{I}^M} := \sup_{\omega\in\mathcal{I}^M}||G_{NM}(j\omega)||$ with $\mathcal{I}^M$ being the union of the intervals $\mathcal{I}^{Mm} := [-\omega_h/2 + m(2M+1)\omega_h, \omega_h/2 + m(2M+1)\omega_h)$, $m \in \mathcal{Z}$; i.e., $\mathcal{I}^M = \cup_{m=-\infty}^{+\infty}\mathcal{I}^{Mm}$.

Proof Noting that $\underline{G}_{[N,M]}(j\varphi)$ is block-diagonal, it is clear from (4.50) that

$$||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} = \sup_{m\in\mathcal{Z}}||\hat{C}_{NM}(E_{Mm}(j\varphi)-Q_M)^{-1}\hat{B}_{NM} + D_M|| \tag{4.63}$$

By the stability assumption, Q_M has no eigenvalues on the imaginary axis. Then $\mathcal{Q}_M$ has no eigenvalues on the imaginary axis since $\mathcal{Q}_M = Q_M - E_{M0}(j0)$ by the definition. Hence,

$$\begin{aligned}
& \max_{\varphi\in\mathcal{I}_0}||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} \\
= {} & \max_{\varphi\in\mathcal{I}_0}\sup_{m\in\mathcal{Z}}||\mathcal{C}_{NM}(j(\varphi + m(2M+1)\omega_h)I - \mathcal{Q}_M)^{-1}\mathcal{B}_{NM} + \mathcal{D}_M|| \\
= {} & ||G_{NM}||_{\infty,\mathcal{I}^M}
\end{aligned} \tag{4.64}$$

On the other hand, by Lemma 4.4 and Lemma 4.5, we obtain

$$\lim_{N\to\infty}\lim_{M\to\infty}||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} = ||\underline{G}(j\varphi)||_{l_2/l_2} \quad (\forall\varphi \in \mathcal{I}_0) \tag{4.65}$$

Therefore, combining this with (4.64) yields

$$\begin{aligned}
||\mathcal{G}||_\infty &= \max_{\varphi\in\mathcal{I}_0}||\underline{G}(j\varphi)||_{l_2/l_2} = \max_{\varphi\in\mathcal{I}_0}\lim_{N\to\infty}\lim_{M\to\infty}||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} \\
&= \lim_{N\to\infty}\lim_{M\to\infty}||G_{NM}||_{\infty,\mathcal{I}^M}
\end{aligned}$$

Note that the order interchanges involved are valid since the convergence of Lemma 4.4 and Lemma 4.5 is uniform with respect to $\varphi \in \mathcal{I}_0$. This completes the proof. **Q.E.D.**

Remark 4.2 *The relation (4.64) suggests an (asymptotic) H_∞ norm computation method by searching over the frequency grid of $\mathcal{I}^M$. It is straightforward to see that the usual H_∞ norm of the* LTI *system G_{NM} is an upper bound of $||G_{NM}||_{\infty,\mathcal{I}^M}$ since $\mathcal{I}^M \subset (-\infty,+\infty)$.*

Theorem 4.4 shows if N and M are big enough, we can get tight estimations of $||\mathcal{G}||_\infty$ by computing $||G_{NM}||_{\infty,\mathcal{I}^M}$. Theorem 4.5 gives a basis by which $||G_{NM}||_{\infty,\mathcal{I}^M}$ can be determined for any fixed N and M through a modified bisection algorithm we will describe shortly.

Theorem 4.5 *Assume that the system (2.1) is asymptotically stable and $A(t) \in L_{\rm PCD}[0,h]$, $B(t), C(t) \in L_{\rm CAC}[0,h]$ and $D(t)$ is a constant matrix. Then, for any fixed N and M satisfying $M \geq N+1$, $||G_{NM}||_{\infty,\mathcal{I}^M} < \gamma$ only if $\bar{\sigma}(D) < \gamma$ and the Hamiltonian matrix*

$$H_{NM} := \begin{bmatrix} \mathcal{Q}_M + \mathcal{B}_{NM}\mathcal{R}_M^{-1}\mathcal{D}_M^*\mathcal{C}_{NM} & \mathcal{B}_{NM}\mathcal{R}_M^{-1}\mathcal{B}_{NM}^* \\ -\mathcal{C}_{NM}^*(I+\mathcal{D}_M\mathcal{R}_M^{-1}\mathcal{D}_M^*)\mathcal{C}_{NM} & -(\mathcal{Q}_M + \mathcal{B}_{NM}\mathcal{R}_M^{-1}\mathcal{D}_M^*\mathcal{C}_{NM})^* \end{bmatrix}$$

has no eigenvalues on the $j\mathcal{I}^M$ portion of the imaginary axis, where $\mathcal{R}_M := \gamma^2 I - \mathcal{D}_M^*\mathcal{D}_M$.

Proof It is clear that

$$\lim_{m\to\infty} ||\mathcal{C}_{NM}(j(\varphi + m(2M+1))I - \mathcal{Q}_M)^{-1}\mathcal{B}_{NM} + \mathcal{D}_M|| = ||\mathcal{D}_M||$$

Hence, $||G_{NM}||_{\infty,\mathcal{I}^M} < \gamma$ implies that $||\mathcal{D}_M|| < \gamma$ (and thus $\mathcal{R}_M^{-1}$ is well-defined). This is equivalent to saying that $||D|| < \gamma$ by the form of $\mathcal{D}_M$. To complete the proof, it remains to show that $||G_{NM}||_{\infty,\mathcal{I}^M} < \gamma$ only if the eigenvalue condition is satisfied. However, this can be completed by following the necessity proof of Lemma 3.7.2 of [32] with $\Delta(s) := \gamma^2 I - G_{NM}^\sim(s)G_{NM}(s)$ where

$$G_{NM}^\sim(s) := \mathcal{B}_{NM}^*(-sI + \mathcal{Q}_M^*)^{-1}\mathcal{C}_{NM}^* + \mathcal{D}_M^*$$

which is a state realization with complex coefficient matrices. **Q.E.D.**

It should be noted that Theorem 4.5 gives only a necessary condition. This is because the range of $j\mathcal{I}^M$ is not connected and hence we cannot employ a continuity argument (in the usual Hamiltonian test, the whole imaginary axis, rather than its $j\mathcal{I}^M$ portion, plays a key role, and an implicit use is made of a continuity argument, based on the fact that the whole imaginary axis is a connected region). Thus, for some γ, the knowledge on the eigenvalues of the Hamiltonian matrix H_{NM} corresponding to γ alone fails to provide an answer as to whether γ is an upper bound or a lower bound of $||G_{NM}||_{\infty,\mathcal{I}^M}$. To cope with such situations, we propose the following modified bisection algorithm for the computation of $||G_{NM}||_{\infty,\mathcal{I}^M}$ for fixed large enough N and M. To facilitate the notation, we introduce

$$\bar{\mathcal{I}}^{Mm} := \Big[\frac{\omega_h}{2} + (m-1)(2M+1)\omega_h, -\frac{\omega_h}{2} + m(2M+1)\omega_h\Big), \quad m \in \mathcal{Z}$$

and $\bar{\mathcal{I}}^M := \bigcup_{m=-\infty}^{+\infty} \bar{\mathcal{I}}^{Mm}$. It is easy to see that $\mathcal{I}^M$ and $\bar{\mathcal{I}}^M$ are disjoint and that the whole imaginary axis is the union of $j\mathcal{I}^M$ and $j\bar{\mathcal{I}}^M$.

Modified Bisection Algorithm for Computing $||G_{NM}||_{\infty,\mathcal{I}^M}$

Step 1. Select an upper bound γ_u and a lower bound γ_l such that $\gamma_l \leq ||G_{NM}||_{\infty,\mathcal{I}^M} \leq \gamma_u$ and $||D|| \leq \gamma_l$. We may simply take $\gamma_l = ||D||$ and $\gamma_u = ||G_{NM}||_\infty$;

Step 2. If $(\gamma_u - \gamma_l)/\gamma_l \leq$ specified error tolerance level, then stop, and let $||G_{NM}||_{\infty,\mathcal{I}^M} = (\gamma_u + \gamma_l)/2$; otherwise, set $\gamma = (\gamma_u + \gamma_l)/2$ and go to the next step;

Step 3. Compute H_{NM} with given γ, and determine the eigenvalues of H_{NM}. If there exists any eigenvalue of H_{NM} on $j\mathcal{I}^M$, it is concluded by Theorem 4.5 that $||G_{NM}||_{\infty,\mathcal{I}^M} \geq \gamma$. This leads to the operation $\gamma_l = \gamma$ and go back to Step 2; if there are no eigenvalues of H_{NM} on $j\mathcal{I}^M$, go to Step 4;

Step 4. There are two cases. *i*) if there are no eigenvalues of H_{NM} on the whole imaginary axis, it follows by the usual Hamiltonian test and Remark 4.2 that $\gamma > ||G_{NM}||_\infty \geq ||G_{NM}||_{\infty,\mathcal{I}^M}$. This leads to the operation $\gamma_u = \gamma$ and go back to Step 2. *ii*) if there are eigenvalues of H_{NM} on the imaginary axis (i.e., on $j\bar{\mathcal{I}}^M$), it remains unknown whether γ is an upper bound or a lower bound of $||G_{NM}||_{\infty,\mathcal{I}^M}$. To resolve such uncertainty, go to the next step;

Step 5. Let $j\bar{\mathcal{I}}^{Mm_1}, \cdots, j\bar{\mathcal{I}}^{Mm_p}$ be the intervals containing the eigenvalues of H_{NM} on $j\bar{\mathcal{I}}^M$. For $i = 1, 2, \cdots, p$, check if $||G_{NM}(j\omega_i^-)|| < \gamma$ and $||G_{NM}(j\omega_i^+)|| < \gamma$, where

$$\begin{aligned} \omega_i^- &:= \inf \bar{\mathcal{I}}^{Mm_i} = \frac{\omega_h}{2} + (i-1)(2M+1)\omega_h \\ \omega_i^+ &:= \sup \bar{\mathcal{I}}^{Mm_i} = -\frac{\omega_h}{2} + i(2M+1)\omega_h \end{aligned}$$

If one of these tests fails, then γ is a lower bound of $||G_{NM}||_{\infty,\mathcal{I}^M}$. Hence, set $\gamma_l = \gamma$ and go back to Step 2; otherwise, we can conclude that $||G_{NM}(j\omega)|| < \gamma \quad (\forall \omega \in \mathcal{I}^M)$ by the continuity argument. Hence, set $\gamma_u = \gamma$ and go back to Step 2.

Recall that the feedthrough matrix of the system (2.1) is assumed to be constant in the above discussions. It is straightforward to show that even if the feedthrough $D(t)$ is time-varying h-periodic, Lemma 4.4 with $\underline{G}_{[N]}(j\varphi)$ redefined suitably by applying the skew truncation even to $\underline{D}$ holds, provided that $D(t)$ belongs to $L_{\mathrm{CAC}}[0, h]$. Furthermore, if we also redefine $\underline{G}_{[N,M]}(j\varphi)$ suitably by applying the staircase truncation to $\underline{D}_{[N]}$, it is straightforward to show that Lemma 4.5 holds with a deteriorated norm bound, where the deterioration is bounded by the norm of $D(t)$. Combining these results, an upper bound for the H_∞ norm of the FDLCP system with a time-varying h-periodic feedthrough matrix can be obtained.

4.2.3 Skew (Staircase) Truncation Size Assessments

It is apparent from Lemma 4.3 (Lemmas 4.4 and 4.5) that the H_2 (H_∞) norm computation accuracy depends on the truncation parameter N in the skew truncation (N and M in the staircase truncation). These parameters determine also the orders of the asymptotically equivalent LTI systems. In this subsection, it is shown that we can assess the computation errors caused by truncations. To facilitate our statement, we define the following numbers about periodic functions $\hat{B}_N(t) := \sum_{|n|\le N} \hat{B}_n e^{jn\omega_h t}$ and $\bar{B}_N(t) := \hat{B}(t) - \hat{B}_N(t)$.

$$\sum_{|n|\le N} ||\hat{B}_n||^2 = \int_0^h ||\hat{B}_N(t)||^2 dt =: \kappa_{BN}^2 \quad \sum_{|n|>N} ||\hat{B}_n||^2 = \int_0^h ||\bar{B}_N(t)||^2 dt =: \bar{\kappa}_{BN}^2$$

$$\sup_{t\in[0,h]} ||\hat{B}_N(t)|| = ||\underline{\hat{B}}_{[N]}||_{l_2/l_2} =: \zeta_{BN} \quad \sup_{t\in[0,h]} ||\bar{B}_N(t)|| = ||\underline{\hat{B}} - \underline{\hat{B}}_{[N]}||_{l_2/l_2} =: \bar{\zeta}_{BN}$$

$$\sup_{t\in[0,h]} ||\hat{B}_N(t) - \hat{B}_0|| =: \beta_{BN}$$

Similarly, $\kappa_{CN}, \bar{\kappa}_{CN}, \zeta_{CN}, \bar{\zeta}_{CN}$ and β_{CN} can be defined in terms of the Fourier coefficients of $\hat{C}(t)$. The following theorem gives upper bounds of errors in the H_2 and H_∞ norm computation formulas established via the skew and staircase truncations.

Theorem 4.6 *Suppose in the system (2.1) that $A(t) \in L_{\rm PCD}[0,h]$, $B(t), C(t) \in L_{\rm CAC}[0,h]$. Then for the H_2 norm computation with the skew truncation parameter N, we have*

$$\left| ||\mathcal{G}||_2^2 - ||G_N||_2^2 \right| < \frac{10K^2}{h} \sqrt{\left(\zeta_{CN}^2 \bar{\kappa}_{BN}^2 + \bar{\kappa}_{CN}^2 \zeta_{B\infty}^2\right)\left(\zeta_{C\infty}^2 \kappa_{B\infty}^2 + \zeta_{CN}^2 \kappa_{BN}^2\right)}$$

For the H_∞ norm computation with the staircase truncation parameters pair (N, M) satisfying $M \ge N+1$, it holds

$$\left| ||\mathcal{G}||_\infty - ||G_{NM}||_{\infty,\mathcal{I}_M} \right| \le K\Big[\bar{\zeta}_{CN}\zeta_{B\infty} + \zeta_{CN}\bar{\zeta}_{BN} + \frac{2(\beta_{CN}\zeta_{BN} + \zeta_{CN}\beta_{BN})}{M-N+1}\Big] \tag{4.66}$$

where K is given in (2.19).

Proof By the definitions of $\kappa_{B\infty}$ and $\zeta_{C\infty}$, together with Appendix $A.1$, it follows from (3.53) and the facts that $\sum_{n=-\infty}^{+\infty} ||\hat{B}_n||^2 = \kappa_{B\infty}^2$ and $||\underline{\hat{C}}||_{l_2/l_2} = \zeta_{C\infty}$ that

$$\mathrm{trace}\{\underline{G}(j\varphi)^* \underline{G}(j\varphi)\} < 5\zeta_{C\infty}^2 K^2 \kappa_{B\infty}^2$$

Similarly, we obtain

$$\mathrm{trace}\{\underline{G}_{[N]}(j\varphi)^* \underline{G}_{[N]}(j\varphi)\} < 5\zeta_{CN}^2 K^2 \kappa_{BN}^2$$

The above two inequalities imply that

$$\begin{aligned} & [\mathrm{trace}\{\underline{G}(j\varphi)^* \underline{G}(j\varphi)\} + \mathrm{trace}\{\underline{G}_{[N]}(j\varphi)^* \underline{G}_{[N]}(j\varphi)\}]^{\frac{1}{2}} \\ < \; & \sqrt{5}K\sqrt{\zeta_{C\infty}^2 \kappa_{B\infty}^2 + \zeta_{CN}^2 \kappa_{BN}^2} \end{aligned} \tag{4.67}$$

It also follows from (4.43) and (4.44) that

$$\left\{ \begin{array}{l} \sum_{n=-\infty}^{+\infty} ||\underline{\hat{C}}_{[N]}(\underline{E}(j\varphi) - \underline{Q})^{-1}[\underline{\hat{B}} - \underline{\hat{B}}_{[N]}]x_n||_{l_2}^2 \le 5K^2 \zeta_{CN}^2 \bar{\kappa}_{BN}^2 \\ \sum_{n=-\infty}^{+\infty} ||[\underline{\hat{C}} - \underline{\hat{C}}_{[N]}](\underline{E}(j\varphi) - \underline{Q})^{-1} \hat{B}_{[N]} x_n||_{l_2}^2 \le 5K^2 \bar{\kappa}_{CN}^2 \zeta_{B\infty}^2 \end{array} \right. \tag{4.68}$$

Substituting (4.67) and (4.68) into (4.39) and recalling (4.42), some simple computations lead to the result.

Next we consider the H_∞ norm computation error. Under the given assumptions, it is straightforward to show that

$$\begin{cases} ||\hat{\underline{C}} - \hat{\underline{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}\hat{\underline{B}}||_{l_2/l_2} \leq K\bar{\zeta}_{CN}\zeta_{B\infty} \\ ||\hat{\underline{C}}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2}||\hat{\underline{B}} - \hat{\underline{B}}_{[N]}||_{l_2/l_2} \leq K\zeta_{CN}\bar{\zeta}_{BN} \end{cases}$$

which implies by (4.52) that

$$||\underline{G}(j\varphi) - \underline{G}_{[N]}(j\varphi)||_{l_2/l_2} \leq K[\bar{\zeta}_{CN}\zeta_{B\infty} + \zeta_{CN}\bar{\zeta}_{BN}] \tag{4.69}$$

where K is given in (3.54). Furthermore, as shown in the proof of Lemma 4.5,

$$||\hat{\underline{B}}_{[N,M]}||_{l_2/l_2} = ||\hat{B}_{NM}|| \leq ||\hat{\underline{B}}_{[N]}||_{l_2/l_2} = \zeta_{BN}$$

Repeating similar arguments leads to

$$||\check{B}_{NMl}|| \leq ||\hat{\underline{B}}_{[N]} - \mathcal{T}\{\hat{B}_0\}||_{l_2/l_2} = \beta_{BN}$$

Therefore, it follows from (4.55) through (4.57) that

$$||(\underline{E}(j\varphi) - \underline{Q})^{-1}\check{\underline{B}}_{[N,M]}||_{l_2/l_2} \leq \frac{2K\beta_{BN}}{M - N + 1} \tag{4.70}$$

Similarly, we can show that

$$||\check{\underline{C}}_{[N,M]}(\underline{E}(j\varphi) - \underline{Q})^{-1}||_{l_2/l_2} \leq \frac{2K\beta_{CN}}{M - N + 1} \tag{4.71}$$

Combining (4.70) and (4.71) with (4.53), it follows that

$$||\underline{G}_{[N]}(j\varphi) - \underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} \leq \frac{2K}{M - N + 1}[\beta_{CN}\zeta_{BN} + \zeta_{CN}\beta_{BN}]$$

Recalling (4.64), together with (4.69), gives the desired result. **Q.E.D.**

4.2.4 Frequency Response Gain Computations

In Subsection 4.2.2, the H_∞ norm computation via the staircase truncation on the frequency response operator is considered. The convergence for such a truncation is guaranteed by Lemmas 4.4 and 4.5. In other words, under the assumptions of Theorem 4.4, we have

$$\lim_{M\to\infty} \lim_{N\to\infty} ||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} = ||\underline{G}(j\varphi)||_{l_2/l_2} \tag{4.72}$$

uniformly over $\varphi \in \mathcal{I}_0$, as we already asserted in (4.65). Now focusing the attention on the block-diagonal structure of the truncated operator $\underline{G}_{[N,M]}(j\varphi)$, it is not hard to see that

$$||\underline{G}_{[N,M]}(j\varphi)||_{l_2/l_2} = \sup_{m\in\mathcal{Z}} ||G_{NM}(j(\varphi + m(2M+1)\omega_h))|| \tag{4.73}$$

which says, together with (4.72), that under the assumptions of Theorem 4.4, the supremum of the frequency response gains, $||G_{NM}(j(\varphi + m(2M+1)\omega_h))||$, of the LTI continuous-time system (4.62) for a fixed $\varphi \in \mathcal{I}_0$ and over $m \in \mathcal{Z}$ can approach that of the frequency response operator of the FDLCP system (2.1) as closely as desired. Or equivalently, the staircase truncation on the frequency response operator $\underline{G}(j\varphi)$ also provides a way to compute the frequency response gains of FDLCP systems. It is worth mentioning that such a formula for the frequency response gains is uniform over the frequency interval $\mathcal{I}_0$, and that the error in such a method is bounded by the right-hand of (4.66) uniformly over $\varphi \in \mathcal{I}_0$.

4.2.5 Numerical Examples

In this subsection, several numerical examples are given to show the efficacy of the suggested computation methods for the H_2 norm, the H_∞ norm and the frequency response gains of the frequency response operator of a given FDLCP system.

Example 4.3 *First we consider the H_2 norm computation for the following π-periodic system by means of the asymptotic trace formula developed in Subsection 4.2.1 when the input weighting parameter β varies from* 0 *to* 0.5.

$$\begin{cases} \dot{x} = \begin{bmatrix} -1-\sin^2(2t) & 2-\frac{1}{2}\sin(4t) \\ -2-\frac{1}{2}\sin(4t) & -1-\cos^2(2t) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1-2\beta\rho(t) \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{cases}$$

Here, the function ρ is given by

$$\rho(t) = \begin{cases} \sin(2t) & (0 \le t \le \frac{\pi}{2}) \\ 0 & (\frac{\pi}{2} < t \le \pi) \end{cases}$$

The transition matrix of the above FDLCP *system has a Floquet factorization of the form*

$$P(t,0) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

It is evident that the system matrices $(A(t), B(t), C(t))$ satisfy the required assumptions in applying the asymptotic trace formula and the asymptotic Hamiltonian test. Noting that the transition matrix is available, the Floquet transformed form of the frequency response operator $\underline{G}(j\varphi)$ can be written explicitly and thus all the above results apply.

Since the transition matrix of the original FDLCP system is available and the Fourier series expansion of $P(t,0)$ and $P^{-1}(t,0)$ only consist of finitely many terms up to the first harmonic element, and since $C(t)$ is constant, it follows that the Fourier series expansion of $\hat{C}(t)$ only involves up to the first harmonic. Then, Corollary 3.2 can be applied directly to get the *exact* value of the H_2 norm of the above system. This is because it suffices to solve

(3.66) for the 6×6 (or 3×3 in the blockwise sense) central sub-matrix of the solution $\underline{\hat{W}}$ and apply the trace formula $||\mathcal{G}||_2^2 = \text{trace}(\underline{\hat{c}}\,\underline{\hat{W}}\,\underline{\hat{c}}^*)$. Noting that $\underline{Q} - \underline{E}(j0)$ is block-diagonal, this sub-matrix of $\underline{\hat{W}}$ can be obtained by solving only a 6×6 Lyapunov equation. The results are given in Table 4.1.

Table 4.1: H_2 Norm Computation: Applying $\text{trace}(\underline{\hat{c}}\,\underline{\hat{W}}\,\underline{\hat{c}}^*)$ of Corollary 3.2

β	0	0.1	0.2	0.3	0.4	0.5
H_2	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611

To verify the efficacy of the asymptotic trace formula of Theorem 4.3, the H_2 norm of the system is computed asymptotically once again. The H_2 norm computation results are listed in Table 4.2 when the skew truncation parameter N varies from 1 to 43. We can assess the errors of the H_2 norm computations by Theorem 4.6 discussed in Subsection 4.2.3. For example, in the case of $\beta = 0.5$, the error of the squared H_2 norm between the exact and its estimated ones is bounded by 0.0262 when $N = 30$. In the last row of Table 4.2, the computation results with $N = 43$ are shown, for which the error of the squared H_2 norm between the exact and estimated ones is bounded by 0.0155 in the case of $\beta = 0.5$. □

Example 4.4 *Now we introduce a feedthrough term $D(t) = 1$ into the* FDLCP *system in Example 4.3 and compute the H_∞ norm of the corresponding system (which is not strictly proper) by the modified bisection algorithm presented in Subsection* 4.2.2. *Here the input weighting parameter β also varies from 0 to 0.5.*

Table 4.3 shows the computation results, where the initial upper and lower bounds for the H_∞ norm are $\gamma_u = 4.62$ and $\gamma_l = 1 (= ||D||)$ while the tolerance error is 0.0001. This upper bound γ_u of $||G_{NM}||_{\infty, \mathcal{I}^M}$ is chosen by working on $||G_{NM}||_\infty$ directly, which is taken such that γ_u is a upper bound of $||G_{NM}||_\infty$ over all $\beta \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ and $N = 1, 2, \cdots, 43$, $M = 6, 7, \cdots, 52$ for simplicity. Such an upper bound of $||G_{NM}||_\infty$ is computed numerically via the usual bisection algorithm of LTI continuous-time systems. The Fourier coefficients involved are computed by a numerical quadrature. In addition, we can also assess the computation errors of the H_∞ norm by Theorem 4.6 of Subsection 4.2.3. For instance, the error of the H_∞ norm between the exact and estimated ones is bounded by 0.2641 when $N = 43$ and $M = 52$ in the case of $\beta = 0.5$. □

Example 4.5 *Finally we consider to compute, based on (4.73) of Subsection 4.2.4, the frequency response gains of the* FDLCP *system given in Example* 4.3 *when the feedthrough term $D(t) = 1$ (that is, the same π-periodic system of Example 4.4 is considered here).*

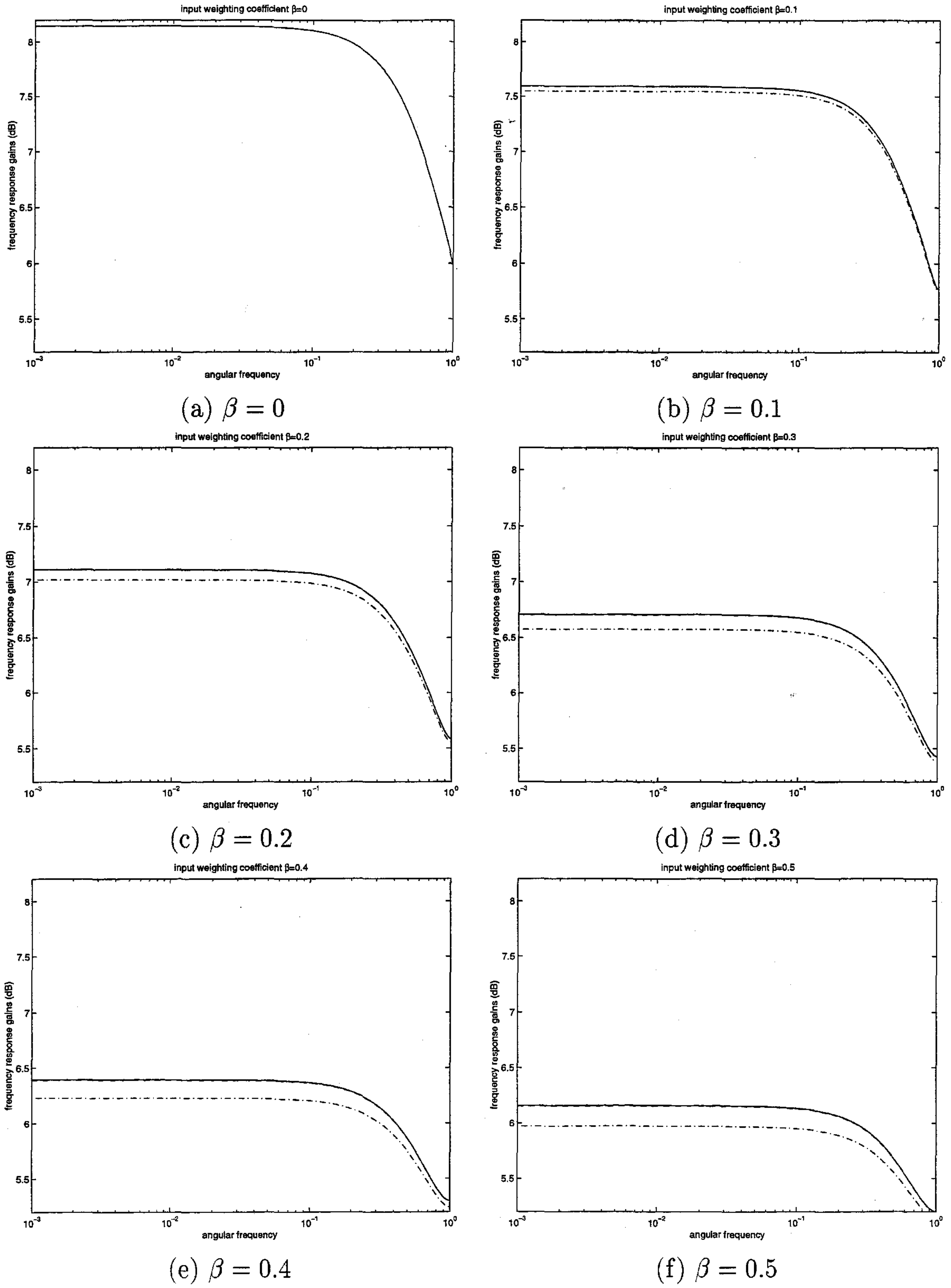
Here three pairs of the skew truncation parameter N and staircase truncation parameter M are considered. In the first case, we choose $N = 2, M = 5$, and in the second and third cases

Table 4.2: H_2 Norm Computation: Skew-Truncating $\underline{G}(j\varphi)$ to $\underline{G}_{[N]}(j\varphi)$

	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$N = 1$	0.7323	0.6825	0.6352	0.5907	0.5500	0.5137
2	0.7323	0.6839	0.6409	0.6046	0.5761	0.5566
3	0.7323	0.6836	0.6408	0.6052	0.5781	0.5608
4	0.7323	0.6836	0.6408	0.6052	0.5781	0.5608
5	0.7323	0.6836	0.6408	0.6052	0.5782	0.5610
6	0.7323	0.6836	0.6408	0.6052	0.5782	0.5610
7	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
8	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
9	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
10	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
11	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
12	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
13	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
14	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
15	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
16	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
17	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
18	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
19	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
20	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
21	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
22	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
23	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
24	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
25	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
26	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
27	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
28	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
29	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
30	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
43	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611

Table 4.3: H_∞ Norm Computation: Staircase-Truncating $\underline{G}(j\varphi)$ to $\underline{G}_{[N,M]}(j\varphi)$

	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$(N = 1, M = 6)$	1.5024	1.4574	1.4158	1.3778	1.3460	1.3230
(2, 6)	1.5024	1.4582	1.4202	1.3893	1.3654	1.3478
(3, 6)	1.5024	1.4518	1.4264	1.3981	1.3761	1.3601
(4, 6)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(5, 6)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(6, 12)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(7, 12)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(8, 12)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(9, 12)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(10, 12)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(11, 17)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(12, 17)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(13, 17)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(14, 17)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(15, 17)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(16, 22)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(17, 22)	1.5024	1.4618	1.4273	1.3981	1.3767	1.3601
(18, 22)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(19, 22)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(20, 22)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(21, 27)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(22, 27)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(23, 27)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(24, 27)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(25, 27)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(26, 32)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(27, 32)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(28, 32)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(29, 32)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(30, 32)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(30, 52)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601
(43, 52)	1.5024	1.4618	1.4273	1.3981	1.3769	1.3601



(a) $\beta = 0$
(b) $\beta = 0.1$
(c) $\beta = 0.2$
(d) $\beta = 0.3$
(e) $\beta = 0.4$
(f) $\beta = 0.5$

Figure 4.3: Frequency response gains, i.e., $||\underline{G}(j\varphi)||_{l_2/l_2}$, over $(0, \omega_h/2] = (0, 1]$ asymptotically computed via $\underline{G}_{[N,M]}(j\varphi)$ (solid curves for $(N = 12, M = 18)$; dashed curves for $(N = 3, M = 7)$; dash-dotted curves for $(N = 2, M = 5)$)

$N = 3, M = 7$ and $N = 12, M = 18$, respectively. In each case, the truncation parameters N and M are kept unchanged for all the input weighting coefficient β varying among $\{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ for simplicity. The computation results are given in Figure 4.3. The solid curves stand for the frequency response gains when the truncation parameters N and M are the third case, i.e., $(N = 12, M = 18)$, and the dashed curves stand for the frequency response gains of the second case, i.e., $(N = 3, M = 7)$, while the dash-dotted curves stand for those of the first case, i.e., $(N = 2, M = 5)$. □

In summary, the computation results in Tables 4.2 and 4.3 clearly show the convergence of the H_2 and H_∞ norm computation algorithms suggested in Subsection 4.2.1 and Subsection 4.2.2. The frequency response gains curves of Figure 4.3 indicate, in particular, that the numerical difference between the second and third cases is quite small so that the suggested computation method of the frequency response gains is of fairly high accuracy even with relatively small truncation parameters. This observation manifests that the staircase truncation is also a useful tool in the frequency response gains computation besides the skew-rectangular truncation technique of [88].

4.3 H_2 and H_∞ Norm Computations via Approximate Modeling

In the preceding section, the H_2 and H_∞ norms computations are dealt with via truncations on the *exact* frequency response operator. Careful observations will reveal that these algorithms heavily rely on the knowledge of the transition matrix of the corresponding FDLCP system. Unfortunately, however, it is generally difficult to find the transition matrix exactly. Therefore, to apply the algorithms effectively to practical FDLCP systems, one has to resort to an approximate modeling technique as we have considered in Section 4.1 for the stability analysis. The general idea is that, if we first construct an approximate model for a given FDLCP system and if the transition matrix of this approximate model (possibly also FDLCP) can be determined explicitly, then all the operations needed in the norm computations become possible for the approximate model. It is expected that the numerical results for the approximate model will approach those of the original FDLCP system if the modeling error is small enough. The main task in this section is to show under what conditions convergence can be guaranteed for the norm computations via such approximate modeling.

To avoid the use of the hard-to-find transition matrix of an FDLCP system in the suggested H_2 and H_∞ norm computation methods, we construct an FDLCP approximate model for the original FDLCP system (2.1) described by

$$G_a : \begin{cases} \dot{\tilde{x}} = A_a(t)\tilde{x} + B(t)u \\ y = C(t)\tilde{x} + D(t)\,u \end{cases} \tag{4.74}$$

It is assumed that the state matrix $A(t)$ of (2.1) belongs to $L_{\mathrm{PCD}}[0, h]$, while $B(t)$ and

$C(t)$ belong to $L_{\mathrm{CAC}}[0,h]$, and $D(t)$ belongs to $L_{\mathrm{PCC}}[0,h]$. Here $A_a(t)$ is taken such that $A_a(t) \in L_{\mathrm{PCD}}[0,h]$, and we define the error matrix $A_\Delta(t) := A(t) - A_a(t)$. The constraints on $A_a(t)$ guarantee that the frequency response operator of the approximate model G_a is well-defined. It should be pointed out that no approximate treatments are imposed on $B(t)$, $C(t)$ and $D(t)$. We also suppose that $A_a(t)$ is taken such that (4.74) has the explicit transition matrix $\Phi_a(t,t_0) = P_a(t,t_0)e^{Q_a(t-t_0)}$. It is well-known [71] that if the approximate state matrix $A_a(t)$ is given by piecewise constant functions, the transition matrix $\Phi_a(t,t_0)$ can be explicitly determined.

Now, since $A_a(t) \in L_{\mathrm{PCD}}[0,h]$, $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$ and $D(t) \in L_{\mathrm{PCC}}[0,h]$, the frequency response operator $\underline{G}_a(j\varphi)$ of the system (4.74) is well-defined by Theorem 3.3 if the approximate model (4.74) is asymptotically stable (this can be checked by the eigenvalues of Q_a), and given by $\underline{G}_a(j\varphi) := \underline{C}(\underline{E}(j\varphi) - \underline{A}_a)^{-1}\underline{B} + \underline{D}$ with $\underline{A}_a := \mathcal{T}\{A_a(t)\}$. It is also clear that $A_\Delta(t) \in L_{\mathrm{PCD}}[0,h]$ and $\underline{A} = \underline{A}_a + \underline{A}_\Delta$ with $\underline{A} := \mathcal{T}\{A(t)\}$ and $\underline{A}_\Delta := \mathcal{T}\{A_\Delta(t)\}$. By the Fourier series expansion operator from $L_2[0,h]$ to l_2, which is an isometric isomorphism, it follows from Lemma 2.8 that

$$||\underline{A}_\Delta||_{l_2/l_2} = ||A_\Delta(\cdot)||_{L_2[0,h]/L_2[0,h]} = \sup_{t\in[0,h]} ||A_\Delta(t)|| =: ||A_\Delta(\cdot)|| \tag{4.75}$$

4.3.1 Convergence Lemmas via Approximate Modeling

In the following arguments, it is assumed that the approximate FDLCP system (4.74) is strictly proper, i.e., $D(t) = 0, \forall t \in [0,h]$, whenever the H_2 norm is concerned. The following lemma shows that the H_2 norm of the approximate FDLCP system (4.74), denoted by $||\mathcal{G}_a||_2$, can approach that of the original FDLCP system, i.e., $||\mathcal{G}||_2$, as close as desired by making the error $||A_\Delta(\cdot)||$ small enough. The convergence property of the H_∞ norm of (4.74), denoted by $||\mathcal{G}_a||_\infty$, to that of the original FDLCP system, i.e., $||\mathcal{G}||_\infty$, will be given in another forthcoming lemma, Lemma 4.7.

Lemma 4.6 *Assume in the system (2.1) that $A(t) \in L_{\mathrm{PCD}}[0,h]$, $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$, and that the system is asymptotically stable. Then if $||A_\Delta(\cdot)||$ is small enough, the approximate model (4.74) is also asymptotically stable and $||\mathcal{G}||_2 = \lim_{||A_\Delta(\cdot)||\to 0} ||\mathcal{G}_a||_2$.*

Proof By Theorem 2.5, Proposition 4.1 and the stability assumption of the original system (2.1), asymptotic stability of the approximate system (4.74) follows readily as $||A_\Delta(\cdot)|| \to 0$. Also in the necessity proof of Theorem 4.1, we have shown that $\sup_{t\in[0,h]} ||P_a(t,0)||$ and $\sup_{t\in[0,h]} ||P_a^{-1}(t,0)||$ are uniformly bounded over $\mathcal{A}_\delta$ for sufficiently small $\delta > 0$. Now we show that there exists $K_\delta > 0$ independent of $A_a(t) \in \mathcal{A}_\delta$ and $\varphi \in \mathcal{I}_0$ such that

$$||(j\varphi_m I - Q_a)^{-1}|| \leq K_\delta f(m) \quad (\forall m \in \mathcal{Z}, \varphi \in \mathcal{I}_0) \tag{4.76}$$

where the function f is defined in Appendix A.1.

To show (4.76), it suffices to show that (4.76) holds for any $A_a(t) \in \overline{\mathcal{A}}_\delta$ and $\varphi \in \mathcal{I}_0$ since $\mathcal{A}_\delta \subset \overline{\mathcal{A}}_\delta$ by the definition given in the necessity proof of Theorem 4.1. To this end, we first fix an $A_{a1}(t) \in \overline{\mathcal{A}}_\delta$, which has the associated matrix Q_{a1}, satisfying

$$||(j\varphi_m I - Q_{a1})^{-1}|| \leq K(Q_{a1})f(m) \quad (\forall m \in \mathcal{Z}, \varphi \in \mathcal{I}_0)$$

for some $K(Q_{a1}) > 0$ and f defined in Appendix A.1. It is well-known that the inverse of a finite-dimensional matrix is continuous with respect to its elements. Therefore, if the associated matrix Q_a with an approximate model $A_a(t) \in \overline{\mathcal{A}}_\delta$ is located in a neighborhood $\mathcal{N}(Q_{a1})$ of Q_{a1}, there exists a number $\tilde{K}(Q_{a1}) > 0$ dependent only on Q_{a1} such that

$$||(j\varphi_m I - Q_a)^{-1}|| \leq \tilde{K}(Q_{a1})f(m) \quad (\forall m \in \mathcal{Z}, \varphi \in \mathcal{I}_0)$$

for all $Q_a \in \mathcal{N}(Q_{a1})$. On the other hand, from the necessity proof of Theorem 4.1, it is already known that the set of all Q_a associated with $A_a(t) \in \overline{\mathcal{A}}_\delta$ is bounded and closed. Therefore, from the Heine-Borel finite-covering theorem [60, p. 36], the result of (4.76) follows.

Next by the assumptions on $A(t)$, $B(t)$ and $C(t)$, stability of G_a implies the frequency response operator of the system (4.74) is well-defined, which we denote by $\underline{G}_a(j\varphi)$. Hence it makes sense to define the H_2 norm of the approximate FDLCP system G_a. Now we are ready to show the main assertion. Since $||\mathcal{G}||_2$ and $||\mathcal{G}_a||_2$ are defined through a finite integral interval, it suffices to show

$$\lim_{||A_\Delta(\cdot)||\to 0} \text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi)) = \text{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) \tag{4.77}$$

uniformly over $\varphi \in \mathcal{I}_0$. Repeating the arguments in (4.39) yields

$$\begin{aligned}
&|\text{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) - \text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi))| \\
\leq\ & \sqrt{2}\Big[\text{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) + \text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi))\Big]^{\frac{1}{2}} \\
&\cdot\Big[\sum_{n=-\infty}^{+\infty} ||(\underline{G}(j\varphi) - \underline{G}_a(j\varphi))\underline{u}_n||_{l_2}^2\Big]^{\frac{1}{2}}
\end{aligned}$$

which implies that to complete the proof, we must show that $\text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi))$ is uniformly bounded over $A_a(t) \in \mathcal{A}_\delta$ and $\varphi \in \mathcal{I}_0$, and that

$$\lim_{||A_\Delta(\cdot)||\to 0} \sum_{n=-\infty}^{+\infty} ||(\underline{G}(j\varphi) - \underline{G}_a(j\varphi))\underline{u}_n||_{l_2}^2 = 0 \tag{4.78}$$

uniformly over $\varphi \in \mathcal{I}_0$. To see the uniform boundedness of $\text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi))$ over $A_a(t) \in \mathcal{A}_\delta$ and $\varphi \in \mathcal{I}_0$, we note from the proof of Lemma 3.2 that

$$\begin{aligned}
&\text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi)) \\
=\ & \sum_{n=-\infty}^{+\infty} ||\underline{C}\,\underline{P}_a(\underline{E}(j\varphi) - \underline{Q}_a)^{-1}\underline{P}_a^{-1}\underline{B}\,\underline{u}_n||_{l_2}^2 \\
\leq\ & ||\underline{C}||_{l_2/l_2}^2 ||\underline{P}_a||_{l_2/l_2}^2 \sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{Q}_a)^{-1}\underline{P}_a^{-1}\underline{B}\,\underline{u}_n||_{l_2}^2 \\
=\ & ||\underline{C}||_{l_2/l_2}^2 ||\underline{P}_a||_{l_2/l_2}^2 \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} ||(j\varphi_m I - Q_a)^{-1}[\underline{P}_a^{-1}\underline{B}]_{m-n}||^2
\end{aligned} \tag{4.79}$$

where $\underline{P}_a := \mathcal{T}\{P_a(t,0)\}$, $\underline{Q}_a = \mathcal{T}\{Q_a\}$, and $[\underline{P}_a^{-1}\underline{B}]_i$ denotes the i-th Fourier coefficient of $P_a^{-1}(t,0)B(t)$. On the other hand, by Theorem 2.2 it follows readily that $\mathcal{T}\{P_a^{-1}(t,0)B(t)\} = \underline{P}_a^{-1}\underline{B}$. Therefore, by using (4.76) in (4.79), it follows that

$$\begin{aligned}
&\text{trace}(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi)) \\
\leq\ & ||\underline{C}||^2_{l_2/l_2}||\underline{P}_a||^2_{l_2/l_2} \sum_{m=-\infty}^{+\infty} ||(j\varphi_m I - Q_a)^{-1}||^2 \sum_{n=-\infty}^{+\infty} ||[\underline{P}_a^{-1}\underline{B}]_{m-n}||^2 \\
\leq\ & ||\underline{C}||^2_{l_2/l_2}||\underline{P}_a||^2_{l_2/l_2} \sum_{m=-\infty}^{+\infty} K_\delta^2 f(m)^2 \frac{1}{h}\int_0^h ||P_a^{-1}(t,0)B(t)||^2 dt \\
\leq\ & 5K_\delta^2||\underline{C}||^2_{l_2/l_2} \sup_{t\in[0,h]} ||P_a(t,0)||^2 \sup_{t\in[0,h]} ||P_a^{-1}(t,0)||^2 \sum_{n=-\infty}^{+\infty} ||B_n||^2 < \infty
\end{aligned}$$

by using the Parseval theorem and Appendix $A.1$. Also some arguments similar to (4.75) are introduced to $\underline{P}_a$ and $\underline{P}_a^{-1}$. Recalling that $\sup_{t\in[0,h]}||P_a(t,0)||$ and $\sup_{t\in[0,h]}||P_a^{-1}(t,0)||$ are uniformly bounded over $A_a(t) \in \mathcal{A}_\delta$, the above inequality clearly implies the assertion that trace$(\underline{G}_a(j\varphi)^*\underline{G}_a(j\varphi))$ is uniformly bounded over $A_a(t) \in \mathcal{A}_\delta$ and $\varphi \in \mathcal{I}_0$ as claimed.

Now we turn to show that (4.78) is true. It is clear that

$$\begin{aligned}
&\sum_{n=-\infty}^{+\infty} ||(\underline{G}(j\varphi) - \underline{G}_a(j\varphi))\underline{u}_n||^2_{l_2} \\
=\ & \sum_{n=-\infty}^{+\infty} ||\underline{C}(\underline{E}(j\varphi) - \underline{A}_a)^{-1}(\underline{A} - \underline{A}_a)(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}\,\underline{u}_n||^2_{l_2} \\
\leq\ & ||\underline{C}||^2_{l_2/l_2}||\underline{P}_a||^2_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q}_a)^{-1}||^2_{l_2/l_2}||\underline{P}_a^{-1}||^2_{l_2/l_2}||\underline{A}_\Delta||^2_{l_2/l_2} \\
& \quad\cdot \sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}\,\underline{u}_n||^2_{l_2} \\
\leq\ & ||\underline{C}||^2_{l_2/l_2} \sup_{t\in[0,h]} ||P_a(t,0)||^2 \sup_{t\in[0,h]} ||P_a^{-1}(t,0)||^2||A_\Delta(\cdot)||^2 K_\delta^2 \\
& \quad\cdot \sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}\,\underline{u}_n||^2_{l_2}
\end{aligned} \tag{4.80}$$

where we used the fact from (4.76) that $||(\underline{E}(j\varphi) - \underline{Q}_a)^{-1}||_{l_2/l_2} \leq K_\delta$ for all $A_a(t) \in \mathcal{A}_\delta$ and over $\varphi \in \mathcal{I}_0$. Since the last factor is uniformly bounded over $\varphi \in \mathcal{I}_0$, the assertion (4.78) follows immediately. **Q.E.D.**

Next we show the convergence of the H_∞ norm computation via approximate modeling.

Lemma 4.7 *Suppose that the system (*2.1*) is stable and that* $A(t) \in L_{\text{PCD}}[0,h]$, $B(t), C(t) \in L_{\text{CAC}}[0,h]$ *and* $D(t) \in L_{\text{PCC}}[0,h]$. *Then if the error* $||A_\Delta(\cdot)||$ *is small enough, the approximate* FDLCP *model (*4.74*) is asymptotically stable and* $\lim_{||A_\Delta(\cdot)||\to 0}||\mathcal{G}_a||_\infty = ||\mathcal{G}||_\infty$.

Proof From the former part of the proof of Lemma 4.6, the stability of the approximate model follows. Moreover, $\sup_{t\in[0,h]}||P_a(t,0)||$ and $\sup_{t\in[0,h]}||P_a^{-1}(t,0)||$ are uniformly bounded over $\mathcal{A}_\delta$ if δ is small enough. Or equivalently, $\underline{P}_a := \mathcal{T}\{P_a(t,0)\}$ and $\underline{P}_a^{-1}$ are uniformly bounded on l_2 over $\mathcal{A}_\delta$ by arguments similar to (4.75). Based on these facts and (4.76), we observe that

$$\begin{aligned}
& ||\underline{G}(j\varphi) - \underline{G}_a(j\varphi)||_{l_2/l_2} \\
= \ & ||\underline{C}((\underline{E}(j\varphi) - \underline{A}_a)^{-1}(\underline{A} - \underline{A}_a)(\underline{E}(j\varphi) - \underline{A})^{-1})\underline{B}||_{l_2/l_2} \\
\leq \ & ||\underline{C}(\underline{E}(j\varphi) - \underline{A}_a)^{-1}||_{l_2/l_2}||\underline{A} - \underline{A}_a||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}||_{l_2/l_2} \\
\leq \ & ||\underline{C}||_{l_2/l_2}||\underline{P}_a||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{Q}_a)^{-1}||_{l_2/l_2}||\underline{P}_a^{-1}||_{l_2/l_2} \\
& \cdot||\underline{A} - \underline{A}_a||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}||_{l_2/l_2} \\
\leq \ & K_\delta||\underline{C}||_{l_2/l_2}||\underline{P}_a||_{l_2/l_2}||\underline{P}_a^{-1}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}||_{l_2/l_2}||A_\Delta(\cdot)||
\end{aligned}$$

which, together with the facts that $(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}$ is bounded on l_2 uniformly over $\varphi \in \mathcal{I}_0$ and $\underline{P}_a$ and $\underline{P}_a^{-1}$ are bounded on l_2 uniformly over $A_a(t) \in \mathcal{A}_\delta$, implies the convergence assertion immediately. **Q.E.D.**

4.3.2 Numerical Examples

Lemmas 4.6 and 4.7 tell us that one can get quite tight H_2 and H_∞ norm estimations of the FDLCP system (2.1) by those of an approximate (possibly FDLCP) model constructed in the form of (4.74), if the transition matrix of (4.74) can be explicitly determined and the modeling error $||A_\Delta(\cdot)||$ is sufficiently small. Unfortunately, however, Lemmas 4.6 and 4.7 only provide the convergence needed in the H_2 and H_∞ norm computations through approximate modeling. In other words, the authentic computations can only be carried out by resorting again to the algorithms as discussed in Section 4.2 with skew or staircase truncations being suitably introduced on the frequency response operator of the corresponding approximate model, which is called the *approximate* frequency response operator for brevity.

Example 4.6 *Here we consider again the H_2 norm computations of the* FDLCP *system given in Example* 4.3 *by introducing the skew truncation on the frequency response operator of an approximate model with $A_a(t)$ being piecewise constant approximation of the state matrix $A(t)$ of the original* FDLCP *system.*

To be more precise, the period π is divided into N_a segments with the same length of h/N_a, during each of which $A(t)$ is treated as a constant matrix, as defined in (4.29). For this kind of approximate FDLCP models, the transition matrices can be computed explicitly [71], and thus the corresponding *approximate* frequency response operator can be explicitly expressed (in the sense that its finitely many entries needed in computations can be exactly determined). Hence, the skew truncation and thus the asymptotic trace formula apply to this *approximate* frequency response operator.

The computation results are given in Table 4.4, in which we consider only three cases of approximation on $A(t)$, i.e., $N_a = 50, 100, 180$, respectively, while the skew truncation parameter N running from 1 to 43 partially. □

Table 4.4: H_2 Norm Computation: Approximately-Modeling G by G_a and Skew-Truncating $\underline{G}_a(j\varphi)$ to $\underline{G}_{a[N]}(j\varphi)$

$N_a = 50$	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$N = 1$	0.7323	0.6826	0.6352	0.5909	0.5501	0.5139
2	0.7323	0.6839	0.6410	0.6047	0.5762	0.5568
3	0.7323	0.6836	0.6409	0.6053	0.5783	0.5611
4	0.7323	0.6836	0.6409	0.6053	0.5783	0.5611
5	0.7323	0.6836	0.6409	0.6054	0.5784	0.5613
10	0.7323	0.6836	0.6409	0.6054	0.5785	0.5613
20	0.7323	0.6836	0.6409	0.6054	0.5785	0.5614
30	0.7323	0.6836	0.6409	0.6054	0.5785	0.5614
43	0.7323	0.6836	0.6409	0.6054	0.5785	0.5614
$N_a = 100$						
$N = 1$	0.7323	0.6825	0.6352	0.5908	0.5500	0.5138
2	0.7323	0.6839	0.6409	0.6046	0.5761	0.5567
3	0.7323	0.6836	0.6408	0.6052	0.5782	0.5609
4	0.7323	0.6836	0.6408	0.6052	0.5782	0.5609
5	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
10	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
20	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
30	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
43	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
$N_a = 180$						
$N = 1$	0.7323	0.6825	0.6352	0.5907	0.5500	0.5137
2	0.7323	0.6839	0.6409	0.6046	0.5761	0.5566
3	0.7323	0.6836	0.6408	0.6052	0.5781	0.5608
4	0.7323	0.6836	0.6408	0.6052	0.5781	0.5608
5	0.7323	0.6836	0.6408	0.6052	0.5782	0.5610
10	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
20	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
30	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
43	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611

Example 4.7 *Now we consider again the* H_∞ *norm computations of the* FDLCP *system given in Example* 4.4 *by introducing the staircase truncation on the frequency response operator of an approximate model constructed from the original* FDLCP *system by piecewise constant approximation of* $A(t)$.

The approximate models needed are constructed in the same way as we explained in Example 4.6. The H_∞ norm computations that are acquired through the modified bisection algorithm are given in Table 4.5, where the staircase truncations are applied for the *approximate* frequency response operator (that is, the frequency response operator of the corresponding approximate FDLCP model). In Table 4.5, three cases of approximate modeling treatments are considered, i.e., $N_a = 50, 100, 180$, respectively. Recall that Table 4.3 gives the H_∞ norm computation results when the staircase truncations are applied directly on the *exact* frequency response operator of the given FDLCP system.

In the computations, the initial values of upper and lower bounds for the H_∞ norm estimations are $\gamma_u = 2.8747$ and $\gamma_l = 1 (= ||D||)$ while the tolerance error is 0.0001. This upper bound γ_u is given by an upper bound estimation formula in Remark 3.7, which is taken invariably over $\beta \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ and $N = 1, 2, \cdots, 43$ for simplicity. The staircase truncation parameter M is taken from 6 to 52 so as to satisfy the truncation parameter constraint of $M \geq N + 1$ for each skew truncation parameter N in the table. The Fourier coefficients needed in computations are obtained by a numerical quadrature. □

In summary, the results in Table 4.4 reflect the fact that convergence in the H_2 norm computation is guaranteed first by Lemma 4.6 when $A(t)$ is approximated as described, and then by Lemma 4.3 when the *approximate* frequency response operator is skew truncated. From Table 4.5, we can also see convergence of the suggested computation methods in the H_∞ norm computation. In particular, the approximate modeling approach does not rely on the transition matrix knowledge of the original FDLCP system. From the observations about Table 4.4 and Table 4.5, the above H_2 and H_∞ norm computation algorithms based on truncations on the *approximate* frequency response operator are implementable in most practical FDLCP systems.

4.4 Truncating the Trace Formula Based on the Harmonic Lyapunov Equation

In Section 3.4, the *exact* trace formula for the H_2 norm is established based on the solution of the harmonic Lyapunov equation, i.e., Theorem 3.7. However, it is apparent that there exist two obstacles for one to apply the trace formula. The first is that one has to determine the solution of the harmonic Lyapunov equation (i.e., (3.58) or (3.59)), which is represented in terms of infinite-dimensional matrices. Strictly speaking, this task cannot be completed even though Theorem 3.1 really gives the closed-form solution in the form of (3.14), since it

Table 4.5: H_∞ Norm Computation: Approximately-Modeling G by G_a and Staircase-Truncating $\underline{G}_a(j\varphi)$ to $\underline{G}_{a[N,M]}(j\varphi)$

$N_a = 50$	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$(N = 1, M = 6)$	1.5021	1.4572	1.4160	1.3794	1.3492	1.3282
(2, 6)	1.5021	1.4581	1.4206	1.3904	1.3666	1.3501
(3, 6)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(4, 6)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(5, 6)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(10, 12)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(20, 22)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(30, 32)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
(43, 52)	1.5021	1.4618	1.4270	1.3996	1.3785	1.3629
$N_a = 100$						
$(N = 1, M = 6)$	1.5021	1.4572	1.4160	1.3785	1.3474	1.3254
(2, 6)	1.5021	1.4581	1.4206	1.3895	1.3657	1.3492
(3, 6)	1.5021	1.4618	1.4270	1.3986	1.3776	1.3620
(4, 6)	1.5021	1.4618	1.4270	1.3986	1.3776	1.3620
(5, 6)	1.5021	1.4618	1.4270	1.3996	1.3776	1.3620
(10, 12)	1.5021	1.4618	1.4270	1.3996	1.3776	1.3620
(20, 22)	1.5021	1.4618	1.4270	1.3996	1.3776	1.3620
(30, 32)	1.5021	1.4618	1.4270	1.3996	1.3776	1.3620
(43, 52)	1.5021	1.4618	1.4270	1.3996	1.3776	1.3620
$N_a = 180$						
$(N = 1, M = 6)$	1.5030	1.4572	1.4160	1.3785	1.3474	1.3245
(2, 6)	1.5030	1.4581	1.4206	1.3895	1.3657	1.3483
(3, 6)	1.5030	1.4618	1.4270	1.3986	1.3767	1.3611
(4, 6)	1.5030	1.4618	1.4270	1.3986	1.3767	1.3611
(5, 6)	1.5030	1.4618	1.4270	1.3986	1.3776	1.3611
(10, 12)	1.5030	1.4618	1.4270	1.3986	1.3776	1.3611
(20, 22)	1.5030	1.4618	1.4270	1.3986	1.3776	1.3611
(30, 32)	1.5030	1.4618	1.4270	1.3986	1.3776	1.3611
(43, 52)	1.5030	1.4618	1.4270	1.3986	1.3776	1.3611

unfortunately relies on the transition matrix expression of the FDLCP system in concern, which is hard to find. The second is that, even if we know the solution of the related harmonic Lyapunov equation, we still face the multiplication of infinite-dimensional matrices in the trace formula itself. To overcome these problems, in this section we discuss trace formulas developed via truncation on this trace formula, which can be used for asymptotic computations of the H_2 norm of the original FDLCP system. To overcome the difficulty in solving the harmonic Lyapunov equation, the trace formula stated via the harmonic Lyapunov equation of an approximate FDLCP model is established in Subsection 4.4.2, which in turn produces applicable trace formulas when truncation technique is used further.

To finish our understanding about the trace formula for the H_2 norm of FDLCP systems, connections between the *exact* trace formula of Corollary 3.2 (or Theorem 3.7) and finite-dimensional trace formulas proposed in Theorem 4.3 of Subsection 4.2.1 will be clarified in the light of truncation on the *exact* trace formula.

4.4.1 Trace Formula Derived via Direct Truncation

In this subsection, we consider to reduce the trace formula of Theorem 3.7 to finite computations. To this end, we truncate the matrix vector $\underline{b}$ into $\underline{b}_N$ which is given by

$$\underline{b}_N := [\cdots, 0, B_{-N}^T, \cdots, B_0^T, \cdots, B_N^T, 0, \cdots]^T \tag{4.81}$$

Now we replace $\underline{b}$ in (3.57) with $\underline{b}_N$, which gives the following relation[1]

$$||\mathcal{G}_N||_2^2 := \text{trace}\Big(\underline{b}_N^* \underline{V}\, \underline{b}_N\Big) \tag{4.82}$$

Note that by the structure of $\underline{b}_N$, only finitely many block matrix entries at the center of $\underline{V}$ are actually involved in the computation of $||\mathcal{G}_N||_2$. Therefore, by truncating $\underline{b}$, we get two benefits: on one hand, we do not need to know all the components of $\underline{V}$ (thereupon, only finitely many variables of the harmonic Lyapunov equation need to be determined); on the other hand, the trace formula computation itself is reduced to some finite operations. However, before we take any advantage of these truncation merits, the convergence problem should be scrutinized first: does $||\mathcal{G}_N||_2$ converge to $||\mathcal{G}||_2$ as $N \to \infty$? The following lemma gives the answer to this convergence question.

Lemma 4.8 *Suppose in the system (2.1) that $A(t)$ belongs to $L_{\text{PCD}}[0,h]$, $B(t)$ and $C(t)$ belong to $L_{\text{CAC}}[0,h]$, and that the system is asymptotically stable and strictly proper. Then, it holds that $||\mathcal{G}||_2 = \lim_{N\to\infty} ||\mathcal{G}_N||_2$.*

Proof Let $B_N(t) := \sum_{|n|\le N} B_n e^{jn\omega_h t}$, where $\{B_n\}_{n=-\infty}^{+\infty}$ is the Fourier coefficients sequence of $B(t)$. Then it is evident that for any N, $B_N(t) \in L_{\text{CAC}}[0,h] \subset L_{\text{PCC}}[0,h]$. Hence, the frequency response operator of the truncated system $(A(t), B_N(t), C(t))$ is well-defined and

[1]Note that $||\mathcal{G}_N||_2$ in (4.82) is defined differently from that in (4.36), simply to avoid a clumsy notation.

can be written as $\underline{G}_N(j\varphi) := \underline{C}(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B}_{[N]}$ with $\underline{B}_{[N]} := \mathcal{T}\{B_N(t)\}$ [2]. Accordingly, the H_2 norm of $\underline{G}_N(j\varphi)$ is also well-defined for each N. To see the convergence, we introduce an orthonormal basis $\{\underline{u}_n\}_{n=-\infty}^{+\infty}$ on l_2 with $\underline{u}_n = [\cdots, 0^T, u_n^T, 0^T, \cdots]^T$. Then by (3.52),

$$\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) = \mathrm{trace}(\underline{G}(j\varphi)\underline{G}(j\varphi)^*) = \sum_{n=-\infty}^{+\infty} ||\underline{G}(j\varphi)^*\underline{u}_n||_{l_2}^2 \tag{4.83}$$

By (4.83) and following similar arguments to (4.39), it follows readily that

$$\begin{aligned}
&\left|\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) - \mathrm{trace}(\underline{G}_N(j\varphi)^*\underline{G}_N(j\varphi))\right| \\
\leq\ & \sqrt{2}\Big[\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) + \mathrm{trace}(\underline{G}_N(j\varphi)^*\underline{G}_N(j\varphi))\Big]^{\frac{1}{2}} \\
&\cdot\Big[\sum_{n=-\infty}^{+\infty} ||\underline{B} - \underline{B}_{[N]}||_{l_2/l_2}||(\underline{E}(j\varphi) - \underline{A})^{-*}\underline{C}^*\underline{u}_n||_{l_2}^2\Big]^{\frac{1}{2}} \\
\leq\ & \sqrt{2}\Big[\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) + \mathrm{trace}(\underline{G}_N(j\varphi)^*\underline{G}_N(j\varphi))\Big]^{\frac{1}{2}} \\
&\cdot\Big[\sum_{|n|>N} ||B_n||\Big]^{\frac{1}{2}}\Big[\sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{A})^{-*}\underline{C}^*\underline{u}_n||_{l_2}^2\Big]^{\frac{1}{2}}
\end{aligned} \tag{4.84}$$

Again, note that $\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi))$ and $\mathrm{trace}(\underline{G}_N(j\varphi)^*\underline{G}_N(j\varphi))$ are bounded uniformly over $\varphi \in \mathcal{I}_0$. This can be easily proved by following the proof of Lemma 3.2, and, in fact, the latter can be shown to be bounded uniformly over N as well as $\varphi \in \mathcal{I}_0$. Treating $\sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{A})^{-*}\underline{C}^*\underline{u}_n||_{l_2}^2$ as the trace of the system $(A(t), I, C(t))$, then the uniform boundedness of $\sum_{n=-\infty}^{+\infty} ||(\underline{E}(j\varphi) - \underline{A})^{-*}\underline{C}^*\underline{u}_n||_{l_2}^2$ over $\varphi \in \mathcal{I}_0$ follows easily. Summarizing the above discussions, it follows from (4.84) that

$$|\mathrm{trace}(\underline{G}(j\varphi)^*\underline{G}(j\varphi)) - \mathrm{trace}(\underline{G}_N(j\varphi)^*\underline{G}_N(j\varphi))| \to 0$$

uniformly over $\varphi \in \mathcal{I}_0$ as $N \to \infty$ since $B(t) \in L_{\mathrm{CAC}}[0, h]$. Finally noting that the H_2 norm is defined via a finite interval integral, the result follows readily. **Q.E.D.**

Equipped with Lemma 4.8, one might optimistically feel that all the difficulties to numerically implement the trace formula of Theorem 3.7 have been removed by truncating $\underline{b}$. However, some careful observations indicate that it is hopeless, strictly speaking, to try to find some finitely many matrix entries of $\underline{V}$ by working only on some finite-dimensional portion of the harmonic Lyapunov equation (3.58). This is because there are multiplications of infinite-dimensional matrices in the harmonic Lyapunov equation (3.58).

To overcome this difficulty, Corollary 3.2 provides us with a solution but under the prerequisite that the transition matrix of the given FDLCP system is explicitly known. If this is the case, by truncating $\hat{\underline{b}}$ to $\hat{\underline{b}}_N$ as we truncate $\underline{b}$ to $\underline{b}_N$ in (4.81) and following the arguments of Lemma 4.8 on the trace formula of Corollary 3.2, it is not hard to see that

$$\lim_{N\to\infty} \mathrm{trace}\left(\hat{\underline{b}}_N^*\hat{\underline{V}}\,\hat{\underline{b}}_N\right) = ||\mathcal{G}||_2^2 \tag{4.85}$$

[2]Note that $\underline{G}_N(j\varphi)$ is different from $\underline{G}_{[N]}(j\varphi)$ defined in (4.35).

where $\hat{\underline{V}}$ is the unique solution of (3.66) of Corollary 3.2. Noting further that in (3.66), $(\underline{Q}-\underline{E}(j0))^*$ and $\underline{Q}-\underline{E}(j0)$ are block-diagonal, it follows that one can solve the $(2N+1)\times(2N+1)$ sub-equation at the center of (3.66) exactly to get the $(2N+1)\times(2N+1)$ submatrix at the center of $\hat{\underline{V}}$, denoted by $(\hat{\underline{V}})_{[N,N]}$, which is actually used in the trace formula of (4.85). Now we extract that $(2N+1)\times(2N+1)$ sub-equation out of (3.66) as follows

$$((\underline{Q}-\underline{E}(j0))^*)_{[N,N]}(\hat{\underline{V}})_{[N,N]} + (\hat{\underline{V}})_{[N,N]}(\underline{Q}-\underline{E}(j0))_{[N,N]} = -(\hat{\underline{C}}^*\hat{\underline{C}})_{[N,N]} \tag{4.86}$$

and define $\hat{b}_N = [\hat{B}_{-N}^T, \cdots, \hat{B}_{-1}^T, \hat{B}_0^T, \hat{B}_1^T, \cdots, \hat{B}_N^T]^T$. Hence, (4.85) can be written as

$$\lim_{N\to\infty} \text{trace}\big(\hat{b}_N^*(\hat{\underline{V}})_{[N,N]}\hat{b}_N\big) = ||\mathcal{G}||_2^2 \tag{4.87}$$

which clearly says that one can get the asymptotic computation of the H_2 norm only through finite-dimensional matrices and the solution of a finite-dimensional Lyapunov equation.

4.4.2 Trace Formula Derived via Approximate Modeling

As we have seen in the preceding subsection, to reduce the solution of the harmonic Lyapunov equation to that of a finite-dimensional algebraic Lyapunov equation, we need to rewrite the harmonic Lyapunov equation (3.58) in terms of the block-diagonal matrix $\underline{Q}-\underline{E}(j0)$ instead of $\underline{A}-\underline{E}(j0)$ as in Corollary 3.2. However, this requires us to have the knowledge about the transition matrix of the original FDLCP system. To avoid this inherent difficulty, we resort to the approximate modeling approach.

It is clear from Corollary 3.2 that the H_2 norm of the approximate FDLCP model (4.74) (when (4.74) is assumed to be stable and strictly proper) can be expressed as

$$||\mathcal{G}_a||_2^2 = \text{trace}\big(\hat{\underline{b}}_a^*\hat{\underline{V}}_a\hat{\underline{b}}_a\big) \tag{4.88}$$

where $\hat{\underline{b}}_a := [\cdots, \hat{B}_{a-1}^T, \hat{B}_{a0}^T, \hat{B}_{a1}^T, \cdots]^T$ with $\{\hat{B}_{an}\}_{n=-\infty}^{+\infty}$ being the Fourier coefficients sequence of $\hat{B}_a(t) := P_a^{-1}(t,0)B(t)$, and $\underline{V}_a$ is the solution of the harmonic Lyapunov equation

$$(\underline{Q}_a-\underline{E}(j0))^*\hat{\underline{V}}_a + \hat{\underline{V}}_a(\underline{Q}_a-\underline{E}(j0)) = -\hat{\underline{C}}_a^*\hat{\underline{C}}_a \tag{4.89}$$

with $\hat{\underline{C}}_a := \mathcal{T}\{P_a(t,0)C(t)\} =: \mathcal{T}\{\hat{C}_a(t)\}$. From (4.88), let us further truncate $\hat{\underline{b}}_a$ to $\hat{\underline{b}}_{aN}$ as we truncate $\underline{b}$ to $\underline{b}_N$ in (4.81). The discussions in Lemma 4.6 have already revealed that if the modeling error $||A_\Delta(\cdot)||$ is sufficiently small, it makes sense to define the H_2 norm of the truncated approximate FDLCP system $(Q_a, \hat{B}_{aN}(t), \hat{C}_a(t))$, which can be written as $||\mathcal{G}_{aN}||_2^2 := \text{trace}(\hat{\underline{b}}_{aN}^*\hat{\underline{V}}_a\hat{\underline{b}}_{aN})$. The following result shows that $||\mathcal{G}_{aN}||_2$ can be an asymptotic computation of the H_2 norm of the original FDLCP system (2.1).

Theorem 4.7 *Assume in the system (2.1) that $A(t) \in L_{\text{PCD}}[0,h]$, $B(t), C(t) \in L_{\text{CAC}}[0,h]$, and that the system is asymptotically stable and strictly proper. Let $||\mathcal{G}_{aN}||_2$ denote the H_2 norm of the stable truncated approximate* FDLCP *model $(Q_a, \hat{B}_{aN}(t), \hat{C}_a(t))$. Then*

$$\lim_{||A_\Delta(\cdot)||\to 0} \lim_{N\to\infty} ||\mathcal{G}_{aN}||_2 = ||\mathcal{G}||_2 \tag{4.90}$$

Furthermore, $||\mathcal{G}_{aN}||_2^2 = \mathrm{trace}(\hat{b}_{aN}^* \hat{V}_{aN} \hat{b}_{aN})$ *with* $\hat{b}_{aN} := [\hat{B}_{a-N}^T, \cdots, \hat{B}_{a-1}^T, \hat{B}_{a0}^T, \hat{B}_{a1}^T, \cdots, \hat{B}_{aN}^T]^T$ *and* $\hat{V}_{aN}$ *being the unique solution of the finite-dimensional algebraic Lyapunov equation*

$$((\underline{Q}_a - \underline{E}(j0))^*)_{[N,N]}\hat{V}_{aN} + \hat{V}_{aN}(\underline{Q}_a - \underline{E}(j0))_{[N,N]} = -(\hat{\underline{C}}_a^* \hat{\underline{C}}_a)_{[N,N]} \tag{4.91}$$

In the above, $(\cdot)_{[N,N]}$ *denotes the* $(2N+1)\times(2N+1)$ *sub-matrix at the center of the infinite-dimensional matrix* $(\cdot)$.

Proof By Corollary 3.2, $||\mathcal{G}_{aN}||_2^2 = \mathrm{trace}(\hat{\underline{b}}_{aN}^* \hat{\underline{V}}_a \hat{\underline{b}}_{aN}) = \mathrm{trace}(\hat{b}_{aN}^* (\hat{\underline{V}}_a)_{[N,N]} \hat{b}_{aN})$. Hence, the convergence of (4.90) follows readily from Lemmas 4.6 and 4.8. On the other hand, since for each fixed N, only the $(2N+1)\times(2N+1)$ sub-matrix at the center of the infinite-dimensional matrix $\hat{\underline{V}}_a$ is actually used in the computation of $||\mathcal{G}_{aN}||_2$, it is enough to determine only that sub-matrix from (4.89), which is denoted by $(\hat{\underline{V}}_a)_{[N,N]} =: \hat{V}_{aN}$. Noting that $\underline{Q}_a - \underline{E}(j0)$ is block-diagonal, the reduced-order Lyapunov equation (4.91) follows. **Q.E.D.**

Remark 4.3 *In the finite-dimensional algebraic Lyapunov equation (*4.91*), there is still another obstacle that should be overcome: i.e., the infinite-dimensional matrices multiplication* $\hat{\underline{C}}_a^* \hat{\underline{C}}_a$. *Fortunately, in the discussions of Section* 2.2, *it is shown that under the assumptions that* $A_a(t) \in L_{\mathrm{PCD}}[0,h]$ *and* $C(t) \in L_{\mathrm{CAC}}[0,h]$, $\mathcal{T}\{\hat{C}_a(t)^*\hat{C}_a(t)\} = \mathcal{T}\{\hat{C}_a(t)^*\}\mathcal{T}\{\hat{C}_a(t)\} = \hat{\underline{C}}_a^* \hat{\underline{C}}_a$. *This implies that* $(\hat{\underline{C}}_a^* \hat{\underline{C}}_a)_{[N,N]}$ *can be constructed directly by computing only finitely many Fourier coefficients of the* h*-periodic matrix function* $\hat{C}_a(t)^*\hat{C}_a(t)$.

4.4.3 Relationships among Various Trace Formulas

From the discussions in Subsection 3.4.3, Subsection 4.2.1 and Subsection 4.4.1, it can be seen that different trace formulas have been established for the H_2 norm computations of FDLCP systems exactly or asymptotically. Recall that the *exact* trace formula of the H_2 norm is represented via the solution of the harmonic Lyapunov equation (Subsection 3.4.3), while the asymptotic trace formulas are derived via the skew truncation on the frequency response operator (Subsection 4.2.1), or derived via truncation on the *exact* trace formula (Subsection 4.4.1). It would be interesting to study their relations. In this subsection, we indicate that the asymptotic trace formula of Theorem 4.3 given in Subsection 4.2.1 can also be viewed as the truncated version of the *exact* trace formula, but applied to the *truncated* FDLCP system $(Q, \hat{B}(t), \hat{C}_N(t))$ (where $\hat{C}_N(t) = \sum_{m=-N}^{N} \hat{C}_m e^{jm\omega_h t}$). This fact tells us that as the truncation parameter N tends to ∞, the limit of the asymptotic trace formula given in Theorem 4.3 goes to that of Corollary 3.2 (or Theorem 3.7), in the light of Lemma 4.8 in Subsection 4.4.1.

In the sequel, we sketch a proof for this limit assertion. From the definitions of the system matrices of the LTI continuous-time system $G_N(s)$ in (4.45) and the definitions of $\hat{\underline{b}}_N, \hat{\underline{C}}_{[N]}$ and $\underline{Q} - \underline{E}(j0)$, it is evident that

$$\mathcal{B}_N = (\hat{\underline{b}}_N)_{[N]}, \quad \mathcal{C}_N = (\hat{\underline{C}}_{[N]})_{[2N,N]}, \quad \mathcal{Q}_N = (\underline{Q} - \underline{E}(j0))_{[N,N]} \tag{4.92}$$

where $(\cdot)_{[N]}$ denotes the $(2N+1)$ sub-vector at the center of an infinite-dimensional matrix vector $(\cdot)$ and $(\cdot)_{[2N,N]}$ denotes the $(4N+1) \times (2N+1)$ submatrix at the center of an infinite-dimensional matrix $(\cdot)$. To describe the connection result mentioned in the above, we consider the following Lyapunov equations:

$$(\underline{Q} - \underline{E}(j0))^* \hat{\underline{V}}(N) + \hat{\underline{V}}(N)(\underline{Q} - \underline{E}(j0)) = -\hat{\underline{C}}^*_{[N]} \hat{\underline{C}}_{[N]} \tag{4.93}$$

$$\mathcal{Q}_N^* \mathcal{V}_N + \mathcal{V}_N \mathcal{Q}_N = -\mathcal{C}_N^* \mathcal{C}_N \tag{4.94}$$

The algebraic Lyapunov equation (4.94) is used for the asymptotic trace formula as stated in Theorem 4.3 and the harmonic Lyapunov equation (4.93) is used for the *exact* trace formula of the FDLCP system $(Q, \hat{B}(t), \hat{C}_N(t))$, in which $\hat{C}(t)$ is truncated to $\hat{C}_N(t)$. Note here that $\hat{\underline{V}}(N)$ is the unique solution of (4.93), which is dependent on N since $\hat{\underline{C}}_{[N]}$ is related to the truncation parameter N. Carefully observing (4.92), (4.93) and (4.94) and noting the block-diagonal structure of $\underline{Q} - \underline{E}(j0)$, it is clear that the algebraic Lyapunov equation (4.94) is nothing but the $(2N+1) \times (2N+1)$ submatrix portion at the center of the harmonic Lyapunov equation (4.93). From this, it follows readily that

$$\text{trace}\{\mathcal{B}_N^* \mathcal{V}_N \mathcal{B}_N\} = \text{trace}\{\hat{\underline{b}}_N^* \hat{\underline{V}}(N) \hat{\underline{b}}_N\} \tag{4.95}$$

which in turn implies from Corollary 3.2 that $\text{trace}\{\mathcal{B}_N^* \mathcal{V}_N \mathcal{B}_N\}$ is just the H_2 norm of the FDLCP system $(Q, \hat{B}_N(t), \hat{C}_N(t))$ that is formed by further truncating the Fourier series expansions of $\hat{B}(t)$ in the FDLCP system $(Q, \hat{B}(t), \hat{C}_N(t))$.

On the other hand, it is straightforward to show that

$$\lim_{N \to \infty} \text{trace}\{\hat{\underline{b}}_N^* \hat{\underline{V}}(N) \hat{\underline{b}}_N\} = ||\mathcal{G}||_2^2 \tag{4.96}$$

by first applying Lemma 4.8 to the truncated FDLCP system $(Q, \hat{B}(t), \hat{C}_N(t))$ and then applying similar convergence arguments to the proof of Lemma 4.8 to the original FDLCP system $(Q, \hat{B}(t), \hat{C}(t))$.

Finally, using (4.96) in (4.95), it follows immediately that

$$\lim_{N \to \infty} \text{trace}\{\mathcal{B}_N^* \mathcal{V}_N \mathcal{B}_N\} = ||\mathcal{G}||_2^2$$

which is nothing but the limit relation of Theorem 4.3. The above arguments show that the results of Theorem 4.3 that are developed via the skew truncation on the frequency response operator can also be verified via truncation on the *exact* trace formula based on the harmonic Lyapunov equation. In particular, Theorem 3.1 guarantees that under the assumptions of Theorem 4.3, the limits of the finite-dimensional Lyapunov equations of Theorem 4.3 do exist, and the limits are just the harmonic Lyapunov equations of Corollary 3.2.

4.4.4 Numerical Examples

In this subsection, we consider to compute the H_2 norm of the π-periodic system of Example 4.3. First, a direct truncation on the *exact* trace formula is considered as suggested in (4.87). Then we apply truncation to the trace formula stated on an approximate model constructed from the given FDLCP system, by adopting Theorem 4.7. These treatments will reduce the trace formulas involved to finite-dimensional computations if the harmonic Lyapunov equations can be solved in a local fashion, as we have already seen in Subsection 4.4.1 and Subsection 4.4.2.

Example 4.8 *First we consider the H_2 norm computations of the* FDLCP *system given in Example* 4.3 *by directly truncating the exact trace formula,* $\mathrm{trace}(\hat{\underline{b}}^* \underline{\hat{V}}\, \hat{\underline{b}})$*, stated via the solution of the harmonic Lyapunov equation, to* $\mathrm{trace}(\hat{\underline{b}}_N^* \underline{\hat{V}}\, \hat{\underline{b}}_N)$*, stated via the solution of a corresponding finite-dimensional Lyapunov equation as indicated in Subsection 4.4.1.*

Obviously, we could develop arguments similar to those in Subsection 4.4.1, in which $\hat{\underline{c}}$ is truncated to $\hat{\underline{c}}_N$ defined accordingly. These dual truncation arguments lead to another but similar method for the H_2 norm computation. However, such a method, when applied to the specific example here, leads to identical computation results to those in Table 4.1 for any truncation parameter $N \geq 1$. This is because the Fourier coefficients of $\hat{C}(t)$ have only up to the first harmonic as discussed in Example 4.3, and thus $\hat{\underline{c}}_N = \hat{\underline{c}}$ for $N \geq 1$. Now in this example, we turn to reduce the trace formula, $\mathrm{trace}(\hat{\underline{b}}^* \underline{\hat{V}}\, \hat{\underline{b}})$, to some finite-dimensional computation by truncating $\hat{\underline{b}}$ to $\hat{\underline{b}}_N$.

Since the transition matrix of the given FDLCP system is available, the truncation on the trace formula can be equivalently converted into truncation on the input matrix $\hat{B}(t)$ as in (4.85). Note also that the harmonic Lyapunov equation $(\underline{E}(j0) - \underline{Q})^* \underline{V} + \underline{V}(\underline{E}(j0) - \underline{Q}) = -\underline{C}^* \underline{C}$ is always solvable in the sense that the $(2N+1) \times (2N+1)$ portion of the solution $\underline{V}$ can be computed (see (4.86)). Hence, the truncated trace formula, i.e., $\mathrm{trace}(\hat{\underline{b}}_N^* \underline{\hat{V}}\, \hat{\underline{b}}_N)$, reduces to finite-dimensional matrices computation. The computation results are given in Table 4.6. □

Example 4.9 *Now we consider the H_2 norm computations of the* FDLCP *system of Example* 4.3 *by truncating the trace formula* $\mathrm{trace}(\hat{\underline{b}}_a^* \underline{\hat{V}}_a \hat{\underline{b}}_a)$ *of the approximate model G_a (that is constructed through piecewise constant approximation of $A(t)$ as described in (4.29)) to* $\mathrm{trace}(\hat{\underline{b}}_{aN}^* \underline{\hat{V}}_a \hat{\underline{b}}_{aN})$*, as described in Theorem 4.7.*

The purpose of this example is to show the effectiveness and convergence of the H_2 norm computation method suggested by Theorem 4.7. Recall that we exploited an explicit form of the Floquet factorization of the transition matrix in the preceding example. However, it is hard to determine the transition matrix of a general FDLCP system. To get around this difficulty in more general cases, one has to resort to approximate modeling so that one can apply Theorem 4.7.

Table 4.6: H_2 Norm Computation: Truncating trace($\hat{\underline{b}}^* \underline{\hat{V}}\, \hat{\underline{b}}$) to trace($\hat{\underline{b}}_N^* \underline{\hat{V}}\, \hat{\underline{b}}_N$)

	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$N = 1$	0.7323	0.6879	0.6486	0.6154	0.5893	0.5713
2	0.7323	0.6836	0.6408	0.6053	0.5783	0.5612
3	0.7323	0.6836	0.6408	0.6053	0.5783	0.5612
4	0.7323	0.6836	0.6408	0.6052	0.5782	0.5610
5	0.7323	0.6836	0.6408	0.6052	0.5782	0.5610
6	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611
7	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611
8	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
9	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
10	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
11	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
12	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
13	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
14	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
15	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
16	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
17	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
18	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
19	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
20	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
21	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
22	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
23	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
24	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
25	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
26	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
27	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
28	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
29	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
30	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611
43	0.7323	0.6836	0.6408	0.6053	0.5783	0.5611

Table 4.7: H_2 Norm Computation: Approximate-Modeling G by G_a and Truncating trace($\hat{\underline{b}}_a^* \hat{\underline{V}}_a \hat{\underline{b}}_a$) to trace($\hat{\underline{b}}_{aN}^* \hat{\underline{V}}_a \hat{\underline{b}}_{aN}$)

$N_a = 50$	$\beta = 0$	0.1	0.2	0.3	0.4	0.5
$N = 1$	0.7321	0.6824	0.6351	0.5908	0.5500	0.5139
2	0.7321	0.6838	0.6409	0.6046	0.5762	0.5568
3	0.7321	0.6835	0.6407	0.6052	0.5782	0.5610
4	0.7321	0.6835	0.6407	0.6052	0.5782	0.5610
5	0.7321	0.6835	0.6408	0.6053	0.5783	0.5612
10	0.7321	0.6835	0.6408	0.6053	0.5784	0.5613
20	0.7321	0.6835	0.6408	0.6053	0.5784	0.5613
30	0.7321	0.6835	0.6408	0.6053	0.5784	0.5613
43	0.7321	0.6835	0.6408	0.6053	0.5784	0.5613
$N_a = 100$						
$N = 1$	0.7323	0.6825	0.6351	0.5907	0.5499	0.5137
2	0.7323	0.6839	0.6409	0.6046	0.5761	0.5567
3	0.7323	0.6835	0.6407	0.6052	0.5781	0.5608
4	0.7323	0.6835	0.6407	0.6052	0.5781	0.5608
5	0.7323	0.6835	0.6408	0.6052	0.5782	0.5610
10	0.7323	0.6835	0.6408	0.6052	0.5783	0.5611
20	0.7323	0.6835	0.6408	0.6052	0.5783	0.5611
30	0.7323	0.6835	0.6408	0.6052	0.5783	0.5611
43	0.7323	0.6835	0.6408	0.6052	0.5783	0.5611
$N_a = 180$						
$N = 1$	0.7323	0.6825	0.6351	0.5907	0.5499	0.5137
2	0.7323	0.6839	0.6409	0.6046	0.5761	0.5566
3	0.7323	0.6835	0.6407	0.6052	0.5781	0.5608
4	0.7323	0.6835	0.6407	0.6052	0.5781	0.5608
5	0.7323	0.6835	0.6408	0.6052	0.5782	0.5610
10	0.7323	0.6835	0.6408	0.6052	0.5783	0.5611
20	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611
30	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611
43	0.7323	0.6836	0.6408	0.6052	0.5783	0.5611

To be more precise, for the given FDLCP system the period π is divided into N_a segments with the same length of h/N_a, on each of which $A(t)$ is treated as a constant matrix. For this kind of approximate FDLCP systems, the trace formula of Theorem 4.7 applies since the transition matrix of the approximate model can be computed. The computation results are listed in Table 4.7, where three cases of the approximation parameter N_a are considered, i.e., $N_a = 50, 100, 180$, respectively. The Fourier coefficients needed in the norm computations are computed by a numerical quadrature. $\square$

In summary, the computation results in Tables 4.6 and 4.7 verify the desired convergence of the truncations on the *exact* trace formula of the given FDLCP system and that of the truncations on the trace formula of the approximate FDLCP models in the H_2 norm sense, respectively. Apparently, the results of Tables 4.6 and 4.7 highly coincide with those in Table 4.1, Table 4.2 and Table 4.4 in the numerical sense.

Chapter 5

Conclusions

The Fourier analysis about periodic functions has been a powerful tool in FDLCP systems analysis and synthesis and produced fruitful results, both technically and historically. Unfortunately, however, the Fourier analysis technique has occasionally been applied to some extent to a much wider group of FDLCP systems than it could be in the mathematically rigorous sense, and there are also ambiguous interpretations due to various convergence problems associated with the Fourier analysis and the specific structure of unbounded operators related to derivative operations. It turns out to be nontrivial for us to extend most well-known properties or characteristics of LTI continuous-time systems to FDLCP systems when these convergence and unboundedness problems are taken into account in theoretical discussions. Bearing these problems in mind, this classical means is utilized once again in this thesis to tackle various analysis problems in FDLCP systems but from an operator-theoretic viewpoint by introducing the Toeplitz transformation. In this thesis, the concentration is focused on the rigorous definitions, derivations and interpretations of basic properties of FDLCP systems, such as stability. In this chapter, the main contributions of this thesis are reviewed briefly, and some possible directions for future research are suggested.

5.1 Summary and Conclusions

In this thesis, a class of finite-dimensional linear continuous-time periodic (FDLCP) systems are attacked via the harmonic analysis, where the Fourier analysis is the main tool but utilized from an operator-theoretic viewpoint. The contributions include the following.

First, the transition matrix properties in terms of Toeplitz operator representations are scrutinized, which lead to the so-called (Floquet) similarity transformation relations stated on the linear spaces l_1 and l_2, i.e., Theorem 2.2 and Theorem 2.4. By means of the similarity transformation relations, asymptotic stability of a class of general FDLCP systems is connected to what we call the harmonic Lyapunov equation, i.e., Theorem 3.1. The similarity transformation relations also reveal the basic characteristics about the eigenvalues of FDLCP systems, i.e., Theorem 2.5, which improves our eigenvalues knowledge about FDLCP systems in a geometric way and this in turn inspires us to the extension of the Gerschgorin theorem to operators defined on l_2, that is, Theorem 3.2. The harmonic Lyapunov equation is stated for FDLCP systems, but in an LTI fashion, with an infinite-dimensional matrix expression. This harmonic Lyapunov equation should be interpreted as an operator-valued Lyapunov equation densely defined on the linear space l_2. This work manifests that FDLCP systems are essentially LTI when their stability is considered. Though we also derived a closed form solution to such a Lyapunov equation, the solution depends, unfortunately, on the transition matrix knowledge and its positive definiteness test remains to be an open problem; these problems constrain the value of the harmonic Lyapunov equation to theoretical analysis. However, as we have shown in Chapter 4, it does help to derive necessary and sufficient stability criteria for FDLCP systems via approximate models. The latter criteria are highly applicable if the transition matrices of such approximate models can be explicitly computed. These stability criteria are summarized in Theorem 4.1, Corollary 4.1 and Theorem 4.2. These results guarantee, in particular, that we may use piecewise-constant-functions treatments in approximating $A(t)$, and as in Section 4.1, the set of such functions forms a practically applicable basis to check stability of FDLCP systems.

Second, the existence conditions of frequency response operators defined through the input/output steady-state analysis are completely clarified and their basic properties are investigated, in particular, in connection with the H_2 and H_∞ norms. This study indicates that because of various convergence problems related to the Fourier analysis and the Toeplitz transformation involved in the definition of the frequency response operator [70], this operator is guaranteed to be densely defined on l_2, i.e., Theorem 3.3, but that it can be extended to have the Hilbert space l_2 as its domain so that we still can define and compute the H_2 and H_∞ norms of FDLCP systems based on this frequency response operator, as argued in Remark 3.3. It is also proved that under standard conditions, the time-domain H_2 norm (respectively, the L_2-induced norm) is equal to the frequency-domain H_2 norm (respectively, the H_∞ norm) of the frequency response operator, i.e., Theorem 3.5 (respectively, Theorem 3.6). Thus the well-known equivalence relations in LTI continuous-time systems are recovered in a class of FDLCP systems for the frequency response operator defined through a way different from the lifting [4], [5], [72]. What has been clarified further is that the frequency response operator defined via the steady-state analysis is well-defined in most practical FDLCP systems and its mathematical expression is similar to that of LTI continuous-time systems. In addition, it is worth mentioning that the frequency response operator thus defined may contain more structural information of FDLCP systems than we had understood in the usual ways prior to this study. For example, it is verified that the frequency response operator defined by the steady-state input-output analysis can also be established as a mapping on l_1 under some strengthened assumptions on the system matrices $\{A(t), B(t), C(t), D(t)\}$, i.e., Theorem 3.4. We believe that through the input/output steady-state analysis to l_p-EMP signals, $2 < p < \infty$, the frequency response operators can be introduced as a mapping (densely defined) on l_p under possibly weaker assumptions than those in the l_2 case.

Finally, numerical implementations of the theoretical results form another group of achievements of this work. Through the skew and staircase truncations, the H_2 and H_∞ norm computations in FDLCP systems are converted to those of asymptotically equivalent LTI continuous-time systems. Hence, the results for the H_2 and H_∞ norm computations in LTI systems are extended to FDLCP systems. More precisely, on one hand, an asymptotic trace formula is established for the H_2 norm computation based on a finite-dimensional algebraic Lyapunov equation, i.e., Theorem 4.3; on the other hand, an asymptotic Hamiltonian test for the H_∞ norm computation is derived, which is stated based on a finite-dimensional LTI continuous-time model, i.e., Theorem 4.5. This Hamiltonian test is useful in developing a modified bisection algorithm for H_∞ norm computation, as discussed in Section 4.2. The implication is that the skew analysis is a useful tool in converting an FDLCP system to an asymptotically equivalent LTI system so that techniques developed for LTI systems can be applied to FDLCP ones asymptotically. In other words, the skew analysis can provide an alternative tool to get insight into the behavior of general FDLCP systems because it converts them into equivalent LTI *continuous-time* systems in an asymptotic sense, while the lifting technique converts them into equivalent LSI *discrete-time* systems. Indeed, the skew analysis on the frequency response operator inspires us strongly to prove the *exact* trace formula for the H_2 norm of FDLCP systems based on the harmonic Lyapunov equation. This study is summarized in Theorem 3.7. In addition, we believe the skew analysis can also convert the H_2 problem into that of an asymptotically equivalent discrete-time system by the well-known impulse modulation formula and the factorization technique [34], although this idea is not pursued in this thesis. It is also worth mentioning that the staircase truncation employed for the H_∞ norm computation in particular gives an alternative method for the frequency response gain computation of FDLCP systems [88], as discussed in Subsection 4.2.4.

In the proposed H_2 and H_∞ norm computation methods, the Floquet transformation is introduced to avoid the invertibility problem of infinite-dimensional operators and provide help in the convergence arguments. Therefore, it becomes necessary to have the transition matrix knowledge before applying the results. Fortunately, the discussion of Section 4.2 establishes in particular that it is enough to have the numerical description of the transition matrix when using the results here in the sense that convergence in the proposed methods is guaranteed theoretically without any analytical assumptions on the transition matrix. The size of asymptotically equivalent LTI systems can be assessed easily in most practical systems according to the accuracy requirement, as discussed in Subsection 4.2.3. Again, approximate models provide help in the H_2 and H_∞ norm computations as discussed in Section 4.3.

5.2 About Future Research

Apparently there are still many problems both theoretical and practical remaining unsolved in the FDLCP world, although numerous efforts have been devoted to them. In the following, we intend to take some space to describe several problems or topics that are worth further probing into. Roughly speaking, what we are going to talk is just what we failed to surmount in our own study, and rough guesses of possible conclusions are given.

Solution of the Harmonic Lyapunov Equation. Since the harmonic Lyapunov equation has an infinite-dimensional matrix expression, its solution is nontrivial in general. Its solution is significant in two aspects: first, it might give a direct way to answer asymptotic stability of FDLCP systems without invoking the solution of periodically time-varying Lyapunov differential equations; second, the H_2 norm can be determined via the trace formula and this, in turn, might give ways to do the H_2 design in the FDLCP setting. The problem is that such a solution must resort to some truncation on the harmonic Lyapunov equation and, therefore, a convergence problem inevitably appears.

Pole/Zero Structure of FDLCP Systems. Since in FDLCP systems the similarity transformation relations and the frequency response operator have similar algebraic expressions to what we have in LTI continuous-time systems, it is natural to extend the pole/zero concepts to FDLCP systems in a similar sense. In fact, there are already works in this effort [56], [70], but the results there are neither unified nor easy to understand in the usual LTI continuous-time sense.

Extended Nyquist Criterion in FDLCP Systems. The Floquet theorem is stated for open-loop FDLCP systems. When feedback is installed, it is hard to check the closed-loop stability by the Floquet theorem. This is also true for the stability criteria developed in this thesis based on approximate modeling. A possible solution is to extend the Nyquist criterion to FDLCP systems based on the frequency response operator. This idea came originally from [39], [69], [70], but the mathematical interpretation of such an extended Nyquist criterion is insufficient or even wrong in some sense. The primary difficulties in this idea is: first, in what class of FDLCP systems it makes sense to establish the Nyquist criterion on the frequency response operator that is infinite-dimensional; second, the numerical implementation of this Nyquist criterion needs truncation, which results in a convergence problem.

Harmonic Riccati Equations in FDLCP Systems. Only by matrix analysis, the harmonic Lyapunov equation is derived. Hence it is reasonable to establish a harmonic Riccati equation similarly from a periodically time-varying Riccati differential equation as in the work of [70]. However, to some extent, the arguments there are not so persuasive and lack rigorous interpretation. Possible obstacles in establishing the so-called harmonic Riccati equation may include: first, there is no closed-form formula for the solution of periodically time-varying Riccati differential equations, and second, the frequency response operator definition needs to be extended to unstable FDLCP systems.

Harmonic Linear Matrix Inequalities in FDLCP Systems. The algebraic Lyapunov and Riccati equations in LTI systems are the basis of the commencement of the linear matrix inequality (LMI) technique. Therefore, the establishment of the harmonic Lyapunov and Riccati equations may pave the way for some harmonic LMI interpretations of properties of FDLCP systems, which in turn may usher in a harmonic LMI approach for analysis and synthesis of FDLCP systems.

Although systematic procedures are adopted and stretched in this thesis for analysis of FDLCP systems, their applications in control engineering remain untouched. Therefore, at this point, it is yet an unanswered question to measure how well the methods developed in this thesis will work for control problems of practical FDLCP systems. Nevertheless, it would be fair to say that the major results completed in this thesis as a whole have succeeded to a considerable degree in establishing a working platform for the further research about FDLCP systems through the harmonic analysis approach.

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Appendix A

A.1 Lemma A

The function $f(n)$ [2] of an integer n is defined by

$$f(n) = \begin{cases} 1 & n = 0 \\ |n|^{-1} & n \neq 0 \end{cases}$$

Then we have $\sum_{n=N+1}^{\infty} f(n)^2 < \frac{1}{N} \quad (N \geq 1)$ and $\sum_{n=-\infty}^{\infty} f(n)^2 < 5$.

A.2 Proof of Lemma 3.1

The discussions just before Lemma 3.1 indicate that the adjoint operator $\underline{E}(j0)^*$ of $\underline{E}(j0)$ is well-defined on its domain $\mathcal{D}(\underline{E}(j0)^*)$. Here we only need to determine the structure of the domain $\mathcal{D}(\underline{E}(j0)^*)$ and the matrix expression of $\underline{E}(j0)^*$.

By the block diagonal structure of $\underline{E}(j0)$, it is obvious that $\underline{E}(j0)$ can be viewed as a weighted sum of projections on l_2. That is, $\underline{E}(j0)\underline{x} = \sum_{m=-\infty}^{+\infty} jm\omega_h \underline{P}_m \underline{x}$ with $\underline{x} \in \mathcal{D}\{\underline{E}(j0)\}$ and $\{\underline{P}_m\}_{m=-\infty}^{+\infty}$ being orthogonal projection operators satisfying $\underline{P}_i \underline{P}_k = 0 \ (i \neq k)$ and $\sum_{m=-\infty}^{+\infty} \underline{P}_m = \underline{I}$. Denote the range of $\underline{P}_m$ by $\mathbf{R}(\underline{P}_m)$. Then the closed linear space $\mathbf{R}(\underline{P}_m)$ are mutually orthogonal and satisfy

$$l_2 = \cdots + \mathbf{R}(\underline{P}_{-1}) + \mathbf{R}(\underline{P}_0) + \mathbf{R}(\underline{P}_1) + \cdots$$

Without loss of generality, we assume that $\mathbf{R}(\underline{P}_m), \forall m \in \mathcal{Z}$ is one-dimensional. Therefore, if $\underline{e}_m$ is a unit vector in $\mathbf{R}(\underline{P}_m)$, then $\{\underline{e}_m\}_{m=-\infty}^{\infty}$ is an orthonormal basis of l_2. The Fourier series theorem (Theorem 1.6.3 of [22]) tells us that for any $\underline{x}, \underline{y} \in l_2$, it holds

$$\underline{x} = \sum_{m=-\infty}^{+\infty} \langle \underline{x}, \underline{e}_m \rangle \underline{e}_m, \quad \langle \underline{x}, \underline{y} \rangle = \sum_{m=-\infty}^{+\infty} \langle \underline{x}, \underline{e}_m \rangle \overline{\langle \underline{y}, \underline{e}_m \rangle}$$

On the other hand, by the definition of the domain $\mathcal{D}\{\underline{E}(j0)\}$, one can say that $\underline{x} \in \mathcal{D}\{\underline{E}(j0)\}$ if and only if

$$\sum_{m=-\infty}^{+\infty} ||\lambda_m [\underline{x}]_m||^2 = \sum_{m=-\infty}^{+\infty} |\lambda_m \langle \underline{x}, \underline{e}_m \rangle|^2 < \infty$$

where $\lambda_m := jm\omega_h$. Noting that $\underline{P}_m \underline{x} = \langle \underline{x}, \underline{e}_m \rangle \underline{e}_m$, simple computations give

$$\underline{E}(j0)\underline{x} = \sum_{m=-\infty}^{+\infty} \lambda_m \underline{P}_m \underline{x} = \sum_{m=-\infty}^{+\infty} \lambda_m \langle \underline{x}, \underline{e}_m \rangle \underline{e}_m$$

Thus, if $\underline{x} \in \mathcal{D}\{\underline{E}(j0)\}$ and $\underline{y} \in l_2$, we obtain

$$\langle \underline{E}(j0)\underline{x}, \underline{y} \rangle = \sum_{m=-\infty}^{+\infty} \lambda_m \langle \underline{x}, \underline{e}_m \rangle \overline{\langle \underline{y}, \underline{e}_m \rangle}$$

Furthermore, if there is a $\underline{z} \in l_2$ such that

$$\langle \underline{E}(j0)\underline{x}, \underline{y} \rangle = \langle \underline{x}, \underline{z} \rangle = \sum_{m=-\infty}^{+\infty} \langle \underline{x}, \underline{e}_m \rangle \overline{\langle \underline{z}, \underline{e}_m \rangle}$$

for all $\underline{x} \in \mathcal{D}\{\underline{E}(j0)\}$, then it follows by the above two equalities that $\langle \underline{z}, \underline{e}_m \rangle = \bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle$. Therefore, if we define $\underline{E}(j0)^*$ by $\underline{z} = \underline{E}(j0)^* \underline{y}$, then $\langle \underline{E}(j0)^* \underline{y}, \underline{e}_m \rangle = \bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle$, which implies that

$$\underline{E}(j0)^* \underline{y} = \sum_{m=-\infty}^{+\infty} \bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle \underline{e}_m = \sum_{m=-\infty}^{+\infty} \bar{\lambda}_m \underline{P}_m \underline{y} = \Big(\sum_{m=-\infty}^{+\infty} \lambda_m \underline{P}_m \Big)^* \underline{y}$$

since $\underline{P}_m^* = \underline{P}_m$ by Theorem 5.16.2 and Theorem 5.23.9 of [55]. This gives us the second assertion.

To see the structure of $\mathcal{D}\{\underline{E}(j0)^*\}$, let us show that $\mathcal{D}\{\underline{E}(j0)^*\} = \mathcal{D}\{\underline{E}(j0)\}$. To this end, suppose that $\underline{x}, \underline{y} \in l_E = \mathcal{D}\{\underline{E}(j0)\}$. Simple computations according to the definition of inner product on l_2 show that

$$\langle \underline{E}(j0)\underline{x}, \underline{y} \rangle = \langle \underline{x}, -\underline{E}(j0)\underline{y} \rangle = \langle \underline{x}, \underline{z} \rangle$$

where $\underline{z} = -\underline{E}(j0)\underline{y} \in l_2$. The above equations clearly say that for $\underline{x}, \underline{y} \in \mathcal{D}\{\underline{E}(j0)\}$, there exists $\underline{z} \in l_2$ such that $\langle \underline{E}(j0)\underline{x}, \underline{y} \rangle = \langle \underline{x}, \underline{z} \rangle$, which implies that $\underline{y} \in \mathcal{D}\{\underline{E}(j0)^*\}$ by the definition of $\mathcal{D}\{\underline{E}(j0)^*\}$. Therefore, we get $\mathcal{D}\{\underline{E}(j0)\} \subset \mathcal{D}\{\underline{E}(j0)^*\}$. Hence to complete the proof, it remains to show that $\mathcal{D}\{\underline{E}(j0)^*\} \subset \mathcal{D}\{\underline{E}(j0)\}$. Now assume $\underline{y} \in \mathcal{D}\{\underline{E}(j0)^*\}$. Then by definition, there is a $\underline{z} \in l_2$ such that for all $\underline{x} \in \mathcal{D}\{\underline{E}(j0)\}$ we have

$$0 = \langle \underline{E}(j0)\underline{x}, \underline{y} \rangle - \langle \underline{x}, \underline{z} \rangle = \sum_{m=-\infty}^{+\infty} \langle \underline{x}, \underline{e}_m \rangle \overline{[\bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle - \langle \underline{z}, \underline{e}_m \rangle]}$$

Since $\underline{e}_m \in \mathcal{D}\{\underline{E}(j0)\}$ for all $m \in \mathcal{Z}$, it follows readily from the above equation that $\bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle = \langle \underline{z}, \underline{e}_m \rangle$ for all $m \in \mathcal{Z}$ so that

$$\underline{z} = \sum_{m=-\infty}^{+\infty} \bar{\lambda}_m \langle \underline{y}, \underline{e}_m \rangle \underline{e}_m = \underline{E}(j0)^* \underline{y}$$

Finally, by the second assertion we just proved, we can rewrite the above relation as $-\underline{z} = \underline{E}(j0)\underline{y}$. This, together with the fact that $-\underline{z} \in l_2$, indicates that $\underline{y} \in \mathcal{D}\{\underline{E}(j0)\}$. **Q.E.D.**

A.3 Auxiliary Arguments of Theorem 3.7

The purpose of this appendix is to give rigorous arguments to validate the order interchanges in (3.62) and (3.63). It is easy to see that if there are p inputs to the system (2.1), then $\hat{\underline{b}}$ consists of p infinite-dimensional vectors. Then, by (3.60), the squared H_2 norm can be given by the summation of all the squared H_2 norms of p single-input subsystems. In other words, it will lose no generality if we assume the system (2.1) has only one input. This will bring convenience in the inner product operations.

First we show the validity of the order interchange of the infinite integral and the limit in the equation (3.62). By the Cauchy-Schwarz inequality

$$
\begin{aligned}
& \Big| \int_{-\infty}^{+\infty} (\hat{\underline{b}} - \hat{\underline{b}}_N)^* (\underline{E}(j\omega) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\omega) - \underline{Q})^{-1} \hat{\underline{b}}\, d\omega \Big| \\
\leq\ & \int_{-\infty}^{+\infty} ||\hat{\underline{C}}(\underline{E}(j\omega) - \underline{Q})^{-1}(\hat{\underline{b}} - \hat{\underline{b}}_N)||_{l_2} ||\hat{\underline{C}}(\underline{E}(j\omega) - \underline{Q})^{-1}\hat{\underline{b}}||_{l_2} d\omega \\
\leq\ & \Big[\int_{-\infty}^{+\infty} ||\hat{\underline{C}}(\underline{E}(j\omega) - \underline{Q})^{-1}(\hat{\underline{b}} - \hat{\underline{b}}_N)||_{l_2}^2 d\omega\Big]^{\frac{1}{2}} \Big[\int_{-\infty}^{+\infty} ||\hat{\underline{C}}(\underline{E}(j\omega) - \underline{Q})^{-1}\hat{\underline{b}}\,||_{l_2}^2 d\omega\Big]^{\frac{1}{2}} \\
\leq\ & ||\hat{\underline{C}}||_{l_2/l_2}^2 \Big[\int_{-\infty}^{+\infty} ||(\underline{E}(j\omega) - \underline{Q})^{-1}(\hat{\underline{b}} - \hat{\underline{b}}_N)||_{l_2}^2 d\omega\Big]^{\frac{1}{2}} \\
& \cdot \Big[\int_{-\infty}^{+\infty} ||(\underline{E}(j\omega) - \underline{Q})^{-1}\hat{\underline{b}}\,||_{l_2}^2 d\omega\Big]^{\frac{1}{2}}
\end{aligned}
\tag{A.1}
$$

Furthermore, by the block-diagonal structure of $(\underline{E}(j\varphi) - \underline{Q})^{-1}$, we have

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} ||(\underline{E}(j\omega) - \underline{Q})^{-1}(\hat{\underline{b}} - \hat{\underline{b}}_N)||_{l_2}^2 d\omega \\
\leq\ & \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} ||(j(\omega + k\omega_h)I - Q)^{-1}||^2 ||[\hat{\underline{b}} - \hat{\underline{b}}_N]_k||^2 d\omega \\
=\ & \sum_{k=-\infty}^{+\infty} ||[\hat{\underline{b}} - \hat{\underline{b}}_N]_k||^2 \int_{-\infty}^{+\infty} ||(j(\omega + k\omega_h)I - Q)^{-1}||^2 d\omega \\
=\ & \sum_{|k|>N} ||\hat{B}_k||^2 \int_0^{+\infty} ||e^{(Q - jk\omega_h I)\tau}||^2 d\tau = \sum_{|k|>N} ||\hat{B}_k||^2 \int_0^{+\infty} ||e^{Q\tau}||^2 d\tau
\end{aligned}
\tag{A.2}
$$

by the Parseval theorem, where $[\cdot]_k$ denotes the k-th block-vector entry of an infinite-dimensional vector $(\cdot)$. Repeating the above arguments on the last factor of $(A.1)$, together with $(A.2)$, the inequality $(A.1)$ gives

$$
\begin{aligned}
& \Big| \int_{-\infty}^{+\infty} (\hat{\underline{b}} - \hat{\underline{b}}_N)^* (\underline{E}(j\omega) - \underline{Q})^{-*} \hat{\underline{C}}^* \hat{\underline{C}} (\underline{E}(j\omega) - \underline{Q})^{-1} \hat{\underline{b}}\, d\omega \Big| \\
\leq\ & ||\hat{\underline{C}}||_{l_2/l_2}^2 \Big[\sum_{|k|>N} ||\hat{B}_k||^2\Big]^{\frac{1}{2}} \Big[\sum_{k=-\infty}^{+\infty} ||\hat{B}_k||^2\Big]^{\frac{1}{2}} \int_0^{+\infty} ||e^{Q\tau}||^2 d\tau =: K_N
\end{aligned}
\tag{A.3}
$$

Now we note by Proposition 2.1 that $\hat{B}(t) = P^{-1}(t,0)B(t)$ and $\hat{C}(t) = C(t)P(t,0)$ belong to $L_{\mathrm{CAC}}[0,h]$. Based on these facts and the stability assumption of the system (2.1) (i.e., all the eigenvalues of Q have negative real parts), one can conclude that $K_N \to 0$ as $N \to \infty$ since the last two factors in $(A.3)$ are finite and $\underline{\hat{C}}$ is bounded on l_2 by Lemma 2.8, while $\sum_{|k|>N} ||\hat{B}_k||^2$ tends to zero as $N \to \infty$. This shows that the order exchange is validated.

We must point out that in the deduction of $(A.2)$, there is still another order interchange between the infinite integral ($\int_{-\infty}^{+\infty}$) and the infinite series ($\sum_{k=-\infty}^{+\infty}$). However, this is validated by the Levi theorem [55, p. 577] under the same assumptions.

Next we show the validity of the order interchange in (3.63). Obviously, the validity of such order interchange can be shown elementwise along the infinite-dimensional vector $(\underline{Q} - \underline{E}(j\omega))^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{Q} - \underline{E}(j\omega))^{-1}\underline{\hat{b}}$. To this purpose, we observe

$$
\begin{aligned}
& \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} ||[(\underline{E}(j\omega) - \underline{Q})^{-*}\underline{\hat{C}}^*\underline{\hat{C}}(\underline{E}(j\omega) - \underline{Q})^{-1}]_{(i,k)}\hat{B}_k||d\omega \\
\leq\ & \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} ||(j(\omega + i\omega_h)I - Q)^{-*}|| \cdot ||[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}|| \\
& \qquad \cdot ||(j(\omega + k\omega_h)I - Q)^{-1}|| \cdot ||\hat{B}_k||d\omega \\
\leq\ & \sum_{k=-\infty}^{+\infty} ||[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}|| \cdot ||\hat{B}_k|| \left(\int_{-\infty}^{+\infty} ||(j(\omega + i\omega_h)I - Q)^{-*}||^2 d\omega\right)^{\frac{1}{2}} \\
& \qquad \cdot \left(\int_{-\infty}^{+\infty} ||(j(\omega + k\omega_h)I - Q)^{-1}||^2 d\omega\right)^{\frac{1}{2}} \\
=\ & \sum_{k=-\infty}^{+\infty} ||[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}|| \cdot ||\hat{B}_k|| \left(\int_{0}^{+\infty} ||e^{Q^*t}e^{-ji\omega_h t}||^2 dt\right)^{\frac{1}{2}} \\
& \qquad \cdot \left(\int_{0}^{+\infty} ||e^{Qt}e^{-jk\omega_h t}||^2 dt\right)^{\frac{1}{2}} \\
\leq\ & \sum_{k=-\infty}^{+\infty} ||[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}|| \sum_{k=-\infty}^{+\infty} ||\hat{B}_k|| \int_{0}^{+\infty} ||e^{Qt}||^2 dt := M_i \quad (i \in \mathcal{Z})
\end{aligned}
$$

by the Cauchy-Schwarz inequality and the Parseval theorem. Here, note that under the given assumptions, $\hat{C}(t)^T\hat{C}(t) \in L_{\text{CAC}}[0, h]$, and thus $\sum_{k=-\infty}^{+\infty} ||[\underline{\hat{C}}^*\underline{\hat{C}}]_{(i,k)}||, i \in \mathcal{Z}$ is finite and independent of i. Hence, M_i is independent of i. This implies by the Levi Theorem [55, p. 577] that the order interchange mentioned above is valid for each $i \in \mathcal{Z}$. **Q.E.D.**

A.4 Proof of Gronwall's Lemma

Gronwall's Lemma is well-known in the literature about asymptotic analysis of solutions of differential equations. It seems highly unnecessary to include a proof for this inequality. However, when we reviewed this lemma from references [25],[38],[61], to our surprise, there are subtle (but important) differences in the assumptions about the functions u, f and the constant number K. This observation alerts us to the strict interpretation of this lemma. Then, we believe that a complete proof for this lemma is important in understanding it properly, at least in this thesis.

The proof given below is quoted from [61, p. 41]. Let us define

$$r(t) = K + \int_{t_1}^{t} f(\tau)u(\tau)d\tau$$

Differentiating this equation gives

$$\dot{r}(t) = f(t)u(t) \leq f(t)r(t) \tag{A.4}$$

since $f(t)$ is nonnegative. Multiplying both sides of (A.4) by the positive function $e^{-\int_{t_1}^{t} f(\tau)d\tau}$, we obtain that

$$\frac{d}{dt}\left[r(t)e^{-\int_{t_1}^{t} f(\tau)d\tau}\right] \leq 0 \quad (\forall t \in [t_1, t_2])$$

Integrating both sides from t_1 to any $t \in [t_1, t_2]$ gives

$$r(t)e^{-\int_{t_1}^{t} f(\tau)d\tau} - K \leq 0 \quad (\forall t \in [t_1, t_2])$$

and this completes the proof. **Q.E.D.**

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