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On the L^2 cohomology of complex spaces II (to appear in Nagoya Math. J.)

By

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Introduction. This is a continuation of the author's previous work [O-6], in which we have settled a conjecture of Cheeger-Goresky-MacPherson [C-G-M] by proving that the L^2 cohomology group of a compact (reduced) complex space is canonically isomorphic to its (middle) intersection cohomology group. Our aim here is, in addition to that result, to extend further the classical L^2 harmonic theory to complex spaces with arbitrary singularities by establishing the following.

Theorem 1. Let X be a compact Kähler space and $H^r_{(2)}(X)$ its r-th L^2 cohomology group. Then every element in $H^r_{(2)}(X)$ is uniquely representable as a sum $\sum_{p+q=r} u^{p,q}$, where $u^{p,q}$ are L^2 harmonic forms of type (p,q). In particular

$$H^r_{(2)}(X) = \bigoplus_{p+q=r} H^{p,q}_{(2),d}(X).$$

Here $H^{p,q}_{(2),d}(X)$ denotes the subspace of $H^r_{(2)}(X)$ consisting of the elements which are representable by (p,q)-forms. Moreover the complex conjugate of $H^{p,q}_{(2),d}(X)$ is equal to $H^{q,p}_{(2),d}(X)$.

Combined with our previous result, Theorem 1 implies that the

intersection cohomology group of a compact Kähler space admits a canonical Hodge structure. Thus we are left with a question whether or not our (L^2-) Hodge structure coincides with another one introduced by M.Saito [S]. It follows from the works of Zucker [Z] and the author [O-5] that they coincide if X admits only isolated singularities.

As for the proof of Theorem 1, a crucial step is in establishing the existence of a family of complete Kähler metrics on $X' := X - \mathrm{Sing}\, X$ converging to the prescribed one on X' such that the L^2 cohomology groups with respect to them are canonically isomorphic to the intersection cohomology group of X. Since one has an axiomatic sheaf theoretic definition of the intersection cohomology, our task is to show the nullity of certain L^2 cohomology, while our complete metrics will be constructed by utilizing a "good" desingularization of X whose existence is assured in general by the celebrated theory of Hironaka. The analytic part of the proof of this sort of vanishing theorem is already contained in our earlier work [O-5], where we proved Theorem 1 under the restriction that X admits only isolated singularities. In order to treat the general case by an obvious induction procedure, we have first to establish an analogue of Leray's theory on the spectral sequences in the L^2 context. We need this work because the theory of equisingular stratification has not developed well enough to fit our sepcific purpose here. Thus our effort will be concentrated to clarify this point (see the splitting lemma in §3). The rest of the proof will be only sketchy because they are essentially the same argument which we have been repeated in [O-1] through [O-6].

§1. Generalized Saper metircs. By generalizing Saper's construction in [S-1,2] we shall introduce a class of Hermitian metrics on the nonsingular parts of complex spaces with arbitrary singularities.

Let X be a (reduced and paracompact) complex space of dimension n and let $X' \subset X$ be the set of regular points. A Hermitian metric of X is by definition a C^∞ Hermitian metric on X' which is the pull-back of some C^∞ Hermitian metric around each point of X via a local holomorphic embedding into $\mathbb{C}^N$ ($N >> 1$). We shall denote a Hermitian metric of X by ds_X^2. By a desingularization of X we shall mean a complex manifold $\tilde{X}$ together with a proper holomorphic map $\tilde{\omega} : \tilde{X} \longrightarrow X$ such that $\tilde{\omega}|\tilde{\omega}^{-1}(X')$ is one-to-one and $E_{\tilde{\omega}} := \tilde{\omega}^{-1}(\mathrm{Sing}\ X)$ is a divisor of simple normal crossings. Let $q \in E_{\tilde{\omega}}$ be a point of multiplicity k. Then we shall denote by $z_1, \ldots, z_k$ a part of a holomorphic local coordinate around q such that $z_1 \cdot \cdots \cdot z_k = 0$ is (set theoretically) a local defining equation of the exceptional set $E_{\tilde{\omega}}$. Let ds^2 be a Hermitian metric on X'. We say that ds^2 satisfies Saper's condition with respect to a desingularization $\tilde{X} \longrightarrow X$ if $\tilde{\omega}^* ds^2$ is quasi-isometrically equivalent to

$$\tilde{\omega}^* ds_X^2 + \frac{d\tilde{s}^2}{-\log|z_1 \cdot \cdots \cdot z_k|} + \sum_{i=1}^{k} \frac{dz_i d\bar{z}_i}{|z_i|^2 (\log|z_1 \cdot \cdots \cdot z_k|)^2}$$

around each point $q \in E_{\tilde{\omega}}$, where $d\tilde{s}^2$ is a Hermitian metric on X and k is the multiplicity of $E_{\tilde{\omega}}$ at q.

We shall say that a desingularization $\tilde{X} \longrightarrow X$ is good if $\tilde{\omega}$ is locally (with respect to X) a projective morphism and there exists a complex analytic stratification $X_n = X \supset X_{n-1} = \mathrm{Sing}\, X \supset \cdots \supset X_0 \supset X_{-1} = \phi$ such that, for each X_α and $x \in X_\alpha \setminus X_{\alpha-1}$ there exist neighbourhoods $U \ni x$ and $V \ni x$ in X and $X_\alpha \setminus X_{\alpha-1}$, respectively, with a holomorphic retraction $f : U \longrightarrow V$ such that $f \circ \tilde{\omega} |\tilde{\omega}^{-1}(U)$ is a holomorphic submersion onto V.

Definition. A Hermitian metric on X' is called a generalized Saper metric if it satisfies Saper's condition with respect to some good desingularization.

Proposition 1.1. Let $X' \subset X$ be as above. Then X' admits a generalized Saper metric.

Proof. Given a complex space X, by Hironaka's theory one can always find a good desingularization. Hence by a patching argument using a nonnegative C^∞ partition of unity we obtain a generalized Saper metric on X'.

From the above construction it is not clear whether a manifold equipped with a generalized Saper metric should enjoy good properties at all. Thus we must begin with describing a property of generalized Saper metrics.

Let ∂ (resp. $\bar{\partial}$) denote the complex exterior derivative of type (1,0) (resp. (0,1)). Given a C^∞ function ψ on a complex manifold, we shall often identify $\partial\bar{\partial}\psi$ with the complex Hessian of ψ by an abuse of notation.

Proposition 1.2. Let X be as above. Then there exist a Hermitian metric ds_0^2 of X and a real-valued C^∞ function ψ on X' such that $ds_0^2 + \partial\bar{\partial}\psi$ is a generalized Saper metric for which the length of $\partial\psi$ is a bounded function on $K \subset X'$ for every compact subset $K \subset X$.

Proof. Let $\tilde{X} \xrightarrow{\tilde{\omega}} X$ be any good desingularization. Since $\tilde{\omega}$ is locally projective, for each point $x \in X$ there exist a neighbourhood $U \ni x$, positive line bundles $L_1, \ldots, L_m$ over $\tilde{U} = \tilde{\omega}^{-1}(U)$ together with holomorphic sections $s_1, \ldots, s_m$ vanishing on $\tilde{U} \cap E_{\tilde{\omega}}$ such that, for any $q \in \tilde{U} \cap E_{\tilde{\omega}}$, of multiplicity k,

$$\partial\bar{\partial}\Big(\sum_{i=1}^{m} -\log(-\log|s_i|)\Big)$$

$$\sim \frac{ds_U^2}{-\log|z_1 \cdot \cdots \cdot z_k|} + \sum_{i=1}^{k} \frac{dz_i d\bar{z}_i}{|z_i|^2(\log|z_1 \cdot \cdots \cdot z_k|)^2}$$

around q (and outside $E_{\tilde{\omega}}$), where $A \sim B$ means that $c^{-1}A \leqq B \leqq cB$ for some positive number c. By patching the functions $-\Sigma \log(\log|s_i|)$ by a partition of unity one obtains a function ψ on X' such that $\rho ds_X^2 + \partial\bar{\partial}\psi$,

for some positive C^∞ function ρ on X, is a Hermitian metric on X' for which $|\partial\psi|$ satisfies the requirement. For detailed estimation of $\partial\bar\partial\psi$ the reader is referred to a computation in [O-2, §1].

Let us summarize the above mentioned local construction of a generalized Saper metric in a more convenient form.

Proposition 1.3. Let ds^2 be a generalized Saper metric on X' associated to a good desingularization $\tilde X \xrightarrow{\tilde\omega} X$. Then for each point $x \in X$ one can find a neighbourhood $U \ni x$ and a finite number of nonnegative C^∞ functions a_i $(i=1,\dots,m)$ on $\tilde\omega^{-1}(U)$ such that

1) $\partial\bar\partial \log a_i$ extends to a C^∞ form on $\tilde\omega^{-1}(U)$.

2) $\log a_i$ is plurisubharmonic for every i.

3) $\tilde\omega^* ds^2 \sim \tilde\omega^* ds_X^2 + \sum_{i=1}^{m} \partial\bar\partial(-\log(-\log a_i))$ on $\tilde\omega^{-1}(U \cap X')$.

Remark. A crucial point in the asymptotics of a generalized Saper metric ds^2 is that it behaves locally like Poincare metrics on the product of the discs and the punctured discs up to the logarithmic factor. By this property the L^2 cohomology classes with proper support conditions on $\tilde\omega^{-1}(U \cap X')$ are "nearly" zero (cf. [O-5]). Additional properties of ds^2 which lead to the precise L^2 cohomology vanishing are summarized as follows. The first one is that it admits a potential of bounded gradient on $\tilde\omega^{-1}(U \cap X')$. The second one is more geometric. Namely, in terms of the above mentioned submersion $f\circ\tilde\omega$: $(\tilde\omega^{-1}(U), \tilde\omega^{-1}(V)) \longrightarrow V$ attached to $x \in X_\alpha \setminus X_{\alpha-1}$, we shall use later that $\tilde\omega^* ds^2$ is quasi-isometrically equivalent to a bundle-like metric on $\tilde\omega^{-1}(U \cap X')$ with respect to a local C^∞ trivialization induced from that of the fibration $f\circ\tilde\omega$. Since this last property is clear from the asymptotics of ds^2 we shall not give any proof here.

§2. L^2 cohomology with boundary conditions. Let (N, ds_N^2) be a Hermitian manifold of pure dimension n and let $\Omega \subset N$ be a domain with C^∞ smooth boundary. We denote by $C_0(\Omega)$ (resp. $C_0(\bar{\Omega})$) the set of compactly supported complex valued C^∞ differential forms on Ω (resp. on $\bar{\Omega}$) and by $C_0^r(\Omega)$ (resp. $C_0^r(\bar{\Omega})$) the subset of $C_0(\Omega)$ (resp. $C_0(\bar{\Omega})$) consisting of the r-forms. Given a real-valued C^∞ function Φ on Ω we put

$$\|u\|_\Phi^2 = \int_\Omega e^{-\Phi}|u|^2 dV \qquad \text{for} \quad u \in C_0(\Omega),$$

where $|u|$ denotes the pointwise norm of u and dV the volume form of N with respect to ds_N^2. The inner product associated to $\| \ \|_\Phi$ will be denoted by $(\ , \ )_\Phi$. The weight function Φ will not be referred to if $\Phi \equiv 0$. Let $L^r(\Omega)$ denote the Hilbert space defined as the completion of $C_0^r(\Omega)$ with respect to $\| \ \|_\Phi$. We are going to define the L^2 cohomology groups of Ω with certain restrictions on their boundary values.

Let d be the exterior derivative operating on the space of currents on Ω, and let δ_Φ be the formal adjoint of d with respect to $(\ , \ )_\Phi$. By using the Hodge's star operator $*$ one has $\delta_\Phi = -e^{\Phi} * d * e^{-\Phi}$. We put $d_0 = d|C_0(\Omega)$ and $\delta_{\Phi,0} = \delta_\Phi|C_0(\Omega)$. These operators will be regarded as linear operators on $L_\Phi(\Omega) := \bigoplus_{r=0}^{2n} L^r(\Omega)$ which have a dense domain $C_0(\Omega)$. Then we put $d_{max} = (\delta_{\Phi,0})_\Phi^*$ and $\delta_{\Phi,max} = (d_0)_\Phi^*$. Here $(\ )_\Phi^*$ denotes the Hilbert space adjoint with respect to $(\ ,)_\Phi$. Similarly we put $d_{min} = (\delta_{\Phi,max})_\Phi^*$ and

$\delta_{\Phi,\min} = (d_{\max})^*_\Phi$. Then the r-th L^2 cohomology group of Ω with respect to ds^2_N and Φ is defined as

$$H^r_{(2),\Phi}(\Omega) := \operatorname{Ker} d_{\max} \cap L^r_\Phi(\Omega) / \operatorname{Im} d_{\max} \cap L^r_\Phi(\Omega).$$

Elements of $\operatorname{Ker} d_{\max} \cap \operatorname{Ker}(d_{\max})^*_\Phi$ will be called harmonic forms. Similarly we put

$$H^r_{(2),\Phi,0}(\Omega) := \operatorname{Ker} d_{\min} \cap L^r_\Phi(\Omega) \ / \ \operatorname{Im} d_{\min} \cap L^r_\Phi(\Omega).$$

Furthermore we put $d_{\mathrm{mid}} := d_{\max}|d_{\max}^{-1}(\operatorname{Dom} d_{\min})$ and

$$H^r_{(2),\Phi,m}(\Omega) := \operatorname{Ker} d_{\min} \cap L^r_\Phi(\Omega) \ / \ \operatorname{Im} d_{\mathrm{mid}} \cap L^r_\Phi(\Omega).$$

Since $d_{\min} \circ d_{\mathrm{mid}} = 0$, $H^r_{(2),\Phi,m}(\Omega)$ is nothing but the image of $H^r_{(2),\Phi,0}(\Omega)$ in $H^r_{(2),\Phi}(\Omega)$ by the natural inclusion homomorphism.

Proposition 2.1. In the above notation we have

$$\operatorname{Dom}(d_{\mathrm{mid}})^*_\Phi \cap C_0(\bar{\Omega}) \subset \{u \in C_0(\bar{\Omega});\ d*(e^{-\Phi}u)|\partial\Omega = 0\}.$$

Here the restriction $|\partial\Omega$ is as a differential form on $\partial\Omega$.

Proof is omitted because it is a direct computation.

The following is also straightforward.

Proposition 2.2. $\mathrm{Dom}(d_{mid})_\Phi \cap C_0(\bar{\Omega})$ is dense in $\mathrm{Dom}(d_{mid})^*_\Phi$ with respect to the graph norm of $(d_{mid})^*_\Phi$. The same is true for d_{min}.

From Hahn-Banach's theorem we have

Proposition 2.3. The following statements are equivalent for any integer r and any positive number C.

(1) $$C(\|d_{min}u\|^2_\Phi + \|(d_{mid})^*_\Phi u\|^2_\Phi) \geq \|u\|^2_\Phi$$

for all $u \in \mathrm{Dom}\, d_{min} \cap \mathrm{Dom}(d_{mid})^*_\Phi \cap L^r_\Phi(\Omega)$.

(2) For any $u \in L^r_\Phi(\Omega)$ there exist $v \in \mathrm{Dom}\, d_{mid} \cap L^{r-1}_\Phi(\Omega)$ and $w \in \mathrm{Dom}(d_{min})^*_\Phi \cap L^{r+1}_\Phi(\Omega)$ such that

$$u = d_{mid}v + (d_{min})^*_\Phi w \quad \text{and} \quad \|u\|^2_\Phi \leq C(\|v\|^2_\Phi + \|w\|^2_\Phi).$$

§3. A splitting lemma. Given a Riemannian submersion of (N, ds_N^2) onto some differentiable manifold, say M, one can naturally expect to compute the L^2 cohomology group of N from that of the fibers and certain local systems on M just as one deals with Leray's spectral sequences in topology or complex analytic geometry. We shall present here a basic lemma which justifies such a procedure in our problem.

From now on we assume that (N, ds_N^2) is a complete Hermitian manifold, $f : (N, ds_N^2) \longrightarrow M$ a holomorphic Riemannian submersion and $\Omega \subset N$ is an open subset with C^∞ smooth boundary such that $f|\partial\Omega$ is also a submersion onto M. We shall assume moreover that for any point $x \in M$ there exists a neighbourhood $B \ni x$ and a C^∞ diffeomorphism $\xi : B \times (f^{-1}(x) \cap \overline{\Omega}) \longrightarrow f^{-1}(B) \cap \overline{\Omega}$ such that $f \circ \xi$ is the projection to the first factor. Let us fix such x, B and ξ. For simplicity we put $F = f^{-1}(x) \cap \Omega$ and $E = B \times F$. By an abuse of notation we put $\overline{E} := B \times \overline{F}$, which is naturally identified with $f^{-1}(B) \cap \overline{\Omega}$. Let p_1 (resp. p_2) be the projection from $\overline{E}$ onto B (resp. onto $\overline{F}$). We shall identify $C_0(B)$ (resp. $C_0(\overline{F})$) with $p_1^* C_0(B)$ (resp. with $p_2^* C_0(\overline{F})$). We assume that the metric $\xi^* ds_N^2$ is of the form $p_1^* ds_B^2 + \Sigma_F$, where Σ_F is a positive semidefinite Hermitian form on E which is smoothly extendable to $\overline{E}$ and annihilates $\operatorname{Ker} p_2$. The Hodge's star operator $*_F$ with respect to Σ_F, which is well-defined on $p^{-1}(y)$ for each $y \in B$, shall be naturally extended by linearity as an operator on $C_0(\overline{E})$. We shall denote by $*_E$ (resp. by $*_B$) the Hodge's star operator with respect to $\xi^* ds_N^2$

(resp. ds_B^2). Then we note that

$$(4) \qquad *_E(u\wedge v) = (-1)^{\deg u \deg v} *_B u \wedge *_F v$$

for all $u \in C_0(B)$ and $v \in C_0(\overline{F})$, since f is a Riemannian submersion. The Laplacian $d\delta + \delta d$ on N will be denoted simply by Δ, which will also stand for the Laplacian on $\overline{E}$.

Let us put

$$\deg_F w = \begin{cases} \inf\{r;\ w \in C_0(\overline{E}) \wedge C_0^r(\overline{F})\} & \text{if } w \in C_0(\overline{E}) \setminus \{0\} \\ 2n & \text{if } w = 0. \end{cases}$$

Then we have the following.

Lemma 3.1. For any $w \in C_0(\overline{E})$, $\deg_F \Delta w \geq \deg_F w$.

Proof. Clearly it suffices to show the inequality for those w of the form $u \wedge v$ with $u \in C_0^r(B)$ and $v \in C_0^s(\overline{F})$. For such a form the result follows from the fact that

$$\deg_F(\delta(u \wedge dv)) \geq s$$

$$\deg_F(\delta(du \wedge v) + (-1)^{r+1} du \wedge *_F d *_F v) \geq s$$

and

$$\deg_F(d\delta(u \wedge v) + (-1)^r du \wedge *_F d *_F v) \geq s.$$

Let ϕ be any C^∞ real-valued function on B and let $\Phi = p_1^* \phi$. Then the weighted Laplacian $\Delta_\Phi := d\delta_\Phi + \delta_\Phi d$ has the same property as above. Namely we have

(5) $$\deg_F \Delta_\Phi w \geq \deg_F w \qquad \text{for any } w \in C_0(\overline{E}).$$

For any $w \in C_0^r(\overline{E})$, one has a canonical decomposition $w = \sum_{s=0}^{r} w_s$ such that $\deg_F w_s = s$ and $\deg_F *_F w_0 > \deg_F *_F w_1 > \cdots > \deg_F *_F w_r$.

Proposition 3.2. Under the notation as above, if $w \in C_0^r(\overline{E}) \cap \mathrm{Dom}\, d_{min} \cap \mathrm{Dom}(d_{mid})_\Phi^*$ then $d_F *_F w_s | \partial\overline{E} = 0$ for all s. Here d_F denotes the exterior derivative along the fiber direction.

Proof is a straightforward computation and may well be omitted.

Using the above mentioned computations we shall prove the following basic lemma.

<u>Splitting Lemma</u> Under the above notation

$$\|dw\|_\Phi^2 + \|\delta_\Phi w\|_\Phi^2 \geq \sum_{s=0}^{r} (\|d_F w_s\|_\Phi^2 + \|\delta_F w_s\|_\Phi^2)$$

for any $w \in C_0^r(\overline{E}) \cap \mathrm{Dom}\, d_{min}$. Here $\delta_F := -*_F d_F *_F$.

Proof. Since $w \in C_0(\overline{E}) \cap \mathrm{Dom}\, d_{min}$,

$$(5) \qquad (\Delta_\Phi w, w)_\Phi = \|dw\|_\Phi^2 + \|\delta_\Phi w\|_\Phi^2 + \int_{\partial E} e^{-\Phi} \delta_\Phi w \wedge \overline{*_E w}.$$

We note that

$$(6) \quad \int_{\partial E} e^{-\Phi} \delta_\Phi w \wedge \overline{*_E w} = \sum_{s=0}^{r} \int_{\partial E} e^{-\Phi} \delta_\Phi w_s \wedge \overline{*_E w_s}$$

since $w \in \mathrm{Dom}\, d_{min}$.

By Lemma 3.1 we have

$$(7) \qquad (\Delta_\Phi w, w)_\Phi = \sum_{s=0}^{r} (\Delta_\Phi w_s, w_s)_\Phi$$

since $(\Delta_\Phi w, w)_\Phi = \lim_{\nu \to \infty} (\Delta_\Phi w, \rho_\nu w)_\Phi$ for cut off functions ρ_ν converging to one.

Thus we obtain from (5), (6) and (7)

$$\|dw\|_\Phi^2 + \|\delta_\Phi w\|_\Phi^2 = \sum_{s=0}^{r} (\|dw_s\|_\Phi^2 + \|\delta_\Phi w_s\|_\Phi^2)$$

from which the desired inequality follows immediately.

Let us choose B in advance so that one has a local C^∞ frame of T^*M, say $\theta_1,\dots,\theta_{2m}$ over B. For each $w \in C_0^r(\bar{E})$ with the canonical splitting $w = \sum w_s$ as above we put

$$w_s = \sum_I \theta_I \wedge w_s^I,$$

where I runs through the increasing multi-indices of length $r-s$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_{r-s}}$ for $I = (i_1,\dots,i_{r-s})$ and $w_s^I \in C_0^0(\bar{E}) \otimes C_0(\bar{F})$. If one has $w \in C_0^r(\bar{E}) \cap \mathrm{Dom}(d_{mid})_\Phi^*$, it is clear that $w_s^I|p_2^{-1}(y) \in \mathrm{Dom}(d_F|p_2^{-1}(y))^*$ for all $y \in B$. Therefore the splitting lemma shows in particular the following.

Proposition 3.3. Under the above situation, suppose moreover that there exists a positive number C such that for every $y \in B$ and $r \in \mathbb{Z}$, the estimate

$$\|d_{min}u\|_\Phi^2 + \|(d_{mid})_\Phi^* u\|_\Phi^2 \geq C\|u\|_\Phi^2$$

holds for all $u \in \mathrm{Dom}\, d_{min} \cap \mathrm{Dom}\,(d_{mid})_\Phi^* \cap L^r(f^{-1}(y) \cap \Omega)$, where the operators d_{min} and d_{mid} represent those on $f^{-1}(y)$ relative to $(f^{-1}(y), ds_N^2|)$. Then one has for every $r \in \mathbb{Z}$,

$$\|d_{min}w\|_\Phi^2 + \|(d_{mid})_\Phi^* w\|_\Phi^2 \geq C\|w\|_\Phi^2$$

for all $w \in \mathrm{Dom}\, d_{min} \cap \mathrm{Dom}\,(d_{mid})_\Phi^* \cap L^r(f^{-1}(B) \cap \Omega)$, provided that the metric $ds_N^2|f^{-1}(B)$ is replaced by a complete metric of the form $ds_N^2 + f^*ds^2$ for some complete metric ds^2 on B.

Given a Riemannian manifold (T, ds_T^2) equpped with a real valued C^∞ function Φ, we say for convenience that the triple (T, ds_T^2, Φ) is L^2-acyclic with magnitude C if

$$\|u\|_\Phi^2 \leq C(\|d_{min}u\|_\Phi^2 + \|(d_{mid})_\Phi u\|_\Phi^2)$$

for all $u \in \mathrm{Dom}\, d_{min} \cap \mathrm{Dom}\, (d_{mid})_\Phi^*$. We say simply that (T, ds_T^2, Φ) is L^2-acyclic if $H^r_{(2),\Phi,m}(T) = 0$ for all r. Let us restate Proposition 3.3 by using this terminology.

Proposition 3.3'. Let (B, ds_B^2) be a complete Riemannian manifold, let Ω_0 be a smoothly bounded domain in a paracompact C^∞ manifold F, let $N = B \times F$, let $ds_N^2 = p^* ds_B^2 + \Sigma_F$ be a bundle-like and complete metric on N with respect to the projection $p : N \longrightarrow B$, and let $\Omega_y = \{y\} \times \Omega_0$. Suppose that $(\Omega_y, \Sigma_F|\Omega_y)$ is L^2-acyclic with magnitude C for all $y \in B$. Then for any C^∞ real valued function ψ on B, $(N, ds_N^2, p^*\psi)$ is also L^2-acyclic with magnitude C.

§4. The main results. Let X be a compact complex space of pure dimension n and let X', ds^2 and $\tilde{X} \xrightarrow{\tilde{\omega}} X$ be as in Proposition 1.3. Let $\{X_\alpha\}$ be a stratification associated to $\tilde{\omega}$, let $x \in X_\alpha \setminus X_{\alpha-1}$ be any point for some α, and let U and V be neighbourhoods of x in X and $X_\alpha \setminus X_{\alpha-1}$, respectively, such that there exists a holomorphic retraction $f : \bar{U} \longrightarrow \bar{V}$ such that $p := f \circ \tilde{\omega} | \tilde{\omega}^{-1}(\bar{U})$ is a holomorphic submersion. For any open set $\Omega \subset X$ we put $\Omega' = \Omega \cap \tilde{\omega}^{-1}(X')$.

Proposition 4.1. There exist a neighbourhood system $\{\Omega_k\}_{k=1}^{\infty}$ of A in $\tilde{\omega}^{-1}(U)$ such that for any complete metric ds_V^2 on V and any C^∞ function $\psi : V \longrightarrow \mathbb{R}$, $(\Omega_k', ds^2|\Omega_k', p^*\psi)$ is L^2 acyclic.

Proof. Let us proceed by induction on n. If $n=0$, there is nothing to prove. Suppose that the assertion is true if $n < k$. Then, from the remark at the end of §1 and Proposition 3.3', the result is true for $n=k$ if $\alpha > 0$. If $\alpha=0$, the result is contained in the author's previous work (cf.[O-6] Theorem. 3.5).

In virtue of the sheaf theoretic characterization of the intersection cohomology group of X (cf.[C-G-M]), Proposition 4.1 implies the following.

Theorem 4.2. With respect to any generalized Saper metric ds^2 on X', we have

$$H^r_{(2)}(X') \cong IH^r(X) \qquad \text{for all } r.$$

Here $IH^r(X)$ denotes the r-th intersection cohomology group of X.

Let us denote by $H^r_{(2)}(X)$ the r-th L^2 cohomology group of X' with respect to a Hermitian metric ds^2_X of X. Then we have $H^r_{(2)}(X) = IH^r(X)$ for all r by [O-6]. Hence, applying Proposition 4.1 to $ds^2_\varepsilon := ds^2_X + \varepsilon ds^2$ for $\varepsilon \in (0,1]$, noting that the magnitude of L^2-acyclicity remains bounded as $\varepsilon \longrightarrow 0$, we obtain the following.

Proposition 4.3. Let $\{\rho_k\}_{k=1}^{\infty}$ be a C^∞ family of compactly supported cut-off functions uniformly converging to 1 on each compact subset of X'. Then, for any harmonic form h on X' with respect to ds^2_X the harmonic parts of $\rho_k h$ with respect to ds^2_ε converge to h as $\varepsilon \longrightarrow 0$ and $k \longrightarrow \infty$.

In case ds^2_X is Kählerian, one can choose ds^2 also to be a Kähler metric. Therefore, from Theorem 4.2 we obtain

Theorem 4.4. If ds^2_X is Kählerian,

$$H^r_{(2)}(X') = \bigoplus_{p+q=r} H^{p,q}_{(2),d}(X')$$

and

$$\overline{H^{p,q}_{(2),d}(X')} = H^{q,p}_{(2),d}(X')$$

with respect to any generalized Saper metric on X'.

Combining Theorem 4.4 with Proposition 4.3 we obtain Theorem 1.

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