# 学位申請論文

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# Spitzer's Markov chains with measurable potentials

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1. Introduction and summary of results. Spitzer [10] has introduced Markov chains, whose space of "time parametres" is an infinite tree T, and whose state space is a set $\{ 1, +1\}$. He investigates Gibbs distributions on T that are Markov chains of such construction. Several works [1], [4] and [8] are made on Gibbs distributions on trees.

In a present paper, we generalize Spitzer's results to a case when the state space is a compact set. If the state space consists of two points as in a case of Spitzer, all Markov chains are reversible. So, in that case, the "time parametre" space T need not be equipped with a direction. But, since Markov chains may not be reversible in our case, we must introduce a direction into T. Thus, we consider Markov chains whose space of "time parametres" is an infinite directed tree T, and whose state space is a compact measure space $(X,B,\mu)$.

Let $F(x,y)$ be a measurable function on $X \times X$, boundedness or symmetry $F(x,y) = F(y,x)$ of which we do not assume. A Markov chain on T, whose transition density we denote by $p(x,y)$, is a Gibbs distribution on T with the potential F, if and only if

$$p(x,y) = \lambda(s,n)u(x)^{\ 1}u(y)^{s}v(y)^{n-1}e^{-F(x,y)},$$

where u and v are positive solutions of integral equations of the Hammerstein type

$$\begin{cases} u(x) = \lambda(s,n) \int_X e^{-F(x,y)}u(y)^{s}v(y)^{n-1}\mu(dy), \\ v(x) = \lambda(s,n) \int_X e^{-F(y,x)}u(y)^{s-1}v(y)^{n}\mu(dy). \end{cases}$$

Numbers s, n and $\lambda(s,n)$ will be defined in the following sections. Let M(F) be the set of Markov chains that are, at the same time, Gibbs distributions with the potential F. Under summability conditions on F, all or no chain in M(F) is reversible. Roughly speaking, all chains in M(F) are reversible if and only if F is nearly symmetric. In a symmetric case, the transition density p(x,y) has the form;

$$p(x,y) = \lambda(s,n)u(x)^{\ 1}u(y)^{n+s-1}e^{-F(x,y)},$$

where u is a positive solution of the integral equation;

$$u(x) = \lambda(s,n)\int_X e^{-F(x,y)}u(y)^{s+n-1}\mu(dy).$$

Existence of positive solutions of the integral equations is proved by applying theory of cones in a Banach space.

Dobrushin and Shlosman [3] proved that all Gibbs distributions in $Z^2$ whose state space is the circle $S^1$, are invariant under rotation of the circle, if the potential is of finite range, of $C^2$-class and rotation-invariant. We present an example of chains in M(F) that are not rotation-invariant although the potential F is rotation-invariant and of $C^\infty$-class.

Next, we consider a potential $\beta F$, where $\beta > 0$ is the reciprocal temparature. We prove uniqueness of $M(\beta F)$ for sufficiently small $\beta$. We present an example in which the number of chains in $M(\beta F)$ is exactly calculated for sufficiently large $\beta$.

2. Potentials and Gibbs distributions. Let X be a compact metric space. Let B be the topological Borel field of X and let $\mu$ be a measure on (X,B). Let T be the infinite directed tree, in which s branches emanate from every vertex and n branches flow into every vertex. Two vertices $a \neq b$ in T are neighbours if they are connected by a branch, which we denote by a-b or b-a. If a branch connecting a and b emanates from a, which is equivalent to that the branch flows into b, we write $a \to b$ or $b \leftarrow a$. We remark $s,n \geq 1$. For a subset V of T, let $\partial V$ be the set of vertices in $V^c$ that are neighbours of vertices in V. Let $\Omega = X^T$. For $\omega \in \Omega$ and $a \in T$, let $x_a(\omega) = \omega_a$. For $V \subset T$, let $x_V(\omega)$ be the restriction $\omega|_V$ of $\omega$ on V, and let $B_V$ be the $\sigma$-algebra of $\Omega$ generated by $x_V$. $B_\Omega$ is the $\sigma$-algebra generated by the cylinder sets.

A potential is a pair $F = (F_1, F_2)$ of real-valued measurable functions $F_1$ and $F_2$, where $F_1$ and $F_2$ are defined on X and on $X \times X$, respectively. For a finite subset V of T and for $\underline{x} \in \Omega$, put

$$H_V(\underline{x}) = H_V^F(\underline{x}) = \sum_{a\in V} F_1(x_a) + \sum_{\substack{a,b\in V\\ a\to b}} F_2(x_a,x_b)$$

$$+ \sum_{\substack{a\in V, b\in\partial V\\ a\to b}} F_2(x_a,x_b) + \sum_{\substack{a\in V, b\in\partial V\\ a\leftarrow b}} F_2(x_b,x_a).$$

The family $\{H_V\}_V$ is called Hamiltonian.

Definition. Two potentials $F = (F_1, F_2)$ and $F' = (F_1', F_2')$ are said to be equivalent, which we denote by $F \cong F'$, if $H_V^F(\underline{x}) \quad H_V^{F'}(\underline{x})$ does not depend on $x_V$ for every finite subset V. We remark that it may depend on $x_{\partial V}$.

Lemma 1. Let $F = (F_1, F_2)$ be a potential and put

$$F_2'(x,y) = F_2(x,y) + \frac{1}{n+s}\{F_1(x) + F_1(y)\},$$

then $F \cong (0, F_2')$. If $F_2$ is symmetric, $F_2'$ is also symmetric.

Proof. Put $F_2''(x,y) = \frac{1}{n+s}\{F_1(x) + F_1(y)\}$. We have

$$\sum_{\substack{a,b\in V\\ a\to b}} F_2''(x_a,x_b) + \sum_{\substack{a\in V,\ b\in\partial V\\ a\to b}} F_2''(x_a,x_b) + \sum_{\substack{a\in V, b\in\partial V\\ a\leftarrow b}} F_2''(x_b,x_a)$$

$$= \sum_{a\in V} F_1(x_a) + \frac{1}{n+s}\sum_{b\in\partial V} \#\{a\in V;\ a\text{-}b\}F_1(x_b).$$

Therefore, $H_V^{(0,F_2')}(\underline{x}) \quad H_V^F(\underline{x}) = \frac{1}{n+s}\sum_{b\in\partial V}\#\{a\in V;\ a\text{-}b\}F_1(x_b)$,

which implies $F \cong (0, F_2')$.

In the following we assume always $F_1 = 0$. We identify a potential $(0, F)$ with the function F.

Definition. 1) A potential F is said to be symmetrizable if there exists a symmetric potential $\hat{F}$ with $F \cong \hat{F}$. We call $\hat{F}$ a symmetrization of F.

2) A potential F is said to be uniformly symmetrizable if there exists a symmetrization $\hat{F}$ of F such that

$$\sup_{x,y}|F(x,y) - \hat{F}(x,y)| < +\infty.$$

We call $\hat{F}$ a uniform symmetrization of F.

Lemma 2. 1) A potential $F$ is symmetrizable if and only if there exists a measurable function $f$ such that

$$F(x,y) - F(y,x) = f(x) - f(y).$$

2) A potential $F$ is uniformly symmetrizable if and only if there exists a bounded measurable function $f$ which satisfies the above equality.

Proof. Assume $F(x,y) - F(y,x) = f(x) - f(y)$. We have

$$F(x,y) = \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{1}{2}\{F(x,y) \quad F(y,x)\}$$

$$= \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{1}{2}\{f(x) - f(y)\}.$$

Put $\hat{F}(x,y) = \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{s-n}{2(n+s)}\{f(x) + f(y)\}$. Since

$$\sum_{\substack{a,b\in V \\ a\to b}} \{f(x_a) - f(x_b)\} + \sum_{\substack{a\in V, b\in \partial V \\ a\to b}} \{f(x_a) - f(x_b)\}$$

$$+ \sum_{\substack{a\in V, b\in \partial V \\ a\leftarrow b}} \{f(x_b) - f(x_a)\}$$

$$= (s - n) \sum_{a\in V} f(x_a) + \sum_{b\in \partial V} [\#\{a\in V; a\leftarrow b\} - \#\{a\in V; a\to b\}] f(x_b),$$

and since

$$\sum_{\substack{a,b\in V \\ a\to b}} \{f(x_a) + f(x_b)\} + \sum_{\substack{a\in V, b\in \partial V \\ a\to b}} \{f(x_a) + f(x_b)\}$$

$$+ \sum_{\substack{a\in V, b\in \partial V \\ a\leftarrow b}} \{f(x_b) + f(x_a)\}$$

$$= (s+n) \sum_{a\in V} f(x_a) + \sum_{b\in \partial V} \#\{a\in V; a-b\} f(x_b),$$

we have $H_V^F(\underline{x}) - H_V^{\hat{F}}(\underline{x}) =$

$$= \frac{1}{2} \sum_{b \in \partial V} [\#\{a \in V; a \leftarrow b\} - \#\{a \in V; a \rightarrow b\} - \frac{s-n}{s+n} \#\{a \in V; a - b\}] f(x_b),$$

which implies $F \cong \hat{F}$. If f is bounded, from an equality

$$F(x,y) - \hat{F}(x,y) = \frac{1}{n+s}\{nf(x) - sf(y)\},$$

it follows $\sup_{x,y} |F(x,y) - \hat{F}(x,y)| < +\infty$.

Conversely, assume $F \cong \hat{F}$, where $\hat{F}$ is symmetric. Let $a_i \rightarrow a$ $(1 \leq i \leq n)$ and $a_j' \leftarrow a$ $(1 \leq j \leq s)$. By the equivalence of potentials, the difference $H_{\{a\}}^F(\underline{x}) - H_{\{a\}}^{\hat{F}}(\underline{x})$ does not depend on $x_a$, which we denote by $\Delta(x_{a_1}, x_{a_2}, \cdots, x_{a_n}, x_{a_1'}, x_{a_2'}, \cdots, x_{a_s'})$. Fixing any $x_0 \in X$, we take arbitrary x and y from X. Put $x_a = y$, $x_{a_1} = x$, $x_{a_i} = x_0$ $(2 \leq i \leq n)$ and $x_{a_j'} = x_0$ $(1 \leq j \leq s)$. Put $\Delta(x) = \Delta(x, x_0, \cdots, x_0)$. We have

$$\begin{aligned}
\Delta(x) &= \Delta(x, x_0, \cdots, x_0) \\
&= H_{\{a\}}^F(\underline{x}) - H_{\{a\}}^{\hat{F}}(\underline{x}) \\
&= \sum_{i=1}^{n} \{F(x_{a_i}, x_a) \quad \hat{F}(x_{a_i}, x_a)\} + \sum_{j=1}^{s} \{F(x_a, x_{a_j'}) - \hat{F}(x_a, x_{a_j'})\} \\
&= \{F(x,y) - \hat{F}(x,y)\} + (n-1)\{F(x_0,y) - \hat{F}(x_0,y)\} \\
&\qquad + s\{F(y,x_0) - \hat{F}(y,x_0)\}.
\end{aligned}$$

Consequently,

$$F(x,y) = \hat{F}(x,y) - (n-1)\{F(x_0,y) - \hat{F}(x_0,y)\} - s\{F(y,x_0) - \hat{F}(y,x_0)\} + \Delta(x).$$

Exchanging x and y, we have

$$F(y,x) = \hat{F}(x,y)-(n-1)\{F(x_0,x)-\hat{F}(x_0,x)\}-s\{F(x,x_0)-\hat{F}(x,x_0)\}+\Delta(y),$$

from which follows an equality

$$F(x,y) - F(y,x) = f(x) - f(y),$$

where $f(x) = \Delta(x) + (n-1)\{F(x_0,x)-\hat{F}(x_0,x)\} + s\{F(x,x_0)-\hat{F}(x,x_0)\}$.

If $\sup_{x,y}|F(x,y)-\hat{F}(x,y)| <+\infty$, then $\Delta(x)$ is bounded, therefore f is also bounded.

For a finite subset V of T, put $\mu_V(dx_V) = \prod_{a\in V}\mu(dx_a)$.

<u>Definition</u>. A potential F is said to be <u>admissible</u> if for any finite subset V of T

$$\Xi(V,x_{\partial V}) \equiv \int_{X^V} e^{-H_V^F(\underline{x})}\mu_V(dx_V) <+\infty \qquad \text{a.e.}(\mu_{\partial V}).$$

<u>Lemma 3</u>. A potential F is admissible, if

$$\text{(A,1)} \qquad \iint e^{(n+s)F(x,y)}\mu(dx)\mu(dy) <+\infty,$$

or if

$$\text{(A,2)} \qquad \sup_x\{\int e^{-F(x,y)}\mu(dy),\int e^{-F(y,x)}\mu(dy)\} <+\infty.$$

<u>Proof</u>. Admissiblity under (A,1) is a direct consequence of 1) in the following Lemma 3'. Under (A,2) we have

$\int e^{-H_V^F(\underline{x})}\mu_{V\cup\partial V}(dx_{V\cup\partial V})<+\infty$ by 2) in Lemma 3', if we put $F_{a,b} = F$

for a b $\in V \cup \partial V$ with $\{a,b\} \notin \partial V$, and if we put $F_{a,b} = 0$ for a-b $\in \partial V$

Lemma 3'. Let be given a family $\{F_{a,b};\ a\to b\in T\}$ of functions $F_{a,b} = F_{a,b}(x,y)$. For a finite subset V of T, put

$$\tilde{H}_V(\underline{x}) = \sum_{\substack{a,b\in V\\ a\to b}} F_{a,b}(x_a,x_b) + \sum_{\substack{a\in V,b\in\partial V\\ a\to b}} F_{a,b}(x_a,x_b) + \sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}} F_{b,a}(x_b,x_a),$$

$$\tilde{\tilde{H}}_V(\underline{x}) = \sum_{\substack{a,b\in V\\ a\to b}} F_{a,b}(x_a,x_b).$$

1) If for each $a\to b\in T$,

$$\text{(A,1)'}\qquad \iint e^{(n+s)F_{a,b}(x,y)}\mu(dx)\mu(dy) < +\infty,$$

then it holds $\int e^{-\tilde{H}_V(\underline{x})}\mu_V(dx_V) < +\infty$ a.e. $(\mu_{\partial V})$.

2) If for each $a\to b\in T$,

$$\text{(A,2)'}\qquad \sup_x\{\int e^{-F_{a,b}(x,y)}\mu(dy), \int e^{-F_{a,b}(y,x)}\mu(dy)\} < +\infty,$$

then it holds $\int e^{-\tilde{\tilde{H}}_V(\underline{x})}\mu_V(dx_V) < +\infty$.

Proof is carried out by induction in #V.

1) Let V be a set consisting of a single vertex a. Let $a_i\to a$ $(1\le i\le n)$ and $a'_j\leftarrow a$ $(1\le j\le s)$. We have

$$\tilde{H}_{\{a\}}(\underline{x}) = \sum_{i=1}^{n} F_{a_i,a}(x_{a_i},x_a) + \sum_{j=1}^{s} F_{a,a'_j}(x_a,x_{a'_j}),$$

$$\int e^{-\tilde{H}_{\{a\}}(\underline{x})}\mu(dx_a) = \int \prod_{i=1}^{n} e^{-F_{a_i,a}(x_{a_i},x_a)} \prod_{j=1}^{s} e^{-F_{a,a'_j}(x_a,x_{a'_j})}\mu(dx_a)$$

$$\leqq \{ \prod_{i=1}^{n} \int e^{-(n+s)F_{a_i,a}(x_{a_i},x_a)} \mu(dx_a) \prod_{j=1}^{s} \int e^{-(n+s)F_{a,a_j'}(x_a,x_{a_j'})} \mu(dx_a)\}^{\frac{1}{n+s}}$$

$$< +\infty \quad \text{a.e.}(\mu_{\partial\{a\}}).$$

We assume that the statement is true if $\#V \leqq k$. Let $\#V = k+1$. Fix any $a_o \in V$ and let $V_o = V \backslash \{a_o\}$. Put

$$F'_{a,a_o}(x) = -\frac{1}{n+s} \log \int e^{-(n+s)F_{a,a_o}(x,z)} \mu(dz), \text{ if } a \to a_o,$$

$$F'_{a_o,a}(x) = -\frac{1}{n+s} \log \int e^{-(n+s)F_{a_o,a}(z,x)} \mu(dz), \text{ if } a \leftarrow a_o,$$

$$F'_{a,b}(x,y) = F_{a,b}(x,y), \text{ if otherwise.}$$

It is clear that $\iint e^{-(n+s)F'_{a,b}(x,y)} \mu(dx)\mu(dy) < +\infty$. We have

$$\tilde{H}_V(\underline{x}) = \sum_{\substack{a \in V_o \cup \partial V \\ a \to a_o}} F_{a,a_o}(x_a, x_{a_o}) + \sum_{\substack{a \in V_o \cup \partial V \\ a \leftarrow a_o}} F_{a_o,a}(x_{a_o}, x_a)$$

$$+ \sum_{\substack{a,b \in V_o \\ a \to b}} F'_{a,b}(x_a, x_b) + \sum_{\substack{a \in V_o, b \in \partial V_o \backslash \{a_o\} \\ a \to b}} F'_{a,b}(x_a, x_b)$$

$$+ \sum_{\substack{a \in V_o, b \in \partial V_o \backslash \{a_o\} \\ a \leftarrow b}} F'_{b,a}(x_b, x_a).$$

Denote the sum of the first two terms and the sum of the last three terms by $\tilde{H}_1(\underline{x})$ and by $\tilde{H}_2(\underline{x})$, respectively. Remark that $\#\{a \in V_o \cup \partial V \; ; \; a - a_o\} = n + s$. We have by Hölder's inequality

$$\int e^{-\tilde{H}_1(\underline{x})}\mu(dx_{a_o}) = \int \prod_{\substack{a\in V_o\cup\partial V\\ a\to a_o}} e^{-F_{a,a_o}(x_a,x_{a_o})} \prod_{\substack{a\in V_o\cup\partial V\\ a\leftarrow a_o}} e^{-F_{a_o,a}(x_{a_o},x_a)}\mu(dx_{a_o})$$

$$\leqq \{ \prod_{\substack{a\in V_o\cup\partial V\\ a\to a_o}} \int e^{-(n+s)F_{a,a_o}(x_a,x_{a_o})}\mu(dx_{a_o}) \prod_{\substack{a\in V_o\cup\partial V\\ a\leftarrow a_o}} \int e^{-(n+s)F_{a,a_o}(x_a,x_{a_o})}\mu(dx_{a_o})\}^{\frac{1}{n+s}}$$

$$= \exp\{ \sum_{\substack{a\in V_o\cup\partial V\\ a\to a_o}} F'_{a,a_o}(x_a) - \sum_{\substack{a\in V_o\cup\partial V\\ a\leftarrow a_o}} F'_{a_o,a}(x_a)\}.$$

On the other hand,

$$\tilde{H}_2(\underline{x}) + \sum_{\substack{a\in V_o\cup\partial V\\ a\to a_o}} F'_{a,a_o}(x_a) + \sum_{\substack{a\in V_o\cup\partial V\\ a\leftarrow a_o}} F'_{a_o,a}(x_a)$$

$$= \tilde{H}'_{V_o}(\underline{x}) + \sum_{\substack{a\in\partial V\\ a\to a_o}} F'_{a,a_o}(x_a) + \sum_{\substack{a\in\partial V\\ a\leftarrow a_o}} F'_{a_o,a}(x_a),$$

where $\tilde{H}'_{V_o}(\underline{x})$ is the Hamiltonian determined by $\{F'_{a,b}\}$, i.e.,

$$\tilde{H}'_{V_o}(\underline{x}) = \sum_{\substack{a,b\in V_o\\ a\to b}} F'_{a,b}(x_a,x_b) + \sum_{\substack{a\in V_o,b\in\partial V_o\\ a\to b}} F'_{a,b}(x_a,x_b)$$

$$+ \sum_{\substack{a\in V_o,b\in\partial V_o\\ a\leftarrow b}} F_{b,a}(x_b,x_a).$$

Therefore, we have

$$\int e^{-\tilde{H}_V(\underline{x})}\mu_V(dx_V) = \int e^{-\tilde{H}_2(\underline{x})}\mu_{V_o}(dx_{V_o})\int e^{-\tilde{H}_1(\underline{x})}\mu(dx_{a_o})$$

$$\leqq \exp\{-\sum_{\substack{a\in\partial V\\ a\to a_o}} F'_{a,a_o}(x_a) \sum_{\substack{a\in\partial V\\ a\leftarrow a_o}} F'_{a_o,a}(x_a)\}\int e^{-\tilde{H}'_{V_o}(\underline{\underline{x}})}\mu_{V_o}(dx_{V_o}).$$

The last integral is finite a.e.$(\mu_{\partial V_o})$ by the assumption of induction.

2) If $\#V = 1$, $\tilde{\tilde{H}}_V(\underline{\underline{x}}) = 0$. Consequently, $\int e^{-\tilde{\tilde{H}}_V(\underline{\underline{x}})}\mu_V(dx_V) < +\infty$ is trivial. We assume that the statement is true if $\#V \leqq k$. Let $\#V = k+1$. It is easy to see that there exists $a_o \in V$ such that $\#(V\cap\partial a_o) = 0$ or $1$. Put $V_o = V\setminus\{a_o\}$. If $\#(V\cap\partial a_o) = 0$, $\tilde{\tilde{H}}_V(\underline{\underline{x}}) = \tilde{\tilde{H}}_{V_o}(\underline{\underline{x}})$. Therefore, by the assumption of induction,

$$\int e^{-\tilde{\tilde{H}}_V(\underline{\underline{x}})}\mu_V(dx_V) = \int\int e^{-\tilde{\tilde{H}}_{V_o}(\underline{\underline{x}})}\mu_{V_o}(dx_{V_o})\mu(dx_{a_o})$$

$$= \mu(X)\int e^{-\tilde{\tilde{H}}_{V_o}(\underline{\underline{x}})}\mu_{V_o}(dx_{V_o}) < +\infty.$$

If $V\cap\partial a_o = \{b\}$ and if, for example, $a_o \to b$, then

$$\tilde{\tilde{H}}_V(\underline{\underline{x}}) = \tilde{\tilde{H}}_{V_o}(\underline{\underline{x}}) + F_{a_o,b}(x_{a_o},x_b).$$

Therefore,

$$\int e^{-\tilde{\tilde{H}}_V(\underline{\underline{x}})}\mu_V(dx_V) = \int\int e^{-\tilde{\tilde{H}}_{V_o}(\underline{\underline{x}}) - F_{a_o,b}(x_{a_o},x_b)}\mu(dx_{a_o})\mu_{V_o}(dx_{V_o})$$

$$\leqq \sup_x \int e^{-F_{a_o,b}(x_{a_o},x)}\mu(dx_{a_o})\int e^{-\tilde{\tilde{H}}_{V_o}(\underline{\underline{x}})}\mu_{V_o}(dx_{V_o}) < +\infty.$$

In the following we consider only admissible potentials without mentioning.

Put

$$q^F_{V,x_{\partial V}}(x_V) = \Xi(V,x_{\partial V})^{-1} e^{-H^F_V(\underline{x})},$$

which is a probability density on $(X^V, \mu_V)$. We call $q^F_{V,x_{\partial V}}$ <u>conditional Gibbs density</u>. We remark that $q^F_{V,x_{\partial V}} = q^{F'}_{V,x_{\partial V}}$ for all finite subset V and for a.a.$(\mu_{\partial V})$ $x_{\partial V}$, if and only if $F \cong F'$.

<u>Definition</u> ([2], [8]). A probability measure P on $(\Omega, B_\Omega)$ is called <u>Gibbs distribution with a potential</u> F, if for each finite subset V of T, conditional probability distribution $P(\ |B_{V^c})$ relative to $B_{V^c}$ is absolutely continuous with respect to $\mu_V$ and

$$\frac{dP(\ |B_{V^c})}{d\mu_V} = q^F_{V,x_{\partial V}} \qquad \text{a.e.}(P).$$

Let G(F) be the set of Gibbs distributions with the potential F.

<u>3. Markov chains on the directed tree T</u>. Let p(x,y) be a positive transition density on $(X,B,\mu)$ and let h(x) be the invariant probability density of p(x,y). Put

$$\hat{p}(x,y) = h(y)p(y,x)h(x)^{-1},$$

which is called <u>reversed transition density</u> of p. We say that p is <u>reversible</u> if $p = \hat{p}$.

Let V be a connected finite subset of T. Let us introduce the second direction $\mapsto$ in V. Fix any $a_0 \in V$. If a - b and there exists a chain $a_0$ $a_1 - \cdots - a_k$ a b, we write $a \mapsto b$ or $b \leftarrow\!\shortmid a$. In particular, $a_0 \mapsto a$ if $a_0$ - a. We remark that if a-b $\in$ V, either $a \mapsto b$ or $a \leftarrow\!\shortmid b$. Put

$$p_V(x_V) = h(x_{a_o}) \prod_{\substack{a,b\in V \\ a\leftrightarrow b \\ a\to b}} p(x_a,x_b) \prod_{\substack{a,b\in V \\ a\leftrightarrow b \\ a\leftarrow b}} \hat{p}(x_a,x_b),$$

$$P_V\{\omega\in\Omega;\ x_V(\omega)\in E\} = \int_E p_V(x_V)\mu_V(dx_V) \text{ for } E \in B_V.$$

It is easy to see that $p_V$ does not depend on the choice of the centre $a_o$ and that $\{P_V\}$ is a consistent cylinder measure. By Kolmogorov's extension theorem, $\{P_V\}$ extends to a measure p on $(\Omega,B_\Omega)$. We identify the measure p with its transition density p(x,y).

Definition. A measure p constructed above is called Spitzer's Markov chain with a potential F if $p \in G(F)$. Denote by M(F) the set of Spitzer's Markov chains with the potential F.

Theorem 1. A transition density p = p(x,y) belongs to M(F), if and only if p(x,y) has the expression;

$$p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^s v(y)^{n-1} e^{-F(x,y)},$$

where $\lambda(s,n)$ is the Perron-Frobenius eigenvalue of the kernel $e^{-F(x,y)}$ if $s = n = 1$, and $\lambda(s,n) = 1$ if otherwise, and u and v are positive measurable functions satisfying

$$(*)\quad \begin{cases} u(x) = \lambda(s,n)\int_X e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy), \\ v(x) = \lambda(s,n)\int_X e^{-F(y,x)}u(y)^{s-1}v(y)^n\mu(dy), \\ \int_X u(x)^s v(x)^n\mu(dx) < +\infty. \end{cases}$$

The invariant probability density $h(x)$ has the form;

$$h(x) = c\ u(x)^s v(x)^n,$$

where $c$ is a normalizing constant.

<u>Proof</u>. 1°. Assume $p(x,y) \in M(F)$. Let $a_i \to a$ $(1 \le i \le n)$ and $a'_j \leftarrow a$ $(1 \le j \le s)$ as before. Choose $a$ as the centre of $\{a, a_1, a_2, \cdots, a_n, a'_1, a'_2, \cdots, a'_s\}$ in the definition of the direction $\mapsto$. We have

$$q_{a,x_{\partial a}}(x) = \Xi(a, x_{\partial a})^{-1} \exp\{ \sum_{i=1}^{n} F(x_{a_i}, x) - \sum_{j=1}^{s} F(x, x_{a'_j})\}$$

$$= Z(x_{\partial a})^{-1} h(x) \prod_{i=1}^{n} \hat{p}(x, x_{a_i}) \prod_{j=1}^{s} p(x, x_{a'_j}),$$

where $Z(x_{\partial a}) = \int h(x) \prod_{i=1}^{n} \hat{p}(x, x_{a_i}) \prod_{j=1}^{s} p(x, x_{a'_j}) \mu(dx)$. Put $U(x,y) = p(x,y) e^{F(x,y)}$. Then,

$$Z(x_{\partial a})^{-1} h(x) \prod_{i=1}^{n} \hat{p}(x, x_{a_i}) \prod_{j=1}^{s} p(x, x_{a'_j})$$

$$= Z(x_{\partial a})^{-1} \prod_{i=1}^{n} h(x_{a_i}) h(x)^{1-n} \prod_{i=1}^{n} U(x_{a_i}, x) \prod_{j=1}^{s} U(x, x_{a'_j}) \times$$

$$\times \exp\{ \sum_{i=1}^{n} F(x_{a_i}, x) \quad \sum_{j=1}^{s} F(x, x_{a'_j})\}.$$

Consequently, $W \equiv h(x)^{1-n} \prod_{i=1}^{n} U(x_{a_i}, x) \prod_{j=1}^{s} U(x, x_{a'_j})$ does not depend on $x$.

Fix $x_o$ in $X$ and take arbitrary $y$ from $X$. Let $x_{a_i} = x_o$ $(1 \le i \le n)$ and let $x_{a'_j} = x_o$ or $y$ $(1 \le j \le s)$. Put $\nu = \#\{j : x_{a'_j} = y\}$.

We have

$$W = h(x)^{1-n}U(x_o,x)^n U(x,y)^\nu U(x,x_o)^{s-\nu}$$

$$= h(x)^{1-n}U(x_o,x)^n U(x,x_o)^s\{\frac{U(x,y)}{U(x,x_o)}\}^\nu .$$

Letting $\nu = 0$, we see that $h(x)^{1-n}U(x_o,x)^n U(x,x_o)^s$ does not depend on x. Next, letting $\nu = 1$, we see that $\frac{U(x,y)}{U(x,x_o)}$ does not depend on x, which we denote by V(y). Putting $U(x) = U(x,x_o)$, we have $U(x,y) = U(x)V(y)$. Therefore, $p(x,y) = U(x)V(y)e^{-F(x,y)}$ and $c_1 \equiv h(x)^{1-n}U(x)^s V(x)^n$ does not depend on x.

Case, $n = 1$. Put

$$u(x) = \begin{cases} U(x)^{-1}, & \text{if } s = 1, \\ c_1^{\frac{1}{s-1}} U(x)^{-1}, & \text{if } s \geqq 2. \end{cases}$$

From $c_1 = U(x)^s V(x)$, it follows that

$$V(x) = c_1 U(x)^{-s} = \begin{cases} c_1 u(x), & \text{if } s = 1, \\ c_1^{\frac{1}{s-1}} u(x)^s, & \text{if } s \geqq 2. \end{cases}$$

We have

$$p(x,y) = U(x)V(y)e^{-F(x,y)}$$

$$= \begin{cases} c_1 u(x)^{1} u(y) e^{-F(x,y)}, & \text{if } s = 1, \\ u(x)^{-1}u(y)^s e^{-F(x,y)}, & \text{if } s \geqq 2. \end{cases}$$

The equality $\int p(x,y)\mu(dy) = 1$ implies that

$$u(x) = \begin{cases} c_1 \int e^{-F(x,y)} u(y) \mu(dy), \text{ if } s = 1, \\ \\ \int e^{-F(x,y)} u(y)^s \mu(dy), \text{ if } s \geqq 2. \end{cases}$$

Since $u(x) > 0$, $c_1$ is the Perron-Frobenius eigenvalue $\lambda(1,1)$ of $e^{-F(x,y)}$. Thus we have

$$p(x,y) = \lambda(s,1) u(x)^{-1} u(y)^s e^{-F(x,y)},$$

$$u(x) = \lambda(s,1) \int e^{-F(x,y)} u(y)^s \mu(dy).$$

Put $v(x) = u(x)^{-s} h(x)$. The equality $h(x) = \int h(y) p(y,x) \mu(dy)$

implies $v(x) = \lambda(s,1) \int e^{-F(y,x)} u(y)^{s-1} v(y) \mu(dy)$.

From $\int h d\mu = 1$, it follows $\int u^s v d\mu = 1$. Thus, the proof is completed in case $n = 1$.

<u>Case</u>, $n \geqq 2$. Put $u(x) = U(x)^{\ 1}$ and $v(y) = \{U(y)^s V(y)\}^{\frac{1}{n-1}}$, i.e.,

$$U(x) = u(x)^{\ 1},\ V(y) = u(y)^s v(y)^{n-1}.$$

Consequently, $p(x,y) = u(x)^{\ 1} u(y)^s v(y)^{n-1} e^{-F(x,y)}$. The equality $\int p(x,y) \mu(dy) = 1$ means

$$u(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy).$$

On the other hand,

$$\begin{aligned} c_1 &= h(x)^{1-n} U(x)^s V(x)^n \\ &= \{h(x)^{\ 1} u(x)^s v(x)^n\}^{n-1}, \end{aligned}$$

which means $h(x) = c_2 u(x)^s v(x)^n$ with a constant $c_2$. The equality $\int h d\mu = 1$ implies $\int u^s v^n d\mu < +\infty$. From $h(x) = \int h(y) p(y,x) \mu(dy)$, it follows that

$$v(x) = \int e^{F(y,x)} u(y)^{s-1} v(y)^n \mu(dy).$$

The proof is completed in case $n \geq 2$.

$2^\circ$. Assume conversely that positive functions u and v satisfy (*). Put

$$p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^s u(y)^{n-1} e^{-F(x,y)},$$

$$h(x) = c\ u(x)^s v(x)^n \quad \text{with } c = (\int u^s v^n d\mu)^{-1}.$$

The reversed transition density $\hat{p}(x,y) = h(y)p(y,x)h(x)^{-1}$ is equal to

$$\hat{p}(x,y) = \lambda(s,n) v(x)^{-1} v(y)^n u(y)^{s-1} e^{-F(y,x)}.$$

Let V be a connected finite subset of T and fix $a_o \in V$ as the centre of $V \cup \partial V$ in the definition of the direction $\mapsto$. We have

$$p_{V\cup\partial V}(x_{V\cup\partial V}) = h(x_{a_o}) \prod_{\substack{a,b\in V\cup\partial V \\ a\mapsto b \\ a\to b}} p(x_a,x_b) \prod_{\substack{a,b\in V\cup\partial V \\ a\mapsto b \\ a\leftarrow b}} \hat{p}(x_a,x_b)$$

$$= c\ \lambda(s,n)^{\#\{a-b\in V\cup\partial V\}} \hat{\Xi}(V,x_{V\cup\partial V})^{-1} \exp\{-\sum_{\substack{a,b\in V\cup\partial V \\ a\to b}} F(x_a,x_b)\},$$

where we put

$$\hat{\Xi}(V,x_{V\cup\partial V})^{-1} = u(x_{a_o})^s v(x_{a_o})^n \prod_{\substack{a,b\in V\cup\partial V \\ a\mapsto b \\ a\to b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \times$$

$$\times \prod_{\substack{a,b\in V\cup\partial V \\ a\mapsto b \\ a\leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}.$$

As usual, let $a_i \to a_o$ $(1\leq i\leq n)$ and $a'_j \leftarrow a_o$ $(1\leq j\leq s)$. Remark that

$\partial a_o = \{a_1,\dots,a_n,a_1',\dots,a_s'\} \subset V\cup\partial V$. We have

$$\hat{\Xi}(V,x_{V\cup\partial V})^{-1} = u(x_{a_o})^s v(x_{a_o})^n \prod_{j=1}^{s} \{u(x_{a_o})^{-1} u(x_{a_j'})^s v(x_{a_j'})^{n-1}\} \times$$

$$\times \prod_{i=1}^{n} \{v(x_{a_o})^{-1} v(x_{a_i})^n u(x_{a_i})^{s-1}\} \prod_{\substack{a,b\in V\cup\partial V, a\neq a_o \\ a\mapsto b \\ a\to b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \times$$

$$\times \prod_{\substack{a,b\in V\cup\partial V, a\neq a_o \\ a\mapsto b \\ a\leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}$$

$$= \prod_{j=1}^{s} \{u(x_{a_j'})^s v(x_{a_j'})^{n-1}\} \prod_{i=1}^{n} \{v(x_{a_i})^n u(x_{a_i})^{s-1}\} \times$$

$$\times \prod_{\substack{a,b\in V\cup\partial U, a\neq a_o \\ a\mapsto b \\ a\to b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \prod_{\substack{a,b\in V\cup\partial V, a\neq a_o \\ a\mapsto b \\ a\leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}$$

Therefore, $\hat{\Xi}(V,x_{V\cup\partial V})^{-1}$ does not depend on $x_{a_o}$. Since $\hat{\Xi}(V,x_{V\cup\partial V})^{-1}$ does not depend on the choice of the centre $a_o \in V$ of the direction $\mapsto$, it does not depend on $x_V$. Thus, we have $p_{V\cup\partial V}(x_{V\cup\partial V}) =$
$= \hat{\hat{\Xi}}(V,x_{\partial V})^{-1} \exp\{-\sum_{\substack{a,b\in V\cup\partial V \\ a\to b}} F(x_a,x_b)\}$, where $\hat{\hat{\Xi}}(V,x_{\partial V})$ depends only on $x_{\partial V}$. It is easy to see that the extension of the cylinder measure $\{p_{V\cup\partial V}\}$ belongs to $G(F)$. The proof of Theorem 1 is completed.

We remark that the expression of $p(x,y)$ in Theorem 1 is not unique. If u and v satisfy (*), then also $\hat{u} = c^{n-1}u$ and $\hat{v} = c^{-(s-1)}v$ satisfy (*) and determine the same $p(x,y)$ as u and v. In order to make the expression unique, we need summability of $u^s v^{n-1}$ and $u^{s-1}v^n$, which does not follow from $\int u^s v^n d\mu < +\infty$.

Lemma 4. Put $X(x,M)=\{y\in X; F(x,y)\leqq M\}$ and $X^*(x,M)=\{y\in X; F(y,x)\leqq M\}$. We assume that there exist M and an integer k such that

$$(A,3)\quad \begin{cases} \mu^k\{(x_1,x_2,\ ..,x_k);\ \mu(X\setminus \bigcup_{i=1}^{k} X(x_i,M)) = 0\} > 0, \\ \\ \mu^k\{(x_1,x_2,\ \ .,x_k);\ \mu(X\setminus \bigcup_{i=1}^{k} X^*(x_i,M)) = 0\} > 0. \end{cases}$$

If u and v satisfy (*) in Theorem 1, it holds that

$$\int u^s v^{n-1} d\mu < +\infty \quad \text{and} \quad \int u^{s-1} v^n d\mu < +\infty.$$

Proof. Since $u(x)=\int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy) \geqq e^{-M} \int_{X(x,M)} u(y)^s v(y)^{n-1} \mu(dy)$,

$$\int u^s v^{n-1} d\mu \leqq \sum_{i=1}^{k} \int_{X(x_i,M)} u^s v^{n-1} d\mu \leqq e^M \sum_{i=1}^{k} u(x_i) < +\infty.$$

Theorem 1'. We assume that there exist M and an integer k such that (A,3) holds. A transition density $p = p(x,y)$ belongs to M(F), if and only if $p(x,y)$ has the expression:

$$p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x,y)},$$

where u and v are positive measurable functions satisfying

$$(*)'\quad \begin{cases} u(x) = \lambda(s,n) \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy), \\ v(x) = \lambda(s,n) \int e^{-F(y,x)} u(y)^{s-1} v(y)^n \mu(dy), \\ \int u(x)^s v(x)^{n-1} \mu(dx) = \int u(x)^{s-1} v(x)^n \mu(dx), \\ \int u(x) \mu(dx) = \int v(x) \mu(dx) = 1, \text{ if } s = n = 1, \\ \int u(x)^s v(x)^n \mu(dx) < +\infty. \end{cases}$$

The expression is unique.

Proof. By Theorem 1, a transition density $p(x,y) \in M(F)$ has the following expression with $\hat{u}$ and $\hat{v}$ satisfying (*)

$$p(x,y) = \lambda(s,n)\hat{u}(x)^{-1}\hat{u}(y)^s\hat{v}(y)^{n-1}e^{-F(x,y)}.$$

In case $n = s = 1$, functions $u = (\int\hat{u}d\mu)^{-1}\hat{u}$ and $v = (\int\hat{v}d\mu)^{-1}\hat{v}$ satisfy $(*)'$, and in case $s+n > 2$, functions $u = c^{n-1}\hat{u}$ and $v = c^{(s-1)}\hat{v}$ with $c = \{(\int\hat{u}^{s-1}\hat{v}^n d\mu)(\int\hat{u}^s\hat{v}^{n-1}d\mu)^{-1}\}^{\frac{1}{s+n-2}}$ satisfy $(*)'$. In both cases, u and v determine the same p(x,y) as $\hat{u}$ and $\hat{v}$.

Next, assume that

$$p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)}$$

$$= \lambda(s,n)\tilde{u}(x)^{-1}\tilde{u}(y)^s\tilde{v}(y)^{n-1}e^{-F(x,y)},$$

where u, v and $\tilde{u}$, $\tilde{v}$ satisfy $(*)'$. We have $\tilde{u}(x)u(x)^{-1} = \tilde{u}(y)^s u(y)^{-s}\tilde{v}(y)^{n-1}v(y)^{-(n-1)}$, which implies $u(x) = c\,\tilde{u}(x)$ in case $n = 1$, and implies $u(x) = c\,\tilde{u}(x)$ and $v(x) = c^{-\frac{s-1}{n-1}}\tilde{v}(x)$ in case $n \geq 2$. From $\int u d\mu = \int\tilde{u}d\mu = 1$ in case $s = n = 1$, or from $\int u^s v^{n-1}d\mu = \int u^{s-1}v^n d\mu$ and $\int\tilde{u}^s\tilde{v}^{n-1}d\mu = \int\tilde{u}^{s-1}\tilde{v}^n d\mu$ in case $s+n > 2$, it follows that $c = 1$. Therefore the expression is unique.

In the following, we indentify a transition density p(x,y) $\in M(F)$ with a pair (u,v) of positive solutions of $(*)'$. The set of pairs of positive solutions of $(*)'$ is denoted also by M(F).

<u>Theorem 2</u>. The set M(F) is not empty,either if

(A,4) $\int e^{-F(x,y)}\mu(dy)$ and $\int e^{-F(y,x)}\mu(dy)$ do not depend on x,

or if

(A,5) $\sup_x\{\int e^{-(n+s)F(x,y)}\mu(dy),\ \int e^{-(n+s)F(y,x)}\mu(dy)\} < +\infty$

and

(A,6) $\sup_x\{\int e^{(n+s)(n+s-2)F(x,y)}\mu(dy), \int e^{(n+s)(n+s-2)F(y,x)}\mu(dy)\} < +\infty$

Proof. We assume (A,4). Put $c_1 = \int e^{-F(x,y)}\mu(dy)$ and $c_2 = \int e^{-F(y,x)}\mu(dy)$. From $\int\int e^{-F(x,y)}\mu(dx)\mu(dy) = c_1\mu(X) = c_2\mu(X)$, it follows $c_1 = c_2$. In case $s = n = 1$, $u(x) = v(x) = \mu(X)^{-1}$ is a positive solution of (*)'. In case $s+n > 2$, $u(x) = v(x) = c_1^{-\frac{1}{n+s-2}}$ is a positive solution of (*)'.

In order to look for positive solutions of (*)' under the assumptions (A,5) and (A,6), we apply theory of cones in a Banach space. In case $s = n = 1$, (*)' is a system of linear equations with positive kernels. Such equations have positive eigenfunctions, if the kernels are square-integrable ([7]), which follows from (A,5). Therefore, it is enough to investigate only a case $s+n > 2$. We first prove existence of positive solutions of (*)' under the assumptions (A,5) and $\sup_{x,y} F(x,y) < +\infty$ instead of (A,6).

Let L be the set of pairs (u,v) of functions u and v such that

$$\|u\| \equiv \{\int |u(x)|^{n+s}\mu(dx)\}^{\frac{1}{n+s}} < +\infty \text{ and } \|v\| \equiv \{\int |v(x)|^{n+s}\mu(dx)\}^{\frac{1}{n+s}} < +\infty.$$

If we put $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in L$, $(L,\|\cdot\|)$ becomes a Banach space. Put for $(u,v) \in L$

$$A_1(u,v)(x) = \int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy),$$

$$A_2(u,v)(x) = \int e^{-F(y,x)}u(y)^{s-1} v(y)^n\mu(dy),$$

$$A(u,v) = (A_1(u,v),\ A_2(u,v)).$$

Lemma 5. (Theorem 3.2 in Ch.1 of Krasnosel'skii [6]). Under the assumption (A,1), A is a completely continuous mapping from L into L.

Put

$$K_1 = \{u(x) = \int e^{-F(x,y)} a(y)\mu(dy);\ a(y) \geqq 0, \|u\| < +\infty\},$$

$$K_2 = \{v(x) = \int e^{-F(y,x)} b(y)\mu(dy);\ b(y) \geqq 0, \|v\| < +\infty\}.$$

Let K be the closure of $K_1 \times K_2$. We remark that K is a cone in $L$, i.e., K is closed and convex, $t K \subset K$ if $t \geq 0$, and $(u,v)$ and $(-u,-v) \in K$ implies $(u,v) = 0$. It is clear that $A(K) \subset K$.

<u>Lemma 6</u>. We assume (A,5) and $\sup_{x,y} F(x,y) < +\infty$. Then, there exists a positive constant c such that $u(x) \geqq c\|u\|$ and $v(x) \geqq c\|v\|$ for all $(u,v) \in K$ and for almost all $x \in X$.

<u>Proof</u>. Let $u(x) = \int e^{-F(x,y)} a(y)\mu(dy) \in K_1$. We have

$$u(x) \geqq e^{-\sup_{x,y} F(x,y)} \int a(y)\mu(dy).$$

On the other hand, by Hölder's inequality

$$u(x) \leqq (\int a d\mu)^{\frac{n+s-1}{n+s}} \{\int e^{(n+s)F(x,y)} a(y)\mu(dy)\}^{\frac{1}{n+s}}.$$

Therefore,

$$\|u\|^{n+s} \leqq (\int a d\mu)^{n+s-1} \int\int e^{(n+s)F(x,y)} a(y)\mu(dx)\mu(dy)$$

$$\leqq (\int a d\mu)^{n+s} \sup_y \int e^{-(n+s)F(x,y)} \mu(dx).$$

Consequently,

$$u(x) \geqq e^{-\sup_{x,y} F(x,y)} \int a d\mu$$

$$\geqq e^{-\sup_{x,y} F(x,y)} \{\sup_y \int e^{(n+s)F(x,y)} \mu(dx)\}^{-\frac{1}{n+s}} \|u\|.$$

Thus, there is a constant $c > 0$ such that $u(x) \geqq c\|u\|$ and $v(x) \geqq c\|v\|$ for $(u,v) \in K_1 \times K_2$. Take any $(u,v) \in K$. There exists a sequence $(u_n,v_n) \in K_1 \times K_2$ such that $\|(u_n,v_n)\ (u,v)\| \to 0$, i.e., $\|u_n-u\|$ and $\|v_n-v\| \to 0$. We can find a subsequence $\{n_j\}$ such that $u_{n_j}(x) \to u(x)$ and $v_{n_j}(x) \to v(x)$ for almost all $x \in X$. Since $\|u_{n_j}\| \to \|u\|$ and $\|v_{n_j}\| \to \|v\|$, we have $u(x) \geqq c\|u\|$ and $v(x) \geqq c\|v\|$.

Lemma 7. (Rothe [10], Krasnosel'skii [6]) Let $A = (A_1,A_2)$ be a completely continuous mapping from a cone $K \subset L$ into itself. Assume $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}} \|A_1(u,v)\| > 0$ and $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}} \|A_2(u,v)\| > 0$. Then there exists $(u_o,v_o) \in K$ such that $\|u_o\| = \|v_o\| = 1$ and

$$(u_o,v_o) = \left(\frac{A_1(u_o,v_o)}{\|A_1(u_o,v_o)\|}, \frac{A_2(u_o,v_o)}{\|A_2(u_o,v_o)\|}\right).$$

Proof. Fix any $(\hat{u}_o,\hat{v}_o) \in K$ with $\hat{u}_o \neq 0$ and $\hat{v}_o \neq 0$. Put

$$\hat{A}_1(u,v) = A_1(u,v) + (1-\|u\|\cdot\|v\|)\hat{u}_o,$$

$$\hat{A}_2(u,v) = A_2(u,v) + (1-\|u\|\cdot\|v\|)\hat{v}_o.$$

Let $\hat{K} = \{(u,v)\in K;\ \|u\|\leqq 1, \|v\|\leqq 1\}$, which is bounded, closed and conex. Our assumption implies $\inf_{(u,v)\in\hat{K}} \|A_1(u,v)\| > 0$ and $\inf_{(u,v)\in\hat{K}} \|A_2(u,v)\| > 0$. Put again

$$B_1(u,v) = \frac{\hat{A}_1(u,v)}{\|\hat{A}_1(u,v)\|},\ B_2(u,v) = \frac{\hat{A}_2(u,v)}{\|\hat{A}_2(u,v)\|}.$$

$B = (B_1,B_2)$ is a completely continuous mapping from $\hat{K}$ into $\hat{K}$.

By Schauder's fixed point theorem, there exists $(u_o,v_o) \in \hat{K}$ such that $(u_o,v_o) = B(u_o,v_o)$, i.e., $u_o = \dfrac{\hat{A}_1(u_o,v_o)}{\|\hat{A}_1(U_o,v_o)\|}$ and $v_o = \dfrac{\hat{A}_2(u_o,v_o)}{\|\hat{A}_2(u_o,v_o)\|}$

Since $\|u_o\| = \|v_o\| = 1$, $\hat{A}_1(u_o,v_o) = A_1(u_o,v_o)$ and $\hat{A}_2(u_o,v_o) = A_2(u_o,v_o)$

Proof of Theorem 2 under the assumptions (A,5) and $\sup_{x,y} F(x,y) < +\infty$.

By Lemma 6, we see that for $(u,v) \in K$

$$A_1(u,v)(x) \geqq c^{s+n-1}\|u\|^s\|v\|^{n-1}\int e^{-F(x,y)}\mu(dy),$$

$$A_2(u,v)(x) \geqq c^{s+n-1}\|u\|^{s-1}\|v\|^n\int e^{-F(y,x)}\mu(dy).$$

Hence, $\inf\limits_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}} \|A_1(u,v)\| > 0$ and $\inf\limits_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}} \|A_2(u,v)\| > 0$. By Lemma 7, there exists $(u_o,v_o) \in K$ with $\|u_o\| = \|v_o\| = 1$ satisfying

$$u_o = \|A_1(u_o,v_o)\|^{-1}A_1(u_o,v_o),$$

$$v_o = \|A_2(u_o,v_o)\|^{-1}A_2(u_o,v_o).$$

Positivity of $u_o$ and $v_o$ follows from $(u_o,v_o) \in K$.

On the other hand, we have

$$\int u_o^s\, v_o^n\, d\mu = \int u_o(x)^{s-1}v_o(x)^n u_o(x)\mu(dx)$$

$$= \|A_1(u_o,v_o)\|^{-1}\int u_o(x)^{s-1}v_o(x)^n A_1(u_o,v_o)(x)\mu(dx)$$

$$= \|A_1(u_o,v_o)\|^{-1}\int\int u_o(x)^{s-1}v_o(x)^n e^{-F(x,y)}u_o(y)^s v_o(y)^{n-1}\,\mu(dx)\mu(dy)$$

$$\int u_o^s\, v_o^n\, d\mu =$$

$$= \|A_2(u_o,v_o)\|^{-1}\int\int u_o(y)^{s-1}v_o(y)^n e^{-F(y,x)}u_o(x)^s v_o(x)^{n-1}\mu(dx)\mu(dy)$$

Integrals above are finite, since

$$\int u_o^s \, v_o^n \, d\mu \leqq \left(\int u_o^{n+s} \, d\mu\right)^{\frac{s}{n+s}} \left(\int v_o^{n+s} d\mu\right)^{\frac{n}{n+s}} < +\infty .$$

Consequently, $\|A_1(u_o,v_o)\| = \|A_2(u_o,v_o)\|$. Put

$$u(x) = \left\{ \|A_1(u_o,v)\|^{-1} \left( \frac{\int u_o^{s-1} \; v_o^n \; d\mu}{\int u_o^s \; v_o^{n-1} \; d\mu} \right)^{n-1} \right\}^{\frac{1}{n+s-2}} u_o(x),$$

$$v(x) = \left\{ \|A_2(u_o,v_o)\|^{\;1} \left( \frac{\int u_o^s \; v_o^{n-1} \; d\mu}{\int u_o^{s-1} \; v_o^n \; d\mu} \right)^{s-1} \right\}^{\frac{1}{n+s-2}} v_o(x).$$

It is easy to see that $(u,v)$ is a positive solution of $(*)'$.

<u>Proof of Theorem 2 under the assumptions (A,5) and (A,6)</u>. Let $F_k(x,y) = \min \{F(x,y),\, k\}$ for $k = 1,2,\cdots$. Let $(u_k,v_k)$ be a positive solution of $(*)'$ with the potential $F_k$. We have

<u>Lemma 8</u>. Under the assumptions (A,5) and (A,6), there exist positive constants $c_1$ and $c_2$ such that $c_1 \leqq u_k(x),\ v_k(x) \leqq c_2$ for all $k$ and almost all $x \in X$.

<u>Proof</u>. Remark that

$$\sup_{k,x} \left\{\int e^{-(n+s)F_k(x,y)}\mu(dy),\ \int e^{-(n+s)F_k(y,x)}\mu(dy)\right\} < +\infty,$$

$$\sup_{k,x} \left\{\int e^{(n+s)(n+s-2)F_k(x,y)}\mu(dy),\ \int e^{(n+s)(n+s-2)F_k(y,x)}\mu(dy)\right\} < +\infty.$$

The proof of Lemma 8 is essentially the same as that of Lemma 12.

Since $u_k$'s and $v_k$'s are bounded, we can extract a subsequence $\{k_j\}$ such that $u_{k_j}$, $v_{k_j}$, $u_{k_j}^s v_{k_j}^{n-1}$ and $u_{k_j}^{s-1} v_{k_j}^n$ are weakly convergent in $L_2$ as $j \to \infty$. Put $u = \text{w-lim}\, u_{k_j}$, $v = \text{w-lim}\, v_{k_j}$, and $\hat{u} = \text{w-lim}\, u_{k_j}^s v_{k_j}^{n-1}$. Remark $c_1 \leqq u(x), v(x) \leqq c_2$ for almost all $x \in X$. Take an arbitrary bounded measurable function f on X. We have

$$\int f(x) u_{k_j}(x) \mu(dx) = \iint f(x) e^{-F_{k_j}(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy)$$

$$= \iint f(x) e^{-F(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy)$$

$$+ \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy).$$

Since $g(y) = \int f(x) e^{-F(x,y)} \mu(dx)$ is a bounded function of y, the first term of the right-hand side converges to

$$\int g(y) \hat{u}(y) \mu(dy) = \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx)\mu(dy).$$

As for the second term, we have

$$\left| \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy) \right|$$

$$\leqq \|f\|_\infty \, c_2^{s+n-1} \iint \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} \mu(dx)\mu(dy).$$

The right-hand side converges to 0 as $j \to \infty$, since $0 \leqq e^{-F_{k_j}} - e^{-F} \leqq e^{-k_j}$. Therefore, we have

$$\int f(x) u(x) \mu(dx) = \lim_{j \to \infty} \int f(x) u_{k_j}(x) \mu(dx)$$

$$= \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx)\mu(dy),$$

from which it follows

$$u(x) = \int e^{F(x,y)} \hat{u}(y)\mu(dy) \qquad \text{a.e.x.}$$

Therefore,

$$u_{k_j}(x) - u(x) = \int e^{-F_{k_j}(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy) - \int e^{-F(x,y)}\hat{u}(y)\mu(dy)$$

$$= \int \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy)$$

$$+ \int e^{-F(x,y)} \{u_{k_j}(y)^s v_{k_j}(y)^{n-1} \quad \hat{u}(y)\}\mu(dy).$$

The first integral converges to 0 as $j \to \infty$ for all x. The second integral also converges to 0, because $e^{-F(x,y)}$ belongs to $L_{(n+s)} \subset L_2 = L_2^*$ as a function of y by the assumption (A,5). Consequently, $\lim_{j\to\infty} u_{k_j}(x) = u(x)$ for almost all x. By the same argument, we have $\lim_{j\to\infty} v_{k_j}(x) = v(x)$. Letting $j \to \infty$ in

$$u_{k_j}(x) = \int e^{-F_{k_j}(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy),$$

$$v_{k_j}(x) = \int e^{-F_{k_j}(y,x)} u_{k_j}(y)^{s-1} v_{k_j}(y)^n\mu(dy),$$

we conclude by Lebesgue's convergence theorem that

$$u(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1}\mu(dy),$$

$$v(x) = \int e^{-F(y,x)} u(y)^{s-1} v(y)^n\mu(dy).$$

<u>4. Reversibility of Markov chains.</u> We say that $p = p(x,y)$ is reversible if $p = \hat{p}$, which means $h(x)p(x,y) = h(y)p(y,x)$. We prove the following

Theorem 3. 1) If there exists a reversible chain in M(F), the potential F is symmetrizable.

2) Let F be a symmetric potential. Assume (A,3) in Lemma 4 and assumme

$$(A,5)\quad \sup_x \int e^{-(n+s)F(x,y)}\mu(dy) < +\infty.$$

Then, all chains in M(F) are reversible.

Proof. 1) Let p be a reversible chain in M(F). By Theorem 1, we have $p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)}$ and $h(x) = c\, u(x)^s v(x)^n$ From $h(x)p(x,y) = h(y)p(y,x)$, it follows $v(x)u(x)^{-1}e^{-F(x,y)} = v(y)u(y)^{-1}e^{-F(y,x)}$, which means $F(x,y) - F(y,x) = \log v(x)u(x)^{-1} - \log v(y)u(y)^{-1}$. By Lemma 2, F is symmetrizable.

2) Let $p = (u,v) \in M(F)$. Put $K(x,y) = e^{-F(x,y)}u(y)^{s-1}v(y)^{n-1}$. We have, by Theorem 1,

$$u(x) = \lambda(s,n)\int K(x,y)u(y)\mu(dy),$$

$$v(x) = \lambda(s,n)\int K(x,y)v(y)\mu(dy).$$

Since $\sup_x u(x) < +\infty$ and $\sup_x v(x) < +\infty$ as will be shown in the following Lemma 9, we have

$$\iint K(x,y)^2\mu(dx)\mu(dy)$$

$$\leq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\iint e^{-2F(x,y)}\mu(dx)\mu(dy)$$

$$\leq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\int\mu(dx)\{\int e^{-(n+s)F(x,y)}\mu(dy)\}^{\frac{2}{n+s}}\mu(X)^{\frac{n+s-2}{n+s}}$$

$$\leq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\{\sup_x \int e^{-(n+s)F(x,y)}\mu(dy)\}^{\frac{2}{n+s}}\mu(X)^{\frac{2(n+s-1)}{n+s}} < +\infty.$$

The kernel $K(x,y)$ being square-integrable, positive eigenfunctions in $L_2$ are unique up to a multiple of constants [7]. Consequently, there is a constant $c_1$ such that $u(x) = c_1 v(x)$. From the equality $\int us\mu = \int v d\mu = 1$ in case $s = n = 1$, or from $\int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu$ in case $s+n > 2$, it follows $c_1 = 1$, i.e., $u = v$. Therefore we have $p(x,y) = \lambda(s,n)u(x)^{\;1}u(y)^{s+n-1}e^{-F(x,y)}$ and $h(x) = c\ u(x)^{s+n}$, which implies $h(x)p(x,y) = h(y)p(y,x)$.

<u>Corollary.</u> Assume that a symmetric potential F satisfies (A,3) and (A,5). Then, a transition density $p = p(x,y)$ belongs to $M(F)$, if and only if $p(x,y)$ has the expression:

$$p(x,y) = \lambda(s,n)u(x)^{\;1}u(y)^{n+s-1}e^{-F(x,y)};$$

where u is a positive measurable function satisfying

$$(**)\quad \begin{cases} u(x) = \lambda(s,n)\int e^{-F(x,y)}u(y)^{s+n-1}\mu(dy), \\ \int u(x)\mu(dx) = 1, \text{ if } s = n = 1, \\ \int u(x)^{s+n}\mu(dx) < +\infty. \end{cases}$$

The invariant probability density $h(x)$ has the form:

$$h(x) = c\ u(x)^{s+n},$$

where c is a normalizing constant. The expression is unique.

<u>Lemma 9.</u> We assume (A,3) and (A,5). Then, $\sup_x u(x) < +\infty$ and $\sup_x v(x) < +\infty$ for each $(u,v) \in M(F)$.

Proof. Put $\sigma = \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu < +\infty$. We have by Hölder's inequality

$$u(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)$$

$$\leqq \sigma^{\frac{n+s-1}{n+s}} \{\int e^{-(n+s)F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)\}^{\frac{1}{n+s}}.$$

Consequently,

$$\int u^{s+n} d\mu \leqq \sigma^{n+s-1} \int\int e^{-(n+s)F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)\mu(dy)$$

$$\leqq \sigma^{n+s} \sup_x \int e^{\ (n+s)F(x,y)} \mu(dy) < +\infty.$$

By the same argument, we have

$$\int v^{s+n} d\mu \leqq \sigma^{n+s} \sup_x \int e^{\ (n+s)F(y,x)} \mu(dy) < +\infty.$$

We have, by Hölder's inequality again,

$$u(x)$$

$$\leqq \{\int e^{-(n+s)F(x,y)} \mu(dy)\}^{\frac{1}{n+s}} \{\int u(y)^{n+s} \mu(dy)\}^{\frac{s}{n+s}} \{\int v(y)^{n+s} \mu(dy)\}^{\frac{n-1}{n+s}}$$

$$\leqq \{\sup_x \int e^{\ (n+s)F(x,y)} \mu(dy)\}^{\frac{1}{n+s}} (\int u^{n+s} d\mu)^{\frac{s}{n+s}} (\int v^{n+s} d\mu)^{\frac{n-1}{n+s}}.$$

As for reversibility of chains in M(F) with a symmetrizable potential F, we have the following

Theorem 3'. We assume (A,3) and

(A,5) $\sup_x \{\int e^{-(n+s)F(x,y)} \mu(dy), \int e^{-(n+s)F(y,x)} \mu(dy)\} < +\infty$,

(A,6)' $\sup_x \{\int e^{(n+s)(n+s-2)'F(x,y)} \mu(dy), \int e^{(n+s)(n+s-2)'F(y,x)} \mu(dy)\} <-$

where $(n+s)(n+s-2)' = \max\{(n+s)(n+s-2), 1\}$. Then the following three statements are equivalent to each other.

1) A potential F is uniformly symmetrizable.

2) There exists a reversible chain in M(F).

3) All chains in M(F) are reversible.

To prove this, we need the the following

Lemma 10. We assume (A,3) and

$$(A,6)'' \qquad \sup_x\{\int e^{F(x,y)}\mu(dy),\int e^{F(y,x)}\mu(dy)\} <+\infty.$$

Then, $\inf_x u(x) > 0$ and $\inf_x v(x) > 0$ for each $(u,v) \in M(F)$.

Proof. We have by Hölder's inequality

$$\int (u^s v^n)^{\frac{n+s-1}{2n+s}} d\mu \leqq \{\int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\}^{\frac{n}{2n+s}} \times$$

$$\times (\int u^{s-1}v^n d\mu)^{\frac{s}{2n+s}}\{\int e^{F(x,y)}\mu(dy)\}^{\frac{n}{2n+s}}$$

$$\leqq u(x)^{\frac{n}{2n+s}}(\int u^{s-1}v^n d\mu)^{\frac{s}{2n+s}}\{\sup_x \int e^{F(x,y)}\mu(dy)\}^{\frac{n}{2n+s}},$$

from which follows $\inf_x u(x) > 0$.

Proof of Theorem 3'. 2)$\Longrightarrow$1). Let $(u,v) \in M(F)$. By the proof of Theorem 3, $F(x,y) - F(y,x) = \log v(x)u(x)^{-1} - \log v(y)u(y)^{1}$. By Lemmas 9 and 10, the function $\log v(x)u(x)^{-1}$ is bounded, hence, F is uniformly symmetrizable by Lemma 2.

1)$\Longrightarrow$3). Let F be a uniformly symmetrizable potential which satisfies (A,3) and (A,5). Then, the uniform symmetrization $\hat{F}$ of F also satisfies (A,3) and (A,5). Therefore, by Theorem 3, all chains in $M(F) = M(\hat{F})$ are reversible.

3)$\Longrightarrow$2) is trivial, since $M(F) \neq \phi$ by Theorem 2.

We present an example in which $M(F)$ contains infinitely many chains. Let $X$ be the unit circle $S^1$ which we identify with the interval $[0,1)$, and let $\mu$ be the Lebesgue measure on $S^1$. Let $s+n = 3$. Let $a_o$, $a_1$ and $a_2$ be positive numbers. Put, for $k = 0,1,2$,

$$\gamma_k = \frac{a_k}{\sum_{j=-2}^{2} a_{|k-j|} a_{|j|}},$$

and put

$$u(x) = \sum_{k=-2}^{2} a_{|k|} e^{2\pi ikx}$$

$$= a_o + 2a_1 \cos 2\pi x + 2a_2 \cos 4\pi x,$$

$$\Gamma(x) = \sum_{k=-2}^{2} \gamma_{|k|} e^{2\pi ikx}$$

$$= \gamma_o + 2\gamma_1 \cos 2\pi x + 2\gamma_2 \cos 4\pi x.$$

It is clear by the definition of $\gamma_k$ that $u(x) = \int_0^1 \Gamma(x-y)u(y)^2 dy$.

If $\gamma_1 - 4\gamma_2 > 0$, then $\min_x \Gamma(x) = \Gamma(x)|_{\cos 2\pi x = -1} = \gamma_o - 2\gamma_1 + 2\gamma_2$, since $\Gamma(x) = 4\gamma_2(\cos 2\pi x + \frac{\gamma_1}{4\gamma_2})^2 + \gamma_o - 2\gamma_2 - \frac{\gamma_1^2}{4\gamma_2}$. We can see

$$\gamma_1 - 4\gamma_2 = \frac{a_1^2 - 6a_o a_2 - 8a_2^2}{2(a_o+a_2)(a_1^2+2a_o a_2)},$$

$$\gamma_o - 2\gamma_1 + 2\gamma_2 = \frac{a_1^2 a_2(a_o+2a_2) + 4a_2^2(a_o^2+a_2^2) + 2(a_o^3 a_2 - a_1^4)}{(a_o^2+2a_1^2+2a_2^2)(a_o+a_2)(a_1^2+2a_o a_2)}.$$

Let $a_1^2 > 8a_2(a_o+a_2)$, $a_1^4 \leqq a_o^3 a_2$ and let $a_1$ and $a_2$ be sufficiently small in comparison with $a_o$. Then, functions u and $\Gamma$ are positive.

Put

$$F(x,y) = -\log \Gamma(x-y),$$

$$u_\alpha(x) = u(x+\alpha) \quad (\alpha \in [0,1)),$$

then $u_\alpha$'s $(0 \leqq \alpha < 1)$ are positive solutions of (**) in Corollary to Theorem 3, that are distinguished from each other.

Dobrushin and Shlosman [3] show that all Gibbs distributions in $Z^2$ with the state space $S^1$, whose potential is of finite range, of $C^2$-class and invariant under rotation of $S^1$, are also rotation-invariant. On the contrary, Spitzer's Markov chains determined by $u_\alpha$ are not rotation-invariant. But, M(F) contains also a rotation-invariant chain, which is determined by a constant solution $\hat{u} = (\int \Gamma(x)dx)^{-1}$ of (**).

5. Uniqueness of Markov chains at high temparature. In the following we consider potentials with the form $\beta F$, where $\beta > 0$ is the reciprocal temparature. We prove

Theorem 4. Assume (A,3), as in Lemma 4, and assume

$$\text{(A,7)} \qquad \sup_x \{\int e^{|F(x,y)|}\mu(dy), \int e^{|F(y,x)|}\mu(dy)\} < +\infty.$$

If $\beta$ is sufficiently small, then $M(\beta F)$ consists of one chain.

Proof. If $\beta$ is sufficiently small, the potential $\beta F$ satisfies (A,5) and (A,6). Therefore $M(\beta F) \neq \phi$ by Theorem 2. In case $s = n = 1$, (*)' in Theorem 1' takes the form

$$
(*)' \begin{cases} u(x) = \lambda \int e^{-\beta F(x,y)} u(y)\mu(dy), \\ v(x) = \lambda \int e^{-\beta F(y,x)} v(y)\mu(dy), \\ \int u(x)\mu(dx) = \int v(x)\mu(dx) = 1, \\ \int u(x)v(x)\mu(dx) < +\infty. \end{cases}
$$

As is shown in Lemma 8, solutions u and v of (*)' are bounded from above if $\beta < \frac{1}{2}$, since (A,5) is satisfied by $\beta F$. Since the kernel $e^{-\beta F(x,y)}$ is square-integrable if $\beta < \frac{1}{2}$, the normalized positive solutions of the Perron-Frobenius equation (*)' are unique ([7]).

To prove in case $s+n > 2$, we need several lemmas.

<u>Lemma 11</u>. Assume (A,7). Put

$$
c_1(\beta) = \sup_x \{|e^{\pm\beta F(x,y)}\mu(dy)-\mu(X)|, |\int e^{\pm\beta F(y,x)}\mu(dy)-\mu(X)|\}.
$$

Then, we have $\lim_{\beta\to 0} c_1(\beta) = 0$.

<u>Proof</u>. By Hölder's inequality, we have

$$
\begin{aligned}
\int e^{\pm\beta F(x,y)}\mu(dy) &\leqq \{\int e^{\pm F(x,y)}\mu(dy)\}^{\beta}\mu(X)^{1-\beta} \\
&\leqq \{\sup_x \int e^{|F(x,y)|}\mu(dy)\}^{\beta}\mu(X)^{1-\beta}.
\end{aligned}
$$

The right-hand side converges to $\mu(X)$ as $\beta \to 0$. By Hölder's inequality again, we have

$$
\begin{aligned}
\mu(X)^2 &= \{\int e^{\pm\frac{\beta}{2}F(x,y)} e^{\mp\frac{\beta}{2}F(x,y)} \mu(dy)\}^2 \\
&\leqq \{\int e^{\pm\beta F(x,y)}\mu(dy)\}\{\int e^{\mp\beta F(x,y)}\mu(dy)\} \\
&\leqq \{\int e^{\pm\beta F(x,y)}\mu(dy)\}\{\sup_x \int e^{|F(x,y)|}\mu(dy)\}^{\beta}\mu(X)^{1-\beta}.
\end{aligned}
$$

Consequently,

$$\int e^{\pm\beta F(x,y)}\mu(dy) \geqq \{\sup_x \int e^{|F(x,y)|}\mu(dy)\}^{-\beta}\mu(X)^{1+\beta},$$

the right-hand side of which converges to $\mu(X)$ as $\beta \to 0$.

Lemma 12. Assume (A,3) and (A,7). Put

$$c_2(\beta) = \sup_{(u,v)\in M(\beta F)} \{\| u-\mu(X)^{-\frac{1}{n+s-2}}\|_\infty, \| v-\mu(X)^{-\frac{1}{n+s-2}}\|_\infty\},$$

$$c_2'(\beta) = \sup_{(u,v)\in M(\beta F)} \{\| u^{s-1}v^{n-1}-\mu(X)^{\ 1}\|_\infty, \| u^s v^{n-2}-\mu(X)^{\ 1}\|_\infty, \| u^{s-2}v^n-\mu(X)^{-1}$$

where $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Then, we have $\lim_{\beta\to 0} c_2(\beta) = \lim_{\beta\to 0} c_2'(\beta) = 0$.

Proof. Take any $(u,v) \in M(\beta F)$. Put $\sigma = \int u^s v^{n-1}d\mu = \int u^{s-1}v^n d\mu$.

1°. $\int u^{s+n}d\mu, \int v^{s+n}d\mu \leqq \sigma^{s+n}\{\mu(X)+c_1(\beta(s+n))\}$.

In fact, we have

$$\begin{aligned} u(x) &= \int e^{-\beta F(x,y)}u(y)^s v(y)^{n-1}\mu(dy) \\ &\leqq \sigma^{\frac{s+n-1}{s+n}}\{\int e^{-\beta(s+n)F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\}^{\frac{1}{n+s}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int u^{s+n}d\mu &\leqq \sigma^{s+n-1}\int\int e^{-\beta(s+n)F(x,y)}u(y)^s v(y)^{n-1}\mu(dx)\mu(dy) \\ &\leqq \sigma^{s+n}\sup_x \int e^{-\beta(s+n)F(x,y)}\mu(dy) \\ &\leqq \sigma^{s+n}\{\mu(X)+c_1(\beta(s+n))\}. \end{aligned}$$

2°. Put $c_3(\beta) = \{\mu(X)+c_1(\beta(s+n))\}^{\frac{s+n-1}{s+n}}\{\mu(X)+c_1(\beta(s+n)(s+n-2))\}^{\frac{1}{s+n}} -\mu($

Then, we have $u(x),v(x) \geqq \{\mu(X)+c_3(\beta)\}^{-\frac{1}{s+n-2}}$ and $\lim_{\beta\to 0} c_3(\beta) = 0$.

To show this, put $p_1 = \frac{s+n-1}{s+n-2}$, $p_2 = (s+n)(s+n-1)$, $p_3 = s\ {}^{1}p_2$, and $p_4 = (n-1)\ {}^{1}p_2$. Remark that $\sum_{i=1}^{4} p_i^{\ 1} = 1$ and $p_3^{\ 1} + p_4^{-1} = (s+n)^{-1}$. We have

$$\sigma = \int u^s v^{n-1} d\mu$$

$$\leqq \{\int e^{-\beta F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)\}^{\frac{1}{p_1}} \{\int e^{\frac{\beta p_2}{p_1} F(x,y)} \mu(dy)\}^{\frac{1}{p_2}} \times$$

$$\times (\int u^{s+n} d\mu)^{\frac{1}{p_3}} (\int v^{s+n} d\mu)^{\frac{1}{p_4}}$$

$$\leqq u(x)^{\frac{1}{p_1}} \{\mu(X)+c_1(\frac{\beta p_2}{p_1})\}^{\frac{1}{p_2}} \sigma^{(s+n)(p_3^{-1}+p_4^{-1})} \{\mu(X)+c_1(\beta(s+n))\}^{p_3^{-1}+p}$$

Hence,

$$u(x) \geqq \{\mu(X)+c_1(\frac{\beta p_2}{p_1})\}^{-\frac{p_1}{p_2}} \{\mu(X)+c_1(\beta(s+n))\}^{-\frac{p_1}{s+n}}$$

$$= \{\mu(X)+c_3(\beta)\}^{-\frac{1}{s+n-2}}.$$

3 . Put $c_4(\beta)=\mu(X)-\mu(X)^{-(s+n-2)}\{\mu(X)+c_3(\beta)\}^{-(n+s-3)}\{\mu(X)-c_1(\beta)\}^{2(s+n-}$

Then, we have $\sigma = \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu \leqq \{\mu(X)-c_4(\beta)\}^{-\frac{1}{s+n-2}}$ and $\lim_{\beta \to 0} c_4(\beta) = 0$.

In fact, we have by 2 ,

$$\{\mu(X)+c_3(\beta)\}^{-\frac{s+n-3}{2(s+n-2)}} \leqq u(x)^{\frac{s}{2}-1} v(x)^{\frac{n-1}{2}}.$$

Therefore,

$$\{\mu(X)+c_3(\beta)\}^{-\frac{s+n-3}{2(s+n-2)}} u(x) \leqq \{u(x)^s v(x)^{n-1}\}^{\frac{1}{2}},$$

$$\{\mu(X)+c_3(\beta)\}^{-\frac{s+n-3}{2(s+n-2)}} \int u d\mu \leqq \int (u^s v^{n-1})^{\frac{1}{2}} d\mu$$

$$\leqq \sigma^{\frac{1}{2}} \mu(X)^{\frac{1}{2}}.$$

On the other hand by Lemma 11,

$$\int u\,d\mu = \int\int e^{-\beta F(x,y)} u(y)^s v(y)^{n-1} \mu(dx)\mu(dy)$$

$$\geqq \{\mu(X)-c_1(\beta)\}\sigma,$$

hence,

$$\{\mu(X)+c_3(\beta)\}^{-\frac{s+n-3}{2(s+n-2)}}\{\mu(X)-c_1(\beta)\}\sigma \leqq \sigma^{\frac{1}{2}}\mu(X)^{\frac{1}{2}}.$$

Thus, we have

$$\sigma \leqq \mu(X)\{\mu(X)+c_3(\beta)\}^{\frac{s+n-3}{s+n-2}}\{\mu(X)-c_1(\beta)\}^{-2}$$

$$= \{\mu(X)-c_4(\beta)\}^{-\frac{1}{s+n-2}}.$$

4°. We have $u(x), v(x) \leqq \{\mu(X)-c_4(\beta)\}^{-\frac{s+n-1}{s+n-2}}\{\mu(X)+c_1(\beta(s+n))\}$.

In fact, we have by Lemma 11, 1° and 3°,

$$u(x) = \int e^{-\beta F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)$$

$$\leqq \{\int e^{-\beta(n+s)F(x,y)}\mu(dy)\}^{\frac{1}{n+s}}(\int u^{s+n}d\mu)^{\frac{s}{n+s}}(\int v^{s+n}d\mu)^{\frac{n-1}{s+n}}$$

$$\leqq \{\mu(X)+c_1(\beta(s+n))\}\sigma^{s+n-1}$$

$$\leqq \{\mu(X)+c_1(\beta(s+n))\}\{\mu(X)-c_4(\beta)\}^{-\frac{s+n-1}{s+n-2}}.$$

The assertions in Lemma 12 follow from 2° and 4°.

<u>Lemma 13</u>. 1) Put

$$R_1(x) \equiv R_1(u_1,v_1;u_2,v_2;x) = u_2^s v_2^{n-1} - \{u_1^s v_1^{n-1} + s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\},$$

$$R_2(x) \equiv R_2(u_1,v_1;u_2,v_2;x) = u_2^{s-1} v_2^n - \{u_1^{s-1} v_1^n + (s-1) u_1^{s-2} v_1^n w_1 + n u_1^{s-1} v_1^{n-1} w_2\},$$

where $w_1 = u_2 - u_1$ and $w_2 = v_2 - v_1$. Then, there exists a constant

$c > 0$ such that

$$\|R_1\|_\infty, \|R_2\|_\infty \leqq c \cdot c_2(\beta) \cdot \max(\|u_2-u_1\|_\infty, \|v_2-v_1\|_\infty)$$

for all $0 < \beta \leqq 1$ and for all $(u_1,v_1)$ and $(u_2,v_2) \in M(\beta F)$.

2) There exists a function $c_5(\beta)$ with $\lim_{\beta\to 0} c_5(\beta) = 0$ such that

$$|\int(u_2-u_1)d\mu - \int(v_2-v_1)d\mu| \leqq c_5(\beta)\max(\|u_2-u_1\|_\infty, \|v_2-v_1\|_\infty)$$

for all $(u_1,v_1)$ and $(u_2,v_2) \in M(\beta F)$.

Proof. 1) The assertion is clear, since

$$R_1 = (u_1+w_1)^s(v_1+w_2)^{n-1} \quad \{u_1^s v_1^{n-1} + s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\}$$

$$= \sum_{\substack{j+k\geq 2 \\ j\leq s,\, k\leq n-1}} \binom{s}{j}\binom{n-1}{k} u_1^{s-j} v_1^{n-1-k} w_1^j w_2^k$$

and since $\sup\{\|u\|_\infty, \|v\|_\infty; (u,v)\in M(\beta F), 0<\beta\leq 1\} < +\infty$ and $\|w_1\|_\infty, \|w_2\|_\infty \leqq 2c_2(\beta)$ by Lemma 12.

2) We have $\mu(X)^{-1}\int(w_1-w_2)d\mu =$

$$= \int[s\{\mu(X)^{-1} - u_1^{s-1}v_1^{n-1}\}w_1 + (n-1)\{\mu(X)^{-1} - u_1^s v_1^{n-2}\}w_2]d\mu$$

$$+ \int[(s-1)\{u_1^{s-2}v_1^n - \mu(X)^{-1}\}w_1 + n\{u_1^{s-1}v_1^{n-1} - \mu(X)^{\;1}\}w_2]d\mu$$

$$+ \int[\{s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\} - \{(s-1) u_1^{s-2} v_1^n w_1 + n u_1^{s-1} v_1^{n-1} w_2\}]d\mu.$$

The first integral in the right-hand side is bounded in the absolute value by

$$\{s\|\mu(X)^{\;1} - u_1^{s-1}v_1^{n-1}\|_\infty \cdot \|w_1\|_\infty + (n-1)\|\mu(X)^{-1} - u_1^s v_1^{n-2}\|_\infty \cdot \|w_2\|_\infty\}\mu(X),$$

which is not less than $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)$ by Lemma 12.

The second integral is also bounded in the absolute value by $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty,\|w_2\|_\infty)$. The third integral is equal to

$$\int\{(u_2^s v_2^{n-1}-u_1^s v_1^{n-1}-R_1)-(u_2^{s-1}v_2^n-u_1^s v_1^{n-1}-R_2)\}d\mu = \int(R_2-R_1)d\mu,$$

since $\int u_i^s v_i^{n-1}d\mu = \int u_i^{s-1}v_i^n d\mu$ $(i=1,2)$. The absolute value of the right-hand side is not less than $(\|R_1\|_\infty+\|R_2\|_\infty)\mu(X)$ $\leqq 2\mu(X)\cdot c\cdot c_2(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty)$. Therefore, we have

$$|\int(w_1-w_2)d\mu| \leqq 2\{(s+n-1)c_2'(\beta)+c\cdot c_2(\beta)\}\mu(X)\max(\|w_1\|_\infty,\|w_2\|_\infty).$$

<u>Proof of Theorem 4 in case</u> $s+n > 2$. Take arbitrary $(u_1,v_1)$ and $(u_2,v_2) \in M(\beta F)$. Put $w_1 = u_2-u_1$ and $w_2 = v_2-v_1$. From $u_i(x) = \int e^{-\beta F(x,y)}u_i(y)^s v_i(y)^{n-1}\mu(dy)$ $(i=1,2)$, it follows that

$$w_1(x) =$$
$$= \int e^{-\beta F(x,y)}\{su_1(y)^{s-1}v_1(y)^{n-1}w_1(y)+(n-1)u_1(y)^s v_1(y)^{n-2}w_2(y)+R_1(y)\}\mu(d$$
$$= (s+n-1)\mu(X)^{-1}\int w_1 d\mu+(n-1)\mu(X)^{-1}\int(w_2-w_1)d\mu$$
$$+s\mu(X)^{-1}\int(e^{-\beta F(x,y)}\ 1)w_1(y)\mu(dy)+(n-1)\mu(X)^{-1}\int(e^{-\beta F(x,y)}\ 1)w_2(y)\mu(dy)$$
$$+s\int e^{-\beta F(x,y)}\{u_1(y)^{s-1}v_1(y)^{n-1}-\mu(X)^{-1}\}w_2(y)\mu(dy)$$
$$+(n-1)\int e^{-\beta F(x,y)}\{u_1(y)^s v_1(y)^{n-2}-\mu(X)^{-1}\}w_2(y)\mu(dy)$$
$$+\int e^{-\beta F(x,y)}R_1(y)\mu(dy).$$

We have

$$|\int(w_2-w_1)d\mu| \leqq c_5(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty) \quad \text{(by Lemma 13)},$$

$$|\int e^{-\beta F(x,y)}\{u_1(y)^{s-1}v_1(y)^{n-1}-\mu(X)^{-1}\}w_1(y)\mu(dy)|$$

$$\leqq \{\mu(X)+c_1(\beta)\}\|u_1^{s-1}v_1^{n-1}-\mu(X)^{-1}\|_\infty \cdot \|w_1\|_\infty \qquad \text{(by Lemma 11)}$$

$$\leqq \{\mu(X)+c_1(\beta)\}c_2'(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty) \qquad \text{(by Lemma 12)},$$

$$\left|\int e^{-\beta F(x,y)}R_1(y)\mu(dy)\right| \leqq \{\mu(X)+c_1(\beta)\}\|R_1\|_\infty \qquad \text{(by Lemma 11)}$$

$$\leqq \{\mu(X)+c_1(\beta)\}c\cdot c_2(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty) \quad \text{(by Lemma 13)}.$$

As for $\int(e^{-\beta F}-1)w_1 d\mu$, we have

$$\left|\int\{e^{-\beta F(x,y)}-1\}w_1(y)\mu(dy)\right|$$

$$\leqq \{\int(e^{-\beta F(x,y)}-1)^2\mu(dy)\}^{\frac{1}{2}}(\int w_1^2 d\mu)^{\frac{1}{2}}$$

$$\leqq \|w_1\|_\infty\cdot\mu(X)^{\frac{1}{2}}\{\int(e^{-2\beta F(x,y)}-2e^{-\beta F(x,y)}+1)\mu(dy)\}^{\frac{1}{2}}.$$

The last integral converges to 0 uniformly in x as $\beta \to 0$ by Lemma 11. Consequently, $w_1(x) = (s+n-1)\mu(X)^{-1}\int w_1 d\mu+R_3(x)$, where $\|R_3\|_\infty \leqq c_6(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty)$ with $\lim_{\beta\to 0} c_6(\beta) = 0$. Hence, we have

$$\int w_1 d\mu = -\frac{1}{s+n-2}\int R_3 d\mu,$$

$$\left|\int w_1 d\mu\right| \leqq \frac{\mu(X)}{s+n-2}\|R_3\|_\infty,$$

$$|w_1|_\infty \leqq (s+n-1)\mu(X)^{-1}\left|\int w_1 d\mu\right|+\|R_3\|_\infty$$

$$\leqq \left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty).$$

By the same argument as above, we have

$$\|w_2\|_\infty \leqq \left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty,\|w_2\|_\infty),$$

from which it follows

$$\max(\|w_1\|_\infty, \|w_2\|_\infty) \leqq \left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty).$$

If $\beta$ is so small that $\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta) < 1$, then $\max(\|w_1\|_\infty, \|w_2\|_\infty) = 0$, which means $u_1 = u_2$ and $v_1 = v_2$.

6. The number of chains at low temparature. An example. We present an example, in which the number of chains in $M(\beta F)$ is exactly calculated for sufficiently large $\beta$. Let X be a finite set and let $\mu_i = \mu(\{i\}) > 0$ for all $i \in X$. We prove

Theorem 5. Let F be a symmetric potential on X satisfying

$$\text{(A,8)} \qquad F(i,j) > F(j,j)+\frac{1}{n+s-1}|F(i,i)-F(j,j)|$$

for all $i \neq j \in X$. Then, the number of chains in $M(\beta F)$ is equal to $2^{\# X}$ 1 for sufficiently large $\beta$, if $n+s > 2$.

Proof. We look for positive solutions of

$$(**) \quad u_i = \sum_{j \in X} e^{-\beta F(i,j)} u_j^{s+n-1} \mu_j \qquad (i \in X).$$

For simplicity we put $p = s+n-1$. If we put

$$x_i = \{e^{-\beta F(i,i)}\mu_i\}^{\frac{1}{p-1}} u_i,$$

the equation $(**)$ is transformed into

$$(**)' \; x_i = x_i^p + \sum_{j:j\neq i} a_{ij} x_j^p \qquad (i \in X),$$

where $a_{ij} = \mu_i^{\frac{1}{p-1}} \mu_j^{-\frac{1}{p-1}} \exp[-\beta\{F(i,j)-F(j,j)-\frac{1}{p-1}(F(j,j)-F(i,i))\}]$.

Under the assumption (A,8), we have $\lim_{\beta\to 0} a_{ij} = 0$. Therefore, Theorem 5 is a corollary to the following

Lemma 14. The number of non-trivial solutions of the equation

$$(***) \qquad x_i = |x_i|^p + \sum_{\substack{1\le j\le N \\ j\ne i}} a_{ij}|x_j|^p \qquad (1\le i\le N)$$

is equal to $2^N-1$, if $p > 1$ and positive coefficients $a_{ij}(1\le i\ne j\le N)$ are sufficiently small.

Proof. Put, for $\underline{\underline{x}} = (x_1, x_2, \cdots, x_N)$ and $\underline{\underline{a}} = (a_{ij} : 1\le i\ne j\le N)$,

$$F_i(\underline{\underline{x}},\underline{\underline{a}}) = |x_i|^p \quad x_i + \sum_{\substack{1\le j\le N \\ j\ne i}} a_{ij}|x_j|^p \qquad (1\le i\le N),$$

$$J(\underline{\underline{x}},\underline{\underline{a}}) = \det\left(\frac{\partial F_i}{\partial x_j}(\underline{\underline{x}},\underline{\underline{a}})\right)_{1\le i,j\le N},$$

where

$$\frac{\partial F_i}{\partial x_j}(\underline{\underline{x}},\underline{\underline{a}}) = p\delta_{ij}|x_i|^{p-1} - \delta_{ij} + p(1-\delta_{ij})a_{ij}|x_j|^{p-1}.$$

1°. The number of non-trivial solutions of (***) is not less than $2^N-1$, if $a_{ij}$'s are sufficiently small.

In fact, let $\hat{\underline{\underline{x}}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N) \ne \underline{\underline{0}}$ with $\hat{x}_i = 0$ or $1$. We have $F_i(\hat{\underline{\underline{x}}},\underline{\underline{0}}) = 0$ $(1\le i\le N)$ and $J(\hat{\underline{\underline{x}}},\underline{\underline{0}}) \ne 0$, since $\frac{\partial F_i}{\partial x_i}(\hat{\underline{\underline{x}}},\underline{\underline{0}}) = p\hat{x}_i \quad 1$ and $\frac{\partial F_i}{\partial x_j}(\hat{\underline{\underline{x}}},\underline{\underline{0}}) = 0$ $(i\ne j)$. Consequently, there exist a constant $A$ and an $R^N$-valued continuous function $\underline{\underline{f}}^{\hat{\underline{\underline{x}}}} = \underline{\underline{f}}^{\hat{\underline{\underline{x}}}}(\underline{\underline{a}})$ defined for $\underline{\underline{a}}$ with $\|\underline{\underline{a}}\| = \max_{i\ne j}|a_{ij}| \le A$, such that

$$\underline{\underline{f}}^{\hat{\underline{\underline{x}}}}(\underline{\underline{0}}) = \hat{\underline{\underline{x}}},$$

$$F_i(\underline{\underline{f}}^{\hat{\underline{\underline{x}}}}(\underline{\underline{a}}),\underline{\underline{a}}) = 0 \quad \text{for } \underline{\underline{a}} \text{ with } \|\underline{\underline{a}}\| \le A \quad (1\le i\le N).$$

Since $\underline{\underline{f}}^{\underline{\underline{x}}}(\underline{\underline{a}}) \ne \underline{\underline{0}}$ if $\underline{\underline{a}}$ is sufficiently small, it is a non-trivial solution of (***). Remark that if $\underline{\underline{x}} \ne \underline{\underline{x}}'$, $\underline{\underline{f}}^{\hat{\underline{\underline{x}}}}(\underline{\underline{a}}) \ne \underline{\underline{f}}^{\hat{\underline{\underline{x}}}'}(\underline{\underline{a}})$ for sufficiently small $\underline{\underline{a}}$. The number of non-trivial solution of (***) is not less than $\#\{\hat{\underline{\underline{x}}};\hat{\underline{\underline{x}}}\ne\underline{\underline{0}}, \hat{x}_i=0 \text{ or } 1\ (1\le i\le N)\} = 2^N - 1$.

$2^\circ$. If $\underline{a}$ is sufficiently small, then $J(\underline{x},\underline{a}) \neq 0$ for any solution $\underline{x} = (x_1,x_2,\cdots,x_N)$ of (***).

In fact, from $x_i - |x_i|^p = \sum_{j\neq i} a_{ij}|x_j|^p \geqq 0$, it follows $0 \leq x_i \leq 1$. From $0 \leqq x_i - |x_i|^p = \sum_{j\neq i} a_{ij}|x_j|^p \leqq \sum_{j\neq i} a_{ij} \leqq (N-1)\|\underline{a}\|$, it follows that $x_i$ is close to 0 or 1 if $\|\underline{a}\|$ is small. Therefore, $|\frac{\partial F_i}{\partial x_i}(\underline{x},\underline{a})| = |px_i^{p-1} - 1| \geqq \frac{1}{2}$ for sufficiently small $\underline{a}$. On the other hand, for $i \neq j$

$$\frac{\partial F_i}{\partial x_j}(\underline{x},\underline{a}) = pa_{ij}x_j^{p-1} \leqq p\|\underline{a}\|.$$

Hence, $J(\underline{x},\underline{a}) \neq 0$ if $\underline{a}$ is sufficiently small.

$3^\circ$. Let $\underline{a}$ be sufficiently small and let $\underline{x} = (x_1,x_2,\cdots,x_N)$ be a solution of (***). There exist continuous functions $f_1(t),f_2(t),\cdots,f_N(t)$ defined on $[0,1]$ such that

$$f_i(1) = x_i \qquad (1\leq i\leq N),$$

$$f_i(t) = |f_i(t)|^p + \sum_{j\neq i} ta_{ij}|f_j(t)|^p \qquad (1\leq i\leq N, 0\leq t\leq 1).$$

In fact, put $\tilde{F}_i(\underline{x};t) = |x_i|^p \quad x_i + \sum_{j\neq i} ta_{ij}|x_j|^p$ $(1\leq i\leq N)$ and let $A_0$ be the infimun of $A$ such that there exists a continuous function $\underline{f}(t) = (f_1(t),f_2(t),\cdots,f_N(t))$ on $[A,1]$ such that

$$\underline{f}(1) = \underline{x},$$

$$\tilde{F}_i(\underline{f}(t);t) = 0 \qquad (1\leq i\leq N, A\leq t\leq 1).$$

Put $\tilde{J}(\underline{x},t) = \det\left(\frac{\partial \tilde{F}_i}{\partial x_j}(\underline{x},t)\right)_{1\leq i,j\leq N}$. Since $\tilde{J}(\underline{x},1) \neq 0$ by $2^\circ$, such a function $\underline{f}(t)$ exists in a neighbourhood of 1. Therefore, $A_0 < 1$.

Suppose $A_o \geqq 0$. Then there exists a sequence $A_n \searrow A_o$ and continuous functions $\underline{\underline{f}}^{(n)}(t)$ on $[A_n,1]$ such that

$$\underline{\underline{f}}^{(n)}(1) = \underline{\underline{x}},$$

$$\tilde{F}_i(\underline{\underline{f}}^{(n)}(t),t) = 0 \qquad (1 \le i \le N, A_n \le t \le 1).$$

Since $\tilde{J}(\underline{\underline{f}}^{(n)}(t);t) \neq 0$ by 2°, uniqueness of implicit functions implies $\underline{\underline{f}}^{(n)}(t) = \underline{\underline{f}}^{(m)}(t)$ for $m > n$ and $A_n \leqq t \le 1$. Put

$$\underline{\underline{f}}(t) = \underline{\underline{f}}^{(n)}(t) \qquad \text{for } A_n \le t \le 1 \quad (n=1,2,\cdots).$$

The function $\underline{\underline{f}}(t)$ satisfies

$$\underline{\underline{f}}(1) = \underline{\underline{x}},$$

$$\tilde{F}_i(\underline{\underline{f}}(t);t) = 0 \qquad (1 \le i \le N, A_o < t \leqq 1).$$

Remark that every component $f_i(t)$ of $\underline{\underline{f}}(t)$ satisfies $0 \le f_i(t) \le 1$. Let $t_n \searrow A_o$. There exists a subsequence $\{t_{n_k}\}$ such that $\underline{\underline{f}}(t_{n_k})$ converges as $k \to \infty$. Put $\underline{\underline{y}} = \lim_{k\to\infty} \underline{\underline{f}}(t_{n_k})$. We have

$$\tilde{F}_i(\underline{\underline{y}};A_o) = 0 \qquad (1 \le i \le N),$$

hence, $\tilde{J}(\underline{\underline{y}};A_o) \neq 0$ by 2°. There exists a unique function $\tilde{\underline{\underline{f}}}(t)$ in some neighbourhood $(A_o-\varepsilon, A_o+\varepsilon)$ of $A_o$ such that

$$\tilde{\underline{\underline{f}}}(A_o) = \underline{\underline{y}},$$

$$\tilde{F}_i(\tilde{\underline{\underline{f}}}(t);t) = 0 \qquad (1 \le i \le N, A_o-\varepsilon < t < A_o+\varepsilon).$$

By uniqueness of implicit functions, we have $\underline{\underline{f}}(t) = \tilde{\underline{\underline{f}}}(t)$ for

$t \in (A_o, A_o+\varepsilon)$. Therefore, $A_o-\varepsilon$ is not less than the infimum of A such that there exists a continuous function $\underline{\underline{f}}(t)$ on $[A,1]$ with

$\underline{\underline{f}}(1) = \underline{\underline{x}}$ and $\tilde{F}_i(\underline{\underline{f}}(t):t) = 0$ $(1 \le i \le N, A \le t \le 1)$, which we have put $A_o$. This is a contradiction. Hence $A_o < 0$.

4°. Let $\underline{\underline{a}}$ be sufficiently small. There is a one-to-one correspondence between non-trivial solutions $\underline{\underline{x}}$ of (***) and $\hat{\underline{\underline{x}}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N) \ne 0$ with $\hat{x}_i = 0$ or 1.

In fact, let $\underline{\underline{x}}$ be a non-trivial solution of (***). There is a continuous function $\underline{\underline{f}}(t)$ on [0,1] such that

$$\underline{\underline{f}}(1) = \underline{\underline{x}},$$

$$f_i(t) = |f_i(t)|^p + \sum_{j \ne i} t\, a_{ij} |f_j(t)|^p \qquad (1 \le i \le N, 0 \le t \le 1).$$

Since $f_i(0) = |f_i(0)|^p$, we have $f_i(0) = 0$ or 1. If $\underline{\underline{f}}(0) = \underline{\underline{0}}$, then $\underline{\underline{f}}(t) = 0$ for all $0 \le t \le 1$ by uniqueness of implict functions.

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