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Approximately Efficient Solutions for Vector Optimization Problems*

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Abstract. The aim of this paper is to present an approach to approximating solutions in vector optimization problems with arbitrary ordering cones. This paper presents a study of approximately efficient points of a given set with respect to a convex cone in an ordered Banach space. Existence results for such approximate points are obtained. Moreover, the domination property related to them is observed, and then it is proved that each element of a given set is approximated by the sum of a point in a convex cone inducing the ordering and a point in a finite set consisting of such approximate points of the set.

Key Words. Vector optimization, multiobjective programming, approximation, approximately efficient solutions, ε-solutions, efficient points, domination property.

1. Introduction

In vector optimization field, theoretical existence results have been studied to conditions which ensure the existence of minimal and/or maximal points (i.e., efficient points) with respect to the partial ordering induced by a convex cone in a topological vector space. Many papers [1, 2, 3, 5, 6, 8, 9, 13, 16, 21] give us some interesting answers.

Furthermore, many researches related to ε-optimal solutions has been made in various areas of applied mathematics, optimization, and mathematical economics as well as vector optimization. The concept of such solutions is considered as a perturbation by $\varepsilon > 0$ or a satisfactory compromise with a given prescribed error $\varepsilon > 0$. This optimal criterion is called an ε-optimal criterion. Recently, Loridan [11] extended the ε-optimal criteria to multiple criteria for multiobjective optimization problems in finite dimensional spaces. Thus, the following questions come to us.

First, what are vector-version's concepts with correspondence to the notions of an infimum and a supremum in $\boldsymbol{R}$? In general, we can define the infimum and supremum of a given subset in an ordered set, but these notions do not harmonize with the notion of an efficient point in a multiple criteria problem. Because their points of a bounded subset in a general ordered set may be far from the set, while those of a bounded subset in $\boldsymbol{R}$ is near the set. Secondly, what is an approximate solution, or ε-optimal solution, in a vector optimization

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problem ? In $\boldsymbol{R}$, approximation solutions are made with satisfaction at conditions that they are near the infimum or supremum of a given subset. Thirdly, is it possible that we construct a sequence of approximate solutions such that it tends to a point with correspondence to an infimal point or a supremum of a given set ? In $\boldsymbol{R}$, we can construct a sequence which tends to an infimal point (or a supremum) of a given set.

The aim of this paper is to present an approach to approximation of solutions of vector optimization problems in partially ordered Banach spaces. In particular, we introduce a notion of an ε-approximately efficient solution of a given set. The concept of approximation herein is different than that studied by Loridan[11], and has the advantage that it excludes certain pathological points. In addition, under certain conditions, given any prescribed error $\varepsilon > 0$, the set of efficient points (if it exists) as well as the set of infimal points is approximated by a finite set of lower ε-approximately efficient points.

This paper is organized as follows. In Section 2, we give the preliminary terminology and define minimal points, infimal points, and approximately efficient solutions of a set in an ordered Banach space, and prove existence results for such solutions. In Section 3, we show that under certain conditions, the set of strict infimal points of a given set is approximated by a finite set of approximately efficient solutions. In Section 4, we consider some connections between efficient points and separation.

2. Efficient Points and Approximately Efficient Points

The definition of approximate solutions is meaningful in any ordered topological vector space X where the vector ordering $\leq_C$ is defined by a convex cone C. For ease of presentation, we shall assume throughout the paper that X is a Banach space.

Let X be an ordered real Banach space, where the norm is denoted by $\|\cdot\|$, with the vector ordering $\leq_C$ induced by a convex cone C, that is, for $x, y \in X$, $x \leq_C y$ if $y - x \in C$. For ease of presentation in this section, the convex cone C is assumed to be pointed, i.e., $C \cap (-C) = \{0\}$, and then the ordering is antisymmetric and $C \ni 0$. Moreover, C is assumed to be acute, that is, $\mathbf{cl}C \cap (-\mathbf{cl}C) = \{0\}$, and hence C is pointed. Hence, $\mathbf{cl}C$ induces another antisymmetric vector ordering $\leq_{\mathbf{cl}C}$ stronger than $\leq_C$ in X. With respect to each of the orderings $\leq_C$ and $\leq_{\mathbf{cl}C}$, we define minimal points and infimal points of a subset A of X.

An element x_0 of a subset A of X is said to be a C-minimal point of A (or an efficient point of A with respect to C) if $\{x \in A \mid x \leq_C x_0, x \neq x_0\} = \emptyset$, which is equivalent to $A \cap (x_0 - C) = \{x_0\}$. Also, we say that an element x_0 of the closure of a subset A of X is a C-infimal point (resp., strictly C-infimal point) of A if $\{x \in \mathbf{cl}A \mid x \leq_C x_0, x \neq x_0\} = \emptyset$ (resp., $\left\{x \in \mathbf{cl}A \,\middle|\, x \leq_{\mathbf{cl}C} x_0, x \neq x_0\right\} = \emptyset$), which is equivalent to $\mathbf{cl}A \cap (x_0 - C) = \{x_0\}$ (resp., $\mathbf{cl}A \cap (x_0 - \mathbf{cl}C) = \{x_0\}$). That is, a C-infimal point and strictly C-infimal point of A are an efficient point of $\mathbf{cl}A$ with respect to C and that of $\mathbf{cl}A$ with respect to $\mathbf{cl}C$, respectively. It is remarked that infimal points of A are near A. We denote the set of all C-minimal (resp., C-infimal, strictly C-infimal) points of A by $\mathbf{Min}A$ (resp., $\mathbf{Inf}A$, $\mathbf{Inf}_{\mathrm{s}}A$). Then, it should be remarked that $\mathbf{Inf}_{\mathrm{s}}A \subset \mathbf{Inf}A = \mathbf{Min}(\mathbf{cl}A)$, and hence that $\mathbf{Inf}A = \mathbf{Min}A$ whenever A is closed.

Let X^* denote the topological dual space of X. We denote the nonnegative polar cone and the positive polar cone by $C^* := \{x^* \in X^* \mid \langle x^*, x\rangle \geq 0, \forall x \in C\}$ and $C^{*+} := \{x^* \in X^* \mid \langle x^*, x\rangle > 0, \forall x \in C \setminus \{0\}\}$, respectively.

Throughout the paper, the axiom of choice is assumed without being mentioned in each instance, and also the ball centered at x with radius r is denoted by $B_r(x)$. Moreover, the symbol **int**C denotes the topological interior of C. Also, the symbol $A \setminus B$ is the set of all x in A which are not in B.

Now, we introduce a concept of approximately efficient solutions.

Definition 2.1. Let A be a nonempty subset of X and $\varepsilon > 0$. A point $x \in X$ is said to be a lower ε-approximately efficient point of A with respect to C if $x \in A$ and $(x - C) \cap (A \setminus B_\varepsilon(x)) = \emptyset$, as illustrated in Figure 2.1. We denote the set of all lower ε-approximately efficient points of A with respect to C by $\mathbf{proxmin}_C\,(A \mid \varepsilon)$.

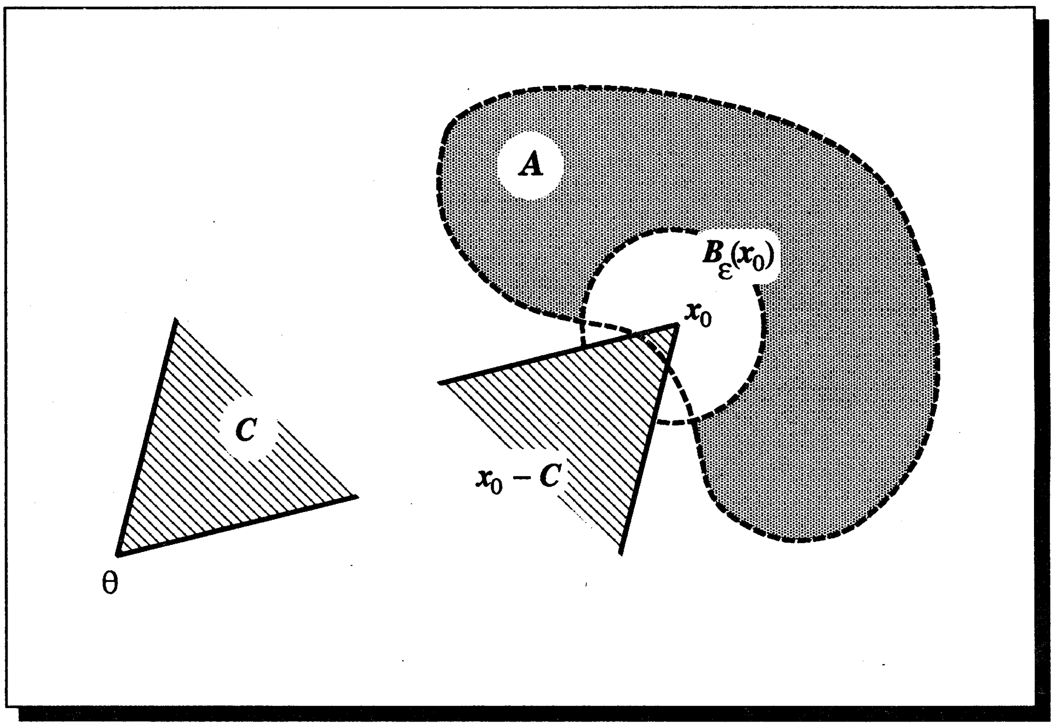

Figure 2.1: **Illustration of a lower ε-approximately efficient point**

For a sufficiently large ε, the lower ε-approximately efficient points are near infimal points of a given set, and we can choose a sequence of lower ε-approximately efficient points tending to an infimal point of the set. This fact is demonstrated in the following propositions.

Proposition 2.1. *Let A be a nonempty subset of X and $\varepsilon_1, \varepsilon_2 > 0$. If $\varepsilon_1 < \varepsilon_2$, then*

$$\mathbf{proxmin}_C\,(A \mid \varepsilon_1) \subset \mathbf{proxmin}_C\,(A \mid \varepsilon_2).$$

Proposition 2.2. *Let A be a nonempty subset of X, then*

$$\mathbf{proxmin}_C\,(A \mid \varepsilon) \supset \mathbf{Min}A, \forall \varepsilon > 0, \tag{2.1}$$

$$\bigcap_{\varepsilon > 0} \mathbf{proxmin}_C\,(A \mid \varepsilon) = \mathbf{Min}A, \tag{2.2}$$

$$\bigcap_{\varepsilon > 0} \mathbf{proxmin}_C\,(\mathbf{cl}A \mid \varepsilon) = \mathbf{Inf}A. \tag{2.3}$$

Since the sets $\mathbf{Min}A$ and $\mathbf{Inf}A$ are possibly empty, we observe the behavior of sets of lower ε-approximately efficient points. In the remainder, the set $A \cap (x - C)$ is called the C-section of A at x.

Definition 2.2. Let A be a nonempty subset of X. We say that A has a compact $\mathbf{cl}C$-section of $\mathbf{cl}A$, if

$$\mathbf{tbd}_C A := \{x \in X \mid \mathbf{cl}A \cap (x - \mathbf{cl}C) \text{ is nonempty and compact } \} \neq \emptyset.$$

Lemma 2.1. *Let A be a nonempty subset of X and $x_0 \in \mathbf{Inf}_\mathbf{s}A$. If there are $r > 0$ and $y_0 \in \mathbf{tbd}_C A$ such that*

$$A \cap B_r(x_0) \subset y_0 - \mathbf{cl}C, \tag{2.4}$$

then there exists $x \in A \cap B_r(x_0)$ such that $A \cap (x - C) \subset B_r(x_0)$.

Proof. Suppose that for any $x \in A \cap B_r(x_0)$, $A \cap (x - C) \setminus B_r(x_0) \neq \emptyset$. Since $x_0 \in \mathbf{Inf}_\mathbf{s}A$ (and hence $x_0 \in \mathbf{cl}A$), $A \cap B_s(x_0) \neq \emptyset$ for all $s > 0$. By the axiom of choice, choose a vector $x(s) \in A \cap B_s(x_0)$ for each $s \in (0, r]$. Then, the net $\{x(s)\}$ converges to x_0. By the axiom of choice again, choose a vector $z(s) \in A \cap (x(s) - C) \setminus B_r(x_0)$ for each $x(s)$. Thus, we get the net $\{z(s)\}$. Since $A \cap B_r(x_0) \subset y_0 - \mathbf{cl}C$, we have

$$\bigcup_{0<s\leq r} A \cap (x(s) - C) \subset \mathbf{cl}A \cap (y_0 - \mathbf{cl}C),$$

and the latter set is compact. Then there is a subnet $\{z(s(k))\}$ which converges to some vector $z_0 \in \mathbf{cl}A \cap (y_0 - \mathbf{cl}C)$. Since $x(s(k)) - z(s(k)) \in C$ and $z(s(k)) \notin B_r(x_0)$ for each k, we have $x_0 - z_0 \in \mathbf{cl}C$ and $x_0 \neq z_0$. This is a contradiction to $x_0 \in \mathbf{Inf}_\mathbf{s}A$. This completes the proof. ∎

Theorem 2.1. *Let A be a nonempty subset of X and $x_0 \in \mathbf{Inf}_\mathbf{s}A$. If there are $r > 0$ and $y_0 \in \mathbf{tbd}_C A$ satisfying condition (2.4), then for each ε, $0 < \varepsilon \leq r$, there exists a lower (2ε)-approximately efficient point of A (approximating x_0).*

Proof. By the assumption, we have $A \cap B_\varepsilon(x_0) \subset y_0 - \mathbf{cl}C$ for each ε, $0 < \varepsilon \leq r$. By Lemma 2.1, there exists $x(\varepsilon) \in A \cap B_\varepsilon(x_0)$ such that $A \cap (x(\varepsilon) - \mathbf{cl}C) \subset B_\varepsilon(x_0)$. Since $B_\varepsilon(x_0) \subset B_{2\varepsilon}(x(\varepsilon))$, we have $(x(\varepsilon) - \mathbf{cl}C) \cap (A \setminus B_{2\varepsilon}(x(\varepsilon))) = \emptyset$, which shows that $x(\varepsilon)$ is a lower (2ε)-approximately efficient point of A. Moreover, the net $\{x(\varepsilon)\}$ converges to x_0 (and so we say that $x(\varepsilon)$ approximates x_0). ∎

Proposition 2.2 and Theorem 2.1 show that given any prescribed accuracy $\varepsilon > 0$, there is a corresponding subset $\mathbf{proxmin}_C\,(A \mid \varepsilon)$ of A which approximates the set $(\mathbf{Inf}_\mathbf{s}A) \cup (\mathbf{Min}A)$. It means that we can construct a sequence of lower ε-approximately efficient points tending to a given infimal point of the set.

Remark 2.1. In Theorem 2.1, the set A is assumed neither to be C-bounded in the sense of downward-directed[19] (that is, there exists a point $a \in X$ such that $A \subset a + C$) nor to be Luc's C-bounded[14] (that is, there is some $\varepsilon > 0$ such that $A \subset B_\varepsilon(0) + C$) includes an example which satisfies the assumption of Theorem 2.1 in spite of satisfying neither notions of C-bounded. Therefore, Theorem 2.1 can be applied to a large class of sets.

In Corollary 2.1 we give sufficient conditions for Eq.(2.4); and hence, for the existence of lower ε-approximately efficient points. Condition (i) of the corollary is the condition given in Loridan[11].

Definition 2.3. A subset A of X is said to be strongly C-compact if the set $(x - \mathbf{cl}C) \cap A$ is empty or compact for each $x \in X$ (not A).

Corollary 2.1. *Let A be a nonempty subset of X and $x_0 \in \mathbf{Inf}_{\mathbf{s}}A$. If either*

(i) $\mathbf{cl}A$ *is C-compact and* $(x_0 + \mathbf{int}C) \cap A \neq \emptyset$, *or*

(ii) $\mathbf{cl}A$ *is strongly C-compact and* $\mathbf{int}C \neq \emptyset$,

then for each $\varepsilon > 0$, there exists a lower ε-approximately efficient point of A (approximating x_0).

Proof. (i) Since $\mathbf{cl}A$ is C-compact, $A \subset \mathbf{tbd}_C A$. Since $(x_0 + \mathbf{int}C) \cap A \neq \emptyset$, there exists $y_0 \in A$ such that $x_0 \in y_0 - \mathbf{int}C$, and hence there is $r > 0$ such that $B_r(x_0) \subset y_0 - C$. This implies that $y_0 \in \mathbf{tbd}_C A$ and $r > 0$ satisfy the condition (2.4). By Theorem 2.1 and Proposition 2.1, we have the conclusion.

(ii) Since $\mathbf{int}C \neq \emptyset$, let $z \in \mathbf{int}C$ and $y_0 := x_0 + z$. Then $x_0 \in y_0 - \mathbf{int}C$. Since $x_0 \in \mathbf{cl}A$ and $\mathbf{cl}A$ is strongly C-compact, we have $y_0 \in \mathbf{tbd}_C A$. In the same way of the proof of (i), we have the conclusion. ∎

Remark 2.2. Clearly, any strongly C-compact set is also a C-compact set. However, the converse is not always true. Moreover, there is no connection between conditions (i) and (ii) of the above corollary.

Corollary 2.2. *Let* $\dim X < \infty$, $\mathbf{int}C \neq \emptyset$, *and A be a nonempty subset of X. If there exists $\lambda \in C^{*+}$ such that*

$$\inf_{a \in A} \langle \lambda, a \rangle > -\infty,$$

then for each $\varepsilon > 0$, $\mathbf{proxmin}_C (A \mid \varepsilon) \neq \emptyset$.

Proof. Let us define the following closed half-space

$$H^{(\lambda,c)} := \{x \in X \mid \langle \lambda, x \rangle \geq c\},$$

where $c := \inf_{a \in A} \langle \lambda, a \rangle$. For each $x \in X$, $(x - \mathbf{cl}C) \cap H^{(\lambda,c)}$ is either empty or a closed bounded set, and hence a compact set. Then, the half-space is strongly C-compact and $\mathbf{cl}A \subset H^{(\lambda,c)}$. Hence, $\mathbf{cl}A$ is also strongly C-compact. By Corollary 2.1, we have the conclusion. ∎

3. Domination Property

In this section, we observe a domination property for lower ε-approximately efficient points. We say that a set A is dominated by a set B with respect to an ordering cone C if $A \subset B+C$. Also, we say that the domination property[13, 14] holds for A if $A \subset \mathbf{Min}A + C$.

Lemma 3.1. *For any $\varepsilon > 0$ and $A \subset X$, $A + \mathbf{cl}C \subset (A + B_\varepsilon(0)) + C$.*

Theorem 3.1. *We assume that $A \subset X$ satisfies the following domination property:*

$$\mathbf{cl}A \subset \mathbf{Inf_s}A + \mathbf{cl}C. \tag{3.1}$$

If there are $r > 0$ and $y_0 \in \mathbf{tbd}_C A$ such that

$$A \cap (\mathbf{Inf_s}A + B_r(0)) \subset y_0 - \mathbf{cl}C, \tag{3.2}$$

then for each ε, $0 < \varepsilon \leq r$, there exists a subset $P(\varepsilon)$ of $\mathbf{proxmin}_C(A \mid \varepsilon)$ such that A is dominated by $P(\varepsilon) + B_\varepsilon(0)$, that is,

$$A \subset (P(\varepsilon) + B_\varepsilon(0)) + C. \tag{3.3}$$

Moreover, if $\mathbf{Inf_s}A$ is compact, the set $P(\varepsilon)$ is a finite set.

Proof. By the assumption and Lemma 3.1, we have

$$A \subset \mathbf{cl}A \subset \mathbf{Inf_s}A + \mathbf{cl}C \subset \mathbf{Inf_s}A + B_{\varepsilon/2}(0) + C. \tag{3.4}$$

By Theorem 2.1, there exists a subset $P(\varepsilon)$ of $\mathbf{proxmin}_C(A \mid \varepsilon/2)$ such that $\mathbf{Inf_s}A \subset P(\varepsilon) + B_{\varepsilon/2}(0)$. Hence, by Proposition 2.1, we get $P(\varepsilon) \subset \mathbf{proxmin}_C(A \mid \varepsilon)$, and thus Eq.(3.3). Furthermore, if $\mathbf{Inf_s}A$ is compact, there is a finite subset $A(\varepsilon)$ of $\mathbf{Inf_s}A$ such that

$$\mathbf{Inf_s}A \subset A(\varepsilon) + B_{\varepsilon/4}(0).$$

By replacing $\varepsilon/2$ in (3.4) by $\varepsilon/4$, we have

$$A \subset A(\varepsilon) + B_{\varepsilon/2}(0) + C.$$

In the same way, there exists a finite subset $P(\varepsilon)$ of $\mathbf{proxmin}_C(A \mid \varepsilon/2)$ such that $A(\varepsilon) \subset P(\varepsilon) + B_{\varepsilon/2}(0)$. This completes the proof. ∎

Corollary 3.1. *Let A be a subset of X. If $\mathbf{cl}A$ is strongly C-compact and $\mathbf{int}C \neq \emptyset$, then for each $\varepsilon > 0$, there exists a subset $P(\varepsilon)$ of $\mathbf{proxmin}_C(A \mid \varepsilon)$ satisfying Eq.(3.3). Moreover, if $\mathbf{Inf_s}A$ is compact, the set $P(\varepsilon)$ is a finite set.*

Proof. By Luc[14], it follows that Eq.(3.1) holds. By Corollary 2.1 and Theorem 3.1, we have the conclusion. ∎

Corollary 3.2. *Let $\dim X < \infty$, $\mathbf{int}C \neq \emptyset$, and A a nonempty subset of X. If there exists $\lambda \in C^{*+}$ such that*

$$\inf_{a \in A} \langle \lambda, a \rangle > -\infty,$$

then for each $\varepsilon > 0$, there exists a subset $P(\varepsilon)$ of $\mathbf{proxmin}_C(A \mid \varepsilon)$ satisfying Eq.(3.3). Moreover, if $\mathbf{Inf_s}A$ is compact, the set $P(\varepsilon)$ is a finite set.

Proof. In the proof of Corollary 2.2, we verify that $\mathbf{cl}A$ is a strongly C-compact set in X, and hence we have the conclusion by the above corollary. ∎

Theorem 3.1 and its corollaries show that under some conditions, each element of a given subset A of X can be approximated by the sum of a point of C and a point in a finite set consisting of lower ε-approximately efficient points of A.

4. Efficient Points and Separation

Given a convex set $A \subset \boldsymbol{R}^n$ and an efficient point (C-minimal point) x_0 of A, there exists a hyperplane that separates A and $x_0 - C$, and hence there exists a nonzero vector p such that $\langle p, x_0\rangle \leq \langle p, y\rangle$ for all $y \in A$. In non-convex case, we can consider a similar situation ([4, 18]); given a set $A \subset \boldsymbol{R}^n$ and an efficient point (C-minimal point) x_0 of A, there exists a nonzero vector $q \in \mathbf{int}C$ such that $z_{x_0,q}(x_0) \leq z_{x_0,q}(y)$ for all $y \in A$, where

$$z_{a,q}(y) := \inf\{t \in \boldsymbol{R} \mid y \in a - \mathbf{cl}C + tq\} \tag{4.1}$$

with parameters $a \in X$ and $q \in \mathbf{int}C$. Thus lower ε-approximately efficient points of A can be also calculated via scalarization.

5. Conclusions

We have proposed an approximate optimal criterion for approximate multiple criteria decision makings, and introduced a harmonious concept (an infimal point), corresponding to an infimum, with the notion of an efficient point. Then, we have defined an approximate solution (a lower ε-approximately efficient point), being near infimal points of a given set.

The approach is based on the concept of approximate solutions of efficient points, but it is different from that of Loridan[11]. The approach taken has the following advantages:

(i) the approximate solutions exclude certain pathological points;

(ii) the approximate solutions are near infimal points of a given set;

(iii) for each infimal point, there is a sequence of approximate solutions converging to the point;

(iv) under certain conditions, the set of efficient points (if it exists) as well as the set of infimal points is approximated by a finite set of approximate solutions corresponding to a given prescribed error $\varepsilon > 0$.

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