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A relation between the size of term and the number of reduction steps in lambda calculus computations

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1. Introduction

If a term has a normal form, there are many ways to obtain its normal form. There could be many reduction paths starting from the given term to its normal form. Then the length varies according to the paths and so does the size of the paths. We understand the size of a path to be the maximum size of the terms in the path. It is considered as the memory size needed by the computation (to obtain the normal form, i.e. the value of the input term).

The time optimal reduction stragies are studied in [4],[6], and an efficient implementation method of reduction is shown in [1]. As for the size, some works has been done in [3] and [5], but they are concerning to the combinatory reduction systems and the formulation of the problem is differrent from the one in this paper. This paper studies a relation, which is stated precisely in the next section, between the length and the size of reduction paths.

The motivation of the problem comes from the fact [2] that we cannot optimize both the length and the size at the same time in general. The term $(\lambda xy.\,pxx(yI))((\lambda x.pxx)A)I$ is such an example [2], where the size of A is supposed to be very large. In fact, the fastest computation of the term requires much more memory size than any other computation. (See the reduction graph of the term in Figure 1.1) We might recognize this fact due to a kind of "time-space tradeoff" in lambda calculus computations.

Figure 1.1 (別紙)

In the paper the author gives a solution to a problem, set by T. Adachi, whether there exists a term which actually has the time-space tradeoff in its reduction process. In other words, whether there exists a term satisfying the condition such that the faster a computation of the term, the more memory it requires. The main theorem shows that only the trivial terms satisfy the condition. Here "trivial" means that the length of reduction of the term is constant and independent on the choice of the reduction path.

The general notions and terminology are referred to [2].

2. Formulation of the problem in lambda calculus

In this section, we give a formulation of time-space tradeoff problem in lambda calculus computations. And we also give the precise definitions specific to this paper.

Definition 2.1 The size $|M|$ of λ-term M is defined inductively as follows: $|x| = 1$, $|(\lambda x\ A)| = |A| + 4$, $|(A\ \ B)| = |A| + |B| + 2$.

The size of a term is the number of symbols in the term including parenthesis and lambda. Note that the size of any redex, say $((\lambda x.M)N)$, is larger than 7.

Definition 2.2 Let $\sigma : M_0 \dashrightarrow M_1 \dashrightarrow \ldots \dashrightarrow M_n$ be a reduction path. Then the length of σ is n and denoted by $|\sigma|$. The size of σ the path is the maximum size of the terms in the path represented by $\|\sigma\|$.

Definition 2.3 Given a term M, the set of all reduction paths of M to its normal form is represented by R(M). If M has no normal form, the set R(M) is empty.
Sometimes we call a path in R(M) normalizing path.

Definition 2.4 A term M is said to be TST if its reduction paths satisfy the following condition:

(TST) if $|\alpha| < |\beta|$ then $\|\beta\| < \|\alpha\|$ for all , in R(M).

We study this condition as a formulation of time-space tradeoff in lambda calculus computations. This condition can be read that if we want to get faster algorithm we have to have more momory size. Or it can be read that if one reduction is faster than another one, then the efficiency is achieved only by the larger memory size.

Example 2.1 It is not the case that any term satisfies the condition whenever it has a normal form. For example, consider the term $M = KI(w_3 w_3)$, where $K=\lambda xy.x$, $I = \lambda x.x$ and $w_3 = \lambda x.xxx$. Figure 2.1 is the reduction graph of the term.

Figure 2.1 (別紙)

It is easy to see that for any n in N there exists a normalizing reduction longer than n. And the size of the reduction increases as the length becomes longer. So the condition is not satisfied.

Example 2.2 We construct the terms Mn inductively, by

$$M_0 = p,$$
$$M_{n+1} = (\lambda x.M_n xx)W,$$

where W= x.xx. Then we can see that M_n is a TST term as follows. Take an arbitrary path α in R(M) and consider the redexes contracted through the reduction. Every such a redex is the unique residual of a redex in M_n. Therefore Card(R(M))=n and we have = n. Hence M_n satisfies the condition TST. See Figure 2.2 for the case of $n \leqslant 3$.

Fugure 2.2

However, these examples are trivial ones. In fact every path of M_n has the same length (and the same size). Thus the condition TST was fulfilled trivially.

Then a question arises naturally. Is there any example in which time-space tradeoff really occurs, i.e., which satisfies the condition TST non-trivially ? The answer is "No", and it is proved in section 4.

3. <u>Structure of the reduction graph of TST term</u>

At the beginning we analyse the structure of the reduction graph of TST term.

Theorem 3.1 There is no cycle in the reduction graph of a TST term, if the term has a normal form.

Proof Let M be a TST term and suppose that there is a reduction cycle in the graph. Let it be $\alpha : M \overset{\alpha_0}{\dashrightarrow\!\!\!\rightarrow} N \overset{\gamma}{\dashrightarrow\!\!\!\rightarrow} N \overset{\alpha_1}{\dashrightarrow\!\!\!\rightarrow} L$, where $|\gamma| \geqslant 1$ and L is the normal form of M. Then consider the reduction $\beta = \alpha_0 + \gamma + \gamma + \alpha_1$. We have $|\beta| = |\alpha| + |\gamma| > |\alpha|$,

and $|\beta| = |\alpha|$. These contradict to TST Q.E.D.

As for theorem 3.2 below, note that even if a term has a normal form, it is not necessarily that every reduction path starting from the term is of finite length. There could be an infinite reduction of the term in general.

Theorem 3.2 If a TST term has a normal, then the length of the normalizing path is uniformly bounded by some constant.

Proof Suppose tat the theorem is not true for a TST term M. Then for each non-negative integer n there is a normalizing path ${}_n$ longer than n. Since every ${}_n$ has no cycle, by theorem 3.1, the reduction ${}_n$ consists of more than n distinct terms. Therefore $\sup\{\ |\beta_n|\ \mid n = 0,1,\ldots\} = \infty$. Then take a shortest normalizing path α. For some large n, we have $|\alpha| < |\beta_n|$ and $|\alpha| < |\beta_n|$.

These contradict to the condition TST for M. Q.E.D.

As we see later in section 4, the length of the normalizing paths is not only uniformly bounded but also constant.

Theorem 3.3 If a TST term has a normal form, then there is no infinite reduction path of the term.

Proof Suppose that a TST term M has an infinite reduction

$$M = M_0 \text{-->} M_1 \text{-->} M_2 \text{-->} \ldots \text{-->} M_n \text{-->} \ldots,$$

and a normal form N. By the Church-Rosser theorem, every M_n is reducible to N by some reduction $_n$. Then the reduction M_0 -->> M_n followed by $_n$ is a normalizing path longer than n. This contradits to the uniformly bondedness of the length. Q.E.D.

Lemma 3.4 Let M be a TST term having a normal form and α , β be reductions in the reduction graph of M. If both reductions start from the same term and terminate at the other same term, then we have $|\alpha| < |\beta| \Longrightarrow |\beta| < |\alpha|$.

Proof Let P and Q be the initial and the terminal point of the reductions respectively and N be a normal form of M. Then there are reductions σ and ρ such that σ :M -->> P and ρ :Q -->> N. Consider the reductions $\alpha_0 = \sigma + \alpha + \rho$ and $\beta_0 = \sigma + \beta + \rho$ in R(M). Then we have $|\alpha_0| - |\beta_0| = |\alpha| - |\beta|$. By the condition TST for α_0 and β_0 ,the theorem holds. Q.E.D.

4. Main result

In this section the main result (theorem 4.3) is proved for lambda terms. It says that only the trivial terms are TST.Here, "trivial" means that the length of the reductions (i.e., the computation time) is constant and is independent on the choice of the reduction method. It includes the case that the term has no normal form.

Lemma 4.1 Let M be a TST term having a normal form, and let σ, ρ be coinitial one step reductions in the reduction graph of the TSTterm M.Then both sides of the elementary diagramof and are of length 2.

Proof Let P be the initial point of σ and ρ, and let Q be the terminating point of the elementary diagram. Let $\sigma' = \sigma/\rho$, $\rho' = \rho/\sigma$ and let Δ_1, Δ_2 be the redexes contracted by $\sigma$and ρ respectively. (See Figure 4.1.)

Figure 4.1 (別紙)

Without loss of generality we can assume that Δ_2 lies left of Δ_1 in P. Then the residual of Δ_2 relative to σ is unique. Therefore $|\rho'| = 1$.

Now it suffices to show that $|\sigma'| = 1$. First suppose that $|\sigma'| = 0$. Then we have $|\rho| = 1 < |\sigma + \rho'| = 2$. Since M is TST, we have $|\rho| > |\sigma + \rho'|$. However, we have another inequality:

$$|\rho| = \max\{|P|, |Q|\} \leqslant \max\{|P|, |Q|, |P_1|\} = |\sigma + \rho'|.$$

These lead to a contadiction. Therefore $|\sigma'| \geqslant 1$.

Next suppose that $|\sigma'| \geqslant 2$. Then P has the form $C[(\lambda x.A)B]$ for some context $C[\ ]$, where $(\lambda x.A)B = \Delta_2$ and Δ_1 is in B. Then A has the free occurrences of x more than twice. Let m ($\geqslant 2$) be the number of the free occurrences of x in A. Since $|\rho + \sigma'| > 3 > |\sigma + \rho'| = 2$, we have $|\sigma + \rho'| < |\rho + \sigma'|$, by TST. Therefore we have the following inequalities: $|P| < |P_1|$, $|Q| < |P_1|$. On the other hand we can evaluate $|P|$, $|P_1|$ and $|Q|$ directly as follows:

$$|P| = |C[(\lambda x.A)B]| = |A| + |B| + 6 + k,$$
$$|P_1| = |C[(\lambda x.A)B']| = |A| + |B'| + 6 + k,$$
$$|Q| = |C[A[x:=B']]| = |A| - m + m|B'| + k,$$

where k is a constant and B' is obtained from B by the contraction of . By $|P| < |P_1|$, we have $|B| < |B'|$. By $|Q| < |P_1|$, we have $(m-1)|B'| < m + 6$. Since $m > 2$, $|B'| < (m + 6)/(m-1) = 1 + 7/(m-1) \leqslant 8$. Therefore $|B| \leqslant 7$. However, B contains a redex $'$, so $|B| > 7$. A contradiction. Therefore $|\sigma'| = 1$. Q.E.D.

Theorem 4.2 Let M be a TST term, and let N be a term reducible from M. If there is a normalizing path of N with length k, then the length of every normalizing reduction path of N to its normal form is k.

Proof By induction on k.

<u>Base Step</u>: k=0.

Then N is in normal form. So the theorem holds trivially.

<u>Inductio</u> <u>Step</u>: (See Figure 4.2.)

Figure 4.2 (別紙)

By the assumption of the theorem, there is a reduction of N to its normal form L such that $|\sigma| = k+1$. Let it be of the following form $\sigma : N \xrightarrow{\sigma_0} N_1 \xrightarrow{\sigma_1}\!\!> L$. Let $\tau : N \xrightarrow{\tau_0} N_2 \xrightarrow{\tau_1}\!\!> L$ be an arbitrary path of N to L. Then by lemma 4.1, both sides $N \xrightarrow{\sigma_0} N_1 \xrightarrow{\tau_0'} N_3$ and $N \xrightarrow{\tau_0} N_2 \xrightarrow{\sigma_0'} N_3$ of the elementary diagram of σ_0 and τ_0 are of length 2. Therefore $|\sigma_0'| = |\tau_0'| = 1$. By the Church-Rosser theorem, there is a reduction $\rho : N_3 \dashrightarrow\!\!> L$. By induction hypothesis for N_1, we have $|\tau_0' + \rho| = k$. Therefore $|\rho| = k - 1$. Thus we have $|\sigma_0' + \rho| = k$. Then by induction hypothesis for N_2, we have $|\tau_1| = k$. Thus $|\tau| = |\tau_0 + \tau_1| = k+1$. Q.E.D.

Recall that a normalizing path is a reduction path starting from a given term and terminating at its normal form.

Theorem 4.3 (Main theorem) A term is TST if and only if every normalizing path has the same length.

Proof "If-part" is trival. When the term has no normal form, the set of all normalizing paths is empty. Then the theorem is trivial. So we can assume that the term has a normal form. Note that every normalizing path starts from the given term and terminates at its normal form. Therefore they are of the same length by theorem 4.2. Q.E.D.

5. Combinatory reduction system

In this section we consider the problem in combinatory reduction system[ ]. The size of a term is defined as follows.

Definition 5.1 $|M| = 1$ if M is a basic combinator,

$$|(M\ N)| = |M| + |N| + 2.$$

Other definitions in the previous sections,including the TST term, are applied to combinatory reduction system without any modification.

Table 5.1 (別紙)

As the basic combinators we have e.g., the ones in table 5.1. In the table, every combinator is classified into two types. For every combinator of each type, the right hand side of the reduction rule has the following property.

type I : Every argument occurs on at most one occasion.

type II : There exists some argument which occurs more than two times.

The last culumn of the table shows the increase of the size of the term by the reduction.

If we read the proof of lemma 5.1, below, we can see that the restriction of the basic combinators to the ones in the table is not necessary. For example, the following condition for the reduction rules is sufficient. Every basic combinators of type II does not decrease the size of the term.

First we prove the lemma 4.1 for the combinatory reduction system.

Lemma 5.1 Let M be a TST term of a combinatory reductin system and let σ, ρ be coinitial one step reductions in the reduction graph of the term. Then both sides of the elementary diagram of σ and ρ are of length 2.

Proof See Figure Figure 4.1. The proof is almost same except the last paragraph. So it suffices to derive a contradiction form the assumption that $|\sigma'| \geqslant 2$. Suppose that $|\sigma'| \geqslant 2$. Then Δ_2 is type II. Since ρ' is the reduction of the residual of Δ_2, it does not decrease the size. Thus we have $|P_1| < |Q|$. Therefore the following inequality holds:

$$\begin{aligned} |\sigma + \rho'| &= \max\{ |P| , |P_1| , |Q| \} \\ &= \max\{ |P| , |Q| \} \\ &\leqslant |\rho + \sigma'| . \end{aligned}$$

While we have another inequality $|\rho + \sigma'| < |\sigma + \rho'|$ by the condition TST and the inequality $|\sigma + \rho'| < |\rho + \sigma'|$. A contradiction. Therefore $|\sigma'| = 1$. Q.E.D.

Theorem 5.2 A term in combinatory reduction system is TST if and only if every normalizing path has the same length.

Proof By lemma 5.1, we can prove the theorem similarly to theorem 4.3. Q.E.D.

Aknowledgement

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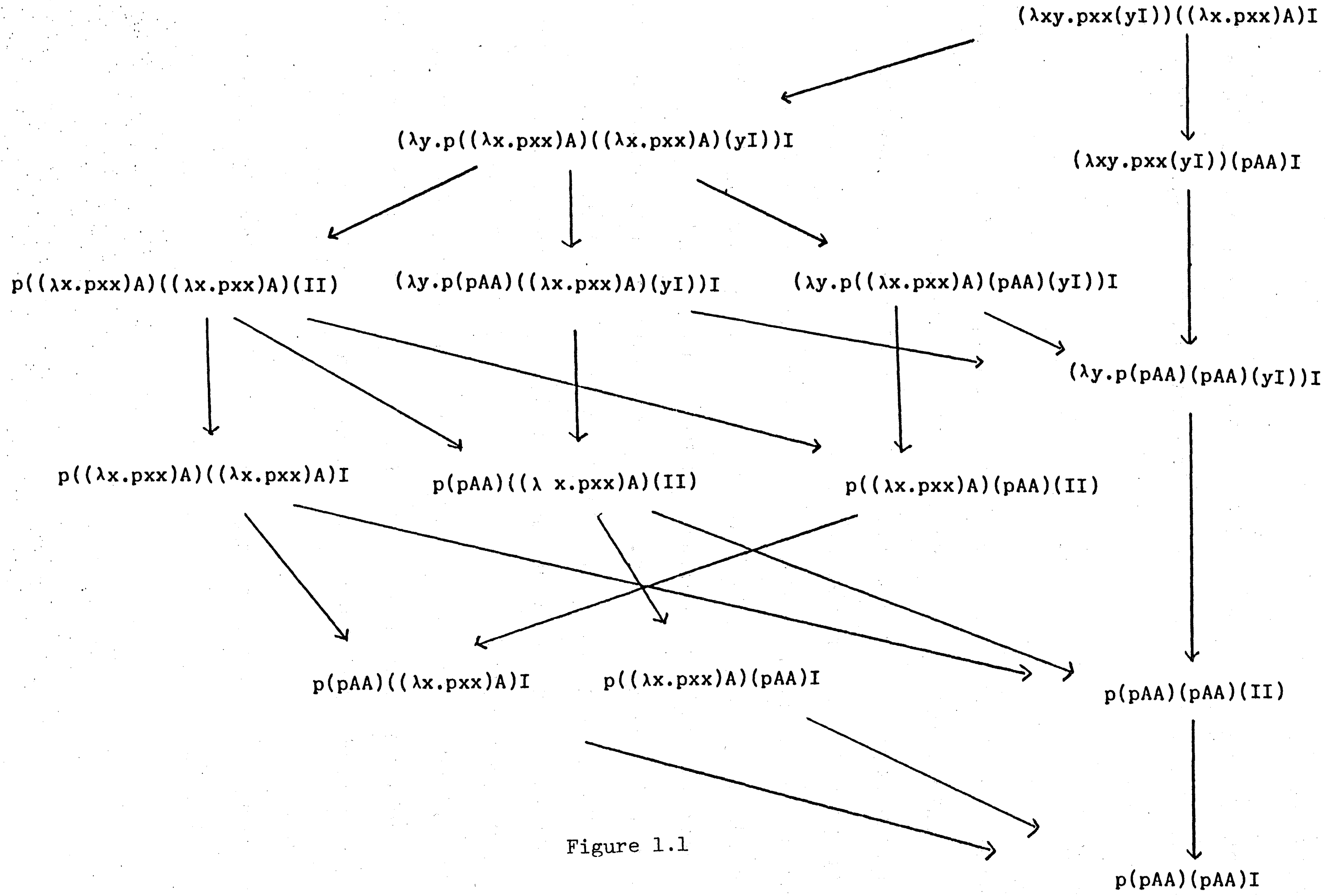
(λxy.pxx(yI))((λx.pxx)A)I
(λy.p((λx.pxx)A)((λx.pxx)A)(yI))I
(λxy.pxx(yI))(pAA)I
p((λx.pxx)A)((λx.pxx)A)(II)
(λy.p(pAA)((λx.pxx)A)(yI))I
(λy.p((λx.pxx)A)(pAA)(yI))I
(λy.p(pAA)(pAA)(yI))I
p((λx.pxx)A)((λx.pxx)A)I
p(pAA)((λ x.pxx)A)(II)
p((λx.pxx)A)(pAA)(II)
p(pAA)((λx.pxx)A)I
p((λx.pxx)A)(pAA)I
p(pAA)(pAA)(II)
p(pAA)(pAA)I

Figure 1.1

n=0 $M_0 = p$

n=1

$$\begin{tikzcd} M_1 = (\lambda x.pxx)\omega \arrow[d] \\ p\omega\omega \end{tikzcd}$$

n=2

$$\begin{tikzcd} & M_2 = (\lambda x.M_1xx)\omega \arrow[dl] \arrow[dr] & \\ M_1\omega\omega \arrow[dr] & & (\lambda x.p\omega\omega xx)\omega \arrow[dl] \\ & p\omega\omega\omega\omega & \end{tikzcd}$$

n=3

$$\begin{tikzcd} & M_3 = (\lambda x.M_2xx)\omega \arrow[dl] \arrow[d] \arrow[dr] & \\ M_2\omega\omega \arrow[d] \arrow[drr] & (\lambda x.M_1\omega\omega xx)\omega \arrow[dl] \arrow[d] & (\lambda x.(\lambda x.p\omega\omega xx)\omega xx)\omega \arrow[dl] \arrow[d] \\ M_1\omega\omega\omega\omega \arrow[dr] & (\lambda x.p\omega\omega\omega\omega xx)\omega \arrow[d] & (\lambda x.p\omega\omega xx)\omega\omega\omega \arrow[dl] \\ & p\omega\omega\omega\omega\omega\omega & \end{tikzcd}$$

Figure 2.2

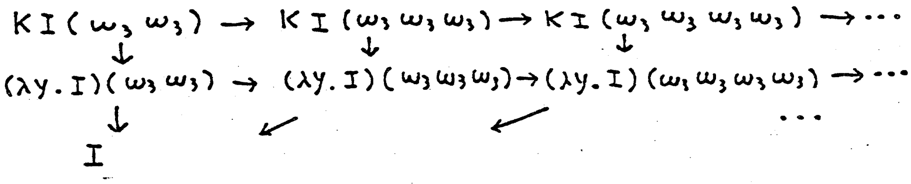

Figure 2.1

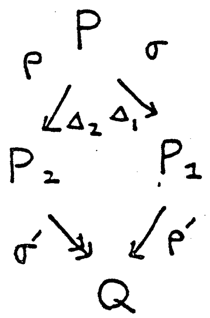

Figure 4.1

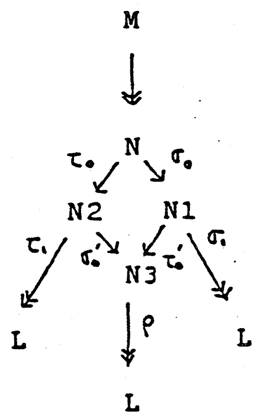

Figure 4.2

type	combinator	reduction rule			increase of the size
I	K	KLM	--→	L	−(‖M‖ + 5)
	I	IL	--→	L	−3
	B	BLMN	--→	L(MN)	−3
	C	CLMN	--→	LNM	−3
II	S	SLMN	--→	LN(MN)	‖M‖ − 1
	W	WLM	--→	LMM	‖M‖ − 1
	J	JLMNO	--→	LM(LON)	‖L‖ − 1
	Φ	ΦLMNO	--→	L(MO)(NO)	‖O‖ − 1
	Ψ	ΨLMNO	--→	L(MN)(MO)	‖M‖ − 1

Table 5.1