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ON A CONJECTURE OF SHIMURA CONCERNING PERIODS OF HILBERT MODULAR FORMS

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Introduction. In this paper, we shall give an affirmative answer to an essential part of the conjecture of Shimura on P -invariants of Hilbert modular forms.

Let F be a totally real algebraic number field of degree n and J_F be the set of all isomorphisms of F into $\mathbf{C}$. Let F_A (resp. $F_A^\times$) be the adele ring (resp. the idele group) of F and $F_\infty^\times$ be the archimedean part of $F_A^\times$. Let χ be a primitive system of eigenvalues of Hecke operators which occurs in the space of holomorphic Hilbert modular cusp forms on $GL(2, F_A)$ of weight k and $\mathbf{f}$ be the primitive form which belongs to χ . In [S1], Shimura introduced an invariant $u(\epsilon, \mathbf{f}) \in \mathbf{C}^\times$ for every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ such that

$$(0) \quad D(m, \mathbf{f}, \varphi) \sim \pi^{mn} u(\epsilon, \mathbf{f})$$

for certain critical values $m \in \mathbf{Z}$ whenever a Hecke character φ of $F_A^\times$ satisfies $\varphi_\infty(x) = \prod_{\tau \in J_F} (\text{sgn } x_\tau)^{m+\epsilon(\tau)}$ for $x = (x_\tau) \in F_\infty^\times$. Here $D(m, \mathbf{f}, \varphi)$ is the standard L -function attached to $\mathbf{f}$ twisted by φ and we write $a \sim b$ for $a, b \in \mathbf{C}$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. Put $U(\chi, \epsilon) = u(\epsilon, \mathbf{f})$.

In [S4], Shimura introduced another invariant $Q(\chi, \delta) \in \mathbf{C}^\times$ for every subset δ of J_F when χ occurs in the space of holomorphic automorphic forms on a quaternion algebra over F of signature $(\delta, J_F \setminus \delta)$ and showed that this invariant appears in critical values of the Rankin-Selberg convolution of two Hilbert modular forms. He conjectured further the following (Conjecture 5.12 of [S4], cf. also [S5], p. 293, (C1), (C2), (C3), (C4) and (C9))

Conjecture P. Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ . Put $k_0 = \max_{\tau \in J_F} (k(\tau))$. Then for every subset δ of J_F and every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^\delta$, there exists a constant $P(\chi, \delta, \epsilon) \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$ which satisfies the following properties.

$$(P1) \quad \pi^{(k_0-2)n/2 - \sum_{\tau \in J_F} k(\tau)/2} U(\chi, \epsilon) \sim P(\chi, J_F, \epsilon).$$

$$(P2) \quad \begin{aligned} Q(\chi, \delta) &\sim \pi^{|\delta|} P(\chi, \delta, \epsilon_1) P(\chi, \delta, \epsilon_2) \\ &\text{if } \epsilon_1(\tau) + \epsilon_2(\tau) \equiv 1 \pmod{2} \text{ for every } \tau \in \delta. \end{aligned}$$

$$(P3) \quad \begin{aligned} P(\chi, \delta_1 \cup \delta_2, \epsilon_1 \cup \epsilon_2) &\sim P(\chi, \delta_1, \epsilon_1) P(\chi, \delta_2, \epsilon_2) \text{ if } \delta_1 \cap \delta_2 = \emptyset, \text{ where} \\ (\epsilon_1 \cup \epsilon_2)(\tau) &= \begin{cases} \epsilon_1(\tau) & \text{if } \tau \in \delta_1, \\ \epsilon_2(\tau) & \text{if } \tau \in \delta_2. \end{cases} \end{aligned}$$

$$(P4) \quad \begin{aligned} &\text{When } \chi \text{ is of CM-type, } P(\chi, \delta, \epsilon) \sim \pi^{-|\delta|} p_K(\xi, \eta) \text{ holds,} \\ &\text{where } p_K \text{ stands for the symbol of CM-periods introduced in [S2].} \end{aligned}$$

The principal result of this paper is:

Main Theorem. Assume $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ . Then, for every $\tau \in J_F$, there exist constants $c_\tau^\pm(\chi) \in \mathbf{C}^\times$ determined uniquely mod $\overline{\mathbf{Q}}^\times$ such that

$$(1) \quad U(\chi, \epsilon) \sim \prod_{\tau \in J_F} c_\tau^{\epsilon(\tau)}(\chi),$$

$$(2) \quad Q(\chi, \delta) \sim \pi^{(k_0-1)|\delta| - \sum_{\tau \in \delta} k(\tau)} \prod_{\tau \in \delta} c_\tau^+(\chi) c_\tau^-(\chi).$$

Here we understand that $c_\tau^0(\chi) = c_\tau^+(\chi)$, $c_\tau^1(\chi) = c_\tau^-(\chi)$ identifying $\mathbf{Z}/2\mathbf{Z}$ with $\{0, 1\}$. By this theorem, it is clear that $P(\chi, \delta, \epsilon)$ satisfying (P1) $\sim$ (P3) is given by

$$(3) \quad P(\chi, \delta, \epsilon) \sim \pi^{(k_0-2)|\delta|/2 - \sum_{\tau \in \delta} k(\tau)/2} \prod_{\tau \in \delta} c_\tau^{\epsilon(\tau)}(\chi).$$

We note that in [Y], §6, we have defined $Q(\chi, \delta) \bmod \overline{\mathbf{Q}}^\times$ assuming only $k(\tau) \geq 3$ for all $\tau \in \delta$.

Let us now outline our ideas of the proof and contents of each section. In §1, we shall review known properties of two basic period invariants $Q(\chi, \delta)$ and $U(\chi, \epsilon)$. In §2, Lemma 1, we shall show that a necessary and sufficient condition for the existence $c_\tau^\pm(\chi)$ as in Main Theorem is the following relations (R1) $\sim$ (R3).

$$(R1) \quad \begin{aligned} U(\chi, \epsilon_1)U(\chi, \epsilon_2) &\sim \pi^{n(1-k_0) + \sum_{\tau \in J_F} k(\tau)} Q(\chi, J_F) \\ &\text{if } \epsilon_1(\tau) + \epsilon_2(\tau) \equiv 1 \bmod 2 \text{ for every } \tau. \end{aligned}$$

$$(R2) \quad Q(\chi, \delta_1)Q(\chi, \delta_2) \sim Q(\chi, \delta_1 \cup \delta_2) \quad \text{if } \delta_1 \cap \delta_2 = \emptyset.$$

$$(R3) \quad \begin{aligned} U(\chi, \epsilon_1)U(\chi, \epsilon_2) &\sim U(\chi, \mu_1)U(\chi, \mu_2) \\ &\text{if } \{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{\mu_1(\tau), \mu_2(\tau)\} \text{ for every } \tau. \end{aligned}$$

We shall also prove (P4) in §2.

Now (R1) is already proved in [S1], Theorem 4.3. Harris [Ha3] proved (R2) under certain conditions, in particular when n , $|\delta_1|$ and $|\delta_2|$ are all even. In §3, using a base change lift of χ to a totally real quadratic extension of F , we shall remove this parity condition and obtain (R2) (Theorem 2).

In §4, we shall prove (R3). By (0), we see that (R3) follows if

$$(4) \quad D(m, \mathbf{f}, \varphi_1)D(m, \mathbf{f}, \varphi_2) \sim D(m, \mathbf{f}, \psi_1)D(m, \mathbf{f}, \psi_2)$$

holds for one choice of a non-vanishing critical value m and of Hecke characters $\varphi_1, \varphi_2, \psi_1, \psi_2$ of $F_A^\times$ whose infinity types correspond to $\epsilon_1, \epsilon_2, \mu_1, \mu_2$ respectively. Let K be a quadratic extension of F such that the Hecke character η of $F_A^\times$ corresponding to K/F satisfies $\eta_\infty = (\varphi_1\varphi_2)_\infty = (\psi_1\psi_2)_\infty$. Again by (0), (4) reduces to

$$(5) \quad D(m, \tilde{\mathbf{f}}, \varphi_1 \circ N_{K/F}) \sim D(m, \tilde{\mathbf{f}}, \psi_1 \circ N_{K/F}),$$

where $\tilde{\mathbf{f}}$ is the base change lift of $\mathbf{f}$ to K . By our choice of K , $(\varphi_1 \circ N_{K/F})_\infty = (\psi_1 \circ N_{K/F})_\infty$ holds and we obtain (5) from a result of Hida [Hi] (§4, Theorem 3).

In §5, we shall prove the invariance of $c_\tau^\pm(\chi)$ under the base change of χ to a totally real cyclic extension of F (Theorem 4). In §6, we shall discuss a possible generalization of Main Theorem including the case where $k(\tau) = 2$ for some τ .

Notation. Throughout the paper, we fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ as the subfield of $\mathbf{C}$. A finite extension of $\mathbf{Q}$ in $\overline{\mathbf{Q}}$ will be called an algebraic number field. For an algebraic number field F , F_v denotes the completion of F at a place v , J_F the set of all isomorphisms of F into $\mathbf{C}$ and I_F the free abelian group generated by J_F . We denote by $\mathfrak{a}_r^F$ (resp. $\mathfrak{a}_c^F$) the set of all real (resp. complex) archimedean places of F and put $\mathfrak{a}^F = \mathfrak{a}_r^F \cup \mathfrak{a}_c^F$. We shall drop the superscript F when the reference to F is clear from the context. When F is totally real, we identify $\mathfrak{a}^F$ with J_F ; a totally imaginary quadratic extension of F will be called a *CM*-extension of F .

For an algebraic group G defined over F , G_A denotes the adelization of G , G_∞ the archimedean part of G_A and $G_{\infty+}$ the identity component of G_∞ . For $x \in F_A^\times$, $|x|_A$ denotes the idele norm of x . For an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $GL(2, F_A)$, $L_f(s, \pi) = \prod_v L(s, \pi_v)$, v extending over all finite places, denotes the finite part of the Jacquet-Langlands L -function attached to π . For $a, b \in \mathbf{C}$, we denote $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$.

§1. Review on Q -invariants and U -invariants

Let F be a totally real algebraic number field of degree n . Let B be a quaternion algebra over F such that B splits (resp. ramifies) at the archimedean places $\tau \in \delta$ (resp. δ'). We call such a B a quaternion algebra of *signature* (δ, δ') . We assume $\delta \neq \emptyset$. Put $G = \text{Res}_{F/\mathbf{Q}}(B^\times)$ and call Z the center of G . We identify Z_A with $F_A^\times$. For $k = \sum_{\tau \in \delta} k(\tau)\tau$ and $\kappa = \sum_{\tau \in \delta'} \kappa(\tau)\tau \in I_F$, we define the space of cusp forms $S_{k, \kappa}(B)$ on G_A of weight (k, κ) as in [S3], II, [Y], §6.

For $\mathbf{f}, \mathbf{g} \in S_{k, \kappa}(B)$, we define the inner product

$$(1.1) \quad \langle \mathbf{f}, \mathbf{g} \rangle = \int_{Z_{\infty+} G_{\mathbf{Q}} \backslash G_A} {}^t \overline{\mathbf{f}(x)} \mathbf{g}(x) dx$$

normalizing the invariant measure so that $\text{vol}(Z_{\infty+} G_{\mathbf{Q}} \backslash G_A) = 1$. If there exists $0 \neq \mathbf{f} \in S_{k, \kappa}(B)$ and a Hecke character ψ of $F_A^\times$ of finite order such that

$$(1.2) \quad \mathbf{f}|T(\mathfrak{p}) = \chi(\mathfrak{p})\mathbf{f} \quad \text{for almost all } \mathfrak{p}, \quad \mathbf{f}(zx) = \psi(z)\mathbf{f}(x), \quad z \in Z_A, x \in G_A,$$

$T(\mathfrak{p})$ being the Hecke operator at the prime ideal $\mathfrak{p}$, we say that a system of eigenvalues of Hecke operators χ occurs in $\mathcal{S}_{k,\kappa}(B)$. Strictly speaking, we should say that (χ, ψ) is a system of eigenvalues of Hecke operators. For simplicity, we shall drop ψ and regard χ accompanying the central character ψ . Let $\mathcal{S}_{k,\kappa}(B, \overline{\mathbf{Q}})$ be the set of all $\overline{\mathbf{Q}}$ -rational elements in $\mathcal{S}_{k,\kappa}(B)$. When χ is given, we set

$$\begin{aligned} W(\chi, B) &= \{\mathbf{f} \in \mathcal{S}_{k,\kappa}(B) \mid \mathbf{f}|T(\mathfrak{p}) = \chi(\mathfrak{p})\mathbf{f} \text{ for almost all } \mathfrak{p}, \\ &\quad \text{and } \mathbf{f}(zx) = \psi(z)\mathbf{f}(x), \quad z \in Z_A, x \in G_A\}, \\ W(\chi, B, \overline{\mathbf{Q}}) &= W(\chi, B) \cap \mathcal{S}_{k,\kappa}(B). \end{aligned}$$

By the Shimizu-Jacquet-Langlands correspondence ([JL]), if χ occurs in $\mathcal{S}_{k,\kappa}(B)$, then it also occurs in $\mathcal{S}_{m,0}(M_2(F))$, where $m(\tau) = k(\tau)$ (resp. $\kappa(\tau) + 2$) if $\tau \in \delta$ (resp. $\tau \in \delta'$). If (1.2) holds for all $\mathfrak{p}$ with the primitive form (the new form) $\mathbf{f} \in \mathcal{S}_{m,0}(M_2(F))$, then we call χ *primitive* (cf. [S3], II, p. 583).

Assume that χ occurs in $\mathcal{S}_{k,0}(M_2(F))$. In [Y], §6, we have shown the following facts sharpening previous results obtained by Shimura [S4], [S5].

$$(1.3) \quad \langle \mathbf{f}, \mathbf{f} \rangle \pmod{\overline{\mathbf{Q}}^\times} \text{ is independent of } 0 \neq \mathbf{f} \in W(\chi, B, \overline{\mathbf{Q}}).$$

$$(1.4) \quad \begin{aligned} &\text{If } B_1 \text{ and } B_2 \text{ are of signature } (\delta, \delta') \text{ and } k(\tau) \geq 2 \text{ for all } \tau \in J_F, \\ &\text{then } \langle \mathbf{f}, \mathbf{f} \rangle \sim \langle \mathbf{g}, \mathbf{g} \rangle \text{ for } \mathbf{f} \in W(\chi, B_1, \overline{\mathbf{Q}}), 0 \neq \mathbf{g} \in W(\chi, B_2, \overline{\mathbf{Q}}). \end{aligned}$$

If $W(\chi, B) \neq \{0\}$ for some quaternion algebra B of signature (δ, δ') , we put

$$(1.5) \quad Q(\chi, \delta) = \langle \mathbf{f}, \mathbf{f} \rangle$$

taking some non-zero form $\mathbf{f} \in W(\chi, B, \overline{\mathbf{Q}})$. By (1.3) and (1.4), $Q(\chi, \delta) \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$ is well defined. Let F_1 be a totally real cyclic extension of degree l of F . We exclude the case where $k(\tau) = 1$ for all $\tau \in J_F$. Then there exists a base change lift $\tilde{\chi}$ of χ which occurs in $\mathcal{S}_{\tilde{k},0}(M_2(F_1))$ where $\tilde{k}(\tau) = k(\tau|F)$, $\tau \in J_{F_1}$. We have

$$(1.6) \quad \text{If } \chi \text{ occurs in } \mathcal{S}_{m,\kappa}(B), \text{ then } \tilde{\chi} \text{ occurs in } \mathcal{S}_{\tilde{m},\tilde{\kappa}}(B \otimes_F F_1).$$

$$(1.7) \quad Q(\tilde{\chi}, \tilde{\delta}) = Q(\chi, \delta)^l \quad \text{if } k(\tau) \geq 3 \text{ for all } \tau \in \delta.$$

Here we have assumed $k(\tau) \geq 3$ for all $\tau \in \delta$ for some technical reasons (cf. §6); $\tilde{m}(\tau) = m(\tau|J_F)$, $\tilde{\kappa}(\tau) = \kappa(\tau|J_F)$, $\tau \in J_{F_1}$ and $\tilde{\delta}$ is the full inverse image of δ under the restriction map $J_{F_1} \rightarrow J_F$. We can use (1.7) to define $Q(\chi, \delta)$ when χ does not occur in any B of signature (δ, δ') . In other words, we can find F_1 and B_1 of signature $(\tilde{\delta}, \tilde{\delta}')$ such that $\tilde{\chi}$ occurs in $\mathcal{S}_{\tilde{m},\tilde{\kappa}}(B_1)$ and put $Q(\chi, \delta) = Q(\tilde{\chi}, \tilde{\delta})^{1/l}$. Then $Q(\chi, \delta) \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$ is well defined and (1.7) holds for this definition. We set $Q(\chi, \emptyset) = 1 \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$.

Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $S_{k,0}(M_2(F))$. Put

$$(1.8) \quad k_0 = \max_{\tau \in J_F} (k(\tau)), \quad k^0 = \min_{\tau \in J_F} (k(\tau)).$$

Let $\mathbf{f} \in W(\chi, M_2(F))$ be the primitive form. We attach a Dirichlet series $D(s, \mathbf{f}) = \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f}) N(\mathfrak{m})^{-s}$ by (2.25) of [S1]. For a Hecke character φ of $F_A^\times$, we put

$$D(s, \mathbf{f}, \varphi) = \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f}) \varphi_*(\mathfrak{m}) N(\mathfrak{m})^{-s}$$

where φ_* denotes the ideal character associated to φ and $\mathfrak{m}$ extends over all integral ideals of F . Set $L(s, \chi, \varphi) = \sum_{\mathfrak{m}} \chi(\mathfrak{m}) \varphi_*(\mathfrak{m}) N(\mathfrak{m})^{-s}$. Then we have $L(s, \chi, \varphi) = D(s + \frac{k_0}{2} - 1, \mathbf{f}, \varphi)$. In [S1], Theorem 4.3, Shimura obtained the following result (cf. also Rohrlich [R]) which we shall recall in a crude form sufficient for our present purpose.

Theorem S. Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ . For every $\epsilon = (\epsilon(\tau)) \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$, there exists a constant $u(\epsilon, \mathbf{f}) \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$ with the following properties.

(I) If φ is a Hecke character of $F_A^\times$ such that

$$\varphi_\infty(x) = \prod_{\tau \in J_F} \text{sgn}(x_\tau)^{\epsilon(\tau) + m}, \quad x = (x_\tau) \in F_\infty^\times,$$

then

$$D(m, \mathbf{f}, \varphi) \sim \pi^{mn} u(\epsilon, \mathbf{f})$$

for every integer m such that

$$\frac{k_0 - k^0}{2} < m < \frac{k_0 + k^0}{2}.$$

(II) If $\epsilon_1, \epsilon_2 \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ satisfy $\epsilon_1(\tau) + \epsilon_2(\tau) \equiv 1 \bmod 2$ for all τ , then

$$u(\epsilon_1, \mathbf{f}) u(\epsilon_2, \mathbf{f}) \sim \pi^{n(1-k_0) + \sum_{\tau \in J_F} k(\tau)} \langle \mathbf{f}, \mathbf{f} \rangle.$$

Put $U(\chi, \epsilon) = u(\epsilon, \mathbf{f})$ taking the primitive form $\mathbf{f} \in W(\chi, M_2(F))$.

Remark. Let $\mathbf{f}$ be as above and let $\pi = \otimes_v \pi_v$ be the irreducible automorphic representation of $GL(2, F_A)$ generated by $\mathbf{f}$. Then π is unitary.

(1) By somewhat laborious computations taking a suitable model of a local component π_v of π and letting the Hecke operator at v defined in [S1], §2 act on the new vector, we can verify the exact equality $D(s, \mathbf{f}) = L_f(s - \frac{k_0-1}{2}, \pi)$. However this is not necessarily so for $D(s, \mathbf{f}, \varphi)$ and $L_f(s - \frac{k_0-1}{2}, \pi \otimes \varphi)$. In fact, some finitely many Euler factors of $L_f(s - \frac{k_0-1}{2}, \pi \otimes \varphi)$ may not appear in $D(s, \mathbf{f}, \varphi)$. The condition for the exact coincidence

is $L(s, \pi_v \otimes \varphi_v) = 1$ whenever φ ramifies at v . This condition is satisfied at v if the exponent of the conductor of φ_v is greater than the exponent of the conductor of π_v .

(2) Let ψ be a Hecke character of $F_A^\times$ such that

$$\psi_\infty(x) = \prod_{\tau \in J_F} \text{sgn}(x_\tau)^{\epsilon_1(\tau)}, \quad x = (x_\tau) \in F_\infty^\times.$$

Let $\mathbf{f}_\psi$ be the primitive form which belongs to $\pi \otimes \psi$. We have

$$(1.9) \quad u(\epsilon, \mathbf{f}_\psi) \sim u(\epsilon + \epsilon_1, \mathbf{f}) \quad \text{for every } \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}.$$

To see this, first choose a critical value m . Take a Hecke character φ of $F_A^\times$ so that φ_∞ is given by the formula in Theorem S, (I) and that the conductor of φ is divisible by $\mathfrak{p}^{e+1}$ whenever $\mathfrak{p}^e$ divides one of the conductors of $\pi, \pi \otimes \psi, \psi$. Then we have

$$D(s, \mathbf{f}_\psi, \varphi) = D(s, \mathbf{f}, \psi\varphi) = L_f(s - \frac{k_0 - 1}{2}, \pi \otimes \psi\varphi).$$

By a theorem of Rohrlich, we can further impose the condition on φ that $L_f(s, \pi \otimes \psi\varphi) \neq 0$ for $s = m - \frac{k_0 - 1}{2}$. Then (1.9) follows from Theorem S. As a result, we see that

$$(1.10) \quad L_f(m - \frac{k_0 - 1}{2}, \pi \otimes \varphi) \sim \pi^{mn} u(\epsilon, \mathbf{f})$$

for a Hecke character φ and critical values m as in Theorem S.

(3) It can be shown, using the unitarity of π_v , that $D(s, \mathbf{f}, \varphi)/L_f(s - \frac{k_0 - 1}{2}, \pi \otimes \varphi)$ is an entire function which has no zeros for $\Re(s) \geq k_0/2$. We can give another proof of (1.9) and (1.10) using this fact and the functional equation of $L(s, \pi \otimes \varphi)$.

§2. Preliminary reduction of Conjecture P

Our main theorem states that 2^{n+1} quantities $U(\chi, \epsilon)$ and $Q(\chi, \delta)$ can be given by $2n$ quantities $c_\tau^\pm(\chi)$, which implies some highly non-trivial relations among $U(\chi, \epsilon)$ and $Q(\chi, \delta)$. We shall analyze these relations by the next Lemma.

Lemma 1. Let $J = \{1, 2, \dots, n\}$ and let Λ_n be the set of all mappings from J to $\{\pm 1\}$. Assume that for every $\epsilon \in \Lambda_n$ and every subset I of J , there are given quantities $p(\epsilon) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ and $q(I) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ which satisfy the following properties:

$$(R1) \quad p(\epsilon)p(-\epsilon) = q(J) \quad \text{where} \quad (-\epsilon)(i) = -\epsilon(i), \quad i \in J.$$

$$(R2) \quad q(I_1 \cup I_2) = q(I_1)q(I_2) \quad \text{if} \quad I_1 \cap I_2 = \emptyset.$$

$$(R3) \quad p(\epsilon_1)p(\epsilon_2) = p(\mu_1)p(\mu_2) \quad \text{if} \quad \{\epsilon_1(i), \epsilon_2(i)\} = \{\mu_1(i), \mu_2(i)\} \quad \text{for every} \quad 1 \leq i \leq n.$$

Then there exist $2n$ constants $c_i^\pm \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$, $1 \leq i \leq n$ such that

$$(2.1) \quad p(\epsilon) = \prod_{i=1}^n c_i^{\epsilon(i)}, \quad \epsilon \in \Lambda_n,$$

$$(2.2) \quad q(I) = \prod_{i \in I} c_i^+ c_i^-, \quad \text{if } I \subseteq J.$$

Moreover $c_i^\pm \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$, $1 \leq i \leq n$ are unique. In (2.1) and (2.2), we understand that $c_i^1 = c_i^+$, $c_i^{-1} = c_i^-$, $\prod_{i \in \emptyset} c_i^+ c_i^- = 1$.

Proof. By (R2), we have $q(\emptyset) = 1$. Hence (2.2) for $I = \emptyset$ holds. If $n = 1$, the assertion holds with

$$c_1^+ = p(\epsilon), \quad c_1^- = p(-\epsilon) \quad \text{for } \epsilon : 1 \longrightarrow 1.$$

Now we assume $n \geq 2$ and that the assertion holds up to $n - 1$. Let $J' = \{1, 2, \dots, n - 1\} = J \setminus \{n\}$ and let Λ_{n-1} be the set of all mappings from J' to $\{\pm 1\}$. Define $\omega_\pm, \omega'_\pm \in \Lambda_n$ by

$$\begin{aligned} \omega_+ &: \{1, 2, \dots, n - 1, n\} \longrightarrow \{1, 1, \dots, 1, 1\}, \\ \omega'_+ &: \{1, 2, \dots, n - 1, n\} \longrightarrow \{-1, -1, \dots, -1, 1\}, \\ \omega_- &: \{1, 2, \dots, n - 1, n\} \longrightarrow \{1, 1, \dots, 1, -1\}, \\ \omega'_- &: \{1, 2, \dots, n - 1, n\} \longrightarrow \{-1, -1, \dots, -1, -1\}. \end{aligned}$$

By (R1), we have

$$(2.3) \quad p(\omega_+)p(\omega'_-) = p(\omega'_+)p(\omega_-) = q(J).$$

For a given $\epsilon \in \Lambda_{n-1}$, choose an extension $\epsilon^* \in \Lambda_n$ so that $\epsilon^*(i) = \epsilon(i)$, $1 \leq i \leq n - 1$ and set

$$(2.4) \quad p'(\epsilon) = p(\epsilon^*) / \sqrt{p(\omega_{\epsilon^*(n)})p(\omega'_{\epsilon^*(n)}) / q(J')} \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times.$$

By (R3), we see that $p'(\epsilon)$ does not depend on the choice of ϵ^* . For $I' \subseteq J'$, we set

$$(2.5) \quad q'(I') = q(I').$$

Then we can verify that the quantities $p'(\epsilon)$, $\epsilon \in \Lambda_{n-1}$ and $q'(I')$ satisfy

$$(R'1) \quad p'(\epsilon)p'(-\epsilon) = q'(J'),$$

$$(R'2) \quad q'(I'_1 \cup I'_2) = q'(I'_1)q'(I'_2) \quad \text{if } I'_1 \cap I'_2 = \emptyset,$$

$$(R'3) \quad \begin{aligned} p'(\epsilon_1)p'(\epsilon_2) &= p'(\mu_1)p'(\mu_2) \quad \text{if} \quad \{\epsilon_1(i), \epsilon_2(i)\} = \{\mu_1(i), \mu_2(i)\} \\ &\text{for every} \quad 1 \leq i \leq n-1. \end{aligned}$$

Relation (R'2) is trivial. To see (R'1), we may choose an extension ϵ^* of ϵ so that $\epsilon^*(n) = 1$ and may apply (2.4). Then we have

$$\begin{aligned} p'(\epsilon)p'(-\epsilon) &= p(\epsilon^*)/\sqrt{p(\omega_+)p(\omega'_+)/q(J')} \cdot p(-\epsilon^*)/\sqrt{p(\omega_-)p(\omega'_-)/q(J')} \\ &= q(J)q(J')/\sqrt{p(\omega_+)p(\omega'_+)p(\omega_-)p(\omega'_-)} = q(J') \end{aligned}$$

by (2.3) and (R1). Similarly (R'3) follows from (R3).

By the hypothesis of induction, there exist $2(n-1)$ quantities $c_i^\pm \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, $1 \leq i \leq n-1$ such that

$$(2.6) \quad p'(\epsilon) = \prod_{i=1}^{n-1} c_i^{\epsilon(i)}, \quad \epsilon \in \Lambda_{n-1},$$

$$(2.7) \quad q(I') = q'(I') = \prod_{i \in I'} c_i^+ c_i^-, \quad I' \subseteq J'.$$

Set

$$(2.8) \quad c_n^+ = \sqrt{p(\omega_+)p(\omega'_+)/q(J')}, \quad c_n^- = \sqrt{p(\omega_-)p(\omega'_-)/q(J')}.$$

To see the relation (2.1), put $\epsilon = \epsilon^*|J$ for $\epsilon^* \in \Lambda_n$. By (2.4), (2.6) and (2.8), we have

$$p(\epsilon^*) = p'(\epsilon) \sqrt{p(\omega_{\epsilon^*(n)})p(\omega'_{\epsilon^*(n)})/q(J')} = \left(\prod_{i=1}^{n-1} c_i^{\epsilon(i)} \right) c_n^{\epsilon^*(n)} = \prod_{i=1}^n c_i^{\epsilon^*(i)}.$$

Hence (2.1) is satisfied.

To see (2.2), we may assume $I \ni n$. Put $I' = I \setminus \{n\}$. By (2.8), (2.3) and (R2), we get

$$c_n^+ c_n^- = q(J)/q(J') = q(\{n\}).$$

Then we obtain

$$q(I) = q(I')q(\{n\}) = \left(\prod_{i \in I'} c_i^+ c_i^- \right) c_n^+ c_n^- = \prod_{i \in I} c_i^+ c_i^-$$

by (R2).

The uniqueness of $c_i^\pm$ is clear since we can express $c_i^\pm$ by a formula similar to (2.8) if (2.1) and (2.2) hold. This completes the proof.

Identify J_F with $\{1, 2, \dots, n\}$ and $\mathbf{Z}/2\mathbf{Z}$ with $\{1, -1\}$. By the above Lemma, we see that our Main Theorem is reduced to (R1) $\sim$ (R3) given in the introduction. We note that (R1) follows from Theorem S, (II) in view of the definition of $Q(\chi, J_F)$.

In the rest of this section, we shall prove (P4). Let K be a CM -extension of F . For $\alpha, \beta \in I_K$, let $p_K(\alpha, \beta) \in \mathbf{C}^\times / \overline{\mathbf{Q}}^\times$ denote the CM -period defined in [S2]. Let Φ be a CM -type of K and set $\xi = \sum_{\tau \in \Phi} \xi_\tau \cdot \tau \in I_K$, $\xi_\tau \geq 0$ for all τ . Let Ξ be a primitive Hecke character of the ideal group of K with conductor $\mathfrak{c}$ such that

$$\Xi((a)) = a^\xi / |a^\xi| \quad \text{if } a \in K, \quad a \equiv 1 \pmod{\mathfrak{c}},$$

where $a^\xi = \prod_{\tau \in \Phi} (a^\tau)^{\xi_\tau}$. Assume $\xi_\tau > 0$ for some τ . Then there exists a primitive system of eigenvalues of Hecke operators χ occurring in $\mathcal{S}_{k,0}(M_2(F))$ such that

$$(2.9) \quad L(s, \chi) = L(s - 1/2, \Xi),$$

where $k(\tau|F) = \xi_\tau + 1$, $\tau \in \Phi$ (cf. [S4], §5). If $\xi_\tau \pmod 2$ is independent of τ and $\xi_\tau > 0$ for all τ , then we have

$$(2.10) \quad U(\chi, \epsilon) \sim \pi^{(\sum_{\tau \in J_F} k(\tau) - nk_0)/2} p_K(\xi, \Phi) \quad \text{for every } \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$$

by [S4], Theorem 5.11, (iii). On the otherhand, we have

$$(2.11) \quad Q(\chi, \delta) \sim \pi^{-|\delta|} p_K(\xi, 2\eta)$$

by [S4], Theorem 5.8, where η is the subset of Φ such that $\text{Res}_{K/F}(\eta) = \delta$. Now for such a χ , (R2) follows from the bilinearity of p_K (cf. [S2], Theorem 1.1) and (R3) is trivially satisfied. We see that the solution to (1) and (2) in the introduction is given by

$$(2.12) \quad c_\tau^+(\chi) = c_\tau^-(\chi) = \pi^{(k(\tau) - k_0)/2} p_K(\xi, \tilde{\tau}), \quad \tau \in J_F$$

from the bilinearity of p_K , where $\tilde{\tau} \in \Phi$ denotes the element such that $\tilde{\tau}|F = \tau$. By (3) in the introduction, we have

$$(2.13) \quad P(\chi, \delta, \epsilon) \sim \pi^{-|\delta|} p_K(\xi, \eta) \quad \text{for every } \epsilon \in (\mathbf{Z}/2\mathbf{Z})^\delta,$$

which is consistent with (C9) of [S5].

§3. Verification of (R2)

We shall use the following result of Harris (cf. [Ha3], §2.6).

Theorem HA. *Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $\mathcal{S}_{k,0}(M_2(F))$. Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \pmod 2$ is independent of τ . Let α and β be subsets of J_F such that $\alpha \cap \beta = \emptyset$. If n , $|\alpha|$ and $|\beta|$ are all even, then*

$$Q(\chi, \alpha \cup \beta) \sim Q(\chi, \alpha) Q(\chi, \beta).$$

By a base change argument, we can remove the parity condition in Theorem HA when $k(\tau) \geq 3$ for all $\tau \in \alpha \cup \beta$.

Theorem 2. Let α and β be subsets of J_F such that $\alpha \cap \beta = \emptyset$. Assume that $k(\tau) \geq 2$ for all $\tau \in J_F$, $k(\tau) \geq 3$ for all $\tau \in \alpha \cup \beta$ and that $k(\tau) \bmod 2$ is independent of τ . Then we have

$$Q(\chi, \alpha \cup \beta) \sim Q(\chi, \alpha)Q(\chi, \beta).$$

Proof. Let F_1 be a totally real quadratic extension of F . Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the full inverse images of α and β under the restriction map $J_{F_1} \rightarrow J_F$ respectively. Let $\tilde{\chi}$ be a base change lift of χ which occurs in $\mathcal{S}_{\tilde{k},0}(M_2(F_1))$, where $\tilde{k}(\tau) = k(\tau|J_F)$, $\tau \in J_{F_1}$. We can apply Theorem HA to $\tilde{\chi}$, $\tilde{\alpha}$, $\tilde{\beta}$ and obtain

$$Q(\tilde{\chi}, \tilde{\alpha} \cup \tilde{\beta}) \sim Q(\tilde{\chi}, \tilde{\alpha})Q(\tilde{\chi}, \tilde{\beta}).$$

By (1.7), we have

$$Q(\chi, \alpha \cup \beta)^2 \sim Q(\chi, \alpha)^2 Q(\chi, \beta)^2.$$

Hence the assertion follows.

By Theorem 2, the condition (R2) is verified.

§4. Verification of (R3)

To present our arguments in a clear-cut way, let us first recall a few facts on representation theory of $GL(2, L)$ for an archimedean field L . Let $\mathcal{H}_L$ denote the Hecke algebra of $GL(2, L)$ defined in Jacquet-Langlands [JL], p. 153, p. 220.

First let $L = \mathbf{R}$. For a positive integer p , let

$$\mu_1(t) = |t|^{p/2}, \quad \mu_2(t) = |t|^{-p/2} \text{sgn}(t)^{\epsilon(p)}, \quad t \in \mathbf{R}^\times$$

where $\epsilon(p) = 0$ or 1 according as p is odd or even. Consider the representation $\sigma_p = \sigma(\mu_1, \mu_2)$ described in [JL], Theorem 5.11. Then σ_p is a unitary discrete series representation of $\mathcal{H}_{\mathbf{R}}$. If an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $GL(2, F_A)$ is generated by $\mathbf{f} \in \mathcal{S}_{k,0}(M_2(F))$, then we have

$$\pi_\infty = \otimes_{\tau \in J_F} \sigma_{k(\tau)-1}$$

if $k(\tau) \geq 2$ for all $\tau \in J_F$. Let ω_p be the character of $\mathbf{C}^\times$ given by

$$\omega_p(z) = z^p (z\bar{z})^{-p/2}, \quad z \in \mathbf{C}^\times.$$

Then we have

$$(4.1) \quad \sigma_p = \pi(\omega_p)$$

in the notation of [JL], p. 176–181. We also have

$$(4.2) \quad L(s, \sigma_p) = L(s, \omega_p) = 2(2\pi)^{-(s+p/2)} \Gamma(s + \frac{p}{2}).$$

Let $L = \mathbf{C}$. For two quasi-characters μ_1, μ_2 of $\mathbf{C}^\times$, let $\pi(\mu_1, \mu_2)$ be the representation of $\mathcal{H}_{\mathbf{C}}$ described in [JL], Theorem 6.2.

Now let $W_{\mathbf{C}} = \mathbf{C}^\times$, $W_{\mathbf{R}} = W_{\mathbf{R}, \mathbf{C}}$ be the Weil groups. We may write (4.1) as $\sigma_p = \pi(\text{Ind}_{W_{\mathbf{C}}}^{W_{\mathbf{R}}} \omega_p)$ in terms of the Langlands parametrization. Hence the base change lift of σ_p to $\mathcal{H}_{\mathbf{C}}$ is given by $\pi((\text{Ind}_{W_{\mathbf{C}}}^{W_{\mathbf{R}}} \omega_p)|W_{\mathbf{C}}) = \pi(\omega_p, \bar{\omega}_p)$ by Langlands [L], p.16, e).

We quote Hida [Hi], Theorem 8.1 in a crude form sufficient for our present purpose.

Theorem HI. Let K be an algebraic number field. Let $\pi = \otimes_w \pi_w$ be an irreducible unitary cuspidal automorphic representation of $GL(2, K_A)$. Assume that

$$\pi_\infty = \otimes_{\tau \in \mathfrak{a}_r} \sigma_{k(\tau)-1} \otimes_{\tau \in \mathfrak{a}_e} \pi(\omega_{k(\tau)-1}, \bar{\omega}_{k(\tau)-1})$$

with $k(\tau) \geq 2$ for all $\tau \in \mathfrak{a}$ and $k(\tau) \bmod 2$ is independent of τ . Put $k_0 = \max_{\tau \in \mathfrak{a}} k(\tau)$. Then for every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{\mathfrak{a}_r}$, there exists a constant $U(\pi, \epsilon) \in \mathbf{C}^\times$ which satisfies the following properties. If φ is a Hecke character of $K_A^\times$ of finite order such that

$$\varphi_\infty(x) = \prod_{\tau \in \mathfrak{a}_r} \text{sgn}(x_\tau)^{\epsilon(\tau)+m}, \quad x = (x_\tau)_{\tau \in \mathfrak{a}} \in K_\infty^\times,$$

then

$$L_f(m - \frac{k_0 - 1}{2}, \pi) \sim \pi^{m[K:\mathbf{Q}]} U(\pi, \epsilon)$$

for every integer m such that

$$\frac{k_0 - k(\tau)}{2} < m < \frac{k_0 + k(\tau)}{2} \quad \text{for every } \tau \in \mathfrak{a}.$$

We are going to verify (R3) using this theorem. It suffices to show

Theorem 3. Let $\mathbf{f} \in \mathcal{S}_{k,0}(M_2(F))$ be a primitive cusp form. We assume $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ . Then we have

$$(4.3) \quad u(\epsilon_1, \mathbf{f})u(\epsilon_2, \mathbf{f}) \sim u(\mu_1, \mathbf{f})u(\mu_2, \mathbf{f})$$

whenever $\epsilon_1, \epsilon_2, \mu_1, \mu_2 \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ satisfy

$$(4.4) \quad \{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{\mu_1(\tau), \mu_2(\tau)\} \quad \text{for all } \tau \in J_F.$$

Proof. We choose an integer m which satisfies the condition of Theorem S, (I). Since we have assumed $k^0 \geq 3$, we can choose such an m so that $m \geq (k_0 + 1)/2$. We fix and denote it by m_0 . Then we have $D(m_0, \mathbf{f}, \varphi) \neq 0$ for every Hecke character φ of $F_A^\times$ of finite order (cf. [S1], Prop. 4.16). Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be Hecke characters of $F_A^\times$ of finite order such that

$$(4.5) \quad \begin{aligned} (\varphi_i)_\infty(x) &= \prod_{\tau \in J_F} (\text{sgn}(x_\tau))^{\epsilon_i(\tau)+m_0}, & i = 1, 2, \\ (\psi_i)_\infty(x) &= \prod_{\tau \in J_F} (\text{sgn}(x_\tau))^{\mu_i(\tau)+m_0}, & i = 1, 2, \end{aligned}$$

for $x = (x_\tau) \in F_\infty^\times$. By Theorem S, (I), (4.3) reduces to

$$(4.6) \quad D(m_0, \mathbf{f}, \varphi_1)D(m_0, \mathbf{f}, \varphi_2) \sim D(m_0, \mathbf{f}, \psi_1)D(m_0, \mathbf{f}, \psi_2).$$

By (4.4), we have $(\varphi_1\varphi_2)_\infty = (\psi_1\psi_2)_\infty$. If $(\varphi_1\varphi_2)_\infty$ is trivial, then we have $\epsilon_1 = \epsilon_2 = \mu_1 = \mu_2$ by (4.4); hence (4.3) holds. We may assume that $(\varphi_1\varphi_2)_\infty$ is non-trivial. Choose $a \in F$ so that $\tau(a) > 0$ (resp. $\tau(a) < 0$) if $(\varphi_1\varphi_2)_{\infty_\tau}$ is trivial (resp. non-trivial). Set $K = F(\sqrt{a})$. Then K is a quadratic extension of F . Let η_K be the Hecke character of $F_A^\times$ which corresponds to the extension K/F . By the choice of a , we have $(\eta_K)_\infty = (\varphi_1\varphi_2)_\infty$.

Let $\pi = \otimes_v \pi_v$ be the irreducible automorphic representation of $GL(2, F_A)$ generated by $\mathbf{f}$ and $\tilde{\pi} = \otimes_w \tilde{\pi}_w$ be the base change lift of π to $GL(2, K_A)$. Then we have

$$(4.7) \quad \begin{aligned} L(s, \tilde{\pi}) &= L(s, \pi)L(s, \pi \otimes \eta_K), & L_f(s, \tilde{\pi}) &= L_f(s, \pi)L_f(s, \pi \otimes \eta_K), \\ \pi_\infty &= \otimes_{\tau \in J_F} \sigma_{k(\tau)-1}, \\ \tilde{\pi}_\infty &= (\otimes_{\tau \in \mathfrak{a}_r} \sigma_{k(\tau|F)-1}) \otimes (\otimes_{\tau \in \mathfrak{a}_c} \pi(\omega_{k(\tau|F)-1}, \bar{\omega}_{k(\tau|F)-1})). \end{aligned}$$

Since the base change lift of $\pi \otimes \varphi_1$ to K is $\tilde{\pi} \otimes (\varphi_1 \circ N_{K/F})$, we have

$$L_f(s, \tilde{\pi} \otimes (\varphi_1 \circ N_{K/F})) = L_f(s, \pi \otimes \varphi_1)L_f(s, \pi \otimes \varphi_1\eta_K).$$

We have $D(m_0, \mathbf{f}, \varphi) \sim L_f(m_0 - \frac{k_0-1}{2}, \pi \otimes \varphi)$ for every Hecke character φ of $F_A^\times$ of finite order. Since $(\varphi_1\eta_K)_\infty = (\varphi_2)_\infty$, we have $L_f(m_0 - \frac{k_0-1}{2}, \pi \otimes \varphi_1\eta_K) \sim L_f(m_0 - \frac{k_0-1}{2}, \pi \otimes \varphi_2)$ by Theorem S, (I). Therefore (4.6) reduces to

$$(4.8) \quad L_f(m_0 - \frac{k_0-1}{2}, \tilde{\pi} \otimes (\varphi_1 \circ N_{K/F})) \sim L_f(m_0 - \frac{k_0-1}{2}, \tilde{\pi} \otimes (\psi_1 \circ N_{K/F})).$$

Assume $\tau \in J_F$ is unramified in K . Then $(\varphi_1\varphi_2)_{\infty_\tau} = (\psi_1\psi_2)_{\infty_\tau} = 1$ and we see that $\{\epsilon_1(\tau), \epsilon_2(\tau)\}$ and $\{\mu_1(\tau), \mu_2(\tau)\}$ are either $\{0, 0\}$ or $\{1, 1\}$. By (4.4), we get $\epsilon_1(\tau) = \mu_1(\tau)$, $(\varphi_1)_{\infty_\tau} = (\psi_1)_{\infty_\tau}$. Therefore we obtain

$$(\varphi_1 \circ N_{K/F})_\infty = (\psi_1 \circ N_{K/F})_\infty.$$

By the consideration given in §2, we may assume that χ is not of CM -type. Then $\tilde{\pi}$ is cuspidal (cf. [L], Lemma 11.3). Now (4.8) follows from Theorem HI. This completes the proof.

Now we have completed our proof of Main Theorem. An identification of $c_\tau^\pm(\chi)$ with Deligne's periods of the motive attached to χ is described in [Y], §4. We note that there is a slight notational difference between [S4] and [S5]. In [S5], p. 293, (C3),

$$P(\chi, \epsilon, J_F) \sim \pi^{-n - \sum_{\tau \in J_F} k(\tau)/2} V(\chi, \epsilon) \sim \pi^{(k_0-2)n/2 - \sum_{\tau \in J_F} k(\tau)/2} U(\chi, (-1)^{k_0/2} \epsilon)$$

is required when $k(\tau)$ is even for all τ . We adjusted our notation to [S4], which is simpler.

Remark. We have

$$(4.9) \quad c_\tau^\pm(\bar{\chi}) \sim \overline{c_\tau^\pm(\chi)} \quad \text{for every } \tau \in J_F$$

where $\overline{}$ denotes the complex conjugation. To see this, let π be the unitary automorphic representation of $GL(2, F_A)$ which corresponds to χ and call ψ the central character of π .

We have $\bar{\pi} \cong \pi \otimes \psi^{-1}$. By definition, it is obvious that $Q(\chi, \delta) \sim \overline{Q(\chi, \delta)}$ for every $\delta \in J_F$. Since $\bar{\chi} = \chi \otimes \psi^{-1}$, we have (cf. [Y], Prop. 6.5)

$$(4.10) \quad Q(\bar{\chi}, \delta) \sim \overline{Q(\chi, \delta)}.$$

As in Theorem S, choose a critical value m and a Hecke character φ of $F_A^\times$ of finite order for $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ so that $L_f(m - \frac{k_0-1}{2}, \pi \otimes \varphi) \neq 0$. By Theorem S, we have

$$\overline{\pi^{mn}U(\chi, \epsilon)} \sim \overline{L_f(m - \frac{k_0-1}{2}, \pi \otimes \varphi)} = L_f(m - \frac{k_0-1}{2}, \bar{\pi} \otimes \varphi^{-1}) \sim \pi^{mn}U(\bar{\chi}, \epsilon).$$

Hence we get

$$(4.11) \quad U(\bar{\chi}, \epsilon) \sim \overline{U(\chi, \epsilon)}.$$

By (4.10), (4.11) and Lemma 2.1, we obtain (4.9) (cf. [S5], p. 293, (C2)).

§5. The invariance of $c_\tau^\pm(\chi)$ under a base change

Theorem 4. *Let F_1 be a totally real cyclic extension of F . Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $\mathcal{S}_{k,0}(M_2(F))$. We assume that $k(\tau) \geq 3$ for all $\tau \in J_F$ and that $k(\tau) \bmod 2$ is independent of τ . Let $\bar{\chi}$ be the base change lift of χ such that $\bar{\chi}$ occurs in $\mathcal{S}_{\bar{k},0}(M_2(F_1))$ and that $\bar{\chi}$ is primitive, where $\bar{k}(\tau) = k(\tau|F)$, $\tau \in J_{F_1}$. Then we have*

$$(5.1) \quad c_\tau^\pm(\bar{\chi}) = c_{\tau|F}^\pm(\chi) \quad \text{for every } \tau \in J_{F_1}.$$

Proof. Let $\tilde{\mathbf{f}} \in W(\bar{\chi}, M_2(F_1), \overline{\mathbf{Q}})$ and $\mathbf{f} \in W(\chi, M_2(F), \overline{\mathbf{Q}})$ be primitive forms. Let $\tilde{\pi}$ (resp. π) be the irreducible automorphic representation of $GL(2, (F_1)_A)$ (resp. $GL(2, F_A)$) generated by $\tilde{\mathbf{f}}$ (resp. $\mathbf{f}$). Then we have

$$(5.2) \quad L_f(s, \tilde{\pi} \otimes \varphi^\sigma) = L_f(s, \tilde{\pi} \otimes \varphi)$$

for every $\sigma \in \text{Gal}(F_1/F)$ and every Hecke character φ of $(F_1)_A^\times$. Here $\varphi^\sigma(x) = \varphi(x^\sigma)$, $x \in (F_1)_A^\times$. Take $m \in \mathbf{Z}$ so that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$. By a theorem of Rohrlich [R], for every $\tilde{\epsilon} \in (\mathbf{Z}/2\mathbf{Z})^{J_{F_1}}$, we can find a Hecke character φ of $(F_1)_A^\times$ such that

$$L_f(m - \frac{k_0-1}{2}, \tilde{\pi} \otimes \varphi) \neq 0, \quad \varphi_\infty(x) = \prod_{\tau \in J_{F_1}} \text{sgn}(x_\tau)^{m+\tilde{\epsilon}(\tau)}, \quad x = (x_\tau) \in (F_1)_\infty^\times.$$

Applying Theorem S, (I) to (5.2) taking $s = m - \frac{k_0-1}{2}$, we obtain

$$(5.3) \quad u(\tilde{\epsilon}^\sigma, \tilde{\mathbf{f}}) \sim u(\tilde{\epsilon}, \tilde{\mathbf{f}}) \quad \text{for every } \sigma \in \text{Gal}(F_1/F),$$

where $\tilde{\epsilon}^\sigma(y) = \tilde{\epsilon}(\sigma y)$, $y \in J_{F_1}$. In a similar way, using Theorem 6.8 of [Y], we can derive the relation

$$(5.4) \quad Q(\tilde{\chi}, \sigma \tilde{\delta}) \sim Q(\tilde{\chi}, \tilde{\delta}) \quad \text{for every } \emptyset \neq \tilde{\delta} \subseteq J_{F_1}.$$

By (5.3) and (5.4), we get

$$(5.5) \quad c_{\sigma\tau}^\pm(\tilde{\chi}) \sim c_\tau^\pm(\tilde{\chi}) \quad \text{for every } \sigma \in \text{Gal}(F_1/F), \quad \tau \in J_{F_1}$$

in view of the uniqueness of the solution to (1) and (2) in the introduction. Taking $\delta = \{\tau|F\}$, $\tau \in J_{F_1}$ in (1.7) and applying (5.5), we get

$$(5.6) \quad c_\tau^+(\tilde{\chi})c_\tau^-(\tilde{\chi}) \sim c_{\tau|F}^+(\chi)c_{\tau|F}^-(\chi), \quad \tau \in J_{F_1}.$$

On the other hand, we have

$$L_f(s, \tilde{\pi} \otimes (\varphi \circ N_{F_1/F})) = \prod_{\eta} L_f(s, \pi \otimes \varphi \eta)$$

for every Hecke character φ of $F_A^\times$. Here η extends over l Hecke characters of $F_A^\times$ which are trivial on $F^\times N_{F_1/F}((F_1)_A^\times)$, l being the degree of F_1 over F . Since $k(\tau) \geq 3$ for all τ , we can apply Theorem S, (I) to this relation in a similar manner to the above and obtain

$$(5.7) \quad u(\tilde{\epsilon}, \tilde{\mathbf{f}}) \sim u(\epsilon, \mathbf{f})^l \quad \text{for every } \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F},$$

where $\tilde{\epsilon}(y) = \epsilon(y|F)$, $y \in J_{F_1}$. By (5.7) and (5.5), we get

$$(5.8) \quad \prod_{\tau \in J_F} c_{\tilde{\tau}}^{\epsilon(\tau)}(\tilde{\chi}) \sim \prod_{\tau \in J_F} c_\tau^{\epsilon(\tau)}(\chi), \quad \text{for every } \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F},$$

where $\tilde{\tau}$ denotes an arbitrary extension of τ to J_{F_1} .

Take any $\tau_0 \in J_F$ and its extension $\tilde{\tau}_0$ to J_{F_1} . Take any $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ and define $\epsilon' \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ by

$$\epsilon'(\tau) = -\epsilon(\tau) \quad \text{if } \tau \neq \tau_0, \quad \epsilon'(\tau_0) = \epsilon(\tau_0).$$

We have

$$\prod_{\tau \in J_F} c_{\tilde{\tau}}^{\epsilon(\tau)}(\tilde{\chi})c_{\tilde{\tau}}^{\epsilon'(\tau)}(\tilde{\chi}) \sim \left(\prod_{\tau \in J_F \setminus \{\tau_0\}} c_\tau^+(\chi)c_\tau^-(\chi) \right) c_{\tilde{\tau}_0}^{\epsilon(\tau_0)}(\tilde{\chi})^2$$

by (5.6) and

$$\prod_{\tau \in J_F} c_{\tilde{\tau}}^{\epsilon(\tau)}(\tilde{\chi})c_{\tilde{\tau}}^{\epsilon'(\tau)}(\tilde{\chi}) \sim \left(\prod_{\tau \in J_F \setminus \{\tau_0\}} c_\tau^+(\chi)c_\tau^-(\chi) \right) c_{\tau_0}^{\epsilon(\tau_0)}(\chi)^2$$

by (5.8). Hence we get

$$c_{\tilde{\tau}_0}^{\epsilon(\tau_0)}(\tilde{\chi})^2 \sim c_{\tau_0}^{\epsilon(\tau_0)}(\chi)^2.$$

This completes the proof.

Remark. In this remark, we use left action of the automorphism group. For $\sigma \in \text{Aut}(\mathbf{C})$, let $\sigma(B)$ be the quaternion algebra over $\sigma(F)$ obtained from B by transporting the algebra structure by the isomorphism $\sigma : F \rightarrow \sigma(F)$. If $B = \sum_{i=1}^4 F e_i$ with $e_i e_j = \sum_{k=1}^4 c_{ijk} e_k$, then $\sigma(B) = \sum_{i=1}^4 \sigma(F) e'_i$ with $e'_i e'_j = \sum_{k=1}^4 \sigma(c_{ijk}) e'_k$. We have the isomorphism of $\mathbf{Q}$ -algebras $\sigma : B \ni \sum a_i e_i \rightarrow \sum \sigma(a_i) e'_i \in \sigma(B)$. If B is of signature (δ, δ') , then $\sigma(B)$ is of signature $(\delta \sigma^{-1}, \delta' \sigma^{-1})$. This isomorphism extends to the isomorphism (we use the same letter) from $G = \text{Res}_{F/\mathbf{Q}}(B^\times)$ to $\sigma(G) = \text{Res}_{\sigma(F)/\mathbf{Q}}(\sigma(B)^\times)$ and also from G_A to $\sigma(G)_A$. For $\mathbf{f} \in \mathcal{S}_{k,\kappa}(B)$, put $\sigma(\mathbf{f})(\sigma x) = \mathbf{f}(x)$, $x \in G_A$. Then $\sigma(\mathbf{f}) \in \mathcal{S}_{k',\kappa'}(\sigma(B))$, where $k'(\tau) = k(\tau\sigma)$, $\kappa'(\tau) = \kappa(\tau\sigma)$, $\tau \in J_{\sigma(F)}$.

If $\mathbf{f} \in W(\chi, B)$, then we see that $\sigma(\mathbf{f}) \in W(\sigma(\chi), \sigma(B))$, where $\sigma(\chi)(\sigma(\mathfrak{m})) = \chi(\mathfrak{m})$ for an integral ideal $\mathfrak{m}$ of F . We can check easily that $\langle \mathbf{f}, \mathbf{f} \rangle = \langle \sigma(\mathbf{f}), \sigma(\mathbf{f}) \rangle$. We can verify that if $\mathbf{f}$ is $\overline{\mathbf{Q}}$ -rational, then $\sigma(\mathbf{f})$ is $\overline{\mathbf{Q}}$ -rational. Therefore, both (5.3) and (5.4) hold under the condition $k_0 \geq 2$.

§6. Comments on the case where $k(\tau) = 2$ for some τ

We expect that our Main Theorem remains true under the weaker condition that $k(\tau) \geq 2$ for all $\tau \in J_F$ and that $k(\tau) \bmod 2$ is independent of τ . Let us first state necessary ingredients to prove Main Theorem in this generality by our method in this paper. Let π be the irreducible unitary cuspidal automorphic representation of $GL(2, F_A)$ which corresponds to χ .

To prove Theorem 2 in this case by base change argument, it suffices to generalize (1.7) for any totally real quadratic extension F_1 of F . For this purpose, the following Hypothesis is sufficient, as remarked in §6.4 of [Y].

Hypothesis 1. There exist a CM-extension K of F and a unitary Hecke character ψ of $K_A^\times$ which satisfy the following conditions.

$$(1) \quad \psi_v(x) = (x/|x|)^{l_v-1}, \quad x \in K_v^\times \cong \mathbf{C}^\times \quad \text{for } v \in \mathfrak{a}^K,$$

where l_v is a positive integer such that $l_\tau < k_\tau$ if $\tau \in \delta$, $l_\tau > k_\tau$ if $\tau \in J_F \setminus \delta$ and that $k_\tau - l_\tau \bmod 2$ is independent of τ . Here we put $l_\tau = l_v$ taking $v \in \mathfrak{a}^K$ such that $v|F = \tau$.

(2) Let π' be the irreducible unitary automorphic representation of $GL(2, F_A)$ which corresponds to ψ . Then

$$L\left(\frac{1}{2}, \pi \times \pi'\right) L\left(\frac{1}{2}, \pi \times \pi' \otimes \eta_{F_1}\right) \neq 0.$$

Here $L(s, \pi \times \pi')$ denotes the L -function obtained by the convolution of π and π' ; η_{F_1} is the Hecke character of $F_A^\times$ which corresponds to the extension F_1/F .

Similarly Theorem 3 can be proved in this generality if the following Hypothesis is valid. Assume $k(\tau) = 2$ for some $\tau \in J_F$.

Hypothesis 2. We use the same notation as in the proof of Theorem 3. There exist a quadratic extension K of F and a Hecke characters φ_1, ψ_1 of $F_A^\times$ which satisfy the following conditions.

$$(1) \quad (\varphi_1)_\infty \text{ and } (\psi_1)_\infty \text{ are given by (4.5) with } m_0 = k_0/2.$$

(2) K is ramified at $\tau \in J_F$ if and only if $\{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{0, 1\}$.

$$(3) \quad L\left(\frac{1}{2}, \pi \otimes \varphi_1\right)L\left(\frac{1}{2}, \pi \otimes \varphi_1\eta_K\right) \neq 0, \quad L\left(\frac{1}{2}, \pi \otimes \psi_1\right)L\left(\frac{1}{2}, \pi \otimes \psi_1\eta_K\right) \neq 0,$$

where η_K is the Hecke character of $F_A^\times$ which corresponds to the extension K/F .

If these two Hypotheses are valid, Main Theorem holds under the weaker condition stated above. These hypotheses, in which we require *simultaneous* non-vanishing, are somewhat beyond our present knowledge. We only mention Harris [Ha4], Rohrlich [R] and Waldspurger [W] as papers treating related subjects.

When $k(\tau) = 2$ for all τ , Shimura proposed a construction of an abelian variety from critical values of $D(s, \chi, \varphi)$ in [S5], §11. If it were shown that these abelian varieties have models over $\overline{\mathbf{Q}}$, as is well known when $F = \mathbf{Q}$, this construction would imply a still deeper assertion on the nature of critical values. If we could prove Main Theorem also in this case, Shimura's periods in [S5], §11 essentially coincide with $c_\tau^\pm(\chi)$, since $P(\chi, \{\tau\}, \epsilon) \sim \pi^{-1} c_\tau^{\epsilon(\tau)}(\chi)$.

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