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Fluid dynamical limit of the Boltzmann equation I

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Abstract

The Boltzmann equation with a small external force is considered. The singular limit of the solution is studied, when the mean free path tends to zero. The macroscopic quantities (mass density, velocity and temparature) associated with the solutions of the Boltzmann equation converge to the solution of the compressible Euler equation with the initial data obtained from that of the Bolzmann equation.

1 Introduction

Let $t>0$ be the time, $x \in \mathbb{R}^3$ the point and $\xi \in \mathbb{R}^3$ the velocity of the gas particles. The change of distribution function $f=f(t,x,\xi)$ is described by the Bolzmann equation (B.1) :

$$\frac{\partial f}{\partial t}+\xi\cdot\nabla_x f+a(x)\cdot\nabla_\xi f=\frac{1}{\epsilon}\,Q[f,f]\;,\quad t>0\;,$$
$$f_{|t=0}=f_0(x,\xi)\;, \tag{1}$$

where $\epsilon\in(0,1]$ is the mean free path, $a(x)=-\nabla b(x)$ is an external force and $Q[\cdot,\cdot]$ is the collision integral acting in the velocity space $\mathbb{R}^3_\xi$:

$$Q[f,g](\xi)=\frac{1}{2}\iint_{\mathbb{R}^3\times S^2} q(|\,\xi-\eta\,|,\theta)\;\{\;f(\xi')\,g(\eta')+f(\eta')\,g(\xi')$$
$$-f(\xi)\,g(\eta)-f(\eta)\,g(\xi)\;\}\;d\eta\,dw\;,$$

$$\xi'=\xi-<\xi-\eta,w>w\;,\quad w\in S^2\;,$$
$$\eta'=\eta-<\xi-\eta,w>w\;,\quad \cos\theta=\frac{<\xi-\eta,w>}{|\,\xi-\eta\,|}\;. \tag{2}$$

The microscopic conservative law holds (even for the complex variables ξ and $\eta \in \mathbb{C}^3$) :

$$\begin{aligned} \frac{\partial(\xi', \eta')}{\partial(\xi, \eta)} &= 1 \quad \text{(mass)} , \\ \xi' + \eta' &= \xi + \eta \quad \text{(momentum)} , \\ <\xi', \xi'> + <\eta', \eta'> &= <\xi, \xi> + <\eta, \eta> \quad \text{(energy)} . \end{aligned}$$

We refer the important properties of the operator $Q[\cdot, \cdot]$:

(i) $Q[f, f] = 0 \Leftrightarrow f(\xi) = \rho(2\pi\theta)^{-3/2} e^{-|\xi - v|^2/2\theta} \; (f \geq 0)$,

(ii) $Q[g, g\varphi] = 0$, $g = e^{-|\xi|^2/2} \Leftrightarrow \varphi(\xi) \in \{1, \xi_1, \xi_2, \xi_3, | \xi |^2\}$,

(iii) $\int_{\mathbb{R}^3} Q[f, h](\xi) \{1, \xi_1, \xi_2, \xi_3, | \xi |^2\} \, d\xi = \{0, 0, 0, 0, 0\}$. (3)

From the distribution function $f(t, x, \xi)$ we define the (macroscopic) fluid dynamical quantities by

(i) $\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi) \, d\xi$ (mass density) ,

(ii) $\rho(t, x) \, u(t, x) = \int f(t, x, \xi) \, \xi \, d\xi$ (velocity) ,

(iii) $\rho(t, x) \, e(t, x) = \int f(t, x, \xi) \frac{1}{2} | \xi - u(t, x) |^2 \, d\xi$ (internal energy) ,

(iii)' $\theta(t, x) = \frac{2}{3} e(t, x)$ (temperature) ,

(iv) $P_{i,j}(t, x) = \int f(t, x, \xi) \{\xi_i - u_i(t, x)\}\{\xi_j - u_j(t, x)\} \, d\xi$ (stress tensor) ,

(iv)' $p(t, x) = \frac{1}{3} P(t, x)$ (pressure) ,

(v) $q_j(t, x) = \int f(t, x, \xi) \frac{1}{2} | \xi - u(t, x) |^2 \{\xi_j - u_j(t, x)\} \, d\xi$ (heat flux) .(4)

We note that if f is a Maxwellian distribution , i.e.

$$f(t, x, \xi) = \frac{\rho(t, x)}{\{2\pi\theta(t, x)\}^{3/2}} e^{-|\xi - u(t,x)|^2/2\theta(t,x)} ,$$

we have the simple relations

(iv) $P_{i,j}(t, x) = p(t, x) \delta_{i,j}$,

(v) $q_j(t, x) = 0$, $1 \leq j \leq 3$,

(vi) $p(t,x) = \rho(t,x)\theta(t,x)$ (ideal gas condition) . (4)'

From (1) we obtain the equation of motion of the Newtonian fluid : (but the system is not closed !)

$$\frac{\partial}{\partial t}\rho + \nabla_x \cdot (\rho u) = 0 \ , \qquad \text{(conservation of mass)}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \sum_j \frac{\partial}{\partial x_j}(\rho u_i u_j) + \sum_j \frac{\partial}{\partial x_j} P_{i,j} = \rho u_i \ , \quad 1 \le i \le 3 \ ,$$

(conservation of momentum)

$$\frac{\partial}{\partial t}\left\{p\left(\frac{1}{2} \mid u \mid^2 + e\right)\right\} + \nabla_x \cdot \left\{\rho\left(\frac{1}{2} \mid u \mid^2 + e\right) v\right\} + \nabla_x \cdot \{Pu + q\} = \rho a \cdot u \,. \quad (5)$$

(conservation of energy)

If $f(t,x,\xi)$ is a Maxwellian of the above form, the equation reduces to the Compressible Euler equation :

$$\frac{\partial}{\partial t}\rho + \nabla_x \cdot (\rho u) = 0 \ ,$$

$$\frac{\partial}{\partial t}u + (u \cdot \nabla_x)u + \frac{1}{\rho}\nabla_x p = a \ ,$$

$$\frac{\partial}{\partial t}\theta + (u \cdot \nabla_x)\theta + (\gamma - 1)\theta \nabla_x u = \frac{2}{3}a \cdot u \ ,$$

$$p = \rho\theta \ , \quad \gamma = 1 + \frac{2}{3} \quad \text{(ideal gas condition) .} \qquad (6)$$

In this paper we show that the solution $f(\epsilon, t, x, \xi)$ of (1) exists in the uniform time interval $[0,T]$ for $\epsilon \in (0,1]$, and it converges to the limit $f(0,t,x,\xi)$ when ϵ tends to zero, if the external force $a(x)$ and the initial data $f_0(x,\xi)$ satisfy some conditions, i.e. analyticity and smallness. We also show that the fluid dynamical part $\{\rho(0,t,x), v(0,t,x), \theta(0,t,x)\}$ of $f(0,t,x,\xi)$ solves (6) with the initial data $\{\rho_0(x), v_0(x), \theta_0(x)\}$ obtained from $f_0(x,\xi)$. In the case where $a(x) = 0$, we have the same results by Nishida [5], Ukai-Asano [6] and Asano-Ukai [2]. Caflisch [3] gave a similiar result without the analycity and smallness of the initial data. His solution does not contain the initial layer.

If we choose another class of the initial data and apply some change of scales, we obtain the solution of the incompressible Navier-Stokes equation from the limit of the solution of (1), when ϵ tends to zero. However this result will be discussed in the succeeding paper, since the approach differs.

2 Compressible Euler limit of the Boltzmann equation

We consider the solution of (1) around the equilibrium

$$g_1(x,\xi) = (2\pi)^{-3/2} e^{-\left(|\xi|^2/2+b(x)\right)} \equiv g_0(\xi) e^{-b(x)} \ .$$

Putting

$$\begin{aligned} f = f(\epsilon,t,x,\xi) &= g_1 + g_1^{1/2} u(\epsilon,t,x,\xi) \ , \\ f_0 = f_0(x,\xi) &= g_1 + g_1^{1/2} u_0(x,\xi) \ , \end{aligned} \tag{7}$$

we obtain the modified Boltzmann equation (B.2):

$$\begin{aligned} \frac{\partial u}{\partial t} + \xi \cdot \nabla_x u + a(x) \cdot \nabla_\xi &= \frac{1}{\epsilon} L_1 u + \frac{1}{\epsilon} \Gamma_1[u,u] \ , \\ u_{|t=0} &= u_0(x,\xi) \ . \end{aligned} \tag{8}$$

Here we use the following notations :

$$\begin{aligned} &\text{(i)} \ L_1 u = 2 g_1^{-1/2} Q[g_1, g_1^{1/2} u] \equiv e^{-b(x)} L u \ , \\ &\quad\ L u = 2 g_0^{-1/2} Q[g_0, g_0^{1/2} u] \\ &\text{(ii)} \ \Gamma_1[u,v] = g_1^{-1/2} Q[g_1^{1/2} u, g_1^{1/2} v] \equiv e^{-b(x)/2} \Gamma[u,v] \ , \\ &\quad\ \Gamma[u,v] = g_0^{-1/2} Q[g_0^{1/2} u, g_0^{1/2} v] \ . \end{aligned} \tag{9}$$

We state the main assumption on $a(x) = - \nabla b(x)$ and $Q[\cdot,\cdot]$. Let $\mathcal{A}(\Omega)$, $(\mathcal{A}_b(\Omega))$ be the set of analytic (and bounded) functions in the domain Ω .

[A] $\partial_x^\alpha b \in \mathcal{A}_b(\Omega_{\rho_0})$, $\Omega_{\rho_0} = I\!R^3 + i(-\rho_0,\rho_0)^3 \equiv I\!R^3 + iI(\rho_0)$, for $|\alpha| \leq$ $\leq l+1$ with $l \geq 3$. $b(x)$ is real for $x \in I\!R^3$.
For simplicity we assume $b(0) = 0$.

[Q] The scattering cross section $q(v,\theta)$ satisfies Grad's condition of "'the angular cutoff hand potential"' i.e.

$$c^{-1}(1+v)^\alpha \leq q(v,\theta) \leq c(1+v)^\alpha \quad \text{for } v \geq 0 \quad \text{with } 0 \leq \alpha \leq 1 \ .$$

Under the condition [Q], the linearized collision operator L has the following properties :

(i) $Lu(\xi+i\eta) = -\nu(\xi+i\eta)u(\xi+i\eta)+\int K(\xi+i\eta,\xi+i\eta)u(\xi+i\eta)\,d\xi$
$\equiv (-\Lambda+K)u$,

$$\nu(\xi+i\eta) = \tfrac{1}{2}\iint q(|\zeta|,\theta)\,g_0(\xi+i\eta-\zeta)\,d\zeta\,dw \in \mathcal{A}(\Omega_\infty) .$$

(ii) There exist ρ_1, σ_0 and $\sigma_1 > 0$ such that

$$\sigma_0(1+|\xi|)^\alpha \le \text{ Re } \nu(\xi+i\eta) \le \sigma_1(1+|\xi|)^\alpha \text{ for } \xi+i\eta\in\Omega_{\rho_1}.$$

(iii) K is a compact operator in $L^p(\mathbb{R}^3+i\eta)$, $1\le p\le\infty$, and K maps

$$L^p_\beta(\Omega_\rho)=\left\{u(\xi+i\eta)\in\mathcal{A}(\Omega_\rho),(1+|\xi|)^\beta u(\xi+i\eta)\in L^\infty(I(\rho));L^p(\mathbb{R}^3_\xi)\right\}$$

into $L^p_{\beta+1}(\Omega_\rho)$ continuously for $\beta\in\mathbb{R}$ and $\rho>0$.

(iv) $L\le 0$ in $L^2(\mathbb{R}^3)$ and L has an isolated eigenvalue 0 with the five dimensional eigenprojection

$$P=\sum_{j=0}^{4}\varphi_j <\cdot,\varphi_j>,\ \{\varphi_j\}=g_0(\xi)^{1/2}\left\{1,\ \xi_1,\ \xi_2,\ \xi_3,\ (|\xi|^2-3)/\sqrt{6}\right\} . \tag{10}$$

We define function spaces :

(i) $X^j_{\rho,\rho'}=\{u(x+iy,\xi+i\eta)\in\mathcal{A}(\Omega_\rho\times\Omega_{\rho'})$;

$$|u|^2_{j,\rho,\rho'}=\sup_{y\in I(\rho),\ \eta\in I(\rho')}\sum_{|\alpha|\le j}|\partial^\alpha_x u(\cdot+iy,\cdot+i\eta)|^2_{L^2(\mathbb{R}^6)}<\infty\} .$$

(ii) Let $\Delta(T)=[0,1]\times[0,T]\setminus(0,0)$.

$Y^j_{\rho,\rho',\gamma}=\{X^j_{\rho-\gamma t,\rho'-\gamma t}$ – valued functions $u(\epsilon,t)$ which are continuous in

$(\epsilon,t)\in\Delta_T$ with the norm

$$|u|_{j,\rho,\rho',\gamma}=\sup_{(\epsilon,t)\in\Delta(T)}|u(\epsilon,t)|_{j,\rho-\gamma t,\rho'-\gamma t}<\infty\}$$

$$\equiv B^0\left(\Delta(T)\ ;\ X^j_{\rho-\gamma t,\rho'-\gamma t}\right) .$$

(ii)' $\tilde{Y}^j_{\rho,\rho',\gamma,\sigma}=\{u(\epsilon,t)\in Y^j_{\rho,\rho',\gamma}$;

$$|u|_{j,\rho,\rho',\gamma,\sigma}=\sup_{(\epsilon,t)\in\Delta(T)}e^{\sigma t/\epsilon}|u(\epsilon,t)|_{j,\rho-\gamma t,\rho'-\gamma t}<\infty\} .$$

(iii) Let $w(t) = (1-t)e^{-t}$. Let $Y_{\rho,\gamma}$ be the space of the same kind as in (ii).

$$Z_\gamma = \{\ u \in Y_{\rho_0-\gamma,\gamma} \text{ for any } \delta > 0\ ;$$

$$\| u \|_\gamma = \sup_{\substack{0 \le \gamma t < \rho_0 - \rho \\ 0 \le \epsilon \le 1}} |\ u(\epsilon, t)\ |_\rho\ w(\tfrac{\gamma t}{\rho_0-\rho}) < \infty\ \}\ . \tag{11}$$

Clearly we have

$$\| \cdot \|_\gamma \le |\cdot|_{\rho_0,\gamma} \le (1-\frac{\gamma'}{\gamma})^{-1} \| \cdot \|'_\gamma\ , \quad \text{for } \gamma > \gamma'\ . \tag{12}$$

We define a (continuous) family of closed operators $B(\epsilon)$ in $X^j_{\rho,\rho'} \equiv X_\rho$ (We omit the index j and ρ' without confusion.) by

$$B(\epsilon) = -\epsilon\,(\xi + i\eta)\cdot\nabla_x + L = -\epsilon\,(\xi+i\eta)\cdot\nabla_x - \Lambda - K\ . \tag{13}$$

In what follows, we show that $B(\epsilon)$ generates a "semigroup" $e^{tB(\epsilon)}$ mapping X_ρ into $X_{\rho-\gamma\epsilon t}$ with a $\gamma > 0$. Using this semigroup, we rewrite (8) as

$$u(t) = e^{tB(\epsilon)/\epsilon}u_0 + \int_0^t e^{(t-s)B(\epsilon)/\epsilon}\left\{\frac{1}{\epsilon}(L_1 - L_0) - a(x)\cdot\nabla_\xi\right\}u(s)\,ds$$

$$+ \frac{1}{\epsilon}\int_0^t e^{(t-s)B(\epsilon)/\epsilon}\Gamma_1[u(s),u(s)]\,ds\ . \tag{14}$$

Hereafter we often omit the parameter ϵ in $u(\epsilon,t)$.

In order to solve (14), we need the abstract Cauchy-Kowalewski theorem and the Ellis-Pinsky theorem on the spectrum of $B(\epsilon)$:

LEMMA 2.1 (Ellis-Pinsky [4]). *Let $\hat{B}(ik) = -ik\cdot\xi + L$ be the symbol operator of $B(1)$ associated with the Fourier transform in x. Then, under the condition* [Q], *$\hat{B}(ik)$ has the following spectral properties :*

(i) There exist positive constants μ_0 and κ_0 such that the spectrum $\hat{B}(-ik)$ in $\{Re\lambda \ge -\mu_0\}$ consists of 5 eigenvalues $\{\lambda_j(ik)\ ;\ 0 \le j \le 4\}$, where $\lambda_j(ik) \in C^\infty(D(2\kappa_0))$, $D(\kappa) = \{k \in I\!R^3\ ;\ |\,k\,| \le \kappa\}$, and

$$\lambda_j(ik) = \pm i|\,k\,|\lambda_{j,1} + (i|\,k\,|)^2\lambda_{j,2} + O(|\,k\,|^3),\ Re\ \lambda_j(ik) \le 0,$$

with $\lambda_{j,1}$ real and $\lambda_{j,2} > 0$, $0 \le j \le 4$.

(ii) There exist eigenfunctions $\{\psi_j(ik);\ 0 \le j \le 4\} \subset C^\infty(D(2\kappa_0); g^{\delta/2}L^2(\mathbb{R}^3),$

$0 < \delta < 1$) *satisfying*

$$\hat{B}(ik)\psi_j(ik) = \lambda_j(ik)\psi_j(ik)\ ,\ 0 \le j \le 4\ ,$$

$$\{\psi_j(0)\ ;\ 0 \le j \le 4\} \equiv \{\varphi_j\ ;\ 0 \le j \le 4\}\quad \textit{(See (10) (iv))},$$

$$\psi_j(ik) - \psi_j(0) = -ik \cdot \psi_{j,1}(ik)\ ,\quad \psi_{j,1}(ik) \in C^\infty(D(2\kappa_0))\ .$$

(iii) Let $\chi_0(\kappa)$ *be the characteristic function of* $D(\kappa_0)$. *Define the projections by*

$$\hat{P}(ik) = \chi_0(\kappa)\sum_{j=0}^{4}\psi_j(ik) < \cdot, \psi_j(-ik) >_{L^2(\mathbb{R}^3)} \equiv \chi_0(k)\sum_{j=0}^{4}\hat{P}_j(ik)\ ,$$

$$\hat{P}'(ik) = 1 - \hat{P}(ik)\ ,$$

$$\tilde{P} = 1 - \sum_{j=0}^{4}\psi_j(0) < \cdot, \psi(0) >_{L^2(\mathbb{R}^3)} = 1 - \hat{P}(0) = 1 - P\ .$$

Then we have (with a κ_0 *replaced by a smaller one if necessary),*

$$(a)\quad e^{t\hat{B}(ik)}\hat{P}(ik) = \chi_0(k)\sum_{j=0}^{4} e^{t\lambda_j(ik)}\hat{P}_j(ik)\ ,$$

$$e^{t\hat{B}(ik)}\hat{P}(ik)\tilde{P} = e^{tB(k)}\hat{P}(ik)\hat{Q}(ik)\cdot ik\tilde{P}\ ,$$

$$\| e^{t\hat{B}(ik)}\hat{P}(ik) \| \le e^{-\mu|k|^2} \le 1 \textit{ with } \mu > 0\ , \textit{ where}$$

$$\hat{Q}(ik) = \sum_{j=0}^{4}\psi_j(ik) < \cdot, \psi_{j,1}(ik) >_{L^2(\mathbb{R}^3)} \equiv \sum_{j=0}^{4}\hat{Q}_j(ik)\ ,$$

and

$$(b)\quad |\, e^{t\hat{B}(ik)}\hat{P}'(ik)u(\xi)\,|_{L^2(\mathbb{R}^3)} \le e^{-t\mu}\ |\, \hat{P}'(ik)u(\xi)\,|_{L^2(\mathbb{R}^3)}\ .$$

We note that the eigenfunctions $\{\varphi_j\}$ in (10) and $\{\psi_j(ik)\}$ in Lemma 2.1 are analytically extended to the complex domain Ω_{ρ_1}, i.e.

LEMMA 2.2 . *(i)* $\{\varphi_j(\xi + i\eta)\}_{j=0}^4 = g_0^{1/2}(\xi + i\eta)\{1,\ \xi_j + i\eta_j\ (1 \le j \le 3),$ $(< \xi + i\eta, \xi + i\eta > -3)/\sqrt{6}\} \subset W^\delta(\Omega_\rho) \equiv \{u \in \mathcal{A}(\Omega_\rho)\ ;\ |u|_\delta^2 \equiv \sup_\eta |g_0^{\delta/2}$

$u(\cdot + i\eta)|^2_{L^2(I\!R^3)} < \infty\}$ *for* $0 \leq \delta < 1$ *and arbitrary* $\rho > 0$.

(ii) $\{\psi_j(ik\,,\,\xi + i\eta)\} \subset W^\delta(\Omega_{\rho_1})$ *for some* $\rho_1 > 0$ *satisfying*

$$B(ik)\,\psi_j(ik\,,\,\xi + i\eta) \equiv \{-i(\xi + i\eta)\cdot k + L\}\psi_j(ik\,,\,\xi + i\eta) = \lambda_j(ik\,,\,i\eta)\psi_j\,,$$
$$\lambda_j(ik\,,\,i\eta) = \lambda_j(ik) - \eta\cdot k + O(|k|^2|\eta|)\,.$$

Proof: Define the Operator $B(ik,\,\xi' + i\eta)$ acting in $L^2(I\!R^3)$ by

$$B(ik,\,\xi' + i\eta)\,u(\xi) = -i(\xi + \xi' + i\eta)\cdot ku(\xi) - \nu(\xi + \xi' + i\eta)u(\xi)$$
$$+ \textstyle\int K(\xi + \xi' + i\eta\,,\,\zeta + \xi' + i\eta)u(\zeta)\,d\zeta\,.$$

Since this is an analytic family of operators, the coresponding Ellis-Pinsky eigenvalues $\{\lambda_j(ik\,,\,\xi' + i\eta)\}$ depend analytically on $\xi' + i\eta$. Recalling the proof of Ellis-Pinsky theorem, we can show that the first order coefficient $\lambda_{j,1}$ of $\lambda_j(ik)$ in Lemma 2.1 (i) are the eigenvalues of $P(\xi + \xi' + i\eta)P$, which do not depend on $\xi' + i\eta$. The eigenfunctions $\{\psi_j(ik\,,\,\xi' + i\eta\,,\cdot)\}$ are calculated from $\{\varphi_j(\xi + \xi' + i\eta)\}$, proving $\psi_j(ik\,,\,\xi' + i\eta\,,\,\cdot) \equiv \psi_j(ik\,,\,\xi + \xi' + i\eta\,,\,0)$. Other properties are easily proved. Q.E.D.

Using Lemma 2.2, we extend the operators $\hat{B}(ik)$, $\hat{P}(ik)$ and $\hat{P}'(ik)$ to the operators in $W^\delta(\Omega_{\rho'}) \equiv W^\delta$, and the operators $B(\epsilon)$, $P(\epsilon)$ and $\bar{P}(\epsilon)$ to $X^l_{\rho,\rho'}$.

LEMMA 2.3 . *(i) There exist constants* ρ_1, μ, $\tilde{\mu} > 0$ *and* $b > 1$ *such that for* $0 \leq \beta \leq 1$ *and* $0 \leq \rho' \leq \rho_1$ $u(t) = e^{t\hat{B}(ik)}u_0$ *satisfies*

$$\frac{d}{dt}|\Lambda^\beta \hat{P}(ik)u(t,\cdot + i\eta)|^2_{L^2(I\!R^3)}$$
$$\leq -2(k\cdot\eta + \mu|k|^2)|\Lambda^\beta \hat{P}(ik)u(t,\cdot + i\eta)|^2_{L^2}\ ,\ \eta \in I(\rho_1)\,, \tag{15}$$

$$\frac{d}{dt}|\Lambda^\beta \hat{P}'(ik)u(t,\cdot + i\eta)|^2_{L^2(I\!R^3)}$$
$$\leq -2(k\cdot\eta + \tilde{\mu})|\Lambda^\beta \hat{P}'(ik)u(t,\cdot + i\eta)|^2_{L^2}$$
$$+ 2b\,|\hat{P}'(ik)u(t,\cdot + i\eta)|^2_{L^2}\ ,\ \eta \in I(\rho_1)\,. \tag{16}$$

The second term on the right hand side is omitted if $\beta = 0$.
(ii) In the space $X_{\rho,\rho'}$, $0 \leq \rho' \leq \rho_1$, *there hold for* $u(t) = e^{t\,B(\epsilon)(ik)}$:

$$|\partial_x^\alpha \Lambda^\beta P(\epsilon)u(t)|^2_{\rho(1-\epsilon t),\rho'} \ \leq\ |\partial_x^\alpha \Lambda^\beta P(\epsilon)u_0|^2_{\rho,\rho'}\,, \tag{17}$$

$$|\partial_x^\alpha \Lambda^\beta \tilde{P}(\epsilon)u(t)|^2_{\rho(1-\epsilon t),\rho'} \leq e^{-2\tilde{\mu}t}\{\ |\partial_x^\alpha \Lambda^\beta \tilde{P}(\epsilon)u_0|^2_{\rho,\rho'} + b\,t|\partial_x^\alpha \tilde{P}(\epsilon)u_0|^2_{\rho,\rho'}\ \}\,, \qquad (18)$$

with the same remark as in (i), $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *can be arbitrary.*

Proof: (i) Since $\hat{B}(ik)$ and $\hat{P}(ik)$ commute, it follows

$$\begin{aligned} \frac{d}{dt}|\hat{P}(ik)u(t,\xi+i\eta)|^2_{L^2} &= 2Re < \hat{B}(ik)\hat{P}(ik)u, \hat{P}(ik)u >_{L^2(I\!R^3)} \\ &= 2Re < \sum \lambda_j(ik,i\eta)\hat{P}_j(ik)u, \hat{P}_j(ik)u >_{L^2(I\!R^3)} \\ &\leq -2\,(\eta\cdot k + \mu|k|^2)|\hat{P}(ik)u|^2_{L^2}\,, \end{aligned}$$

if $|\eta| \leq \rho_1$ and $|k| \leq \kappa_0$ with sufficiently small ρ_1 and κ_0. There also holds

$$Re\ < Lv, v >_{L^2(I\!R^3+i\eta)} \leq -\sigma|\Lambda^{1/2}\tilde{P}v|_{L^2}\ , \sigma > 0\,.$$

Hence we have

$$\frac{d}{dt}|\hat{P}'(ik)u(t,\cdot+i\eta)|^2_{L^2} = 2\,Re < \{-i(\xi+i\eta)\cdot k + L\}\,\hat{P}'(ik)u, \hat{P}'(ik)u >$$

$$\leq -2\,\eta\cdot k|\hat{P}'(ik)u|^2_{L^2} - 2\,\sigma|(1+|\xi|)^{\alpha/2}\tilde{P}\hat{P}'u|^2_{L^2}\,,$$

$$\leq -2\,(\eta\cdot k + \tilde{\mu})|\hat{P}'(ik)u(t,\cdot+i\eta)|^2_{L^2} - 2\,(\sigma - \tilde{\mu} - C|k|)|\hat{P}'(ik)u|^2_{L^2}\,.$$

We have only to choose μ and κ_0 so that $\sigma - \tilde{\mu} - C\kappa_0 > 0$. The last estimate with Λ^β is easily shown if we note that (10) (ii) implies

$$|(\Lambda^{2\beta}K - K\Lambda^{2\beta})\,v|_{L^2} \leq C|\Lambda^\beta v|_{L^2}\,.$$

(ii) Note that

$$\frac{\partial}{\partial t}u(t, x+iy(1-\gamma t), \xi+i\eta) = -i\gamma y\cdot\nabla_x u + B(\epsilon)u(t, x+iy(1-\gamma t, \xi+i\eta)\,.$$

For a function $u(x+iy) \in L^2_0(I\!R + i(-\rho,\rho))$ (See (10) (iii)), we have

$$|u(\cdot+iy)|^2_{L^2} = |e^{-ky}\hat{u}(k)|^2_{L^2}\,.$$

Let $\{e_j\,;\,1 \leq j \leq 3\}$ be the standard unit vectors in $I\!R^3$. Then, we obtain

$$\frac{d}{dt}\sum_{j=1}^3\{\,|u(t,\cdot+iy-iy_j\gamma te_j,\cdot+i\eta|^2_{L^2} + |u(t,x+iy-iy_j(2-\gamma t)e_j,\cdot+i\eta|^2_{L^2}$$

$$= 2 \sum_{j=1}^{3} \{ \left(\gamma(y_j k_j e^{-2y\cdot k} - y_j k_j e^{+2y\cdot k}) e^{-2(y\cdot k)_j} \hat{u}(t,k,\cdot + i\eta)\, ,\, \hat{u}(t,k,\cdot + i\eta)\right)_{L^2}$$
$$+ Re\, (B(\epsilon)u(t,..,\cdot + iy_j(1-\gamma t),..,..)\, ,\, u(t,..,\cdot + iy_j(1-\gamma t),..,..))_{L^2}$$
$$+ Re\, (B(\epsilon)u(t,..,\cdot - iy_j(1-\gamma t),..,..)\, ,\, u(t,..,\cdot - iy_j(1-\gamma t),..,..))_{L^2} \}\, , \tag{19}$$

where $(y\cdot k)_j = y\cdot k - y_j k_j$. Using a trivial inequality:

$$|s|\, e^{|s|} \leq s e^{s} - s e^{-s} + 1/e\, , \tag{20}$$

we have

$$\begin{aligned} -\epsilon\, \eta_j k_j (e^{-2y_j k_j} + e^{2y_j k_j} \ &\leq\ -2\,\epsilon\, |\eta_j / y_j| - \left(y_j k_j e^{-2y_j k_j} + y_j k_j e^{2y_j k_j}\right) \\ &\quad +\epsilon\, |\eta_j/y_j|/e \\ &\leq\ \gamma\left(y_j k_j e^{-2y_j k_j} + y_j k_j e^{2y_j k_j}\right) + \epsilon\, , \end{aligned} \tag{21}$$

if $|\eta_j| \leq \rho'$, $\rho/2 \leq |y_j| \leq \rho$ and $\gamma \geq 4\,\epsilon\,\rho'/\rho$.

Combining (15) and (21) with (19), we obtain (17). In the same way we obtain (18) from (16), (19) and (21). Q.E.D.

Now we construct " the semigroups " generated by:

$$\begin{aligned} A_0 &= -\xi\cdot\nabla_x - a(x)\cdot\nabla_\xi\, , \\ A(\epsilon) &= \epsilon A_0 - e^{-b(x)}\Lambda = -\epsilon\,(-\xi\cdot\nabla_x - a(x)\cdot\nabla_\xi) - e^{-b(x)}\nu(\xi)\, , \\ D(\epsilon) &= A(\epsilon) + e^{-b(x)}K = \epsilon A_0 + L_1\, . \end{aligned}$$

First we define an analytic diffeomosphism $S(t)(x,\xi) = (X(t,x,\xi)\, ,\, \Xi(t,x\xi))$ by solving

$$\frac{dX}{dt} = -\,\Xi \quad , \quad X_{|t=0} = x \in I\!R^3 \subset \Omega_\rho\, ,$$
$$\frac{d\,\Xi}{dt} = -a(X) \quad , \quad \Xi_{|t=0} = \xi \in I\!R^3 \subset \Omega_{\rho'}\, .$$

Then it follows

$$\begin{aligned} e^{tA_0}u_0(x,\xi) &= u_0(S(t)(x,\xi))\, , \\ e^{tA(\epsilon)}u_0(x,\xi) &= u_0(S(t)(x,\xi))\, e^{-\nu(t,x,\xi)}, \\ \nu(t,x,\xi) &= \int_0^t \nu_1(S(\tau)(x,\xi))\, d\tau\, ,\ \nu_1(x,\xi) = e^{-b(x)}\nu(\xi)\, . \end{aligned}$$

Since $K_1 = e^{-b(x)}K$ is a bounded operator in $X^l_{\rho,\rho'}$, $e^{tD(\epsilon)}$ is constructed by the formula

$$e^{tD(\epsilon)} = e^{tA(\epsilon)} + \int_0^t e^{(t-s)A(\epsilon)} K_1 e^{sD(\epsilon)}\, ds\,.$$

In order to give the simple estimates of these semigroups, we use the modified norm $|u|'_{l,\rho,\rho'}$ in $X^l_{\rho,\rho'}$ defined by

$$|u|'^2_{\rho,\rho'} = |u|^2_{l,\rho,\rho'} + |\,\nabla_\xi\, u|^2_{l-1,\rho,\rho'}\,.$$

LEMMA 2.4 . *(i) Let* $|a| = \sum_{0<|\alpha|\le l} \sup_x |\partial_x^\alpha a|$, $|a|' = \sum_{0<|\alpha|\le l} \sup |Re\,\partial_x^\alpha a(x+iy)|$ *and* $\|a\| = \sum_{|\alpha|\le l} \sup |Im\,\partial_x^\alpha a(x+iy)|$. *Let* $\gamma \ge 4\rho'/\rho$ *and* $\gamma' \ge 4\|a\|\rho/\rho'$. *Then there exist* ρ_0 *and* $\rho_1 > 0$ *such that for* $\rho_0/2 \le \rho \le \rho_0$ *and* $\rho_1/2 \le \rho' \le \rho_1$ *there hold*

$$(1)\qquad |e^{tA_0}u_0|'_{\rho-\gamma t,\rho'-\gamma' t} \le e^{t|a|'}|u_0|'_{\rho,\rho'} + t\, e^{t|a|}|u_0|'_{0,0}\,,$$

$$(2)\quad |e^{tA(\epsilon)}u_0|'_{\rho-\gamma\epsilon t,\rho'-\gamma'\epsilon t} \le e^{-\sigma t}\{\, e^{\epsilon t|a|'}|u_0|'_{\rho,\rho'} + \epsilon\, t\, e^{\epsilon t|a|}|u_0|'_{0,0}\,\}\,,$$

$$(3)\quad |e^{tD(\epsilon)}u_0|'_{\rho-\gamma\epsilon t,\rho'-\gamma'\epsilon t} \le e^{\epsilon t|a|'}|u_0|'_{\rho,\rho'} + \epsilon\, t\, e^{\epsilon t|a|}|u_0|'_{0,0}\,.$$

(ii) For $0 \le \beta \le 1$, *there hold*

$$\begin{aligned}(4)\qquad |\Lambda^\beta e^{tA(\epsilon)}u_0|'_{\rho-\gamma\epsilon t,\rho'-\gamma'\epsilon t} &\le e^{-\sigma t}\{\, e^{\epsilon t|a|'}|\Lambda^\beta u_0|'_{\rho,\rho'} \\ &\quad + b\,\epsilon\, t\, e^{\epsilon t|a|}|\Lambda^\beta u_0|'_{0,0}\,\}\,,\end{aligned}$$

$$\begin{aligned}(5)\qquad |\Lambda^\beta e^{tD(\epsilon)}u_0|'_{\rho-\gamma\epsilon t,\rho'-\gamma'\epsilon t} &\le e^{\epsilon t|a|'}|\Lambda^\beta u_0|'_{\rho,\rho'} \\ &\quad + b\,\epsilon\, t\, e^{\epsilon t|a|}|\Lambda^\beta u_0|'_{0,0}\,.\end{aligned}$$

The proof is similar to that of Lemma 2.3 .

Finally we give the L^2-estimate of $\Gamma[u,v]$.

LEMMA 2.5 . *There exists a constant* $d > 0$ *such that for* $0 \le \beta \le 1$

$$\begin{aligned}|\Lambda^{-\frac{1}{2}+\beta}\Gamma[u,v]|_{L^2(\mathbb{R}^3+i\eta)} &\le d\{\, |\Lambda^{\frac{1}{2}+\beta}u|_{L^2}|\Lambda^\beta v|_{L^2} \\ &\quad + |\Lambda^\beta u|_{L^2}\,|\Lambda^{\frac{1}{2}+\beta}v|_{L^2}\,\}\,. \qquad (22)\end{aligned}$$

Proof: According to the definition formula of Γ, we write

$$\Gamma = \Gamma^1 + \Gamma^2 + \Gamma^3 + \Gamma^4 \ ,$$

$$\Gamma^1[u,v] = \tfrac{1}{2}\iint q(|\xi-\eta|,\theta) g_0^{1/2}(\eta) u(\xi')v(\eta')\, d\eta dw = \Gamma^2[u,v] \ ,$$

$$\Gamma^3[u,v] = \tfrac{1}{2}\iint q(|\xi-\eta|,\theta) g_0^{1/2}(\eta) u(\xi)v(\eta)\, d\eta dw = \Gamma^4[u,v] \ .$$

We have only to estimate $\Gamma^1[u,v]$. Clearly the following L^∞ and L^1 estimates hold:

$$\begin{aligned} |\Lambda^{-1+\beta}\Gamma^1[u,v]|_{L^\infty} &\le d|\Lambda^\beta u|_{L^\infty}|\Lambda^\beta v|_{L^\infty} \ , \\ |\Lambda^\beta \Gamma^1[u,v]|_{L^1} &\le d|\Lambda^{1+\beta} u|_{L^1}|\Lambda^\beta v|_{L^1} \ . \end{aligned}$$

We have only to use the following properties:

$$q(|\xi-\eta|) \sim (1+|\xi-\eta|)^\alpha \le (1+|\xi|)^\alpha (1+|\eta|)^\alpha \sim \nu(\xi)\nu(\eta) \ ,$$

$$\nu(\xi')\,\nu(\eta') \sim (1+|\xi'|)^\alpha (1+|\eta'|)^\alpha \ge (1+|\xi|)^\alpha \sim \nu(\xi) \ ,$$

measure preserving of the transform $(\xi,\eta) \leftrightarrow (\xi',\eta')$.

The Riesz-Thorin complex interpolation theory gives our L^2 estimate. Q.E.D.

In order to solve (8), first we replace Γ_1 by $\Gamma_{1,R}$ with a mollified cross-section q_Λ:

$$q_\Lambda(v,\theta) = \chi\left(\tfrac{v}{R}\right) q(v,\theta), \ \chi(v) = (1+v)^{-\alpha} \ .$$

Then $\Gamma_{1,R}$ satisfies the estimate

$$|\Lambda^\beta \Gamma_{1,R}[u,v]|_{L^2(I\!R^3+i\eta)} \le d\,(1+R)^\alpha |\Lambda^\beta u|_{L^2}\, |\Lambda^\beta v|_{L^2} \ . \tag{23}$$

We rewrite (8) as

$$u(t) = e^{tD(\epsilon)/\epsilon} u_0 + \int_0^t e^{(t-s)D(\epsilon)/\epsilon}\, \tfrac{1}{\epsilon}\, \Gamma_{1,R}[u(s),u(s)]\, ds \ . \tag{24}$$

This equation can be solved in $X^l_{\rho_0-\gamma t,\rho_1-\gamma t}$ by the succesive approximation in a short interval $[\,0,\epsilon\,(1+R)^{-\alpha}T_0]$. In fact, the a priori estimate of (24) results to

$$\begin{aligned} |\Lambda^\beta u|_{l,\rho,\rho',\gamma} \ &\le \ e^{t\,(|a|'+b|a|)}|\Lambda^\beta u_0|_{l,\rho,\rho',\gamma} \\ &\qquad + \frac{t}{\epsilon}\, d_l(1+R)^\alpha e^{2t(|a|'+b|a|)}|\Lambda^\beta u|'^2_{l,\rho,\rho',\gamma} \ . \end{aligned} \tag{25}$$

The succesive approximation converges in $\Lambda^\beta X^l_{\rho,\rho'}$ in $[\,0,T\,]$, if

$$D = 1 - 4e^{T\,(|a|'+b|a|)}\tfrac{T}{\epsilon}\, d_l(1+R)^\alpha e^{2T\,(|a|'+b|a|)}|\Lambda^\beta u_0|'_{l,\rho,\rho'} \ > \ 0 \ , \tag{26}$$

i.e. we have the solution $u(t) \equiv u_R(t)$ of the mollified equation (24). From (26) we obtain

$$T \;\geq\; min\,\{1,\,\epsilon\,(1+R)^{-\alpha}T_0\} \quad \text{with}$$

$$T_0 \;=\; \tfrac{1}{8}\left\{d_l\, e^{3(|a|'+b|a|)}|\Lambda^\beta u_0|'_{l,\rho,\rho'}\right\}^{-1}$$

To obtain the better estimate of the existing interval, we decompose the solution as below (We omit subindex R of Γ_1.):

$$u(t) = v(t) + w(t), \quad v \in Y^l_{\rho,\rho',\gamma}\,, \; w \in \tilde{Y}^l_{\rho,\rho',\gamma,\tau}\,, \; 0 < \tau < \tilde{\mu}\,, \tag{27}$$

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{1}{\epsilon}B(\epsilon)w + \frac{1}{\epsilon}\tilde{P}(\epsilon)(L_1 - L)w + \frac{1}{\epsilon}\tilde{P}(\epsilon)\Gamma_1[w,w]\,, \\ &w_{|t=0} = \tilde{P}(\epsilon)w_0\,. \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{\epsilon}B(\epsilon)v + \left\{-a(x)\cdot\nabla_\xi + \frac{1}{\epsilon}(L_1 - L)\right\}v \;+\; \frac{1}{\epsilon}P(\epsilon)\Gamma_1[v+w, v+w] \\ &\quad + \{-a(x)\cdot\nabla_\xi + \frac{1}{\epsilon}P(\epsilon)(L_1 - L_0)\}w \;+\; \frac{1}{\epsilon}\tilde{P}(\epsilon)\{\Gamma_1[v,v] + 2\Gamma_1[v,w]\}\,, \\ v_{|t=0} &= P(\epsilon)u_0\,. \end{aligned} \tag{29}$$

If we solve (27), then $v = u - w$ solves (28). We have only to get the uniform estimate of v with respect to ϵ and R. In this way we have

THEOREM 2.1 *Assume* [A] *and* [Q]. *Then: (i) there exist* $\rho_1 > 0$, $0 < b_0 \ll 1$ *and* $0 < a_0 \ll 1$ *and* $\gamma > 1$ *such that if* $|b(x+iy)| = |b(x+iy) - b(0)| \leq b_0$ *and* $|\Lambda u_0|_{l,\rho,\rho'} \leq a_0$ *with* $0 < \rho \leq \rho_0$ *and* $0 < \rho' \leq \rho_1$, *then the solution* $u(t)$ *of (14) exists uniquely in* $Y_{\rho,\rho',\gamma}$ *on* $[0, T]$.

(ii) $u(t) = u(\epsilon, t)$ *is decomposed as in (27). If* $f_0 > 0$ *then* $f > 0$ *and* $f(0,t,x,\xi) = g_1 + g_1^{1/2}\, v(0,t,x,\xi)$ *is a Maxwellian:*

$$f(0,t,x,\xi) = \rho(t,x)\,\{2\pi\theta(t,x)\}^{-3/2}\, e^{-|\xi - u(t,x)|^2/2\theta(t,x)}\,.$$

$\{\rho(t,x),\, u(t,x),\, \theta(t,x)\}$ *solves the compressible Euler equation (6) with the initial data obtained from* $f_0(x,\xi)$.

Proof: (of Theorem) Step 1. Rewrite (28) as

$$w(t) = e^{tB(\epsilon)/\epsilon}\tilde{P}(\epsilon)u_0 + \frac{1}{\epsilon}\int_0^t e^{(t-s)B(\epsilon)/\epsilon}\tilde{P}(\epsilon)(L_1 - L)w(s)\,ds$$

$$+\frac{1}{\epsilon}\int_0^t e^{(t-s)B(\epsilon)/\epsilon}\Gamma_1[w(s),w(s)]\,ds\,. \qquad (30)$$

By virtue of Lemma 2.3 (ii), we obtain an a priori estimate for w(t):

$$|\Lambda w|_{l,\rho,\rho',\gamma,t} \le |\Lambda\tilde{P}(\epsilon)u_0|_{l,\rho,\rho'} + \tfrac{C}{\tau}(b_0+\delta b_1)|\Lambda w|_{l,\rho,\rho',\gamma,\tau}$$
$$+\tfrac{C}{2\tau}d_l(b_0+\delta b_1)|\Lambda w|^2_{l,\rho,\rho',\gamma,\tau}\,,\ 0\le\tau<\tilde{\mu}\,. \qquad (31)$$

Here $b_1 = \sum_{|\alpha|\le l}\sup|\partial_x^\alpha\nabla b(x+iy)|$, and we apply the modified norm of $X^l_{\rho,\delta}$:

$$|u|^2_{l,\rho,\rho'} = \sum_{|\alpha|\le l}\delta^{|\alpha|}|\partial_x^\alpha|^2_{0,\rho,\rho'}\ ,0<\delta\ll 1\,, \qquad (32)$$

the Banach scale w.r.t. ρ and the following Lemma.

Lemma 2.6 *Put* $g(t)=\int_0^t e^{(t-s)B(\epsilon)/\epsilon}\tilde{P}(\epsilon)\,h(s)\,ds$. *Then,*

$$|\Lambda^\beta g|_{l,\rho,\rho',\gamma,\tau} \le \tfrac{C\epsilon}{\tau}|h|_{l,\rho,\rho',\gamma,\tau}\,,\ 0\le\beta\le 1,\ 0<\tau<\tilde{\mu}\,. \qquad (33)$$

Proof: (of Lemma 2.6) We give a rough sketch. Using the argument in the proof of Lemma 2.3, we obtain

$$\frac{1}{2}\frac{d}{dt}|g(t)|^2_{0,\rho-\gamma t,\rho'} \sim Re\left(\frac{1}{\epsilon}B(\epsilon)\tilde{P}(\epsilon)g\,,\,\tilde{P}(\epsilon)g\right)+Re\,(h,g)$$
$$\le -\frac{\tilde{\mu}}{\epsilon}|\Lambda^{1/2}\tilde{P}(\epsilon)g|^2+|\tilde{P}(\epsilon)g|^2+|h|\,|g|\,,\ \text{i.e.}$$
$$\frac{d}{dt}\{\,e^{\tau t/\epsilon}|g(t)|_{0,\rho-\gamma t,\rho'}\,\} \le |h(t)|_{0,\rho-\gamma t,\rho'}\ \text{for } 0<\epsilon\le\epsilon_0<1\,.$$

This proves

$$e^{\tau t/\epsilon}|g(t)|_{0,\rho-\gamma t,\rho'} \le \int_0^t |h(s)|_{0,\rho-\gamma t,\rho'}\,ds \le \frac{\epsilon}{\tau}|h|_{0,\rho,\rho',\gamma,\tau}\,.$$

Similary we have

$$\frac{1}{2}\frac{d}{dt}|\Lambda^\beta g|^2 \sim Re\left(\frac{1}{\epsilon}\Lambda^\beta B(\epsilon)\tilde{P}(\epsilon)g\Lambda^\beta\tilde{P}(\epsilon)g\right)+Re\,(h,\Lambda^\beta g)$$
$$+\left|\left(|\Lambda^\beta[\Lambda^\beta K-K\Lambda^\beta]g\,,\,g\right)\right|+|\Lambda^\beta g|^2$$
$$\le -\frac{\tilde{\mu}}{\epsilon}|\Lambda^{\beta+\frac{1}{2}}g|^2+C\,|\Lambda^{\beta-\frac{1}{2}}g|^2+|\Lambda^\beta g|^2+|h|\,|\Lambda^\beta g|\,.$$

For $\beta=1/2$ and $0<\epsilon\le\epsilon_0\ll 1$, this implies

$$\frac{1}{2}\frac{d}{dt}\left\{e^{\tau t/\epsilon}(|\Lambda^\beta g|^2_{0,\rho-\gamma t,\rho'}+C'|g|^2_{0,\rho-\gamma t,\rho'}\right\}$$

$$\leq (1+C')|h|_{0,\rho-\gamma t,\rho'} .$$

The case $\beta = 1$ is treated in the same way. Q.E.D.

Step 2. (Continued to the proof of Theorem.) Assume

$$\frac{C}{\tau}(b_0 + \delta b_1) \leq 1/2 \quad \text{and}$$

$$D_1 = \left\{1 - \frac{C}{\tau}(b_0 + \delta b_1)\right\}^2 - 4\frac{C}{2\tau} d_l |\Lambda\tilde{P}(\epsilon)u_0|_{l,\rho,\rho'} > 0 . \tag{34}$$

Then the successive approximation of (30) converges in $\tilde{Y}^l_{\rho,\rho',\gamma,\tau}$, and the limit $w(t)$ solves (30). The estimate is uniform w.r.t R:

$$|\Lambda w(t)|_{l,\rho,\rho',\gamma,\tau} \leq 2\,|\Lambda\tilde{P}(\epsilon)u_0|_{l,\rho,\rho'} . \tag{35}$$

Sep 3. Since $v(t)$ satisfies (29) in the strong sence for $u_0 \in D(B(\epsilon))$ we rewrite (19) as

$$\begin{aligned} v(t) \;=\; & e^{tB(\epsilon)/\epsilon}P(\epsilon)u_0 + \int_0^t e^{(t-s)B(\epsilon)/\epsilon}\left\{-a(x)\cdot\nabla_\xi + \frac{1}{\epsilon}(L_1 - L_0)\right\} v(s)\,ds \\ & + \int_0^t e^{(t-s)B(\epsilon)/\epsilon}\left\{-a(x)\cdot\nabla_\xi + \frac{1}{\epsilon}P(\epsilon)(L_1 - L_0)\right\} w(s)\,ds \\ & + \frac{1}{\epsilon}\int_0^t e^{(t-s)B(\epsilon)/\epsilon}\{P(\epsilon)\Gamma_1[v+w, v+w] \\ & \qquad + \tilde{P}(\epsilon)\Gamma_1[v,v] + 2\tilde{P}(\epsilon)\Gamma_1[v,w]\}\,ds . \end{aligned} \tag{36}$$

Applying the same method in [1] together with Lemma 2.1, and assuming the Banach scale parameter ω separately in x and ξ, we obtain with $\omega \geq 2\gamma e$

$$\begin{aligned} \|v\|_{l,\omega} \;\leq\; & |P(\epsilon)u_0|_{l,\rho,\rho'} + C\left\{\frac{1}{\omega}b_1 + \left(\frac{1}{\omega}+1\right)(b_0 + \delta b_1)\right\}\|v\|_{l,\omega} \\ & + C\left\{\frac{1}{\omega}b_1 + \left(\frac{1}{\omega}+1\right)(b_0 + \delta b_1)\right\}\|w\|_{l,\omega} \\ & + C\left\{\frac{2}{\omega}R_1(\|v\|_{l,\omega} + \|w\|_{l,\omega})\right\} + \frac{C}{\tau}d_l\frac{3}{2}R_1 , \end{aligned}$$

under the conditions

$$\begin{aligned} \|v\|_{l,\rho,\rho',\omega} &= \sup_t |v(t)|_{l,\rho-\omega t,\rho'-\omega t} \leq R_1 , \\ \|w\|_{l,\rho,\rho',\omega} &\leq |w|_{l,\rho,\rho',\gamma} \leq 2\,|\tilde{P}(\epsilon)u_0| \leq \frac{1}{4}R_1 . \end{aligned} \tag{37}$$

If we choose ω sufficiently large so that the coefficient of $\|v\|_{l,\omega}$ does not exceed 1/2 on the right hand side, we obtain

$$\|v\|_{l,\omega} \;\leq\; 2\,|P(\epsilon)u_0|_{l,\rho,\rho'} + \|w\|_{l,\omega} + 3\,\tfrac{C}{\tau}\,d_l R_1^2\,.$$

From (12) it follows

$$\begin{aligned} |v|_{l,\rho,\rho',2\omega} &\leq\; 2\,\|v\|_{l,\omega} \\ &\leq\; 4\,|P(\epsilon)u_0|_{l,\rho,\rho'} + \tfrac{1}{2}R_1 + 6\,\tfrac{C}{\tau}\,d_l R_1^2\,. \end{aligned}$$

If we choose a small R_1, and then $|P(\epsilon)u_0|_{l,\rho,\rho'}$ sufficiently small, the right hand side does not exceed R_1, i.e.

$$|v|_{l,\rho,\rho',2\omega} \leq R_1 \quad \text{for} \quad |P(\epsilon)u_0|_{l,\rho,\rho'} \leq a_0\,,\; |\tilde{P}(\epsilon)u_0| \leq \tfrac{1}{8}\,R_1\,.$$

Since this estimate is independent of the mollifying parameter R, we have the desired estimate for $|v|_{l,\rho,\rho',2\omega}$. $|\Lambda v|_{l,\rho,\rho',\omega}$ is estimated similary. Clearly v converges to a strong solution of (36) with the original Γ_1 as R tends to ∞. The existing time T is determined by the condition $\rho - \omega T \;\leq\; \rho/2$ and $\rho' - \omega T \;\leq\; \rho'/2$. Uniqueness of the solution is shown in a similar way.

Step 4. Fix $t > 0$. We can show easily

$$\tilde{P}(\epsilon)v(t) = -\left[B(\epsilon)\tilde{P}(\epsilon)\right]^{-1}\{(L_1 - L_0)v(t) + \Gamma_1[v(t), v(t)]\} + o(1)$$

as ϵ tends to zero. $v(\epsilon, t) \in \Lambda^{-1}Y^l_{\rho,\rho'}$ has a limit and satisfies

$$L_1\,v(0,t) + \Gamma_1\,[v(0,t)\,,\, v(0,t)] = 0\,.$$

This shows $f(0,t,x,\xi) = g_1 + g_1^{1/2}v(0,t,x,\xi)$ is a Maxwellian. Other parts of the theorem are proved in a standard way. Q.E.D.

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