

An E_∞ Ring G -Spectrum Representing
the Equivariant Algebraic K -Theory
of a Bipermutative G -Category

—Summary—

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Introduction

In [5], May showed that the algebraic K -theory of a bipermutative category C can be represented by an E_∞ ring spectrum functorially constructed from C . In this article, we establish a G -equivariant generalization of this result for arbitrary finite group G . The precise statement of the main result is given by Theorem A below.

Recall from [8] that pairs of (simplicial) permutative G -categories (C, C') functorially give rise to G -prespectra $K_G(C, C')$. In particular, $K_G C = K_G(C, C)$ represents the equivariant algebraic K -theory of C ; that is, its coefficient groups $\pi_*^H K_G C$, $H < G$, are naturally isomorphic, as Mackey functors, to the higher equivariant K -groups (in the sense of [2], [10]) of C . (Compare Corollary 6 of [9].) Also, if we take the pair (CAT, GL) , where CAT denotes O , PL or Top , then $K_G(CAT, GL)$ gives an equivariant infinite delooping of the classifying space $BCAT(G)$.

Theorem A. *There exists a functor E_G from pairs of simplicial permutative G -categories to G -spectra enjoying the following properties.*

(a) *For any (C, C') , $E_G(C, C')$ is equivalent in the stable category to the G -spectrum associated to $K_G(C, C')$.*

(b) *If (C, C') is a pair of a bipermutative G -category $C = (C, \oplus, \otimes)$ and its G -stable subcategory C' closed under the operations \oplus and \otimes , then $E_G(C, C')$ has a natural structure of an E_∞ ring G -spectrum in the sense of May [3].*

§1. External Operad Actions on G -Prespectra

We introduce the notion of external operad actions on a G -prespectrum, and describe the passage from external operad actions on a G -prespectrum to (internal) operad actions on its associated G -spectrum. In this section, G will denote an arbitrary compact Lie group.

Let V be a real inner product space on which G acts through linear isometries. We assume that V contains every irreducible representation of G and denote by \mathcal{A} the indexing set $\{V^n \mid n \geq 0\}$ in the universe $V^\infty = \text{Colim } V^n$. Here $V^n = V \oplus \cdots \oplus V$ is identified with the G -invariant subspace $V^n \oplus 0$ of V^{n+1} .

For each positive integer n , let Σ_n denote the symmetric group on n letters. Since Σ_n acts on V^n by permutations, there is an embedding of Σ_n into $O_n = \text{Aut}^G(V^n)$; the group of self linear G -isometries of V^n . Denote by SV^n the one-point compactification of V^n equipped with the induced $(G \times O_n)$ -action.

Definition 1.1. Let D be a G -prespectrum with structure maps $\delta: DV^n \wedge SV^p \rightarrow DV^{n+p}$. D is called a Σ_* -prespectrum (indexed on \mathcal{A}) if each DV^n admits a base-point preserving Σ_n -action subject to the following axioms.

(1) Σ_n acts on DV^n through G -homeomorphisms. (Thus each DV^n is a based $(G \times \Sigma_n)$ -space.)

(2) For any $\sigma \in \Sigma_n$ and $\tau \in \Sigma_p$, the following diagram commutes:

$$\begin{array}{ccc} DV^n \wedge SV^p & \xrightarrow{\delta} & DV^{n+p} \\ \sigma \wedge \tau \downarrow & & \downarrow \sigma \oplus \tau \\ DV^n \wedge SV^p & \xrightarrow{\delta} & DV^{n+p}; \end{array}$$

here V^p is identified with the orthogonal complement of V^n in V^{n+p} and $\sigma \oplus \tau \in \Sigma_{n+p}$ acts on $V^{n+p} = V^n \oplus V^p$ as the product linear isometry.

A Σ_* -prespectrum $D = (D, \delta)$ is called a Σ_* -spectrum if the G -maps $\tilde{\delta}: DV^n \rightarrow \Omega^V DV^{n+1}$ adjoint to δ are homeomorphisms. A map $f: D \rightarrow D'$ of Σ_* -prespectra is a

map of G -prespectra such that each $f: DV^n \rightarrow D'V^n$ is compatible with Σ_n -actions.

Following [3], let $L: G\mathcal{P}\mathcal{A} \rightarrow G\mathcal{S}\mathcal{A}$ denote the left adjoint to the inclusion $G\mathcal{S}\mathcal{A} \subset G\mathcal{P}\mathcal{A}$ of G -spectra into G -prespectra indexed on \mathcal{A} . It can be shown that for any G -prespectrum $D \in G\mathcal{P}\mathcal{A}$, LD has a unique structure of Σ_* -prespectrum, and the unit $\eta: D \rightarrow LD$ of the adjunction is a map of Σ_* -prespectra with respect to any Σ_* -prespectrum structure on D .

We now introduce the notion of external G -operad actions on Σ_* -prespectra. Recall from [3] that a G -operad is a sequence of $(G \times \Sigma_j)$ -spaces \mathcal{C}_j for $j \geq 0$, with G acting on the left and Σ_j acting on the right and with \mathcal{C}_0 a single point, together with a G -fixed unit element $1 \in \mathcal{C}_1^G$ and suitably associative, unital, and equivariant structure maps

$$\gamma: \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \rightarrow \mathcal{C}_j, \quad j = j_1 + \cdots + j_k.$$

Let \mathcal{C}_j^+ denote the union of \mathcal{C}_j and a disjoint basepoint.

Definition 1.2. Let $D = (D, \delta)$ be a Σ_* -prespectrum and let \mathcal{C} be a G -operad. We say that D is an external \mathcal{C} -prespectrum, or \mathcal{C} acts externally on D , if there are based G -maps

$$\xi_j: \mathcal{C}_j^+ \wedge DV^{n_1} \wedge \cdots \wedge DV^{n_j} \rightarrow DV^{n_1 + \cdots + n_j}$$

for $j \geq 0$ and $n_1, \dots, n_j \geq 0$ satisfying the following conditions.

(1.1) For any $c \in \mathcal{C}_j$, $x_s \in DV^{n_s}$ and $\sigma_s \in \Sigma_{n_s}$ ($1 \leq s \leq j$) we have

$$\xi_j(c \wedge \sigma_1 x_1 \wedge \cdots \wedge \sigma_j x_j) = (\sigma_1 \oplus \cdots \oplus \sigma_j) \xi_j(c \wedge x_1 \wedge \cdots \wedge x_j).$$

(1.2) For given $\sigma \in \Sigma_j$, let $\sigma(n_1, \dots, n_j) \in \Sigma_{n_1 + \cdots + n_j}$ denote the permutation of j blocks $(v_1, \dots, v_j) \mapsto (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(j)})$, $v_s \in V^{n_s}$, $1 \leq s \leq j$. Then we have

$$\xi_j(c \sigma^{-1} \wedge x_{\sigma^{-1}(1)} \wedge \cdots \wedge x_{\sigma^{-1}(j)}) = \sigma(n_1, \dots, n_j) \xi_j(c \wedge x_1 \wedge \cdots \wedge x_j).$$

for all $c \in \mathcal{C}_j$ and $x_s \in DV^{n_s}$, $1 \leq s \leq j$.

(1.3) For $c \in \mathcal{C}_j$, $x_s \in DV^{n_s}$ and $v_s \in SV^{p_s}$ ($1 \leq s \leq j$) we have

$$\begin{aligned} \xi_j(c \wedge \delta(x_1 \wedge v_1) \wedge \cdots \wedge \delta(x_j \wedge v_j)) \\ = \tau(n_1, \dots, n_j; p_1, \dots, p_j) \delta(\xi_j(c \wedge x_1 \wedge \cdots \wedge x_j) \wedge (v_1 \wedge \cdots \wedge v_j)) \end{aligned}$$

where $\tau(n_1, \dots, n_j; p_1, \dots, p_j) \in \Sigma_{n_1 + \cdots + n_j + p_1 + \cdots + p_j}$ represents the shuffle isomorphism $\left(\bigoplus_{s=1}^j V^{n_s} \right) \oplus \left(\bigoplus_{s=1}^j V^{p_s} \right) \cong \bigoplus_{s=1}^j V^{n_s + p_s}$.

(1.4) The composite $\xi_1 \iota_1: DV^n \rightarrow \mathcal{C}_1^+ \wedge DV^n \rightarrow DV^n$ is the identity map, where ι_1 denotes the G -map $x \mapsto 1 \wedge x$, $x \in DV^n$.

(1.5) The diagram

$$\begin{array}{ccc} \mathcal{C}_k^+ \wedge \left(\bigwedge_{s=1}^k \mathcal{C}_{j_s}^+ \wedge \left(\bigwedge_{t=1}^{j_s} DV^{n_{st}} \right) \right) & \xrightarrow{1 \wedge \left(\bigwedge_{s=1}^k \xi_{j_s} \right)} & \mathcal{C}_k^+ \wedge \left(\bigwedge_{s=1}^k DV^{n_s} \right) \\ \zeta \downarrow & & \downarrow \xi_k \\ \mathcal{C}_j^+ \wedge \left(\bigwedge_{s=1}^k \bigwedge_{t=1}^{j_s} DV^{n_{st}} \right) & \xrightarrow{\xi_j} & DV^n \end{array}$$

commutes, where $j = j_1 + \cdots + j_k$, $n_s = n_{s1} + \cdots + n_{sj_s}$, $n = n_1 + \cdots + n_k$ and ζ denotes the composite of $\gamma \wedge 1$ with the evident shuffle isomorphism.

Just as external smash products determine internal smash products, external operad actions determine (internal) operad actions in the sense of [3, Chapter VII, §2]. Precisely, we have

Theorem 1.3. *Let G be a compact Lie group and let \mathcal{C} be a G -operad. Suppose there is a morphism of G -operads $\chi: \mathcal{C} \rightarrow \mathcal{L}$ into the linear isometries operad of V^∞ . Then every external \mathcal{C} -action on a Σ_* -prespectrum D functorially determines on LD a structure of \mathcal{C} -spectrum in the sense of May.*

In particular, by taking χ to be the projection $\mathcal{L} \times \mathcal{C} \rightarrow \mathcal{L}$, we see that any external \mathcal{C} -action on D gives rise to an $(\mathcal{L} \times \mathcal{C})$ -action on LD .

Corollary 1.4. *Let \mathcal{C} be an E_∞ G -operad. If D is an external \mathcal{C} -prespectrum then LD is an E_∞ ring G -spectrum.*

§2. G -Prespectra Associated to Γ_G^∞ -Spaces

We introduce the notion of operad ring Γ_G^∞ -space and describe the passage from operad ring Γ_G^∞ -spaces to external operad ring G -prespectra. Throughout, G -equivalence means weak G -equivalence.

Let \mathcal{T}_G be the category of non-degenerately based G -spaces and basepoint preserving maps with G acting on morphisms by conjugation, and let Γ_G denote the G -stable full subcategory of \mathcal{T}_G consisting of those based finite G -sets \mathbf{s}_ρ with underlying set $\mathbf{s} = \{0, 1, \dots, s\}$ and with G -action given by a homomorphism $\rho: G \rightarrow \Sigma_s$ (cf. [8]). If n is a positive integer, a G -equivariant functor from $\Gamma_G^n = \prod_{i=1}^n \Gamma_G$ to \mathcal{T}_G is called a Γ_G^n -space. We also denote by Γ_G^0 the singleton set $\{1\}$; thus any G -space $A \in \mathcal{T}_G$ can be identified with a G -functor $\Gamma_G^0 \rightarrow \mathcal{T}_G$, called Γ_G^0 -space, which assigns A to 1 .

Definition 2.1. Let A be a Γ_G^n -space and let X be a (contravariant) G -functor $\Gamma_G^n \rightarrow \mathcal{T}_G^{\text{op}}$. We put

$$E_n(A; X) = B(X, \Gamma_G^n, A) / B(*, \Gamma_G^n, A) \cup B(X, \Gamma_G^n, *),$$

where $B(X, \Gamma_G^n, A) \in \mathcal{T}_G$ denotes the geometric realization of the two-sided bar complex $B_*(X, \Gamma_G^n, A)$, and the inclusion $B(*, \Gamma_G^n, A) \cup B(X, \Gamma_G^n, *) \rightarrow B(X, \Gamma_G^n, A)$ is a G -cofibration induced by the morphisms $* \rightarrow A$ and $* \rightarrow X$ defined by the inclusions of the non-degenerate base-points of $A(S_1, \dots, S_n)$ and $X(S_1, \dots, S_n)$. Evidently we have $E_0(A; X) = X \wedge A$ for any G -spaces A and X .

Given G -functors $A: \Gamma_G^m \rightarrow \mathcal{T}_G$, $A': \Gamma_G^n \rightarrow \mathcal{T}_G$ and $X: \Gamma_G^m \rightarrow \mathcal{T}_G^{\text{op}}$, $X': \Gamma_G^n \rightarrow \mathcal{T}_G^{\text{op}}$, there is a G -map

$$\begin{aligned} \mu': B(X, \Gamma_G^m, A) \times B(X', \Gamma_G^n, A') &\cong B(X \times X', \Gamma_G^m \times \Gamma_G^n, A \times A') \\ &\rightarrow B(X \wedge X', \Gamma_G^{m+n}, A \wedge A') \end{aligned}$$

induced by the evident natural transformations $A \times A' \rightarrow A \wedge A'$ and $X \times X' \rightarrow X \wedge X'$.

It is easily checked that μ' induces a natural G -map

$$\mu: E_m(A; X) \wedge E_n(A'; X') \rightarrow E_{m+n}(A \wedge A'; X \wedge X')$$

such that

Lemma 2.2. *The following diagram commutes for any A, A', A'' and X, X', X'' .*

$$\begin{array}{ccc} E_m(A; X) \wedge E_n(A'; X') \wedge E_p(A''; X'') & \xrightarrow{1 \wedge \mu} & E_m(A; X) \wedge E_{n+p}(A' \wedge A''; X' \wedge X'') \\ \mu \wedge 1 \downarrow & & \downarrow \mu \\ E_{m+n}(A \wedge A'; X \wedge X') \wedge E_p(A''; X'') & \xrightarrow{\mu} & E_{m+n+p}(A \wedge A' \wedge A''; X \wedge X' \wedge X''). \end{array}$$

Now let $\Gamma_G^\infty = \text{Colim } \Gamma_G^n$ be the colimit of the inclusions $\Gamma_G^n \rightarrow \Gamma_G^{n+1}, (S_1, \dots, S_n) \mapsto (S_1, \dots, S_n, \mathbf{1})$. Clearly, any G -functor $A: \Gamma_G^\infty \rightarrow \mathcal{T}_G$ determines and is determined by Γ_G^n -spaces $A^{(n)} = A|_{\Gamma_G^n}$ together with natural isomorphisms $\varepsilon: A^{(n)} \cong A^{(n+1)}|_{\Gamma_G^n}$.

Definition 2.3. A (symmetric) Γ_G^∞ -space is a G -functor $A: \Gamma_G^\infty \rightarrow \mathcal{T}_G$ together with natural isomorphisms $\theta_\sigma: A^{(n)} \rightarrow A^{(n)}\sigma$, i.e. G -homeomorphisms

$$A^{(n)}(S_1, \dots, S_n) \rightarrow A^{(n)}\sigma(S_1, \dots, S_n) = A^{(n)}(S_{\sigma^{-1}(1)}, \dots, S_{\sigma^{-1}(n)})$$

natural in (S_1, \dots, S_n) , such that for any $\sigma, \sigma' \in \Sigma_n, \tau \in \Sigma_p$ and $a \in A^{(n)}(S_1, \dots, S_n)$ we have

$$\theta_1 a = a, \quad \theta_{\sigma\sigma'} a = \theta_\sigma \theta_{\sigma'} a, \quad \theta_{\sigma \oplus \tau} \varepsilon^p a = \varepsilon^p \theta_\sigma a$$

where $\varepsilon^p: A^{(n)} \cong A^{(n+p)}|_{\Gamma_G^n}$ denotes the p -fold composite of ε . A morphism of Γ_G^∞ -spaces $(A, \{\theta_\sigma\}) \rightarrow (A', \{\theta'_\sigma\})$ is a natural transformation $F: A \rightarrow A'$ such that $F\theta_\sigma = \theta'_\sigma F$ holds for every $\sigma \in \Sigma_n$. A Γ_G^∞ -space A is called to be *special* if for every $(S_1, \dots, S_n) \in \Gamma_G^n$ the Γ_G -space $S \mapsto A(S_1, \dots, S_n, S)$ is special in the sense of [8, Definition 1.3]; that is, the canonical maps

$$A(S_1, \dots, S_n, S) \rightarrow A(S_1, \dots, S_n, \mathbf{1})^S \cong A(S_1, \dots, S_n)^S$$

induced by the projections $\text{pr}_s: S \rightarrow \mathbf{1}$, $\text{pr}_s^{-1}(1) = \{s\}$ ($s \neq 0$) are G -equivalences.

We now assign to each Γ_G^∞ -space A a G -prespectrum $T_G A \in G\mathcal{P}\mathcal{A}$. For $n \geq 1$, let $SV^{(n)}$ denote the contravariant G -functor $(S_1, \dots, S_n) \mapsto SV^{S_1} \wedge \dots \wedge SV^{S_n}$, and let $SV^{(0)} = S^0$. Then we put

$$T_G A(V^n) = E_n(A^{(n)}; SV^{(n)});$$

in particular, $T_G A(0) = A^{(0)} = A(\mathbf{1})$. The structure map $T_G A(V^n) \wedge SV \rightarrow T_G A(V^{n+1})$ is defined to be the composite

$$\begin{aligned} E_n(A^{(n)}; SV^{(n)}) \wedge SV &\xrightarrow{\mu} E_n(A^{(n)}; SV^{(n)} \wedge SV) \\ &\cong E_n(A^{(n+1)} | \Gamma_G^n; SV^{(n+1)} | \Gamma_G^n) \xrightarrow{\Sigma} E_{n+1}(A^{(n+1)}; SV^{(n+1)}), \end{aligned}$$

where Σ denotes the natural G -map induced by the inclusion $\Gamma_G^n \subset \Gamma_G^{n+1}$.

$T_G A$ becomes a Σ_* -prespectrum if we let Σ_n act on $T_G A(V^n)$ through the Σ_n -action on the bar complex $B_*(SV^{(n)}, \Gamma_G^n, A^{(n)})$;

$$\sigma(v_1 \wedge \dots \wedge v_n, (f_1, \dots, f_n), a) = (v_{\sigma^{-1}(1)} \wedge \dots \wedge v_{\sigma^{-1}(n)}, (f_{\sigma^{-1}(1)}, \dots, f_{\sigma^{-1}(n)}), \theta_\sigma a)$$

for $\sigma \in \Sigma_n$, $v_i \in SV^{T_i}$, $f_i: S_i \rightarrow \dots \rightarrow T_i \in N_* \Gamma_G$, $1 \leq i \leq n$, and $a \in A(S_1, \dots, S_n)$.

Clearly the assignment $A \mapsto T_G A$ is natural in A , and we get a functor T_G from Γ_G^∞ -spaces to Σ_* -prespectra.

Recall from [8, §2] that the restriction $A^{(1)} = A | \Gamma_G$ of a Γ_G^∞ -space A determines a G -prespectrum $S_G A^{(1)} \in G\mathcal{P}\mathcal{A}$ with

$$S_G A^{(1)}(V^n) = B((SV^n)^{(1)}, \Gamma_G, A^{(1)}) / B(*, \Gamma_G, A^{(1)})$$

and with structure maps $S_G A^{(1)}(V^n) \wedge SV \rightarrow S_G A^{(1)}(V^{n+1})$ induced by the natural transformation $\alpha: (SV^n)^{(1)} \wedge SV \rightarrow (SV^{n+1})^{(1)}$ which takes each $f \wedge v \in (SV^n)^S \wedge SV$, $S \in \Gamma_G$, to $\alpha(f \wedge v) \in (SV^{n+1})^S$, $\alpha(f \wedge v)(s) = f(s) \wedge v$ for $s \in S$. Theorem B of

[8] implies that if $A^{(1)}$ is special then $S_G A^{(1)}$ is an almost Ω - G -spectrum, that is, the structure maps $S_G A^{(1)}(V^n) \rightarrow \Omega^V S_G A^{(1)}(V^{n+1})$ are G -equivalences for $n > 0$, and that the adjoint $A^{(1)}(\mathbf{1}) \rightarrow \Omega^V S_G A^{(1)}(V)$ to the natural G -map $A^{(1)}(\mathbf{1}) \wedge SV \rightarrow B(SV^{(1)}, \Gamma_G, A^{(1)})/B(*, \Gamma_G, A^{(1)})$ is a G -equivalence if $A^{(1)}(\mathbf{1})$ is group-like.

Theorem 2.4. *Let A be a special Γ_G^∞ -space. Then $T_G A$ is an almost Ω - G -spectrum and there is a natural equivalence $LS_G A^{(1)} \simeq LT_G A$ defined in the stable category $\bar{h}GSA$.*

Definition 2.5. Let \mathcal{C} be a G -operad. A \mathcal{C} -action on a Γ_G^∞ -space A consists of morphisms of Γ_G^n -spaces

$$\tilde{\xi}_j: \mathcal{C}_j^+ \wedge A^{(n_1)} \wedge \cdots \wedge A^{(n_j)} \rightarrow A^{(n)}, \quad n = n_1 + \cdots + n_j,$$

for $j \geq 0$ and $n_1, \dots, n_j \geq 0$ satisfying the following conditions.

(2.1) For $c \in \mathcal{C}_j$, $a_s \in A(S_{s_1}, \dots, S_{s_{n_s}})$ and $\sigma_s \in \Sigma_{n_s}$ ($1 \leq s \leq j$), we have

$$\tilde{\xi}_j(c \wedge \theta_{\sigma_1} a_1 \wedge \cdots \wedge \theta_{\sigma_j} a_j) = \theta_{\sigma_1 \oplus \cdots \oplus \sigma_j} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.2) For $\sigma \in \Sigma_j$, we have

$$\tilde{\xi}_j(c \sigma^{-1} \wedge a_{\sigma^{-1}(1)} \wedge \cdots \wedge a_{\sigma^{-1}(j)}) = \theta_{\sigma(n_1, \dots, n_j)} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.3) For $p_1, \dots, p_j \geq 0$, we have

$$\tilde{\xi}_j(c \wedge \varepsilon^{p_1} a_1 \wedge \cdots \wedge \varepsilon^{p_j} a_j) = \theta_{\tau(n_1, \dots, n_j; p_1, \dots, p_j)} \varepsilon^{p_1 + \cdots + p_j} \tilde{\xi}_j(c \wedge a_1 \wedge \cdots \wedge a_j).$$

(2.4) The composite $\tilde{\xi}_1 \iota_1: A^{(n)} \rightarrow \mathcal{C}_1^+ \wedge A^{(n)} \rightarrow A^{(n)}$ is the identity map, where ι_1 denotes the natural transformation $A(S_1, \dots, S_n) \rightarrow \mathcal{C}_1^+ \wedge A(S_1, \dots, S_n)$, $a \mapsto 1 \wedge a$.

(2.5) The following diagram commutes, where $j = j_1 + \cdots + j_k$, $n_s = n_{s_1} + \cdots + n_{s_{j_s}}$,

$$n = n_1 + \cdots + n_k:$$

$$\begin{array}{ccc} \mathcal{C}_k^+ \wedge \left(\bigwedge_{s=1}^k \mathcal{C}_{j_s}^+ \wedge \left(\bigwedge_{t=1}^{j_s} A^{(n_{st})} \right) \right) & \xrightarrow{1 \wedge \left(\bigwedge_{s=1}^k \tilde{\xi}_{j_s} \right)} & \mathcal{C}_k^+ \wedge \left(\bigwedge_{s=1}^k A^{(n_s)} \right) \\ \downarrow \zeta & & \downarrow \tilde{\xi}_k \\ \mathcal{C}_j^+ \wedge \left(\bigwedge_{s=1}^k \bigwedge_{t=1}^{j_s} A^{(n_{st})} \right) & \xrightarrow{\tilde{\xi}_j} & A^{(n)}. \end{array}$$

For given \mathcal{C} -action on A , we define G -maps

$$\xi_j: \mathcal{C}_j^+ \wedge T_G A(V^{n_1}) \wedge \cdots \wedge T_G A(V^{n_j}) \rightarrow T_G A(V^n)$$

by the following composites:

$$\begin{aligned} E_0(\mathcal{C}_j^+; S^0) \wedge E_{n_1}(A^{(n_1)}; SV^{(n_1)}) \wedge \cdots \wedge E_{n_j}(A^{(n_j)}; SV^{(n_j)}) \\ \xrightarrow{\mu} E_n(\mathcal{C}_j^+ \wedge A^{(n_1)} \wedge \cdots \wedge A^{(n_j)}; SV^{(n)}) \xrightarrow{E_n(\tilde{\xi}_j, 1)} E_n(A^{(n)}; SV^{(n)}). \end{aligned}$$

It is now easy to see that these ξ_j define an external \mathcal{C} -action on $T_G A$. Thus we have

Theorem 2.6. *Let \mathcal{C} be a E_∞ G -operad. If \mathcal{C} acts on a Γ_G^∞ -space A then $LT_G A$ is an E_∞ ring G -spectrum (with natural $(\mathcal{L} \times \mathcal{C})$ -action).*

§3. Proof of Theorem A

We are now ready to prove Theorem A. First recall from [8] the definition of the functor K_G .

Given a pair of permutative G -categories (C, C') , let $B_G(C, C')$ be the full subcategory of the functor category $\text{Cat}(EG, C)$ with objects those functors $F: EG \rightarrow C$ which factor through C' . Here EG denotes the translation category of G , and $B_G(C, C')$ is regarded as a permutative G -category with respect to the G -action

$$(gF)(x) = gF(xg) \quad \text{for } g \in G, x \in \text{ob}EG = G$$

and the sum operation $(F, F') \mapsto F \oplus F'$;

$$(F \oplus F')(x) = F(x) \oplus F'(x) \quad \text{for } x \in \text{ob}EG.$$

For any permutative G -category M , let M^\wedge denote the Γ_G -category

$$S \mapsto M^\wedge(S) = \text{Moncat}(\mathcal{P}S, M),$$

where $\mathcal{P}S$ is the set of subsets of $S - \{0\}$ viewed as a partial G -monoid under disjoint union $(U, U') \mapsto U \sqcup U'$ for $U, U' \in \mathcal{P}S$ with $U \cap U' = \emptyset$, and $\text{Moncat}(\mathcal{P}S, M)$ denotes the category of monoidal functors from $\mathcal{P}S$ to M with G acting by conjugation of morphisms.

In contrast with the non-equivariant case, the associated Γ_G -space

$$|M^\wedge|: S \mapsto |M^\wedge(S)|$$

is not necessarily special. However, we can prove that $|B_G(C, C')^\wedge|$ is special for every (C, C') , and $K_G(C, C')$ is defined to be the almost Ω - G -spectrum $S_G|B_G(C, C')^\wedge|$.

Now, Theorem A is a consequence of Theorems 2.4, 2.6 and the following

Proposition 3.1. *There exists a functor A from permutative G -categories to Γ_G^∞ -spaces and an E_∞ -operad \mathcal{D} enjoying the following properties:*

- (a) *For any permutative G -category M , $AM^{(1)} = |M^\wedge|$.*
- (b) *For any pair of permutative G -categories (C, C') , $AB_G(C, C')$ is a special Γ_G^∞ -space.*
- (c) *If (C, C') is a pair of bipermutative G -categories then $AB_G(C, C')$ admits a natural \mathcal{D} -action.*

The remainder of this section will be devoted to the proof of the proposition above. As in [7, §2, 2.2], we assign to each permutative G -category M and $(S_1, \dots, S_n) \in \Gamma_G^n$ a category $M^{(n)}(S_1, \dots, S_n)$ defined as follows.

Objects of $M^{(n)}(S_1, \dots, S_n)$ are functors $F: \mathcal{P}S_1 \times \dots \times \mathcal{P}S_n \rightarrow M$ together with natural isomorphisms

$$\begin{aligned} \delta_i: F(U_1, \dots, U_{i-1}, U'_i, U_{i+1}, \dots, U_n) \oplus F(U_1, \dots, U_{i-1}, U''_i, U_{i+1}, \dots, U_n) \\ \xrightarrow{\cong} F(U_1, \dots, U_{i-1}, U'_i \sqcup U''_i, U_{i+1}, \dots, U_n) \end{aligned}$$

for $1 \leq i \leq n$ satisfying the following two conditions.

(C1) F is monoidal in each variable, that is, if we fix objects $U_k \in \mathcal{PS}_k$ for $k \neq i$, and write $F_i(U) = F(U_1, \dots, U_{i-1}, U, U_{i+1}, \dots, U_n)$, then the following diagrams are commutative:

$$\begin{array}{ccc} F_i(U) \oplus F_i(U') \oplus F_i(U'') & \xrightarrow{1 \oplus \delta_i} & F_i(U) \oplus F_i(U' \sqcup U'') \\ \delta_i \oplus 1 \downarrow & & \downarrow \delta_i \\ F_i(U' \sqcup U') \oplus F_i(U'') & \xrightarrow{\delta_i} & F_i(U \sqcup U' \sqcup U''), \end{array}$$

$$\begin{array}{ccc} F_i(U) \oplus 0 & \xlongequal{\quad} & F_i(U) \oplus F_i(\emptyset) & F_i(U) \oplus F_i(U') & \xrightarrow{\delta_i} & F_i(U \sqcup U') \\ \parallel & & \downarrow \delta_i = \text{id} & c \downarrow & & F_i(c) \downarrow \\ F_i(U) & \xlongequal{\quad} & F_i(U \sqcup \emptyset), & F_i(U') \oplus F_i(U) & \xrightarrow{\delta_i} & F_i(U' \sqcup U). \end{array}$$

(C2) For $1 \leq i < j \leq n$ and $U_k \in \mathcal{PS}_k$ with $k \neq i, j$, write

$$F_{ij}(U, W) = F(U_1, \dots, U_{i-1}, U, U_{i+1}, \dots, U_{j-1}, W, U_{j+1}, \dots, U_n).$$

Then the following diagram commutes, where $U \cap U' = \emptyset$ and $W \cap W' = \emptyset$:

$$\begin{array}{ccc} F_{ij}(U, W) \oplus F_{ij}(U, W') \oplus F_{ij}(U', W) \oplus F_{ij}(U', W') & \xrightarrow{\delta_j \oplus \delta_j} & F_{ij}(U, W \sqcup W') \oplus F_{ij}(U', W \sqcup W') \\ (\delta_i \oplus \delta_i)(1 \oplus c \oplus 1) \downarrow & & \downarrow \delta_i \\ F_{ij}(U \sqcup U', W) \oplus F_{ij}(U \sqcup U', W') & \xrightarrow{\delta_j} & F_{ij}(U \sqcup U', W \sqcup W'). \end{array}$$

Equivalently, the assignment $W \mapsto (U \mapsto F_{ij}(U, W))$ defines a monoidal functor from \mathcal{PS}_j to $\text{Moncat}(\mathcal{PS}_i, M)$.

Given objects $F = (F; \delta_1, \dots, \delta_n)$ and $F' = (F'; \delta'_1, \dots, \delta'_n)$ in $M^{(n)}(S_1, \dots, S_n)$, morphisms from F to F' are natural transformations $\alpha: F \rightarrow F'$ such that the following diagrams commute for $1 \leq i \leq n$:

$$\begin{array}{ccc} F_i(\emptyset) & \xlongequal{\quad} & 0 & F_i(U) \oplus F_i(U') & \xrightarrow{\delta_i} & F_i(U \sqcup U') \\ \alpha = \text{id} \downarrow & & \parallel & \alpha \oplus \alpha \downarrow & & \downarrow \alpha \\ F'_i(\emptyset) & \xlongequal{\quad} & 0, & F'_i(U) \oplus F'_i(U') & \xrightarrow{\delta'_i} & F'_i(U \sqcup U'). \end{array}$$

The $M^{(n)}(S_1, \dots, S_n)$ becomes a permutative G -category under the G -action

$$(gF)(U_1, \dots, U_n) = gF(g^{-1}U_1, \dots, g^{-1}U_n) \quad \text{for } g \in G,$$

and the sum operation $(F, F') \mapsto F \oplus F'$;

$$(F \oplus F')(U_1, \dots, U_n) = F(U_1, \dots, U_n) \oplus F'(U_1, \dots, U_n).$$

Moreover, any morphism $f = (f_1, \dots, f_n): (S_1, \dots, S_n) \rightarrow (T_1, \dots, T_n)$ in Γ_G^n induces a morphism of permutative G -categories

$$M^{(n)}(S_1, \dots, S_n) \rightarrow M^{(n)}(T_1, \dots, T_n), \quad F \mapsto F(\mathcal{P}f_1 \times \dots \times \mathcal{P}f_n),$$

where $\mathcal{P}f_i$ denotes the G -map $\mathcal{P}T_i \rightarrow \mathcal{P}S_i$, $W \mapsto f_i^{-1}(W)$ for $1 \leq i \leq n$. Thus we get a G -equivariant functor $M^{(n)}$ from Γ_G^n to the category of permutative G -categories and (non-equivariant) morphisms with G acting by conjugation of morphisms.

One easily observes that the adjunction

$$\text{Cat}(\mathcal{P}S_1 \times \dots \times \mathcal{P}S_n \times \mathcal{P}S_{n+1}, M) \cong \text{Cat}(\mathcal{P}S_{n+1}, \text{Cat}(\mathcal{P}S_1 \times \dots \times \mathcal{P}S_n, M))$$

induces an isomorphism

$$M^{(n+1)}(S_1, \dots, S_n, S_{n+1}) \cong \text{Moncat}(\mathcal{P}S_{n+1}, M^{(n)}(S_1, \dots, S_n))$$

natural in S_1, \dots, S_{n+1} . Thus we have

Lemma 3.2. *For any permutative G -category M , $M^{(1)} = M^\wedge$ and there are isomorphisms of permutative G -categories*

$$M^{(n+1)}(S_1, \dots, S_n, S_{n+1}) \cong M^{(n)}(S_1, \dots, S_n)^\wedge(S_{n+1})$$

natural in S_1, \dots, S_{n+1} and M .

In particular, $M^{(n+1)}(S_1, \dots, S_n, \mathbf{1})$ can be identified with $M^{(n)}(S_1, \dots, S_n)$ via the natural isomorphism

$$M^{(n)}(S_1, \dots, S_n) \hat{=} M^{(n)}(S_1, \dots, S_n).$$

Definition 3.3. For given permutative G -category M , we denote by AM the Γ_G^∞ -space with

$$AM(S_1, \dots, S_n) = |M^{(n)}(S_1, \dots, S_n)|,$$

and with $\theta_\sigma: AM(S_1, \dots, S_n) \rightarrow AM(S_{\sigma^{-1}(1)}, \dots, S_{\sigma^{-1}(n)})$ induced by the permutation $\mathcal{P}S_{\sigma^{-1}(1)} \times \dots \times \mathcal{P}S_{\sigma^{-1}(n)} \rightarrow \mathcal{P}S_1 \times \dots \times \mathcal{P}S_n$.

We will show that the A enjoys the properties stated in Proposition 3.1. It is clear, by the definition, that (a) holds. The property (b) follows from [8, Proposition 2.2] and the fact that there are evidently defined natural isomorphisms

$$\psi: B_G(C, C')^{(n)}(S_1, \dots, S_n) \cong B_G(C^{(n)}(S_1, \dots, S_n), C'^{(n)}(S_1, \dots, S_n)).$$

To see that (c) holds, let us take an E_∞ G -operad \mathcal{D} with

$$\mathcal{D}_j = |\text{Cat}(EG, E\Sigma_j)| \quad \text{for } j \geq 0,$$

and with structure maps $\gamma: \mathcal{D}_k \times \mathcal{D}_{j_1} \times \dots \times \mathcal{D}_{j_k} \rightarrow \mathcal{D}_{j_1 + \dots + j_k}$ induced by the evident maps $\tilde{\gamma}: \Sigma_k \times \Sigma_{j_1} \times \dots \times \Sigma_{j_k} \rightarrow \Sigma_{j_1 + \dots + j_k}$ (cf. [8, §2, Remark]). We need show that any pair of bipermutative G -categories (C, C') functorially determines morphisms of Γ_G^n -spaces

$$\tilde{\xi}_j: \mathcal{D}_j^+ \wedge AB_G(C, C')^{(n_1)} \wedge \dots \wedge AB_G(C, C')^{(n_j)} \rightarrow AB_G(C, C')^{(n_1 + \dots + n_j)}$$

satisfying the conditions of Definition 2.5.

First let

$$\tilde{\xi}_j'': E\Sigma_j \times C^{(n_1)} \times \dots \times C^{(n_j)} \rightarrow C^{(n)}, \quad n = n_1 + \dots + n_j$$

be a morphism of Γ_G^n -categories defined on objects by

$$\tilde{\xi}_j''(\nu, F_1, \dots, F_j) = \theta_{\nu(n_1, \dots, n_j)^{-1}}(F_{\nu^{-1}(1)} \otimes \dots \otimes F_{\nu^{-1}(j)})$$

for $\nu \in \Sigma_j = \text{ob}E\Sigma_j$ and $F_s \in C^{(n_s)}(S_{s1}, \dots, S_{sn_s})$, $1 \leq s \leq j$. That $\tilde{\xi}_j''(\nu, F_1, \dots, F_j)$ defines an object of $C^{(n)}(S_{11}, \dots, S_{1n_1}, \dots, S_{j1}, \dots, S_{jn_j})$ follows from the fact that the j -fold multiplication $C^j \rightarrow C$, $(x_1, \dots, x_j) \mapsto x_1 \otimes \dots \otimes x_j$, together with the isomorphisms

$$\begin{aligned} (x_1 \otimes \dots \otimes x_{i-1} \otimes x'_i \otimes x_{i+1} \otimes \dots \otimes x_j) \oplus (x_1 \otimes \dots \otimes x_{i-1} \otimes x''_i \otimes x_{i+1} \otimes \dots \otimes x_j) \\ \cong x_1 \otimes \dots \otimes x_{i-1} \otimes (x'_i \oplus x''_i) \otimes x_{i+1} \otimes \dots \otimes x_j \end{aligned}$$

induced by the distributive laws satisfy the conditions similar to (C1) and (C2) with $\mathcal{P}S_1 \times \dots \times \mathcal{P}S_n$ and \sqcup replaced by C^j and \oplus respectively. (Here we need not assume that the right or left distributive law holds strictly; cf. [5].) Because $B_G(C, C')^{(m)}$ is naturally isomorphic to $B_G(C^{(m)}, C'^{(m)})$ under ψ , $\tilde{\xi}_j''$ induces

$$\tilde{\xi}_j': \text{Cat}(EG, E\Sigma_j) \times B_G(C, C')^{(n_1)} \times \dots \times B_G(C, C')^{(n_j)} \rightarrow B_G(C, C')^{(n)}.$$

It is evident that $\tilde{\xi}_j''(\sigma; F_1, \dots, F_j) = 0$ if some $F_s = 0$; hence its realization $|\tilde{\xi}_j''|$ induces a morphism of Γ_G^n -spaces

$$\tilde{\xi}_j: \mathcal{D}_j^+ \wedge AB_G(C, C')^{(n_1)} \wedge \dots \wedge AB_G(C, C')^{(n_j)} \rightarrow AB_G(C, C')^{(n)}.$$

The next lemma implies that the morphisms $\tilde{\xi}_j$ define a \mathcal{D} -action on $AB_G(C, C')$, and completes the proof of Proposition 3.1.

Lemma 3.4. *The morphisms $\tilde{\xi}_j''$ enjoy the following properties.*

(3.1) For $\nu \in E\Sigma_j$, $F_s \in C^{(n_s)}(S_{s1}, \dots, S_{sn_s})$ and $\sigma_s \in \Sigma_{n_s}$ ($1 \leq s \leq j$),

$$\tilde{\xi}_j''(\nu, \theta_{\sigma_1} F_1, \dots, \theta_{\sigma_j} F_j) = \theta_{\sigma_1 \oplus \dots \oplus \sigma_j} \tilde{\xi}_j''(\nu, F_1, \dots, F_j).$$

(3.2) For $\sigma \in \Sigma_j$, $\tilde{\xi}_j''(\nu\sigma^{-1}, F_{\sigma^{-1}(1)}, \dots, F_{\sigma^{-1}(j)}) = \theta_{\sigma(n_1, \dots, n_j)} \tilde{\xi}_j''(\nu, F_1, \dots, F_j)$.

(3.3) $\tilde{\xi}_j''(\nu, \varepsilon^{p_1} F_1, \dots, \varepsilon^{p_j} F_j) = \theta_{\tau(n_1, \dots, n_j; p_1, \dots, p_j)} \varepsilon^{p_1 + \dots + p_j} \tilde{\xi}_j''(\nu, F_1, \dots, F_j)$.

(3.4) The composite $\tilde{\xi}_1'' \iota_1: C^{(n)} \rightarrow E\Sigma_1 \times C^{(n)} \rightarrow C^{(n)}$ is the identity map, where $\iota_1: C^{(n)}(S_1, \dots, S_n) \rightarrow \{1\} \times C^{(n)}(S_1, \dots, S_n)$, $F \mapsto 1 \times F$.

(3.5) The following diagram commutes, where $j = j_1 + \dots + j_k$, $n_s = n_{s1} + \dots + n_{sj_s}$, $n = n_1 + \dots + n_k$.

$$\begin{array}{ccc} E\Sigma_j \times \left(\prod_{s=1}^k E\Sigma_{j_s} \times \left(\prod_{t=1}^{j_s} C^{(n_{st})} \right) \right) & \xrightarrow{1 \times \left(\prod_{s=1}^k \tilde{\xi}_{j_s}'' \right)} & E\Sigma_k \times \left(\prod_{s=1}^k C^{(n_s)} \right) \\ \downarrow \wr & & \downarrow \tilde{\xi}_k'' \\ E\Sigma_j \times \left(\prod_{s=1}^k \prod_{t=1}^{j_s} C^{(n_{st})} \right) & \xrightarrow{\tilde{\xi}_j''} & C^{(n)}. \end{array}$$

Proof. It is evident that (3.4) holds. The property (3.1) follows from the equality

$$\begin{aligned} \tilde{\xi}_j''(\nu, \theta_{\sigma_1} F_1, \dots, \theta_{\sigma_j} F_j) &= \theta_{\nu(n_1, \dots, n_j)}^{-1} \left(\bigotimes_{s=1}^j \theta_{\sigma_{\nu^{-1}(s)}} F_{\nu^{-1}(s)} \right) \\ &= \theta_{\nu(n_1, \dots, n_j)}^{-1} \theta_{\sigma_{\nu^{-1}(1)} \oplus \dots \oplus \sigma_{\nu^{-1}(j)}} \left(\bigotimes_{s=1}^j F_{\nu^{-1}(s)} \right) \\ &= \theta_{\sigma_1 \oplus \dots \oplus \sigma_j} \theta_{\nu(n_1, \dots, n_j)}^{-1} \left(\bigotimes_{s=1}^j F_{\nu^{-1}(s)} \right) \\ &= \theta_{\sigma_1 \oplus \dots \oplus \sigma_j} \tilde{\xi}_j''(\nu, F_1, \dots, F_j). \end{aligned}$$

Here we used the identity

$$\nu(n_1, \dots, n_j)^{-1} (\sigma_{\nu^{-1}(1)} \oplus \dots \oplus \sigma_{\nu^{-1}(j)}) = (\sigma_1 \oplus \dots \oplus \sigma_j) \nu(n_1, \dots, n_j)^{-1}.$$

Similarly, we can prove (3.2) and (3.5) by using the respective identities

$$(\nu\sigma^{-1})(n_{\sigma^{-1}(1)}, \dots, n_{\sigma^{-1}(j)})^{-1} = \sigma(n_1, \dots, n_j) \nu(n_1, \dots, n_j)^{-1}$$

and

$$\begin{aligned} \nu(n_1, \dots, n_j)^{-1} \left(\bigoplus_{s=1}^k \sigma_{\nu^{-1}(s)}(n_{\nu^{-1}(s)1}, \dots, n_{\nu^{-1}(s)j_{\nu^{-1}(s)}})^{-1} \right) \\ = \left(\left(\bigoplus_{s=1}^k \sigma_{\nu^{-1}(s)}(n_{\nu^{-1}(s)1}, \dots, n_{\nu^{-1}(s)j_{\nu^{-1}(s)}})^{-1} \right) \nu(n_1, \dots, n_j) \right)^{-1} \\ = E\tilde{\gamma}(\nu; \sigma_1, \dots, \sigma_k)(n_{11}, \dots, n_{1j_1}, \dots, n_{k1}, \dots, n_{kj_k})^{-1} \end{aligned}$$

where $E\tilde{\gamma}$ denotes the functor $E\Sigma_k \times E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k} \rightarrow E\Sigma_{j_1+\cdots+j_k}$ induced by $\tilde{\gamma}$.

Finally, (3.3) follows from the equality

$$\begin{aligned} \tilde{\xi}_j''(\nu, \varepsilon^{p_1} F_1, \dots, \varepsilon^{p_j} F_j) &= \theta_{\nu(n_1+p_1, \dots, n_j+p_j)}^{-1} \left(\bigotimes_{s=1}^j \varepsilon^{p_{\nu^{-1}(s)}} F_{\nu^{-1}(s)} \right) \\ &= \theta_{\nu(n_1+p_1, \dots, n_j+p_j)}^{-1} \theta_{\tau(n_{\nu^{-1}(1)}, \dots, n_{\nu^{-1}(j)}; p_{\nu^{-1}(1)}, \dots, p_{\nu^{-1}(j)})} \\ &\quad \varepsilon^{p_{\nu^{-1}(1)} + \cdots + p_{\nu^{-1}(j)}} \left(\bigotimes_{s=1}^j F_{\nu^{-1}(s)} \right) \\ &= \theta_{\tau(n_1, \dots, n_j; p_1, \dots, p_j)} \varepsilon^{p_1 + \cdots + p_j} \theta_{\nu(n_1, \dots, n_j)}^{-1} \left(\bigotimes_{s=1}^j F_{\nu^{-1}(s)} \right) \\ &= \theta_{\tau(n_1, \dots, n_j; p_1, \dots, p_j)} \varepsilon^{p_1 + \cdots + p_j} \tilde{\xi}_j''(\nu, F_1, \dots, F_j). \end{aligned}$$

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