SYSTEMS OF A HYPERBOLIC-PARABOLIC COMPOSITE TYPE, WITH APPLICATIONS TO THE EQUATIONS OF MAGNETOHYDRODYNAMICS

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ABSTRACT

The global (in time) existence and asymptotic stability of smooth solutions to the initial value problem are proved for a general class of quasilinear symmetric hyperbolic-parabolic composite systems, under the smallness assumptions on the initial data and the *dissipation condition* on the linearized systems. In the special case of hyperbolic-parabolic systems of conservation laws with a convex entropy, it is also proved that for time $t \to \infty$, the solutions of the nonlinear systems are asymptotic to those of the linear ones if the space-dimension $n \geq 2$, and to those of the semi-linear ones if $n = 1$. These results are applicable to the equations of compressible viscous fluids, the equations of magnetohydrodynamics (or electro-magneto-fluid dynamics) for electrically conducting compressible viscous fluids, the equations of heat conduction with finite speed of propagation, and so on.

Furthermore hyperbolic systems of conservation laws with small viscosity are investigated on the relation to the limit systems without viscosity. It is proved that as viscosity tends to zero, the smooth solutions of the systems with viscosity converge on a finite time interval to the smooth solutions of the limit systems.

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CHAPTER I

INTRODUCTION AND BACKGROUND

1.1 HISTORICAL BACKGROUND

Many physical phenomena arising in mathematical physics are described by the quasilinear symmetric hyperbolic-parabolic systems of composite type which consist of first-order hyperbolic equations and second-order parabolic ones, when the effects of dissipative mechanisms (such as viscosity, heat conduction etc.) are taken into account. If the dissipative effects are neglected, then these systems degenerate to the first-order quasilinear symmetric hyperbolic systems.

In recent years, these systems describing physical laws have been studied intensively from a mathematical point of view. Vol'pert and Hudjaev [85] considered the initial value problem for a general class of symmetric hyperbolic-parabolic composite systems and established in a unified way the local (in time) existence and uniqueness of solutions in the L^2-Sobolev spaces. Their results remain valid for the two special cases, quasilinear symmetric hyperbolic systems and quasilinear symmetric parabolic ones. Similar existence and uniqueness results were also obtained by Fisher and Marsden [16] and Kato $[37]_4$ (see also [30]) in the case of quasilinear symmetric hyperbolic systems.

However, there has been nothing about a unified treatment of the global (in time) existence problem for these general systems. Concerning the global existence problem, we should note the following: smooth solutions of the first-order quasilinear hyperbolic systems (without dissipation) in general develop singularities in the first derivatives in finite time no matter how smooth the initial data are, which was proved by Lax $[50]_2$, John $[35]_1$ and others $[52]_7$, [53], [9], [46] (see also $[35]_{2,3}$, $[37]_7$). This fact suggests that at least for these hyperbolic systems (without dissipation) global solutions must be sought in a class of non-smooth (discontinuous) functions. This approach was pursued for the first-order nonlinear strictly hyperbolic systems of conservation laws in one space-dimension; the global existence of weak solutions in the space of bounded variation was shown by Glimm [22] and Kuznetsov and Tupchiev [48] for small initial data; the structure of weak solutions was studied by DiPerna $[13]_3$; asymptotic behaviors of weak solutions as time $t \to \infty$ were investigated by Glimm and Lax [23], DiPerna $[13]_{2,4}$ and by Liu $[52]_{3-6}$. These results are applicable to the system of gas dynamics in one space-dimension. But it is not straightforward to generalize these results to higher dimensions.

Another approach for the global existence problem is to still seek smooth solutions. Some authors have succeed to prove the global existence, uniqueness and asymptotic stability of smooth solutions for the physical systems in which the effects of dissipative mechanisms are taken into account. Among these systems, we mention:

(a) the equations of compressible viscous fluids,

(b) the equations of magnetohydrodynamics (or electro-magneto-fluid dynamics) for electrically conducting compressible viscous fluids,

(c) the equations of nonlinear viscoelasticity (or thermoelasticity),

(d) the equations of heat conduction with finite speed of propagation,

(e) the equations for discrete velocity models of the Boltzmann equation.

The systems (a), (b) and (c) (resp. (d) and (e)) are typical examples of quasilinear symmetric hyperbolic-parabolic composite systems (resp. quasilinear symmetric hyperbolic systems with dissipation). Global existence theorems for these systems were proved by a combination of the local existence results and the a priori estimates of solutions. In many cases, a priori estimates are derived by the L^2-energy method which makes use of the *energy integral* associated with the physical structure of the systems; see Kanel' $[36]_1$ and Okada and Kawashima [66] (and also $[36]_2$, [43], $[42]_2$) for the system (a), Kawashima and Okada [41] (and also $[38]_3$) for (b), Greenberg, MacCamy and Mizel [27] (and also [25], [12], [26], [2], [3]) for (c), and Kawashima $[38]_4$ (cf. $[64]_1$) for (e). It is also effective, especially in the case of higher dimensions, to make use of the decay estimates for linearized equations (with constant coefficients); see Matsumura and Nishida $[55]_2$ for (a), Browne [4] and Potier-Ferry [67] for (c), Matsumura $[54]_1$ (and also [71]) for (d), and Inoue and Nishida [33] (and also $[38]_4$) for (e); for another class of equations, the Boltzmann equation and the nonlinear wave equations, we refer to Nishida and Imai [62] (and also [79], $[80]_{1,2}$) and Klainerman [45] (and also [47], [70]). The existence of energy integral and the decay estimates (for linearized equations) are, however, not known for a general class of quasilinear symmetric hyperbolic-parabolic composite systems.

Usually we have two different systems for a physical phenomenon, one of which corresponds to the dissipative case and the other to the non-dissipative case. It is then expected that in the limit, as the coefficients of dissipations tend to zero, the solutions of the dissipative system converge to the solutions of the non-dissipative system. This convergence problem has been solved in the case of a single equation, but it is still open for general systems. The convergence of progressive-wave solutions to shock-wave solutions in one space-dimension was shown by Foy [17] and Conley and Smoller $[10]_1$ (and also [73], [74]) for general systems; we also refer to [86], [20], [21] (resp. $[10]_{2,3}$, [29]) for the equations of compressible fluids (resp. magnetohydrodynamics). The convergence (on a finite time interval) of smooth solutions in higher dimensions was proved by Nishida and Kawashima [63] for the equations of compressible fluids; this is a generalization of the results for incompressible fluids, see [76], $[37]_2$ (cf. [24], [57]). Similar convergence results were also obtained for the Boltzmann equation (resp. its discrete velocity models), see $[61]_3$, [6], $[80]_3$ (resp. [33], [7]). Finally we should note the recent work of DiPerna $[13]_6$. He has established a general convergence result for a model system of one-dimensional nonlinear elasticity: smooth solutions of the dissipative system converge (for all time $t \geq 0$) to the weak solutions of the non-dissipative system.

1.2 AIM OF THE PRESENT WORK

The main purpose of the present work is to show, in a unified way, the global (in time) existence and asymptotic stability of smooth solutions to the initial value problem for a class of quasilinear symmetric hyperbolic-parabolic composite systems. We are only concerned with small amplitude solutions because the systems treated here are general enough and so the global existence of large amplitude solutions can not be expected in general in a class of smooth functions. Therefore in this situation we are sufficient to show that the constant equilibrium solutions for these systems are asymptotically stable (in time) for small perturbations at the initial time.

In order to establish these results, we usually need decay estimates for the equations linearized around the equilibrium state. For our general class of systems, we shall formulate a condition which guarantees the decay structure for linearized systems. This *dissipation condition* enables us to conclude the global existence and asymptotic stability of solutions when the space-dimension $n \geq 3$.

In the case $n \leq 2$, additional considerations are needed. We shall restrict our attention to a class of hyperbolic-parabolic composite systems of conservation laws with a convex entropy. These systems enjoy the *energy integral* associated with the entropy. This energy integral together with the dissipation condition gives the global existence and asymptotic stability results for all $n \geq 1$.

We then show that our results are applicable to the physical systems (a), (b) and (d) mentioned in section 1.1 (though, in some cases, slight modifications are needed); the applications to the systems (c) and (e) will be omitted in this dissertation, we refer to Kawashima $[38]_4$ for (e). In applications a key point is to verify the dissipation condition for each system, and this can be done rather algebraically.

The second aim of this work is to justify the vanishing viscosity method locally in time for a class of hyperbolic systems of conservation laws. We shall assume that our systems with small viscosity are of hyperbolic-parabolic composite type and possess a convex entropy. In this situation a convergence theorem (as the viscosity tends to zero) is established for local smooth solutions and this theorem can be applied to the physical systems such as (a) and (b).

1.3 SUMMARY

The contents of this dissertation are as follows. In chapter II we shall consider the initial value problem for a class of quasilinear symmetric hyperbolic-parabolic composite systems and prove a local (in time) existence theorem in the Sobolev space $H^s(\mathbb{R}^n)$. The global (in time) existence problem for these general systems is studied in chapter III. Making use of the dissipation condition, we shall derive a priori estimates (with decay rate $t^{-n/4}$ as time $t \to \infty$) of small solutions when the initial data are in $H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $n \geq 3$. Combining these a priori estimates with the local existence result, we can show the global existence and asymptotic stability of smooth solutions if $n \geq 3$ and the initial data are sufficiently small in $H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. In the last section of chapter III, we shall apply these results to the equations of heat conduction with finite speed of propagation.

In chapter IV we treat a rather restricted class of systems, hyperbolic-parabolic composite systems of conservation laws with a convex entropy. By the technical energy method based on the energy integral, we shall establish similar global existence and asymptotic stability results for all $n \geq 1$. In this case it is also proved that for $t \to \infty$, the solutions of the nonlinear systems are asymptotic to those of the linear ones if $n \geq 2$, and to those of the semi-linear ones if $n = 1$.

In chapter V we shall investigate the convergence problem for a class of hyperbolic systems of conservation laws with vanishing viscosity. It is proved that in the limit, as viscosity tends to zero, the smooth solutions of the systems with small viscosity converge on a finite time interval to the smooth solutions of the limit systems without viscosity.

Chapter VI contains the applications to the equations of magnetohydrodynamics (or electro-magneto-fluid dynamics) for electrically conducting compressible fluids. Global existence and stability results are established for the following cases:

(1) two special type systems of electro-magneto-fluid dynamics in $\mathbb{R}^2$,

(2) the equations of magnetohydrodynamics in $\mathbb{R}^3$ or $\mathbb{R}^2$,

(3) the equations of magnetohydrodynamics in $\mathbb{R}^1$.

The dissipation condition is verified for (1) and (2) if all the effects of dissipative mechanisms are assumed, while for (3) this can be done by assuming only one or two of them. As a special case, we also discuss briefly the system of fluid mechanics in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and $\mathbb{R}^1$.

CHAPTER II

QUASILINEAR SYMMETRIC HYPERBOLIC-PARABOLIC SYSTEMS, I
(LOCAL EXISTENCE)

2.1 INTRODUCTION

In this chapter we shall consider the initial value problem for a system of quasilinear partial differential equations of the form

$$\text{(2.1)}\qquad \begin{cases} A_1^0(u,v)u_t + \sum_{j=1}^{n} A_{11}^j(u,v)u_{x_j} = f_1(u,v,D_x v) , \\ A_2^0(u,v)v_t - \sum_{j,k=1}^{n} B_2^{jk}(u,v)v_{x_j x_k} = f_2(u,v,D_x u,D_x v) , \end{cases}$$

where $t \geq 0$ and $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$ $(n \geq 1)$; $u = u(t,x)$ and $v = v(t,x)$ are vectors with m' and m'' components, respectively, and the pair $(u,v)(t,x)$ takes its values in an open convex set $\mathcal{O}$ in $\mathbb{R}^m$ $(m = m'+m'' \geq 1)$; A_1^0 and A_{11}^j $(j = 1,\cdots,n)$ (resp. A_2^0 and B_2^{jk} $(j,k = 1,\cdots,n)$) are square matrices of order m' (resp. m''); f_1 (resp. f_2) is a $\mathbb{R}^{m'}$-valued (resp. $\mathbb{R}^{m''}$-valued) function; D_x denotes the derivatives $(\partial/\partial x)^\alpha$ with $|\alpha| = 1$. The initial data are prescribed at $t = 0$:

$$\text{(2.2)}\qquad (u,v)(0,x) = (u_0,v_0)(x) .$$

We assume that the system (2.1) is *symmetric hyperbolic-parabolic* in the following sense.

Condition 2.1 The functions $A_1^0(u,v)$, $A_2^0(u,v)$, $A_{11}^j(u,v)$ $(j=1,\cdots,n)$ and $B_2^{jk}(u,v)$ $(j,k=1,\cdots,n)$ are sufficiently smooth in $(u,v) \in \mathcal{O}$ such that

(i) $A_1^0(u,v)$ and $A_2^0(u,v)$ are real symmetric and positive definite for $(u,v) \in \mathcal{O}$,

(ii) $A_{11}^j(u,v)$ are real symmetric for $(u,v) \in \mathcal{O}$,

(iii) $B_2^{jk}(u,v)$ are real symmetric and satisfy $B_2^{jk}(u,v) = B_2^{kj}(u,v)$ for $(u,v) \in \mathcal{O}$; $\sum_{jk} B_2^{jk}(u,v)\omega_j\omega_k$ is (real symmetric) positive definite for all $(u,v) \in \mathcal{O}$ and $\omega = (\omega_1,\cdots,\omega_n) \in S^{n-1}$.

Under these conditions $f_1(u,v,D_xv)$ and $f_2(u,v,D_xu,D_xv)$ in the right hand side can be regarded as lower order terms of the system. Let $\eta \in \mathbb{R}^{nm'}$ and $\zeta \in \mathbb{R}^{nm''}$ denote the vectors corresponding to D_xu and D_xv. We assume:

Condition 2.2 The functions $f_1(u,v,\zeta)$ and $f_2(u,v,\eta,\zeta)$ are sufficiently smooth in $(u,v,\zeta) \in \mathcal{O} \times \mathbb{R}^{nm''}$ and $(u,v,\eta,\zeta) \in \mathcal{O} \times \mathbb{R}^{nm}$, respectively, and satisfy $f_1(\bar{u},\bar{v},0) = f_2(\bar{u},\bar{v},0,0) = 0$ for some constant state $(\bar{u},\bar{v}) \in \mathcal{O}$.

Remark 2.1 In the special case $m'' = 0$, the system (2.1) is reduced to

$$(2.1)' \qquad A_1^0(u)u_t + \sum_j A_{11}^j(u)u_{x_j} = f_1(u) ,$$

which is symmetric hyperbolic; while in the case $m' = 0$,

$$(2.1)'' \qquad A_2^0(v)v_t - \sum_{jk} B_2^{jk}(v)v_{x_j x_k} = f_2(v, D_x v) ,$$

which is parabolic in the sense indicated above.

In a similar situation Vol'pert and Hudjaev [85] have proved the existence of local solutions (in the $L^2(\mathbb{R}^n)$-Sobolev spaces) to the initial value problem (2.1),(2.2), by use of the Schauder's fixed point theorem. Their results are applicable to the special case $m' = 0$ or $m'' = 0$. On the other hand Fischer and Marsden [16] and Kato $[37]_4$ have established similar existence results for symmetric hyperbolic systems (2.1)' (i.e., $m'' = 0$), by using the general theory of abstract evolution equations (see Kato $[37]_{1,3}$). The purpose of this chapter is to reconstruct a local solution of (2.1),(2.2).

The contents of this chapter are as follows. In section 2.2 we introduce some function spaces. The basic inequalities in the Sobolev spaces are also given. In section 2.3 we consider the linearized equations (with variable coefficients) for (2.1). The existence result is obtained as an application of Theorem II of Kato $[37]_3$. We also derive the energy inequalities in the $L^2(\mathbb{R}^n)$-Sobolev spaces. In section 2.4 we construct a local solution of (2.1),(2.2) as a limit of successive approximation sequence.

2.2 PRELIMINARIES

We shall first introduce some function spaces. $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ denotes the space of measurable functions whose p-th powers are integrable on $\mathbb{R}^n$, with the norm

$$\|f\|_{L^p} = \left(\int |f(x)|^p \, dx \right)^{1/p} .$$

We sometimes write $\|f\|$ instead of $\|f\|_{L^2}$. $L^\infty(\mathbb{R}^n)$ denotes the space of bounded measurable functions on $\mathbb{R}^n$, with the norm

$$\|f\|_{L^\infty} = \operatorname*{ess.sup}_x |f(x)| .$$

$H^s(\mathbb{R}^n)$ $(s \ge 0 : \text{integer})$ denotes the space of $L^2(\mathbb{R}^n)$-functions f whose derivatives (in the sense of distribution) $D_x^k f$ $(k \le s)$ are also $L^2(\mathbb{R}^n)$-functions, with the norm

$$\|f\|_s = \left(\sum_{k=0}^{s} \|D_x^k f\|^2 \right)^{1/2} .$$

Here D_x^k denotes the derivatives $(\partial/\partial x)^\alpha$ with $|\alpha| = k$; we write D_x instead of D_x^1. Note that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $\|f\|_0 = \|f\|$. $V^s(\mathbb{R}^n)$ $(s \ge 0 : \text{integer})$ denotes the space of $L^\infty(\mathbb{R}^n)$-functions f whose derivatives $D_x f$ are $H^{s-1}(\mathbb{R}^n)$-functions, with the norm

$$\|f\|_{V^s} = \|f\|_{L^\infty} + \|D_x f\|_{s-1} .$$

In the case of $s = 0$ we define $V^0(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ and $\|f\|_{V^0} = \|f\|_{L^\infty}$.

$\mathcal{B}^\ell(\mathbb{R}^n)$ ($\ell \geq 0$: integer) denotes the space of bounded continuous functions f whose derivatives $D_x^k f$ ($k \leq \ell$) are also bounded continuous, with the norm

$$|f|_\ell = \sum_{k=0}^{\ell} \sup_x |D_x^k f(x)| .$$

$\mathcal{B}^{\ell+\sigma}(\mathbb{R}^n)$ ($\ell \geq 0$: integer, $0 < \sigma < 1$) denotes the space of $\mathcal{B}^\ell(\mathbb{R}^n)$-functions such that their ℓ-th order derivatives are σ-Hölder continuous, with the norm

$$|f|_{\ell+\sigma} = |f|_\ell + \sup_{x \neq y} \frac{|D_x^\ell f(x) - D_x^\ell f(y)|}{|x - y|^\sigma}$$

Let X be a Banach space and let $t' < t$. $L^p(t',t\,;X)$ $(1 \leq p \leq \infty)$ denotes the space of L^p-functions $f(t)$ on $[t',t]$ with the values in X. $\mathcal{C}^\ell(t',t\,;X)$ ($\ell \geq 0$: integer) denotes the space of ℓ-th times continuously differetiable functions $f(t)$ on $[t',t]$ with the values in X.

The following interpolation inequalities for $L^p(\mathbb{R}^n)$-norm of the derivatives $D_x^j u$ are found in [60].

<u>Lemma 2.1</u> ([60]) *Let* $1 \leq q, r \leq \infty$, *and let* $k > 0$ *be an integer. Assume that* $u \in L^q(\mathbb{R}^n)$ *and* $D_x^k u \in L^r(\mathbb{R}^n)$. *Then for the derivatives* $D_x^j u$, $0 \leq j < k$, *the following inequalities hold:*

$$(2.3) \qquad \|D_x^j u\|_{L^p} \leq C\, \|D_x^k u\|_{L^r}^a\, \|u\|_{L^q}^{1-a} ,$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{k}{n}\right) + (1 - a)\frac{1}{q}$$

for all a *satisfying* $j/k \le a \le 1$, *and* C *is a positive constant; there are the following exceptional cases:*

(i) If $j = 0$, $rk < n$ *and* $q = \infty$, *then we made the additional assumption that either* $u(x) \to 0$ *as* $|x| \to \infty$ *or* $u \in L^{q'}(\mathbb{R}^n)$ *for some* $0 < q' < \infty$.

(ii) If $1 < r < \infty$, *and* $k - j - n/r$ *is a non-negative integer, then* (2.3) *holds only for* a *satisfying* $j/k \le a < 1$.

As a consequence of (2.3) we have:

<u>Corollary 2.2</u> *Let* $s \ge 0$ *be an integer and assume that* $u \in H^s(\mathbb{R}^n)$. *Then the following two statements are true.*

(i) For any p *with* $\max\{0, 1/2 - s/n\} \le 1/p \le 1/2$, *we have* $u \in L^p(\mathbb{R}^n)$ *and*

$$(2.4) \qquad \|u\|_{L^p} \le C \|D_x^s u\|^a \|u\|^{1-a} \le C \|u\|_s ,$$

where $a = (n/s)(1/2 - 1/p)$, *with the exceptional case: if* $s = n/2$, *then* (2.4) *holds only for* $0 < 1/p \le 1/2$ *(i.e.,* $2 \le p < \infty$*).*

(ii) If $s > n/2$, *then for* $\ell = s - s_0 \ge 0$ *(*$s_0 = [n/2] + 1$*) and for any* σ *with* $0 < \sigma < s_0 - n/2$, *we have* $u \in B^{\ell+\sigma}(\mathbb{R}^n)$ *and* $|u|_{\ell+\sigma} \le C \|u\|_s$.

Remark 2.2 It follows from (ii) that for $s > n/2$ there is an imbedding of $H^s(\mathbb{R}^n)$ into $V^s(\mathbb{R}^n)$, and $\|u\|_{V^s} \le C\,\|u\|_s$.

By virtue of Corollary 2.2, Lemma 2.1 and Leibniz's formula we get the following estimates for composite functions.

Lemma 2.3 *Let* $s \ge 0$ *and* $\ell \ge 0$ *be integers satisfying* $s + \ell \ge s_0$ $(s_0 = [n/2]+1)$. *Assume that* $u \in H^s(\mathbb{R}^n)$ *(resp.* $u \in V^s(\mathbb{R}^n)$ *) and* $v \in H^\ell(\mathbb{R}^n)$. *Then for* $k = \min\{s, \ell, s+\ell-s_0\}$ *we have* $uv \in H^k(\mathbb{R}^n)$ *and*

$$(2.5) \qquad \|uv\|_k \le C\,\|u\|_s\,\|v\|_\ell \qquad (\textit{resp.}\ \ \|uv\|_k \le C\,\|u\|_{V^s}\,\|v\|_\ell\,).$$

Note that if $s \ge s_0$ and $0 \le \ell \le s$, the estimate (2.5) holds for $k = \ell$.

Lemma 2.4 ([85]) *Let* $s \ge 1$ *be an integer and assume that* $v = (v_1,\cdots,v_m) \in V^s(\mathbb{R}^n)$. *Let* $F = F(v)$ *be a* C^∞*-function of* $v \in \mathbb{R}^m$. *Then for* $1 \le j \le s$, *we have* $D_x F(v) \in H^{j-1}(\mathbb{R}^n)$ *and*

$$(2.6) \qquad \|D_x F(v)\|_{j-1} \le CM(1+\|v\|_{L^\infty})^{j-1}\,\|D_x v\|_{j-1} \,,$$

where C *is a positive constant and* $M = \sum_{k=1}^{j} \sup_v |D_v^k F(v)|$ *(* $\sup_v$ *is taken over all* v *with* $|v| \le \|v\|_{L^\infty}$ *).*

Finally we shall give the estimates for commutators (for the proof, see [58] and $[54]_3$).

<u>Lemma 2.5</u> *Let* $s \geq s_0 + 1$ *(*$s_0 = [n/2]+1$*) be an integer and assume that* $u \in V^s(\mathbb{R}^n)$.

(i) Let $1 \leq \ell \leq s$ *be an integer and let* $v \in H^{\ell-1}(\mathbb{R}^n)$. *Then for* $0 \leq k \leq \ell$ *, we have* $[D_x^k, u]v \equiv D_x^k(uv) - uD_x^k v \in L^2(\mathbb{R}^n)$ *and*

$$\sum_{k=0}^{\ell} \| [D_x^k, u]v \| \leq C \|D_x u\|_{s-1} \|v\|_{\ell-1} . \tag{2.7}$$

(ii) Let $0 \leq \ell \leq s$ *be an integer and let* $v \in H^{\ell}(\mathbb{R}^n)$. *Let* $\phi_\delta *$ *denote the Friedrichs mollifier. Then we have* $[\phi_\delta *, u]D_x v \equiv \phi_\delta * (uD_x v) - u(\phi_\delta * D_x v) \in H^\ell(\mathbb{R}^n)$ *,*

$$\| [\phi_\delta *, u]D_x v \|_\ell \leq C \|u\|_{V^s} \|v\|_\ell , \tag{2.8}$$

and $\| [\phi_\delta *, u]D_x v \|_\ell \to 0$ *as* $\delta \to 0$.

<u>2.3</u> <u>LINEARIZED EQUATIONS</u>

In this section we shall prove the existence of solutions for the linearized equations of the form

$$A_1^0(u,v)\hat{u}_t + \sum_j A_{11}^j(u,v)\hat{u}_{x_j} = f_1 , \tag{2.9$_1$}$$

$$A_2^0(u,v)\hat{v}_t - \sum_{jk} B_2^{jk}(u,v)\hat{v}_{x_j x_k} = f_2 . \tag{2.9$_2$}$$

Let $Q_T = [0,T] \times \mathbb{R}^n$ (T is a positive constant) and $s_0 = [n/2] + 1$, and let $s \geq s_0 + 1$ and $0 \leq \ell \leq s$ be integers. For $(u,v)(t,x)$ and $(f_1, f_2)(t,x)$, given functions on Q_T, we assume the following conditions.

$$(2.10)_1 \qquad u - \bar{u} \in C^0(0,T;H^s(\mathbb{R}^n)), \quad \partial_t u \in C^0(0,T;H^{s-1}(\mathbb{R}^n)),$$

$$(2.10)_2 \quad \begin{cases} v - \bar{v} \in C^0(0,T;H^s(\mathbb{R}^n)), \\ \partial_t v \in C^0(0,T;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T;H^{s-1}(\mathbb{R}^n)), \end{cases}$$

$$(2.11) \qquad (u,v)(t,x) \in \mathcal{O}_1 \quad \text{for any} \quad (t,x) \in Q_T,$$

where $(\bar{u},\bar{v}) \in \mathcal{O}$ is the constant state in Condition 2.2 and $\mathcal{O}_1$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_1 \subset \mathcal{O}$;

$$(2.12)_1 \qquad f_1 \in C^0(0,T;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T;H^{\ell}(\mathbb{R}^n)),$$

$$(2.12)_2 \qquad f_2 \in C^0(0,T;H^{\ell-1}(\mathbb{R}^n)).$$

First we shall obtain the energy estimates for $(2.9)_{1,2}$. For this purpose we only require the conditions which are obtained by replacing $C^0(\cdots)$ in $(2.10)_{1,2}$ and $(2.12)_{1,2}$ by $L^\infty(\cdots)$. That is,

$$(2.13)_1 \qquad u - \bar{u} \in L^\infty(0,T;H^s(\mathbb{R}^n)), \quad \partial_t u \in L^\infty(0,T;H^{s-1}(\mathbb{R}^n)),$$

$$(2.13)_2 \quad \begin{cases} v - \bar{v} \in L^\infty(0,T;H^s(\mathbb{R}^n)), \\ \partial_t v \in L^\infty(0,T;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T;H^{s-1}(\mathbb{R}^n)), \end{cases}$$

$$(2.14)_1 \qquad f_1 \in L^\infty(0,T;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T;H^{\ell}(\mathbb{R}^n)),$$

$$(2.14)_2 \qquad f_2 \in L^\infty(0,T\,;H^{\ell-1}(\mathbb{R}^n))\,.$$

Then we have:

Lemma 2.6 (energy estimates for linearized equations) *Let us assume Condition* 2.1. *Let* $n \geq 1$ *and* $s \geq s_0 + 1$ *(*$s_0 = [n/2]+1$*) be integers and let* $(u,v)(t,x)$ *satisfy the conditions* $(2.13)_{1,2}$ *and* (2.11). *Put*

$$M = \sup_{0\le t\le T} \|(u-\bar{u},\ v-\bar{v})(t)\|_s\,, \qquad M_1 = \left(\int_0^T \|\partial_t(u,v)(t)\|_{s-1}^2\,dt\right)^{1/2}.$$

(i) Let $0 \leq \ell \leq s$ *be an integer and let* $f_1(t,x)$ *satisfy* $(2.14)_1$. *Assume that* $\hat{u}(t,x)$ *is a solution of* $(2.9)_1$ *satisfying*

$$(2.15)_1 \qquad \hat{u} \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n))\,, \qquad \partial_t\hat{u} \in L^\infty(0,T\,;H^{\ell-1}(\mathbb{R}^n))\,.$$

Then we have $\hat{u} \in C^0(0,T\,;H^\ell(\mathbb{R}^n))$. *Furthermore there exist constants* $C_1 = C_1(\mathcal{O}_1) > 1$ *and* $C_2 = C_2(\mathcal{O}_1, M) > 0$ *such that the following energy inequality holds for* $t \in [0,T]$.

$$(2.16)_1 \qquad \|\hat{u}(t)\|_\ell^2 \leq C_1^2 e^{C_2(Mt+M_1t^{1/2})}\{\|\hat{u}(0)\|_\ell^2 + C_2t\int_0^t \|f_1(\tau)\|_\ell^2\,d\tau\}\,.$$

(ii) Let $1 \leq \ell \leq s$ *be an integer and let* $f_2(t,x)$ *satisfy* $(2.14)_2$. *Assume that* $\hat{v}(t,x)$ *is a solution of* $(2.9)_2$ *satisfying*

$$(2.15)_2 \qquad \hat{v} \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n))\,, \qquad \partial_t\hat{v} \in L^\infty(0,T\,;H^{\ell-2}(\mathbb{R}^n))\,.$$

Then $\hat{v} \in C^0(0,T\,;H^\ell(\mathbb{R}^n)) \cap L^2(0,T\,;H^{\ell+1}(\mathbb{R}^n))$, *and the following energy inequality holds for* $t \in [0,T]$.

$$(2.16)_2 \qquad \|\hat{v}(t)\|_\ell^2 + \int_0^t \|\hat{v}(\tau)\|_{\ell+1}^2 d\tau$$
$$\le C_1^2 e^{C_2(t+M_1 t^{1/2})} \{\|\hat{v}(0)\|_\ell^2 + C_2\int_0^t \|f_2(\tau)\|_{\ell-1}^2 d\tau\},$$

where C_1 *and* C_2 *are constants as in (i).*

Proof. This lemma can be proved by the standard energy method (see [63] or $[54]_3$, for example). We devide the proof of (i) into 4 steps.

<u>step 1</u> We first show the estimate $(2.16)_1$ under the assumptions that u satisfies $(2.13)_1$, v and f_1 satisfy

$$(2.13)_2' \qquad v - \bar{v} \in L^\infty(0,T\,;H^s(\mathbb{R}^n)), \qquad \partial_t v \in L^\infty(0,T\,;H^{s-1}(\mathbb{R}^n)),$$

$$(2.14)_1' \qquad f_1 \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n)),$$

respectively, and that $\hat{u}$ is a solution of $(2.9)_1$ satisfying

$$(2.15)_1' \qquad \hat{u} \in L^\infty(0,T\,;H^{\ell+1}(\mathbb{R}^n)), \qquad \partial_t\hat{u} \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n)).$$

Applying D_x^k $(k \le \ell)$ to the system $(2.9)_1$, we have

$$(2.17)_1 \qquad A_1^0(u,v)D_x^k\hat{u}_t + \sum_j A_{11}^j(u,v)D_x^k\hat{u}_{x_j} = F_1^k,$$

where

$$F_1^k = A_1^0(u,v)D_x^k\{A_1^0(u,v)^{-1}f_1\} - A_1^0(u,v)\sum_j [D_x^k, A_1^0(u,v)^{-1}A_{11}^j(u,v)]\hat{u}_{x_j} .$$

Take the inner product (in $\mathbb{R}^{m'}$) of $(2.17)_1$ with $D_x^k\hat{u}$. Integrating it over $\mathbb{R}^n$ and adding for $k = 0,1,\cdots,\ell$, we obtain

$$(2.18)_1 \qquad \sum_{k=0}^{\ell}\int < A_1^0(u,v)D_x^k\hat{u}_t + \sum_j A_{11}^j(u,v)D_x^k\hat{u}_{x_j},\ D_x^k\hat{u} > dx$$

$$= \sum_{k=0}^{\ell}\int < F_1^k,\ D_x^k\hat{u} > dx ,$$

where $<\ ,\ >$ denotes the standard inner product in $\mathbb{R}^{m'}$. We introduce here the energy norm:

$$E_1[\hat{u}] = \left(\sum_{k=0}^{\ell}\int < A_1^0(u,v)D_x^k\hat{u},\ D_x^k\hat{u} > dx\right)^{1/2} .$$

Since $A_1^0(u,v)$ is real symmetric and positive definite, $E_1[\hat{u}]$ is equivalent to $\|\hat{u}\|_\ell$, that is, there are constants $c_0 = c_0(\mathcal{O}_1) > 0$ and $C_0 = C_0(\mathcal{O}_1) > 0$ such that $c_0\|\hat{u}\|_\ell \le E_1[\hat{u}] \le C_0\|\hat{u}\|_\ell$. By integration by parts we find that the left member of $(2.18)_1$ is bounded from below by

$$\frac{\partial}{\partial t}\left(\frac{1}{2}E_1[\hat{u}]^2\right) - C_0(M + \|\partial_t(u,v)\|_{s-1})E_1[\hat{u}]^2$$

for some constant $C_0 = C_0(\mathcal{O}_1)$, where (2.4) (with $p=\infty$ and $s \to s-1 \ge s_0$) was used. On the other hand (2.7) and (2.6) (with $j=s$) yield the estimate

$$\sum_{k=0}^{\ell} \| [D_x^k,\ A_1^0(u,v)^{-1}A_{11}^j(u,v)]\hat{u}_{x_j} \| \le CM \|\hat{u}\|_\ell ,$$

while (2.5) (with $k=\ell$) and (2.6) (with $j=s$) give the estimate $\|A_1^0(u,v)^{-1}f_1\|_\ell \le C\|f_1\|_\ell$, where $C = C(O_1,M)$. Therefore the right hand side of $(2.18)_1$ is majorized by

$$\sum_{k=0}^{\ell} \|F_1^k\| \|D_x^k\hat{u}\| \le CM E_1[\hat{u}]^2 + C\|f_1\|_\ell E_1[\hat{u}] .$$

Thus we arrive at

$$\frac{\partial}{\partial t} E_1[\hat{u}] \le C(M + \|\partial_t(u,v)\|_{s-1})E_1[\hat{u}] + C\|f_1\|_\ell .$$

Applying the Gronwall's inequality, we obtain

$$E_1[\hat{u}](t) \le e^{C(Mt+M_1t^{1/2})}\{E_1[\hat{u}](0) + Ct^{1/2}(\int_0^t \|f_1(\tau)\|_\ell^2 d\tau)^{1/2}\}$$

for some constant $C = C(O_1,M)$. The estimate $(2.16)_1$ is an immediate consequence of the last inequality.

<u>step 2</u> Next we show the estimate $(2.16)_1$ for (u,v), f_1 and $\hat{u}$ satisfying $(2.13)_{1,2}$, $(2.14)_1'$ and $(2.15)_1'$, respectively. Let $\phi_\delta*$ denotes the Friedrichs mollifier and put $v_\delta = \phi_\delta * v$. Then v_δ satisfies $(2.13)_2'$. The system $(2.9)_1$ is rewritten in the form

$$(2.19)_1 \qquad A_1^0(u,v_\delta)\hat{u}_t + \sum_j A_{11}^j(u,v_\delta)\hat{u}_{x_j} = f_1 + R_1^\delta ,$$

where

$$R_1^\delta = A_1^0(u,v_\delta)\{A_1^0(u,v)^{-1} - A_1^0(u,v_\delta)^{-1}\}f_1 -$$

$$- A_1^0(u,v_\delta)\sum_j \{A_1^0(u,v)^{-1}A_{11}^j(u,v) - A_1^0(u,v_\delta)^{-1}A_{11}^j(u,v_\delta)\}\hat{u}_{x_j} .$$

The results in step 1 are applicable to $(2.19)_1$. So we get the estimate $(2.16)_1$ with f_1 replaced by $f_1 + R_1^\delta$. Since $\|R_1^\delta\|_\ell \le C\|v - v_\delta\|_s \cdot (\|f_1\|_\ell + \|\hat{u}\|_{\ell+1})$ by (2.5) and (2.6), we have $\sup_\tau \|R_1^\delta(\tau)\|_\ell \to 0$ as $\delta \to 0$. Therefore, letting $\delta \to 0$, we conclude the estimate $(2.16)_1$ for (u,v), f_1 and $\hat{u}$ mentioned above.

<u>step 3</u> Finally we show that $(2.16)_1$ also holds for (u,v), f_1 and $\hat{u}$ satisfying $(2.13)_{1,2}$, $(2.14)_1$ and $(2.15)_1$. Applying $\phi_\delta *$ to the system $(2.9)_1$, we obtain

$$(2.20)_1 \qquad A_1^0(u,v)\hat{u}_{\delta,t} + \sum_j A_{11}^j(u,v)\hat{u}_{\delta,x_j} = f_{1,\delta} + Q_1^\delta ,$$

where $\hat{u}_\delta = \phi_\delta * \hat{u}$, $f_{1,\delta} = \phi_\delta * f_1$ and

$$Q_1^\delta = A_1^0(u,v)\,[\phi_\delta *, A_1^0(u,v)^{-1}]f_1 -$$

$$- A_1^0(u,v)\sum_j [\phi_\delta *, A_1^0(u,v)^{-1}A_{11}^j(u,v)]\hat{u}_{x_j} .$$

The results in step 2 are applicable to $(2.20)_1$ because $f_{1,\delta} + Q_1^\delta$ and $\hat{u}_\delta$ satisfy $(2.14)_1'$ (cf. (2.8)) and $(2.15)_1'$, respectively. So we get the estimate $(2.16)_1$ with $\hat{u}$ and f_1 replaced by $\hat{u}_\delta$ and $f_{1,\delta} + Q_1^\delta$. It is easy to see that as $\delta \to 0$, $\hat{u}_\delta \to \hat{u}$ strongly in $L^\infty(0,T;H^\ell(\mathbb{R}^n))$, $f_{1,\delta} \to f_1$ strongly in $L^2(0,T;H^\ell(\mathbb{R}^n))$, and $Q_1^\delta \to 0$ strongly in $L^2(0,T;H^\ell(\mathbb{R}^n))$; the last convergence is a consequence of Lemma 2.5 (ii). Therefore, letting $\delta \to 0$, we obtain $(2.16)_1$ for (u,v), f_1 and $\hat{u}$ mentioned in Lemma 2.6.

step 4 It remains to prove $\hat{u} \in C^0(0,T\,;H^\ell(\mathbb{R}^n))$. Note that $\hat{u}_\delta \in C^0(0,T\,;H^\ell(\mathbb{R}^n))$ for $\delta > 0$. From $(2.20)_1$ we have

$$A_1^0(u,v)(\hat{u}_\delta - \hat{u}_{\delta'})_t + \sum_j A_{11}^j(u,v)(\hat{u}_\delta - \hat{u}_{\delta'})_{x_j}$$

$$= (f_{1,\delta} + Q_1^\delta) - (f_{1,\delta'} + Q_1^{\delta'}) \equiv f_1^{\delta,\delta'} .$$

Since the estimate $(2.16)_1$ is applicable to this system, we get $(2.16)_1$ with $\hat{u}$ and f_1 replaced by $\hat{u}_\delta - \hat{u}_{\delta'}$ and $f_1^{\delta,\delta'}$, respectively. It is easily seen that as $\delta, \delta' \to 0$, $(\hat{u}_\delta - \hat{u}_{\delta'})(0) \to 0$ strongly in $H^\ell(\mathbb{R}^n)$ and $f_1^{\delta,\delta'} \to 0$ strongly in $L^2(0,T\,;H^\ell(\mathbb{R}^n))$. This implies that $\hat{u}_\delta$ $(\delta > 0)$ is a Cauchy sequence in $C^0(0,T\,;H^\ell(\mathbb{R}^n))$. Therefore the limit $\hat{u}$ belongs to $C^0(0,T\,;H^\ell(\mathbb{R}^n))$. Thus the proof of (i) is completed.

The proof of (ii) is almost similar to that of (i). Since the arguments on the mollifier are also applicable, it suffices to prove the estimate $(2.16)_2$ only in the case that u, v and f_2 satisfy $(2.13)_1'$, $(2.13)_2'$ and

$$(2.14)_2' \qquad f_2 \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n)) ,$$

respectively, and that $\hat{v}$ is a solution of $(2.9)_2$ satisfying

$$(2.15)_2' \qquad \hat{v} \in L^\infty(0,T\,;H^{\ell+2}(\mathbb{R}^n)) , \qquad \partial_t\hat{v} \in L^\infty(0,T\,;H^\ell(\mathbb{R}^n)) .$$

Applying D_x^k $(k \le \ell)$ to $(2.9)_2$, we have

$$(2.17)_2 \qquad A_2^0(u,v)D_x^k\hat{v}_t - \sum_{ij} B_2^{ij}(u,v)D_x^k\hat{v}_{x_ix_j} = F_2^k ,$$

where

$$F_2^k = A_2^0(u,v)D_x^k\{A_2^0(u,v)^{-1}f_2\} + A_2^0(u,v)\sum_{ij}[D_x^k, A_2^0(u,v)^{-1}B_2^{ij}(u,v)]\hat{v}_{x_ix_j} .$$

In the same way as in step 1 we obtain as a counterpart of $(2.18)_1$:

$$(2.18)_2 \qquad \sum_{k=0}^{\ell}\int < A_2^0(u,v)D_x^k\hat{v}_t - \sum_{ij} B_2^{ij}(u,v)D_x^k\hat{v}_{x_ix_j}, D_x^k\hat{v} > dx$$

$$= \sum_{k=0}^{\ell}\int < F_2^k, D_x^k\hat{v} > dx .$$

Define the energy norm

$$E_2[\hat{v}] = \Big(\sum_{k=0}^{\ell}\int < A_2^0(u,v)D_x^k\hat{v}, D_x^k\hat{v} > dx\Big)^{1/2} ,$$

which is equivalent to $\|\hat{v}\|_\ell$. By integration by parts and the Gårding's inequality for the strongly elliptic operator $\sum_{ij} B_2^{ij}(u,v)(\partial^2/\partial x_i\partial x_j)$, we find two positive constants $c_0 = c_0(\mathcal{O}_1)$ and $C = C(\mathcal{O}_1,M)$ such that the left hand side of $(2.18)_2$ is bounded from below by

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}E_2[\hat{v}]^2\Big) + c_0\|\hat{v}\|_{\ell+1}^2 - C(1+\|\partial_t(u,v)\|_{s-1})E_2[\hat{v}]^2 .$$

On the other hand the right member of $(2.18)_2$ is majorized by

$$C \|\hat{v}\|_{\ell+1} E_2[\hat{v}] + C \|f_2\|_{\ell-1} \|\hat{v}\|_{\ell+1} ,$$

if we estimate the terms containing $D_x^\ell f_2$ by integration by parts and the other terms by using (2.5), (2.6) and (2.7), where $C = C(\mathcal{O}_1, M)$ is a constant. Thus we arrive at

$$\frac{\partial}{\partial t} E_2[\hat{v}]^2 + c_0 \|\hat{v}\|_{\ell+1}^2 \le C(1 + \|\partial_t(u,v)\|_{s-1}) E_2[\hat{v}]^2 + C \|f_2\|_{\ell-1}^2 ,$$

from which follows the desired estimate $(2.16)_2$ by virtue of the Gronwall's inequality.

The regularity result, $\hat{v} \in C^0(0,T; H^\ell(\mathbb{R}^n)) \cap L^2(0,T; H^{\ell+1}(\mathbb{R}^n))$, can be shown in the same way as in step 4 of (i). So we omit the details. This completes the proof of Lemma 2.6.

Next we state the existence results for $(2.9)_{1,2}$.

<u>Proposition 2.7</u> (existence of solutions for linearized equations) *Let us assume Condition* 2.1. *Let* $n \ge 1$ *and* $s \ge s_0 + 1$ *(*$s_0 = [n/2] + 1$*) be integers and let* $(u,v)(t,x)$ *satisfy* $(2.10)_{1,2}$ *and* (2.11).

(i) Let $1 \le \ell \le s$ *be an integer and let* $f_1(t,x)$ *satisfy* $(2.12)_1$. *If the initial data satisfy* $\hat{u}(0) \in H^\ell(\mathbb{R}^n)$*, then the system* $(2.9)_1$ *has a unique solution* $\hat{u} \in C^0(0,T; H^\ell(\mathbb{R}^n)) \cap C^1(0,T; H^{\ell-1}(\mathbb{R}^n))$ *satisfying the estimate* $(2.16)_1$.

(ii) Let $2 \le \ell \le s$ *be an integer and let* $f_2(t,x)$ *satisfy* $(2.12)_2$. *If* $\hat{v}(0) \in H^\ell(\mathbb{R}^n)$*, then* $(2.9)_2$ *has a unique solution* $\hat{v} \in C^0(0,T; H^\ell(\mathbb{R}^n)) \cap C^1(0,T; H^{\ell-2}(\mathbb{R}^n)) \cap L^2(0,T; H^{\ell+1}(\mathbb{R}^n))$ *satisfying the estimate* $(2.16)_2$.

Proof. The system $(2.9)_1$ is written in the form

$$(2.21)_1 \qquad \frac{d\hat{u}}{dt} + \hat{A}_1(t)\hat{u} = \hat{f}_1(t) , \qquad t \in [0,T] ,$$

where $\hat{A}_1(t) = \sum_j A_1^0(u,v)^{-1}A_{11}^j(u,v)\partial_{x_j}$, $\hat{f}_1(t) = A_1^0(u,v)^{-1}f_1$. We apply Theorem II of Kato $[37]_3$ to the system $(2.21)_1$. Let $X = L^2(\mathbb{R}^n)$, $Y = H^\ell(\mathbb{R}^n)$ and $S(t) = S = (1-\Delta)^{\ell/2}$. It is not difficult to verify the conditions (i)', (ii)''', (iii) of Theorem I in $[37]_3$. Since $\hat{f}_1 \in L^1(0,T;H^\ell(\mathbb{R}^n)) \cap C^0(0,T;H^{\ell-1}(\mathbb{R}^n))$ by $(2.12)_1$, Theorem II in $[37]_3$ gives a solution $\hat{u} \in C^0(0,T;H^\ell(\mathbb{R}^n)) \cap C^1(0,T;L^2(\mathbb{R}^n))$ of $(2.21)_1$ (and consequently $(2.9)_1$). Moreover it follows that $\partial_t\hat{u} \in C^0(0,T;H^{\ell-1}(\mathbb{R}^n))$. Therefore $\hat{u}$ is the desired solution. The estimate $(2.16)_1$ is an immediate consequence of Lemma 2.6 (i). Thus the proof of (i) is completed.

The system $(2.9)_2$ is written in the form

$$(2.21)_2 \qquad \frac{d\hat{v}}{dt} + \hat{A}_2(t)\hat{v} = \hat{f}_2(t) , \qquad t \in [0,T] ,$$

where $\hat{A}_2(t) = -\sum_{jk} A_2^0(u,v)^{-1}B_2^{jk}(u,v)\partial^2_{x_jx_k}$, $\hat{f}_2(t) = A_2^0(u,v)^{-1}f_2$. We give the proof for $\ell = 2$. Letting $X = L^2(\mathbb{R}^n)$, $Y = H^2(\mathbb{R}^n)$ and $S(t) = \hat{A}_2(t) + \beta + 1$ (with a sufficiently large $\beta > 0$), we can verify the conditions (i)', (ii)''', (iii) of Theorem I. Suppose now that $f_2 \in L^2(0,T;H^2(\mathbb{R}^n)) \cap C^0(0,T;L^2(\mathbb{R}^n))$. Then $\hat{f}_2 \in L^1(0,T;H^2(\mathbb{R}^n)) \cap C^0(0,T;L^2(\mathbb{R}^n))$, and therefore Theorem II gives a solution $\hat{v} \in C^0(0,T;H^2(\mathbb{R}^n)) \cap C^1(0,T;L^2(\mathbb{R}^n))$ of $(2.21)_2$ (and consequently $(2.9)_2$). For general f_2 satisfying $(2.12)_2$ with $\ell = 2$, we consider the problem

$$(2.22)^{\delta} \qquad A_2^0(u,v)\hat{v}_t^{\delta} - \sum_{jk} B_2^{jk}(u,v)\hat{v}_{x_jx_k}^{\delta} = f_{2,\delta}\,, \qquad \hat{v}^{\delta}(0) = \hat{v}(0)\,,$$

where $f_{2,\delta} = \phi_\delta * f_2$. Since $f_{2,\delta} \in L^2(0,T;H^2(\mathbb{R}^n)) \cap C^0(0,T;L^2(\mathbb{R}^n))$, the above consideration shows the existence of a solution $\hat{v}^{\delta} \in C^0(0,T;H^2(\mathbb{R}^n)) \cap C^1(0,T;L^2(\mathbb{R}^n))$ of $(2.22)^{\delta}$. Applying $(2.16)_2$ (with $\ell = 2$) to the system $(2.22)^{\delta} - (2.22)^{\delta'}$, we have

$$\| (\hat{v}^{\delta} - \hat{v}^{\delta'})(t) \|_2^2 \le C(T)\int_0^t \| (f_{2,\delta} - f_{2,\delta'})(\tau) \|_1^2 d\tau$$

for some constant $C(T)$. Since $f_{2,\delta} - f_{2,\delta'} \to 0$ (as $\delta, \delta' \to 0$) strongly in $C^0(0,T;H^1(\mathbb{R}^n))$, $\hat{v}^{\delta}$ is a Cauchy sequence in $C^0(0,T;H^2(\mathbb{R}^n))$. Therefore there is a $\hat{v} \in C^0(0,T;H^2(\mathbb{R}^n))$ such that $\hat{v}^{\delta} \to \hat{v}$ strongly in $C^0(0,T;H^2(\mathbb{R}^n))$ as $\delta \to 0$. This limit $\hat{v}$ is a solution of $(2.9)_2$, and so we know $\partial_t\hat{v} \in C^0(0,T;L^2(\mathbb{R}^n))$. Hence, by Lemma 2.6 (ii), we have a regularity $\hat{v} \in L^2(0,T;H^3(\mathbb{R}^n))$ and the estimate $(2.16)_2$ with $\ell = 2$. Thus the proof for $\ell = 2$ is completed. We can give the proof for $2 < \ell \le s$ by induction, but we omit it. This completes the proof of Proposition 2.7.

2.4 LOCAL EXISTENCE

First we shall consider the linearized system of (2.1):

$$(2.23) \quad \begin{cases} A_1^0(u,v)\hat{u}_t + \sum_j A_{11}^j(u,v)\hat{u}_{x_j} = f_1(u,v,D_xv)\,, \\ \end{cases}$$

$$A_2^0(u,v)\hat{v}_t - \sum_{jk} B_2^{jk}(u,v)\hat{v}_{x_jx_k} = f_2(u,v,D_xu,D_xv) ,$$

with the initial data

(2.24) $$(\hat{u},\hat{v})(0,x) = (u,v)(0,x) = (u_0,v_0)(x) .$$

Let Conditions 2.1 and 2.2 be assumed and let $s \geq s_0 + 1$ ($s_0 = [n/2]+1$) be an integer. For $(u_0,v_0)(x)$ we assume that $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$ and

(2.25) $$(u_0,v_0)(x) \in \mathcal{O}_0 \qquad \text{for any} \quad x \in \mathbb{R}^n ,$$

where $\mathcal{O}_0$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_0 \subset \mathcal{O}$. For $(u,v)(t,x)$, given functions on Q_T, we assume that

$(2.10)_1$ $$u - \bar{u} \in C^0(0,T;H^s(\mathbb{R}^n)) , \qquad \partial_t u \in C^0(0,T;H^{s-1}(\mathbb{R}^n)) ,$$

$(2.10)'_2$ $$\begin{cases} v - \bar{v} \in C^0(0,T;H^s(\mathbb{R}^n)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^n)) , \\ \partial_t v \in C^0(0,T;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T;H^{s-1}(\mathbb{R}^n)) , \end{cases}$$

(2.11) $$(u,v)(t,x) \in \mathcal{O}_1 \qquad \text{for any} \quad (t,x) \in Q_T ,$$

$(2.26)_1$ $$\sup_{0\leq\tau\leq t} \|(u-\bar{u}, v-\bar{v})(\tau)\|_s^2 + \int_0^t \|(v-\bar{v})(\tau)\|_{s+1}^2 d\tau \leq M^2 ,$$

$(2.26)_2$ $$\int_0^t \|\partial_t(u,v)(\tau)\|_{s-1}^2 d\tau \leq M_1^2 \qquad \text{for} \quad t \in [0,T] ,$$

where $\mathcal{O}_1$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\overline{\mathcal{O}}_1 \subset \mathcal{O}$, and M and M_1 are constants. We denote by $X_T^s(\mathcal{O}_1, M, M_1)$ the set of functions $(u,v)(t,x)$ satisfying $(2.10)_1$, $(2.10)_2'$, (2.11), $(2.26)_1$ and $(2.26)_2$.

We shall determine $\mathcal{O}_1$, M, M_1 and T so that for $(u,v) \in X_T^s(\mathcal{O}_1, M, M_1)$, the initial value problem (2.23),(2.24) has a unique solution $(\hat{u},\hat{v})$ in the same $X_T^s(\mathcal{O}_1, M, M_1)$. That is, the set $X_T^s(\mathcal{O}_1, M, M_1)$ is invariant under the mapping defined by $(u,v) \to (\hat{u},\hat{v})$. To state more precisely, we need some preparations. Let $(u,v) \in X_T^s(\mathcal{O}_1, M, M_1)$. Then Condition 2.2 together with the estimates (2.5) (with $k = \ell = s-1$ and $s \to s-1 \geq s_0$) and (2.6) (with $j = s-1$) gives

$$\|f_1(u,v,D_x v)\|_{s-1} + \|f_2(u,v,D_x u, D_x v)\|_{s-1} \leq CM \tag{2.27}$$

for some constant $C = C(\mathcal{O}_1, M)$. Let $(\hat{u},\hat{v})(t,x)$ be a solution of (2.23) satisfying $(2.10)_1$, $(2.10)_2'$ and $(2.26)_1$ with M replaced by $\hat{M}$. Then, by using (2.5) (with $k = \ell = s-1$), (2.6) (with $j = s$) and (2.27), we have

$$\int_0^t \|\partial_t(\hat{u},\hat{v})(\tau)\|_{s-1}^2 d\tau \leq C_3^2\{\hat{M}^2 + (\hat{M}^2 + M^2)t\} \tag{2.28}$$

with some constant $C_3 = C_3(\mathcal{O}_1, M)$. Now fix a constant d_1 so that $0 < d_1 < d_0 \equiv \mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O})$, and take $\mathcal{O}_1$, M and M_1 as follows:

$$\begin{cases} \mathcal{O}_1 = d_1\text{-neighborhood of } \mathcal{O}_0, \\ M = 2C_1 \|u_0 - \bar{u}, v_0 - \bar{v}\|_s, \qquad M_1 = 2C_3 M, \end{cases} \tag{2.29}$$

where $C_1 = C_1(O_1)$ and $C_3 = C_3(O_1, M)$ are constants in Lemma 2.6 and (2.28), respectively. Then we have:

<u>Proposition 2.8</u> (invariant set under iterations) *Let Conditions* 2.1 *and* 2.2 *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 1$ *(*$s_0 = [n/2] + 1$*) be integers. Suppose that the initial data satisfy* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$ *and* (2.25). *Then there exists a positive constant* T_0*, depending only on* O_0*,* d_1 *and* $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_s$*, such that if* $(u,v) \in X^s_{T_0}(O_1, M, M_1)$ *with* O_1*,* M *and* M_1 *defined by* (2.29)*, the initial value problem* (2.23)*,* (2.24) *has a unique solution* $(\hat{u}, \hat{v})$ *in the same* $X^s_{T_0}(O_1, M, M_1)$.

Proof. The existence of a solution $(\hat{u}, \hat{v})$ to (2.23), (2.24) follows from Proposition 2.7 with $\ell = s$. So it suffices to estimate the solution. By $(2.16)_{1,2}$ (with $\ell = s$) we have

$$(2.30) \qquad \|(\hat{u} - \bar{u},\ \hat{v} - \bar{v})(t)\|_s^2 + \int_0^t \|(\hat{v} - \bar{v})(\tau)\|_{s+1}^2 d\tau$$
$$\leq C_1^2 e^{C(t + M_1 t^{1/2})} \{ \|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 + CM^2 t(1+t)\}$$

for some constant $C = C(O_1, M)$. Here we have used the estimates $\|f_1(u, v, D_x v)\|_s \leq C(M + \|D_x v\|_s)$ and (2.27). Take T_0 so that

$$e^{C(T_0 + M_1 T_0^{1/2})} \leq 2, \quad CM^2 T_0(1 + T_0) \leq \|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 .$$

Then the right hand side of (2.30) is majorized by $4C_1^2 \|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 = M^2$. Therefore the solution $(\hat{u}, \hat{v})$ satisfies $(2.26)_1$, which also gives

(2.28) with $\hat{M} = M$:

$$\int_0^t \|\partial_t(\hat{u},\hat{v})(\tau)\|_{s-1}^2 d\tau \le C_3^2 M^2 (1+2t) \ .$$

The right hand side of the above inequality is bounded by $4C_3^2 M^2 = M_1^2$ provided $T_0 \le 3/2$. So the estimate $(2.26)_2$ is proved to be satisfied for the solution. On the other hand this estimate gives

$$|(\hat{u},\hat{v})(t,x) - (u_0,v_0)(x)| \le C\int_0^t \|\partial_t(\hat{u},\hat{v})(\tau)\|_{s-1} d\tau \le CM_1 t^{1/2} \ ,$$

where C is the constant in (2.4). Take T_0 so that $CM_1 T_0^{1/2} \le d_1$. Then the last inequality implies that the solution satisfies (2.11). This completes the proof of Proposition 2.8.

Based on Proposition 2.8, we shall introduce the successive approximation sequence $\{(u^n,v^n)(t,x)\}_{n=0}^{\infty}$ for the initial value problem (2.1), (2.2) as follows:

$$(u^0,v^0)(t,x) = (\bar{u},\bar{v}) \ ,$$

and for $n \ge 0$,

$$(2.31)^{n+1} \begin{cases} A_1^0(u^n,v^n)u_t^{n+1} + \sum_j A^j(u^n,v^n)u_{x_j}^{n+1} = f_1(u^n,v^n,D_x v^n) \ , \\ \\ A_2^0(u^n,v^n)v_t^{n+1} - \sum_{jk} B_2^{jk}(u^n,v^n)v_{x_j x_k}^{n+1} = f_2(u^n,v^n,D_x u^n,D_x v^n) \ , \end{cases}$$

$$(2.32)^{n+1} \qquad (u^{n+1},v^{n+1})(0,x) = (u_0,v_0)(x) \ .$$

By Proposition 2.8 the sequence $(u^n,v^n)(t,x)$ is well defined on Q_{T_0} for all $n \geq 0$, and is uniformly bounded with respect to $n \geq 0$, i.e., $(u^n,v^n) \in X^s_{T_0}(\mathcal{O}_1,M,M_1)$. We will show the convergence of the sequence (u^n,v^n) as $n \to \infty$. Consider the difference $(2.31)^{n+1} - (2.31)^n$. Let $(\hat{u}^n,\hat{v}^n) = (u^{n+1}-u^n, v^{n+1}-v^n)$ for $n \geq 1$. Then we obtain

$$(2.33)^n \quad \begin{cases} A_1^0(u^n,v^n)\hat{u}^n_t + \sum_j A^j_{11}(u^n,v^n)\hat{u}^n_{x_j} = \hat{f}^n_1 , \\ \\ A_2^0(u^n,v^n)\hat{v}^n_t - \sum_{jk} B^{jk}_2(u^n,v^n)\hat{v}^n_{x_jx_k} = \hat{f}^n_2 , \end{cases}$$

$$(2.34)^n \qquad (\hat{u}^n,\hat{v}^n)(0,x) = (0,\ 0) ,$$

where

$$\begin{aligned} \hat{f}^n_1 = {} & A_1^0(u^n,v^n)\{A_1^0(u^n,v^n)^{-1}f_1(u^n,v^n,D_xv^n) - A_1^0(u^{n-1},v^{n-1})^{-1}\cdot \\ & \cdot f_1(u^{n-1},v^{n-1},D_xv^{n-1})\} - A_1^0(u^n,v^n)\sum_j\{A_1^0(u^n,v^n)^{-1}A^j_{11}(u^n,v^n) - \\ & - A_1^0(u^{n-1},v^{n-1})^{-1}A^j_{11}(u^{n-1},v^{n-1})\}u^n_{x_j} , \end{aligned}$$

$$\begin{aligned} \hat{f}^n_2 = {} & A_2^0(u^n,v^n)\{A_2^0(u^n,v^n)^{-1}f_2(u^n,v^n,D_xu^n,D_xv^n) - A_2^0(u^{n-1},v^{n-1})^{-1}\cdot \\ & \cdot f_2(u^{n-1},v^{n-1},D_xu^{n-1},D_xv^{n-1})\} - A_2^0(u^n,v^n)\sum_{jk}\{A_2^0(u^n,v^n)^{-1}\cdot \\ & \cdot B^{jk}_2(u^n,v^n) - A_2^0(u^{n-1},v^{n-1})^{-1}B^{jk}_2(u^{n-1},v^{n-1})\}v^n_{x_jx_k} . \end{aligned}$$

By the estimates (2.5), (2.6) and $(u^n,v^n) \in X^s_{T_0}(\mathcal{O}_1,M,M_1)$, we find a constant $C = C(\mathcal{O}_1,M)$ independent of $n \geq 1$ such that

$$\|\hat{f}_1^n\|_{s-1} \le C(\|\hat{u}^{n-1}\|_{s-1} + \|\hat{v}^{n-1}\|_s) ,$$

$$\|\hat{f}_2^n\|_{s-2} \le C \|\hat{u}^{n-1},\hat{v}^{n-1}\|_{s-1} .$$

Therefore, applying $(2.16)_{1,2}$ (with $\ell = s-1$) to the system $(2.33)^n$, we get

$$(2.35) \qquad \sup_{0\le\tau\le t} \|(\hat{u}^n,\hat{v}^n)(\tau)\|_{s-1}^2 + \int_0^t \|\hat{v}^n(\tau)\|_s^2 d\tau$$

$$\le Ct(1+t)e^{C(t+M_1 t^{1/2})} \{ \sup_{0\le\tau\le t} \|(\hat{u}^{n-1},\hat{v}^{n-1})(\tau)\|_{s-1}^2 +$$

$$+ \int_0^t \|\hat{v}^{n-1}(\tau)\|_s^2 d\tau \} ,$$

where $C = C(\mathcal{O}_1,M)$ is a constant independent of $n \ge 1$. Take T_1 so small that

$$T_1 \le T_0 , \qquad CT_1(1+T_1)e^{C(T_1+M_1T_1^{1/2})} < 1 .$$

Then it follows from (2.35) that $(u^n-\bar{u},v^n-\bar{v})$ is a Cauchy sequence in $\mathcal{C}^0(0,T_1 ; H^{s-1}(\mathbb{R}^n))$. Therefore there is a $(u,v)(t,x)$ with $(u-\bar{u},v-\bar{v}) \in \mathcal{C}^0(0,T_1 ; H^{s-1}(\mathbb{R}^n))$ such that $(u^n-u,v^n-v) \to 0$ strongly in $\mathcal{C}^0(0,T_1 ; H^{s-1}(\mathbb{R}^n))$ as $n \to \infty$. On the other hand it follows from the uniform estimate $(u^n,v^n) \in X^s_{T_0}(\mathcal{O}_1,M,M_1) \subset X^s_{T_1}(\mathcal{O}_1,M,M_1)$ that there is a subsequence $\{n'\}$ of $\{n\}$ such that $v^{n'}-v \to 0$ weakly in $L^2(0,T_1 ; H^{s+1}(\mathbb{R}^n))$. Furthermore there is a subsequence $\{n''\} = \{n''(t)\}$ of $\{n'\}$, depending on $t \in [0,T_1]$, such that $(u^{n''}-u,v^{n''}-v) \to 0$ weakly in $H^s(\mathbb{R}^n)$ for every fixed $t \in [0,T_1]$. Thus we have a solution $(u,v)(t,x)$ of (2.1), (2.2) satisfying

$$u - \bar{u} \in L^\infty(0,T_1\,;H^s(\mathbb{R}^n)) \,,$$

$$v - \bar{v} \in L^\infty(0,T_1\,;H^s(\mathbb{R}^n)) \cap L^2(0,T_1\,;H^{s+1}(\mathbb{R}^n)) \,.$$

Moreover it follows that

$$\partial_t u \in L^\infty(0,T_1\,;H^{s-1}(\mathbb{R}^n)) \,,$$

$$\partial_t v \in L^\infty(0,T_1\,;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T_1\,;H^{s-1}(\mathbb{R}^n)) \,.$$

Therefore, by Lemma 2.6, we have a regularity $(u-\bar{u},v-\bar{v}) \in C^0(0,T_1\,; H^s(\mathbb{R}^n))$, and consequently $\partial_t u \in C^0(0,T_1\,;H^{s-1}(\mathbb{R}^n))$ and $\partial_t v \in C^0(0,T_1\,; H^{s-2}(\mathbb{R}^n))$. Thus we have proved:

Theorem 2.9 (local existence) *Let Conditions* 2.1 *and* 2.2 *be assumed. Let* $n \ge 1$ *and* $s \ge s_0 + 1$ $(s_0 \ge [n/2]+1)$ *be integers. Suppose that the initial data satisfy* $(u_0-\bar{u},v_0-\bar{v}) \in H^s(\mathbb{R}^n)$ *and* (2.25). *Then there exists a positive constant* T_1 $(\le T_0)$, *depending only on* $\mathcal{O}_0$, d_1 *and* $\|u_0-\bar{u},\ v_0-\bar{v}\|_s$, *such that the initial value problem* (2.1), (2.2) *has a unique solution* $(u,v) \in X^s_{T_1}(\mathcal{O}_1,M,M_1)$, *where* $\mathcal{O}_1$, M *and* M_1 *are determined by* (2.29). *In particular, the solution satisfies*

$$u-\bar{u} \in C^0(0,T_1\,;H^s(\mathbb{R}^n)) \cap C^1(0,T_1\,;H^{s-1}(\mathbb{R}^n)) \,,$$

$$v-\bar{v} \in C^0(0,T_1\,;H^s(\mathbb{R}^n)) \cap C^1(0,T_1\,;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T_1\,;H^{s+1}(\mathbb{R}^n)) \,,$$

$$\sup_{0 \leq \tau \leq t} \| (u-\bar{u},\ v-\bar{v})(\tau) \|_s^2 + \int_0^t \| (u-\bar{u})(\tau) \|_s^2 + \| (v-\bar{v})(\tau) \|_{s+1}^2 \, d\tau \tag{2.36}$$

$$\leq C_4^2 \| u_0 - \bar{u},\ v_0 - \bar{v} \|_s^2 \qquad \textit{for} \quad t \in [0, T_1] ,$$

where $C_4 > 1$ *is a constant depending only on* O_0, d_1 *and* $\| u_0 - \bar{u},\ v_0 - \bar{v} \|_s$.

Remark 2.3 (i) In the special case $m'' = 0$ (resp. $m' = 0$), we can show a similar local existence result for the symmetric hyperbolic system (2.1)' (resp. symmetric parabolic system (2.1)").

(ii) The proof and the statement of Theorem 2.9 remain valid in the case when B_2^{jk} $(j,k = 1, \cdots, n)$ depend on $D_x v$ as well as (u,v), i.e., $B_2^{jk} = B_2^{jk}(u,v,D_x v)$, provided that $s \geq s_0 + 2$ is assumed.

CHAPTER III

QUASILINEAR SYMMETRIC HYPERBOLIC-PARABOLIC SYSTEMS, II
(GLOBAL EXISTENCE)

3.1 INTRODUCTION

Let $(\bar{u},\bar{v})$ be the constant state in Condition 2.2. Then $(u,v)(t,x) = (\bar{u},\bar{v})$ is a constant equilibrium solution of the system (2.1). In this chapter we shall prove that under appropriate conditions a solution to the initial value problem (2.1),(2.2) exists for all time in a small neighborhood of $(\bar{u},\bar{v})$ and decays to $(\bar{u},\bar{v})$ as $t \to \infty$, that is, the equilibrium state $(\bar{u},\bar{v})$ is asymptotically stable as $t \to \infty$.

Our analysis below is based on a study of the *dissipative structure* of the linearized system for (2.1) around the equilibrium state $(\bar{u},\bar{v})$.

$$(3.1) \qquad A^0(\bar{u},\bar{v})U_t + \sum_j A^j(\bar{u},\bar{v})U_{x_j} - \sum_{jk} B^{jk}(\bar{u},\bar{v})U_{x_j x_k} + L(\bar{u},\bar{v})U = 0$$

(cf. (3.16)), where all the coefficients $A^0(\bar{u},\bar{v})$, $A^j(\bar{u},\bar{v})$, $B^{jk}(\bar{u},\bar{v})$ and $L(\bar{u},\bar{v})$ are constant square matrices of order m; they are given explicitly by

$$
(3.2)\quad
\begin{aligned}
A^0(u,v) &= \begin{pmatrix} A_1^0(u,v) & 0 \\ 0 & A_2^0(u,v) \end{pmatrix}, \quad A^j(u,v) = \begin{pmatrix} A_{11}^j(u,v) & A_{12}^j(u,v) \\ A_{21}^j(u,v) & A_{22}^j(u,v) \end{pmatrix}, \\
B^{jk}(u,v) &= \begin{pmatrix} 0 & 0 \\ 0 & B_2^{jk}(u,v) \end{pmatrix}, \quad L(u,v) = \begin{pmatrix} L_{11}(u,v) & L_{12}(u,v) \\ L_{21}(u,v) & L_{22}(u,v) \end{pmatrix},
\end{aligned}
$$

where

$$
(3.3)_1\quad \begin{cases} A_{12}^j(u,v) = -D_{\zeta_j} f_1(u,v,0), \\ A_{21}^j(u,v) = -D_{\eta_j} f_2(u,v,0,0), \quad A_{22}^j(u,v) = -D_{\zeta_j} f_2(u,v,0,0), \end{cases}
$$

$$
(3.3)_2\quad \begin{cases} L_{11}(u,v) = -D_u f_1(u,v,0), \quad L_{12}(u,v) = -D_v f_1(u,v,0), \\ L_{21}(u,v) = -D_u f_2(u,v,0,0), \quad L_{22}(u,v) = -D_v f_2(u,v,0,0). \end{cases}
$$

Here $\eta = (\eta_1,\cdots,\eta_n) \in \mathbb{R}^{nm'}$ and $\zeta = (\zeta_1,\cdots,\zeta_n) \in \mathbb{R}^{nm''}$ are vectors corresponding to $D_x u$ and $D_x v$, respectively; D_{ζ_j} denotes the differentiation with respect to ζ_j and so on.

We assume the following conditions on the linearized system (3.1); these conditions guarantee the *dissipative structure* for the system (3.1) (see Proposition 3.A.4).

<u>Condition 3.1</u>

(i) $A^0(\bar{u},\bar{v})$ is real symmetric and positive definite,

(ii) $A^j(\bar{u},\bar{v})$ $(j=1,\cdots,n)$ are real symmetric,

(iii) $B^{jk}(\bar{u},\bar{v})$ $(j,k=1,\cdots,n)$ are real symmetric and satisfy $B^{jk}(\bar{u},\bar{v})$ =

$B^{jk}(\bar{u},\bar{v})$; $\sum_{jk} B^{jk}(\bar{u},\bar{v})\omega_j\omega_k$ is (real symmetric) positive semi-definite for any $\omega = (\omega_1,\cdots\cdots,\omega_n) \in S^{n-1}$,

(iv) $L(\bar{u},\bar{v})$ is real symmetric and positive semi-definite.

Condition 3.2 There exist (real) constant square matrices K^j $(j=1,\cdots\cdots,n)$ of order m such that

(i) $K^jA^0(\bar{u},\bar{v})$ $(j=1,\cdots\cdots,n)$ are real anti-symmetric,

(ii) the symmetric part of the matrix $\sum_{jk}\{K^jA^k(\bar{u},\bar{v}) + B^{jk}(\bar{u},\bar{v})\}\omega_j\omega_k + L(\bar{u},\bar{v})$ is positive definite for any $\omega = (\omega_1,\cdots\cdots,\omega_n) \in S^{n-1}$.

It is noted that, in view of (3.2), Condition 2.1 (i) and (iii) imply Condition 3.1 (i) and (iii) respectively.

Under these conditions it has been proved by Umeda, Kawashima and Shizuta [81] that the solution of the linearized system (3.1) with the initial data $U(0) \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ $(1 \le p \le 2)$ decays at the rate $t^{-\gamma}$ (with $\gamma = n(1/2p - 1/4)$) as $t \to \infty$ (see appendix 3.A.1). A combination of this decay estimate with the energy inequalities for (2.1) gives the main result of this chapter. "Let $n \ge 3$. If the initial data are close to the constant equilibrium state $(\bar{u},\bar{v})$ in $H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ (with $s \ge [n/2]+3$ and $1 \le p < 2n/(n+2)$), then the initial value problem (2.1),(2.2) has a unique global solution in a small neighborhood of $(\bar{u},\bar{v})$ and the solution tends to $(\bar{u},\bar{v})$ at the rate $t^{-\gamma}$ (with $\gamma = n(1/2p - 1/4)$) as $t \to \infty$." (Theorem 3.6).

The plan of this chapter is as follows. In section 3.2 we give the a priori estimates for higher order derivatives of small solutions of (2.1) by a somewhat technical energy method, which make use of the K^j

in Condition 3.2. For similar energy methods, see [54],[66],[41], $[38]_{3,4}$. In section 3.3 we shall prove that small solutions of (2.1) have the decay rate $t^{-\gamma}$, $\gamma = n(1/2p - 1/4)$, if $n \geq 3$ and if $1 \leq p < 3/2$ for $n = 3$ and $1 \leq p < 2$ for $n \geq 4$; this is an immediate consequence of the same decay result for linearized system (3.1). This decay rate also gives the a priori estimates for lower order derivatives of solutions when $n \geq 3$ and $1 \leq p < 2n/(n+2)$. The global existence of a solution of (2.1) is proved in section 3.4 by the standard continuation argument, based on the a priori estimates derived in sections 3.2 and 3.3. Section 3.5 contains some global existence results for (2.1) in the case that the nonlinear terms satisfy additional conditions. In section 3.6, as an application of our results, we shall treat the equations of heat conduction with finite speed of propagation.

In the appendix the linearized system (3.1) is investigated on the decay estimates and the spectral analysis. The decay estimates in appendix 3.A.1 are used in section 3.3. The eigenvalue problem associated with (3.1) is discussed in appendix 3.A.2.

3.2 A PRIORI ESTIMATES, I (ENERGY ESTIMATE)

Let $s \geq s_0 + 1$ ($s_0 = [n/2] + 1$) be an integer and let $T > 0$ be a constant, and consider a solution $(u,v)(t,x)$ of (2.1),(2.2) satisfying

$$\begin{cases} u - \bar{u} \in C^0(0,T; H^s(\mathbb{R}^n)) \cap C^1(0,T; H^{s-1}(\mathbb{R}^n)) , \end{cases}$$

$$(3.4) \qquad \begin{cases} v - \bar{v} \in C^0(0,T;H^s(\mathbb{R}^n)) \cap C^1(0,T;H^{s-2}(\mathbb{R}^n)) , \\ D_x u \in L^2(0,T;H^{s-1}(\mathbb{R}^n)) , \qquad D_x v \in L^2(0,T;H^s(\mathbb{R}^n)) , \end{cases}$$

$$(3.5) \qquad (u,v)(t,x) \in \mathcal{O}_2 \quad \text{for any} \quad (t,x) \in Q_T ,$$

where $\mathcal{O}_2$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_2 \subset \mathcal{O}$. For the solution we introduce

$$(3.6) \qquad N_s(t',t)^2 = \sup_{t' \le \tau \le t} \|(u-\bar{u},\ v-\bar{v})(\tau)\|_s^2 + \\ + \int_{t'}^{t} \|D_x u(\tau)\|_{s-1}^2 + \|D_x v(\tau)\|_s^2 d\tau \quad \text{for} \quad 0 \le t' < t \le T ,$$

and we simply write $N_s(t) = N_s(0,t)$. Then by (2.4) (with $p=\infty$ and $s=s_0$) we find a positive constant a_0 such that

(3.7) if $N_{s_0}(T) \le a_0$, then (3.5) is satisfied automatically.

So it is convienient to assume $N_s(T) \le a_0$ instead of (3.5).

The purpose of this and the next sections is to obtain the a priori estimate for $N_s(T)$ when $N_s(T)$ satisfies a smallness assumption. Our first result is stated as follows:

<u>Lemma 3.1</u> *Assume Conditions* 2.1, 2.2 *and* 3.1. *Let* $n \ge 1$ *and* $s \ge s_0 + 1$ ($s_0 = [n/2]+1$) *be integers, and let the initial data satisfy* $(u_0 - \bar{u},\ v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$. *Let* $(u,v)(t,x)$ *be a solution of the problem* (2.1), (2.2) *satisfying* (3.4) *and* $N_s(T) \le a_0$. *Then there is a constant* $C =$

$C(a_0) > 1$ *such that the following a priori estimate holds for* $t \in [0,T]$.

$$(3.8) \qquad \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 + \|P^+ D_x(u,v)(\tau)\|_{s-1}^2 d\tau$$

$$\le C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + N_s(T)^3 \} ,$$

where P^+ *is the orthogonal projection onto the range of* $L(\bar{u},\bar{v})$ *(= the orthogonal complement of the null space of* $L(\bar{u},\bar{v})$ *).*

Proof. We first rewrite the system (2.1). Since $f_1(\bar{u},\bar{v},0) = f_2(\bar{u},\bar{v},0,0) = 0$ by Condition 2.2, the lower order terms f_1 and f_2 can be written in the form (see $(3.3)_{1,2}$)

$$(3.9)_1 \qquad f_1(u,v,D_x v) = - \sum_j A_{12}^j(\bar{u},\bar{v}) v_{x_j} -$$

$$- \{L_{11}(\bar{u},\bar{v})(u-\bar{u}) + L_{12}(\bar{u},\bar{v})(v-\bar{v})\} + \tilde{f}_1(u,v,D_x v) ,$$

$$(3.9)_2 \qquad f_2(u,v,D_x u,D_x v) = - \sum_j \{A_{21}^j(\bar{u},\bar{v}) u_{x_j} + A_{22}^j(\bar{u},\bar{v}) v_{x_j}\} -$$

$$- \{L_{21}(\bar{u},\bar{v})(u-\bar{u}) + L_{22}(\bar{u},\bar{v})(v-\bar{v})\} + \tilde{f}_2(u,v,D_x u,D_x v) ,$$

where the remainders $\tilde{f}_1$ and $\tilde{f}_2$ are smooth in each argument and satisfy

$$(3.10) \qquad \begin{cases} \tilde{f}_1(u,v,D_x v) = O(\{|u-\bar{u},\ v-\bar{v}| + |D_x v|\}^2) , \\ \tilde{f}_2(u,v,D_x u,D_x v) = O(\{|u-\bar{u},\ v-\bar{v}| + |D_x(u,v)|\}^2) \end{cases}$$

for $|u-\bar{u}, v-\bar{v}| + |D_x(u,v)| \to 0$. Substitute $(3.9)_{1,2}$ into (2.1) to obtain

$$(3.11) \qquad A^0(u,v)U_t + \sum_j \tilde{A}^j(u,v)U_{x_j} - \sum_{jk} B^{jk}(u,v)U_{x_j x_k} + L(\bar{u},\bar{v})U$$

$$= \tilde{f}(u,v,D_x u,D_x v) \ ,$$

where $U = {}^t(u-\bar{u}, v-\bar{v})$, $\tilde{f}(u,v,D_x u,D_x v) = {}^t(\tilde{f}_1(u,v,D_x v), \tilde{f}_2(u,v,D_x u,D_x v))$,

$$\tilde{A}^j(u,v) = \begin{pmatrix} A^j_{11}(u,v) & A^j_{12}(\bar{u},\bar{v}) \\ A^j_{21}(\bar{u},\bar{v}) & A^j_{22}(\bar{u},\bar{v}) \end{pmatrix} .$$

It suffices to prove the lemma for sufficiently smooth solutions because the arguments on the mollifier are also applicable. Apply D_x^ℓ $(1 \le \ell \le s)$ to (3.11). The resulting system can be written in the form

$$(3.12)^\ell \qquad \tilde{A}^0(u,v)D_x^\ell U_t + \sum_j \tilde{A}^j(u,v)D_x^\ell U_{x_j} - \sum_{jk} B^{jk}(\bar{u},\bar{v})D_x^\ell U_{x_j x_k} + L(\bar{u},\bar{v})D_x^\ell U$$

$$= {}^t(\tilde{F}_1^\ell, D_x^\ell h_2(u,v,D_x u,D_x v,D_x^2 v)) \ ,$$

where

$$\tilde{A}^0(u,v) = \begin{pmatrix} A^0_1(u,v) & 0 \\ 0 & A^0_2(\bar{u},\bar{v}) \end{pmatrix} ,$$

$$(3.13)_1 \qquad \tilde{F}_1^\ell = A^0_1(u,v)D_x^\ell\{A^0_1(u,v)^{-1}\tilde{f}_1(u,v,D_x v)\} -$$

$$- A_1^0(u,v) \sum_j \{[D_x^\ell, A_1^0(u,v)^{-1} A_{11}^j(u,v)] u_{x_j} + [D_x^\ell, A_1^0(u,v)^{-1}] A_{12}^j(\bar{u},\bar{v}) v_{x_j}\}$$

$$- A_1^0(u,v) [D_x^\ell, A_1^0(u,v)^{-1}] \{L_{11}(\bar{u},\bar{v})(u-\bar{u}) + L_{12}(\bar{u},\bar{v})(v-\bar{v})\} ,$$

$$(3.13)_2 \qquad h_2(u,v,D_x u,D_x v,D_x^2 v) = A_2^0(\bar{u},\bar{v}) A_2^0(u,v)^{-1} \tilde{f}_2(u,v,D_x u,D_x v) -$$

$$- A_2^0(\bar{u},\bar{v}) \{A_2^0(u,v)^{-1} - A_2^0(\bar{u},\bar{v})^{-1}\} \sum_j \{A_{21}^j(\bar{u},\bar{v}) u_{x_j} + A_{22}^j(\bar{u},\bar{v}) v_{x_j}\} +$$

$$+ A_2^0(\bar{u},\bar{v}) \sum_{jk} \{A_2^0(u,v)^{-1} B_2^{jk}(u,v) - A_2^0(\bar{u},\bar{v})^{-1} B_2^{jk}(\bar{u},\bar{v})\} v_{x_j x_k} -$$

$$- A_2^0(\bar{u},\bar{v}) \{A_2^0(u,v)^{-1} - A_2^0(\bar{u},\bar{v})^{-1}\} \{L_{21}(\bar{u},\bar{v})(u-\bar{u}) + L_{22}(\bar{u},\bar{v})(v-\bar{v})\} .$$

It should be noticed that the equations for $D_x^\ell u$ are regarded as a linear hyperbolic system whose principal part is of variable (not constant) coefficients. Take the inner product (in $\mathbb{R}^m$) of $(3.12)^\ell$ with $D_x^\ell U$, integrate the resulting equality over $Q_t = [0,t] \times \mathbb{R}^n$ and then add for $\ell = 1,\cdots,s$. Noting that $A_1^0(u,v)$, $A_2^0(\bar{u},\bar{v})$ and $\sum_{jk} B_2^{jk}(\bar{u},\bar{v}) \omega_j \omega_k$ are positive definite and that $L(\bar{u},\bar{v})$ is positive semi-definite, we have by integration by parts

$$(3.14) \qquad \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 + \|P^+ D_x(u,v)(\tau)\|_{s-1}^2 \, d\tau$$

$$\leq C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + \int_0^t R_1(\tau)\, d\tau \} ,$$

where $C = C(a_0) > 1$ is a constant and

$$R_1(t) = \sum_{\ell=1}^s \int (|A_1^0(u,v)_t| + \sum_j |A_{11}^j(u,v)_{x_j}|) |D_x^\ell u|^2 + |\tilde{F}_1^\ell| |D_x^\ell u| \, dx +$$

$$+ \int |D_x h_2(u,v,D_x u,D_x v,D_x^2 v)| \, |D_x v| \, dx +$$

$$+ \sum_{\ell=2}^{s} \int |D_x^{\ell-1} h_2(u,v,D_x u,D_x v,D_x^2 v)| \, |D_x^{\ell+1} v| \, dx \, .$$

By the estimates (2.5), (2.6) and (2.7) we have

$$R_1(t) \le C \, \|u-\bar{u}, \ v-\bar{v}\|_s (\|D_x u\|_{s-1}^2 + \|D_x v\|_s^2)$$

with a constant $C = C(a_0)$, where $N_s(T) \le a_0$ is assumed. Therefore the desired estimate (3.8) follows from (3.14) easily. This completes the proof of Lemma 3.1.

Next we show the estimate for $L^2(0,T\,;H^{s-2}(\mathbb{R}^n))$-norm of the derivatives $D_x^2 u$.

<u>Lemma 3.2</u> *Assume Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $(u_0,v_0)(x)$ *and* $(u,v)(t,x)$ *be the same as in Lemma* 3.1. *Then there is a positive constant* $C = C(a_0)$ *such that the following a priori estimate holds for* $t \in [0,T]$.

$$(3.15) \qquad \int_0^t \|D_x^2 u(\tau)\|_{s-2}^2 d\tau - C\{ \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 +$$

$$+ \|P^+ D_x(u,v)(\tau)\|_{s-1}^2 d\tau \} \le C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + N_s(T)^3 \} \, .$$

Proof. We again rewrite the system (2.1) such that the linear pars at the constant state $(\bar{u},\bar{v})$ appear in the left hand side and the nonlinear parts in the right hand side:

$$\text{(3.16)} \qquad A^0(\bar{u},\bar{v})U_t + \sum_j A^j(\bar{u},\bar{v})U_{x_j} - \sum_{jk} B^{jk}(\bar{u},\bar{v})U_{x_jx_k} + L(\bar{u},\bar{v})U$$

$$= h(u,v,D_xu,D_xv,D_x^2v) \ ,$$

where $U = {}^t(u-\bar{u},v-\bar{v})$ and $h = {}^t(h_1,h_2)$; $h_2 = h_2(u,v,D_xu,D_xv,D_x^2v)$ is given by $(3.13)_2$ and $h_1 = h_1(u,v,D_xu,D_xv)$ by

$$\text{(3.17)} \qquad h_1(u,v,D_xu,D_xv) = A_1^0(\bar{u},\bar{v})A_1^0(u,v)^{-1}\tilde{f}_1(u,v,D_xv) -$$

$$- A_1^0(\bar{u},\bar{v}) \sum_j \{A_1^0(u,v)^{-1}A_{11}^j(u,v) - A_1^0(\bar{u},\bar{v})^{-1}A_{11}^j(\bar{u},\bar{v})\}u_{x_j} -$$

$$- A_1^0(\bar{u},\bar{v})\{A_1^0(u,v)^{-1} - A_1^0(\bar{u},\bar{v})^{-1}\} \sum_j A_{12}^j(\bar{u},\bar{v})v_{x_j} -$$

$$- A_1^0(\bar{u},\bar{v})\{A_1^0(u,v)^{-1} - A_1^0(\bar{u},\bar{v})^{-1}\}\{L_{11}(\bar{u},\bar{v})(u-\bar{u}) + L_{12}(\bar{u},\bar{v})(v-\bar{v})\} \ .$$

Apply D_x^ℓ $(1 \le \ell \le s-1)$ to (3.16) and then multiply the resulting system by the matrices K^j (in Condition 3.2). Take the inner product (in $\mathbb{R}^m$) of these equations with the vectors $D_x^\ell U_{x_j}$, integrate them over Q_t and then add for both $j = 1,\cdots,n$ and $\ell = 1,\cdots,s-1$. By Condition 3.2 we have

$$< K^jA^0(\bar{u},\bar{v})D_x^\ell U_t, \ D_x^\ell U_{x_j} >$$

$$= \frac{1}{2}\{ < K^jA^0(\bar{u},\bar{v})D_x^\ell U, \ D_x^\ell U_{x_j} > \}_t - \{ \cdots\cdots \}_{x_j}$$

and the estimate

$$\sum_{jk} \int < K^j A^k(\bar{u},\bar{v}) D_x^\ell U_{x_k}, \; D_x^\ell U_{x_j} > dx$$

$$\geq c \, \|D_x^{\ell+1} U\|^2 - C\{ \|D_x^{\ell+1} V\|^2 + \|P^+ D_x^{\ell+1} U\|^2 \}$$

with constants c and C $(0 < c < C)$. Therefore the above calculation yields the estimate

$$\int_0^t \|D_x^2(u,v)(\tau)\|_{s-2}^2 - C\{ \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 + \tag{3.18}$$

$$+ \|P^+ D_x(u,v)(\tau)\|_{s-1}^2 d\tau \} \leq C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + \int_0^t R_2(\tau)\, d\tau \} ,$$

where C is a constant and

$$R_2(t) = \sum_{\ell=1}^{s-1} \int_0^t |D_x^\ell h(u,v,D_x u, D_x v, D_x^2 v)| \, |D_x^{\ell+1}(u,v)| \, dx .$$

By use of (2.5), (2.6) and $N_s(T) \leq a_0$, we know a constant $C = C(a_0)$ such that

$$R_2(t) \leq C \, \|u - \bar{u}, \; v - \bar{v}\|_s (\|D_x u\|_{s-1}^2 + \|D_x v\|_s^2) .$$

This estimate and (3.18) yield (3.15). This completes the proof of Lemma 3.2.

3.3 A PRIORI ESTIMATES, II (DECAY ESTIMATE)

In this section we shall get the decay estimates of small solutions to the problem (2.1),(2.2). Let $(u,v)(t,x)$ be a solution of (2.1), (2.2) and put $U(t,x) = {}^t(u-\bar{u},v-\bar{v})(t,x)$. Then, noting (3.16), we arrive at the expression

$$(3.19) \qquad U(t) = (A^0)^{-1/2}\{e^{-tS}(A^0)^{1/2}U_0 + \int_0^t e^{-(t-\tau)S}(A^0)^{-1/2}h(\tau)\,d\tau\},$$

where $U_0 = {}^t(u_0-\bar{u},v_0-\bar{v})$, and $h = {}^t(h_1,h_2)$ is given by (3.17) and $(3.13)_2$; e^{-tS} is defined by (3.A.13) (for the definition of $S(\xi)$, see (3.A.4)) in appendix 3.A.1. Applying the decay estimate (3.A.14) to the expression (3.19), we have:

Proposition 3.3 (a priori decay estimate) *Assume Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \geq 3$ *and* $s \geq s_0 + 2$ ($s_0 = [n/2]+1$) *be integers, and let* $p \in [1,3/2]$ *for* $n = 3$, $p \in [1,2)$ *for* $n = 4$ *and* $p \in [1,2]$ *for* $n \geq 5$. *Suppose that* $(u_0-\bar{u},v_0-\bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and put*

$$(3.20) \qquad \|u_0-\bar{u},\ v_0-\bar{v}\|_{\ell,p} = \|u_0-\bar{u},\ v_0-\bar{v}\|_{\ell} + \|u_0-\bar{u},\ v_0-\bar{v}\|_{L^p}$$

for $\ell \leq s$. *Let* $(u,v)(t,x)$ *be a solution of* (2.1), (2.2) *satisfying* (3.4). *Then there exist positive constants* a_1 ($\leq a_0$), $\delta_1 = \delta_1(a_1)$ *and* $C_5 = C_5(a_1,\delta_1) > 1$ *such that if* $N_s(T) \leq a_1$ *and* $\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p} \leq \delta_1$, *then the following decay estimate holds for* $t \in [0,T]$.

$$(3.21) \qquad \|(u-\bar{u},\ v-\bar{v})(t)\|_{s-1} \le C_5(1+t)^{-\gamma} \|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p}$$

where $\gamma = n(1/2p - 1/4)$.

<u>Corollary 3.4</u> *Assume the conditions of Proposition* 3.3 *with* p *satisfying* $1 \le p < 2n/(n+2)$. *Then the following energy estimate holds for* $t \in [0,T]$.

$$(3.22) \qquad \|(u-\bar{u},\ v-\bar{v})(t)\|_{s-1}^2 + \int_0^t \|(u-\bar{u},\ v-\bar{v})(\tau)\|_{s-1}^2 d\tau$$

$$\le C\,\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p}^2 ,$$

where $C = C(a_1,\delta_1) > 1$ *is a constant.*

<u>Proof of Proposition 3.3</u> Applying (3.A.14) (with $0 \le \ell \le s-1$) to (3.19), we obtain the inequality

$$(3.23) \qquad \|U(t)\|_{s-1} \le C(1+t)^{-\gamma} \|U_0\|_{s-1,p} +$$

$$+ C\int_0^t e^{-c_2(t-\tau)} \|h(\tau)\|_{s-1} + (1+t-\tau)^{-n/4} \|h(\tau)\|_{L^1} d\tau$$

with $\gamma = n(1/2p - 1/4)$ and a constant C. From (3.17) and $(3.13)_2$ we see that for $|u-\bar{u},\ v-\bar{v}| + |D_x(u,v)| \to 0$,

$$(3.24) \qquad h(u,v,D_xu,D_xv,D_x^2v) = O(\{|u-\bar{u},\ v-\bar{v}| + |D_x(u,v)|\}^2 +$$

$$+ |u-\bar{u},\ v-\bar{v}|\,|D_x^2v|)$$

Therefore if $N_s(T) \le a_0$ with $s \ge s_0 + 2$, then

$$(3.25)_1 \qquad \|h\|_{s-1} \le C\,\|U\|_{s-1}(\,\|U\|_s + \|D_x v\|_s)\,,$$

$$(3.25)_2 \qquad \|h\|_{L^1} \le C\,\|U\|_2^2$$

hold for some constant $C = C(a_0)$. Substitution of $(3.25)_{1,2}$ to (3.23) yields

$$\|U(t)\|_{s-1} \le C(1+t)^{-\gamma}\,\|U_0\|_{s-1,p} +$$

$$+ C \sup_{0\le\tau\le t} \|U(\tau)\|_s \int_0^t e^{-c_2(t-\tau)} \|U(\tau)\|_{s-1}\, d\tau +$$

$$+ C\Big(\int_0^t \|D_x v(\tau)\|_s^2\, d\tau\Big)^{1/2}\Big(\int_0^t e^{-2c_2(t-\tau)} \|U(\tau)\|_{s-1}^2\, d\tau\Big)^{1/2} +$$

$$+ C\int_0^t (1+t-\tau)^{-n/4}\,\|U(\tau)\|_{s-1}^2\, d\tau\,.$$

Put $|||U(t)|||_{s-1,\gamma} = \sup_{0\le\tau\le t} (1+\tau)^{\gamma}\,\|U(\tau)\|_{s-1}$. Then it follows that

$$(3.26) \qquad |||U(t)|||_{s-1,\gamma} \le C\,\|U_0\|_{s-1,p} +$$

$$+ C\mu_1(t) N_s(t)\,|||U(t)|||_{s-1,\gamma} + C\mu_2(t)\,|||U(t)|||_{s-1,\gamma}^2\,,$$

where

$$\mu_1(t) = \sup_{0\le\tau\le t} (1+\tau)^{\gamma} \int_0^{\tau} e^{-c_2(\tau-\tau_1)} (1+\tau_1)^{-\gamma}\, d\tau_1 +$$

$$+ \sup_{0\le\tau\le t} (1+\tau)^{\gamma}\{\int_0^{\tau} e^{-2c_2(\tau-\tau_1)}(1+\tau_1)^{-2\gamma}\,d\tau_1\}^{1/2} ,$$

$$\mu_2(t) = \sup_{0\le\tau\le t} (1+\tau)^{\gamma}\int_0^{\tau}(1+\tau-\tau_1)^{-n/4}(1+\tau_1)^{-2\gamma}\,d\tau_1 .$$

Here we note that $\mu_1(t)$ and $\mu_2(t)$ are majorized by a constant independent of $t \in [0,\infty)$ for any n and p indicated in Proposition 3.3. Therefore we can deduce from (3.26) that there are positive constants a_1 $(\le a_0)$ and $\delta_1 = \delta_1(a_1)$ such that if $N_s(T) \le a_1$ and $\|U_0\|_{s-1,p} \le \delta_1$, then $|||U(t)|||_{s-1,\gamma} \le C\,\|U_0\|_{s-1,p}$ holds with some constnat $C = C(a_1,\delta_1)$. This implies (3.21). Thus the proof of Proposition 3.3 is completed.

We next prove Corollary 3.4. Since $\gamma = n(1/2p-1/4) > 1/2$ for $n \ge 3$ and $1 \le p < 2n/(n+2)$, the function $(1+t)^{-\gamma}$ is square integrable with respect to $t \in [0,\infty)$. Therefore (3.22) is proved as a direct consequence of (3.21). This completes the proof.

Finally in this section we make a summary of the a priori estimates of small solutions to the initial value problem (2.1),(2.2). Let us combine the estimates in Lemmas 3.1 and 3.2 and Corollary 3.4 so as to make (3.8) + (3.15) $\times\alpha$ + (3.22) with a positive constant α satisfying $\alpha C < 1$. Then we obtain the inequality for $N_s(T)$:

$$N_s(T)^2 + \int_0^T \|(u-\bar{u},\ v-\bar{v})(t)\|^2\,dt$$

$$\le C\{\|u_0-\bar{u},\ v_0-\bar{v}\|_{s,p}^2 + N_s(T)^3\} ,$$

from which we can deduce that both $N_s(T)$ and the $L^2(0,T\,;L^2(\mathbb{R}^n))$-norm of $(u-\bar{u}, v-\bar{v})$ are bounded by $C\,\|u_0-\bar{u},\ v_0-\bar{v}\|_{s,p}$ if $N_s(T)$ is suitably small. Thus we have proved:

Proposition 3.5 (a priori estimate) *Assume Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \geq 3$ *and* $s \geq s_0 + 2$ $(s_0 = [n/2]+1)$ *be integers, and let* $1 \leq p < 2n/(n+2)$. *Suppose that* $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. *Let* $(u,v)(t,x)$ *be a solution of* (2.1), (2.2) *with* (3.4). *Then there are positive constants* a_2 $(\leq a_1)$ *and* $C_6 = C_6(a_2,\delta_1) > 1$ $(\delta_1$ *is the constant in Proposition* 3.3) *such that if* $N_s(T) \leq a_2$ *and* $\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p} \leq \delta_1$, *then the following a priori estimate holds for any* $t \in [0,T]$.

$$(3.27) \qquad N_s(t)^2 + \int_0^t \|(u-\bar{u},\ v-\bar{v})(\tau)\|^2 d\tau \leq C_6^2\,\|u_0-\bar{u},\ v_0-\bar{v}\|_{s,p}^2 \,.$$

Remark 3.1 (i) In special cases $m' = 0$ and $m'' = 0$, Proposition 3.3 and Corollary 3.4 (and therefore Proposition 3.5) are valid for $s \geq s_0 + 1$ because the estimate $(3.25)_1$ for h is replaced by

$$\|h\|_{s-1} \leq C\,\|v\|_{s-1}\,\|D_x v\|_s \qquad \text{for } m' = 0\,,$$

$$\|h\|_{s-1} \leq C\,\|u\|_{s-1}\,\|u\|_s \qquad \text{for } m'' = 0\,,$$

where $N_s(T) \leq a_0$ (with $s \geq s_0+1$) is assumed.

(ii) The a priori estimate (3.27) has been derived by a combination of the energy inequalities for quasilinear equations and the decay estimates for linearized equations (with constant coefficients). This method was previously employed by Matsumura and Nishida $[55]_2$ for the equations of compressible viscous fluids in $\mathbb{R}^3$.

3.4 GLOBAL EXISTENCE

Based on the a priori estimates (Propositions 3.5 and 3.3) and the local existence result (Theorem 2.9), we can conclude the global existence and asymptotic stability of a solution to the problem (2.1),(2.2).

Theorem 3.6 (global existence and asymptotic decay) *Assume Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \ge 3$ *and* $s \ge s_0 + 2$ *(*$s_0 = [n/2]+1$*) be integers, and let* $1 \le p < 2n/(n+1)$. *Suppose that* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and define* $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_{\ell,p}$ *(with* $\ell \le s$*) by* (3.20). *Then there exists a positive constant* δ_2 *(*$\le \delta_1, a_2$*) such that if* $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s,p} \le \delta_2$, *then the initial value problem* (2.1),(2.2) *has a unique global solution* $(u,v)(t,x)$ *with*

$$u - \bar{u} \in C^0(0,\infty; H^s(\mathbb{R}^n)) \cap C^1(0,\infty; H^{s-1}(\mathbb{R}^n)) \cap L^2(0,\infty; H^s(\mathbb{R}^n)),$$

$$v - \bar{v} \in C^0(0,\infty; H^s(\mathbb{R}^n)) \cap C^1(0,\infty; H^{s-2}(\mathbb{R}^n)) \cap L^2(0,\infty; H^{s+1}(\mathbb{R}^n)).$$

The solution satisfies the following estimates for $t \in [0,\infty)$.

$$\|(u - \bar{u},\ v - \bar{v})(t)\|_s^2 + \int_0^t \|(u - \bar{u})(\tau)\|_s^2 + \|(v - \bar{v})(\tau)\|_{s+1}^2 \, d\tau \tag{3.28}$$

$$\leq C_6^2 \, \|u_0 - \bar{u}, \; v_0 - \bar{v}\|_{s,p}^2 \, ,$$

(3.29) $$\|(u - \bar{u}, \; v - \bar{v})(t)\|_{s-1} \leq C_5(1+t)^{-\gamma} \, \|u_0 - \bar{u}, \; v_0 - \bar{v}\|_{s-1,p} \, ,$$

where $\gamma = n(1/2p - 1/4)$ *;* C_6 *and* C_5 *are constants in* (3.27) *and* (3.21)*, respectively.*

Remark 3.2 In special cases $m' = 0$ and $m'' = 0$, the above results also hold for $s \geq s_0 + 1$, see Remark 3.1 (i).

Proof of Theorem 3.6 Take δ_2 so that

$$\delta_2 = \min\{\delta_1, \; a_2/C_4, \; a_2/C_6(1+C_4^2)^{1/2}\} \, .$$

Then the solution of (2.1),(2.2) can be continued globally in time provided the smallness condition $\|u_0 - \bar{u}, \; v_0 - \bar{v}\|_{s,p} \leq \delta_2$ is satisfied. In fact we have $\|u_0 - \bar{u}, \; v_0 - \bar{v}\|_s \leq \delta_2 \leq a_2 \leq a_0$. Therefore, by Theorem 2.9, there are constants $T_1 = T_1(a_0) > 0$ and $C_4 = C_4(a_0) > 1$ such that a solution exists on $[0,T_1]$ and satisfies $N_s(T_1) \leq C_4 \, \|u_0 - \bar{u}, \; v_0 - \bar{v}\|_s$. Since $N_s(T_1) \leq C_4\delta_2 \leq a_2$ $(\leq a_1)$ and $\|u_0 - \bar{u}, \; v_0 - \bar{v}\|_{s-1,p} \leq \delta_2 \leq \delta_1$ by the definition of δ_2, Propositions 3.5 and 3.3 give the estimates (3.27) and (3.21) for $t \in [0,T_1]$.

Noting that $\|(u - \bar{u}, \; v - \bar{v})(T_1)\|_s \leq N_s(T_1) \leq a_2 \leq a_0$, we apply Theorem 2.9 by taking $t = T_1$ as the new initial time. Then we have a solution on $[T_1, 2T_1]$ with a estimate $N_s(T_1, 2T_1) \leq C_4 \, \|(u - \bar{u}, \; v - \bar{v})(T_1)\|_s$. By the estimate (3.27) (for $t \in [0,T_1]$), we have

$$N_s(2T_1) \le \{N_s(T_1)^2 + N_s(T_1,2T_1)^2\}^{1/2}$$

$$\le (1+C_4^2)^{1/2} N_s(T_1) \le C_6(1+C_4^2)^{1/2} \|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s,p} ,$$

from which we conclude that $N_s(2T_1) \le C_6(1+C_4^2)^{1/2}\delta_2 \le a_2$. On the other hand the condition $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s-1,p} \le \delta_1$ was already checked. Therefore Propositions 3.5 and 3.3 again give the estimates (3.27) and (3.21) for $t \in [0,2T_1]$. In the same way we can extend the solution to the interval $[0,nT_1]$ successively $n = 1,2,\cdots\cdots$, and get a global solution. This completes the proof of Theorem 3.6.

3.5 SOME FURTHER REMARKS

In this section we shall treat the case when the remainder term $\tilde{f}(u,v,D_x u,D_x v) = {}^t(\tilde{f}_1(u,v,D_x v),\ \tilde{f}_2(u,v,D_x u,D_x v))$ ($\tilde{f}_1$ and $\tilde{f}_2$ are defined by $(3.9)_{1,2}$) satisfies the additional condition

$$(3.30) \qquad (I - P^+)\tilde{f}(u,v,D_x u,D_x v) = O(|u - \bar{u},\ v - \bar{v}|^3 + $$
$$+ |u - \bar{u},\ v - \bar{v}|\,|D_x(u,v)| + |D_x(u,v)|^2)$$

for $|u - \bar{u},\ v - \bar{v}| + |D_x(u,v)| \to 0$, where P^+ is the orthogonal projection onto the range of $L(\bar{u},\bar{v})$. Note that $(I - P^+)\tilde{f}$ does not contain the quadratic term $|u - \bar{u},\ v - \bar{v}|^2$ (compare (3.30) with (3.10)).

For general $\tilde{f}(u,v,D_x u,D_x v)$ the existence of a global solution has been proved for the initial data near the constant state in $H^s(\mathbb{R}^n) \cap$

$L^p(\mathbb{R}^n)$, where $n \geq 3$, $s \geq s_0 + 2$ ($s_0 = [n/2]+1$) and $1 \leq p < 2n/(n+2)$ (see Theorem 3.6). In the case (3.30), we will show a similar existence result for a class of initial data near the constant state in $H^s(\mathbb{R}^n)$ with $n \geq 2$ and $s \geq s_0 + 1$.

We start out to improve the a priori estimates in Lemmas 3.1 and 3.2.

Lemma 3.7 *Let* $n \geq 2$ *be an integer and assume the same conditions as in Lemma* 3.1. *We further assume the additional condition* (3.30). *Then the following a priori estimate holds for* $t \in [0,T]$.

$$\| (u-\bar{u},\ v-\bar{v})(t)\|_s^2 + \int_0^t \|D_x v(\tau)\|_s^2 + \|P^+(u-\bar{u}, v-\bar{v})(\tau)\|_s^2 d\tau \tag{3.31}$$

$$\leq C\{\|u_0-\bar{u},\ v_0-\bar{v}\|_s^2 + N_s(T)^3\} .$$

Proof. We put $U = {}^t(u-\bar{u}, v-\bar{v})$ and use the equation (3.16). It follows from (3.17) and (3.13)$_2$ that the right member h is expressed as

$$h = A^0(\bar{u},\bar{v})A^0(u,v)^{-1}\tilde{f} - A^0(\bar{u},\bar{v})\{A^0(u,v)^{-1} - A^0(\bar{u},\bar{v})^{-1}\}\cdot \tag{3.32}$$

$$\cdot L(\bar{u},\bar{v})U + O(|U|\{|D_xU| + |D_x^2 v|\})$$

for $|U| + |D_xU| \to 0$. The first term in the right hand side of (3.32) is of the form $\tilde{f} + O(|U||\tilde{f}|) = \tilde{f} + O(|U|\{|U| + |D_xU|\}^2)$, where (3.10) was used. While the second term is dominated by $O(|U||P^+U|)$. Therefore we get

$$h = \tilde{f} + O(|U||P^+U| + |U|^3 + |U|\{|D_xU| + |D_x^2 v|\}) \tag{3.32$'$}$$

for $|U| + |D_x U| \to 0$. Now we take the inner product of (3.16) with U and integrate it over Q_t. By the same arguments as in Lemma 3.1 we have

$$(3.33)\qquad \|(u-\bar{u},\ v-\bar{v})(t)\|^2 + \int_0^t \|D_x v(\tau)\|^2 + \|P^+(u-\bar{u},v-\bar{v})(\tau)\|^2 d\tau$$

$$\le C\{\|u_0-\bar{u},\ v_0-\bar{v}\|^2 + \int_0^t R_3(\tau)\,d\tau\}$$

with some constant C, where

$$R_3(t) = \int |\tilde{f}||P^+U| + |(I-P^+)\tilde{f}||U| + |h-\tilde{f}||U|\,dx\ .$$

It follows from (3.10), (3.30) and (3.32)' that for $|U| + |D_x U| \to 0$, the integrand in $R_3(t)$ is dominated by

$$O(|U|^2|P^+U| + |U|^4 + |U|^2\{|D_x U| + |D_x^2 v|\} + |U||D_x U|^2)\ .$$

Therefore if $N_s(T) \le a_0$, we have

$$(3.34)\qquad R_3(t) \le C\{\|P^+U\|\|U\|^2_{L^4} + \|U\|^4_{L^4} + \|U\|^2_{L^4}(\|D_x U\| + \|D_x^2 v\|) +$$

$$+ \|U\|_{L^\infty}\|D_x U\|^2\} \le \varepsilon\|P^+U\|^2 + C_\varepsilon\|U\|_{s_0}\|D_x U\|^2_{s_0-1}$$

for any $\varepsilon > 0$ and a positive constant $C_\varepsilon = C_\varepsilon(a_0)$, where we have used the estimate (2.4) for $n \ge 2$ (with $p=4$ and $s=s_0-1$; $p=\infty$ and $s=s_0$). Choose ε so small that $\varepsilon C < 1$. Then (3.33) together with (3.34) yields

$$(3.35)\qquad \|(u-\bar{u},\ v-\bar{v})(t)\|^2 + \int_0^t \|D_x v(\tau)\|^2 + \|P^+(u-\bar{u},v-\bar{v})(\tau)\|^2 d\tau$$

$$\leq C\{ \|u_0 - \bar{u},\ v_0 - \bar{v}\|^2 + N_{s_0}(T)^3 \} .$$

The desired estimate (3.31) is a consequence of (3.35) and (3.8). This completes the proof of Lemma 3.7.

Lemma 3.8 *Let* $n \geq 2$ *be an integer and assume the same conditions as in Lemma* 3.2. *Then the following a priori estimate holds for* $t \in [0,T]$.

$$(3.36) \qquad \int_0^t \|D_x u(\tau)\|_{s-1}^2 d\tau - C\{ \|(u-\bar{u}, v-\bar{v})(t)\|_s^2 + \int_0^t \|D_x v(\tau)\|_s^2 + \|P^+(u-\bar{u}, v-\bar{v})(\tau)\|_s^2 d\tau \} \leq C\{ \|u_0 - \bar{u}, v_0 - \bar{v}\|_s^2 + N_s(T)^3 \} .$$

Proof. Multiply (3.16) by K^j and take the inner product of the resulting equation by U_{x_j}. Integrate it over Q_t and add for $j = 1, \cdots, n$. The arguments in Lemma 3.2 give the estimate

$$(3.37) \qquad \int_0^t \|D_x(u,v)(\tau)\|^2 d\tau - C\{ \|(u-\bar{u}, v-\bar{v})(t)\|_1^2 + \int_0^t \|D_x v(\tau)\|_1^2 + \|P^+(u-\bar{u}, v-\bar{v})(\tau)\|_1^2 d\tau \} \leq C\{ \|u_0 - \bar{u}, v_0 - \bar{v}\|_1^2 + \int_0^t R_4(\tau)\, d\tau \} ,$$

where

$$R_4(t) = \int |h(u,v,D_x u, D_x v, D_x^2 v)|\,|D_x(u,v)|\, dx .$$

Taking (3.24) into account, we have in the same way as in (3.34)

(3.38) $$R_4(t) \le C\,\|u-\bar{u},\ v-\bar{v}\|_{s_0}\,\|D_x(u,v)\|^2_{s_0-1}\,,$$

provided that $n \ge 2$ and $N_s(T) \le a_0$ are assumed. Here we again used the estimate (2.4) (with $p=4$ and $s=s_0-1$). Substituting (3.38) into (3.37), we obtain the estimate (3.36) for $s=1$, with $N_1(T)$ replaced by $N_{s_0}(T)$. This estimate together with (3.15) implies the desired estimate (3.36). This completes the proof of Lemma 3.8.

A combination of Lemmas 3.7 and 3.8 completes the a priori estimates of small solutions to the problem (2.1),(2.2) when the additional condition (3.30) is satisfied. Indeed, we combine Lemmas 3.7 and 3.8 so as to make (3.31) + (3.36) $\times\alpha$ with a positive constant α satisfying $\alpha C < 1$. Then we obtain

$$N_s(T)^2 + \int_0^T \|P^+(u-\bar{u},\ v-\bar{v})(t)\|^2\,dt$$

$$\le C\{\|u_0-\bar{u},\ v_0-\bar{v}\|^2_s + N_s(T)^3\}\,,$$

where $C=C(a_0)$ is a constant and $N_s(T) \le a_0$ is assumed. From this inequality we can deduce:

<u>Proposition 3.9</u> (a priori estimate) *Let Conditions* 2.1, 2.2, 3.1 *and* 3.2 *as well as* (3.30) *be assumed. Let* $n \ge 2$ *and* $s \ge s_0+1$ *(*$s_0=[n/2]+1$*) be integers. Suppose that* $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n)$, *and* $(u,v)(t,x)$ *is a solution of* (2.1),(2.2) *satisfying* (3.4). *Then there exist positive constants* a_3 *(*$\le a_0$*) and* $C_7 = C_7(a_3) > 1$ *such that if* $N_s(T) \le a_3$,

then the following a priori estimate holds for $t \in [0,T]$.

$$(3.39) \qquad N_s(t)^2 + \int_0^t \|P^+(u-\bar{u},\ v-\bar{v})(\tau)\|^2 d\tau \le C_7^2 \|u_0-\bar{u},\ v_0-\bar{v}\|_s^2 .$$

Combining Theorem 2.9 with Proposition 3.9, we can conclude the existence of a global solution to (2.1),(2.2).

<u>Theorem 3.10</u> (global existence) *Assume* (3.30) *in addition to Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \ge 2$ *and* $s \ge s_0 + 1$ $(s_0 = [n/2]+1)$ *be integers and suppose that* $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n)$. *Then there exists a positive constant* δ_3 $(\le a_3)$ *such that if* $\|u_0-\bar{u},\ v_0-\bar{v}\|_s \le \delta_3$, *then the problem* (2.1), (2.2) *has a unique global solution* $(u,v)(t,x)$ *satisfying* (3.4) *with* $T = \infty$. *The solution satisfies the estimate*

$$(3.40) \qquad \|(u-\bar{u},\ v-\bar{v})(t)\|_s^2 + \int_0^t \|D_x u(\tau)\|_{s-1}^2 + \|D_x v(\tau)\|_s^2 +$$

$$+ \|P^+(u-\bar{u},\ v-\bar{v})(\tau)\|^2 d\tau \le C_7^2 \|u_0-\bar{u},\ v_0-\bar{v}\|_s^2$$

for $t \in [0,\infty)$, *where* C_7 *is the constant in* (3.39). *Furthermore the solution decays to the constant state* $(\bar{u},\bar{v})$ *(uniformly in* $x \in \mathbb{R}^n$*) as* $t \to \infty$:

$$(3.41) \qquad |(u-\bar{u},\ v-\bar{v})(t)|_{s-(s_0+1)} \to 0 \qquad \text{as} \quad t \to \infty .$$

Proof. Take δ_3 so that $\delta_3 = \min\{a_3/C_4,\ a_3/C_7(1+C_4^2)^{1/2}\}$. Then the existence of a global solution can be proved in the same way as in Theorem 3.6. So we omit it. We only prove the decay law (3.41). Put $\Phi(t) = \|D_x(u,v)(t)\|_{s-2}^2$. Then it follows from (3.40) and (2.1) that

$$\int_0^\infty |\Phi(t)|\,dt + \int_0^\infty |\partial_t\Phi(t)|\,dt \le C\,\|u_0-\bar{u},\ v_0-\bar{v}\|_s^2$$

with some constant C. From this estimate we can deduce that $\Phi(t) = \|D_x(u,v)(t)\|_{s-2}^2 \to 0$ as $t \to \infty$. This and the inequality

$$|u|_{s-(s_0+1)} \le C\,\|D_x u\|_{s-2}^a\,\|u\|_{s-(s_0+1)}^{1-a} \quad \text{with} \quad a = n/2s_0$$

(which follows from (2.4)) give the decay result in (3.41). Thus the proof is completed.

It is easy to get the decay rate of the solution (constructed in Theorem 3.10) when the initial data satisfy $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Theorem 3.11 (asymptotic decay) *Assume* (3.30) *in addition to Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \ge 3$ *and* $s \ge s_0 + 2$ $(s_0 = [n/2]+1)$ *be integers, and let* $p \in [1,3/2]$ *for* $n = 3$ *and* $p \in [1,2)$ *for* $n \ge 4$. *Suppose that* $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and define* $\|u_0-\bar{u},\ v_0-\bar{v}\|_{\ell,p}$ *(with* $\ell \le s$*) by* (3.20). *Then there exists a positive constant* δ_4 $(\le \delta_1, \delta_3)$ *such that if* $\|u_0-\bar{u},\ v_0-\bar{v}\|_{s,p} \le \delta_4$, *the solution constructed in Theorem* 3.10 *satisfies the decay estimate* (3.21) *for* $t \in [0,\infty)$:

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_{s-1} \le C_5(1+t)^{-\gamma}\,\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p}\,,$$

where $\gamma = n(1/2p - 1/4)$.

Remark 3.3 In special cases $m' = 0$ and $m'' = 0$, the above decay results also hold for $s \geq s_0 + 1$, see Remark 3.1 (i).

Proof of Theorem 3.11 Take δ_4 so that $\delta_4 = \min\{\delta_1, \delta_3, a_1/C_7\}$. Then, by the estimate (3.40), the solution satisfies $N_s(t) \leq C_7 \|u_0 - \bar{u}, v_0 - \bar{v}\|_s \leq C_7\delta_4 \leq a_1$ for $t \in [0,\infty)$. On the other hand the condition $\|u_0 - \bar{u}, v_0 - \bar{v}\|_{s-1,p} \leq \delta_4 \leq \delta_1$ is obvious. Therefore Proposition 3.3 proves the assertion of Theorem 3.11. This completes the proof.

Finally in this section we shall make a slight modification of Proposition 3.3 (and consequently of Theorem 3.11), when the nonlinear term $h(u,v,D_xu,D_xv,D_x^2v) = {}^t(h_1(u,v,D_xu,D_xv), h_2(u,v,D_xu,D_xv,D_x^2v))$ (h_1 and h_2 are defined by (3.17) and $(3,13)_2$ respectively) satisfies the additional condition

$$(3.42) \qquad (I - P^+)h(u,v,D_xu,D_xv,D_x^2v) = 0$$

for any $(u,v,D_xu,D_xv,D_x^2v) \in \mathcal{O} \times \mathbb{R}^{nm} \times \mathbb{R}^{n^2m''}$.

Proposition 3.12 (a priori decay estimate) *Assume Conditions* 2.1, 2.2, 3.1 *and* 3.2 *as well as* (3.42). *Let* $n \geq 1$ *and* $s \geq s_0 + 2$ ($s_0 = [n/2] + 1$) *be integers, and let* $p = 1$ *for* $n = 1$, $p \in [1,2)$ *for* $n = 2$ *and* $p \in [1,2]$ *for* $n \geq 3$. *Suppose that* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, *and* $(u,v)(t,x)$ *is a solution of* (2.1), (2.2) *satisfying* (3.4).

Then there exist positive constants a_1' $(\le a_0)$, $\delta_1' = \delta_1'(a_1')$ *and* $C_5' = C_5'(a_1', \delta_1') > 1$ *such that if* $N_s(T) \le a_1'$ *and* $\|u_0 - \bar{u}, v_0 - \bar{v}\|_{s-2,p} \le \delta_1'$, *then the following decay estimate holds for* $t \in [0,T]$.

$$(3.43) \qquad \|(u - \bar{u},\ v - \bar{v})(t)\|_{s-2} \le C_5'(1+t)^{-\gamma} \|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s-2,p},$$

where $\gamma = n(1/2p - 1/4)$.

Proof. Let $U = {}^t(u - \bar{u}, v - \bar{v})$. Noting (3.24), we have

$$(3.44) \qquad \|h\|_{s-2} \le C \|U\|_{s-2} \|U\|_s$$

and $(3.25)_2$, where $N_s(T) \le a_0$ $(s \ge s_0 + 2)$ is assumed. Let $n = 1$. Applying (3.A.16) (with $0 \le \ell \le s-2$ and $q = 1$) to the solution of (3.16) and using (3.44) and $(3.25)_2$, we obtain as a counterpart of (3.26):

$$(3.45) \qquad |||U(t)|||_{s-2,\gamma} \le C \|U_0\|_{s-2,p} + $$

$$+ C\mu_3(t) N_s(t) |||U(t)|||_{s-2,\gamma} + C\mu_4'(t) |||U(t)|||^2_{s-2,\gamma},$$

where $\gamma = 1/2p - 1/4$ and

$$\mu_3(t) = \sup_{0 \le \tau \le t} (1+\tau)^{\gamma} \int_0^{\tau} e^{-c_2(\tau - \tau_1)} (1+\tau_1)^{-\gamma} d\tau_1,$$

$$\mu_4'(t) = \sup_{0 \le \tau \le t} (1+\tau)^{\gamma} \int_0^{\tau} (1+\tau-\tau_1)^{-3/4} (1+\tau_1)^{-2\gamma} d\tau_1.$$

It is easy to see that $\mu_3(t)$ and $\mu_4'(t)$ are bounded by a constant independent of $t \in [0,\infty)$ for $p = 1$ (note that $\gamma = 1/4$ for $p=1$). Therefore we deduce from (3.45) that $|||U(t)|||_{s-2,1/4} \le C\,\|U_0\|_{s-2,1}$ if $N_s(T)$ and $\|U_0\|_{s-2,1}$ are sufficiently small. Thus the proof for $n = 1$ is completed.

Let $n \ge 2$. Apply (3.A.12) (with $0 \le \ell \le s-2$ and $q=1$) to the solution of (3.16). Following the above arguments we obtain

$$(3.46)\qquad |||U(t)|||^2_{s-2,\gamma} \le C\,\|U_0\|^2_{s-2,p} + \\ + C\mu_5(t)N_s(t)^2|||U(t)|||^2_{s-2,\gamma} + C\mu_6(t)\,|||U(t)|||^4_{s-2,\gamma}\,,$$

where $\gamma = n(1/2p - 1/4)$ and

$$\mu_5(t) = \sup_{0\le\tau\le t}(1+\tau)^{2\gamma}\int_0^\tau e^{-c_1(\tau-\tau_1)}(1+\tau_1)^{-2\gamma}\,d\tau_1\,,$$

$$\mu_6(t) = \sup_{0\le\tau\le t}(1+\tau)^{2\gamma}\int_0^\tau (1+\tau-\tau_1)^{-n/2}(1+\tau_1)^{-4\gamma}\,d\tau_1\,.$$

It is easy to see that $\mu_5(t)$ and $\mu_6(t)$ are uniformly bounded with respect to $t \in [0,\infty)$ for $n \ge 2$ and $p \in [1,2)$. Therefore the desired estimate (3,43) for $n \ge 2$ follows from (3.46) in the same way as in the proof for $n = 1$. This completes the proof of Proposition 3.12.

Proposition 3.12 gives the following modification of Theorem 3.11.

Theorem 3.13 (asymptotic decay) *Assume* (3.30) *and* (3.42) *in addition to Conditions* 2.1, 2.2, 3.1 *and* 3.2. *Let* $n \geq 2$ *and* $s \geq s_0 + 2$ *(*$s_0 = [n/2]+1$*) be integers, and let* $p \in [1,2)$. *Suppose that* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. *Then there exists a positive constant* δ_4' *(*$\leq \delta_1', \delta_3$*) such that if* $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s,p} \leq \delta_4'$, *the solution of Theorem* 3.10 *satisfies the decay estimate* (3.43) *for* $t \in [0,\infty)$*:*

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_{s-2} \leq C_5'(1+t)^{-\gamma}\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-2,p},$$

where $\gamma = n(1/2p - 1/4)$.

Remark 3.4 For symmetric hyperbolic systems $(m'' = 0)$, Proposition 3.12 is valid for $s \geq s_0 + 1$ and the estimate (3.43) is replaced by

$$(3.43)' \qquad \|(u-\bar{u})(t)\|_{s-1} \leq C_5'(1+t)^{-\gamma}\|u_0-\bar{u}\|_{s-1,p},$$

if the condition $\|u_0 - \bar{u},\ v_0 - \bar{v}\|_{s-2,p} \leq \delta_1'$ is replaced by $\|u_0 - \bar{u}\|_{s-1,p} \leq \delta_1'$. Therefore, in this case, the results in Theorem 3.13 also hold for $s \geq s_0 + 1$ and the decay estimate is improved as in (3.43)'.

Proof of Theorem 3.13 Take δ_4' so that $\delta_4' = \min\{\delta_1', \delta_3, a_1/C_7\}$. Then the theorem is proved in the same way as in Theorem 3.11. The details are omitted.

3.6 EXAMPLE

As an application of our results, we treat here the equations of heat conduction in an anisotropic rigid body of constant density at rest (cf. [56],[84],[8],[51], and also [28]):

$$(3.47) \qquad \begin{cases} e_t = -\operatorname{div} q \ , \\ \\ \tau q_t + q = -\kappa \nabla \theta \ , \end{cases}$$

where the unknown functions e and q represent the (real-valued) internal energy and the ($\mathbb{R}^n$-valued) heat flux respectively, which are the functions of time $t \geq 0$ and position $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$; $\theta = \theta(e)$, the given function of e, is the absolute temperature; $\tau = \tau(e,q)$ and $\kappa = \kappa(e,q)$, the given functions of (e,q), are the (scalar) relaxation time and the heat conductivity tensor, respectively. The first equation of (3.47) describes the energy balance, where the external heat supply is neglected; while the second one of (3.47) is the generalized Fourier's law.

We assume the following conditions on the system (3.47).

$(3.48)_1$ The function $\theta(e)$ is smooth in $e \in \mathbb{R}$ and satisfies $\theta(e) > 0$ and $d\theta(e)/de > 0$ for $e \in \mathbb{R}$.

$(3.48)_2$ The functions $\tau(e,q)$ and $\kappa(e,q)$ are smooth in $(e,q) \in \mathbb{R}^{n+1}$; $\tau(e,q) > 0$ holds for $(e,q) \in \mathbb{R}^{n+1}$, while $\kappa(e,q)$ is real symmetric and positive definite for every fixed

$(e,q) \in \mathbb{R}^{n+1}$.

Let us rewrite (3.47) in the form

$$(3.49) \qquad A^0(e,q)\begin{pmatrix} e \\ q \end{pmatrix}_t + \sum_{j=1}^{n} A^j \begin{pmatrix} e \\ q \end{pmatrix}_{x_j} + L(e,q)\begin{pmatrix} e \\ q \end{pmatrix} = 0 ,$$

where

$$A^0(e,q) = \begin{pmatrix} 1 & 0 \\ 0 & \tau c_V \kappa^{-1} \end{pmatrix}, \qquad \sum_j A^j \omega_j = \begin{pmatrix} 0 & \omega \\ {}^t\omega & 0 \end{pmatrix},$$

$$L(e,q) = \begin{pmatrix} 0 & 0 \\ 0 & c_V \kappa^{-1} \end{pmatrix},$$

$c_V = (d\theta/de)^{-1}$ is the heat capacity at constant volume, and $\omega = (\omega_1,\cdots,\omega_n) \in S^{n-1}$. We can show that if $(3.48)_{1,2}$ are assumed, then Conditions 2.1, 2.2, 3.1 and 3.2 as well as the conditions (3.30) and (3.42) are satisfied for the system (3.49). Indeed, it is easily seen that $A^0(e,q)$ is real symmetric and positive definite, A^j $(j=1,\cdots,n)$ are real symmetric, and $L(e,q)$ is real symmetric and positive semi-definite. Therefore Conditions 2.1, 2.2 and 3.1 are verified with $m'' = 0$, $\mathcal{O} = \{(e,q);\ e \in \mathbb{R},\ q \in \mathbb{R}^n\}$ and with a constant state $(e,q) = (\bar{e}, 0)$, where $\bar{e} \in \mathbb{R}$ is an arbitrarily fixed constant. To check Condition 3.2, we define the matrices K^j $(j=1,\cdots,n)$ by

$$(3.50)\qquad \sum_j K^j\omega_j = \alpha \begin{pmatrix} 0 & \omega \\ -{}^t\omega & 0 \end{pmatrix} A^0(\bar{e},0)^{-1} ,$$

where α is a positive constant determined below. For the above K^j, it is obvious that $K^jA^0(\bar{e},0)$ $(j=1,\cdots,n)$ are real anti-symmetric. Furthermore a simple calculation shows that

$$\sum_{jk} K^jA^k\omega_j\omega_k + L(\bar{e},0) = \begin{pmatrix} \alpha < \omega\Lambda_1, \omega > & 0 \\ 0 & \Lambda_2 - \alpha\,{}^t\omega\omega \end{pmatrix} ,$$

where $\Lambda_1 = (\tau^{-1}c_V^{-1}\kappa)(\bar{e},0)$ and $\Lambda_2 = (c_V\kappa^{-1})(\bar{e},0)$ are real symmetric and positive definite; $< , >$ denotes the standard inner product in $\mathbb{R}^n$ and ${}^t\omega\omega$ the symmetric matrix with elements $\omega_i\omega_j$. Therefore $\sum_{jk} K^jA^k\omega_j\omega_k + L(\bar{e},0)$ is proved to be positive definite for any $\omega \in S^{n-1}$ if $\alpha > 0$ is suitably small. Thus Condition 3.2 is verified. Since the projection P^+ onto the range of $L(\bar{e},0)$ is given by

$$P^+ = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} ,$$

we can verify $(I - P^+)\tilde{f} = (I - P^+)h = 0$ which implies (3.30) and (3.42).

Summarizing the above considerations, we have:

<u>Lemma 3.14</u> *Let* $(3.48)_{1,2}$ *be assumed. Then the system* (3.47) *satisfies Conditions* 2.1, 2.2, 3.1 *and* 3.2 *as well as* (3.30), (3.40), *with* $m'' = 0$, $O = \{(e,q) \in \mathbb{R}^{n+1}\}$ *and a constant state* $(e,q) = (\bar{e},0)$. *In particular,*

the matrices K^j *(*$j=1,\cdots,n$*) in Condition* 3.2 *are taken as in* (3.50) *with a suitably small constant* $\alpha > 0$.

By this lemma we can apply all the theorems (in particular, Theorems 2.9, 3.10, 3.13 and also Remarks 2.4 (i), 3.3, 3.4) in chapters II and III to the system (3.47). Consequently, we obtain the local solution for $n \geq 1$ and the global solution for $n \geq 2$.

Finally in this section we shall remark that the system (3.47) admits a global solution even if $n = 1$. Let

$$\tilde{N}_s(t',t)^2 = N_s(t',t)^2 + \int_{t'}^{t} \|P^+(u-\bar{u},\ v-\bar{v})(\tau)\|^2 d\tau ,$$

where $N_s(t',t)$ is defined by (3.6). For the system (3.47), $\tilde{N}_s(t',t)$ takes the form

$$\tilde{N}_s(t',t)^2 = \sup_{t'\leq\tau\leq t} \|(e-\bar{e},q)(\tau)\|_s^2 + \int_{t'}^{t} \|D_x e(\tau)\|_{s-1}^2 + \|q(\tau)\|_s^2 d\tau .$$

It is not difficult to see that for our system (3.47), Lemmas 3.7 and 3.8 also hold for $n \geq 1$ with $N_s(T)$ replaced by $\tilde{N}_s(T)$. Hence Proposition 3.9 (with $\tilde{N}_s(T)$ instead of $N_s(T)$) and Theorems 3.10, 3.13 (cf. Proposition 3.12) are also valid for $n \geq 1$. Therefore we obtain the following results for the system (3.47).

Theorem 3.15 (local existence, global existence and asymptotic decay) *Let* $(3.48)_{1,2}$ *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 1$ *(*$s_0 = [n/2]+1$*) be integers.*

(i) (cf. Theorem 2.9) Suppose that the initial data $(e-\bar{e},q)(0) \in H^s(\mathbb{R}^n)$. *Then there is a positive constant* T_1 *such that the initial value problem for* (3.47) *has a unique solution* $(e-\bar{e},q) \in C^0(0,T_1;H^s(\mathbb{R}^n)) \cap C^1(0,T_1;H^{s-1}(\mathbb{R}^n))$.

(ii) (cf. Theorems 3.10 and 3.13) If $(e-\bar{e},q)(0) \in H^s(\mathbb{R}^n)$ *and* $\|(e-\bar{e},q)(0)\|_s$ *is small, then the solution of* (3.47) *exists for all time* $t \geq 0$ *and decays to the constant state* $(\bar{e},0)$ *as* $t \to \infty$: $|(e-\bar{e},q)(t)|_{s-(s_0+1)} \to 0$ *as* $t \to \infty$. *Furthermore if* $(e-\bar{e},q)(0) \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *(with* $p=1$ *for* $n=1$ *and* $p \in [1,2)$ *for* $n \geq 2$*) and* $\|(e-\bar{e},q)(0)\|_{s,p}$ *is small, then* $\|(e-\bar{e},q)(t)\|_{s-1}$ *decays at the rate* $t^{-\gamma}$ *(with* $\gamma = n(1/2p-1/4)$*) as* $t \to \infty$.

<u>Remark 3.5</u> If τ is a constant and $\kappa = \kappa(e)$ is independent of q, the system (3.47) is reduced to the second-order single equation

$$\tau e_{tt} + e_t = \operatorname{div}(c_V^{-1}\kappa\nabla e) ,$$

which is called the dissipative wave equation. For these equations, the global existence results are known; see $[61]_2$, $[54]_2$, [71].

APPENDIX TO CHAPTER III

LINEARIZED SYSTEMS AT THE CONSTANT EQUILIBRIUM STATE

3.A.1 DECAY ESTIMATES

We shall treat here the linearized system (3.1) with the right hand side:

$$A^0 U_t + \sum_j A^j U_{x_j} - \sum_{jk} B^{jk} U_{x_j x_k} + LU = h \,, \tag{3.A.1}$$

where $A^0 = A^0(\bar{u},\bar{v})$, $A^j = A^j(\bar{u},\bar{v})$, $B^{jk} = B^{jk}(\bar{u},\bar{v})$ and $L = L(\bar{u},\bar{v})$ are (real) constant square matrices of order m, and $h = h(t,x)$ is a given function on $Q_T = [0,T] \times \mathbb{R}^n$. We assume Conditions 3.1 and 3.2 on the system (3.A.1). Under these conditions the decay rate of solutions of (3.A.1) with $h = 0$ has been obtained in [81] by the method of estimating the Fourier image of solutions. The purpose of this section is to extend the results to the case that

$$(I - P^+)h(t,x) = 0 \qquad \text{for any} \quad (t,x) \in Q_T \,, \tag{3.A.2}$$

where P^+ is the orthogonal projection onto the range of $L = L(\bar{u},\bar{v})$.

Let $\hat{f}(\xi)$ denote the Fourier image of $f(x)$:

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int e^{-ix\cdot\xi} f(x)\,dx\ .$$

Taking the Fourier transform of (3.A.1), we have

$$A^0\hat{U}_t + \{i|\xi|A(\omega) + |\xi|^2B(\omega) + L\}\hat{U} = \hat{h}\ , \tag{3.A.3}$$

where $A(\omega) = \sum_j A^j\omega_j$, $B(\omega) = \sum_{jk} B^{jk}\omega_j\omega_k$ and $\omega = \xi/|\xi|$. It should be noted that (3.A.3) can be reduced to a symmetric system with $A^0 = I$ if we consider $(A^0)^{1/2}\hat{U}$ instead of $\hat{U}$. In fact we have

$$(A^0)^{1/2}\hat{U}_t + S(\xi)(A^0)^{1/2}\hat{U} = (A^0)^{-1/2}\hat{h}\ , \tag{3.A.3)'}$$

where $S(\xi)$ is defined by

$$S(\xi) = (A^0)^{-1/2}\{iA(\xi) + B(\xi) + L\}(A^0)^{-1/2} \tag{3.A.4}$$

with $A(\xi) = |\xi|A(\omega)$ and $B(\xi) = |\xi|^2B(\omega)$.

The estimate for $\hat{U}(t,\xi)$ is given by the following.

Lemma 3.A.1 *Assume Conditions* 3.1 *and* 3.2. *If* $h(t,x)$ *satisfy* (3.A.2), *then the solution of* (3.A.3) *has the estimate*

$$|\hat{U}(t,\xi)|^2 \le C\,e^{-2t\rho(|\xi|)}|\hat{U}(0,\xi)|^2 + \tag{3.A.5}$$

$$+\ C\int_0^t e^{-2(t-\tau)\rho(|\xi|)}|\hat{h}(\tau,\xi)|^2\,d\tau \qquad \textit{for}\quad (t,\xi)\in[0,T]\times\mathbb{R}^n\ ,$$

where $\rho(r) = c_1 r^2/(1+r^2)$ *;* C *and* c_1 *are positive constants.*

Remark 3.A.1 Since the solution of (3.A.3) with $h = 0$ is represented by the formula $(A^0)^{1/2}\hat{U}(t,\xi) = e^{-tS(\xi)}(A^0)^{1/2}\hat{U}(0,\xi)$, the estimate (3.A.5) with $h = 0$ gives

$$(3.A.5)' \qquad |e^{-tS(\xi)}\hat{f}(\xi)| \le Ce^{-t\rho(|\xi|)}|\hat{f}(\xi)| \quad \text{for any } f.$$

Proof of Lemma 3.A.1 Take the inner product (in $\mathbb{C}^m$) of (3.A.3) with $\hat{U}$. Since A^0, $A(\omega)$, $B(\omega)$ and L are real symmetric, its real part is

$$(3.A.6) \qquad \{\frac{1}{2}(A^0\hat{U}, \hat{U})\}_t + |\xi|^2(B(\omega)\hat{U}, \hat{U}) + (L\hat{U}, \hat{U}) = \mathrm{Re}\,(\hat{h}, \hat{U}) ,$$

where $(\ ,\ )$ denotes the standard inner product in $\mathbb{C}^m$. Next multiply (3.A.3) by $-i|\xi|K(\omega)$ $(K(\omega) = \sum_j K^j\omega_j)$ and then take the inner product with $\hat{U}$. Since $iK(\omega)A^0$ is hermitian, the real part of the resulting equality is

$$(3.A.7) \qquad \{-\frac{1}{2}|\xi|(iK(\omega)A^0\hat{U}, \hat{U})\}_t + |\xi|^2([K(\omega)A(\omega)]'\hat{U}, \hat{U})$$

$$= \mathrm{Re}\,\{i|\xi|^3(K(\omega)B(\omega)\hat{U}, \hat{U}) + i|\xi|(K(\omega)L\hat{U}, \hat{U})\} - \mathrm{Re}\,i|\xi|(K(\omega)\hat{h}, \hat{U}) ,$$

where $[K(\omega)A(\omega)]'$ denotes the symmetric part of $K(\omega)A(\omega)$. Noting that $B(\omega)$ and L are positive semi-definite and the condition (3.A.2), we have from (3.A.6) and (3.A.7)

$$\{(A^0\hat{U},\ \hat{U})\}_t + |\xi|^2(B(\omega)\hat{U},\ \hat{U}) + (L\hat{U},\ \hat{U}) \le C|\hat{h}|^2\ , \tag{3.A.8}$$

$$\{-\tfrac{1}{2}|\xi|(iK(\omega)A^0\hat{U},\ \hat{U})\}_t + |\xi|^2([K(\omega)A(\omega)]'\hat{U},\ \hat{U})$$

$$\le \varepsilon|\xi|^2|\hat{U}|^2 + C_\varepsilon\{|\xi|^4(B(\omega)\hat{U},\ \hat{U}) + (L\hat{U},\ \hat{U}) + |\hat{h}|^2\} \tag{3.A.9}$$

for any $\varepsilon > 0$ and for some constants C and C_ε. Combine (3.A.8) with (3.A.9) so as to make $(3.A.8) \times (1+|\xi|^2) + (3.A.9) \times \alpha$ with a constant $\alpha > 0$ (α will be determined later). Then we have

$$\begin{aligned}(1+|\xi|^2)E^\alpha_t &+ |\xi|^2(\{\alpha[K(\omega)A(\omega)]' + B(\omega) + L\}\hat{U},\ \hat{U}) + \\ &+ |\xi|^4(B(\omega)\hat{U},\ \hat{U}) + (L\hat{U},\ \hat{U}) \le \alpha\varepsilon|\xi|^2|\hat{U}|^2 + \\ &+ \alpha C_\varepsilon\{|\xi|^4(B(\omega)\hat{U},\ \hat{U}) + (L\hat{U},\ \hat{U})\} + (C+\alpha C_\varepsilon)(1+|\xi|^2)|\hat{h}|^2\ ,\end{aligned} \tag{3.A.10}$$

where we set

$$E^\alpha = (A^0\hat{U},\ \hat{U}) - \frac{\alpha}{2}\frac{|\xi|}{1+|\xi|^2}(iK(\omega)A^0\hat{U},\ \hat{U})\ .$$

It is easy to see that there exists a constant $\alpha_0 > 0$ such that E^α is equivalent to $|\hat{U}|^2$ for $\alpha \in (0,\alpha_0]$. On the other hand, by Condition 3.2, there exists a constant $c > 0$ such that the second term in the left member of (3.A.10) is bounded from below by $\alpha c|\xi|^2|\hat{U}|^2$, where $\alpha \le 1$ is assumed. Now choose ε and α so that $\varepsilon = c/2$ and $\alpha = \min\{1,\ \alpha_0,\ 1/C_\varepsilon\}$. Then (3.A.10) implies

$$E^{\alpha}_{t} + 2\rho(|\xi|)E^{\alpha} \leq C|\hat{h}|^2 \tag{3.A.11}$$

with $\rho(r) = c_1 r^2/(1+r^2)$; c_1 and C are positive constants. Integration of (3.A.11) with respect to t gives the desired estimate (3.A.5) by virtue of the Gronwall's inequality. This completes the proof of Lemma 3.A.1.

Integrating (3.A.5) over $\mathbb{R}^n_\xi$ and applying the Plancherel's theorem, we can obtain the following decay estimate.

Theorem 3.A.2 (decay estimate) *Assume Conditions* 3.1 *and* 3.2. *Let* $n \geq 1$ *and* $\ell \geq 0$ *be integers and let* $p, q \in [1,2]$. *Suppose that* $U(0) \in H^{\ell}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *and that* $h \in C^0(0,T; H^{\ell}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ *satisfies* (3.A.2). *Then the solution of* (3.A.1) *has the estimate*

$$\|D^{\ell}_x U(t)\|^2 \leq C\{e^{-c_1 t}\|D^{\ell}_x U(0)\|^2 + (1+t)^{-(2\gamma+\ell)}\|U(0)\|^2_{L^p}\} + C\int_0^t e^{-c_1(t-\tau)}\|D^{\ell}_x h(\tau)\|^2 + (1+t-\tau)^{-(2\gamma'+\ell)}\|h(\tau)\|^2_{L^q}\, d\tau \tag{3.A.12}$$

for $t \in [0,T]$, *where* $\gamma = n(1/2p - 1/4)$ *and* $\gamma' = n(1/2q - 1/4)$, *and* c_1 *is the constant in* (3.A.5).

Remark 3.A.2 Let us define e^{-tS} by

$$(e^{-tS}f)(x) = (2\pi)^{-n/2}\int e^{ix\cdot\xi}e^{-tS(\xi)}\hat{f}(\xi)\, d\xi \quad \text{for } f \in L^2(\mathbb{R}^n). \tag{3.A.13}$$

Then the estimate

$$(3.A.14) \qquad \|D_x^\ell(e^{-tS}f)\| \le C\{e^{-c_2 t}\|D_x^\ell f\| + (1+t)^{-(\gamma+\ell/2)}\|f\|_{L^p}\}$$

holds for $f \in H^\ell(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, where $c_2 = c_1/2$ and $\gamma = n(1/2p - 1/4)$. This implies that the $L^2(\mathbb{R}^n)$-norm of solutions of (3.A.1) with $h = 0$ decays at the rate $t^{-\gamma}$ as $t \to \infty$. This decay rate coincides with the one for the dissipative wave equations ($[54]_1$), the Boltzmann equation ($[62],[80]_1$) and the equations of compressible viscous fluids ($[55]_2$).

Proof of Theorem 3.A.2 It suffices to prove that (3.A.5)' implies (3.A.14). Multiply the square of (3.A.5)' by $|\xi|^{2\ell}$ and integrate it over $\mathbb{R}^n_\xi$. By the Plancherel's theorem we have

$$\|D_x^\ell(e^{-tS}f)\|^2 \le C\int |\xi|^{2\ell} e^{-2t\rho(|\xi|)}|\hat{f}(\xi)|^2\,d\xi .$$

We divide the integral in the right hand side into two parts I_1 and I_2 according to the regions $|\xi| \le 1$ and $|\xi| \ge 1$. By use of the Holder's inequality we have

$$\begin{aligned} I_1 &\le C\int_{|\xi|\le 1} |\xi|^{2\ell} e^{-c_1|\xi|^2 t}|\hat{f}(\xi)|^2\,d\xi \\ &\le C\left(\int_{|\xi|\le 1}|\xi|^{2\ell r} e^{-rc_1|\xi|^2 t}\,d\xi\right)^{1/r}\left(\int_{|\xi|\le 1}|\hat{f}(\xi)|^{2r'}\,d\xi\right)^{1/r'} \\ &\le C(1+t)^{-(n/2r+\ell)}\|\hat{f}\|^2_{L^{2r'}} , \end{aligned}$$

where $r \in [1,\infty]$ and $1/r + 1/r' = 1$. Apply to the last term the Hausdorff-Young inequality

$$\|\hat{f}\|_{L^{p'}} \le C\,\|f\|_{L^p}\,, \qquad p \in [1,2],\ \ 1/p + 1/p' = 1,$$

with taking $2r' = p'$ (i.e., $1/2r = 1/p - 1/2$). Then we obtain

$$I_1 \le C(1+t)^{-(2\gamma+\ell)}\,\|f\|_{L^p}^2$$

with $\gamma = n(1/2p - 1/4)$. As for I_2 we have

$$I_2 \le C\,e^{-c_1 t}\int_{|\xi|\ge 1} |\xi|^{2\ell}\,|\hat{f}(\xi)|^2\,d\xi \le C\,e^{-c_1 t}\,\|D_x^\ell f\|^2 .$$

Thus the desired estimate (3.A.14) is obtained. This completes the proof of Theorem 3.A.2.

For general $f \in H^\ell(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we have prove the decay estimate (3.A.14). Next we shall show that in some case the decay rate $t^{-\gamma}$ is improved to $t^{-(\gamma+1/2)}$.

Theorem 3.A.3 (decay estimate) *Assume Conditions* 3.1 *and* 3.2. *Let* $n = 1$, $\ell \ge 0$ *(an integer) and* $p \in [1,2]$. *Assume that* $f \in H^\ell(\mathbb{R}^1) \cap L^p(\mathbb{R}^1)$ *and that for each* $x \in \mathbb{R}^1$ *the vector* $f(x)$ *is orthogonal to the null space of* $(A^0)^{-1/2}L(A^0)^{-1/2}$. *Then the decay estimate* (3.A.14) *is improved to*

$$\text{(3.A.15)} \qquad \|D_x^\ell(e^{-tS}f)\| \le C\{e^{-c_2t}\|D_x^\ell f\| + (1+t)^{-(\gamma+1/2+\ell/2)}\|f\|_{L^p}\} ,$$

where $\gamma = 1/2p - 1/4$ *(because of* $n=1$*).*

<u>Remark 3.A.3</u> It follows from (3.A.3)' that

$$U(t) = (A^0)^{-1/2}\{e^{-tS}(A^0)^{1/2}U(0) + \int_0^t e^{-(t-\tau)S}(A^0)^{-1/2}h(\tau)\,d\tau\} .$$

Note that (3.A.2) is equivalent to the condition that the vector $(A^0)^{-1/2}\cdot h(t,x)$ is orthogonal to the null space of $(A^0)^{-1/2}L(A^0)^{-1/2}$ for each $(t,x) \in Q_T$. Therefore, applying (3.A.14) (with $n=1$) and (3.A.15) to the above expression, we conclude the estimate

$$\text{(3.A.16)} \qquad \|D_x^\ell U(t)\| \le C\{e^{-c_2t}\|D_x^\ell U(0)\| + (1+t)^{-(\gamma+\ell/2)}\|U(0)\|_{L^p}\} +$$

$$+ C\int_0^t e^{-c_2(t-\tau)}\|D_x^\ell h(\tau)\| + (1+t-\tau)^{-(\gamma'+1/2+\ell/2)}\|h(\tau)\|_{L^q}\,d\tau$$

for $t \in [0,T]$, provided that the conditions (with $n=1$) of Theorem 3.A.2 are satisfied, where $\gamma = 1/2p - 1/4$ and $\gamma' = 1/2q - 1/4$.

<u>Proof of Theorem 3.A.3</u> When $n = 1$,

$$S(\xi) = (A^0)^{-1/2}(i\xi A + \xi^2 B + L)(A^0)^{-1/2}$$

is an one-parameter family of matrices, where $\xi = \xi_1 \in \mathbb{R}^1$, $A = A^1(\bar{u},\bar{v})$ and $B = B^{11}(\bar{u},\bar{v})$. Therefore the perturbation theory of matrices (see

Kato $[37]_6$) is applicable to $S(\xi)$. This enables us to represent the matrix exponential $e^{-tS(\xi)}$ explicitly for $\xi \to 0$. We have to estimate this expression carefully as in [33] or [15] (see also [62]). Noting that $\hat{f}(\xi)$ is orthogonal to the null space of $(A^0)^{-1/2}L(A^0)^{-1/2}$ for $\xi \in \mathbb{R}^1$, we can get

$$|e^{-tS(\xi)}\hat{f}(\xi)| \le C|\xi|e^{-c|\xi|^2 t}|\hat{f}(\xi)| \qquad \text{for} \quad \xi \to 0 ,$$

where c and C are positive constants. This inequality together with (3.A.5)' gives the desired decay estimate (3.A.15) in the same way as in the proof of Theorem 3.A.2. We omit the details.

3.A.2 SPECTRAL ANALYSIS

In our analysis in chapter III, Condition 3.2 has played a crucial role. We discuss in this section the eigenvalue problem associated with the linearized system (3.1):

$$\text{(3.A.17)} \qquad \lambda A^0\phi + \{i|\xi|A(\omega) + |\xi|^2 B(\omega) + L\}\phi = 0 ,$$

where $\lambda = \lambda(i|\xi|,\omega) \in \mathbb{C}$ and $\phi = \phi(i|\xi|,\omega) \in \mathbb{C}^m \setminus \{0\}$. We prove that (under Condition 3.1) Condition 3.2 guarantees the *dissipative structure* for the system (3.1) in the following sense.

<u>Proposition 3.A.4</u> *Let Condition* 3.1 *be assumed. Let* $\lambda = \lambda(i|\xi|,\omega)$ *satisfy* (3.A.17) *for some* $\phi = \phi(i|\xi|,\omega) \in \mathbb{C}^m \setminus \{0\}$. *Then the following statements are true.*

(i) Condition 3.2 *implies that*

$$(3.A.18)_1 \quad \mathrm{Re}\,\lambda(i|\xi|,\omega) \le -\rho(|\xi|) \quad \textit{for any} \quad |\xi| \ge 0 \quad \textit{and} \quad \omega \in S^{n-1},$$

where $\rho(r) = cr^2/(1+r^2)$ *with some constant* $c > 0$.

(ii) The condition $(3.A.18)_1$ *and the following two conditions* $(3.A.18)_2$ *and* $(3.A.18)_3$ *are equivalent with each other.*

$$(3.A.18)_2 \quad \mathrm{Re}\,\lambda(i|\xi|,\omega) < 0 \quad \textit{for any} \quad |\xi| \neq 0 \quad \textit{and} \quad \omega \in S^{n-1}.$$

$(3.A.18)_3$ *Let* ψ *satisfy* $B(\omega)\psi = L\psi = 0$ *for some* $\omega \in S^{n-1}$. *Then for this* ω *and for any* $\mu \in \mathbb{R}$, $\mu A^0\psi + A(\omega)\psi \neq 0$ *holds. (Here* $\psi \neq 0$ *is assumed.)*

<u>Remark 3.A.4</u> The estimate $(3.A.18)_1$ was proved in [81]. The proof of (ii) was owing to the private communication with Y. Shizuta.

<u>Proof of Proposition 3.A.4</u> The proof of (i) is essentially the same as that of Lemma 3.A.1. The equation (3.A.17) is equal to (3.A.3) if ∂_t, $\hat{U}$ and $\hat{h}$ are replaced by λ, ϕ and 0, respectively. Therefore we obtain (3.A.6)-(3.A.10) with ∂_t, $\hat{U}$ and $\hat{h}$ replaced by $2\,\mathrm{Re}\,\lambda$, ϕ and 0, respectively. Hence, as a counterpart of (3.A.11), we have $2\,\mathrm{Re}\,\lambda + 2\rho(|\xi|) \le 0$, which is the desired estimate $(3.A.18)_1$ with $c = c_1$. This

completes the proof of (i).

We next prove (ii). First note that the implication $(3.A.18)_1 \Rightarrow (3.A.18)_2$ is trivial. The remaining part of the proof is devided into 4 steps.

step 1 We prove $(3.A.18)_2 \Longleftrightarrow (3.A.18)_3$ by contradiction. Let $B(\omega)\psi = L\psi = 0$ and $\mu A^0\psi + A(\omega)\psi = 0$ hold for some $\omega \in S^{n-1}$, $\psi \neq 0$ and $\mu \in \mathbb{R}$. Then we have (3.A.17) with $\lambda = i|\xi|\mu$ and $\phi = \psi$. This contradicts $(3.A.18)_2$. Thus $(3.A.18)_2 \Rightarrow (3.A.18)_3$ is proved.

From (3.A.3) with ∂_t, $\hat{U}$ and $\hat{h}$ replaced by $2\,\mathrm{Re}\,\lambda$, ϕ and 0, we have

$$\text{(3.A.19)} \qquad \mathrm{Re}\,\lambda(A^0\phi,\ \phi) + |\xi|^2(B(\omega)\phi,\ \phi) + (L\phi,\ \phi) = 0\ .$$

Since A^0 is positive definite, $B(\omega)$ and L are positive semi-definite, we know $\mathrm{Re}\,\lambda \le 0$. Now assume that $\mathrm{Re}\,\lambda = 0$ holds for some $|\xi| \neq 0$ and $\omega \in S^{n-1}$. Then, from (3.A.19), we obtain $B(\omega)\phi = L\phi = 0$. Using these relations in (3.A.17), we get $\lambda A^0\phi + i|\xi|A(\omega)\phi = 0$. This gives $\mu A^0\psi + A(\omega)\psi = 0$ with $\mu = \mathrm{Im}\,\lambda/|\xi|$ and $\psi = \phi$ because $\mathrm{Re}\,\lambda = 0$. This is a contradiction. Thus the proof is completed.

step 2 We prove that $(3.A.18)_3$ implies $(3.A.18)_1$ for any $r \le |\xi| \le R$ and $\omega \in S^{n-1}$, where r and R are arbitrary positive constants. This is a easy consequence of $(3.A.18)_3 \Rightarrow (3.A.18)_2$ because $\lambda(i|\xi|,\omega)$ is a continuous function of $|\xi|$ and ω.

step 3 We prove the implication $(3.A.18)_3 \Rightarrow (3.A.18)_1$ for $|\xi| \to 0$ and $\omega \in S^{n-1}$ by contradiction. Assume that there are sequences $|\xi_j| \to 0$ $(j \to \infty)$ and $\omega_j \in S^{n-1}$ such that $\lambda_j = \lambda(i|\xi_j|,\omega_j)$ satisfies

$\mathrm{Re}\,\lambda_j/|\xi_j|^2 \to 0$ $(j \to \infty)$. Let $\phi_j \in \mathbb{C}^m \setminus \{0\}$ be the vector corresponding to λ_j:

$$(3.A.20) \qquad \lambda_j A^0 \phi_j + \{i|\xi_j|A(\omega_j) + |\xi_j|^2 B(\omega_j) + L\}\phi_j = 0 .$$

We normalize ϕ_j by the relation $(A^0\phi_j, \phi_j) = 1$. By choosing subsequences, we may assume without loss of generality that $\omega_j \to \omega_0$ and $\phi_j \to \phi_0$ as $j \to \infty$. Take the inner product of (3.A.20) with ϕ_j. From its real part,

$$(3.A.21) \qquad \mathrm{Re}\,\lambda_j + |\xi_j|^2 (B(\omega_j)\phi_j, \phi_j) + (L\phi_j, \phi_j) = 0 .$$

Divide (3.A.21) by $|\xi_j|^2$ and take the limit along $j \to \infty$. Then, since $B(\omega)$ and L are positive semi-definite, we get $B(\omega_0)\phi_0 = 0$ and $L\phi_j/|\xi_j| \to 0$ as $j \to \infty$. From the latter we know $L\phi_0 = 0$. On the other hand we have from the imaginary part

$$\mathrm{Im}\,\lambda_j + i|\xi_j|(A(\omega_j)\phi_j, \phi_j) = 0 .$$

Divide the above equality by $i|\xi_j|$ and let $j \to \infty$. Then we obtain

$$(3.A.22) \qquad \lim_{j\to\infty} \mathrm{Im}\,\lambda_j/|\xi_j| = \mu_0 \equiv -(A(\omega_0)\phi_0, \phi_0) \in \mathbb{R} .$$

Now devide (3.A.20) by $i|\xi_j|$ and let $j \to \infty$. Using the relations obtained above, we conclude the equality $\mu_0 A^0 \phi_0 + A(\omega_0)\phi_0 = 0$, which contradicts $(3.A.18)_3$.

step 4 We prove $(3.A.18)_3 \Rightarrow (3.A.18)_1$ for $|\xi| \to \infty$ and $\omega \in S^{n-1}$ by contradiction. Assume that there are sequences $|\xi_j| \to \infty$ $(j \to \infty)$ and $\omega_j \in S^{n-1}$ such that $\lambda_j = \lambda(i|\xi_j|, \omega_j)$ satisfies $\mathrm{Re}\,\lambda_j \to 0$. Let $\phi_j \in \mathbb{C}^m \setminus \{0\}$ be the vector satisfying (3.A.20) and $(A^0\phi_j, \phi_j) = 1$. By choosing subsequences, we may assume that $\omega_j \to \omega_0$ and $\phi_j \to \phi_0$ as $j \to \infty$. Letting $j \to \infty$ in (3.A.21), we have $|\xi_j|B(\omega_j)\phi_j \to 0$ as $j \to \infty$ and $L\phi_0 = 0$. In particular, we get $B(\omega_0)\phi_0 = 0$. Therefore, deviding (3.A.20) by $i|\xi_j|$ and letting $j \to \infty$, we obtain $\mu_0 A^0\phi_0 + A(\omega_0)\phi_0 = 0$ in the same way as in step 3, where μ_0 is given by (3.A.22). This is a contradiction.

By steps 2-4, we have proved the implication $(3.A.18)_3 \Rightarrow (3.A.18)_1$. Thus the proof of (ii) is completed.

CHAPTER IV

HYPERBOLIC-PARABOLIC SYSTEMS OF CONSERVATION LAWS WITH A CONVEX EXTENSION

4.1 INTRODUCTION

Many equations in mathematical physics are described by conservation laws. In this chapter we shall consider the initial value problem for the following system of conservation laws.

$$(4.1) \qquad f^0(w)_t + \sum_{j=1}^{n} f^j(w)_{x_j} = \sum_{j,k=1}^{n} \{G^{jk}(w)w_{x_k}\}_{x_j} ,$$

$$(4.2) \qquad w(0,x) = w_0(x) .$$

Here $t \geq 0$ and $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$; $w = w(t,x)$ takes its values in an open convex set $\mathcal{O}$ in $\mathbb{R}^m$; $f^j(w)$ $(j=0,1,\cdots,n)$ are $\mathbb{R}^m$-valued functions and $G^{jk}(w)$ $(j,k=1,\cdots,n)$ are square matrices of order m.

Keeping applications to the equations of fluid mechanics (or elasticity) in mind, we assume the following conditions on the system (4.1).

Condition 4.1 The functions $f^j(w)$ $(j=0,1,\cdots,n)$ and $G^{jk}(w)$ $(j,k=1,\cdots,n)$ are sufficiently smooth in $w \in \mathcal{O}$ such that

(i) $f_w^0(w)$ is non-singular for $w \in \mathcal{O}$.

There exist smooth functions $\eta(z)$ $(z \in \mathcal{O}' = f^0(\mathcal{O}))$ and $q^j(w)$ $(w \in \mathcal{O})$ $(j = 1, \cdots, n)$ such that

(ii) $\eta(z)$ is strictly convex for $z \in \mathcal{O}'$,

(iii) $q_w^j(w) = {}^t f_w^j(w) \eta_z(f^0(w))$ $(j = 1, \cdots, n)$ hold for all $w \in \mathcal{O}$,

(iv) $\tilde{B}^{jk}(w) \equiv {}^t f_w^0(w) \eta_{zz}(f^0(w)) G^{jk}(w)$ $(j,k = 1, \cdots, n)$ satisfy ${}^t\tilde{B}^{jk}(w) = \tilde{B}^{kj}(w)$ for $w \in \mathcal{O}$; $\sum_{jk} \tilde{B}^{jk}(w)\omega_j\omega_k$ is (real symmetric) positive semi-definite for all $w \in \mathcal{O}$ and $\omega \in S^{n-1}$.

Here and in the sequel we sometimes use the abbreviations $f_w^0(w) = D_w f^0(w)$, $\eta_z(f^0(w)) = D_z\eta(f^0(w))$, $\eta_{zz}(f^0(w)) = D_z^2\eta(f^0(w))$ etc. The functions η and $(q_1, \cdots, q_n)$ in Condition 4.1 are called the *convex entropy* and the associated *entropy flux*, respectively. The notion of the convex entropy was introduced by Friedrichs and Lax [19] for the first-order systems of conservation laws (i.e., (4.1) with $G^{jk}(w) \equiv 0$); for related topics of the convex entropy, see $[50]_3$, $[13]_{4-6}$ (and also $[18]_2$, [68]).

It was proved in [19] that the first-order systems of conservation laws with a convex entropy can be put into a symmetric hyperbolic form. A similar result still holds for our system (4.1). Indeed, we have from (4.1)

$$f_w^0(w)w_t + \sum_j f_w^j(w)w_{x_j} - \sum_{jk} G^{jk}(w)w_{x_j x_k} = \sum_{jk} G^{jk}(w)_{x_j} w_{x_k} .$$

Multiplying the above equation by ${}^t f_w^0(w)\eta_{zz}(f^0(w))$, we get

$$A^0(w)w_t + \sum_j A^j(w)w_{x_j} - \sum_{jk} B^{jk}(w)w_{x_j x_k} = g(w, D_x w) , \tag{4.3}$$

where

$$(4.4)_1 \qquad A^j(w) = {}^t f_w^0(w)\eta_{zz}(f^0(w)) f_w^j(w) , \qquad j = 0,1,\cdots,n ,$$

$$(4.4)_2 \qquad B^{jk}(w) = \frac{1}{2}\{\tilde{B}^{jk}(w) + \tilde{B}^{kj}(w)\} , \qquad j,k = 1,\cdots,n ,$$

$$(4.4)_3 \qquad g(w,D_x w) = \sum_{jk} f_w^0(w)\eta_{zz}(f^0(w)) G_w^{jk}(w)(w_{x_j}, w_{x_k}) ,$$

Under Condition 4.1, the coefficient matrices in (4.3) satisfy the following:

$(4.5)_1$ $A^0(w)$ is real symmetric and positive definite for $w \in \mathcal{O}$,

$(4.5)_2$ $A^j(w)$ $(j=1,\cdots,n)$ are real symmetric for $w \in \mathcal{O}$,

$(4.5)_3$ $B^{jk}(w)$ $(j,k=1,\cdots,n)$ are real symmetric and satisfy $B^{jk}(w) = B^{kj}(w)$ for $w \in \mathcal{O}$; $\sum_{jk} B^{jk}(w)\omega_j\omega_k$ is (real symmetric) positive semi-definite for $w \in \mathcal{O}$ and $\omega \in S^{n-1}$.

It follows from Condition 4.1 (ii) that $\eta_{zz}(z)$ is real symmetric and positive definite, which together with Condition 4.1 (i) implies $(4.5)_1$. The property $(4.5)_3$ is a consequence of Condition 4.1 (iv) while $(4.5)_2$ is proved in [19].

For the symmetric system (4.3) we assume:

<u>Condition 4.2</u> There is a partition $w = {}^t(u,v)$ with $u \in \mathbb{R}^{m'}$ and $v \in \mathbb{R}^{m''}$ $(m = m' + m'')$ such that

(i) $$A^0(w) = \begin{pmatrix} A_1^0(u,v) & 0 \\ 0 & A_2^0(u,v) \end{pmatrix},$$

(ii) $$\tilde{B}^{jk}(w) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B}_2^{jk}(u,v) \end{pmatrix}, \qquad j,k = 1,\cdots,n,$$

where $\sum_{jk} B_2^{jk}(u,v)\omega_j\omega_k$ is (real symmetric) positive definite for $(u,v) \in \mathcal{O}$ and $\omega \in S^{n-1}$,

(iii) $$g(w,D_x w) = {}^t(g_1(u,v,D_x v),\ g_2(u,v,D_x u,D_x v)) .$$

Here we note that Condition 4.2 (ii) gives

$$B^{jk}(w) = \begin{pmatrix} 0 & 0 \\ 0 & B_2^{jk}(u,v) \end{pmatrix},$$

where $B_2^{jk}(u,v) = \frac{1}{2}\{\tilde{B}_2^{jk}(u,v) + \tilde{B}_2^{kj}(u,v)\}$. According to the partition in Condition 4.2, let us denote

$$A^j(w) = \begin{pmatrix} A_{11}^j(u,v) & A_{12}^j(u,v) \\ A_{21}^j(u,v) & A_{22}^j(u,v) \end{pmatrix}.$$

Then the system (4.3) can be written in the form (2.1) with the right members

$$(4.6)\quad \begin{cases} f_1(u,v,D_x v) = -\sum_j A_{12}^j(u,v)v_{x_j} + g_1(u,v,D_x v), \\ f_2(u,v,D_x u,D_x v) = -\sum_j \{A_{21}^j(u,v)u_{x_j} + A_{22}^j(u,v)v_{x_j}\} + \\ \qquad + g_2(u,v,D_x u,D_x v) . \end{cases}$$

The above system satisfies Conditions 2.1 and 2.2. In fact Condition 2.1 (i), (ii) follow from $(4.5)_{1,2}$ while Condition 2.1 (iii) from $(4.5)_3$ and Condition 4.2 (ii). Since $G^{jk}(w) = \{{}^t f^0_w(w)\eta_{zz}(f^0(w))\}^{-1}\tilde{B}^{jk}(w)$, $(4.4)_3$ together with Condition 4.2 (ii) and (iii) implies that

$$(4.7) \quad \begin{cases} g_1(u,v,D_x v) = O(|D_x v|^2) , \\ g_2(u,v,D_x u,D_x v) = O(|D_x(u,v)||D_x v|) \end{cases}$$

for $|u-\bar{u},\ v-\bar{v}| \to 0$, where $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ is an arbitrarily fixed constant state. So Condition 2.2 is satisfied for arbitrary $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$. Thus we have proved that under Conditions 4.1 and 4.2 the system (4.1) can be reduced to a *symmetric hyperbolic-parabolic* form in the sense indicated in chapter II. Hence, by Theorem 2.9, we have a local solution to the problem (4.1),(4.2).

To discuss the global existence and asymptotic stability of solutions, we shall require, as in chapter III, the conditions which guarantee the *dissipative structure* for the linearized system of (4.3):

$$(4.8) \quad A^0(\bar{w})U_t + \sum_j A^j(\bar{w})U_{x_j} - \sum_{jk} B^{jk}(\bar{w})U_{x_j x_k} = 0$$

(cf. (4.18)), where we have used $L(\bar{w}) = D_w g(\bar{w},0) = 0$. We note that Condition 3.1 is satisfied automatically because $(4.5)_{1,2,3}$ and $L(\bar{w}) = 0$. So we only require:

<u>Condition 4.3</u> There are (real) constant square matrices K^j $(j=1,\cdots$

$\cdots,n)$ of order m such that

(i) $K^j A^0(\bar{w})$ $(j=1,\cdots\cdot,n)$ are real anti-symmetric,

(ii) the symmetric part of the matrix $\sum_{jk}\{K^j A^k(\bar{w}) + B^{jk}(\bar{w})\}\omega_j\omega_k$ is positive definite for any $\omega \in S^{n-1}$.

Since the remainder term $\tilde{f}(u,v,D_x u,D_x v)$ associated with $f = {}^t(f_1,f_2)$ in (4.6) satisfies (3.30), we can apply Theorems 3.10 and 3.11 to the system (4.1). Hence we obtain the global existence and asymptotic stability results for (4.1),(4.2) if $n \geq 2$.

The purpose of this chapter is to establish the global existence results for (4.1),(4.2) for all dimensions $n \geq 1$, and to get an asymptotic form of the solution as $t \to \infty$. In order to get the a priori estimates of solutions of (4.1) in one space-dimension, we employ a technical energy method, which makes use of the quadratic function associated with the convex entropy (see the proof of Lemma 4.1). For similar energy methods, see $[36]_{1,2}$, [66], [41], $[38]_{3,4}$, [27]. It seems to the author that in one-dimensional case the convex entropy plays a crucial role in a study of the global existence problem for (4.1).

The contents of this chapter are as follows. In section 4.2 we shall derive the a priori estimates of small solutions of (4.1) by the technical energy method. As a consequence we get a global solution to (4.1), (4.2) for small initial data with $w_0 - \bar{w} \in H^s(\mathbb{R}^n)$ (with $n \geq 1$ and $s \geq [n/2]+2$). In section 4.3 we shall prove the decay rate of the solution: if $w_0 - \bar{w} \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ (with $n \geq 1$ and $s \geq [n/2]+3$; $p=1$ for $n=1$ and $p \in [1,2)$ for $n \geq 2$) is small, then the solution of (4.1),(4.2) decays to the constant state $\bar{w}$ at the rate $t^{-\gamma}$ (with $\gamma = n(1/2p - 1/4)$) as t

$\to \infty$. In section 4.4 the asymptotic behavior of the solution is discussed more precisely. It is proved that if $n \geq 2$ (resp. $n = 1$), the solution of (4.1), (4.2) is asymptotic to that of the linearized system (4.8) (resp. the semi-linear system (4.39)) with the corresponding initial conditions at the rate $t^{-\beta}$ (with some $\beta > \gamma$) as $t \to \infty$. The results in sections 4.3 and 4.4 are based on the conservation form of the system (4.1).

4.2 GLOBAL EXISTENCE

First, following [19], we write down the equation of the convex entropy. Differentiating $\eta(f^0(w))$ with respect to t and using (4.1), we get

$$\eta(f^0(w))_t = -\sum_j < \eta_z(f^0(w)), f^j_w(w)w_{x_j} > +$$

$$+ \sum_{jk} \{ < \eta_z(f^0(w)), G^{jk}(w)w_{x_k} > \}_{x_j} - \sum_{jk} < \eta_z(f^0(w))_{x_j}, G^{jk}(w)w_{x_k} > ,$$

where $< , >$ denotes the inner product in $\mathbb{R}^m$. It follows from Condition 4.1 (iii) that the first term in the right hand side is equal to $\sum_j q^j(w)_{x_j}$. Since $\eta_{zz}(z)$ is real symmetric, the last term in the right member is rewritten as $\sum_{jk} < \tilde{B}^{jk}(w)w_{x_k}, w_{x_j} >$ (cf. the definition of $\tilde{B}^{jk}(w)$ in Condition 4.1 (iv)). Therefore we obtain the equation

(4.9) $\qquad \eta(f^0(w))_t + \sum_j q^j(w)_{x_j}$

$$= \sum_{jk} \{ < \eta_z(f^0(w)),\ G^{jk}(w)w_{x_k} > \}_{x_j} - \sum_{jk} < \tilde{B}^{jk}(w)w_{x_k},\ w_{x_j} > .$$

This equation has a physical meaning if the latter half of Condition 4.1 (iv) is replaced by the following stronger condition:

$$\sum_{jk} < \tilde{B}^{jk}(w)\phi_k,\ \phi_j > \ \geq 0 \quad \text{holds for all} \quad w \in \mathcal{O} \quad \text{and} \quad \phi_j \in \mathbb{R}^m \quad (j=1,\cdots,n).$$

In fact in this case (4.9) implies that the integral $-\int \eta(f^0(w(t,x)))dx$ is non-decreasing in t, which corresponds to the second law in thermodynamics.

Now let us introduce

$$(4.10) \qquad \eta^*(z,\bar{z}) = \eta(z) - \eta(\bar{z}) - < \eta_z(\bar{z}),\ z-\bar{z} >$$

for $z,\ \bar{z} \in \mathcal{O}' = f^0(\mathcal{O})$. Since $\eta(z)$ is strictly convex, $\eta^*(z,\bar{z})$ is positive definite; $\eta^*(z,\bar{z}) = 0$ holds if and only if $z = \bar{z}$. In particular, $\eta^*(z,\bar{z})$ is equivalent to the quadratic function $|z-\bar{z}|^2$ in $B_r(\bar{z}) = \{ z \in \mathbb{R}^m ;\ |z-\bar{z}| < r \}$, where $r > 0$ is arbitrary as long as $B_r(\bar{z}) \in \mathcal{O}'$ is satisfied. Let $z = f^0(w)$ and $\bar{z} = f^0(\bar{w})$, where $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ is an arbitrarily fixed constant state. Since $f^0_w(w)$ is non-singular, the inverse mapping theorem shows that there exists a neighborhood $B_{r_0}(\bar{w}) = \{ w \in \mathbb{R}^m ;\ |w-\bar{w}| < r_0 \} \subset \mathcal{O}$ with some $r_0 > 0$ such that

$$(4.11)_1 \qquad c|w-\bar{w}| \leq |f^0(w) - f^0(\bar{w})| \leq C|w-\bar{w}| \quad \text{holds for} \quad w \in B_{r_0}(\bar{w}),$$

with some positive constants c and C,

$(4.11)_2$ the inverse function $w = w(z)$ exists and satisfies $D_z w(z) = f_w^0(w(z))^{-1}$ for $z \in f^0(B_{r_0}(\bar{w}))$.

Without loss of generality we can assume that $\overline{B_{r_0}(\bar{w})} \subset \mathcal{O}$ and $f^0(B_{r_0}(\bar{w})) \subset B_r(f^0(\bar{w})) \subset \mathcal{O}'$. So we get the following estimate for any $w \in B_{r_0}(\bar{w})$.

$$c|w-\bar{w}|^2 \le \eta^*(f^0(w), f^0(\bar{w})) \le C|w-\bar{w}|^2 \tag{4.12}$$

with some positive constants $c = c(r_0)$ and $C = C(r_0)$.

Now, letting $s \ge s_0 + 1$ ($s_0 = [n/2]+1$) be an integer and $T > 0$ be a constant, we consider a solution $w(t,x) = {}^t(u,v)(t,x)$ of (4.1) satisfying (3.4) and (3.5) with $\mathcal{O}_2 = B_{r_0}(\bar{w}) = \{w \in \mathbb{R}^m;\ |w-\bar{w}| < r_0\}$. Let $N_s(t',t)$ be defined by (3.6). Then there is a positive constant $a_4 = a_4(r_0)$ such that

(4.13) if $N_{s_0}(T) \le a_4$, then (3.5) with $\mathcal{O}_2 = B_{r_0}(\bar{w})$ is satisfied automatically.

We will derive the a priori estimate for $N_s(T)$. The a priori estimate for $L^2(\mathbb{R}^n)$-norm of the solution is obtained by means of the quadratic function $\eta^*(z,\bar{z})$:

<u>Lemma 4.1</u> *Let* $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ *be an arbitrarily fixed constant state. Assume Conditions* 4.1 *and* 4.2. *Let* $n \ge 1$ *and* $s \ge s_0 + 1$ *(*$s_0 = [n/2] + 1$*) be integers. Suppose that the initial data* $w_0(x) = {}^t(u_0,v_0)(x)$ *satisfy* $w_0 - \bar{w} = {}^t(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$ *and that* $w(t,x) = {}^t(u,v)(t,x)$

is a solution of the problem (4.1), (4.2) *satisfying* (3.4) *and* $N_{s_0}(T) \leq a_4$. *Then there is a constant* $C = C(a_4) > 1$ *such that the following a priori estimate holds for* $t \in [0,T]$:

$$(4.14) \qquad \|(u-\bar{u},\ v-\bar{v})(t)\|^2 + \int_0^t \|D_x v(\tau)\|^2 d\tau$$

$$\leq C\{\|u_0-\bar{u},\ v_0-\bar{v}\|^2 + N_{s_0}(T)^3\} .$$

Proof. For the solution $w = w(t,x)$, we consider the quadratic function $\eta^*(f^0(w), f^0(\bar{w}))$. From (4.9) and (4.1) we get the equation of $\eta^*(f^0(w), f^0(\bar{w}))$:

$$\eta^*(f^0(w), f^0(\bar{w}))_t + \sum_j \{q^j(w) - q^j(\bar{w}) - \langle \eta_z(f^0(\bar{w})),\ f^j(w) - f^j(\bar{w}) \rangle\}_{x_j}$$

$$= \sum_{jk} \{\langle \eta_z(f^0(w)) - \eta_z(f^0(\bar{w})),\ G^{jk}(w) w_{x_k} \rangle\}_{x_j} - \sum_{jk} \langle \tilde{B}^{jk}(w) w_{x_k},\ w_{x_j} \rangle .$$

Integration of this equality over $Q_t = [0,t] \times \mathbb{R}^n$ yields

$$(4.15) \qquad \Big[\int \eta^*(f^0(w), f^0(\bar{w}))\, dx\Big]_{\tau=0}^{\tau=t} + \sum_{jk} \int_0^t \int \langle \tilde{B}^{jk}(w) w_{x_k},\ w_{x_j} \rangle\, dx\, d\tau = 0 .$$

Since $\sum_{jk} \tilde{B}_2^{jk}(\bar{u},\bar{v}) \omega_j \omega_k$ is real symmetric and positive definite, the second term in (4.15) is bounded from below by

$$c\int_0^t \|D_x v(\tau)\|^2 d\tau - C N_{s_0}(T)^3$$

with some positive constants c and $C = C(a_4)$. Therefore (4.15) together with (4.12) gives the desired estimate (4.14). This completes the proof of Lemma 4.1.

We proceed to estimate the derivatives of the solution. Under Conditions 4.1 and 4.2 (resp. 4.1-4.3), Lemma 3.1 (resp. 3.2) with $P^{+} = 0$ is applicable to the system (4.1). Therefore, if $N_s(T) \leq a_4$, we obtain the estimates

$$(4.16) \qquad \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 d\tau$$

$$\leq C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + N_s(T)^3 \} ,$$

$$(4.17) \qquad \int_0^t \|D_x^2 u(\tau)\|_{s-2}^2 d\tau - C\{ \|D_x(u,v)(t)\|_{s-1}^2 + \int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 d\tau \}$$

$$\leq C\{ \|D_x(u_0,v_0)\|_{s-1}^2 + N_s(T)^3 \}$$

for $t \in [0,T]$, where $C = C(a_4)$ is a positive constant.

To complete the estimate for $N_s(T)$, it suffices to estimate the $L^2(0,T;L^2(\mathbb{R}^n))$-norm of the derivatives $D_x u$. Let $U = w - \bar{w} = {}^t(u-\bar{u}, v-\bar{v})$. The system (4.1) is rewritten in the form

$$(4.18) \qquad A^0(\bar{w})U_t + \sum_j A^j(\bar{w})U_{x_j} - \sum_{jk} B^{jk}(\bar{w})U_{x_j x_k} = h(u,v,D_x u,D_x v,D_x^2 v) ,$$

where $h = {}^t(h_1,h_2)$ is the nonlinear term of the form

$$(4.19)_1 \qquad h_1(u,v,D_x u,D_x v) = A_1^0(\bar{u},\bar{v})A_1^0(u,v)^{-1}g_1(u,v,D_x v) -$$

$$- A_1^0(\bar{u},\bar{v}) \sum_j \{A_1^0(u,v)^{-1}A_{11}^j(u,v) - A_1^0(\bar{u},\bar{v})^{-1}A_{11}^j(\bar{u},\bar{v})\}u_{x_j} -$$

$$- A_1^0(\bar{u},\bar{v}) \sum_j \{A_1^0(u,v)^{-1}A_{12}^j(u,v) - A_1^0(\bar{u},\bar{v})^{-1}A_{12}^j(\bar{u},\bar{v})\}v_{x_j} ,$$

$$(4.19)_2 \qquad h_2(u,v,D_xu,D_xv,D_x^2v) = A_2^0(\bar{u},\bar{v})A_2^0(u,v)^{-1}g_2(u,v,D_xu,D_xv) -$$

$$- A_2^0(\bar{u},\bar{v}) \sum_j \{A_2^0(u,v)^{-1}A_{21}^j(u,v) - A_2^0(\bar{u},\bar{v})^{-1}A_{21}^j(\bar{u},\bar{v})\}u_{x_j} -$$

$$- A_2^0(\bar{u},\bar{v}) \sum_j \{A_2^0(u,v)^{-1}A_{22}^j(u,v) - A_2^0(\bar{u},\bar{v})^{-1}A_{22}^j(\bar{u},\bar{v})\}v_{x_j} +$$

$$+ A_2^0(\bar{u},\bar{v}) \sum_{jk} \{A_2^0(u,v)^{-1}B_2^{jk}(u,v) - A_2^0(\bar{u},\bar{v})^{-1}B_2^{jk}(\bar{u},\bar{v})\}v_{x_jx_k} .$$

Therefore, following the arguments in Lemma 3.8, we obtain the estimate (3.37) with $P^+ = 0$. Since

$$(4.20) \qquad h(u,v,D_xu,D_xv,D_x^2v) = O(|D_x(u,v)|^2 + + |u-\bar{u},\ v-\bar{v}|\{|D_x(u,v)| + |D_x^2v|\})$$

for $|u-\bar{u},\ v-\bar{v}| \to 0$ (compare this with (3.24)), the estimate (3.38) also holds for $n \geq 1$. So we get

$$(4.21) \qquad \int_0^t \|D_xu(\tau)\|^2 d\tau - C\{\|(u-\bar{u},\ v-\bar{v})(t)\|_1^2 + \int_0^t \|D_xv(\tau)\|_1^2 d\tau\}$$
$$\leq C\{\|u_0-\bar{u},\ v_0-\bar{v}\|_1^2 + N_{s_0}(T)^3\} .$$

Now consider the combination $(4.14) + (4.16) + \{(4.17) + (4.21)\} \times \alpha$ with a positive constant α satisfying $2\alpha C < 1$. Then we obtain

$$N_s(T)^2 \leq C\{\ \|u_0-\bar{u},\ v_0-\bar{v}\|_s^2 + N_s(T)^3\ \}\ ,$$

whenever $N_s(T) \leq a_4$ is satisfied. Hence we have:

Proposition 4.2 (a priori estimate) *Let* $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ *be arbitrary and let Conditions* 4.1-4.3 *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 1$ *(*$s_0 = [n/2]+1$*) be integers. Suppose that* $w_0 - \bar{w} = {}^t(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n)$, *and* $w(t,x) = {}^t(u,v)(t,x)$ *is a solution of* (4.1), (4.2) *with* (3.4). *Then there exist positive constants* a_5 *(*$\leq a_4$*) and* $C_8 = C_8(a_5) > 1$ *such that if* $N_s(T) \leq a_5$, *the following a priori estimate holds for* $t \in [0,T]$.

$$N_s(t) \leq C_8\|u_0-\bar{u},\ v_0-\bar{v}\|_s\ . \tag{4.22}$$

The global existence result for (4.1), (4.2) (for all $n \geq 1$) is now follows from Theorem 2.9 and Proposition 4.2 by the continuation arguments as in Theorem 3.10.

Theorem 4.3 (global existence) *Let* $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ *be arbitrary and let Conditions* 4.1-4.3 *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 1$ *(*$s_0 = [n/2]+1$*) be integers. Suppose that* $w_0 - \bar{w} = {}^t(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\mathbb{R}^n)$. *Then there exists a positive constant* δ_5 *(*$\leq a_5$*) such that if* $\|u_0-\bar{u},\ v_0-\bar{v}\|_s \leq \delta_5$, *the problem* (4.1), (4.2) *has a unique global solution* $w(t,x) = {}^t(u,v)(t,x)$ *satisfying* (3.4) *with* $T = \infty$. *The solution satisfies* (4.22) *for* $t \in [0,\infty)$*:*

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_s^2 + \int_0^t \|D_x u(\tau)\|_{s-1}^2 + \|D_x v(\tau)\|_s^2\, d\tau \tag{4.23}$$

$$\leq C_8^2 \, \|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 \ .$$

Furthermore the solution decays to the constant state $\bar{w} = {}^t(\bar{u},\bar{v})$ *as* $t \to \infty$*:* $|(u - \bar{u},\ v - \bar{v})(t)|_{s-(s_0+1)} \to 0$ *as* $t \to \infty$.

Remark 4.1 The constant δ_5 in the theorem is determined by $\delta_5 = \min \{ a_5/C_4,\ a_5/C_8(1 + C_4^2)^{1/2} \}$.

4.3 ASYMPTOTIC DECAY

In this section we shall show that if $w_0 - \bar{w} \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, the solution of Theorem 4.3 decays at the rate $t^{-\gamma}$ (with $\gamma = n(1/2p - 1/4)$) as $t \to \infty$. If $n \geq 3$, this decay law was already proved in Theorem 3.11. The following arguments including the case $n = 1, 2$ are based on the conservation form of the system (4.1).

Theorem 4.4 (asymptotic decay) *Let* $\bar{w} \in O$ *be arbitrary and let Conditions* 4.1-4.3 *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 2$ *(*$s_0 = [n/2] + 1$*) be integers, and let* $p = 1$ *for* $n = 1$ *and* $p \in [1,2)$ *for* $n \geq 2$*. Suppose that* $w_0 - \bar{w} \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$*. Then there is a positive constant* δ_6 *(*$\leq \delta_5$*) such that if* $\|w_0 - \bar{w}\|_{s,p} \leq \delta_6$*, then the solution* $w(t,x)$ *constructed in Theorem 4.3 satisfies*

$$(4.24) \qquad \|f^0(w(t)) - f^0(\bar{w})\|_{s-2} \leq C(1+t)^{-\gamma} \|w_0 - \bar{w}\|_{s-2,p}$$

for $t \in [0,\infty)$, *where* $\gamma = n(1/2p - 1/4)$, *and* $C = C(\delta_6)$ *is a positive constant; the norm* $\|\cdot\|_{\ell,p}$ *is defined by* (3.20).

Remark 4.2 The solution $w(t,x)$ itself satisfies

$$(4.24)' \qquad \|(w-\bar{w})(t)\|_{s-2} \le C(1+t)^{-\gamma}\|w_0 - \bar{w}\|_{s-2,p} .$$

Proof of Theorem 4.4 We consider $z = f^0(w)$ as the unknown and linearize (4.1) at the constant state $\bar{z} = f^0(\bar{w})$. Noting $(4.11)_2$, we obtain

$$(4.25) \qquad f^0(w)_t + \sum_j f^j_w(\bar{w})f^0_w(\bar{w})^{-1}f^0(w)_{x_j} - \sum_{jk} G^{jk}(\bar{w})f^0_w(\bar{w})^{-1}f^0(w)_{x_jx_k}$$

$$= \sum_j \tilde{h}^j(w)_{x_j} + \sum_{jk}\{\tilde{H}^{jk}(w)w_{x_k}\}_{x_j} ,$$

where we set

$$\tilde{h}^j(w) = -\{f^j(w) - f^j(\bar{w}) - f^j_w(\bar{w})f^0_w(\bar{w})^{-1}\{f^0(w) - f^0(\bar{w})\}\} ,$$

$$\tilde{H}^{jk}(w) = \{G^{jk}(w)f^0_w(w)^{-1} - G^{jk}(\bar{w})f^0_w(\bar{w})^{-1}\}f^0_w(w) .$$

Put

$$(4.26) \qquad V = f^0_w(\bar{w})^{-1}\{f^0(w) - f^0(\bar{w})\} .$$

Multiplying (4.25) by ${}^t f^0_w(\bar{w})\eta_{zz}(f^0(\bar{w}))$ and noting $(4.4)_{1,2}$ (see also Condition 4.1 (iv)), we have

$$(4.27)\qquad A^0(\bar{w})V_t + \sum_j A^j(\bar{w})V_{x_j} - \sum_{jk} B^{jk}(\bar{w})V_{x_jx_k}$$

$$= \sum_j h^j(w)_{x_j} + \sum_{jk} \{H^{jk}(w)w_{x_k}\}_{x_j} ,$$

where

$$(4.28)_1\qquad h^j(w) = {}^tf^0_w(\bar{w})\eta_{zz}(f^0(\bar{w}))\tilde{h}^j(w) ,$$

$$(4.28)_2\qquad H^{jk}(w) = {}^tf^0_w(\bar{w})\eta_{zz}(f^0(\bar{w}))\tilde{H}^{jk}(w) .$$

It should be noticed that the linear part of (4.27) coincides with that of (4.18) and the nonlinear term of (4.27) is of a conservation form.

Let $A(\xi) = \sum_j A^j(\bar{w})\xi_j$ and $B(\xi) = \sum_{jk} B^{jk}(\bar{w})\xi_j\xi_k$. Let $S(\xi)$ be defined by (3.A.4) with $A^0 = A^0(\bar{w})$ and $L = 0$, and let e^{-tS} by (3.A.13). Then the solution V of (4.27) has the expression

$$(4.29)\qquad V(t) = (A^0)^{-1/2}[e^{-tS}(A^0)^{1/2}V(0) +$$

$$+ \int_0^t e^{-(t-\tau)S}(A^0)^{-1/2}\{\sum_j h^j(w)_{x_j}(\tau) + \sum_{jk}\{H^{jk}(w)w_{x_k}\}_{x_j}(\tau)\}\,d\tau\,] .$$

Applying (3.A.14) to (4.29) and using the conservation form of the nonlinear term, we obtain

$$(4.30)\qquad \|V(t)\|_{s-2} \le C(1+t)^{-\gamma}\|V(0)\|_{s-2,p} +$$

$$+ C\int_0^t e^{-c_2(t-\tau)}\{\sum_j \|h^j(w)_{x_j}(\tau)\|_{s-2} + \sum_{jk}\|\{H^{jk}(w)w_{x_k}\}_{x_j}(\tau)\|_{s-2}\}\,d\tau +$$

$$+ C\int_0^t (1+t-\tau)^{-(n/4+1/2)}\{\sum_j \|h^j(w)(\tau)\|_{L^1} + \sum_{jk}\|\{H^{jk}(w)w_{x_k}\}(\tau)\|_{L^1}\}d\tau$$

with $\gamma = n(1/2p - 1/4)$ and a constant C. We estimate the norms in the right hand side of (4.30). It follows from $(4.28)_{1,2}$ that $h^j(w) = O(|w-\bar{w}|^2)$ and $H^{jk}(w) = O(|w-\bar{w}|)$ for $|w-\bar{w}| \to 0$. On the other hand $(4.11)_1$ gives $c|w-\bar{w}| \le |V| \le C|w-\bar{w}|$ for $w \in B_{r_0}(\bar{w})$, and $(4.11)_2$ shows the existence of the inverse function $w = w(V)$ of (4.26) in the neighborhood $V(B_{r_0}(\bar{w}))$. Therefore if $\|V\|_s$ (with $s \ge s_0+2$) is suitably small (this condition is satisfied if $\|w-\bar{w}\|_s \le a_4'$ with sufficiently small $a_4' \le a_4$), then

$$(4.31)_1 \qquad \sum_j \|h^j(w(V))_{x_j}\|_{s-2} + \sum_{jk}\|\{H^{jk}(w(V))w(V)_{x_k}\}_{x_j}\|_{s-2} \le C\,\|V\|_{s-2}\,\|V\|_s\,,$$

$$(4.31)_2 \qquad \sum_j \|h^j(w(V))\|_{L^1} + \sum_{jk}\|H^{jk}(w(V))w(V)_{x_k}\|_{L^1} \le C\,\|V\|_1^2$$

hold with some constant C. Substituting $(4.31)_{1,2}$ to (4.30) and using $\|V\|_{s-2,p} \le C\,\|w-\bar{w}\|_{s-2,p}$, $\|V\|_s \le C\,\|w-\bar{w}\|_s$ and (4.23), we obtain the inequality for $|||V(t)|||_{s-2,\gamma} = \sup_{0\le\tau\le t}(1+\tau)^{\gamma}\|V(\tau)\|_{s-2}$:

$$(4.32) \qquad |||V(t)|||_{s-2,\gamma} \le C\,\|w_0-\bar{w}\|_{s-2,p} + $$

$$+ C\mu_3(t)\,\|w_0-\bar{w}\|_s\,|||V(t)|||_{s-2,\gamma} + C\mu_4(t)\,|||V(t)|||_{s-2,\gamma}^2\,,$$

where $\mu_3(t)$ appeared in (3.45) and $\mu_4(t)$ is given by

$$\mu_4(t) = \sup_{0\le\tau\le t} (1+\tau)^{\gamma}\int_0^{\tau} (1+\tau-\tau_1)^{-(n/4+1/2)}(1+\tau_1)^{-2\gamma}\,d\tau_1 .$$

Note that $\mu_4(t)$ is identical with $\mu_4'(t)$ in (3.45) if $n = 1$. Since $\mu_3(t)$ and $\mu_4(t)$ are uniformly bounded with respect to $t \in [0,\infty)$ for any n and p indicated in Theorem 4.4, the desired estimate (4.24) follows from (4.32) as in Proposition 3.3 (or 3.12). Thus the proof of the theorem is completed.

4.4 ASYMPTOTIC BEHAVIOR

In this section we shall study the asymptotic behavior of solutions of (4.1) in detail. We first consider the case $n \ge 2$. Let $w^*(t,x)$ be a solution of the linearized system (4.8):

$$A^0(\bar{w})w^*_t + \sum_j A^j(\bar{w})w^*_{x_j} - \sum_{jk} B^{jk}(\bar{w})w^*_{x_jx_k} = 0 , \tag{4.33}$$

with the initial conditions

$$w^*(0,x) = f^0_w(\bar{w})^{-1}\{f^0(w_0(x)) - f^0(\bar{w})\} , \tag{4.34}$$

where $w_0(x)$ is the initial data in (4.2). Let $w(t,x)$ be the solution of (4.1),(4.2) constructed in Theorems 4.3 and 4.4. Since $w^*(t) = (A^0)^{-1/2}e^{-tS}(A^0)^{1/2}w^*(0)$ and $w^*(0) = V(0)$, (4.29) yields

$$f^0_w(\bar{w})^{-1}\{f^0(w(t)) - f^0(\bar{w})\} - w^*(t) \tag{4.35}$$

$$= \int_0^t (A^0)^{-1/2} e^{-(t-\tau)S} (A^0)^{-1/2} \{ \sum_j h^j(w)_{x_j}(\tau) + \sum_{jk} \{H^{jk}(w)w_{x_k}\}_{x_j}(\tau) \} d\tau \ .$$

Let $s \geq s_0 + 2$ (note that $s_0 = [n/2]+1 \geq 2$ for $n \geq 2$). Then for $\|w - \bar{w}\|_s \leq a_4$,

$$(4.36) \qquad \sum_j \|h^j(w)_{x_j}\|_{s-4} + \sum_{jk} \|\{H^{jk}(w)w_{x_k}\}_{x_j}\|_{s-4} \leq C\,\|w - \bar{w}\|^2_{s-2}$$

holds with some constant $C = C(a_4)$. Applying (3.A.14) to (4.35) and using the estimates (4.36), $(4.31)_2$ (with $\|V\|_1$ replaced by $\|w - \bar{w}\|_1$) and (4.24)', we obtain

$$\|f^0(w(t)) - f^0(\bar{w}) - f^0_w(\bar{w})w^*(t)\|_{s-4}$$

$$\leq C\mu_7(t)(1+t)^{-\beta}\,\|w_0 - \bar{w}\|^2_{s-2,p} \ ,$$

with $\beta = 2\gamma - \varepsilon$ (for any small $\varepsilon > 0$) for $n = 2$ and $\beta = \min\{n/4 + 1/2, 2\gamma\}$ for $n \geq 3$; $\mu_7(t)$ is defined by

$$\mu_7(t) = (1+t)^{\beta} \int_0^t (1+t-\tau)^{-(n/4+1/2)}(1+\tau)^{-2\gamma}\,d\tau \ .$$

It is not difficult to see that $\mu_7(t) \leq C_\varepsilon$ holds for every small $\varepsilon > 0$ if $n = 2$ and $\mu_7(t) \leq C$ if $n \geq 3$. Thus we have proved:

<u>Theorem 4.5</u> (asymptotic behavior) *Let* $n \geq 2$ *be an integer and assume the same conditions as in Theorem* 4.4. *Let* $w(t,x)$ *and* $w^*(t,x)$ *be the solutions of* (4.1), (4.2) *and* (4.33), (4.34), *respectively. Then the fol-*

lowing estimate holds for $t \in [0,\infty)$.

$$\|f^0(w(t)) - f^0(\bar{w}) - f^0_w(\bar{w})w^*(t)\|_{s-4} \le C(1+t)^{-\beta}\|w_0 - \bar{w}\|^2_{s-2,p}, \tag{4.37}$$

where $\beta > \gamma = n(1/2p - 1/4)$ *is determined by*

$$\beta = \begin{cases} 2\gamma - \varepsilon \quad (\textit{for any small } \varepsilon > 0) & \textit{if} \quad n = 2, \\ \min\{n/4 + 1/2,\ 2\gamma\} & \textit{if} \quad n \ge 3, \end{cases}$$

and $C = C(\delta_6)$ *is a constant (in the case of* $n=2$ *it depends on* $\varepsilon > 0$ *too).*

Remark 4.3 The estimate (4.37) implies that the solutions of the non-linear equations (4.1) are asymptotic to those of the linear ones (4.33). A similar result was obtained by Kawashima, Matsumura and Nishida [39] for the equations of compressible viscous fluids in $\mathbb{R}^3$. Based on this estimate, it was proved in [39] that the Boltzmann equation can be approximated by the equations of compressible viscous fluids for $t \to \infty$ and that the latter equations by the Navier-Stokes equation for incompressible viscous fluids.

Next we consider the case $n = 1$. Let $A(\bar{w}) = A^1(\bar{w})$, $B(\bar{w}) = B^{11}(\bar{w})$, $h(w) = h^1(w)$, $H(w) = H^{11}(w)$ and $x = x_1 \in \mathbb{R}^1$. Then, in this case, the system (4.27) is reduced to a simple one:

$$A^0(\bar{w})V_t + A(\bar{w})V_x - B(\bar{w})V_{xx} = h(w)_x + \{H(w)w_x\}_x. \tag{4.27'}$$

Let $w = w(V)$ be the inverse function of (4.26). Since $h(w) = O(|w - \bar{w}|^2)$ for $|w - \bar{w}| \to 0$, we find that $D_V^k h(w(0)) = 0$ for $k = 0,1$. Therefore $h(w(V))$ has the form

$$h(w(V)) = h^*(V) + h_R(V) , \tag{4.38}$$

where $h^*(V) = D_V^2 h(w(0))(V, V)$, and $h_R(V)$ is the remainder term with $h_R(V) = O(|V|^3)$ for $|V| \to 0$.

We will show that for $t \to \infty$, the solution of (4.1),(4.2) (with $n = 1$) is approximated by the solution of the semi-linear equations

$$A^0(\bar{w})w_t^* + A(\bar{w})w_x^* - B(\bar{w})w_{xx}^* = h^*(w^*)_x \tag{4.39}$$

with the initial conditions (4.34). We need some preparations. First we show that in the case of $n = 1$ the $L^2(\mathbb{R}^1)$-norm of the derivatives of solutions to (4.1),(4.2) decays at the rate $t^{-\beta}$ (with $\beta = 3/4 - \varepsilon$ for any small $\varepsilon > 0$) as $t \to \infty$.

<u>Lemma 4.6</u> *Let* $n = 1$ *and* $s \ge 5$ *(an integer), and assume that* $w_0 - \bar{w} \in H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$. *Then for any small* $\varepsilon > 0$, *there exists a positive constant* $\delta_7 = \delta_7(\varepsilon)$ *(* $\le \delta_6$*) such that if* $\|w_0 - \bar{w}\|_{s,1} \le \delta_7$, *then the solution constructed in Theorems* 4.3 *and* 4.4 *satisfies*

$$\|D_x f^0(w(t))\|_{s-5} \le C(1+t)^{-\beta}\|w_0 - \bar{w}\|_{s-4,1} , \tag{4.40}$$

for $t \in [0,\infty)$, *where* $\beta = 3/4 - \varepsilon$, *and* $C = C(\varepsilon,\delta_7)$ *is a constant.*

Proof. Apply D_x to (4.29) (with $n=1$). Application of (3.A.14) to the resulting equation yields

$$(4.41)\qquad \|D_x V(t)\|_{s-5} \le C(1+t)^{-3/4}\|V(0)\|_{s-4,1} +$$

$$+ C\int_0^t e^{-c_2(t-\tau)}\{\|D_x^2 h(w)(\tau)\|_{s-5} + \|D_x^2\{H(w)w_x\}(\tau)\|_{s-5}\}\,d\tau +$$

$$+ C\int_0^t (1+t-\tau)^{-3/4}\|D_x h(w)(\tau)\|_{L^1} +$$

$$+ (1+t-\tau)^{-5/4}\|\{H(w)w_x\}(\tau)\|_{L^1}\,d\tau\ .$$

By use of the estimate (2.4) (with $p=\infty$ and $s=n=1$), we have in the same way as in $(4.31)_{1,2}$,

$$(4.42)_1\qquad \|D_x^2 h(w(V))\|_{s-5} + \|D_x^2\{H(w(V))w(V)_x\}(\tau)\|_{s-5}$$

$$\le C\,\|D_x V\|_{s-5}^{1/2}\,\|V\|_{s-4}^{1/2}\,\|V\|_{s-2}\ ,$$

$$(4.42)_2\qquad \|D_x h(w(V))\|_{L^1} + \|H(w(V))w(V)_x\|_{L^1} \le C\,\|V\|\,\|D_x V\|\ .$$

Substituting $(4.42)_{1,2}$ to (4.41) and using the decay estimate (4.24) (with $\gamma=1/4$ for $n=p=1$), we obtain

$$(4.43)\qquad |||D_x V(t)|||_{s-5,\beta} \le C\,\|w_0-\bar{w}\|_{s-4,1} +$$

$$+ C\mu_8(t)\,\|w_0-\bar{w}\|_{s-2,1}\|w_0-\bar{w}\|_{s-4,1}^{1/2}|||D_x V(t)|||_{s-5,\beta}^{1/2} +$$

$$+ \; C\mu_9(t)\,\|w_0-\bar{w}\|_{s-4,1}\,|||D_xV(t)|||_{s-5,\beta} \;,$$

where $\beta = 3/4 - \varepsilon$ (for any small $\varepsilon > 0$) and

$$\mu_8(t) = \sup_{0\le\tau\le t}(1+\tau)^{\beta}\int_0^{\tau} e^{-c_2(\tau-\tau_1)}(1+\tau_1)^{-(\beta/2+3/8)}\,d\tau_1 \;,$$

$$\mu_9(t) = \sup_{0\le\tau\le t}(1+\tau)^{\beta}\int_0^{\tau}(1+\tau-\tau_1)^{-3/4}(1+\tau_1)^{-(\beta+1/4)}\,d\tau_1 \;.$$

For every small $\varepsilon > 0$, $\mu_8(t)$ and $\mu_9(t)$ are bounded by a constant C_ε independent of $t \in [0,\infty)$. Therefore from (4.43) we can deduce the desired estimate (4.40). This completes the proof of Lemma 4.6.

Next we investigate the initial value problem for the semi-linear system (4.39). In the same way as in Theorems 4.3 and 4.4, and as in Lemma 4.6, we can obtain:

<u>Lemma 4.7</u> *Assume Conditions* 4.1-4.3 *for* $n=1$.

(i) Let $s \ge 1$ *and assume* $w_0 - \bar{w} \in H^s(\mathbb{R}^1)$. *Then there is a positive constant* δ_5^* *such that if* $\|w_0-\bar{w}\|_s \le \delta_5^*$, *the problem* (4.39), (4.34) *is solved globally in time as in Theorem* 4.3.

(ii) Let $s \ge 2$ *and assume* $w_0 - \bar{w} \in H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$. *Then there is a positive constant* δ_6^* $(\le \delta_5^*)$ *such that if* $\|w_0-\bar{w}\|_{s,1} \le \delta_6^*$, *the solution* $w^*(t,x)$ *of (i) satisfies*

$$(4.44) \qquad \|f^0(w^*(t)) - f^0(\bar{w})\|_{s-1} \le C(1+t)^{-1/4}\|w_0-\bar{w}\|_{s-1,1} \;.$$

(iii) Let $s \geq 3$ *and assume* $w_0 - \bar{w} \in H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$. *Then for any small* $\varepsilon > 0$, *there is a positive constant* $\delta_7^* = \delta_7^*(\varepsilon)$ $(\leq \delta_6^*)$ *such that if* $\|w_0 - \bar{w}\|_{s,1} \leq \delta_7^*$, *the solution* $w^*(t,x)$ *satisfies*

$$(4.45) \qquad \|D_x f^0(w^*(t))\|_{s-3} \leq C(1+t)^{-\beta}\|w_0 - \bar{w}\|_{s-2,1},$$

with $\beta = 3/4 - \varepsilon$ *and a constant* $C = C(\varepsilon, \delta_7^*)$.

Now we state the result on the asymptotic behavior (as $t \to \infty$) of the solution of (4.1),(4.2) in one-dimensional case.

Theorem 4.8 (asymptotic behavior) *Assume Conditions* 4.1-4.3. *Let* $n = 1$ *and* $s \geq 6$ *(an integer), and assume that* $w_0 - \bar{w} \in H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$. *Then for any small* $\varepsilon > 0$, *there exists a positive constant* $\delta_8 = \delta_8(\varepsilon)$ $(\leq \delta_7, \delta_7^*)$ *such that if* $\|w_0 - \bar{w}\|_{s,1} \leq \delta_8$, *then* $w(t,x)$ *and* $w^*(t,x)$, *the solutions of* (4.1), (4.2) *and* (4.39), (4.34) *respectively, satisfy*

$$(4.46) \qquad \|f^0(w(t)) - f^0(\bar{w}) - f^0_w(\bar{w})w^*(t)\|_{s-6} \leq C(1+t)^{-\beta}\|w_0 - \bar{w}\|^2_{s-4,1},$$

with $\beta = 3/4 - \varepsilon$ *and a constant* $C = C(\varepsilon, \delta_8)$. *(For the existence of* $w(t,x)$ *and* $w^*(t,x)$, *see Theorem* 4.3 *and Lemma* 4.7. *See also Theorem* 4.4 *and Lemma* 4.6.*)*

Remark 4.4 A similar result was proved by Kawashima $[38]_1$ for one-dimensional model equations of a viscous compressible fluid, which are derived from the Broadwell model of the Boltzmann equation by the Chapman-Enskog expansion. Using this result, he proved the following: in one-dimensional case, the Broadwell model of the Boltzmann equation can be approximated by the model equations of a viscous compressible fluid for $t \to \infty$.

Proof of Theorem 4.8 We express the solution $w^*(t,x)$ of (4.39),(4.34) by means of e^{-tS} and subtruct it from (4.29). Noting (4.26) and (4.38), we have

$$(V - w^*)(t) = \int_0^t (A^0)^{-1/2} e^{-(t-\tau)S} (A^0)^{-1/2} \{h^*(V) - h^*(w^*) +$$

$$+ h_R(V) + H(w(V))w(V)_x\}_x(\tau)\, d\tau .$$

Applying (3.A.14) to the above equation, we get

$$(4.47) \qquad \|(V - w^*)(t)\|_{s-6} \le C \int_0^t e^{-c_2(t-\tau)} \{\|h^*(V)_x(\tau)\|_{s-6} +$$

$$+ \|h^*(w^*)_x(\tau)\|_{s-6} + \|h_R(V)_x(\tau)\|_{s-6} + \|\{H(w(V))w(V)_x\}_x(\tau)\|_{s-6}\}\, d\tau +$$

$$+ C\int_0^t (1+t-\tau)^{-3/4} \|h^*(V)(\tau) - h^*(w^*)(\tau)\|_{L^1}\, d\tau +$$

$$+ C\int_0^t (1+t-\tau)^{-3/4} \{\|h_R(V)(\tau)\|_{L^1} + \|\{H(w(V))w(V)_x\}(\tau)\|_{L^1}\}\, d\tau .$$

The norms in the right member can be estimated in the same way as in $(4.31)_{1,2}$:

$$\|h^*(V)_x\|_{s-6} + \|h^*(w^*)_x\|_{s-6} + \|h_R(V)_x\|_{s-6} +$$

$$+ \; \| \{H(w(V))w(V)_x\}_x \|_{s-6} \le C\, \|V, w^*\|_{s-5}\, \|D_x(V,w^*)\|_{s-5} ,$$

$$\|h^*(V) - h^*(w^*)\|_{L^1} \le C\, \|V, w^*\|\, \|V - w^*\| ,$$

$$\|h_R(V)\|_{L^1} \le C\, \|V\|^{5/2}\, \|D_x V\|^{1/2} .$$

Substitute the above estimates and $(4.42)_2$ to (4.47). Then by virtue of the decay estimates (4.24) (with $\gamma = 1/4$ for $n = p = 1$), (4.40), (4.44) and (4.45), we have the inequality for $|||(V - w^*)(t)|||_{s-6,\beta}$ (with $\beta = 3/4 - \varepsilon$):

$$(4.48) \qquad |||(V - w^*)(t)|||_{s-6,\beta} \le C\{\mu_9(t) + \mu_{10}(t)\}\|w_0 - \bar{w}\|^2_{s-4,1} +$$

$$+ \; C\mu_9(t)\,\|w_0 - \bar{w}\|_{s-4,1}|||(V - w^*)(t)|||_{s-6,\beta} ,$$

where $\mu_9(t)$ appeared in (4.43) and $\mu_{10}(t)$ is given by

$$\mu_{10}(t) = \sup_{0 \le \tau \le t} (1+\tau)^{\beta} \int_0^{\tau} (1+\tau-\tau_1)^{-3/4}(1+\tau_1)^{-(\beta/2+5/8)}\, d\tau_1 .$$

Both of the first and the last integrals in the right member of (4.47) are corresponding to the first term in the right hand side of (4.48). It is easy to see that $\mu_9(t) + \mu_{10}(t) \le C_\varepsilon$ holds for every small $\varepsilon > 0$. Therefore we can deduce from (4.48) that the estimate $|||(V - w^*)(t)|||_{s-6,\beta} \le C_\varepsilon \|w_0 - \bar{w}\|^2_{s-4,1}$ holds for suitably small $\|w_0 - \bar{w}\|_{s,1}$. Thus the proof of Theorem 4.8 is completed.

CHAPTER V

HYPERBOLIC SYSTEMS OF CONSERVATION LAWS WITH VANISHING VISCOSITY

5.1 INTRODUCTION

In this chapter we shall consider the initial value problem for the system of conservation laws with a parameter $\varepsilon \in (0,1]$:

$$(5.1)\qquad f^0(w)_t + \sum_j f^j(w)_{x_j} = \varepsilon \sum_{jk} \{G^{jk}(w)w_{x_k}\}_{x_j} ,$$

$$(5.2)\qquad w(0,x) = w_0(x) ,$$

where $t \geq 0$ and $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$; $w = w(t,x)$, $f^j(w)$ $(j=0,1,\cdots,n)$ and $G^{jk}(w)$ $(j,k=1,\cdots,n)$ are the same as in chapter IV. We also assume Conditions 4.1 and 4.2. Then it was examined in chapter IV that the system (5.1) can be reduced to

$$(5.3)\qquad A^0(w)w_t + \sum_j A^j(w)w_{x_j} - \varepsilon \sum_{jk} B^{jk}(w)w_{x_j x_k} = \varepsilon g(w,D_x w) ,$$

which is symmetric hyperbolic-parabolic (in the sense indicated in chapter II) for every fixed $\varepsilon \in (0,1]$, where $A^j(w)$ $(j=0,1,\cdots,n)$, $B^{jk}(w)$

$(j,k=1,\cdots,n)$ and $g(w,D_x w)$ are defined by $(4.4)_{1,2,3}$, respectively. It should be also noted that the system (5.3) with $\varepsilon = 0$ is a symmetric hyperbolic one, as was pointed out by Friedrichs and Lax [19]; this fact is seen from $(4.5)_{1,2}$.

In the following we are interested in the asymptotic problem as $\varepsilon \to 0$. Since (5.3) with $\varepsilon \in (0,1]$ is symmetric hyperbolic-parabolic, Theorem 2.9 gives a local solution of (5.1),(5.2) on a time interval $[0,T^{\varepsilon}]$. However in this case we only know $T^{\varepsilon} = O(\varepsilon)$ for $\varepsilon \to 0$ and so we can not take the limit as $\varepsilon \to 0$. To prove the existence of solutions to (5.1),(5.2) on a time interval $[0,T]$ independent of ε, we must utilize the property that $A^j(w)$ $(j=1,\cdots,n)$ are real symmetric; in Theorem 2.9 we only require that the block matrices $A_{11}^j(w)$ $(j=1,\cdots,n)$ are real symmetric. As is expected, we can establish the following results. "The initial value problem (5.1),(5.2) has a unique smooth solution $w^{\varepsilon} = w^{\varepsilon}(t,x)$ on a time interval $[0,T]$ independent of ε if the initial data satisfy $w_0 - \bar{w} \in H^s(\mathbb{R}^n)$ (with $n \geq 1$ and $s \geq [n/2]+2$) with a given constant state $\bar{w} \in \mathcal{O}$. Furthermore, as $\varepsilon \to 0$, the solution w^{ε} converges on $[0,T]$ to a limit w^0, which is a unique smooth solution of the limit system (i.e., (5.1) with $\varepsilon = 0$) for the same initial data." (see Theorems 5.4 and 5.5).

Similar results were obtained by many authors. In one-dimensional case the system (5.1) with $\varepsilon = 0$ admits shock-wave solutions and (5.1) with $\varepsilon \in (0,1]$ admits progressive-wave solutions, if (5.1) with $\varepsilon = 0$ is strictly hyperbolic and genuinely nonlinear in the sense of Lax $[50]_1$. The convergence of progressive waves to the shock waves as $\varepsilon \to 0$ was proved in [17], $[10]_1$, [73], [74] for general systems, in [86], [20], [21] for the equations of compressible fluids, and in $[10]_{2,3}$, [29] for the equations of magnetohydrodynamics. We also refer to DiPerna $[13]_6$, where general convergence theorems are established for some simplest systems in one space-dimension: smooth solutions w^ε converge to a limit w^0 for all time $t \geq 0$, and w^0 is a weak solution of the limit system. For convergence results (local in time) in higher dimensions $(n \geq 2)$, we refer to [63], where the equations of compressible fluids in $\mathbb{R}^3$ are considered.

Similar convergence problems were also considered for the equations of incompressible fluids and for the Boltzmann equation. We refer to [24], [57], [76], $[37]_2$ for the former equations and $[61]_3$, [6], $[80]_3$ (and also [33], [7]) for the latter one.

The contents of this chapter are as follows. In section 5.2 we consider the linearized system (with variable coefficients) for (5.1). The energy estimates and the existence results, which are valid uniformly in $\varepsilon \in (0,1]$, are established by a similar method as in section 2.3 of chapter II. By virtue of these results, we can show that (5.1), (5.2) has a smooth solution $w^\varepsilon = w^\varepsilon(t,x)$ on a time interval $[0,T]$ independent of $\varepsilon \in (0,1]$, see section 5.3. The solution is constructed by the successive approximation method. In section 5.4 we shall prove that as $\varepsilon \to 0$, w^ε converges on $[0,T]$ to a smooth solution of the limit system. Section 5.5 contains some remarks on the global existence problem for (5.1), (5.2). In particular, it is proved that if the amplitude of the initial data is $O(\varepsilon)$, then (5.1), (5.2) admits a global solution whose amplitude is of the same order.

5.2 UNIFORM STABILITY FOR LINEARIZED EQUATIONS

In this section we shall consider the linearized equations of the form

$$(5.4) \qquad A^0(w)\hat{w}_t + \sum_j A^j(w)\hat{w}_{x_j} - \varepsilon \sum_{jk} B^{jk}(w)\hat{w}_{x_j x_k} = f + \varepsilon g ,$$

where coefficient matrices $A^j(w)$ $(j=0,1,\cdots,n)$ and $B^{jk}(w)$ $(j,k=1,\cdots,n)$ are the same as in chapter IV (cf. $(4.5)_{1,2,3}$ and Condition 4.2); $w(t,x) = {}^t(u,v)(t,x)$, $f(t,x) = {}^t(f_1,f_2)(t,x)$ and $g(t,x) = {}^t(g_1,g_2)(t,x)$ are given functions on $Q_T = [0,T]\times \mathbb{R}^n$. Let $s \geq s_0 + 1$ $(s_0 = [n/2]+1)$ and $1 \leq \ell \leq s$ be integers. Let $w = {}^t(u,v)$ satisfy $(2.10)_{1,2}$ and (2.11):

$$(5.5)_1 \qquad u - \bar{u} \in C^0(0,T\,;H^s(\mathbb{R}^n)) , \qquad \partial_t u \in C^0(0,T\,;H^{s-1}(\mathbb{R}^n)) ,$$

$$(5.5)_2 \qquad \begin{cases} v - \bar{v} \in C^0(0,T\,;H^s(\mathbb{R}^n)) , \\ \partial_t v \in C^0(0,T\,;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{s-1}(\mathbb{R}^n)) , \end{cases}$$

$$(5.6) \qquad w(t,x) = {}^t(u,v)(t,x) \in \mathcal{O}_1 \quad \text{for any} \quad (t,x) \in Q_T ,$$

where $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ is an arbitrarily fixed constant state and $\mathcal{O}_1$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_1 \subset \mathcal{O}$. For $f = {}^t(f_1,f_2)$ and $g = {}^t(g_1,g_2)$, we assume:

$$(5.7) \qquad f = {}^t(f_1,f_2) \in C^0(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{\ell}(\mathbb{R}^n)) ,$$

$$(5.8)_1 \qquad \varepsilon^{1/2} g_1 \in C^0(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{\ell}(\mathbb{R}^n)) \ ,$$

$$(5.8)_2 \qquad g_2 \in C^0(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \ .$$

As in chapter II, we also consider the conditions $(5.5)_{1,2}$, (5.7) and $(5.8)_{1,2}$ with $C^0(\cdots)$ replaced by $L^\infty(\cdots)$.

$$(5.9)_1 \qquad u - \bar{u} \in L^\infty(0,T\,;H^s(\mathbb{R}^n)) \ , \qquad \partial_t u \in L^\infty(0,T\,;H^{s-1}(\mathbb{R}^n)) \ ,$$

$$(5.9)_2 \qquad \begin{cases} v - \bar{v} \in L^\infty(0,T\,;H^s(\mathbb{R}^n)) \ , \\ \partial_t v \in L^\infty(0,T\,;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{s-1}(\mathbb{R}^n)) \ , \end{cases}$$

$$(5.10) \qquad f = {}^t(f_1,f_2) \in L^\infty(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{\ell}(\mathbb{R}^n)) \ ,$$

$$(5.11)_1 \qquad \varepsilon^{1/2} g_1 \in L^\infty(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \cap L^2(0,T\,;H^{\ell}(\mathbb{R}^n)) \ ,$$

$$(5.11)_2 \qquad g_2 \in L^\infty(0,T\,;H^{\ell-1}(\mathbb{R}^n)) \ .$$

Under these assumptions we can prove the energy estimates for (5.4), which are valid uniformly in $\varepsilon \in (0,1]$.

<u>Lemma 5.1</u> *Let* $A^j(w)$ $(j=0,1,\cdots,n)$ *and* $B^{jk}(w)$ $(j,k=1,\cdots,n)$ *are the same as in chapter* IV. *Let* $n \ge 1$, $s \ge s_0+1$ $(s_0 = [n/2]+1)$ *and* $1 \le \ell \le s$ *be integers. Let* $w = {}^t(u,v)$ *satisfy the conditions* $(5.9)_{1,2}$ *and* (5.6) *and put*

$$M = \sup_{0\le t\le T} \|(w-\bar{w})(t)\|_s \ , \qquad M_1 = \left(\int_0^T \|\partial_t w(t)\|_{s-1}^2\, dt\right)^{1/2} .$$

Let $f = {}^t(f_1,f_2)$ *and* $g = {}^t(g_1,g_2)$ *satisfy the conditions* (5.10) *and* $(5.11)_{1,2}$, *respectively. Further assume that* $\hat{w} = {}^t(\hat{u},\hat{v})$ *is a solution of* (5.4) *(with* $\varepsilon \in (0,1]$*) satisfying*

$$\hat{u} \in L^\infty(0,T;H^\ell(\mathbb{R}^n)), \qquad \partial_t\hat{u} \in L^\infty(0,T;H^{\ell-1}(\mathbb{R}^n)),$$

$$\hat{v} \in L^\infty(0,T;H^\ell(\mathbb{R}^n)); \qquad \partial_t\hat{v} \in L^\infty(0,T;H^{\ell-2}(\mathbb{R}^n)).$$

Then it follows that $\hat{w} = {}^t(\hat{u},\hat{v}) \in C^0(0,T;H^\ell(\mathbb{R}^n))$ *and* $\varepsilon^{1/2}\hat{v} \in L^2(0,T;H^{\ell+1}(\mathbb{R}^n))$. *Furthermore we have the energy estimate which is valid uniformly in* $\varepsilon \in (0,1]$*:*

$$(5.12) \qquad \|(\hat{u},\hat{v})(t)\|_\ell^2 + \varepsilon\int_0^t \|\hat{v}(\tau)\|_{\ell+1}^2 d\tau$$

$$\leq C_9^2 e^{C_{10}(t/\alpha + M_1 t^{1/2})} [\, \|(\hat{u},\hat{v})(0)\|_\ell^2 + \int_0^t \|(f_1,f_2)(\tau)\|_\ell^2 d\tau +$$

$$+ \varepsilon\{\alpha\varepsilon\int_0^t \|g_1(\tau)\|_\ell^2 d\tau + C_{10}\int_0^t \|g_2(\tau)\|_{\ell-1}^2 d\tau\}\,]$$

for $t \in [0,T]$ *and for any* $\alpha \in (0,1]$, *where* $C_9 = C_9(\mathcal{O}_1) > 1$ *and* $C_{10} = C_{10}(\mathcal{O}_1,M)$ *are constants independent of* ε *and* α.

Proof. We use the arguments in Lemma 2.6. Apply D_x^k $(k \leq \ell)$ to the system (5.4) and take the inner product (in $\mathbb{R}^m$) of the resulting equation by $D_x^k\hat{w}$. Integrating it over $\mathbb{R}^n$ and adding for $k = 0,1,\cdots,\ell$, we obtain

$$(5.13)\qquad \sum_{k=0}^{\ell}\int < A^0(w)D_x^k\hat{w}_t + \sum_j A^j(w)D_x^k\hat{w}_{x_j} - \varepsilon\sum_{ij} B^{ij}(w)D_x^k\hat{w}_{x_ix_j},\ D_x^k\hat{w} > dx$$

$$= \sum_{k=0}^{\ell}\int < F^k + \varepsilon G^k,\ D_x^k\hat{w} > dx ,$$

where

$$F^k = A^0(w)D_x^k\{A^0(w)^{-1}f\} - A^0(w)\sum_j [D_x^k,\ A^0(w)^{-1}A^j(w)]\hat{w}_{x_j} ,$$

$$G^k = A^0(w)D_x^k\{A^0(w)^{-1}g\} + A^0(w)\sum_{ij} [D_x^k,\ A^0(w)^{-1}B^{ij}(w)]\hat{w}_{x_ix_j} .$$

Let us introduce the energy norm

$$E[\hat{w}] = \Big(\sum_{k=0}^{\ell}\int < A^0(w)D_x^k\hat{w},\ D_x^k\hat{w} > dx\Big)^{1/2} ,$$

which is equivalent to $\|\hat{w}\|_\ell$. By the arguments in Lemma 2.6, we can deduce that the left hand side of (5.13) is bounded from below by

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}E[\hat{w}]^2\Big) + \varepsilon c_0\|\hat{v}\|_{\ell+1}^2 - C(1+\|\partial_t w\|_{s-1})E[\hat{w}]^2 ,$$

where $c_0 = c_0(\mathcal{O}_1)$ and $C = C(\mathcal{O}_1, M)$ are positive constants independent of ε. On the other hand the right member of (5.13) is majorized by

$$C(E[\hat{w}]^2 + \|f_1, f_2\|_\ell E[\hat{w}]) +$$

$$+ \varepsilon C(E[\hat{w}]\|\hat{v}\|_{\ell+1} + \|g_1\|_\ell\|\hat{u}\|_\ell + \|g_2\|_{\ell-1}\|\hat{v}\|_{\ell+1})$$

with a constant $C = C(\mathcal{O}_1, M)$ independent of ε. Therefore, noting that

$\varepsilon C\|g_1\|_\ell \|\hat{u}\|_\ell \le \alpha\varepsilon^2\|g_1\|_\ell^2 + \alpha^{-1}C\,E[\hat{w}]^2$ for any $\alpha > 0$, we can get the inequality

$$\frac{\partial}{\partial t} E[\hat{w}]^2 + \varepsilon c_0\|\hat{v}\|_{\ell+1}^2 \le \alpha^{-1}C(1 + \|\partial_t w\|_{s-1})E[\hat{w}]^2 +$$

$$+ \|f_1, f_2\|_\ell^2 + \varepsilon(\alpha\varepsilon\|g_1\|_\ell^2 + C\|g_2\|_{\ell-1}^2) ,$$

where $\alpha \le 1$ is assumed. The desired estimate (5.12) is an immediate consequence of the above inequality. We omit the proof of the regularity results, because these results can be obtained in the same way as in Lemma 2.6. This completes the proof of Lemma 5.1.

The existence of a solution of (5.4) follows from Proposition 2.7. Indeed, (5.4) can be written in the form

$$(5.14) \quad \begin{cases} A_1^0(u,v)\hat{u}_t + \sum_j A_{11}^j(u,v)\hat{u}_{x_j} = f_1^\varepsilon(t,x\,;D_x\hat{v}) , \\ A_2^0(u,v)\hat{v}_t - \varepsilon\sum_{jk} B_2^{jk}(u,v)\hat{v}_{x_jx_k} = f_2^\varepsilon(t,x\,;D_x\hat{u},D_x\hat{v}) , \end{cases}$$

where

$$f_1^\varepsilon(t,x\,;D_x\hat{v}) = f_1 + \varepsilon g_1 - \sum_j A_{12}^j(u,v)\hat{v}_{x_j} ,$$

$$f_2^\varepsilon(t,x\,;D_x\hat{u},D_x\hat{v}) = f_2 + \varepsilon g_2 - \sum_j \{A_{21}^j(u,v)\hat{u}_{x_j} + A_{22}^j(u,v)\hat{v}_{x_j}\} .$$

Therefore, based on Proposition 2.7, we can define the successive approximation sequence $\{\hat{w}^n\} = \{(\hat{u}^n,\hat{v}^n)\}$ for (5.14) as follows:

$$(\hat{u}^0,\hat{v}^0)(t,x) = (\bar{u},\bar{v}) ,$$

and for $n \geq 0$,

$$A_1^0(u,v)\hat{u}_t^{n+1} + \sum_j A_{11}^j(u,v)\hat{u}_{x_j}^{n+1} = f_1^\varepsilon(t,x\,;D_x\hat{v}^{n+1}) ,$$

$$A_2^0(u,v)\hat{v}_t^{n+1} - \varepsilon\sum_{jk} B_2^{jk}(u,v)\hat{v}_{x_jx_k}^{n+1} = f_2^\varepsilon(t,x\,;D_x\hat{u}^n,D_x\hat{v}^n) ,$$

$$(\hat{u}^{n+1},\hat{v}^{n+1})(0,x) = (\hat{u},\hat{v})(0,x) .$$

Note that the last variable of f_1^ε is not $D_x\hat{v}^n$ but $D_x\hat{v}^{n+1}$. We apply the energy estimates $(2.16)_{1,2}$ to the equations for the difference $\hat{w}^{n+1} - \hat{w}^n$. Then it follows that for every fixed $\varepsilon \in (0,1]$, $\hat{w}^n = {}^t(\hat{u}^n,\hat{v}^n)$ and $\hat{v}^n$ are the Cauchy sequences in $C^0(0,T\,;H^\ell(\mathbb{R}^n))$ and $L^2(0,T\,;H^{\ell+1}(\mathbb{R}^n))$, respectively. So we have a solution of (5.14) (and therefore (5.4)) as a strong limit of the sequence $\hat{w}^n$. Summarizing the above considerations, we have:

<u>Proposition 5.2</u> (uniform stability for linearized equations) *Let* $A^j(w)$ *(*$j=0,1,\cdots,n$*) and* $B^{jk}(w)$ *(*$j,k=1,\cdots,n$*) are the same as in chapter* IV. *Let* $n \geq 1$, $s \geq s_0+1$ *(*$s_0=[n/2]+1$*) and* $1 \leq \ell \leq s$ *be integers. Let* $w={}^t(u,v)$, $f={}^t(f_1,f_2)$ *and* $g={}^t(g_1,g_2)$ *satisfy the conditions* $(5.5)_{1,2}-(5.8)_{1,2}$. *If the initial data satisfy* $\hat{w}(0)={}^t(\hat{u},\hat{v})(0) \in H^\ell(\mathbb{R}^n)$, *then* (5.4) *with* $\varepsilon \in (0,1]$ *admits a unique solution* $\hat{w}(t,x) = {}^t(\hat{u},\hat{v})(t,x)$ *with*

$$\hat{u} \in C^0(0,T;H^\ell(\mathbb{R}^n)) \cap C^1(0,T;H^{\ell-1}(\mathbb{R}^n)) ,$$

$$\hat{v} \in C^0(0,T;H^\ell(\mathbb{R}^n)) \cap C^1(0,T;H^{\ell-2}(\mathbb{R}^n)) ,$$

$$\varepsilon^{1/2}\hat{v} \in L^2(0,T;H^{\ell+1}(\mathbb{R}^n)) ,$$

which satisfies the energy estimate (5.12) *for* $t \in [0,T]$ *and for any* $\varepsilon, \alpha \in (0,1]$.

5.3 UNIFORM STABILITY FOR NONLINEAR EQUATIONS

In this section we shall construct a solution of (5.3) (and consequently (5.1)) on a time interval independent of $\varepsilon \in (0,1]$ by the successive approximation method. To prove the existence of a invariant set (uniformly bounded with respect to ε) under iterations, we first consider the linearized system for (5.3):

$$A^0(w)\hat{w}_t + \sum_j A^j(w)\hat{w}_{x_j} - \varepsilon \sum_{jk} B^{jk}(w)\hat{w}_{x_j x_k} = \varepsilon g(w,D_x w) , \tag{5.15}$$

with the initial data

$$\hat{w}(0,x) = w(0,x) = w_0(x) . \tag{5.16}$$

Let Conditions 4.1 and 4.2 be assumed and let $s \geq s_0 + 1$ $(s_0 = [n/2]+1)$ be an integer. For $w_0(x) = {}^t(u_0,v_0)(x)$ we assume that $w_0 - \bar{w} = {}^t(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$ and

$$(5.17) \qquad w_0(x) = {}^t(u_0,v_0)(x) \in \mathcal{O}_0 \qquad \text{for any} \quad x \in \mathbb{R}^n ,$$

where $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ is an arbitrarily fixed constant state and $\mathcal{O}_0$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_0 \subset \mathcal{O}$. For given functions $w(t,x) = {}^t(u,v)(t,x)$, we assume that

$$(5.5)_1 \qquad u - \bar{u} \in C^0(0,T;H^s(\mathbb{R}^n)) , \qquad \partial_t u \in C^0(0,T;H^{s-1}(\mathbb{R}^n)) ,$$

$$(5.5)_2' \qquad \begin{cases} v - \bar{v} \in C^0(0,T;H^s(\mathbb{R}^n)) , \qquad \varepsilon^{1/2}(v - \bar{v}) \in L^2(0,T;H^{s+1}(\mathbb{R}^n)) , \\ \partial_t v \in C^0(0,T;H^{s-2}(\mathbb{R}^n)) \cap L^2(0,T;H^{s-1}(\mathbb{R}^n)) , \end{cases}$$

$$(5.6) \qquad w(t,x) = {}^t(u,v)(t,x) \in \mathcal{O}_1 \qquad \text{for any} \quad (t,x) \in Q_T ,$$

$$(5.18)_1 \qquad \sup_{0\le\tau\le t} \| (u-\bar{u},\ v-\bar{v})(\tau) \|_s^2 + \varepsilon\int_0^t \| (v-\bar{v})(\tau) \|_{s+1}^2 d\tau \le M^2 ,$$

$$(5.18)_2 \qquad \int_0^t \| \partial_t (u,v)(\tau) \|_{s-1}^2 d\tau \le M_1^2 \qquad \text{for} \quad t \in [0,T] ,$$

where $\mathcal{O}_1$ is a bounded open convex set in $\mathbb{R}^m$ satisfying $\bar{\mathcal{O}}_1 \subset \mathcal{O}$, and M and M_1 are constants; $\mathcal{O}_1$, M and M_1 will be determined later. We denote by $X_T^s(\mathcal{O}_1,M,M_1;\varepsilon)$ the set of functions $w(t,x) = {}^t(u,v)(t,x)$ satisfying $(5.5)_1$, $(5.5)_2'$, (5.6) and $(5.18)_{1,2}$.

Let $w = {}^t(u,v) \in X_T^s(\mathcal{O}_1,M,M_1;\varepsilon)$. Then $\|g(w,D_x w)\|_{s-1} \le CM^2$ holds with some positive constant $C = C(\mathcal{O}_1,M)$ (see (4.7)). Therefore, if $\hat{w} = {}^t(\hat{u},\hat{v})$ is a solution of (5.15) satisfying $(5.5)_1$, $(5.5)_2'$ and $(5.18)_1$ with M replaced by $\hat{M}$, then there is a constant $C_{11} = C_{11}(\mathcal{O}_1,M)$ such that

$$(5.19) \qquad \int_0^t \|\partial_t(\hat{u},\hat{v})(\tau)\|_{s-1}^2 \, d\tau \le C_{11}^{\,2}\{\varepsilon \hat{M}^2 + (\hat{M}^2 + \varepsilon^2 M^4)t\}$$

holds for $t \in [0,T]$. Now fix d_1 so that $0 < d_1 < d_0 \equiv \mathrm{dist}(\mathcal{O}_0, \partial\mathcal{O})$, and then take $\mathcal{O}_1$, M and M_1 as follows:

$$(5.20) \qquad \begin{cases} \mathcal{O}_1 = d_1\text{-neighborhood of } \mathcal{O}_0 , \\ M = 2C_9 \|u_0 - \bar{u}, v_0 - \bar{v}\|_s , \qquad M_1 = 2C_{11}M , \end{cases}$$

where $C_9 = C_9(\mathcal{O}_1)$ and $C_{11} = C_{11}(\mathcal{O}_1, M)$ are constants in Lemma 5.1 and (5.19), respectively. For this choice of $\mathcal{O}_1$, M and M_1, we can show that the set $X_T^s(\mathcal{O}_1, M, M_1; \varepsilon)$ is invariant under the mapping $(u,v) \to (\hat{u},\hat{v})$ if T is sufficiently small (but independent of ε).

<u>Proposition 5.3</u> (invariant set under iterations) *Let Conditions* 4.1 *and* 4.2 *be assumed. Let* $n \ge 1$ *and* $s \ge s_0 + 1$ ($s_0 = [n/2]+1$) *be integers, and let* $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ *be an arbitrarily fixed constant state. Suppose that the initial data satisfy* $w_0 - \bar{w} = {}^t(u_0 - \bar{u}, v_0 - \bar{v}) \in H^s(\mathbb{R}^n)$ *and* (5.17). *Then there exists a positive constant* T_2 ($\le T$)*, depending on* $\mathcal{O}_0$, d_1 *and* $\|u_0 - \bar{u}, v_0 - \bar{v}\|_s$ *but not on* $\varepsilon \in (0,1]$*, such that if* $w = {}^t(u,v) \in X_{T_2}^s(\mathcal{O}_1, M, M_1; \varepsilon)$ *with* $\mathcal{O}_1$, M *and* M_1 *defined by* (5.20)*, the initial value problem* (5.15), (5.16) *has a unique solution* $\hat{w} = {}^t(\hat{u},\hat{v})$ *in the same* $X_{T_2}^s(\mathcal{O}_1, M, M_1; \varepsilon)$.

Proof. This lemma can be proved in the same way as in Proposition 2.8. It suffices to show $\hat{w} = {}^t(\hat{u},\hat{v}) \in X_{T_2}^s(\mathcal{O}_1, M, M_1; \varepsilon)$, because the existence of

$\hat{w}$ follows immediately from Proposition 5.2 with $\ell = s$. Since $\|g_1(u, v, D_x v)\|_s \le CM\|D_x v\|_s$ and $\|g_2(u,v,D_x u, D_x v)\|_{s-1} \le CM^2$, the energy estimate (5.12) (with $\ell = s$) yields

$$(5.21)\qquad \|(\hat{u}-\bar{u}, \hat{v}-\bar{v})(t)\|_s^2 + \varepsilon\int_0^t \|(\hat{v}-\bar{v})(\tau)\|_{s+1}^2 d\tau$$

$$\le C_9^2 e^{C_{10}(t/\alpha + M_1 t^{1/2})}\{\|u_0-\bar{u}, v_0-\bar{v}\|_s^2 + \varepsilon CM^4(\alpha+t)\},$$

where $C = C(\mathcal{O}_1, M)$ is a constant independent of $\varepsilon \in (0,1]$ and $\alpha \in (0,1]$. Take α so that $2\alpha CM^4 \le \|u_0-\bar{u}, v_0-\bar{v}\|_s^2$. For this choice of α, we take T_2 so small that

$$e^{C_{10}(T_2/\alpha + M_1 T_2^{1/2})} \le 2, \quad 2CM^4 T_2 \le \|u_0-\bar{u}, v_0-\bar{v}\|_s^2 .$$

Then the right hand side of (5.21) is majorized by $4C_9^2\|u_0-\bar{u}, v_0-\bar{v}\|_s^2 = M^2$. Therefore it is proved that $\hat{w} = {}^t(\hat{u},\hat{v})$ satisfies $(5.18)_1$. Since we have (5.19), the remainig estimates $(5.18)_2$ and (5.6) can be verified in the same way as in Proposition 2.8. This completes the proof of Proposition 5.3.

By virtue of Proposition 5.3, the initial value problem (5.1),(5.2) can be solved on a time interval independent of ε:

<u>Theorem 5.4</u> (uniform stability for nonlinear equations) *Let Conditions* 4.1 *and* 4.2 *be assumed. Let* $n \ge 1$ *and* $s \ge s_0 + 1$ $(s_0 = [n/2]+1)$ *be integers, and let* $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ *be an arbitrarily fixed constant state.*

Suppose that the initial data satisfy $w_0 - \overline{w} = {}^t(u_0 - \overline{u}, v_0 - \overline{v}) \in H^s(\mathbb{R}^n)$ *and* (5.17). *Then there exists a positive constant* T_3 *(* $\le T_2$*), depending on* $\mathcal{O}_0$, d_1 *and* $\|u_0 - \overline{u},\ v_0 - \overline{v}\|_s$ *but not on* $\varepsilon \in (0,1]$, *such that the initial value problem* (5.1), (5.2) *has a unique solution* $w = {}^t(u,v) \in X^s_{T_3}(\mathcal{O}_1, M, M_1; \varepsilon)$ *for any* $\varepsilon \in (0,1]$, *where* $\mathcal{O}_1$, M *and* M_1 *are determined by* (5.20). *In particular, the solution satisfies*

$$u - \overline{u} \in C^0(0,T_3\,;H^s(\mathbb{R}^n)) \cap C^1(0,T_3\,;H^{s-1}(\mathbb{R}^n)) ,$$

$$v - \overline{v} \in C^0(0,T_3\,;H^s(\mathbb{R}^n)) \cap C^1(0,T_3\,;H^{s-2}(\mathbb{R}^n)) ,$$

$$\varepsilon^{1/2}(v - \overline{v}) \in L^2(0,T_3\,;H^{s+1}(\mathbb{R}^n)) ,$$

$$\text{(5.22)} \qquad \sup_{0\le\tau\le t} \|(u-\overline{u},\ v-\overline{v})(\tau)\|_s^2 + \varepsilon\int_0^t \|(u-\overline{u})(\tau)\|_s^2 +$$

$$+ \|(v-\overline{v})(\tau)\|_{s+1}^2\, d\tau \le C_{12}^2 \|u_0 - \overline{u},\ v_0 - \overline{v}\|_s^2 \qquad \textit{for } t \in [0,T_3] ,$$

where $C_{12} > 1$ *is a constant depending on* $\mathcal{O}_0$, d_1 *and* $\|u_0 - \overline{u},\ v_0 - \overline{v}\|_s$ *but not on* ε.

Proof. Let us introduce the successive approximation sequence $\{w^n\} = \{(u^n, v^n)\}$ for (5.3), (5.2) as follows:

$$w^0(t,x) = {}^t(u^0, v^0)(t,x) = {}^t(\overline{u}, \overline{v}) ,$$

and for $n \ge 0$,

$$A^0(w^n)w^{n+1}_t + \sum_j A^j(w^n)w^{n+1}_{x_j} - \varepsilon \sum_{jk} B^{jk}(w^n)w^{n+1}_{x_j x_k} = \varepsilon g(w^n, D_x w^n) ,$$

$$w^{n+1}(0,x) = w_0(x) .$$

By Proposition 5.3 the sequence $w^n = {}^t(u^n, v^n)$ is well defined on Q_{T_2} for $n \geq 0$, and is uniformly bounded with respect to $n \geq 0$ and $\varepsilon \in (0,1]$, i.e., $w^n = {}^t(u^n, v^n) \in X^s_{T_2}(\mathcal{O}_1, M, M_1; \varepsilon)$. To prove the convergence of w^n to a solution of (5.3), (5.2), we consider the difference $\hat{w}^n = w^{n+1} - w^n = {}^t(u^{n+1} - u^n, v^{n+1} - v^n) = {}^t(\hat{u}^n, \hat{v}^n)$:

$$(5.23) \qquad A^0(w^n)\hat{w}^n_t + \sum_j A^j(w^n)\hat{w}^n_{x_j} - \varepsilon \sum_{jk} B^{jk}(w^n)\hat{w}^n_{x_j x_k} = \hat{f}^n + \varepsilon \hat{g}^n ,$$

$$(5.24) \qquad \hat{w}^n(0,x) = 0 ,$$

where

$$\hat{f}^n = - A^0(w^n) \sum_j \{A^0(w^n)^{-1}A^j(w^n) - A^0(w^{n-1})^{-1}A^j(w^{n-1})\} w^n_{x_j} ,$$

$$\hat{g}^n = A^0(w^n)\{A^0(w^n)^{-1}g(w^n, D_x w^n) - A^0(w^{n-1})^{-1}g(w^{n-1}, D_x w^{n-1})\} +$$

$$+ A^0(w^n) \sum_{jk} \{A^0(w^n)^{-1}B^{jk}(w^n) - A^0(w^{n-1})^{-1}B^{jk}(w^{n-1})\} w^n_{x_j x_k} .$$

Since $w^n \in X^s_{T_2}(\mathcal{O}_1, M, M_1; \varepsilon)$, the right members $\hat{f}^n = {}^t(\hat{f}^n_1, \hat{f}^n_2)$ and $\hat{g}^n = {}^t(\hat{g}^n_1, \hat{g}^n_2)$ satisfy

$$\|\hat{f}^n_1, \hat{f}^n_2\|_{s-1} + \|\hat{g}^n_2\|_{s-2} \leq CM\, \|\hat{w}^{n-1}\|_{s-1} ,$$

$$\|\hat{g}^n_1\|_{s-1} \leq CM(\|\hat{u}^{n-1}\|_{s-1} + \|\hat{v}^{n-1}\|_s)$$

for a constant $C = C(\mathcal{O}_1, M)$ independent of n and ε. Therefore application of (5.12) (with $\ell = s-1$) to (5.23),(5.24) yields

$$(5.25)\qquad \sup_{0\le\tau\le t} \|\hat{w}^n(\tau)\|_{s-1}^2 + \varepsilon\int_0^t \|\hat{v}^n(\tau)\|_s^2 d\tau$$

$$\le CM^2 e^{C(t/\alpha + M_1 t^{1/2})}\{t \sup_{0\le\tau\le t} \|\hat{w}^{n-1}(\tau)\|_{s-1}^2 + \alpha\varepsilon^2\int_0^t \|\hat{v}^{n-1}(\tau)\|_s^2 d\tau\}$$

with a constant $C = C(\mathcal{O}_1, M)$ independent of n, ε and α, where $\varepsilon \le 1$ and $\alpha \le 1$ are used. Take α so that $2\alpha CM^2 < 1$. And for this choice of α, we take T_3 so small that

$$T_3 \le T_2\,, \qquad e^{C(T_3/\alpha + M_1 T_3^{1/2})} \le 2\,, \qquad 2CM^2 T_3 < 1\,.$$

Then it follows from (5.25) that $w^n - \bar{w}$ is a Cauchy sequence in $C^0(0,T_3\,;H^{s-1}(\mathbb{R}^n))$, and hence there is a $w = {}^t(u,v)$ with $w - \bar{w} = {}^t(u-\bar{u}, v-\bar{v}) \in C^0(0,T_3\,;H^{s-1}(\mathbb{R}^n))$ such that $w^n - w \to 0$ strongly in $C^0(0,T_3\,;H^{s-1}(\mathbb{R}^n))$ as $n \to \infty$. Moreover by the arguments in Theorem 2.9 it is easily seen that the limit $w = {}^t(u,v)$ is the desired solution to the problem (5.3), (5.2) (and therefore (5.1),(5.2)). Thus the proof of Theorem 5.4 is completed.

5.4 LIMIT AS VISCOSITY TENDS TO ZERO

The solution of (5.1),(5.2) constructed in Theorem 5.4 is depending on $\varepsilon \in (0,1]$. So, in this section, it is denoted by $w^\varepsilon = {}^t(u^\varepsilon, v^\varepsilon)$. We shall show that there is a time interval $[0,T]$ independent of ε such that as $\varepsilon \to 0$, w^{ε} converges on $[0,T]$ to a limit w^0, which is a smooth solution of the limit system (5.1) with $\varepsilon = 0$. To see this we consider the difference $\hat{w} = w^{\delta} - w^{\varepsilon} = {}^t(u^{\delta} - u^{\varepsilon}, v^{\delta} - v^{\varepsilon}) = {}^t(\hat{u},\hat{v})$, where $0 < \delta < \varepsilon \leq 1$. Since w^{ε} satisfies

$$A^0(w^{\varepsilon})w^{\varepsilon}_t + \sum_j A^j(w^{\varepsilon})w^{\varepsilon}_{x_j} - \varepsilon \sum_{jk} B^{jk}(w^{\varepsilon})w^{\varepsilon}_{x_j x_k} = \varepsilon g(w^{\varepsilon}, D_x w^{\varepsilon}) ,$$

$$w^{\varepsilon}(0,x) = w_0(x) ,$$

the system for the difference $\hat{w} = w^{\delta} - w^{\varepsilon}$ is

$$A^0(w^{\delta})\hat{w}_t + \sum_j A^j(w^{\delta})\hat{w}_{x_j} - \delta \sum_{jk} B^{jk}(w^{\delta})\hat{w}_{x_j x_k} \tag{5.26}$$

$$= f^{\varepsilon,\delta} + \delta g^{\varepsilon,\delta} + (\varepsilon - \delta)h^{\varepsilon,\delta} ,$$

$$w(0,x) = 0 , \tag{5.27}$$

where

$$f^{\varepsilon,\delta} = - A^0(w^{\delta}) \sum_j \{A^0(w^{\delta})^{-1}A^j(w^{\delta}) - A^0(w^{\varepsilon})^{-1}A^j(w^{\varepsilon})\}w^{\varepsilon}_{x_j} ,$$

$$g^{\varepsilon,\delta} = A^0(w^{\delta})\{A^0(w^{\delta})^{-1}g(w^{\delta},D_x w^{\delta}) - A^0(w^{\varepsilon})^{-1}g(w^{\varepsilon},D_x w^{\varepsilon})\} +$$

$$+ A^0(w^{\delta}) \sum_{jk} \{A^0(w^{\delta})^{-1}B^{jk}(w^{\delta}) - A^0(w^{\varepsilon})^{-1}B^{jk}(w^{\varepsilon})\}w^{\varepsilon}_{x_j x_k} ,$$

$$h^{\varepsilon,\delta} = - A^0(w^{\delta})A^0(w^{\varepsilon})^{-1}\{g(w^{\varepsilon},D_x w^{\varepsilon}) + \sum_{jk} B^{jk}(w^{\varepsilon})w^{\varepsilon}_{x_j x_k}\} .$$

In Theorem 5.4 we have already proved that $w^{\varepsilon} \in X^{s}_{T_3}(\mathcal{O}_1, M, M_1; \varepsilon)$ for any $\varepsilon \in (0,1]$. So $f^{\varepsilon,\delta} = {}^t(f_1^{\varepsilon,\delta}, f_2^{\varepsilon,\delta})$, $g^{\varepsilon,\delta} = {}^t(g_1^{\varepsilon,\delta}, g_2^{\varepsilon,\delta})$ and $h^{\varepsilon,\delta} = {}^t(h_1^{\varepsilon,\delta}, h_2^{\varepsilon,\delta})$ respectively satisfy

$$(5.28)_1 \qquad \|f_1^{\varepsilon,\delta},\ f_2^{\varepsilon,\delta}\|_{s-1} + \|g_2^{\varepsilon,\delta}\|_{s-2} \le CM\,\|\hat{w}\|_{s-1}\ ,$$

$$(5.28)_2 \qquad \|g_1^{\varepsilon,\delta}\|_{s-1} \le CM(\,\|\hat{u}\|_{s-1} + \|\hat{v}\|_s)\ ,$$

$$(5.28)_3 \qquad \|h_1^{\varepsilon,\delta},\ h_2^{\varepsilon,\delta}\|_{s-1} \le C(M^2 + \|v^{\varepsilon}\|_{s+1})\ ,$$

where $C = C(\mathcal{O}_1, M)$ is a constant independent of ε and δ. Therefore, applying (5.12) (with $\ell = s-1$, $f = f^{\varepsilon,\delta} + (\varepsilon-\delta)h^{\varepsilon,\delta}$ and $g = g^{\varepsilon,\delta}$) to (5.26), (5.27) and using $(5.28)_{1,2,3}$ and $w^{\varepsilon} \in X^{s}_{T_3}(\mathcal{O}_1, M, M_1; \varepsilon)$, we obtain

$$(5.29) \qquad \sup_{0\le\tau\le t} \|\hat{w}(\tau)\|^2_{s-1} + \delta\int_0^t \|\hat{v}(\tau)\|^2_s\, d\tau$$

$$\le CM^2 e^{C(t/\alpha + M_1 t^{1/2})}\{\varepsilon^{-1}|\varepsilon-\delta|^2(1+t) +$$

$$+\, t \sup_{0\le\tau\le t}\|\hat{w}(\tau)\|^2_{s-1} + \alpha\delta^2\int_0^t \|\hat{v}(\tau)\|^2_s\, d\tau\}$$

for any $0 < \delta < \varepsilon \le 1$ and $\alpha \in (0,1]$, where $C = C(\mathcal{O}_1, M)$ is a constant independent of ε, δ and α. Take α so that $2\alpha CM^2 < 1$. For this α, we take T_4 so small that

$$T_4 \le T_3\ , \qquad e^{C(T_4/\alpha + M_1 T_4^{1/2})} \le 2\ , \qquad 2CM^2 T_4 < 1\ .$$

Then (5.29) gives for $t \in [0, T_4]$,

$$\sup_{0\le\tau\le t} \|(w^\varepsilon - w^\delta)(\tau)\|_{s-1}^2 + \delta\int_0^t \|(v^\varepsilon - v^\delta)(\tau)\|_s^2 d\tau \le C\varepsilon^{-1}|\varepsilon-\delta|^2 , \tag{5.30}$$

where $0<\delta<\varepsilon\le 1$, and $C = C(\mathcal{O}_1, M)$ is independent of ε and δ. Estimate (5.30) implies that $w^\varepsilon - \bar{w}$ is a Cauchy sequence in $C^0(0,T_4; H^{s-1}(\mathbb{R}^n))$, and so there is a w^0 with $w^0 - \bar{w} \in C^0(0,T_4; H^{s-1}(\mathbb{R}^n))$ such that $w^\varepsilon - w^0 \to 0$ strongly in $C^0(0,T_4; H^{s-1}(\mathbb{R}^n))$ as $\varepsilon \to 0$. Moreover, since $w^\varepsilon \in X^s_{T_4}(\mathcal{O}_1, M, M_1; \varepsilon)$, by the arguments in Theorem 2.9 we can see that this limit w^0 is a solution of (5.3) with $\varepsilon = 0$, satisfying $w^0 - \bar{w} \in L^\infty(0,T_4; H^s(\mathbb{R}^n))$ and $\partial_t w^0 \in L^\infty(0,T_4; H^{s-1}(\mathbb{R}^n))$. Therefore, by Lemma 2.6 (i), we have a regularity $w^0 - \bar{w} \in C^0(0,T_4; H^s(\mathbb{R}^n))$. Thus we have proved:

<u>Theorem 5.5</u> (limit as $\varepsilon \to 0$) *Let* $n \ge 1$ *and* $s \ge s_0 + 1$ *(*$s_0 = [n/2]+1$*) be integers. Assume the same conditions as in Theorem* 5.4. *Let* $w^\varepsilon = w^\varepsilon(t,x)$ *be the solution (on* Q_{T_3}*) of* (5.1), (5.2) *constructed in Theorem* 5.4. *Then there is a positive constant* T_4 *(*$\le T_3$*) independent of* $\varepsilon \in (0,1]$, *such that* $w^0 - \bar{w} = \lim_{\varepsilon\to 0}(w^\varepsilon - \bar{w})$ *exists strongly in* $C^0(0,T_4; H^{s-1}(\mathbb{R}^n))$. *This limit function* w^0 *is a unique solution of the limit system* (5.1) *with* $\varepsilon = 0$, *satisfying* $w^0 - \bar{w} \in C^0(0,T_4; H^s(\mathbb{R}^n)) \cap C^1(0,T_4; H^{s-1}(\mathbb{R}^n))$. *Moreover, as a consequence of* (5.30), *we have*

$$\|(w^\varepsilon - w^0)(t)\|_{s-1} \le \varepsilon^{1/2} C \tag{5.31}$$

for $t \in [0,T_4]$ *and for any* $\varepsilon \in (0,1]$, *where* C *is a constant depending on the initial data but not on* ε.

<u>Remark</u> (i) In the case of $s \geq s_0 + 2$ $(s_0 = [n/2]+1)$, we can get a rapid convergence result:

$$(5.31)' \qquad \|(w^\varepsilon - w^0)(t)\|_{s-2} \leq \varepsilon C .$$

Because in this case we have the estimates $(5.28)_{1,2}$ with s replaced by $s-1$ as well as the estimate $\|h_1^{\varepsilon,\delta}, h_2^{\varepsilon,\delta}\|_{s-2} \leq CM$. Therefore as a counterpart of (5.30), we have

$$\sup_{0\leq\tau\leq t} \|(w^\varepsilon - w^\delta)(\tau)\|_{s-2}^2 + \delta\int_0^t \|(v^\varepsilon - v^\delta)(\tau)\|_{s-1}^2 d\tau \leq C|\varepsilon-\delta|^2 ,$$

which implies (5.31)'.

(ii) In order to discuss the convergence $w^\varepsilon \to w^0$ for all time $t \geq 0$, we have to consider weak solutions (see DiPerna $[13]_6$). Because smooth solutions of the limit system (5.1) with $\varepsilon = 0$ in general develop singularities in the first derivatives in finite time.

5.5 REMARKS ON THE GLOBAL EXISTENCE

In Theorem 4.3 we have proved that if $\|u_0-\bar{u}, v_0-\bar{v}\|_s \leq \delta_5$, a solution of (5.1) with $\varepsilon = 1$ exists for all time $t \geq 0$. In this section we shall show that if the smallness condition $\|u_0-\bar{u}, v_0-\bar{v}\|_s \leq \varepsilon\delta_5$ is satisfied, then (5.1) admits a global smooth solution for any $\varepsilon \in (0,1]$.

We assume that the initial data (5.2) are of the form

(5.32) $$w_0(x) = \bar{w} + \varepsilon\,\tilde{w}_0(x) \;,$$

where $\bar{w} = {}^t(\bar{u},\bar{v}) \in \mathcal{O}$ is an arbitrarily fixed constant state. Then we have:

Theorem 5.6 (global existence) *Let Conditions* 4.1-4.3 *be assumed. Let* $n \geq 1$ *and* $s \geq s_0 + 1$ ($s_0 = [n/2] + 1$) *be integers. Suppose that the initial data satisfy* (5.32) *with* $\tilde{w}_0 = {}^t(\tilde{u}_0,\tilde{v}_0) \in H^s(\mathbb{R}^n)$. *Let* δ_5 *be the constant in Theorem* 4.3. *If* $\|\tilde{u}_0,\tilde{v}_0\|_s \leq \delta_5$, *then the initial value problem* (5.1), (5.2) *has a unique global solution* $w(t,x) = \bar{w} + \varepsilon\,\tilde{w}(t,x)$ *for any* $\varepsilon \in (0,1]$, *with* $\tilde{w} = {}^t(\tilde{u},\tilde{v})$ *satisfying*

$$\tilde{u} \in C^0(0,\infty\,;H^s(\mathbb{R}^n)) \cap C^1(0,\infty\,;H^{s-1}(\mathbb{R}^n)) \;,$$

$$\tilde{v} \in C^0(0,\infty\,;H^s(\mathbb{R}^n)) \cap C^1(0,\infty\,;H^{s-2}(\mathbb{R}^n)) \;,$$

$$\varepsilon^{1/2}D_x\tilde{u} \in L^2(0,\infty\,;H^{s-1}(\mathbb{R}^n)) \;, \qquad \varepsilon^{1/2}D_x\tilde{v} \in L^2(0,\infty\,;H^s(\mathbb{R}^n)) \;.$$

Furthermore we have the estimate for $\tilde{w} = {}^t(\tilde{u},\tilde{v})$, *which is valid uniformly in* $\varepsilon \in (0,1]$:

(5.33) $$\|(\tilde{u},\tilde{v})(t)\|_s^2 + \varepsilon\int_0^t \|D_x\tilde{u}(\tau)\|_{s-1}^2 + \|D_x\tilde{v}(\tau)\|_s^2\,d\tau$$

$$\leq C_8^2\,\|\tilde{u}_0,\tilde{v}_0\|_s^2 \qquad \textit{for } t \in [0,\infty) \;,$$

where $C_8 > 1$ *is the constant in Theorem* 4.3. *We also have the decay law:* $|(u,v)(t)|_{s-(s_0+1)} \to 0$ *uniformly in* $\varepsilon \in (0,1]$ *as* $t \to \infty$.

Proof. First we note the local existence result (Theorem 2.9). "There exists a positive constant T_1^{ε} $(=O(\varepsilon)$ as $\varepsilon \to 0)$ such that a solution $w(t,x) = {}^t(u,v)(t,x)$ of (5.1), (5.2) exists on the time interval $[0,T_1^{\varepsilon}]$ and satisfies

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_s^2 + \varepsilon\int_0^t \|(u-\bar{u})(\tau)\|_s^2 + \|(v-\bar{v})(\tau)\|_{s+1}^2 d\tau$$

$$\le C_4^2 \|u_0-\bar{u},\ v_0-\bar{v}\|_s^2 \qquad \text{for} \quad t \in [0,T_1^{\varepsilon}] \text{ ."}$$

Next we shall prove the a priori estimates for the solutions. We shall modify $N_s(t',t)$ as follows:

$$N_s(t',t;\varepsilon)^2 = \sup_{t'\le\tau\le t} \|(u-\bar{u},\ v-\bar{v})(\tau)\|_s^2 +$$

$$+ \varepsilon\int_{t'}^t \|D_x u(\tau)\|_{s-1}^2 + \|D_x v(\tau)\|_s^2 d\tau \ .$$

We put $N_s(t;\varepsilon) = N_s(0,t;\varepsilon)$. Then we have the following modification of Proposition 4.2. "Let a_5 and C_8 be the constants in Proposition 4.2. Assume that $N_s(T;\varepsilon) \le \varepsilon a_5$. Then the following a priori estimate holds for $t \in [0,T]$:

(5.34) $$N_s(t;\varepsilon) \le C_8\|u_0-\bar{u},\ v_0-\bar{v}\|_s \text{ ."}$$

Indeed, as a counterpart of (4.14), we have

(5.35) $$\|(u-\bar{u},\ v-\bar{v})(t)\|^2 + \varepsilon\int_0^t \|D_x v(\tau)\|^2 d\tau$$

$$\leq C\{\|u_0 - \bar{u},\ v_0 - \bar{v}\|^2 + N_{s_0}(T;\varepsilon)^3\}\ ,$$

where $N_{s_0}(T;\varepsilon) \leq a_4$ is assumed. Moreover if $N_s(T;\varepsilon) \leq a_4$, then

$$\|D_x(u,v)(t)\|_{s-1}^2 + \varepsilon\int_0^t \|D_x^2 v(\tau)\|_{s-1}^2 d\tau \tag{5.36}$$

$$\leq C\{\|D_x(u_0,v_0)\|_{s-1}^2 + \varepsilon^{-1}N_s(T;\varepsilon)^3\}\ ,$$

$$\int_0^t \|D_x u(\tau)\|_{s-1}^2 d\tau - C\{\|(u-\bar{u},\ v-\bar{v})(t)\|_s^2 + \int_0^t \|D_x v(\tau)\|_s^2 d\tau\} \tag{5.37}$$

$$\leq C\{\|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 + \varepsilon^{-1}N_s(T;\varepsilon)^3\}\ .$$

The estimate (5.36) (resp. (5.37)) is corresponding to (4.16) (resp. (4.17) + (4.21)). Combining the estimates (5.35)-(5.37) as in Proposition 4.2, we get the inequality

$$N_s(T;\varepsilon)^2 \leq C\{\|u_0 - \bar{u},\ v_0 - \bar{v}\|_s^2 + \varepsilon^{-1}N_s(T;\varepsilon)^3\}\ ,$$

whenever $N_s(T;\varepsilon) \leq a_4$ is satisfied. The desired estimate (5.34) is an immediate consequence of the above inequality.

A combination of the local existence result and the a priori estimate stated above gives the theorem, see the proof of Theorem 3.10. This completes the proof.

CHAPTER VI

APPLICATIONS TO THE EQUATIONS OF ELECTRICALLY CONDUCTING FLUIDS

6.1 INTRODUCTION AND EQUATIONS

In this chapter, as applications of the general theory developed in chapters II - V, we shall deal with the system of equations describing the motion of an electrically conducting fluid in the presence of an electromagnetic field. The state of the fluid motion is specified by the mass density ρ, the velocity $\mathbb{u} = (u^1, u^2, u^3)$ and the absolute temperature θ, while the electromagnetic field by the electric field $\mathbb{E} = (E^1, E^2, E^3)$, the magnetic induction $\mathbb{B} = (B^1, B^2, B^3)$ and the electric charge density ρ_e. All these quantities are functions of time $t \geq 0$ and position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Since the flow and the electromagnetic field are closely connected with each other, the system of fundamental equations of the fluid becomes a coupled system of conservation laws for hydrodynamical quantities and Maxwell's law for electromagnetic ones (see [32], $[5]_1$):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbb{u}) = 0 , \\ \\ \rho(\mathbb{u}_t + (\mathbb{u}\cdot\nabla)\mathbb{u}) + \nabla p = \operatorname{div}(2\mu\mathbb{P} + \mu' I \operatorname{div} \mathbb{u}) + \rho_e \mathbb{E} + \mathbb{J} \times \mathbb{B} , \end{cases}$$

$$(6.1)\quad \begin{cases} \rho e_\theta(\theta_t + \mathbf{u}\cdot\nabla\theta) + \theta p_\theta \operatorname{div}\mathbf{u} \\ \qquad\qquad = \operatorname{div}(\kappa\nabla\theta) + \Psi + (\mathbb{J} - \rho_e\mathbf{u})\cdot(\mathbb{E} + \mathbf{u}\times\mathbb{B}) \ , \\ \varepsilon_0\mathbb{E}_t - (1/\mu_0)\operatorname{rot}\mathbb{B} + \mathbb{J} = 0 \ , \\ \mathbb{B}_t + \operatorname{rot}\mathbb{E} = 0 \ , \qquad (\rho_e)_t + \operatorname{div}\mathbb{J} = 0 \ , \end{cases}$$

$$(6.2)\quad \varepsilon_0 \operatorname{div}\mathbb{E} = \rho_e \ , \qquad \operatorname{div}\mathbb{B} = 0 \ .$$

Here the pressure p and the internal energy e are expressed with the aid of the thermodynamic quantities ρ and θ by the equations of state, i.e., $p = p(\rho,\theta)$ and $e = e(\rho,\theta)$ (the abbreviations such as $p_\theta = \partial p/\partial\theta$, $e_\theta = \partial e/\partial\theta$, $\cdots\cdots$ are used); the fluid under consideration is an isotropic Newtonian fluid, i.e., the stress tensor $-pI + (2\mu\mathbb{P} + \mu' I \operatorname{div}\mathbf{u})$ is a linear function of the deformation tensor $\mathbb{P} = (1/2)(u^i_{x_j} + u^j_{x_i})_{1\le i,j\le 3}$, where I is the unit matrix of order 3, and $\mu = \mu(\rho,\theta)$ and $\mu' = \mu'(\rho,\theta)$ are the coefficients of viscosity; Ψ is called the viscous dissipation function and is given explicitly by

$$\Psi = \frac{\mu}{2}\sum_{ij}(u^i_{x_j} + u^j_{x_i})^2 + \mu'(\operatorname{div}\mathbf{u})^2 \ ;$$

the heat flux $\mathbf{q}$ is given by Fourier's law, i.e., $\mathbf{q} = -\kappa\nabla\theta$ (cf. the second equation of (3.47)), where $\kappa = \kappa(\rho,\theta)$ is the coefficient of heat conductivity; the electric current density $\mathbb{J}$ is given by Ohm's law, i.e.,

$$(6.3)\quad \mathbb{J} - \rho_e\mathbf{u} = \sigma(\mathbb{E} + \mathbf{u}\times\mathbb{B}) \ ,$$

where $\sigma = \sigma(\rho,\theta)$ is called the coefficient of electrical conductivity; the dielectric constant ε_0 and the magnetic permeability μ_0 are assumed to be positive constants.

We list up the conditions for (6.1),(6.2).

[C1] The thermodynamic quantities p and e are smooth functions of $\rho > 0$ and $\theta > 0$ such that

1. the relation $de = \theta dS - pd(1/\rho)$ holds for some smooth function $S = S(\rho,\theta)$; this relation expresses the first law of thermodynamics and S is called the entropy (see [11],$[49]_1$),
2. p_ρ $(= \partial p/\partial\rho) > 0$ and $e_\theta > 0$ for $\rho > 0$, $\theta > 0$.

[C2] The coefficients μ, μ' and κ are smooth functions of $\rho > 0$ and $\theta > 0$, and satisfy one of the following four conditions for $\rho > 0$, $\theta > 0$. $(\nu = 2\mu + \mu')$

1. $\mu, \nu > 0$, $\kappa > 0$, 2. $\mu \equiv \nu \equiv 0$, $\kappa > 0$,
3. $\mu, \nu > 0$, $\kappa \equiv 0$, 4. $\mu \equiv \nu \equiv 0$, $\kappa \equiv 0$.

[C3] The coefficient σ is a smooth function of $\rho > 0$ and $\theta > 0$ such that $\sigma > 0$ for $\rho > 0$, $\theta > 0$.

Under these conditions, the equations (6.1),(6.2) form a closed system of 14 equations for 12 unknowns $(\rho, \mathbb{u}, \theta, \mathbb{E}, \mathbb{B}, \rho_e)$, which is called the system of electro-magneto-fluid dynamics.

If letting $\varepsilon_0 \to 0$ formally in (6.1),(6.2), we have $\rho_e = 0$ and $\mathbb{J} = (1/\mu_0)\,\mathrm{rot}\,\mathbb{B}$. These relations together with (6.3) yield $\mathbb{E} = -\mathbb{u} \times \mathbb{B} +$

$(1/\sigma\mu_0)\,\mathrm{rot}\,\mathbb{B}$. Therefore the system (6.1),(6.2) can be reduced to (see [49]$_2$, [32], [5]$_1$)

$$(6.4)\quad \begin{cases} \rho_t + \mathrm{div}(\rho\,\mathbf{u}) = 0\,, \\ \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u}) + \nabla p - (1/\mu_0)\,\mathrm{rot}\,\mathbb{B}\times\mathbb{B} = \mathrm{div}(2\mu\mathbb{P} + \mu'\mathbb{I}\,\mathrm{div}\,\mathbf{u})\,, \\ \rho e_\theta(\theta_t + \mathbf{u}\cdot\nabla\theta) + \theta p_\theta \mathrm{div}\,\mathbf{u} = \mathrm{div}(\kappa\nabla\theta) + \Psi + (1/\sigma\mu_0^2)\,|\mathrm{rot}\,\mathbb{B}|^2\,, \\ \mathbb{B}_t - \mathrm{rot}(\mathbf{u}\times\mathbb{B}) = -\,\mathrm{rot}\{(1/\sigma\mu_0)\,\mathrm{rot}\,\mathbb{B}\}\,, \end{cases}$$

$$(6.5)\quad \mathrm{div}\,\mathbb{B} = 0\,.$$

For this system, it is convenient to replace the condition [C3] by the following.

[C3]' The coefficient $1/\sigma$ is a smooth function of $\rho > 0$ and $\theta > 0$, and satisfies either

1. $1/\sigma > 0$ or 2. $1/\sigma \equiv 0$ for $\rho > 0,\ \theta > 0$.

Under these conditions, we can consider (6.4),(6.5) as a closed system of 9 equations for 8 unknowns $(\rho, \mathbf{u}, \theta, \mathbb{B})$, which is called the system of magnetohydrodynamics.

Furthermore, in the special case when the magnetic induction is neglected (i.e., $\mathbb{B} = 0$) in (6.4),(6.5), we get the usual system of fluid mechanics (see [49]$_1$):

$$(6.6)\quad \begin{cases} \rho_t + \mathrm{div}(\rho\,\mathbf{u}) = 0\ , \\ \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u}) + \nabla p = \mathrm{div}(2\mu\mathbb{P} + \mu'\mathrm{I}\,\mathrm{div}\,\mathbf{u})\ , \\ \rho e_\theta(\theta_t + \mathbf{u}\cdot\nabla\theta) + \theta p_\theta \mathrm{div}\,\mathbf{u} = \mathrm{div}(\kappa\nabla\theta) + \Psi\ . \end{cases}$$

This is a closed system of 5 equations for 5 unknowns $(\rho, \mathbf{u}, \theta)$.

Here we briefly discuss the difference between the two systems (6.1), (6.2) and (6.4),(6.5). The second system (6.4),(6.5) of magnetohydrodynamics (and therefore the system (6.6) of fluid mechanics) is of the desired type, i.e., it is transformed into a symmetric hyperbolic-parabolic system in the sense of chapter II. The system (6.1),(6.2) of electro-magneto-fluid dynamics is also of hyperbolic-parabolic composite type. However it is not of symmetric form. In fact the electromagnetic part $(\mathbb{E}, \mathbb{B}, \rho_e)$ of (6.1) is considered as a first-order hyperbolic system but it is neither symmetric hyperbolic nor strictly hyperbolic (cf. $[18]_2$). The situation is same for the first-order hyperbolic system of $(\mathbb{E}, \mathbb{B})$ which is derived from the above system by eliminating ρ_e with the aid of the first equation of (6.2). Hence the existence problem for (6.1),(6.2) (in $\mathbb{R}^3$) is still open even if local in time.

In this chapter we first restrict ourselves to the two-dimensional equations of (6.1),(6.2); this means that all the quantities appearing in (6.1),(6.2) do not depend on the space-variable x_3. There are two interesting cases (see Kawashima $[38]_3$): the first case is

$$(6.7)_1 \qquad \mathbf{u} = (u^1, u^2, 0)\ , \quad \mathbb{E} = (0,0,E^3)\ , \quad \mathbb{B} = (B^1, B^2, 0)\ ,$$

and the second one is

$$(6.7)_2 \qquad \mathbb{u} = (u^1,u^2,0) \ , \qquad \mathbb{E} = (E^1,E^2,0) \ , \qquad \mathbb{B} = (0,0,B^3) \ .$$

It was pointed out in $[38]_3$ that in the case $(6.7)_1$, the system (6.1), (6.2) becomes a symmetric hyperbolic-parabolic form (in the sense of chapter II) in the domain $\{\rho > 0,\ \theta > 0\}$, provided that $[C1]_2$ and one of $[C2]_j$ $(j = 1,2,3,4)$ are satisfied; while in the case $(6.7)_2$, (6.1),(6.2) can be reduced to a symmetric hyperbolic-parabolic system in the non-relativistic domain $\{\rho > 0,\ |u| < c_0,\ \theta > 0\}$, provided that $[C1]_2$ and one of $[C2]_j$ $(j = 1,3)$ are satisfied, where $u = (u^1,u^2)$, and $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light. Therefore, in these two-dimensional cases, the local existence result of Theorem 2.9 can be applied to the system (6.1), (6.2). Furthermore it is seen that in both cases $(6.7)_{1,2}$, the linearized system for (6.1),(6.2) satisfies Conditions 3.1 and 3.2 (with a slight modification) if $[C1]_2$, $[C2]_1$ and [C3] are assumed. Using this property and the energy integral associated with $-\rho S$ (see $[C1]_1$), we can establish the global existence and asymptotic stability results for these two-dimensional system of (6.1),(6.2).

We next consider the system (6.4),(6.5) of magnetohydrodynamics. This system can be transformed into a symmetric hyperbolic-parabolic system (in the sense of chapter II), provided that $[C1]_2$, one of $[C2]_j$ $(j = 1,2,3,4)$ and one of $[C3]'_k$ $(k = 1,2)$ are satisfied. Therefore by Theorem 2.9 we have a local solution. Furthermore the corresponding linearized system satisfies Conditions 3.1 and 3.2 if $[C1]_2$, $[C2]_1$ and $[C3]'_1$ are assumed. Moreover, if (6.5) holds, then the system (6.4) can be put into a conservation form, provided that the condition $[C1]_1$ is satisfied. Hence, in the same way as in Theorems 4.3 - 4.5, we have the global existence and asymptotic stability results for (6.4), (6.5).

Finally in this chapter, following Kawashima and Okada [41], we consider the one-dimensional equations of (6.4), (6.5) in Lagrangian coordinates. Let $[C1]_{1,2}$ be assumed. Then this one-dimensional system satisfies Conditions 4.1 and 4.2 if one of $[C2]_j$ $(j=1,2,3,4)$ and one of $[C3]'_k$ $(k=1,2)$ are assumed. Moreover, if $\kappa > 0$, then the system also satisfies Condition 4.3 in each of the following three cases:

$$1^\circ \quad \mu, \nu > 0 , \quad 1/\sigma > 0 , \qquad 2^\circ \quad \mu \equiv \nu \equiv 0 , \quad 1/\sigma > 0 ,$$
$$3^\circ \quad \mu, \nu > 0 , \quad 1/\sigma \equiv 0 .$$

Therefore we can get the global existence and asymptotic stability results in these three cases. On the other hand Condition 4.3 is not satisfied if $\kappa \equiv 0$. In this case we take (p,S), in place of (ρ,θ), as the thermodynamic unknowns. Then the corresponding linearized system separates into two parts; the first part consists of a single equation $S_t = 0$ and the second part forms a system which satisfies Condition 4.3 in each of the above three cases. Using this property, we can also establish the global existence results even if $\kappa \equiv 0$.

The plan of this chapter is as follows. Section 6.2 contains some basic properties on the system (6.1), (6.2) of electro-magneto-fluid dynamics. The two-dimensional systems of (6.1), (6.2) are studied in sections 6.3 and 6.4 (the cases $(6.7)_1$ and $(6.7)_2$ are treated in sections 6.3 and 6.4, respectively). The system (6.4), (6.5) of magnetohydrodynamics in $\mathbb{R}^3$ is considered in section 6.5, while the one-dimensional system of (6.4),(6.5) in Lagrangian coordinates is studied in sections 6.6 and 6.7. In sections 6.5 and 6.7, as a special case, we also discuss briefly the system (6.6) of fluid mechanics.

6.2 BASIC PROPERTIES

In this section we shall summarize some basic properties on the system (6.1),(6.2) of electro-magneto-fluid dynamics.

[P1] A smooth solution of (6.1) satisfies (6.2) for all time $t > 0$ if it satisfies (6.2) at $t = 0$.

Indeed, applying div to the equation of $\mathbb{E}$ and subtracting from it the equation of ρ_e, we obtain $(\varepsilon_0 \operatorname{div} \mathbb{E} - \rho_e)_t = 0$. On the other hand the application of div to the equation of $\mathbb{B}$ gives $(\operatorname{div} \mathbb{B})_t = 0$. These equalities prove the assertion.

We introduce here the total momentum $\mathbb{M}_{EM}$ and the total energy ρE_{EM} of the electrically conducting fluid:

$$(6.8)_1 \qquad \mathbb{M}_{EM} = \rho \mathbf{u} + \varepsilon_0 (\mathbb{E} \times \mathbb{B}) \ ,$$

$$(6.8)_2 \qquad \rho E_{EM} = \rho (e + \frac{1}{2} |\mathbf{u}|^2) + \frac{1}{2} (\varepsilon_0 |\mathbb{E}|^2 + \frac{1}{\mu_0} |\mathbb{B}|^2) \ .$$

[P2] Let $(\rho, \mathbf{u}, \theta, \mathbb{E}, \mathbb{B}, \rho_e)$ be a solution of (6.1). Then, under the condition $[C1]_1$, the quantities ρ, ρE_{EM}, $\mathbb{B}$ and ρ_e respectively satisfy conservation laws. Moreover, if the solution satisfies (6.2), the equation for $\mathcal{M}_{EM}$ also becomes a conservation form.

This fact directly follows from (6.1) for ρ, $\mathbb{B}$ and ρ_e. As for $\mathcal{M}_{EM}$, we have the following equation (see [32]).

(6.9) $$(\mathcal{M}_{EM})_t + \mathrm{div}\{\rho\,{}^t\mathbf{u}\mathbf{u} + p\mathrm{I} - (\varepsilon_0/2)(2\,{}^t\mathbb{E}\mathbb{E} - |\mathbb{E}|^2\mathrm{I}) - (1/2\mu_0)(2\,{}^t\mathbb{B}\mathbb{B} - |\mathbb{B}|^2\mathrm{I})\} + \mathbb{E}(\varepsilon_0 \mathrm{div}\,\mathbb{E} - \rho_e) + (1/\mu_0)\mathbb{B}\,\mathrm{div}\,\mathbb{B} = \mathrm{div}(2\mu\mathbb{P} + \mu'\mathrm{I}\,\mathrm{div}\,\mathbf{u}) ,$$

where we have used the equality $\mathbb{B}\times \mathrm{rot}\,\mathbb{B} = -\frac{1}{2}\mathrm{div}(2\,{}^t\mathbb{B}\mathbb{B} - |\mathbb{B}|^2\mathrm{I}) + \mathbb{B}\,\mathrm{div}\,\mathbb{B}$ (the same equality for $\mathbb{E}$ was also used). Here ${}^t\mathbf{u}\mathbf{u}$ denotes the matrix with elements $u^i u^j$. The equation (6.9) becomes a conservation form if (6.2) holds. This proves the assertion for $\mathcal{M}_{EM}$. Next we can deduce from $[\mathrm{C1}]_1$ that

(6.10) $$e_\rho = (p - \theta p_\theta)/\rho^2 , \quad s_\rho = -p_\theta/\rho^2 , \quad s_\theta = e_\theta/\theta .$$

By use of (6.1) and the first relation of (6.10) we get the equation of ρE_{EM} (see [32]).

(6.11) $$(\rho E_{EM})_t + \mathrm{div}\{\rho\mathbf{u}(e + |\mathbf{u}|^2/2) + p\mathbf{u} + (1/\mu_0)(\mathbb{E}\times\mathbb{B})\} = \mathrm{div}(2\mu\,\mathbf{u}\mathbb{P} + \mu'\,\mathbf{u}\,\mathrm{div}\,\mathbf{u} + \kappa\nabla\theta) ,$$

which proves the assertion for ρE_{EM}. This completes the proof of [P2].

[P3] Under the conditions $[C1]_{1,2}$ the negative entropy $-\rho S$ (resp. $-S$) is a strictly convex function of $(\rho, \rho\mathbf{u}, \rho E_{EM}, \mathbb{E}, \mathbb{B})$ (resp. $(1/\rho, \mathbf{u}, E_{EM}, \mathbb{E}/\rho, \mathbb{B}/\rho)$). The total energy ρE_{EM} (resp. E_{EM}) is also a strictly convex function of $(\rho, \rho\mathbf{u}, \rho S, \mathbb{E}, \mathbb{B})$ (resp. $(1/\rho, \mathbf{u}, S, \mathbb{E}/\rho, \mathbb{B}/\rho)$).

The assertion of [P3] is a consequence of the strict convexity of the internal energy e as a function of $(1/\rho, S)$. The strict convexity of e can be shown by a direct calculation of the Hessian of e with respect to $(1/\rho, S)$ (see [66]). Indeed, regarding the quantities e, p and θ as smooth functions of (V, S) (where $V = 1/\rho$), we get the relations

$$\frac{\partial e}{\partial V} = -p\,, \qquad \frac{\partial e}{\partial S} = \theta$$

$$\frac{\partial p}{\partial V} = -(\rho^2 p_\rho + \theta p_\theta^2/e_\theta)\,, \qquad \frac{\partial p}{\partial S} = \theta p_\theta/e_\theta\,,$$

$$\frac{\partial \theta}{\partial V} = -\theta p_\theta/e_\theta\,, \qquad \frac{\partial \theta}{\partial S} = \theta/e_\theta\,,$$

and hence we have

$$\frac{\partial^2 e}{\partial V^2} = \rho^2 p_\rho + \theta p_\theta^2/e_\theta\,,$$

$$\frac{\partial^2 e}{\partial V \partial S} = -\theta p_\theta/e_\theta\,, \qquad \frac{\partial^2 e}{\partial S^2} = \theta/e_\theta\,.$$

These relations together with the condition $[C1]_2$ shows that the Hessian of $e = e(V,S)$ is positive definite for $\rho > 0$ and $\theta > 0$. This completes the proof of [P3].

Here we remark that the second and the third relations of (6.10) together with (6.1) yield the equation of the entropy.

$$(6.12) \qquad (\rho S)_t + \mathrm{div}(\rho \mathbb{u} S) = \mathrm{div}\{(\kappa/\theta)\nabla\theta\} + \\ + (1/\theta)\{\Psi + (\kappa/\theta)|\nabla\theta|^2 + \sigma|\mathbb{E} + \mathbb{u}\times\mathbb{B}|^2\} .$$

Next, as in chapter IV (4.10), we shall introduce the quadratic functions associated with the convex functions in [P3]. We first consider $\eta = E_{EM}$ relative to the states $z = {}^t(1/\rho, \mathbb{u}, S, \mathbb{E}/\rho, \mathbb{B}/\rho)$ and $\bar{z} = {}^t(1/\bar{\rho}, 0, \bar{S}, 0, \bar{\mathbb{B}}/\bar{\rho})$, where $\bar{S} = S(\bar{\rho},\bar{\theta})$. Let $\eta^* = E^*_{EM}$ be the quadratic function associated with $\eta = E_{EM}$. By direct calculations we have

$$(6.13) \qquad \rho E^*_{EM} = \rho\{e - \bar{e} + \bar{p}(1/\rho - 1/\bar{\rho}) - \bar{\theta}(S - \bar{S}) + \frac{1}{2}|\mathbb{u}|^2\} + \\ + \frac{1}{2}(\varepsilon_0|\mathbb{E}|^2 + \frac{1}{\mu_0}|\mathbb{B} - \bar{\mathbb{B}}|^2) ,$$

where $\bar{e} = e(\bar{\rho},\bar{\theta})$ and $\bar{p} = p(\bar{\rho},\bar{\theta})$. In the same way we can see that the quadratic functions associated with ρE_{EM}, $-S$ and $-\rho S$ are respectively given by ρE^*_{EM}, $(1/\bar{\theta})E^*_{EM}$ and $(1/\bar{\theta})\rho E^*_{EM}$.

Now suppose that $\bar{\rho} > 0$, $\bar{\theta} > 0$ and $\bar{\mathbb{B}} \in \mathbb{R}^3$ are constant states. Then, from (6.11), (6.12) and (6.1), we have the equation of ρE^*_{EM} :

$$(6.14)\qquad (\rho E^*_{EM})_t + \mathrm{div}\{\rho\,\mathbf{u}\{e - \bar{e} + \bar{p}(1/\rho - 1/\bar{\rho}) - \bar{\theta}(S - \bar{S}) + |\mathbf{u}|^2/2\} + (p - \bar{p})\,\mathbf{u} + (1/\mu_0)\{\mathbb{E}\times(\mathbb{B} - \bar{\mathbb{B}})\}\} + (\bar{\theta}/\theta)\{\Psi + (\kappa/\theta)|\nabla\theta|^2 + \sigma|\mathbb{E} + \mathbf{u}\times\mathbb{B}|^2\} = \mathrm{div}\{2\mu\,\mathbf{u}\mathbb{P} + \mu'\,\mathbf{u}\,\mathrm{div}\,\mathbf{u} + \kappa(1 - \bar{\theta}/\theta)\nabla\theta\}\ .$$

6.3 ELECTRO-MAGNETO-FLUID DYNAMICS IN $\mathbb{R}^2$, I

We shall consider the two-dimensional motion of an electrically conducting fluid. We assume that the flow is uniform in the x_3-axis, i.e., all the quantities in (6.1),(6.2) do not depend on the space-variable x_3. We further assume $(6.7)_1$, that is, the velocity and the magnetic induction are parallel to the (x_1,x_2)-plane and the electric field is parallel to the x_3-axis. Under these assumptions we have $\rho_e \equiv 0$ (the exact neutrality) by the first equation of (6.2). Therefore the system (6.1),(6.2) and Ohm's law (6.3) are simplified as follows:

$$(6.15)\qquad \begin{cases} \rho_t + \mathrm{div}(\rho u) = 0\ , \\ \rho(u_t + (u\cdot\nabla)u) + \nabla p = \mathrm{div}(2\mu P + \mu' I\,\mathrm{div}\,u) + J\times B\ , \\ \rho e_\theta(\theta_t + u\cdot\nabla\theta) + \theta p_\theta \mathrm{div}\,u = \mathrm{div}(\kappa\nabla\theta) + \Psi + J(E + u\times B)\ , \\ \varepsilon_0 E_t - (1/\mu_0)\mathrm{rot}\,B + J = 0\ , \end{cases}$$

$$\left\{ B_t + \mathrm{rot}\, E = 0 \ , \right.$$

$$\text{(6.16)} \qquad \mathrm{div}\, B = 0 \ ,$$

$$\text{(6.17)} \qquad J = \sigma(E + u \times B) \ ,$$

where $u = (u^1,u^2)$, $E = E^3$, $B = (B^1,B^2)$, $P = (1/2)(u^i_{x_j} + u^j_{x_i})_{1\le i,j\le 2}$, and I is the unit matrix of order 2. Here we have used the following notations for 2-vectors $u = (u^1,u^2)$ and $v = (v^1,v^2)$ and a scalor α in addition to the ordinary ones.

$$\text{(6.18)}_1 \quad \begin{cases} u \times v = -\, v \times u = u^1v^2 - u^2v^1 \ , \\ u \times \alpha = -\, \alpha \times u = (\alpha u^2, -\alpha u^1) \ , \end{cases}$$

$$\text{(6.18)}_2 \quad \begin{cases} \mathrm{rot}\, u = \nabla \times u = u^2_{x_1} - u^1_{x_2} \ , \\ \mathrm{rot}\, \alpha = \nabla \times \alpha = (\alpha_{x_2}, -\alpha_{x_1}) \ . \end{cases}$$

The equations (6.15), (6.16) form a closed system of 8 equations for 7 unknowns (ρ,u,θ,E,B). For this system, the properties [P1]-[P3] hold with a trivial modification. Moreover we have the following.

<u>Lemma 6.1</u> *We assume* $[C1]_2$ *and one of* $[C2]_j$ $(j=1,2,3,4)$. *Let* σ *be a smooth function of* (ρ,θ), *and let* $\bar\rho > 0$, $\bar\theta > 0$ *and* $\bar B \in \mathbb{R}^2$ *be arbitrarily fixed constants. Then the system* (6.15) *satisfies Conditions* 2.1 *and* 2.2 *for* $\mathcal{O} = \{(\rho,u,\theta,E,B) \in \mathbb{R}^7 \,;\, \rho > 0,\ \theta > 0\}$ *and a constant state*

$(\bar{\rho},0,\bar{\theta},0,\bar{B})$, *i.e.,* (6.15) *is symmetric hyperbolic-parabolic in the sense of chapter* II. *(In the case of* $[C2]_4$, (6.15) *is a symmetric hyperbolic system* $(m''=0)$*.)*

Proof. Put $w = {}^t(\rho,u,\theta,E,B)$. The system (6.15) can be written in the form

$$A^0(w)w_t + \sum_{j=1}^{2} A^j(w)w_{x_j} - \sum_{j,k=1}^{2} B^{jk}(w)w_{x_jx_k} = f^1(w,D_xw) + f^2(w) , \tag{6.19}$$

where $A^0(w)$, $A^j(w)$ and $B^{jk}(w)$ are square matrices of order 7, and $f^1(w,D_xw)$ and $f^2(w)$ are $\mathbb{R}^7$-valued functions; they are given explicitly by

$$(6.20)_1 \qquad A^0(w) = \left(\begin{array}{ccc|cc} p_\rho/\rho & & & & \\ & \rho I & & & 0 \\ & & \rho e_\theta/\theta & & \\ \hline & 0 & & \varepsilon_0 & \\ & & & & (1/\mu_0)I \end{array}\right) ,$$

$$(6.20)_2 \qquad \sum_j A^j(w)\xi_j$$

$$= \left(\begin{array}{ccc|cc} (p_\rho/\rho)(u\cdot\xi) & p_\rho\xi & 0 & & \\ p_\rho{}^t\xi & \rho(u\cdot\xi)I & p_\theta{}^t\xi & & 0 \\ 0 & p_\theta\xi & (\rho e_\theta/\theta)(u\cdot\xi) & & \\ \hline & 0 & & 0 & (1/\mu_0)\xi^* \\ & & & (1/\mu_0){}^t\xi^* & 0 \end{array}\right) ,$$

$$(6.20)_3 \qquad \sum_{jk} B^{jk}(w)\xi_j\xi_k = \left(\begin{array}{ccc|cc} 0 & & & & \\ & \mu|\xi|^2 I + (\mu+\mu')\,{}^t\xi\xi & & 0 & \\ & & (\kappa/\theta)|\xi|^2 & & \\ \hline & & & 0 & \\ & 0 & & & 0 \end{array}\right),$$

$$(6.20)_4 \qquad f^1(w, D_x w) = \left(\begin{array}{c} 0 \\ 2(\nabla\mu)P + \nabla\mu' \operatorname{div} u \\ (1/\theta)(\Psi + \nabla\kappa\cdot\nabla\theta) \\ \hline 0 \\ 0 \end{array}\right);$$

$f^2(w)$ is the term which does not contain the derivatives $D_x w$. Here $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\xi^* = (\xi_2, -\xi_1)$. Note that the symbols w, $A^0(w)$, $\cdots$ used here don't always agree with the previous ones (in chapters II and III). It is easily seen that $A^0(w)$ is (real) diagonal and positive definite for $\rho > 0$, $\theta > 0$, and $A^j(w)$ and $B^{jk}(w) = B^{kj}(w)$ are real symmetric. Furthermore a simple calculation shows that

$$(6.21) \qquad \langle \{\sum_{jk} B^{jk}(w)\omega_j\omega_k\}\hat{w}, \hat{w}\rangle \geq \min\{\mu, \nu\}|\hat{u}|^2 + (\kappa/\theta)|\hat{\theta}|^2$$

for $\hat{w} = {}^t(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E}, \hat{B}) \in \mathbb{R}^7$ and $\omega = (\omega_1, \omega_2) \in S^1$, where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^7$. On the other hand the right members $f^1(w, D_x w)$ and $f^2(w)$ are regarded as lower order terms in every case of $[C2]_j$ $(j = 1,2,3,4)$, and satisfy $f^1(\bar{w}, 0) = f^2(\bar{w}) = 0$ for $\bar{w} = {}^t(\bar{\rho}, 0, \bar{\theta}, 0, \bar{B})$. All these considerations prove the lemma.

Now we shall consider the initial value problem for (6.15),(6.16) with the initial data

(6.22) $$(\rho,u,\theta,E,B)(0,x) = (\rho_0,u_0,\theta_0,E_0,B_0)(x) .$$

We first note that Lemma 6.1, Theorem 2.9 and the property [P1] give the following local existence results for the problems (6.15),(6.22) and (6.15),(6.16),(6.22).

<u>Theorem 6.2</u> ($[38]_3$) (local existence) *Let* $[C1]_2$ *and one of* $[C2]_j$ *(*$j = 1,2,3,4$*) be assumed and let* σ *be a smooth function of* (ρ,θ). *Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$ *and* $\bar{B} \in \mathbb{R}^2$ *be arbitrarily fixed constants. Suppose that the initial data satisfy* $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}, E_0, B_0 - \bar{B}) \in H^s(\mathbb{R}^2)$ *(for* $s \geq 3$*) and* $\inf_x\{\rho_0(x), \theta_0(x)\} > 0$. *Then the problem* (6.16), (6.22) *has a unique solution* $(\rho,u,\theta,E,B)(t,x)$ *(in the Sobolev spaces) on* Q_T *with some* $T > 0$, *which satisfies* $\inf_{Q_T}\{\rho(t,x), \theta(t,x)\} > 0$ *(for the solution space, see Theorem* 2.9*). Furthermore if* $\operatorname{div} B_0(x) = 0$ *for* $x \in \mathbb{R}^2$, *then* $(\rho,u,\theta,E,B)(t,x)$ *becomes a solution of the original problem* (6.15), (6.16), (6.22).

Next we shall study the global existence problem for (6.15),(6.16), (6.22). As a preliminary we will show that the linearized system for (6.15) at the constant equilibrium state $w = \bar{w} = {}^t(\bar{\rho},0,\bar{\theta},0,\bar{B})$ satisfies Conditions 3.1 and 3.2 (with a slight modification). To this end we consider the linearization of (6.15) around $w = \bar{w}$:

$$(6.23) \qquad A^0(\overline{w})U_t + \sum_j A^j(\overline{w})U_{x_j} - \sum_{jk} B^{jk}(\overline{w})U_{x_j x_k} + L(\overline{w})U = 0 ,$$

where $A^0(\overline{w})$, $A^j(\overline{w})$ and $B^{jk}(\overline{w})$ are given by $(6.20)_{1-3}$ with $w = \overline{w}$, and

$$(6.24) \qquad L(\overline{w}) = \overline{\sigma} \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & {}^t\overline{B}^*\overline{B}^* & 0 & {}^t\overline{B}^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \overline{B}^* & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

where $\overline{\sigma} = \sigma(\overline{\rho},\overline{\theta})$ and $\overline{B}^* = (\overline{B^2}, -\overline{B^1})$. Note that (6.23) is a symmetric system. While, as a linearized form of (6.16), we have

$$(6.25) \qquad \sum_j R^j U_{x_j} = 0 ,$$

where

$$\sum_j R^j \xi_j = (0,0,0,0,\xi) \qquad \text{for } \xi = (\xi_1,\xi_2) \in \mathbb{R}^2 .$$

(R^j are (1,7)-matrices). Since the solution of (6.23) is subordinate to (6.25), we introduce

$$(6.26) \qquad X_\omega = \{ \hat{w} = {}^t(\hat{\rho},\hat{u},\hat{\theta},\hat{E},\hat{B}) \in \mathbb{R}^7 ; \ (\sum_j R^j\omega_j)\hat{w} \equiv \omega\cdot\hat{B} = 0 \}$$

for $\omega \in S^1$, and modify Condition 3.2 as follows.

Condition 3.2' (cf. [81]) There exist (real) constant square matrices K^j $(j=1,\cdots,n)$ of order m such that

(i) $K^jA^0(\bar{w})$ are real anti-symmetric,

(ii) for any $\omega \in S^{n-1}$, the symmetric part of $\sum_{jk}\{K^jA^k(\bar{w}) + B^{jk}(\bar{w})\}\omega_j\omega_k + L(\bar{w})$ is positive definite on a linear subspace X_ω of $\mathbb{R}^m$.

Then we have:

Lemma 6.3 *Let the conditions* $[C1]_2$, $[C2]_1$ *and* [C3] *be assumed. Then the linearized system* (6.23) *of* (6.15) *satisfies Conditions* 3.1 *and* 3.2'. *In particular,* X_ω *and* K^j $(j=1,2)$ *in Condition* 3.2' *are taken as in* (6.26) *and* (6.27) *(with a suitably small constant* $\alpha > 0$*), respectively.*

Proof. It is easy to verify Condition 3.1. So we omit it. We only check Condition 3.2'. Let α be a positive constant and let K^j $(j=1,2)$ to be

$$\sum_j K^j\xi_j = \alpha\left(\begin{array}{ccc|cc} 0 & \bar{p}_\rho\xi & 0 & & \\ -\bar{p}_\rho{}^t\xi & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ \hline & & & 0 & -(1/\mu_0)\xi^* \\ & 0 & & (1/\mu_0)^t\xi^* & 0 \end{array}\right) A^0(\bar{w})^{-1}, \tag{6.27}$$

where $\bar{p}_\rho = p_\rho(\bar{\rho},\bar{\theta})$, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ and $\xi^* = (\xi_2,-\xi_1)$. Then it is seen that $K^jA^0(\bar{w})$ are real anti-symmetric. Furthermore, for $\hat{w} = {}^t(\hat{\rho},\hat{u},\hat{\theta},\hat{E},\hat{B}) \in X_\omega$, we have

$$(6.28)\qquad \langle \{\sum_{jk}[K^jA^k(\bar{w})]'\omega_j\omega_k\}\hat{w},\ \hat{w}\rangle$$

$$\geq \frac{\alpha}{2}\{(\bar{p}_\rho^{\,2}/\bar{\rho})|\hat{\rho}|^2 + (c_0^2/\mu_0)|\hat{B}|^2 - C|\hat{u},\hat{\theta},\hat{E}|^2\}\ ,$$

where ${}^t\xi^*\xi^* = |\xi|^2I - {}^t\xi\xi$ and $\omega\cdot\hat{B} = 0$ are used. In (6.28), $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light and C is a constant independent of $\omega \in S^1$ and α; $[K^jA^k(\bar{w})]'$ denotes the symmetric part of $K^jA^k(\bar{w})$. On the other hand

$$(6.29)\qquad \langle L(\bar{w})\hat{w},\ \hat{w}\rangle = \bar{\sigma}|\hat{E} + \hat{u}\times\bar{B}|^2$$

holds for $\hat{w} \in \mathbb{R}^7$. Combining (6.21) (with $w = \bar{w}$), (6.28) and (6.29), we can deduce that $\sum_{jk}\{[K^jA^k(\bar{w})]' + B^{jk}(\bar{w})\}\omega_j\omega_k + L(\bar{w})$ is positive definite on X_ω for a suitably small $\alpha > 0$. Thus the proof of Lemma 6.3 is completed.

Let $[C1]_2$, $[C2]_1$ and [C3] be assumed. Then, by virtue of Lemma 6.3 and $n = 2$, we can apply the results of Lemmas 3.1 and 3.8 to the solution of (3.15),(3.16), and consequently we obtain the following a priori estimates for $t \in [0,T]$ (see (3.8) and (3.36)).

$$(6.30)\qquad \|D_x(\rho,u,\theta,E,B)(t)\|_{s-1}^2 + \int_0^t \|D_x^2(u,\theta)(\tau)\|_{s-1}^2 +$$

$$+ \|D_x(E+u\times\bar{B})(\tau)\|_{s-1}^2\,d\tau \leq C\{\|D_x(\rho_0,u_0,\theta_0,E_0,B_0)\|_{s-1}^2 + N_s(T)^3\}\ ,$$

$$(6.31)\qquad \int_0^t \|D_x(\rho,E,B)(\tau)\|_{s-1}^2\,d\tau - C\{\|(\rho-\bar{\rho},\ u,\ \theta-\bar{\theta},\ E,\ B-\bar{B})(t)\|_s^2 +$$

$$+ \int_0^t \|D_x(u,\theta)(\tau)\|_s^2 + \|(E + u\times\overline{B})(\tau)\|_s^2 d\tau\}$$

$$\le C\{\|\rho_0 - \overline{\rho}, u_0, \theta_0 - \overline{\theta}, E_0, B_0 - \overline{B}\|_s^2 + N_s(T)^3\} ,$$

where $s \ge 3$, C is a constant, and $N_s(T)$ is assumed to be suitably small; note that for our system (6.15),(6.16), $N_s(t)$ is given by

$$N_s(t)^2 = \sup_{0\le\tau\le t} \|(\rho - \overline{\rho}, u, \theta - \overline{\theta}, E, B - \overline{B})(\tau)\|_s^2 +$$

$$+ \int_0^t \|D_x(\rho,E,B)(\tau)\|_{s-1}^2 + \|D_x(u,\theta)(\tau)\|_s^2 d\tau .$$

On the other hand, using the quadratic function ρE^*_{EM} (see (6.13)), we obtain as in Lemma 4.1

(6.32) $$\|(\rho - \overline{\rho}, u, \theta - \overline{\theta}, E, B - \overline{B})(t)\|^2 + \int_0^t \|D_x(u,\theta)(\tau)\|^2 +$$

$$+ \|(E + u\times\overline{B})(\tau)\|^2 d\tau$$

$$\le C\{\|\rho_0 - \overline{\rho}, u_0, \theta_0 - \overline{\theta}, E_0, B_0 - \overline{B}\|^2 + N_2(T)^3\} .$$

Indeed, for the solution of (6.15),(6.16), the equation (6.14) is valid with a trivial modification. Integrating it over $Q_t = [0,t]\times\mathbb{R}^2$, we obtain after integration by parts

(6.33) $$[\int \rho E^*_{EM} dx]_{\tau=0}^{\tau=t} + \int_0^t\int (\overline{\theta}/\theta)\{\Psi + (\kappa/\theta)|\nabla\theta|^2 +$$

$$+\ \sigma(E+u\times B)^2\}\,dx\,d\tau = 0\ .$$

Using the estimate (2.4) (with $p=4$, $s=1$ and $n=2$), we see that the second term of the left member of (6.33) is bounded from below by

$$c\int_0^t \|D_x(u,\theta)(\tau)\|^2 + \|(E+u\times\overline{B})(\tau)\|^2\,d\tau - CN_2(T)^3$$

for some positive constants c and C. Therefore the desired estimate (6.32) follows from (6.33) because ρE^*_{EM} is equivalent to the quadratic function $|\rho-\overline{\rho},\, u,\, \theta-\overline{\theta},\, E,\, B-\overline{B}|^2$.

Combining the estimates (6.30)-(6.32), we get

$$(6.34)\qquad N_s(t)^2 + \int_0^t \|(E+u\times\overline{B})(\tau)\|^2\,d\tau$$

$$\le C\,\|\rho_0-\overline{\rho},\, u_0,\, \theta_0-\overline{\theta},\, E_0,\, B_0-\overline{B}\|_s^2 \qquad \text{for} \quad t\in[0,T]\ ,$$

which is corresponding to (3.39) in Proposition 3.9. Therefore we can establish the global existence result for (6.15),(6.16),(6.22) in the same way as in Theorem 3.10.

Theorem 6.4 ([38]$_3$) (global existence) *Let the conditions* [C1]$_{1,2}$, [C2]$_1$ *and* [C3] *be assumed. Suppose that* $(\rho_0-\overline{\rho},\, u_0,\, \theta_0-\overline{\theta},\, E_0,\, B_0-\overline{B}) \in H^s(\mathbb{R}^2)$ *(for* $s\ge 3$*),* $\mathrm{div}\, B_0(x) = 0$ *for* $x\in\mathbb{R}^2$ *and* $\|\rho_0-\overline{\rho},\, u_0,\, \theta_0-\overline{\theta},\, E_0,\, B_0-\overline{B}\|_s$ *is sufficiently small. Then the problem* (6.15), (6.16), (6.22) *has a unique global solution (in the Sobolev spaces) satisfying* (6.34) *(for the solution space, see Theorem* 3.10*). The solution decays,* *in the* $B^{s-3}(\mathbb{R}^2)$*-norm, to the constant state* $(\bar{\rho},0,\bar{\theta},0,\bar{B})$ *as* $t \to \infty$.

6.4 ELECTRO-MAGNETO-FLUID DYNAMICS IN $\mathbb{R}^2$, II

In this section we study another two-dimensional flow. We assume that all the quantities in (6.1),(6.2) are independent of x_3. We further assume $(6.7)_2$. Then (6.1),(6.2) and (6.3) are reduced to

$$(6.35)\qquad \begin{cases} \rho_t + \mathrm{div}(\rho u) = 0\ , \\ \rho(u_t + (u\cdot\nabla)u) + \nabla p = \mathrm{div}(2\mu P + \mu' I\,\mathrm{div}\,u) + \rho_e E + J\times B\ , \\ \rho e_\theta(\theta_t + u\cdot\nabla\theta) + \theta p_\theta \mathrm{div}\,u = \mathrm{div}(\kappa\nabla\theta) + \Psi + (J-\rho_e u)\cdot(E + u\times B)\ , \\ \varepsilon_0 E_t - (1/\mu_0)\,\mathrm{rot}\,B + J = 0\ , \\ B_t + \mathrm{rot}\,E = 0\ , \qquad (\rho_e)_t + \mathrm{div}\,J = 0\ , \end{cases}$$

$$(6.36)\qquad \varepsilon_0 \mathrm{div}\,E = \rho_e\ ,$$

$$(6.37)\qquad J - \rho_e u = \sigma(E + u\times B)\ ,$$

where $u = (u^1,u^2)$, $E = (E^1,E^2)$, $B = B^3$ and $P = (1/2)(u^i_{x_j} + u^j_{x_i})_{1\le i,j\le 2}$. Here we again used the notations in $(6.18)_{1,2}$. The equations (6.35), (6.36) form a closed system of 9 equations for 8 unknowns $(\rho,u,\theta,E,B,\rho_e)$.

If ρ_e is eliminated by $\rho_e = \varepsilon_0 \operatorname{div} E$, the system (6.35) is transformed to

$$
(6.35)' \quad \begin{cases}
\rho_t + \operatorname{div}(\rho u) = 0 , \\
\rho(u_t + (u\cdot\nabla)u) + \nabla p = \operatorname{div}(2\mu P + \mu' I \operatorname{div} u) + \\
\qquad\qquad + \varepsilon_0 (E + u \times B) \operatorname{div} E + \sigma(E + u \times B) \times B , \\
\rho e_\theta (\theta_t + u\cdot\nabla\theta) + \theta p_\theta \operatorname{div} u = \operatorname{div}(\kappa\nabla\theta) + \Psi + \sigma|E + u \times B|^2 , \\
\varepsilon_0 E_t - (1/\mu_0) \operatorname{rot} B + \varepsilon_0 u \operatorname{div} E + \sigma(E + u \times B) = 0 , \\
B_t + \operatorname{rot} E = 0 .
\end{cases}
$$

This is a closed system of 7 equations for 7 unknowns (ρ, u, θ, E, B). Conversely, for a given (6.35)', we put $\rho_e = \varepsilon_0 \operatorname{div} E$ and $J = \rho_e u + \sigma(E + u \times B)$. Then, applying div to the equation of E of (6.35)', we get $(\rho_e)_t + \operatorname{div} J = 0$. Hence (6.35)' is equivalent to the system (6.35), (6.36).

For the system (6.35)', we have the following lemma (compare it with Lemma 6.1).

<u>Lemma 6.5</u> *Let* $[C1]_2$ *and one of* $[C2]_j$ $(j = 1,3)$ *be assumed and let* σ *be a smooth function of* (ρ, θ). *Suppose that* $\bar\rho > 0$, $\bar\theta > 0$ *and* $\bar B \in \mathbb{R}^1$ *are arbitrarily fixed constants. Then the system* (6.35)' *satisfies Con-*

ditions 2.1 *and* 2.2 *for* $\mathcal{O} = \{(\rho,u,\theta,E,B) \in \mathbb{R}^7;\ \rho > 0,\ |u| < c_0,\ \theta > 0\}$ *and a constant state* $(\bar{\rho},0,\bar{\theta},0,\bar{B})$, *where* $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ *is the speed of light.*

Proof. Set $w = {}^t(\rho,u,\theta,E,B)$. Then (6.35)' can be written in the form (6.19), where $B^{jk}(w)$ and $f^1(w,D_xw)$ are the same as the previous ones (see $(6.20)_{3,4}$), $A^0(w)$ and $A^j(w)$ are given by

$$(6.38)_1 \qquad A^0(w) = \left(\begin{array}{ccc|cc} p_\rho/\rho & & & & \\ & \rho I & & 0 & \\ & & \rho e_\theta/\theta & & \\ \hline & & & \varepsilon_0 I & \varepsilon_0\,{}^t u^* \\ & 0 & & \varepsilon_0 u^* & 1/\mu_0 \end{array}\right),$$

$$(6.38)_2 \qquad \sum_j A^j(w)\xi_j$$

$$= \left(\begin{array}{ccc|cc} (p_\rho/\rho)(u\cdot\xi) & p_\rho\xi & 0 & 0 & 0 \\ p_\rho\,{}^t\xi & \rho(u\cdot\xi)I & p_\theta\,{}^t\xi & -\varepsilon_0\,{}^t(E+u^*B)\xi & 0 \\ 0 & p_\theta\xi & (\rho e_\theta/\theta)(u\cdot\xi) & 0 & 0 \\ \hline & & & \varepsilon_0({}^t u\xi - {}^t u^*\xi^*) & -(1/\mu_0)\,{}^t\xi^* \\ & 0 & & -(1/\mu_0)\xi^* & -(1/\mu_0)(u\cdot\xi) \end{array}\right),$$

and $f^2(w)$ is the term which does not contain the derivatives D_xw. Here we used the notations $u^* = (u^2, -u^1)$, $\xi^* = (\xi_2, -\xi_1)$. It is seen that $A^0(w)$ is real symmetric and positive definite in the domain $\{\rho > 0,\ |u| < c_0,\ \theta > 0\}$. While $A^j(w)$ is real symmetric if the element corresponding to $-\varepsilon_0{}^t(E+u*B)\xi$ is absent, and the term $-\varepsilon_0{}^t(E+u\times B)\operatorname{div}E$ (associated with it) can be regarded as a lower order term if $\mu, \nu > 0$ is assumed. Moreover we have $f^2(\bar{w}) = 0$ for $\bar{w} = {}^t(\bar{\rho},0,\bar{\theta},0,\bar{B})$. These considerations together with the properties of $B^{jk}(w)$ and $f^1(w,D_x w)$ (see Lemma 6.1) prove the lemma.

Thus the results of Theorem 2.9 are applicable to the problem (6.35)',(6.22), and consequently we have the following local existence results as in Theorem 6.2.

Theorem 6.6 ($[38]_3$) (local existence) *Let* $[C1]_2$ *and one of* $[C2]_j$ *(*$j=1,3$*) be assumed and let* σ *be a smooth function of* (ρ,θ). *Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$ *and* $\bar{B} \in \mathbb{R}^1$ *be arbitrarily fixed constants. Suppose that* $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}, E_0, B_0 - \bar{B}) \in H^s(\mathbb{R}^2)$ *(for* $s \geq 3$*) and* $\inf_x\{\rho_0(x), \theta_0(x)\} > 0$, $\sup_x|u_0(x)| < c_0 = 1/\sqrt{\varepsilon_0\mu_0}$. *Then the problem* (6.35)',(6.22) *has a unique solution* $(\rho,u,\theta,E,B)(t,x)$ *(in the Sobolev spaces) on* Q_T *(with some* $T > 0$*), which satisfies* $\inf_{Q_T}\{\rho(t,x), \theta(t,x)\} > 0$ *and* $\sup_{Q_T}|u(t,x)| < c_0$.

Next we consider the global existence problem for (6.35)',(6.22). The linearized system for (6.35)' at the constant state $w = \bar{w} = {}^t(\bar{\rho},0,\bar{\theta},0,\bar{B})$ is written in the form (6.23), where $A^0(\bar{w})$, $A^j(\bar{w})$ and $B^{jk}(\bar{w})$ are given by $(6.38)_1$, $(6.38)_2$ and $(6.20)_3$ (with $w=\bar{w}$), respectively, and

$$(6.39) \qquad L(\overline{w}) = \overline{\sigma} \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}^2 I & 0 & \overline{B}^t I^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \overline{B} I^* & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \text{with} \quad I^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Note that this linearized system satisfies Condition 3.1 (and therefore it is a symmetric system). It also satisfies Condition 3.2. In fact, in this case, we may take K^j $(j=1,2)$ to be

$$(6.40) \qquad \sum_j K^j \xi_j = \alpha \left(\begin{array}{ccc|cc} 0 & \overline{p}_\rho \xi & 0 & & \\ -\overline{p}_\rho {}^t\xi & 0 & 0 & & 0 \\ 0 & 0 & 0 & & \\ \hline & & & 0 & (1/\mu_0)\, {}^t\xi^* \\ & 0 & & -(1/\mu_0)\, \xi^* & 0 \end{array} \right) A^0(\overline{w})^{-1}$$

with a suitably small constant $\alpha > 0$. Then after a simple calculation we get (6.28) for $\hat{w} \in \mathbb{R}^7$. Moreover we have (6.29) by use of the expression (6.39). These estimates together with (6.21) (with $w=\overline{w}$) shows Condition 3.1 (ii).

These considerations are summarized as follows.

<u>Lemma 6.7</u> *Let* $[C1]_2$, $[C2]_1$ *and* [C3] *be assumed. Then the linearized system of* (6.35)' *satisfies Conditions* 3.1 *and* 3.2. *In particular,* K^j $(j=1,2)$ *in Condition* 3.2 *are taken as in* (6.40) *with a suitably small constant* $\alpha > 0$.

Thus, in the same way as in Theorem 6.4, we have:

Theorem 6.8 ($[38]_3$) (global existence) *Assume the same conditions as in Theorem* 6.4. *Then the problem* (6.35)', (6.22) *can be solved globally in time as in Theorem* 6.4.

6.5 MAGNETOHYDRODYNAMICS IN $\mathbb{R}^3$

We shall consider the system (6.4), (6.5) of magnetohydrodynamics. We first summarize the basic propertites on (6.4), (6.5). As a counterpart of [P1], we have:

[Q1] A smooth solution of (6.4) satisfies (6.5) for all time $t > 0$ if it satisfies (6.5) at $t = 0$.

The total momentum $\mathbb{M}_M$ and the total energy ρE_M of the magnetohydrodynamical system are given by $(6.8)_{1,2}$ with $\varepsilon_0 = 0$:

$$\mathbb{M}_M = \rho \mathbb{u} , \qquad \rho E_M = \rho(e + \frac{1}{2}|\mathbb{u}|^2) + \frac{1}{2\mu_0}|\mathbb{B}|^2 .$$

Then, as a counterpart of [P2], we have:

[Q2] Let $(\rho, \mathbb{u}, \theta, \mathbb{B})$ be a solution of (6.4). Then, under the condition $[C1]_1$, the quantities ρ, ρE_M and $\mathbb{B}$ respectively satisfy conservation laws. Moreover, if the solution satisfies (6.5), the equation for $\rho\mathbb{u}$ also becomes a conservation form.

In fact it is obvious for ρ and $\mathbb{B}$. As for $\rho\mathbb{u}$ and ρE_M, we have the equations (6.9) and (6.11) with $\varepsilon_0 = \rho_e = 0$ and $\mathbb{E} = -\mathbb{u}\times\mathbb{B} + (1/\sigma\mu_0)\operatorname{rot}\mathbb{B}$:

$$(\rho\mathbb{u})_t + \operatorname{div}\{\rho\,{}^t\mathbb{u}\mathbb{u} + pI - (1/2\mu_0)(2\,{}^t\mathbb{B}\mathbb{B} - |\mathbb{B}|^2 I)\} +$$

$$+ (1/\mu_0)\mathbb{B}\operatorname{div}\mathbb{B} = \operatorname{div}(2\mu\mathbb{P} + \mu' I\operatorname{div}\mathbb{u}) \;,$$

$$(\rho E_M)_t + \operatorname{div}\{\rho\mathbb{u}(e+|\mathbb{u}|^2/2) + p\mathbb{u} - (1/\mu_0)(\mathbb{u}\times\mathbb{B})\times\mathbb{B}\}$$

$$= \operatorname{div}\{2\mu\mathbb{u}\mathbb{P} + \mu'\mathbb{u}\operatorname{div}\mathbb{u} + \kappa\nabla\theta + (1/\sigma\mu_0^2)\mathbb{B}\times\operatorname{rot}\mathbb{B}\} \;,$$

from which follows the assertion for $\rho\mathbb{u}$ and ρE_M.

The property [P3] is modified as follows.

[Q3] Under the conditions $[C1]_{1,2}$ the negative entropy $-\rho S$ (resp. $-S$) is a strictly convex function of the conserved quantities $(\rho, \rho\mathbb{u}, \rho E_M, \mathbb{B})$ (resp. $(1/\rho, \mathbb{u}, E_M, \mathbb{B}/\rho)$). The total energy ρE_M (resp. E_M) is also a strictly convex function of $(\rho, \rho\mathbb{u}, \rho S, \mathbb{B})$ (resp. $(1/\rho, \mathbb{u}, S, \mathbb{B}/\rho)$).

We note that the equation of the entropy is given by (6.12) with $\mathbb{E} + \mathbb{u}\times\mathbb{B} = (1/\sigma\mu_0)\operatorname{rot}\mathbb{B}$:

$$(\rho S)_t + \operatorname{div}(\rho\mathbb{u}S) = \operatorname{div}\{(\kappa/\theta)\nabla\theta\} +$$

$$+ (1/\theta)\{\Psi + (\kappa/\theta)|\nabla\theta|^2 + (1/\sigma\mu_0^2)|\mathrm{rot}\,\mathbb{B}|^2\} .$$

Let E_M^* be the quadratic function associated with E_M. Then ρE_M^* is given by (6.13) with $\varepsilon_0 = 0$:

$$\rho E_M^* = \rho\{e - \bar{e} + \bar{p}(1/\rho - 1/\bar{\rho}) - \bar{\theta}(S - \bar{S}) + \frac{1}{2}|\mathbf{u}|^2\} + \frac{1}{2\mu_0}|\mathbb{B} - \bar{\mathbb{B}}|^2 ,$$

where $\bar{\rho} > 0$, $\bar{\theta} > 0$ and $\bar{\mathbb{B}} \in \mathbb{R}^3$ are arbitrarily fixed constants, and $\bar{e} = e(\bar{\rho}, \bar{\theta})$ etc. It satisfies the equation (6.14) with $\varepsilon_0 = 0$ and $\mathbb{E} = -\mathbf{u} \times \mathbb{B} + (1/\sigma\mu_0)\mathrm{rot}\,\mathbb{B}$, i.e.,

$$\begin{aligned}
&(\rho E_M^*)_t + \mathrm{div}\{\rho\mathbf{u}\{e - \bar{e} + \bar{p}(1/\rho - 1/\bar{\rho}) - \bar{\theta}(S - \bar{S}) + \\
&+ |\mathbf{u}|^2/2\} + (p - \bar{p})\mathbf{u} - (1/\mu_0)(\mathbf{u} \times \mathbb{B}) \times (\mathbb{B} - \bar{\mathbb{B}})\} + \\
&+ (\bar{\theta}/\theta)\{\Psi + (\kappa/\theta)|\nabla\theta|^2 + (1/\sigma\mu_0^2)|\mathrm{rot}\,\mathbb{B}|^2\} \\
&= \mathrm{div}\{2\mu\mathbf{u}\mathbb{P} + \mu'\mathbf{u}\,\mathrm{div}\,\mathbf{u} + \kappa(1 - \bar{\theta}/\theta)\nabla\theta + (1/\sigma\mu_0^2)(\mathbb{B} - \bar{\mathbb{B}}) \times \mathrm{rot}\,\mathbb{B}\} .
\end{aligned}$$

Now, taking (6.5) into account, we transform (6.4) into the following symmetric system:

$$(6.4)' \quad \begin{cases}
\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 , \\
\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p - (1/\mu_0)\mathrm{rot}\,\mathbb{B} \times \mathbb{B} = \mathrm{div}(2\mu\mathbb{P} + \mu' I\,\mathrm{div}\,\mathbf{u}) , \\
\rho e_\theta(\theta_t + \mathbf{u}\cdot\nabla\theta) + \theta p_\theta \mathrm{div}\,\mathbf{u} = \mathrm{div}(\kappa\nabla\theta) + \Psi + (1/\sigma\mu_0^2)|\mathrm{rot}\,\mathbb{B}|^2 ,
\end{cases}$$

$$\begin{cases} \mathbb{B}_t + (\mathbf{u}\cdot\nabla)\mathbb{B} + \mathbb{B}\,\mathrm{div}\,\mathbf{u} - (\mathbb{B}\cdot\nabla)\,\mathbf{u} \\ \qquad\qquad = (1/\sigma\mu_0)\Delta\mathbb{B} - \nabla(1/\sigma\mu_0)\times\mathrm{rot}\,\mathbb{B}\,. \end{cases}$$

Here we have used the equalities

$$(6.41)_1 \qquad \mathrm{rot}(\mathbf{u}\times\mathbb{B}) = \{\mathbf{u}\,\mathrm{div}\,\mathbb{B} - (\mathbf{u}\cdot\nabla)\mathbb{B}\} - \{\mathbb{B}\,\mathrm{div}\,\mathbf{u} - (\mathbb{B}\cdot\nabla)\,\mathbf{u}\}\,,$$

$$(6.41)_2 \qquad -\mathrm{rot}\{(1/\sigma\mu_0)\mathrm{rot}\,\mathbb{B}\} = (1/\sigma\mu_0)(\Delta\mathbb{B} - \nabla\mathrm{div}\,\mathbb{B}) - \nabla(1/\sigma\mu_0)\times\mathrm{rot}\,\mathbb{B}\,.$$

It is easy to see that the system (6.4)',(6.5) is equivalent to the original system (6.4),(6.5). As for (6.4)', we also have the following property.

[Q1]' A smooth solution of (6.4)' (in the Sobolev spaces) satisfies (6.5) for all time $t > 0$ if it satisfies (6.5) at $t = 0$.

Indeed, applying div to the equation of $\mathbb{B}$ of (6.4)' and using $(6.41)_1$ and $(6.41)_2$, we obtain $(\mathrm{div}\,\mathbb{B})_t + \mathrm{div}(\mathbf{u}\,\mathrm{div}\,\mathbb{B}) = \mathrm{div}\{(1/\sigma\mu_0)\nabla\,\mathrm{div}\,\mathbb{B}\}$, which proves the assertion.

To show the existence of a local solution, we prepare the following lemma.

<u>Lemma 6.9</u> *Let* $[\mathrm{C1}]_2$, *one of* $[\mathrm{C2}]_j$ *(*$j = 1,2,3,4$*) and one of* $[\mathrm{C3}]'_k$ *(*$k = 1, 2$*) be assumed. Let* $\bar\rho > 0$, $\bar\theta > 0$ *and* $\bar{\mathbb{B}} \in \mathbb{R}^3$ *be arbitrarily fixed constants. Then the system* (6.4)' *satisfies Conditions* 2.1 *and* 2.2 *for* $\mathcal{O} = \{(\rho,\mathbf{u},\theta,\mathbb{B}) \in \mathbb{R}^8\,;\ \rho > 0,\ \theta > 0\}$ *and a constant state* $(\bar\rho, 0, \bar\theta, \bar{\mathbb{B}})$,

i.e., it is symmetric hyperbolic-parabolic in the sense of chapter II. *(In the case where* $[C2]_4$ *and* $[C3]_2'$ *hold,* (6.4)' *is a symmetric hyperbolic system* $(m''=0)$*.)*

Proof. Put $w = {}^t(\rho, \mathbf{u}, \theta, \mathbf{B})$. Then the system (6.4)' is written in the form

$$(6.42) \qquad A^0(w)w_t + \sum_{j=1}^{3} A^j(w)w_{x_j} - \sum_{j,k=1}^{3} B^{jk}(w)w_{x_j x_k} = g(w, D_x w) ,$$

where $A^0(w)$, $A^j(w)$ and $B^{jk}(w)$ are square matrices of order 8, and $g(w, D_x w)$ is a $\mathbb{R}^8$-valued function; they are given explicitly by

$$(6.43)_1 \qquad A^0(w) = \begin{pmatrix} p_\rho/\rho & & & \\ & \rho I & & \\ & & \rho e_\theta/\theta & \\ & & & (1/\mu_0)I \end{pmatrix} ,$$

$$(6.43)_2 \qquad \sum_j A^j(w)\xi_j$$

$$= \begin{pmatrix} (p_\rho/\rho)(\mathbf{u}\cdot\xi) & p_\rho \xi & 0 & 0 \\ p_\rho {}^t\xi & \rho(\mathbf{u}\cdot\xi)I & p_\theta {}^t\xi & (1/\mu_0)({}^t\xi\mathbf{B} - (\mathbf{B}\cdot\xi)I) \\ 0 & p_\theta \xi & (\rho e_\theta/\theta)(\mathbf{u}\cdot\xi) & 0 \\ 0 & (1/\mu_0)({}^t\mathbf{B}\xi - (\mathbf{B}\cdot\xi)I) & 0 & (1/\mu_0)(\mathbf{u}\cdot\xi)I \end{pmatrix} ,$$

$$(6.43)_3 \qquad \sum_{jk} B^{jk}(w)\xi_j\xi_k$$

$$= \begin{pmatrix} 0 & & & \\ & \mu|\xi|^2 I + (\mu+\mu')\,{}^t\xi\xi & & \\ & & (\kappa/\theta)|\xi|^2 & \\ & & & (1/\sigma\mu_0^2)|\xi|^2 \end{pmatrix},$$

$$(6.43)_4 \qquad g(w,D_x w) = \begin{pmatrix} 0 \\ 2(\nabla\mu)\mathbb{P} + \nabla\mu' \operatorname{div}\mathbb{u} \\ (1/\theta)\{\Psi + \nabla\kappa\cdot\nabla\theta + (1/\sigma\mu_0^2)|\operatorname{rot}\mathbb{B}|^2\} \\ -\nabla(1/\sigma\mu_0^2)\times\operatorname{rot}\mathbb{B} \end{pmatrix}.$$

Here the relation $\mathbb{B}\times\operatorname{rot}\mathbb{B} = \frac{1}{2}\nabla(|\mathbb{B}|^2) - (\mathbb{B}\cdot\nabla)\mathbb{B}$ is used. It is seen that $A^0(w)$ is (real) diagonal and positive definite for $\rho > 0$, $\theta > 0$, and $A^j(w)$ and $B^{jk}(w) = B^{kj}(w)$ are real symmetric. Furthermore we have

$$(6.44) \qquad <\sum_{jk} B^{jk}(w)\omega_j\omega_k\hat{w},\ \hat{w}> \ \geq \min\{\mu,\ \nu\}|\hat{\mathbb{u}}|^2 + $$
$$+ (\kappa/\theta)|\hat{\theta}|^2 + (1/\sigma\mu_0^2)|\hat{\mathbb{B}}|^2$$

for $\hat{w} = {}^t(\hat{\rho}, \hat{\mathbb{u}}, \hat{\theta}, \hat{\mathbb{B}}) \in \mathbb{R}^8$ and $\omega \in S^2$. On the other hand $g(w,D_x w)$ can be regarded as a lower order term in every case of $[C2]_j$ and $[C3]'_k$ $(j=1,2,3,4\,;\,k=1,2)$. Moreover $g(\bar{w},0) = 0$ holds for $\bar{w} = {}^t(\bar{\rho},0,\bar{\theta},\bar{\mathbb{B}})$. Therefore the proof of Lemma 6.9 is completed.

We prescribe the initial data at $t = 0$:

(6.45) $\quad (\rho, \mathbf{u}, \theta, \mathbb{B})(0,x) = (\rho_0, \mathbf{u}_0, \theta_0, \mathbb{B}_0)(x)$.

By virtue of Lemma 6.9, Theorem 2.9 and the property [Q1]', we have:

Theorem 6.10 (local existence) *Let* $[C1]_2$, *one of* $[C2]_j$ *(*$j=1,2,3,4$*) and one of* $[C3]'_k$ *(*$k=1,2$*) be assumed. Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$ *and* $\bar{\mathbb{B}} \in \mathbb{R}^3$ *be arbitrarily fixed constants. Suppose that* $(\rho_0 - \bar{\rho}, \mathbf{u}_0, \theta_0 - \bar{\theta}, \mathbb{B}_0 - \bar{\mathbb{B}}) \in H^s(\mathbb{R}^3)$ *(for* $s \geq 3$*) and* $\inf_x\{\rho_0(x), \theta_0(x)\} > 0$. *Then the problem* (6.4)', (6.45) *has a unique solution* $(\rho, \mathbf{u}, \theta, \mathbb{B})(t,x)$ *(in the Sobolev spaces) on* Q_T *with some* $T > 0$, *which satisfies* $\inf_{Q_T}\{\rho(t,x), \theta(t,x)\} > 0$. *Furthermore if* $\operatorname{div} \mathbb{B}_0(x) = 0$ *for* $x \in \mathbb{R}^3$, *then* $(\rho, \mathbf{u}, \theta, \mathbb{B})(t,x)$ *becomes a solution of the problem* (6.4)', (6.5), (6.45) *(and consequently* (6.4), (6.5), (6.45)*).*

Remark 6.1 As a special case ($\mathbb{B} = \bar{\mathbb{B}} = 0$), we get a local solution (in the Sobolev spaces) to the initial value problem for the system (6.6) of fluid mechanics in every case of $[C2]_j$ ($j=1,2,3,4$).

Here we briefly survey the local existence results for the system (6.6). In the case $[C2]_1$ (i.e., viscosity and heat conductivity are assumed), the initial value problem for (6.6) was solved locally in time by Nash [59] and Itaya $[34]_{1,3}$ in the Hölder spaces, and by Vol'pert and Hudjaev [85] (see also $[55]_1$) in the Sobolev spaces. The existence results to the initial boundary value problems were established by Tani $[77]_{1,2}$ in the Hölder spaces, and by Matsumura and Nishida $[55]_{3,4}$ (see also [82], [69]) in the Sobolev spaces.

In the non-viscous case $[C2]_4$, the initial value problem was also solved locally in time (in the Sobolev spaces) by Vol'pert and Hudjaev [85] and Kato $[38]_4$. However the initial boundary value problems (in the general situation) are still open in this case; see [14], [83] and [1], where the existence results were obtained under the assumption that $p = p(\rho)$ is independent of θ (the barotropic case).

In the case $[C2]_2$ or $[C2]_3$, we don't know the existence results to the initial boundary value problems for (6.6).

Now we consider the global existence problem for (6.4), (6.5), (6.45). We need the following lemma.

Lemma 6.11 *Let the conditions* $[C1]_2$, $[C2]_1$ *and* $[C3]_1'$ *be assumed. Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$ *and* $\bar{\mathbb{B}} \in \mathbb{R}^3$ *be arbitrarily fixed constants. Then the linearized system of* (6.4)' *at the constant state* $\bar{w} = {}^t(\bar{\rho}, 0, \bar{\theta}, \bar{\mathbb{B}})$ *satisfies Conditions* 3.1 *and* 3.2 *(with* $L(\bar{w}) = 0$*). In particular, the matrices* K^j *(*$j = 1,2,3$*) in Condition* 3.2 *are taken as in* (6.46) *with a suitably small constant* $\alpha > 0$.

Proof. The condition 3.1 is easily verified. Here we only check Condition 3.2. We may take K^j to be

$$(6.46) \qquad \sum_j K^j \xi_j = \alpha \left(\begin{array}{c|ccc} 0 & \bar{p}_\rho \xi & 0 & 0 \\ \hline -\bar{p}_\rho {}^t\xi & & & \\ 0 & & 0 & \\ 0 & & & \end{array} \right) A^0(\bar{w})^{-1}$$

with a suitably small constant $\alpha > 0$, where $\bar{p}_\rho = p_\rho(\bar{\rho},\bar{\theta})$ and $\bar{w} = {}^t(\bar{\rho}, 0, \bar{\theta}, \bar{\mathbb{B}})$. For this choice of K^j, Condition 3.2 (i) is obvious. Moreover we have

$$(6.47) \qquad < \{\sum_{jk}[K^jA^k(\bar{w})]'\omega_j\omega_k\}\hat{w}, \hat{w} >$$

$$\geq \frac{\alpha}{2}\{(\bar{p}_\rho^{\,2}/\bar{\rho})|\hat{\rho}|^2 - C|\hat{u}, \hat{\theta}, \hat{\mathbb{B}}|^2\}$$

for $\hat{w} = {}^t(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{\mathbb{B}}) \in \mathbb{R}^8$, where C is a constant independent of ω and α, and $[K^jA^k(\bar{w})]'$ denotes the symmetric part of $K^jA^k(\bar{w})$. The estimates (6.44) (with $w = \bar{w}$) and (6.47) imply Condition 3.2 (with $L(\bar{w}) = 0$) for a suitably small $\alpha > 0$. This completes the proof of Lemma 6.11.

Under the conditions $[C1]_2$, $[C2]_1$ and $[C3]_1'$, the results of Theorems 3.10 and 3.11 are applicable to the problem (6.4)',(6.45) (and therefore (6.4),(6.5),(6.45)) because the condition (3.30) is satisfied for the system (6.4)'. Hence we can get the global existence and asymptotic stability results. Moreover in [P2] and [P3] we have proved that the system (6.4),(6.5) can be put into a conservation form and has a convex entropy if $[C1]_{1,2}$ are assumed. So the arguments in Theorems 4.3, 4.4 and 4.5 are valid for the present system (6.4),(6.5). Thus we have:

<u>Theorem 6.12</u> (global existence and asymptotic stability) *Let* $[C1]_{1,2}$, $[C2]_1$ *and* $[C3]_1'$ *be assumed. Suppose that* $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}, \mathbb{B}_0 - \bar{\mathbb{B}}) \in H^s(\mathbb{R}^3)$ *(for* $s \geq 3$*),* $\operatorname{div}\mathbb{B}_0(x) = 0$ *for* $x \in \mathbb{R}^3$ *and* $\|\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}, \mathbb{B}_0 - \bar{\mathbb{B}}\|_s$ *is sufficiently small. Then the problem* (6.4),(6.5),(6.45) *has*

a unique global solution $(\rho, \mathbb{u}, \theta, \mathbb{B})(t,x)$ *(in the Sobolev spaces), which converges, in the* $B^{s-3}(\mathbb{R}^3)$*-norm, to the constant state* $(\bar{\rho}, 0, \bar{\theta}, \bar{\mathbb{B}})$ *as* $t \to \infty$.

Moreover we assume that $(\rho_0 - \bar{\rho}, \mathbb{u}_0, \theta_0 - \bar{\theta}, \mathbb{B}_0 - \bar{\mathbb{B}}) \in H^s(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ *(for* $s \geq 4$ *and* $p \in [1,2)$*) and* $\|\rho_0 - \bar{\rho}, \mathbb{u}_0, \theta_0 - \bar{\theta}, \mathbb{B}_0 - \bar{\mathbb{B}}\|_{s,p}$ *is sufficiently small. Then* $\|(\rho - \bar{\rho}, \mathbb{u}, \theta - \bar{\theta}, \mathbb{B} - \bar{\mathbb{B}})(t)\|_{s-2} \to 0$ *at the rate* $t^{-\gamma}$ *(*$\gamma = 3(1/2p - 1/4)$*) as* $t \to \infty$. *Furthermore this solution satisfies the asymptotic relation similar to* (4.37).

<u>Remark 6.2</u> As a special case ($\mathbb{B} = \bar{\mathbb{B}} = 0$), we obtain similar global existence and asymptotic stability results to the initial value problem for (6.6); these results were previously obtained by Matsumura and Nishida $[55]_{1,2}$ and Matsumura $[54]_4$. The global existence and asymptotic decay (without decay rate) of solutions to the initial boundary value problems for (6.6) were shown by Matsumura and Nishida $[55]_{3,4}$.

<u>6.6 MAGNETOHYDRODYNAMICS IN $\mathbb{R}^1$ (LOCAL EXISTENCE)</u>

In this section we shall consider the one-dimensional equations of magnetohydrodynamics. We assume that all the quantities appearing in (6.4),(6.5) are independent of (x_2, x_3). Then the first component of $\mathbb{B}$ becomes a constant ($B^1 = \overline{B^1} \in \mathbb{R}^1$) and the system (6.4),(6.5) is reduced to

$$\begin{cases} \rho_t + (\rho u)_x = 0 , \end{cases}$$

$$
(6.48)\quad \begin{cases}
\rho(u_t + uu_x) + p_x + (1/\mu_0)B\cdot B_x = (\nu u_x)_x\,, \\[2ex]
\rho(v_t + uv_x) - (1/\mu_0)\overline{B^1}B_x = (\mu v_x)_x\,, \\[2ex]
\rho e_\theta(\theta_t + u\theta_x) + \theta p_\theta u_x = (\kappa\theta_x)_x + \Psi + (1/\sigma\mu_0^2)\,|B_x|^2\,, \\[2ex]
B_t + (uB - v\overline{B^1})_x = \{(1/\sigma\mu_0)B_x\}_x\,,
\end{cases}
$$

where $x = x_1$, $u = u^1$, $v = (u^2,u^3)$, $B = (B^2,B^3)$ and $\Psi = \nu u_x^2 + \mu|v_x|^2$ (recall that $\nu = 2\mu + \mu'$).

We now consider the transformation $(t,x) \to (\tau,\xi)$:

$$
\tau = t\,, \qquad \xi = \int_0^x \rho(t,y)\,dy - \int_0^t (\rho u)(s,0)\,ds\,;
$$

(τ,ξ) is called a system of Lagrangian coordinates. Since $\partial/\partial t = \partial/\partial\tau - \rho u(\partial/\partial\xi)$ and $\partial/\partial x = \rho(\partial/\partial\xi)$, the system (6.48) is transformed to

$$
(6.49)\quad \begin{cases}
(1/\rho)_t - u_x = 0\,, \\[2ex]
u_t + \{p + (1/2\mu_0)\,|B|^2\}_x = (\nu\rho u_x)_x\,, \\[2ex]
v_t - (1/\mu_0)\overline{B^1}B_x = (\mu\rho v_x)_x\,, \\[2ex]
e_\theta\theta_t + \theta p_\theta u_x = (\kappa\rho\theta_x)_x + \rho\{\Psi + (1/\sigma\mu_0^2)\,|B_x|^2\}\,, \\[2ex]
B_t + \rho(Bu_x - \overline{B^1}v_x) = \rho\{(1/\sigma\mu_0)\rho B_x\}_x\,,
\end{cases}
$$

where (τ,ξ) is again denoted by (t,x). The equations (6.49) form a closed system of 7 equations for 7 unknowns (ρ,u,v,θ,B).

For the system (6.49), we have the following modification of [Q2].

[R2] Under the condition $[C1]_1$ the system (6.49) can be put into a conservation form; the conserved quantities are $(1/\rho,\mathbf{u},E_M,B/\rho)$, where $\mathbf{u}=(u,v)$ and $\rho E_M = \rho(e+|\mathbf{u}|^2/2) + (1/2\mu_0)|B|^2$.

It suffices to derive the conservation laws for E_M and B/ρ; they are given by

$$(6.50)_1 \qquad (E_M)_t + \{(p+(1/2\mu_0)|B|^2)u - (1/\mu_0)\overline{B^1}B\cdot v\}_x$$
$$= [\rho\{\nu u u_x + \mu v\cdot v_x + \kappa\theta_x + (1/\sigma\mu_0^2)B\cdot B_x\}]_x ,$$

$$(6.50)_2 \qquad (B/\rho)_t - (\overline{B^1}v)_x = \{(1/\sigma\mu_0)\rho B_x\}_x .$$

The property [Q3] is also valid in this case.

[R3] Under the conditions $[C1]_{1,2}$ the negative entropy $-S$ (resp. $-\rho S$) is a strictly convex function of the conserved quantities $(1/\rho,\mathbf{u},E_M,B/\rho)$ (resp. $(\rho,\rho\mathbf{u},\rho E_M,B)$). The total energy E_M (resp. ρE_M) is also a strictly convex function of $(1/\rho,\mathbf{u},S,B/\rho)$ (resp. $(\rho,\rho\mathbf{u},\rho S,B)$).

The equation of the entropy is given by

$$(6.51)\qquad S_t = \{(\kappa/\theta)\rho\theta_x\}_x + (\rho/\theta)\{\Psi + (\kappa/\theta)\theta_x^2 + (1/\sigma\mu_0^2)|B_x|^2\} .$$

The quadratic function E_M^* associated with the convex function E_M is given by

$$\rho E_M^* = \rho\{e - \bar{e} + \bar{p}(1/\rho - 1/\bar{\rho}) - \bar{\theta}(S - \bar{S}) + \frac{1}{2}|u|^2\} + \frac{1}{2\mu_0}|B - \bar{B}|^2 ,$$

where $\bar{\rho} > 0$, $\bar{\theta} > 0$ and $\bar{B} \in \mathbb{R}^2$ are arbitrarily fixed constants, and $\bar{e} = e(\bar{\rho},\bar{\theta})$ etc. It satisfies

$$(6.52)\qquad \begin{aligned}&(E_M^*)_t + [\{(p + (1/2\mu_0)|B|^2) - (\bar{p} + (1/2\mu_0)|\bar{B}|^2)\}u - \\ &- (1/\mu_0)\overline{B^1}(B - \bar{B})\cdot v]_x + \rho(\bar{\theta}/\theta)\{\Psi + (\kappa/\theta)\theta_x^2 + (1/\sigma\mu_0^2)|B_x|^2\} \\ &= [\rho\{\nu u u_x + \mu v\cdot v_x + \kappa(1 - \bar{\theta}/\theta)\theta_x + (1/\sigma\mu_0^2)(B - \bar{B})\cdot B_x\}]_x .\end{aligned}$$

In the case $\kappa \equiv 0$, we also take (p,u,v,S,B) as the unknowns of the system (6.49) and change (6.49) into

$$(6.53)\qquad \begin{cases} p_t + a^2 u_x = (\rho p_\theta/e_\theta)\{\Psi + (1/\sigma\mu_0^2)|B_x|^2\} , \\ u_t + \{p + (1/2\mu_0)|B|^2\}_x = (\nu\rho u_x)_x , \\ v_t - (1/\mu_0)\overline{B^1}B_x = (\mu\rho v_x)_x , \\ S_t = (\rho/\theta)\{\Psi + (1/\sigma\mu_0^2)|B_x|^2\} , \end{cases}$$

$$\left| \; B_t + \rho(Bu_x - \overline{B^1}v_x) = \rho\{(1/\sigma\mu_0)\rho B_x\}_x \right. ,$$

where $a = (\rho^2 p_\rho + \theta p_\theta^2/e_\theta)^{1/2}$ is the sound speed in Lagrangian coordinates.

Now we will show that Conditions 2.1 and 2.2 are satisfied for the one-dimensional system (6.49) or (6.53).

<u>Lemma 6.13</u> *Let* $[C1]_2$, *one of* $[C2]_j$ $(j=1,2,3,4)$ *and one of* $[C3]'_k$ $(k=1,2)$ *be assumed. Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$, $\overline{B^1} \in \mathbb{R}^1$ *and* $\bar{B} \in \mathbb{R}^2$ *be arbitrarily fixed constants. Then the system* (6.49) *satisfies Conditions* 2.1 *and* 2.2 *for* $\mathcal{O} = \{(\rho,u,v,\theta,B) \in \mathbb{R}^7 ;\ \rho > 0,\ \theta > 0\}$ *and a constant state* $(\bar{\rho},0,0,\bar{\theta},\bar{B})$. *Moreover, in the case* $\kappa \equiv 0$, *the system* (6.53) *also satisfies Conditions* 2.1 *and* 2.2 *for* $\mathcal{O} = \{(p,u,v,S,B) \in \mathbb{R}^7 ;\ p = p(\rho,\theta)$ *and* $S = S(\rho,\theta)$ *for* $\rho > 0,\ \theta > 0\}$ *and a constant state* $(\bar{p},0,0,\bar{S},\bar{B})$ *with* $\bar{p} = p(\bar{\rho},\bar{\theta})$ *and* $\bar{S} = S(\bar{\rho},\bar{\theta})$.

Proof. Put $w = {}^t(\rho,u,v,\theta,B)$. Then (6.49) is written in the form

$$A^0(w)w_t + A(w)w_x - B(w)w_{xx} = g(w,w_x) , \tag{6.54}$$

where $A^0(w)$, $A(w)$ and $B(w)$ are square matrices of order 7 and $g(w,w_x)$ is a $\mathbb{R}^7$-valued function; they are given explicitly by

$$(6.55)_1 \qquad A^0(w) = \frac{1}{\rho}\begin{pmatrix} p_\rho/\rho & & & & \\ & \rho & & & \\ & & \rho I & & \\ & & & \rho e_\theta/\theta & \\ & & & & (1/\mu_0)I \end{pmatrix},$$

$$(6.55)_2 \qquad A(w) = \begin{pmatrix} 0 & p_\rho & 0 & 0 & 0 \\ p_\rho & 0 & 0 & p_\theta & (1/\mu_0)\,{}^tB \\ 0 & 0 & 0 & 0 & -(1/\mu_0)\bar{B}^1 I \\ 0 & p_\theta & 0 & 0 & 0 \\ 0 & (1/\mu_0)B & -(1/\mu_0)\bar{B}^1 I & 0 & 0 \end{pmatrix},$$

$$(6.55)_3 \qquad B(w) = \rho\begin{pmatrix} 0 & & & & \\ & \nu & & & \\ & & \mu I & & \\ & & & \kappa/\theta & \\ & & & & (1/\sigma\mu_0^2)I \end{pmatrix},$$

$$(6.55)_4 \qquad g(w,w_x) = \begin{pmatrix} 0 \\ (\nu\rho)_x u_x \\ (\mu\rho)_x v_x \\ (1/\theta)\{\rho\Psi + (\kappa\rho)_x\theta_x + \rho(1/\sigma\mu_0^2)|B_x|^2\} \\ (\rho/\sigma\mu_0^2)_x B_x \end{pmatrix},$$

where I is the unit matrix of order 2. Compare $(6.55)_{1-4}$ with $(6.43)_{1-4}$. The assertion for the system (6.49) easily follows from the expressions $(6.55)_{1-4}$.

In the case $\kappa \equiv 0$ we put $w = {}^t(\rho,u,v,S,B)$ and $w' = {}^t(\rho,u,v,B)$. Then the system (6.53) nearly separates into two parts: $S_t = (\rho/\theta)\{\Psi + (1/\sigma\mu_0^2)|B_x|^2\}$ and

$$A^0(w)w'_t + A(w)w'_x - B(w)w'_{xx} = g(w,w_x) \,, \tag{6.56}$$

where $A^0(w)$, $A(w)$ and $B(w)$ are square matrices of order 6, and $g(w,w_x)$ is a $\mathbb{R}^6$-valued function; they are given explicitly by

$$(6.57)_1 \qquad A^0(w) = \frac{1}{\rho}\begin{pmatrix} \rho/a^2 & & & \\ & \rho & & \\ & & \rho I & \\ & & & (1/\mu_0)I \end{pmatrix},$$

$$(6.57)_2 \qquad A(w) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & (1/\mu_0)\,{}^tB \\ 0 & 0 & 0 & -(1/\mu_0)\overline{B^1}I \\ 0 & (1/\mu_0)B & -(1/\mu_0)\overline{B^1}I & 0 \end{pmatrix},$$

$$(6.57)_3 \qquad B(w) = \rho\begin{pmatrix} 0 & & & \\ & \nu & & \\ & & \mu I & \\ & & & (1/\sigma\mu_0^2)I \end{pmatrix},$$

$$(6.57)_4 \qquad g(w,w_x) = \begin{pmatrix} (\rho p_\theta/a^2 e_\theta)\{\Psi + (1/\sigma\mu_0^2)|B_x|^2\} \\ (\nu\rho)_x u_x \\ (\mu\rho)_x v_x \\ (\rho/\sigma\mu_0^2)_x B_x \end{pmatrix},$$

The assertion for (6.53) follows from $(6.57)_{1-4}$. This completes the proof of Lemma 6.13.

We prescribe the initial data at $t = 0$:

$$(6.58) \qquad (\rho,u,v,\theta,B)(0,x) = (\rho_0,u_0,v_0,\theta_0,B_0)(x) .$$

By virtue of Lemma 6.13 the initial value problem (6.49),(6.58) is solved locally in time as follows.

<u>Theorem 6.14</u> (local existence) *Let* $[C1]_2$, *one of* $[C2]_j$ *(*$j=1,2,3,4$*) and one of* $[C3]_k'$ *(*$k=1,2$*) be assumed. Let* $\bar{\rho} > 0$, $\bar{\theta} > 0$, $\overline{B^1} \in \mathbb{R}^1$ *and* $\bar{B} \in \mathbb{R}^2$ *be arbitrarily fixed constants. Suppose that* $(\rho_0 - \bar{\rho}, u_0, v_0, \theta_0 - \bar{\theta}, B_0 - \bar{B}) \in H^s(\mathbb{R}^1)$ *(for* $s \geq 2$*) and* $\inf_x\{\rho_0(x),\ \theta_0(x)\} > 0$. *Then the initial value problem* (6.49),(6.58) *has a unique solution* $(\rho,u,v,\theta,B)(t,x)$ *(in the Sobolev spaces) on* Q_T, *with some* $T > 0$, *which satisfies* $\inf_{Q_T}\{\rho(t,x),\ \theta(t,x)\} > 0$.

6.7 MAGNETOHYDRODYNAMICS IN $\mathbb{R}^1$ (GLOBAL EXISTENCE)

We shall consider the global existence problem for (6.49),(6.58). We first show the following lemma.

Lemma 6.15 *Let* $[C1]_{1,2}$ *be assumed and let* $\bar{\rho} > 0$, $\bar{\theta} > 0$, $\overline{B^1} \in \mathbb{R}^1$ *and* $\bar{B} \in \mathbb{R}^2$ *be constants.*

(i) If one of $[C2]_j$ *(j = 1,2,3,4) and one of* $[C3]'_k$ *(k = 1,2) are assumed, then the system* (6.49) *satisfies Conditions* 4.1 *and* 4.2 *for* $f^0(w) = {}^t(1/\rho, u, v, E_M, B/\rho)$, $\eta = -S$ *and* $\mathcal{O} = \{(\rho,u,v,\theta,B) \in \mathbb{R}^7 ; \rho > 0, \theta > 0\}$.

(ii) We assume one of the following three conditions:

$$1^\circ \quad \mu, \nu > 0, \quad \kappa > 0, \quad 1/\sigma > 0,$$

$$2^\circ \quad \mu \equiv \nu \equiv 0, \quad \kappa > 0, \quad 1/\sigma > 0,$$

$$3^\circ \quad \mu, \nu > 0, \quad \kappa > 0, \quad 1/\sigma \equiv 0.$$

In the case 2° *(resp. case* 3°*) we also assume* $|p_\theta(\bar{\rho},\bar{\theta})| + |\bar{B}| \neq 0$ *and* $\overline{B^1} \neq 0$ *(resp.* $\overline{B^1} \neq 0$*). Then the linearized system of* (6.49) *at the constant state* $\bar{w} = {}^t(\bar{\rho},0,0,\bar{\theta},\bar{B})$ *satisfies Condition* 4.3 *; the matrix* K *is taken as in* $(6.62)_{1-3}$ *below.*

(iii) We assume one of the following three conditions:

$$4^\circ \quad \mu, \nu > 0, \quad \kappa \equiv 0, \quad 1/\sigma > 0,$$

$$5^\circ \quad \mu \equiv \nu \equiv 0, \quad \kappa \equiv 0, \quad 1/\sigma > 0,$$

$$6^\circ \quad \mu, \nu > 0, \quad \kappa \equiv 0, \quad 1/\sigma \equiv 0.$$

In the case 5° *(resp. case* 6°*) we also assume* $|\bar{B}| \neq 0$ *and* $\overline{B^1} \neq 0$

(resp. $\overline{B^1} \neq 0$*). Then the linearized system of* (6.56) *at the constant state* $\overline{w} = {}^t(\overline{p},0,0,\overline{S},\overline{B})$ *satisfies Condition* 4.3, *where* $\overline{p} = p(\overline{\rho},\overline{\theta})$ *and* $\overline{S} = S(\overline{\rho},\overline{\theta})$*; the matrix* K *is taken as in* $(6.62)_{4-6}$ *below.*

Remark 6.3 (i) The condition $\overline{B^1} \neq 0$ is not essential in the case 2° or 5° (i.e., $\mu \equiv \nu \equiv 0$ and $1/\sigma > 0$). Indeed, if $\overline{B^1} = 0$ is assumed in the case 2°, the system (6.49) completely separates into two parts; the first part consists of $v_t = 0$ and the second part forms a system

$$
(6.59)\quad \begin{cases}
(1/\rho)_t - u_x = 0\ , \\
u_t + \{p + (1/2\mu_0)|B|^2\}_x = 0\ , \\
e_\theta\theta_t + \theta p_\theta u_x = (\kappa\rho\theta_x)_x + \rho(1/\sigma\mu_0^2)|B_x|^2\ , \\
B_t + \rho B u_x = \rho\{(1/\sigma\mu_0)\rho B_x\}_x\ ,
\end{cases}
$$

whose linearized system satisfies Condition 4.3 if $|p_\theta(\overline{\rho},\overline{\theta})| + |\overline{B}| \neq 0$. The case 5° is considered similarly. In fact, if $\overline{B^1} = 0$, the equations (6.56) separates into $v_t = 0$ and the remaining part whose linearized system satisfies Condition 4.3 if $|\overline{B}| \neq 0$.

(ii) In the case 3° or 6° (where $1/\sigma \equiv 0$), additional considerations are necessary if $\overline{B^1} = 0$. In fact, if $\overline{B^1} = 0$, the equation $(6.50)_2$ implies that $(B/\rho)(t,x) = (B_0/\rho_0)(x)$, and so the system (6.49) is reduced to

$$
\begin{cases}
(1/\rho)_t - u_x = 0\ ,
\end{cases}
$$

$$
(6.60)\quad \begin{cases} u_t + \{p + (1/2\mu_0)\,|(B_0/\rho_0)(x)|^2\rho^2\}_x = (\nu\rho u_x)_x\,, \\[2ex] v_t = (\mu\rho v_x)_x\,, \\[2ex] e_\theta\theta_t + \theta p_\theta u_x = (\kappa\rho\theta_x)_x + \rho\Psi\,. \end{cases}
$$

This system depends explicitly on the space variable x (unless $(B_0/\rho_0)(x)$ is a constant) and our results are not applicable.

<u>Proof of Lemma 6.15</u> (i) Condition 4.1 follows from the properties [R2] and [R3]; note that (6.51) is corresponding to (4.9). As a counterpart of (4.3) we have (6.54), and so Condition 4.2 was already checked in Lemma 6.13. Thus the proof of (i) is finished.

(ii) We first note that $(6.55)_3$ yields

$$
(6.61)\qquad \langle B(w)\hat{w}, \hat{w}\rangle = \rho\{\nu|\hat{u}|^2 + \mu|\hat{v}|^2 + (\kappa/\theta)|\hat{\theta}|^2 + (1/\sigma\mu_0^2)|\hat{B}|^2\}
$$

for $\hat{w} = {}^t(\hat{\rho},\hat{u},\hat{v},\hat{\theta},\hat{B}) \in \mathbb{R}^7$. In the case 1° we may take the matrix K in the same way as in Lemma 6.11:

$$
(6.62)_1\qquad K = \alpha \left(\begin{array}{c|cccc} 0 & \bar{p}_\rho & 0 & 0 & 0 \\ \hline -\bar{p}_\rho & & & & \\ 0 & & & & \\ 0 & & & 0 & \\ 0 & & & & \\ 0 & & & & \end{array}\right) A^0(\bar{w})^{-1}\,,
$$

where $\alpha > 0$ is a suitably small constant and $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$. Then, using $(6.55)_{1,2}$, we have

$$(6.63)_1 \qquad \langle [KA(\bar{w})]'\hat{w}, \hat{w} \rangle \geq \frac{\alpha}{2}\{\bar{p}_\rho^{\,2}|\hat{\rho}|^2 - C|\hat{u}, \hat{v}, \hat{\theta}, \hat{B}|^2\}$$

with some constant C independent of α, where $[KA(\bar{w})]'$ denotes the symmetric part of $KA(\bar{w})$. The estimates (6.61) and $(6.63)_1$ imply Condition 4.3.

In the case 2° we may take K to be

$$(6.62)_2 \qquad K = \alpha \left(\begin{array}{ccc|cc} 0 & \beta\bar{p}_\rho & -(2/\mu_0)(\overline{B^1}\,{}^t\bar{B}/\bar{\rho}) & 0 & 0 \\ -\beta\bar{p}_\rho & 0 & 0 & \bar{p}_\theta & (1/\mu_0)\,{}^t\bar{B} \\ (2/\mu_0)(\overline{B^1}\bar{B}/\bar{\rho}) & 0 & 0 & 0 & -(1/\mu_0)\overline{B^1}I \\ \hline 0 & -\bar{p}_\theta & 0 & & \\ 0 & -(1/\mu_0)\bar{B} & (1/\mu_0)\overline{B^1}I & & 0 \end{array} \right) A^0(\bar{w})^{-1}$$

with suitably small constants $\alpha > 0$ and $\beta > 0$, where $\bar{p}_\theta = p_\theta(\bar{\rho}, \bar{\theta})$. Then a simple calculation shows that

$$(6.63)_2 \qquad \langle [KA(\bar{w})]'\hat{w}, \hat{w} \rangle \geq \frac{\alpha}{2}\{\beta\bar{p}_\rho^{\,2}|\hat{\rho}|^2 + (-\beta\bar{\rho}^{\,2}\bar{p}_\rho +$$

$$+ \bar{\theta}\,\bar{p}_\theta^{\,2}/\bar{e}_\theta + (1/\mu_0)\bar{\rho}|\bar{B}|^2)|\hat{u}|^2 + (1/\mu_0)\bar{\rho}|\overline{B^1}|^2|\hat{v}|^2 - C|\hat{\theta}, \hat{B}|^2\}$$

with some constant C independent of α and β, where the conditions $|\bar{p}_\theta| + |\bar{B}| \neq 0$ and $\overline{B^1} \neq 0$ are used. The estimates (6.61) (with $\mu \equiv \nu$

$\equiv 0)$ and $(6.63)_2$ imply Condition 4.3 for small α and β.

In the case 3° we may take K to be

$$(6.62)_3 \qquad K = \alpha \left(\begin{array}{c|cccc|c} 0 & \bar{p}_\rho & & 0 & 0 & 0 \\ \hline -\bar{p}_\rho & & & & & -(1/\mu_0)\,{}^t\bar{B} \\ 0 & & & 0 & & (1/\mu_0)\overline{B^1}I \\ 0 & & & & & 0 \\ \hline 0 & (1/\mu_0)\bar{B} & & -(1/\mu_0)\overline{B^1}I & 0 & 0 \end{array}\right) A^0(\bar{w})^{-1}$$

with a suitably small constant $\alpha > 0$. After a simple calculation we have

$$(6.63)_3 \qquad \langle [KA(\bar{w})]'\hat{w}, \hat{w} \rangle \geq \frac{\alpha}{2}\{|\bar{p}_\rho\hat{\rho} + (1/\mu_0)\bar{B}\cdot\hat{B}|^2 + $$

$$+ (1/\mu_0)^2|\overline{B^1}|^2|\hat{B}|^2 - C|\hat{u}, \hat{v}, \hat{\theta}|^2\}$$

with some constant C independent of α. This estimate together with (6.61) (with $1/\sigma \equiv 0$) implies Condition 4.3 if $\overline{B^1} \neq 0$ is satisfied. This completes the proof of (ii).

(iii) Letting $\alpha > 0$ and $\beta > 0$ be suitably small constants, we take the matrix K as follows: in the case 4°

$$(6.62)_4 \qquad K = \alpha \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline -1 & & & \\ 0 & & 0 & \\ 0 & & & \end{array}\right) A^0(\bar{w})^{-1} ,$$

in the case 5°

$$
(6.62)_5 \qquad K = \alpha \left(\begin{array}{ccc|c}
0 & \beta & -(2/\mu_0)(\bar\rho\overline{B^1}\,{}^t\bar B/\bar a^2) & 0 \\
-\beta & 0 & 0 & (1/\mu_0)\,{}^t\bar B \\
(2/\mu_0)(\bar\rho\overline{B^1}\bar B/\bar a^2) & 0 & 0 & -(1/\mu_0)\overline{B^1}I \\
\hline
0 & -(1/\mu_0)\bar B & (1/\mu_0)\overline{B^1}I & 0
\end{array}\right) A^0(\bar w)^{-1},
$$

with $\bar a = a(\bar\rho,\bar\theta) = (\bar\rho^{2}\bar p_\rho + \bar\theta\bar p_\theta^2/\bar e_\theta)^{1/2}$, and in the case 6°

$$
(6.62)_6 \qquad K = \alpha \left(\begin{array}{c|cc|c}
0 & 1 & 0 & 0 \\
\hline
-1 & & & -(1/\mu_0)\,{}^t\bar B \\
 & & 0 & \\
0 & & & (1/\mu_0)\overline{B^1}I \\
\hline
0 & (1/\mu_0)\bar B & -(1/\mu_0)\overline{B^1}I & 0
\end{array}\right) A^0(\bar w)^{-1}.
$$

Then, by the similar arguments as in (ii), we get the conclusion of (iii). The details are omitted. This completes the proof of Lemma 6.15.

By virtue of Lemma 6.15 we have the following results conserning the global existence and asymptotic stability of the solution to (6.49), (6.58).

<u>Theorem 6.16</u> ([41]) (global existence and asymptotic stability) *Let* $[C1]_{1,2}$ *be assumed and let* $\bar\rho > 0$ $\bar\theta > 0$ $\overline{B^1} \in \mathbb{R}^1$ *and* $\bar B \in \mathbb{R}^2$ *be constants.*

(i) *We consider one of the cases* 1° − 3° *of Lemma* 6.15 *(ii); the addi-*

tional conditions specified in Lemma 6.15 *are also assumed. Suppose that* $(\rho_0-\bar\rho, u_0, v_0, \theta_0-\bar\theta, B_0-\bar B) \in H^s(\mathbb{R}^1)$ *(for* $s \geq 2$*) and* $\|\rho_0-\bar\rho, u_0, v_0, \theta_0-\bar\theta, B_0-\bar B\|_s$ *is sufficiently small. Then the problem* (6.49), (6.58) *has a unique global solution* $(\rho, u, v, \theta, B)(t,x)$ *(in the Sobolev spaces), which converges, in the* $B^{s-2}(\mathbb{R}^1)$*-norm, to the constant state* $(\bar\rho, 0, 0, \bar\theta, \bar B)$ *as* $t \to \infty$.

Moreover, if $(\rho_0-\bar\rho, u_0, v_0, \theta_0-\bar\theta, B_0-\bar B)$ *is small in* $H^s(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$ *(for* $s \geq 3$*), then* $\|(\rho-\bar\rho, u, v, \theta-\bar\theta, B-\bar B)(t)\|_{s-2}$ *converges to zero at the rate* $t^{-1/4}$ *as* $t \to \infty$. *This solution also satisfies the asymptotic relation similar to* (4.46), *provided that the above smallness condition is satisfied for* $s \geq 6$.

(ii) We consider one of the cases $4° - 6°$ *of Lemma* 6.15 *(iii), with the additional conditions specified there. Suppose that* $(\rho_0-\bar\rho, u_0, v_0, \theta_0-\bar\theta, B_0-\bar B) \in H^s(\mathbb{R}^1)$ *(for* $s \geq 2$*) and* $\|\rho_0-\bar\rho, u_0, v_0, \theta_0-\bar\theta, B_0-\bar B\|_s$ *is sufficiently small. Then the problem* (6.49), (6.58) *has a unique global solution* $(\rho, u, v, \theta, B)(t,x)$ *(in the Sobolev spaces), which satisfies the following decay law:* $|(p(\rho,\theta)-p(\bar\rho,\bar\theta), u, v, B-\bar B)(t)|_{s-2} \to 0$ *as* $t \to \infty$.

Remark 6.4 In the case $\mu \equiv \nu \equiv \kappa \equiv 1/\sigma \equiv 0$ the system (6.49) becomes a nonlinear hyperbolic system of conservation laws, and so we cannot expect in general the global existence of smooth solutions (see $[57]_7$, for example). In this case, however, the global existence of weak solutions is well known, see [22], [48]. While in the last case where $\mu \equiv \nu \equiv 1/\sigma \equiv 0$ and $\kappa > 0$ hold, global existence (or non-existence) problems are open.

Proof of Theorem 6.16 The results of (i) directly follow from Theorems 4.3, 4.4 and 4.8. Here we give the proof of (ii). We consider the case 4° and omit the arguments for the cases 5° and 6°. Noting that Condition 4.3 is satisfied only for $w' = {}^t(p,u,v,B)$, we modify $N_s(t)$ as follows:

$$N_s(t)^2 = \sup_{0\le\tau\le t} \|(\rho-\bar{\rho}, u, v, \theta-\bar{\theta}, B-\bar{B})(\tau)\|_s^2 +$$

$$+ \int_0^t \|D_x p(\rho,\theta)(\tau)\|_{s-1}^2 + \|D_x(u,v,B)(\tau)\|_s^2 d\tau .$$

Firstly, integrating (6.52) (with $\kappa \equiv 0$) by parts, we have

$$\|(\rho-\bar{\rho}, u, v, \theta-\bar{\theta}, B-\bar{B})(t)\|^2 + \int_0^t \|D_x(u,v,B)(\tau)\|^2 d\tau$$

$$\le C\|\rho_0-\bar{\rho}, u_0, v_0, \theta_0-\bar{\theta}, B_0-\bar{B}\|^2 .$$

Secondly, applying the arguments of Proposition 4.2 to the equations (6.56) for $w' = {}^t(p,u,v,B)$, we get the estimates

$$\|D_x(p,u,v,B)(t)\|_{s-1}^2 + \int_0^t \|D_x^2(u,v,B)(\tau)\|_{s-1}^2 d\tau$$

$$\le C\{\|D_x(p_0,u_0,v_0,B_0)\|_{s-1}^2 + N_s(T)^3\} ,$$

$$\int_0^t \|D_x p(\tau)\|_{s-1}^2 d\tau - C\{\|(p-\bar{p}, u, v, B-\bar{B})(t)\|_s^2 +$$

$$+ \int_0^t \|D_x(u,v,B)(\tau)\|_s^2 d\tau\}$$

$$\le C\{\|p_0-\bar{p}, u_0, v_0, B_0-\bar{B}\|_s^2 + N_s(T)^3\} ,$$

where $p_0 = p(\rho_0,\theta_0)$ and $\bar{p} = p(\bar{\rho},\bar{\theta})$. Here we have used the fact that the right hand side of (6.56) satisfies $g(w,D_x w) = O(|D_x w'|^2 + |D_x(\rho,\theta)||D_x w'|)$ for $|\rho-\bar{\rho}, u, v, \theta-\bar{\theta}, B-\bar{B}| \to 0$ (see $(6.57)_4$). Finally, from the equation of S, we deduce

$$\|D_x S(t)\|_{s-1}^2 \le C\{\|D_x S_0\|_{s-1}^2 + N_s(T)^3\} ,$$

where $S_0 = S(\rho_0,\theta_0)$. Here we have used the fact that the right member of the equation of S is dominated by $O(|D_x w'|^2)$ for $|\rho-\bar{\rho}, \theta-\bar{\theta}| \to 0$. Since $|p-\bar{p}, S-\bar{S}|$ is equivalent to $|\rho-\bar{\rho}, \theta-\bar{\theta}|$, a combination of the above estimates gives the desired a priori estimate for $N_s(T)$, from which follow the results of (ii) (cf. Theorem 4.3). This completes the proof.

Finally in this section, we briefly survey the global existence results for the system of fluid mechanics in one space-dimension. If the magnetic induction and the second and the third components of the velocity are neglected (i.e., $B=v=0$) in (6.49), we are led to the one-dimensional system of fluid mechanics in Lagrangian coordinates.

$$(6.64) \quad \begin{cases} (1/\rho)_t - u_x = 0 , \qquad u_t + p_x = (\nu\rho u_x)_x , \\ \\ e_\theta\theta_t + \theta p_\theta u_x = (\kappa\rho\theta_x)_x + \rho\Psi . \end{cases}$$

For this system, Theorem 6.16 is simplified as follows:

Corollary 6.17 ([41]) *Let* $[C1]_{1,2}$ *and one of the three conditions* $[C2]_j$ *(*$j=1,2,3$*) be assumed; in the case* $[C2]_2$ *we also assume* $|p_\theta(\bar{\rho},\bar{\theta})| \neq 0$*, where* $\bar{\rho}$ *and* $\bar{\theta}$ *are positive constants. Then, in the case* $[C2]_1$ *or* $[C2]_2$ *(resp.* $[C2]_3$*), the initial value problem for* (6.64) *is solved globally in time as in Theorem* 6.16 *(i) (resp. (ii)).*

Remark 6.5 In the case $[C2]_4$ (i.e., $\mu \equiv \nu \equiv \kappa \equiv 0$), smooth solutions of (6.64) in general develop singularities in the first derivatives in finite time (see $[52]_7$, for example). However, weak solutions (in the space of bounded variation) of (6.64) exist for all time $t \geq 0$ if the initial data have small total variation, see [22],[48]. Global weak solutions for large initial data were obtained by Nishida $[61]_1$, Nishida and Smoller $[65]_1$, DiPerna $[13]_1$ and Liu $[52]_1$ for ideal polytropic gases where the equations of state are given by

$$(6.65) \qquad p = R\rho\theta , \qquad e = c_V\theta + \text{constant} .$$

Here $R > 0$ is the gas constant and c_V (: positive constant) denotes the heat capacity at constant volume; the relation $c_V = R/(\gamma - 1)$ holds, where $\gamma \geq 1$ is the adiabatic exponent. They established the global existence results under the condition that the quantity $Q_1 \equiv (\gamma - 1)\cdot$ {total variation of the initial data} is sufficiently small. For initial boundary value problems, similar global existence results were also obtained in $[61]_1$, $[65]_2$, $[52]_2$. For asymptotic behaviors of these weak solutions, see $[13]_4, [52]_{5,6}$.

<u>Remark 6.6</u> In the case $[C2]_1$ (i.e., $\mu, \nu, \kappa > 0$) there are many results conserning global smooth solutions of (6.64). The general fluids satisfying $[C1]_{1,2}$ were considered by Okada and Kawashima [66],[41] (see the results of Corollary 6.17). In particular, the following result was proved in [66]. When μ, ν and κ are independent of θ, (6.64) has a solution in the Hölder spaces (which tends, in the maximum norm, to the constant state as $t \to \infty$) if the initial data belong to the corresponding Hölder spaces and are small in $H^1(\mathbb{R}^1)$. This result remain valid for the initial boundary value problems in a finite interval; in this case the solution decays at the exponential rate as $t \to \infty$.

Global smooth solutions to the initial value problem with large initial data were obtained by Kanel' $[36]_1$, Itaya $[34]_{2,4}$, Kawashima and Nishida [40], Kazhikhov $[42]_2$, and Okada and Kawashima [66] for ideal polytropic gases. Notice that (6.65) together with $[C1]_1$ gives

$$(6.66) \qquad p = C\rho^\gamma e^{(\gamma-1)S/R} \qquad \text{with some constant } C.$$

Kanel' $[36]_1$ considered the case $p = C\rho^\gamma$ (i.e., $S = \text{constant}$) and showed the global existence and asymptotic decay of solution (in the Hölder spaces) under the condition that the initial data belong to both $H^1(\mathbb{R}^1)$ and the Hölder spaces. These results were extended in [40] and [66] to the case (6.66), where the quantity $Q_2 \equiv (\gamma - 1)\cdot\{H^1(\mathbb{R}^1)\text{-norm of the initial data}\}$ is assumed to be sufficiently small. Kazhikhov [42] showed such global existence result without restriction on the quantity

Q_2. On the other hand, Itaya $[34]_{2,4}$ considered the case $p = C\rho$ (i.e., $\gamma = 1$ or equivalently $\theta =$ constnat) and proved the global existence of solution in the Hölder spaces when the initial data are in the corresponding Hölder spaces but not necessarily in the Sobolev spaces. It is an open problem to investigate asymptotic behaviors of the solutions obtained by Kazhikhov and Itaya.

The initial boundary value problems (in a finite interval) for the equations of ideal polytropic gases were also solved globally in time for large initial data; see Kazhikhov $[42]_{1,3}$, Kazhikhov and Shelukhin [44], Itaya $[34]_{5,6}$, and Okada and Kawashima [66] for the global existence, and also $[42]_3$ and [66] for asymptotic behaviors.

The monotonicity condition $[C1]_2$ on the pressure can not be satisfied for the Van der Waals gas, for which the relation

$$p = R\theta/(V-b) - a/V^2 \quad \text{holds for} \quad V \equiv 1/\rho > b ,$$

where a and b are positive constants. In this case the global existence problem for (6.64) is still open, see Kazhikhov and Nikolaev [43] and Kawohl [89]. We also refer to [12], [2], [3], [88] and [87], where similar problems in viscoelasticity were discussed.

Remark 6.7 In [72], Slemrod considered the initial boundary value problem for the system (6.64) of thermoelasticity under the condition $[C2]_2$ (i.e., $\mu \equiv \nu \equiv 0$, $\kappa > 0$). He showed the global existence of smooth solutions for small initial data. But his result can not be applied to the initial value problem in $\mathbb{R}^1$.

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