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WKB analysis and deformation of Schrödinger equations

Takahiro KAWAI

Research Institute for Mathematical Sciences

Kyoto University

Kyoto, 606-01 JAPAN

and

Yoshitsugu TAKEI

Research Institute for Mathematical Sciences

Kyoto University

Kyoto, 606-01 JAPAN

§0. Introduction

The purpose of this report is to make a survey of our recent trials to develop the exact WKB analysis of the Painlevé transcendents. The motivation of such trials is two-fold: On one hand, in view of the WKB-theoretic expression of a monodromy group ([SAKT],[AKT3]) we want to analyze isomonodromic deformations from the viewpoint of the exact WKB analysis, which naturally leads us to the study of the Painlevé equations with a large parameter (Table A.1); on the other hand, such equations admit instanton-type formal solutions, which seem to be crucially important to describe the connection formula for Painlevé transcendents with a large parameter, and hence to describe the monodromic structure in question. Unfortunately we have been unable to obtain complete solutions to these problems. Still we believe that the results which we report here will convince the reader of the importance of the subject.

This report is organized as follows:

In §1, we review some basic facts concerning the WKB analysis of isomonodromic deformations of some Schrödinger equations, which we call (SL_J) (cf. [AKT2]). One important feature of our analysis is that an isomonodromic deformation inevitably brings a double turning point into the theory (Proposition 1.2 (i)). This is really an unpleasant feature of the problem, but it is the starting point of the new approach to the Painlevé transcendents; first of all, this degeneracy is a counterpart of our way of constructing new formal solutions of the Painlevé equations (Proposition 1.2 (ii)), and furthermore a subtle and interesting analyticity property of a WKB solution of (SL_J) at such a double turning point is established (Proposition 1.2 (iii)). The analyticity property seems to reflect the fact that such a point is tied up with the apparent singular point of (SL_J).

In §2 we first introduce some basic notions such as a Stokes curve for the Painlevé equations. They are closely related to the Stokes geometry of (SL_J) in a (currently still) mysterious way through an intriguing relation (2.4). We then establish an important transformation theorem for some particular Painlevé transcendents to the effect that they can be mutually transformed locally (Theorem 2.3). It is probably worth emphasizing that the transformation is constructed with the aid of the transformation of (SL_J); although we are interested in the transformation of the Painlevé equations, it is achieved by studying the underlying Schrödinger equations (together with their deformation equations). Slightly weaker version of these results are announced in [KT].

The formal solutions discussed in §2 generate another class of formal solutions of the Painlevé equations; one of their characteristic properties is that they are infinite sums of terms containing exponential factors, which give rise to a periodic structure concerning the location of the singular points of their Borel transforms. We call such solutions "instanton-type solutions". These solutions are the main subject of §3. As is easily imagined, instanton-type solutions are important ingredients of the connection formula for Painlevé transcendents with a large parameter. We exemplify this fact by a concrete computation of the connection formula for the solution discussed in §2 in the case of Painlevé I.

In the final section (§4), we discuss the exact WKB analysis of (SL_J), that is, some of our conjectures on the transformation of (SL_J) whose coefficients contain instanton-type solutions. Our expectation, though not yet fully confirmed, is that the connection formula for all instanton-type Painlevé transcendents should be deduced through the transformation of the sort from the results for Painlevé I, a part of which is given in §3.

In ending this introduction, we would like to express our heartiest thanks to Professor T. Aoki for the stimulating discussions with him.

§1. Deformation of WKB solutions.

First let us fix our notations.

Throughout this article we use the symbol (P_J) to denote the J-th Painlevé equation with a large parameter η. As is stated in the Introduction, the Painlevé equations (P_J) $(J = I, \cdots, VI)$ naturally arise as conditions for isomonodromic deformations (in the sense of [JMU]) of the relevant Schrödinger equations

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_J(x,t,\eta)\right)\psi_J(x,t,\eta) = 0, \tag{1.1}$$

which will be denoted by (SL_J). As a matter of fact, these Schrödinger equations (SL_J) can be isomonodromically deformed if the unknown function ψ_J satisfies the deformation equation

$$\frac{\partial \psi_J}{\partial t} = A_J(x,t,\lambda)\frac{\partial \psi_J}{\partial x} - \frac{1}{2}\frac{\partial A_J(x,t,\lambda)}{\partial x}\psi_J \tag{1.2}$$

where A_J is a certain rational function. The Painlevé equation (P_J) can be obtained as the compatibility condition for the simultaneous equations (1.1) and (1.2).

As for the explicit forms of (P_J), we refer the reader to Table A.1 in the Appendix below.

Definition 1.1. *Let $F_J(\lambda, t)$ denote the coefficient of η^2 in (P_J). Then $F_J(\lambda, t)$ is a rational function of (λ, t), i.e., a ratio of two polynomials of (λ, t). We denote by $F_J^\dagger(\lambda, t)$ the polynomial in the numerator, which is normalized as in Table A.2. (We note that the normalization factor is slightly different from [KT].)*

We also list up Q_J (resp. A_J) in Table A.3 (resp. Table A.5). In writing down the potential Q_J, we use in Table A.3 the symbol K_J (whose list can be seen in Table A.4), which is the t-dependent Hamiltonian with (λ, ν) obeying the Hamiltonian system

$$(1.3) \qquad \begin{cases} \dfrac{d\lambda}{dt} = \eta \dfrac{\partial K_J}{\partial \nu} \\[2mm] \dfrac{d\nu}{dt} = -\eta \dfrac{\partial K_J}{\partial \lambda}. \end{cases}$$

This system is known to be equivalent to the Painlevé equation (P_J) (cf. [O] and references cited there).

Note that the equations (P_J) to be discussed here differ from the original Painlevé equations in that they contain a large parameter η. In fact, the original one can be obtained by substituting $\eta = 1$ in the expression of (P_J) (cf. Table A.1). The situation is the same also for the other equations, i.e., (1.1), (1.2) and (1.3). In particular, in this formulation λ and ν are, in addition to satisfying (1.3), supposed to have the following expansions in η^{-1}:

$$(1.4) \qquad \lambda = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots$$

$$(1.5) \qquad \nu = \nu_0(t) + \eta^{-1}\nu_1(t) + \eta^{-2}\nu_2(t) + \cdots,$$

while the variables (i.e., x and t) and the other constants (such as α_0, α_1, etc. in Table A.1) are supposed to be independent of η. This way of introducing the parameter η is compatible with the ordinary procedure of confluence of singularities in the Schrödinger equations (SL_J) and the Painlevé equations (P_J) (cf. e.g. [O]).

Now let us consider WKB solutions of the Schrödinger equation (SL_J):

$$\psi_J(x,t,\eta) = \exp \int^x S_J(x,t,\eta)dx, \tag{1.6}$$

where $S_J = \eta S_{J,-1} + S_{J,0} + \eta^{-1}S_{J,1} + \cdots$ is a formal power series (in η^{-1}) solution of the Riccati equation with a parameter t:

$$S_J(x,t,\eta)^2 + \frac{\partial S_J(x,t,\eta)}{\partial x} = \eta^2 Q_J(x,t,\eta). \tag{1.7}$$

When a non-zero WKB solution satisfies the deformation equation (1.2), λ and ν should obey the Hamiltonian system (1.3), hence the Painlevé equation (P_J) appears. Furthermore we then encounter several interesting phenomena concerning the structure of (SL_J) itself and the logarithmic derivative S_J of ψ_J. In the remaining part of this section we explain these phenomena which will be used in the WKB analysis of the Painlevé equations below. Let us begin with the following proposition.

Proposition 1.1. *Suppose that a non-zero WKB solution of (SL_J) $(J = I, \cdots, VI)$ satisfies (1.2). Then S_J, i.e., the logarithmic derivative of the WKB solution satisfies the following equation:*

$$\frac{\partial S_J}{\partial t} = \frac{\partial}{\partial x}\Big(A_J S_J - \frac{1}{2}\frac{\partial A_J}{\partial x}\Big). \tag{1.8}$$

Remark 1.1. The relation (1.8) entails that

$$\omega_J = S_J dx + \Big(A_J S_J - \frac{1}{2}\frac{\partial A_J}{\partial x}\Big)dt \tag{1.9}$$

is a closed form. Hence we can construct a WKB solution ψ_J of (SL_J) satisfying (1.2) by setting $\psi_J = \exp \int^{(x,t)} \omega_J$.

Making use of Proposition 1.1, we can actually analyze the structure of (SL_J) and the logarithmic derivative S_J of ψ_J. We summarize them as follows:

Proposition 1.2. *Under the same assumption as in Proposition 1.1 the following hold:*
(i) *Let $Q_{J,0}$ denote the leading part of Q_J, that is, the part which is homogeneous of degree 0 with respect to η. Then we find the following:*

$$Q_{J,0}(x,t)\Big|_{x=\lambda_0(t)} = \frac{\partial}{\partial x}Q_{J,0}(x,t)\Big|_{x=\lambda_0(t)} = 0. \tag{1.10}$$

(ii) *The top terms $\nu_0(t)$ and $\lambda_0(t)$ in the expansions (1.5) and (1.4) respectively satisfy*

$$\nu_0(t) = 0 \quad \textit{and} \quad F_J^{\dagger}(\lambda_0(t),t) = 0, \tag{1.11}$$

while $\nu_j(t)$ and $\lambda_j(t)$ $(j \geqq 1)$ are uniquely determined in a recursive manner.
(iii) *For any odd integer j $S_{J,j}(x,t)$ is holomorphic near $(x,t) = \big(\lambda_0(t_0), t_0\big)$ if t_0 is not contained in*

$$\Delta_J \overset{\text{def}}{=} \{t \in \mathbb{C};\ \textit{there exists } \lambda \textit{ such that } F_J^{\dagger}(\lambda,t) = \partial F_J^{\dagger}(\lambda,t)/\partial\lambda = 0\}. \tag{1.12}$$

Proposition 1.2 (i) asserts that $x = \lambda_0(t)$ is a double turning point of (SL_J): This means that the Riemann surface of $\sqrt{Q_{J,0}(x,t)}$ is degenerate (for each fixed t). The degeneracy of this sort is a real problem in the study of the Schrödinger equation (1.1) from the viewpoint of the exact WKB analysis (cf. [AKT2]) and an isomonodromic deformation inevitably causes such a degeneracy; it is a tragedy. This phenomenon, however, is a starting point of our subject "the exact WKB analysis of the Painlevé transcendents": As Proposition 1.2 (ii) claims, starting with $\lambda_0(t)$ determined in (1.11), we can obtain $\lambda_j(t)$ recursively which solves the equation (P_J) formally. Such a solution λ_J of (P_J) is our main concern in this article.

§2. A local transformation theorem for the solution λ_J of the Painlevé equation (P_J).

In the subsequent part of this report we discuss mainly the formal power series solution $\lambda_J = \sum_{j\geq 0} \lambda_j(t)\eta^{-j}$ of the Painlevé equation (P_J). First we establish a local transformation theorem in this section: As we will see later (Theorem 2.3), the solution λ_J of (P_J) can be locally transformed to λ_I in the formal sense.

Let us begin with introducing the following terminologies.

Definition 2.1. *(i) A turning point for λ_J is, by definition, a point t which satisfies*

$$F_J(\lambda_0(t), t) = \frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t) = 0. \tag{2.1}$$

Such a point t is said to be simple if

$$\frac{\partial^2 F_J}{\partial \lambda^2}(\lambda_0(t), t) \neq 0. \tag{2.2}$$

(ii) For a turning point τ for λ_J a (real one-dimensional) curve defined by

$$\operatorname{Im} \int_\tau^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}\, dt = 0 \tag{2.3}$$

is called a Stokes curve for λ_J (emanating from τ).

For example, in the case of (P_I), $t = 0$ is the unique turning point and the configuration of the Stokes curves is as follows:

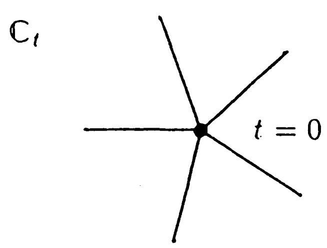

Figure2.1.

The relevance of these notions to the exact WKB analysis of λ_J, in particular, to the location of the singular points of its Borel transform, shall be discussed in §3. The following Proposition 2.1 shows they are also closely related to the Stokes geometry of the Schrödinger equation (SL_J).

Proposition 2.1. *Let τ be a simple turning point for λ_J. Then there exists a simple turning point $a(t)$ of SL_J, i.e., a simple zero $x = a(t)$ of $Q_{J,0}(x,t,\lambda_0(t))$, which satisfies the following:*

(i) At $t = \tau, a(t)$ merges with the double turning point $x = \lambda_0(t)$.

(ii) The following relation holds:

$$\int_{a(t)}^{\lambda_0(t)} \sqrt{Q_{J,0}(x,t,\lambda_0(t))}\, dt = \frac{1}{2}\int_{\tau}^{t} \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(s),s)}\, ds. \tag{2.4}$$

The statement (i) implies that at a (simple) turning point τ for λ_J the associated Schrödinger equation (SL_J) has a triple turning point. Furthermore, considering the imaginary part of both sides of (2.4), we find that at a point σ in a Stokes curve for λ_J there exists a Stokes curve of (SL_J) that connects two turning points of SL_J, i.e., $\lambda_0(\sigma)$ and $a(\sigma)$.

Taking account of this relationship between the Stokes geometry of (P_J) and that of (SL_J), we first consider the transformation of (SL_J) to obtain the transformation of (P_J) in the following way.

Let $\tilde{\tau}$ be a simple turning point for $\tilde{\lambda}_J(\tilde{t})$ and $\tilde{\sigma}$ be a point in a Stokes curve emanating from $\tilde{\tau}$. Here we assume $\tilde{\sigma}$ is distinct from $\tilde{\tau}$. Then Proposition 2.1 tells us that there exists a simple turning point $\tilde{a}(\tilde{t})$ of (SL_J) that merges with $\tilde{\lambda}_0(\tilde{t})$ at $\tilde{t} = \tilde{\tau}$, and that, at $\tilde{t} = \tilde{\sigma}, \tilde{a}(\tilde{\sigma})$ and $\tilde{\lambda}_0(\tilde{\sigma})$ are connected by a Stokes curve of (SL_J). Let $\tilde{\gamma}$ denote the portion of the Stokes curve that begins at $\tilde{a}(\tilde{\sigma})$ and ends at $\tilde{\lambda}_0(\tilde{\sigma})$. Note that in the case of (SL_I) there exists only one simple turning point, i.e., $-2\lambda_0(t)$. (Hence $a(t)$ should coincide with $-2\lambda_0(t)$.)

Theorem 2.1. *There exist a neighborhood $\tilde{V}$ of $\tilde{\sigma}$, a neighborhood $\tilde{U}$ of $\tilde{\gamma}$, and holomorphic functions $x_j(\tilde{x},\tilde{t})$ $(j = 0,1,2,\ldots)$ on $\tilde{U}\times\tilde{V}$ and $t_j(\tilde{t})$ $(j = 0,1,2,\ldots)$ on $\tilde{V}$ so that the following relation may hold:*

$$\tilde{Q}_J(\tilde{x},\tilde{t},\eta) = \left(\frac{\partial x(\tilde{x},\tilde{t},\eta)}{\partial \tilde{x}}\right)^2 Q_I(x(\tilde{x},\tilde{t},\eta), t(\tilde{t},\eta),\eta) - \frac{1}{2}\eta^{-2}\{x(\tilde{x},\tilde{t},\eta);\tilde{x}\}. \tag{2.5}$$

Here $x(\tilde{x},\tilde{t},\eta)$ *and* $t(\tilde{t},\eta)$ *respectively denote the formal series* $\sum_{j\geq 0} x_j(\tilde{x},\tilde{t})\eta^{-j}$ *and* $\sum_{j\geq 0} t_j(\tilde{t})\eta^{-j}$, *and* $\{x;\tilde{x}\}$ *denotes the Schwarzian derivative* $\dfrac{\partial^3 x/\partial\tilde{x}^3}{\partial x/\partial\tilde{x}} - \dfrac{3}{2}\left(\dfrac{\partial^2 x/\partial\tilde{x}^2}{\partial x/\partial\tilde{x}}\right)^2$. *Furthermore we can construct* $x_j(\tilde{x},\tilde{t})$ *and* $t_j(\tilde{t})$ *so that they should satisfy the following:*

(i) $x_0(\tilde{a}(\tilde{t})) = -2\lambda_0(t_0(\tilde{t})), x_0(\tilde{\lambda}_0(\tilde{t})) = \lambda_0(t_0(\tilde{t}))$, *and* $\partial x_0/\partial\tilde{x} \neq 0$ *on* $\tilde{U}\times\tilde{V}$.

(ii) *We find*

(2.6)
$$\int_{\tilde{\tau}}^{\tilde{t}} \sqrt{\frac{\partial\tilde{F}_J}{\partial\tilde{\lambda}}(\tilde{\lambda}_0(\tilde{s}),\tilde{s})}d\tilde{s} = \int_0^t \sqrt{\frac{\partial F_I}{\partial\lambda}(\lambda_0(s),s)}ds\Bigg|_{t=t_0(\tilde{t})},$$

and, in particular, $dt_0/d\tilde{t} \neq 0$ *on* $\tilde{V}$.

(iii) *For odd* j*'s the function* x_j *and* t_j *vanish identically.*

Remark 2.1. The roles of (SL_J) and (SL_I) are symmetric in the above result, that is, they can be interchanged.

The relation (2.5) means that we can transform (SL_J) into (SL_I) by the (formal) transformation

(2.7)
$$\begin{cases} x = x(\tilde{x},\tilde{t},\eta) = \sum_{j\geq 0} x_{2j}(\tilde{x},\tilde{t})\eta^{-2j} \\ t = t(\tilde{t},\eta) = \sum_{j\geq 0} t_{2j}(\tilde{t})\eta^{-2j} \\ \psi_I(x,t,\eta)\Big|_{\substack{x=x(\tilde{x},\tilde{t},\eta)\\ t=t(\tilde{t},\eta)}} = \left(\dfrac{\partial x}{\partial\tilde{x}}(\tilde{x},\tilde{t},\eta)\right)^{1/2}\tilde{\psi}_J(\tilde{x},\tilde{t},\eta) \end{cases}$$

on a neighborhood $\tilde{U}$ of $\tilde{\gamma}$ (for each fixed $\tilde{t}$ in $\tilde{V}$). Furthermore this transformation automatically brings the deformation equation for $\tilde{\psi}_J$ to that for ψ_I through the following result.

Theorem 2.2. *Let* $x(\tilde{x},\tilde{t},\eta)$ *and* $t(\tilde{t},\eta)$ *be the formal series constructed in Theorem 2.1. Then the following relation holds:*

(2.8)
$$A_I(x,t,\lambda_I(t,\eta))\Big|_{\substack{x=x(\tilde{x},\tilde{t},\eta)\\ t=t(\tilde{t},\eta)}} = \left(\frac{\partial t}{\partial\tilde{t}}(\tilde{t},\eta)\right)^{-1}\{\tilde{A}_J(\tilde{x},\tilde{t},\tilde{\lambda}_J(\tilde{t},\eta))\frac{\partial x}{\partial\tilde{x}}(\tilde{x},\tilde{t},\eta) - \frac{\partial x}{\partial\tilde{t}}(\tilde{x},\tilde{t},\eta)\}.$$

In fact, using (2.8) we can easily verify that the deformation equation (1.2) for $\tilde{\psi}_J$ should be transformed to that for ψ_I by the transformation (2.7). Thus we have obtained the formal transformation which transforms the system (1.1) and (1.2) for any J to that for $J = I$.

Making use of this transformation (2.7), we find the following local transformation theorem for the solution λ_J of the Painlevé equations (P_J).

Theorem 2.3. *Using the series $x(\tilde{x}, \tilde{t}, \eta)$ and $t(\tilde{t}, \eta)$ in Theorem 2.1, we find the following:*

$$\lambda_I(t(\tilde{t}, \eta), \eta) = x(\tilde{x}, \tilde{t}, \eta)\Big|_{\tilde{x} = \lambda_J(\tilde{t}, \eta)}. \tag{2.9}$$

§3. Instanton-type solutions and the exact WKB analysis of (P_I).

Theorem 2.3 gives us a hope that basic properties of λ_J should be deduced from those of λ_I. Having such a hope in mind, we now study the solution λ_I of the first Painlevé equation (P_I) from the viewpoint of the exact WKB analysis. Our main concern is the connection formula for λ_I.

For the sake of notational simplicity let us discuss the following equation $(P_I)'$ instead of (P_I).

$$\frac{d^2\lambda}{dt^2} = \eta^2(\lambda^2 - t). \tag{$(P_I)'$}$$

Note that the equation $(P_I)'$ is obtained from (P_I) through a change of scales, i.e., considering $\alpha\lambda_I(\beta t)$ for suitable constants α and β. For $(P_I)'$ the formal power series solution corresponding to λ_I (which is denoted by λ throughout this section) is expressed as follows:

$$\lambda = \sum_{j \geq 0} \lambda_{2j}(t)\eta^{-2j} = \sum_{j \geq 0} a_j t^{\frac{1}{2} - \frac{5}{2}j} \eta^{-2j}, \tag{3.1}$$

where the coefficient a_j is given by the following recursive formula:

$$\begin{cases} a_0 = 1, \ a_1 = -1/8, \ a_2 = -49/128 \\ a_j = \frac{25}{8}(j-1)^2 a_{j-1} - \frac{1}{2}\Big(\sum\limits_{\substack{k+l=j \\ k,l \geq 2}} a_k a_l\Big) \quad (j \geq 3). \end{cases} \tag{3.2}$$

By using the relation (3.2) we immediately find that the solution λ does not converge in the usual sense. To overcome this difficulty we apply the Borel resummation technique to the solution λ, which is actually the basic idea in the theory of the exact WKB analysis. For the definition of the Borel resummation we refer [V], [AKT1], etc. Our expectation is that the Borel sum of λ should define an analytic solution of $(P_I)'$ except on Stokes curves and that on a Stokes curve it has a kind of discontinuity which is described by the so-called connection formula.

Figure 3.1 shows the configuration of Stokes curves for $(P_I)'$. We want to determine the explicit form of the connection formula for λ on each of these Stokes curves. First of all, let us recall that the Borel sum of λ is defined as a Laplace integral of the Borel transform $\lambda_B(t,y)$ of λ. Then it is reasonable to guess that the connection formula for λ should be of the following form:

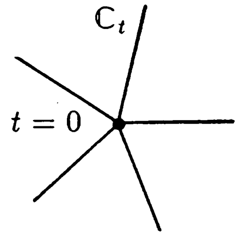

Figure 3.1.

$$\lambda \longrightarrow \lambda + e^{-\phi(t)\eta}\lambda_1 + \cdots, \tag{3.3}$$

where $\phi(t)$ designates the location of a singular point of the Borel transform $\lambda_B(t,y)$ and λ_1 corresponds to its singular part at $y=\phi(t)$. (The meaning of the formula (3.3) is that, if we take the Borel sum of λ in one of the sectorial regions in Figure 3.1 and consider its analytic continuation to the adjacent region across a Stokes curve, then the resulting solution should have the expansion given in the right-hand side of (3.3) in that adjacent region.) In view of (3.3) we find that a new class of formal solutions described in the following theorem is an important ingredient of the connection formula for λ.

Theorem 3.1. *The equation $(P_I)'$ admits the following formal solution:*

$$\lambda = \lambda^{(0)} + e^{-\phi(t)\eta}\lambda^{(1)} + e^{-2\phi(t)\eta}\lambda^{(2)} + \cdots, \tag{3.4}$$

where each $\lambda^{(j)} = \sum_{k\geq 0}\lambda_k^{(j)}(t)\eta^{-k}$ is a formal power series of η^{-1}. Furthermore, in order that the formal series (3.4) may define a solution of $(P_I)'$, it is necessary and sufficient that the following conditions should be satisfied:

(i) $\lambda^{(0)}$ *itself satisfies* $(P_I)'$. *(Hence* $\lambda^{(0)}$ *coincides with the solution mentioned above.)*

(ii) $\phi'(t)^2 = 2\sqrt{t}$

(iii) $\lambda^{(1)}$ *satisfies the following:*

$$(3.5)_1 \qquad -2\left(\sum_{k\geq 2}\lambda_k^{(0)}\eta^{-k}\right)\lambda^{(1)} - \eta^{-1}(2\phi'\frac{d\lambda^{(1)}}{dt} + \phi''\lambda^{(1)}) + \eta^{-2}\frac{d^2\lambda^{(1)}}{dt^2} = 0.$$

This implies that each $\lambda_k^{(1)}$ $(k \geq 0)$ *is a solution of a first order linear ordinary differential equation with a regular singular point at* $t = 0$.

(iv) The other $\lambda^{(j)}$ $(j \geq 2)$ *satisfies*

$$(3.5)_j \quad 2(j^2\lambda_0^{(0)} - \lambda^{(0)})\lambda^{(j)} - j\eta^{-1}(2\phi'\frac{d\lambda^{(j)}}{dt} + \phi''\lambda^{(j)}) + \eta^{-2}\frac{d^2\lambda^{(j)}}{dt^2} = \sum_{\substack{l+m=j\\ l,m\geq 1}}\lambda^{(l)}\lambda^{(m)}.$$

In particular, all $\lambda_k^{(j)}$ $(j \geq 2, k \geq 0)$ *are determined recursively and uniquely. Otherwise stated, "j-instanton contributions* $\lambda^{(j)}$*"* $(j \geq 2)$ *are uniquely determined by* $\lambda^{(0)}$ *and* $\lambda^{(1)}$.

Note that in this Theorem 3.1 we consider the exponential term $-e^{-\phi(t)\eta}$ to be very small compared with the formal power series part $\lambda^{(0)}$. We call such formal solutions "instanton-type" solutions.

Remark 3.1. The explicit form of the equation that $\lambda_k^{(1)}$ $(k \geq 0)$ should satisfy is the following:

$$(3.6) \qquad \begin{cases} (8t\dfrac{d}{dt} + 1)\lambda_0^{(1)} = 0 \\ (8t\dfrac{d}{dt} + 1)\lambda_1^{(1)} = \dfrac{\sqrt{2}}{2}t^{-\frac{5}{4}}(4t^2\dfrac{d^2}{dt^2} + 1)\lambda_0^{(1)} \\ \qquad \cdots \end{cases}$$

Hence we find $\lambda^{(1)}$ has the expansion

$$(3.7) \qquad \lambda^{(1)} = (c_0 + c_1\eta^{-1} + \cdots)t^{-\frac{1}{8}}(1 - \frac{5}{64}\sqrt{2}\,t^{-\frac{5}{4}}\eta^{-1} + \cdots)$$

with arbitrary constants c_l $(l = 0, 1, 2, \cdots)$.

Remark 3.2 Let μ denote $e^{-\phi(t)\eta}\lambda^{(1)}$. Then it can be easily seen that μ satisfies the following linearized equation of $(P_I)'$ at the solution $\lambda^{(0)}$:

$$\frac{d^2}{dt^2}\mu = 2\eta^2\lambda^{(0)}\mu. \tag{3.8}$$

The simplest way to verify statements (ii) and (iii) in Theorem 3.1 is to use this linearized equation (3.8).

Remark 3.3. If we replace (ii) by

$$\phi'_J(t)^2 = \frac{\partial F_J}{\partial \lambda}(\lambda^{(0)}_{J,0}(t), t) \tag{ii$_J$}$$

and modify (iii) and (iv) in an appropriate way, we find Theorem 3.1 holds for every Painlevé equation (P_J). That is, even for (P_J) the following instanton-type solutions should exist:

$$\lambda^{(0)}_J + e^{-\phi_J(t)\eta}\lambda^{(1)}_J + e^{-2\phi_J(t)\eta}\lambda^{(2)}_J + \cdots . \tag{3.4$_J$}$$

In particular, this implies that the Borel transform of the formal power series solution λ_J has a singularity at

$$y = m\left\{\int_\tau^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}dt + y_0\right\} \qquad (y_0 : \text{const}) \tag{3.9}$$

where τ is a turning point and m is an arbitrary integer. Our expectation is that the constant y_0 should be equal to 0 (in the case of $(P_I)'$ this can be really confirmed, as we shall see below) and the definition of Stokes curves for λ_J given in §2 was motivated by this expectation.

It follows from Theorem 3.1 that the explicit form of the connection formula for λ (cf. the expression (3.3)) can be uniquely determined except some constants of integration for $\phi(t)$ and $\lambda^{(1)}_k(t)$ ($k = 0, 1, 2, \cdots$). In what follows we try to fix these constants and find the exact form of the connection formula for the solution λ of the first Painlevé equation $(P_I)'$.

Let us denote by $\lambda_B(t,y)$ the Borel transform of λ as usual. Its explicit form is given as follows by the definition:

$$(3.10) \qquad \begin{aligned}\lambda_B(t,y) &= \sum_{j=0}^{\infty} \frac{a_j}{(2j)!} t^{\frac{1}{2}-\frac{5}{2}j} y^{2j} \\ &= t^{\frac{1}{2}} \sum_{j=0}^{\infty} b_j z^j,\end{aligned}$$

where $\{a_j\}$ is a series of constants given by the recursive relation (3.2) and z and $\{b_j\}$ are defined as follows:

$$(3.11) \qquad z = \frac{25}{32} t^{-\frac{5}{2}} y^2, \ b_j = \frac{a_j}{(2j)!} (\frac{32}{25})^j \ \ (j = 0, 1, 2, \cdots).$$

It is easily verified that the sum $\sum b_j z^j$ actually converges for sufficiently small z and that the Borel transform $\lambda_B(t,y)$ defines an analytic function near $y = 0$ for each fixed $t \neq 0$. Here let us further **assume** that $\lambda_B(t,y)$ can be extended as (multi-valued) analytic function of y to the whole complex plane except at most countable number of isolated singular points. Taking account of the fact that $t^{-1/2}\lambda_B(t,y)$ is described as an analytic function of one variable z, we find Theorem 3.1 implies the following:

(i) The radius of convergence of $\sum b_j z^j$ is equal to 1.

(ii) On the unit circle, the point $z = 1$ is the unique singular point of $\sum b_j z^j$.

(iii) At $z = 1$ we have the following expression:

$$(3.12) \qquad \sum b_j z^j = (1-z)^{1/2} f(z) + g(z),$$

where f and g are holomorphic functions in a neighborhood of $z = 1$.

It follows from (ii) that $\lambda_B(t,y)$ is singular at $y = \pm\frac{4\sqrt{2}}{5} t^{5/4}$. This means that the definition of Stokes curves given in §2 is 'good' in the case of $(P_I)'$, that is, the Borel sum of λ actually has a discontinuity on Stokes curves $\operatorname{Im} t^{5/4} = 0$. Furthermore, noting that the term $(1-z)^{1/2} f(z)$ in the expression (3.12) corresponds to (the Borel transform of) $\lambda^{(1)}$ in Theorem 3.1, we can conclude that on a Stokes curve $\operatorname{Im} t^{5/4} = 0$ the following connection formula holds for λ:

$$(3.3)' \qquad \lambda \longrightarrow \lambda + e^{-\phi(t)\eta}\lambda^{(1)} + e^{-2\phi(t)\eta}\lambda^{(2)} + \cdots,$$

where

$$\phi(t) = \sqrt{2}\int_0^t t^{1/4}dt = \frac{4\sqrt{2}}{5}t^{5/4}, \tag{3.13}$$

$$\lambda^{(1)} = \theta(-i)2^{-3/4}\sqrt{5\pi}\,\eta^{-1/2}t^{-1/8}(1 - \frac{5}{64}\sqrt{2}t^{-5/4}\eta^{-1} + \cdots) \tag{3.14}$$

($\theta = f(z)|_{z=1}$) and $\lambda^{(j)}$ ($j \geq 2$) are uniquely determined by the recursive formula $(3.5)_j$.

This is the connection formula for the formal power series solution λ. As is clear from $(3.3)'$, instanton-type solutions play an important role in this expression.

Finally we show an explicit formula for $\theta = f(1)$ in the following way: Applying the celebrated Darboux method (cf. e.g. [CH], pp.532-533) to the analytic function $\sum b_j z^j$ (roughly speaking, its essence consists in comparing $\sum b_j z^j$ with a linear combination of $(1-z)^{k+1/2}$ ($k = 0, 1, \cdots$)), we obtain the following characterization of the constant θ.

Proposition 3.1. *Let $\{\theta_j\}$ be a sequence of positive numbers defined by the following recursive relation:*

$$\theta_0 = 1,\ \theta_1 = 4/25,\ \theta_2 = 392/1875,$$
$$\theta_j = \frac{(j-1)^2}{(j-\frac{1}{2})(j-\frac{3}{2})}\theta_{j-1} + \frac{1}{2}\sum_{\substack{k+l=j\\ k,l\geq 2}} \frac{(2k-1)!!(2k-3)!!(2l-1)!!(2l-3)!!}{(2j-1)!!(2j-3)!!}\theta_k\theta_l \quad (j \geq 3)$$

where $(2n-1)!! = (2n-1)\cdot(2n-3)\cdot \cdots \cdot 1$. Then $\theta = \lim_{j\to\infty} \theta_j$.

§4. Toward the exact WKB analysis of (P_J) and (SL_J).

As we have seen so far, the formal power series solution λ_J of (P_J) can be locally transformed to the solution λ_I of (P_I) and the connection formula for λ_I is explicitly described in terms of instanton-type solutions. However, in order to obtain the explicit form of the connection formula for λ_J, it should be more efficient if we could establish a local transformation theorem not only for λ_J but also for the instanton-type solutions of (P_J). Though we have not yet succeeded in proving it, we briefly discuss this problem in this section and present a conjecture on the transformation of the instanton-type solutions of (P_J) as well as the transformation of (SL_J) with instanton-type coefficients.

Our discussion will be done in a similar way as in §2. First by substituting instanton-type solutions $(3.4)_J$ of (P_J) into the explicit expression of the potential Q_J we obtain

$$\left(-\frac{d^2}{dx^2}+\eta^2(Q_J^{(0)}+e^{-\phi_J(t)\eta}Q_J^{(1)}+\cdots)\right)\psi_J=0. \tag{4.1}$$

To study transformations of the Schrödinger equations (4.1) with "instanton-type" coefficients, we consider transformations of instanton-type solutions of the Riccati equation associated with (4.1) (instead of WKB solutions of (4.1)). That is, we consider the following Riccati equation

$$S_J^2+\frac{\partial S_J}{\partial x}=\eta^2(Q_J^{(0)}+e^{-\phi_J(t)\eta}Q_J^{(1)}+\cdots), \tag{4.2}$$

and discuss its instanton-type solution S_J of the following form:

$$S_J(x,t,\eta)=S_J^{(0)}(x,t,\eta)+e^{-\phi_J(t)\eta}S_J^{(1)}(x,t,\eta)+\cdots. \tag{4.3}$$

Using these instanton-type solutions (4.3), we try to construct a transformation in such a way that it should satisfy the following:

$$\tilde{S}_J(\tilde{x},\tilde{t},\eta)=-\frac{1}{2}\frac{\frac{\partial^2 x}{\partial\tilde{x}^2}(\tilde{x},\tilde{t},\eta)}{\frac{\partial x}{\partial\tilde{x}}(\tilde{x},\tilde{t},\eta)}+\frac{\partial x}{\partial\tilde{x}}(\tilde{x},\tilde{t},\eta)S_I(x(\tilde{x},\tilde{t},\eta),t(\tilde{t},\eta),\eta), \tag{4.4}$$

where $\tilde{S}_J$ and S_I are instanton-type solutions of the Riccati equation associated with (SL_J) and (SL_I) respectively, and the transformations $x(\tilde{x},\tilde{t},\eta)$ and $t(\tilde{t},\eta)$ are assumed to have the following "instanton-type" expansions:

$$\begin{cases} x(\tilde{x},\tilde{t},\eta)=x^{(0)}(\tilde{x},\tilde{t},\eta)+e^{-\phi_I(t^{(0)}(\tilde{t},\eta))\eta}x^{(1)}(\tilde{x},\tilde{t},\eta)+\cdots, \\ t(\tilde{t},\eta)=t^{(0)}(\tilde{t},\eta)+e^{-\phi_I(t^{(0)}(\tilde{t},\eta))\eta}t^{(1)}(\tilde{t},\eta)+\cdots. \end{cases} \tag{4.5}$$

For example, $x^{(1)}$ and $t^{(1)}$ should satisfy the following:

$$\begin{aligned} (4.6)\quad & e^{\eta^{-1}\Delta\phi}\tilde{S}_J^{(1)}(\tilde{x},\tilde{t},\eta) \\ &=-\frac{1}{2}\frac{x^{(0)''}}{x^{(0)'}}\left(\frac{x^{(1)''}}{x^{(0)''}}-\frac{x^{(1)'}}{x^{(0)'}}\right)+x^{(1)'}S_I^{(0)}(x^{(0)}(\tilde{x},\tilde{t},\eta),t^{(0)}(\tilde{t},\eta),\eta) \\ &\quad +x^{(0)'}\Big[\frac{\partial S_I^{(0)}}{\partial x}(x^{(0)}(\tilde{x},\tilde{t},\eta),t^{(0)}(\tilde{t},\eta),\eta)x^{(1)}(\tilde{x},\tilde{t},\eta) \\ &\quad +\frac{\partial S_I^{(0)}}{\partial t}(x^{(0)}(\tilde{x},\tilde{t},\eta),t^{(0)}(\tilde{t},\eta),\eta)t^{(1)}(\tilde{t},\eta)+S_I^{(1)}(x^{(0)}(\tilde{x},\tilde{t},\eta),t^{(0)}(\tilde{t},\eta),\eta)\Big], \end{aligned}$$

where $\Delta\phi = (\phi_I(t^{(0)}(\tilde{t},\eta)) - \phi_I(t_0^{(0)}(\tilde{t})))\eta^2 = (\phi_I(t^{(0)}(\tilde{t},\eta)) - \tilde{\phi}_J(\tilde{t}))\eta^2$ and $'$ designates the derivative with respect to $\tilde{x}$. Now our conjecture is that, just as in Theorem 2.1, there should exist formal power series $x^{(j)}(\tilde{x},\tilde{t},\eta)$ and $t^{(j)}(\tilde{t},\eta)$ $(j = 0, 1, 2, \cdots)$ which solves the equation (4.4) and each coefficient of which is holomorphic on $\tilde{U} \times \tilde{V}$ and on $\tilde{V}$ respectively (recall that $\tilde{V}$ (resp., $\tilde{U}$) denotes a neighborhood of a point in a Stokes curve for $\tilde{\lambda}_J$ (resp., of the portion of the Stokes curve that begins at a simple turning point and ends at the double one)), and further that in terms of $x^{(1)}(\tilde{x},\tilde{t},\eta)$ and $t^{(1)}(\tilde{t},\eta)$ we have the following transformation formula between $\tilde{\lambda}_J^{(1)}$ and $\lambda_I^{(1)}$:

$$\begin{aligned}(4.7)\qquad &\frac{d\lambda_I^{(0)}}{dt}(t^{(0)}(\tilde{t},\eta),\eta)t^{(1)}(\tilde{t},\eta) + e^{\eta^{-1}\Delta\phi}\lambda_I^{(1)}(t^{(0)}(\tilde{t},\eta),\eta)\\ &=\frac{\partial x^{(0)}}{\partial\tilde{x}}(\tilde{\lambda}_J^{(0)}(\tilde{t},\eta),\tilde{t},\eta)\tilde{\lambda}_J^{(1)}(\tilde{t},\eta) + x^{(1)}(\tilde{\lambda}_J^{(0)}(\tilde{t},\eta),\tilde{t},\eta).\end{aligned}$$

Note that (4.7) can be regarded as the concrete expression of the alien derivative (in the sense of Ecalle) of (2.9) (cf. [E], [P]).

Our expectation is that, once the transformation (4.7) is established, the connection formula for (P_J) can be deduced through it from the results for (P_I). Furthermore we hope that it should be also possible to determine the explicit form of the connection formula for all instanton-type Painlevé transcendents. Concerning these subjects, as our study is still in progress, we would like to discuss them elsewhere in some future.

Appendix. Tables of the Painlevé equations and the associated Schrödinger equations with a large parameter.

Table A.1. (*Painlevé equations with a large parameter* η).

$$(P_I)\qquad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2+t)\,.$$

$$(P_{II})\qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3+t\lambda+\alpha)\,.$$

$$(P_{III})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right].$$

$$(P_{IV})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + \left(t^2+4\alpha_1\right)\lambda - \frac{4\alpha_0}{\lambda}\right].$$

$$(P_V)\qquad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda-\frac{1}{2\lambda}\right)$$
$$+\eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0+\alpha_\infty) - \alpha_0\frac{1}{\lambda^2} + \alpha_2\frac{t}{(\lambda-1)^2} - \alpha_1 t^2\frac{\lambda+1}{(\lambda-1)^3}\right].$$

$$(P_{VI})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+\frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1-\frac{\lambda^2-2t\lambda+t}{4\lambda^2(\lambda-1)^2}\right.$$
$$\left.+\eta^2\left\{(\alpha_0+\alpha_1+\alpha_t+\alpha_\infty) - \alpha_0\frac{t}{\lambda^2} + \alpha_1\frac{t-1}{(\lambda-1)^2} - \alpha_t\frac{t(t-1)}{(\lambda-t)^2}\right\}\right].$$

Table A.2. ($F_J(\lambda,t)$: *coefficient of* η^2 *in* (P_J)).

$$F_I^\dagger(\lambda,t) = 6\lambda^2+t\,.$$

$$F_{II}^\dagger(\lambda,t) = 2\lambda^3+t\lambda+\alpha\,.$$

$$F_{III}^\dagger(\lambda,t) = 2\alpha_\infty t\lambda^4 + \alpha'_\infty\lambda^3 - \alpha'_0\lambda - 2\alpha_0 t\,.$$

$$F_{IV}^\dagger(\lambda,t) = \frac{3}{4}\lambda^4 + 2t\lambda^3 + \left(t^2+4\alpha_1\right)\lambda^2 - 4\alpha_0\,.$$

$$
\begin{aligned}
F_V^\dagger(\lambda,t) &= (\alpha_0+\alpha_\infty)\,\lambda^2(\lambda-1)^3-\alpha_0(\lambda-1)^3+\alpha_2 t\lambda^2(\lambda-1)-\alpha_1 t^2\lambda^2(\lambda+1)\,.\\
F_{VI}^\dagger(\lambda,t) &= (\alpha_0+\alpha_1+\alpha_t+\alpha_\infty)\,\lambda^2(\lambda-1)^2(\lambda-t)^2-\alpha_0 t(\lambda-1)^2(\lambda-t)^2\\
&\quad +\alpha_1(t-1)\lambda^2(\lambda-t)^2-\alpha_t t(t-1)\lambda^2(\lambda-1)^2\,.
\end{aligned}
$$

$$
\begin{aligned}
F_I(\lambda,t) &= F_I^\dagger(\lambda,t)\,.\\
F_{II}(\lambda,t) &= F_{II}^\dagger(\lambda,t)\,.\\
F_{III}(\lambda,t) &= \frac{8}{t\lambda}\,F_{III}^\dagger(\lambda,t)\,.\\
F_{IV}(\lambda,t) &= \frac{2}{\lambda}\,F_{IV}^\dagger(\lambda,t)\,.\\
F_V(\lambda,t) &= \frac{2}{t^2\lambda(\lambda-1)}\,F_V^\dagger(\lambda,t)\,.\\
F_{VI}(\lambda,t) &= \frac{2}{t^2(t-1)^2\lambda(\lambda-1)(\lambda-t)}\,F_{VI}^\dagger(\lambda,t)\,.
\end{aligned}
$$

Table A.3. (*Schrödinger equations with a large parameter* η).

$$
(SL_J)\qquad \left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q_J(x,t,\eta)\right)\psi_J(x,t,\eta)=0\,.
$$

$$
\begin{aligned}
Q_I &= 4x^3+2tx+2K_I-\eta^{-1}\frac{\nu}{x-\lambda}+\eta^{-2}\frac{3}{4(x-\lambda)^2}\,.\\
Q_{II} &= x^4+tx^2+2\alpha x+2K_{II}-\eta^{-1}\frac{\nu}{x-\lambda}+\eta^{-2}\frac{3}{4(x-\lambda)^2}\,.\\
Q_{III} &= \frac{\alpha_0 t^2}{x^4}+\frac{\alpha_0' t}{x^3}+\frac{\alpha_\infty' t}{x}+\alpha_\infty t^2+\frac{tK_{III}}{2x^2}\\
&\quad +\eta^{-1}\left(\frac{1}{2x^2}-\frac{1}{x(x-\lambda)}\right)\lambda\nu+\eta^{-2}\frac{3}{4(x-\lambda)^2}\,.\\
Q_{IV} &= \frac{\alpha_0}{x^2}+\alpha_1+\left(\frac{x+2t}{4}\right)^2+\frac{K_{IV}}{2x}-\eta^{-1}\frac{\lambda\nu}{x(x-\lambda)}+\eta^{-2}\frac{3}{4(x-\lambda)^2}\,.\\
Q_V &= \frac{\alpha_0}{x^2}+\frac{\alpha_1 t^2}{(x-1)^4}+\frac{\alpha_2 t}{(x-1)^3}+\frac{\alpha_\infty}{(x-1)^2}+\frac{tK_V}{x(x-1)^2}\\
&\quad -\eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)}+\eta^{-2}\frac{3}{4(x-\lambda)^2}\,.
\end{aligned}
$$

$$
\begin{aligned}
Q_{VI} &= \frac{\alpha_0}{x^2} + \frac{\alpha_1}{(x-1)^2} + \frac{\alpha_\infty}{x(x-1)} + \frac{\alpha_t}{(x-t)^2} + \frac{t(t-1)K_{VI}}{x(x-1)(x-t)} \\
&\quad - \eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}.
\end{aligned}
$$

Table A.4. (*Hamiltonian K_J in (SL_J)*).

$$
\begin{aligned}
K_I &= \frac{1}{2}\left[\nu^2 - \left(4\lambda^3 + 2t\lambda\right)\right]. \\
K_{II} &= \frac{1}{2}\left[\nu^2 - \left(\lambda^4 + t\lambda^2 + 2\alpha\lambda\right)\right]. \\
K_{III} &= \frac{2\lambda^2}{t}\left[\nu^2 - \eta^{-1}\frac{3\nu}{2\lambda} - \left(\frac{\alpha_0 t^2}{\lambda^4} + \frac{\alpha_0' t}{\lambda^3} + \frac{\alpha_\infty' t}{\lambda} + \alpha_\infty t^2\right)\right]. \\
K_{IV} &= 2\lambda\left[\nu^2 - \eta^{-1}\frac{\nu}{\lambda} - \left(\frac{\alpha_0}{\lambda^2} + \alpha_1 + \left(\frac{\lambda+2t}{4}\right)^2\right)\right]. \\
K_V &= \frac{\lambda(\lambda-1)^2}{t} \\
&\quad \times \left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\nu - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1 t^2}{(\lambda-1)^4} + \frac{\alpha_2 t}{(\lambda-1)^3} + \frac{\alpha_\infty}{(\lambda-1)^2}\right)\right]. \\
K_{VI} &= \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \\
&\quad \times \left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\nu - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1}{(\lambda-1)^2} + \frac{\alpha_\infty}{\lambda(\lambda-1)} + \frac{\alpha_t}{(\lambda-t)^2}\right)\right].
\end{aligned}
$$

Table A.5. (*Deformation equations*).

$$
\frac{\partial\psi_J}{\partial t} = A_J(x,t,\lambda)\frac{\partial\psi_J}{\partial x} + B_J(x,t,\lambda)\psi_J, \qquad \left(B_J = -\frac{1}{2}\frac{\partial A_J}{\partial x}\right).
$$

$$
A_I = A_{II} = \frac{1}{2(x-\lambda)}, \qquad A_{III} = \frac{2\lambda x}{t(x-\lambda)} + \frac{x}{t}, \qquad A_{IV} = \frac{2x}{x-\lambda},
$$

$$
A_V = \frac{\lambda-1}{t}\frac{x(x-1)}{x-\lambda}, \qquad A_{VI} = \frac{\lambda-t}{t(t-1)}\frac{x(x-1)}{x-\lambda}.
$$

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