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C^0-sufficiency via blowing-up

by Satoshi Koike

Let $\mathcal{E}_{[q]}(n,1)$ denote the set of C^q function germs : $(\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$. For a function $f \in \mathcal{E}_{[q]}(n,1)$, $j^s f$ denotes the s-jet of f at $0 \in \mathbb{R}^n$ ($q \geq s$) . $J^s(n,1)$ is the set of s-jets of function germs in $\mathcal{E}_{[q]}(n,1)$. An s-jet $z \in J^s(n,1)$ is called <u>C^0-sufficient</u> in $\mathcal{E}_{[q]}(n,1)$, if for any functions f , g in $\mathcal{E}_{[q]}(n,1)$ such that $j^s f = j^s g = z$, there exists a local homeomorphism $\sigma : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $f \circ \sigma = g$. Thus C^0-sufficiency amounts to saying that all terms of degree $> s$ can be omitted without changing the local topological behavior of the realizations. Concerning C^0-sufficiency in $\mathcal{E}_{[s]}(n,1)$ or $\mathcal{E}_{[s+1]}(n,1)$, the following "Kuiper-Kuo Theorem" is well-known :

<u>Basic Results for C^0-sufficiency of jets</u> ([3], [7]). For $z \in J^s(n,1)$, the following conditions are equivalent.

(1) z is C^0-sufficient in $\mathcal{E}_{[s]}(n,1)$.
(resp. z is C^0-sufficient in $\mathcal{E}_{[s+1]}(n,1)$.)

(2) There exists $C > 0$ (resp. $\delta > 0$) such that

$$|\mathrm{grad}\ z(x)| \geq C\ |x|^{s-1}$$

(resp. $|\mathrm{grad}\ z(x)| \geq C\ |x|^{s-\delta}$) around $0 \in \mathbb{R}^n$.

A similar result for C^0-sufficiency of "complex" jets has been obtained ([1], [4]).

In this note, we shall give another characterization of C^0-sufficiency by using the "after blowing-up" functions. In practice, this criteria is often easier to check than the above one. Here we describe the results about C^0-sufficiency in $\mathcal{E}_{[s]}(n,1)$ only. Of course, similar results hold also for C^0-sufficiency in $\mathcal{E}_{[s+1]}(n,1)$ and C^0-sufficiency of complex jets.

§1. Observations.

First consider the case $n = 2$. Then, due to Lu Theorems ([10]), we can assume the given jet is in Weierstrass form (see [5]) :

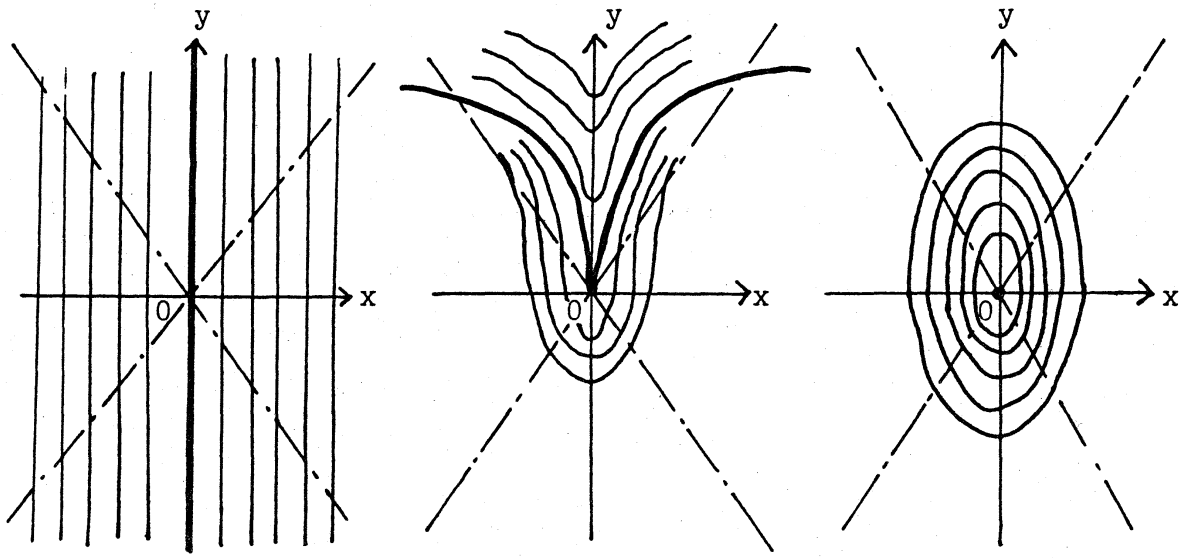
$$(*) \quad w(x,y) = x^k + H_{k+1}(x,y) + \cdots + H_{k+r}(x,y) \quad ,$$

where $H_j(x,y)$ is a homogeneous polynomial of degree j ($k + 1 \leq j \leq k + r$). Then

$$w(XY,Y) = Y^k H(X,Y) \quad ,$$

where $H(X,Y) = X^k + Y H_{k+1}(X,1) + \cdots + Y^r H_{k+r}(X,1)$.

<u>Observation 1.</u> Consider the level surfaces of the following three functions around $0 \in \mathbb{R}^2$:



$f_0(x,y) = x^2$, $f_1(x,y) = x^2 - y^3$, $f_2(x,y) = x^2 + y^4$.

Notice that although the paterns of their level surfaces are different from one another, they are almost (intrinsically) the same outside the sector of $\mathbb{R}^2$ containing their tangent cone $\{ x = 0 \}$ (the variety of the initial form). That is to say, <u>the initial form controls the behavior of a function outside the sector of $\mathbb{R}^2$ containing its tangent cone.</u>

Let us consider the blowing-up of $\mathbb{R}^2$ at 0 : $\pi : \mathcal{M}_2 \longrightarrow \mathbb{R}^2$. <u>The sector of $\mathbb{R}^2$ containing $\{ x = 0 \}$ is the image by π of an open set containing the intersection of the exceptional variety and the strict transform of $\{ x = 0 \}$ in $\mathcal{M}_2$.</u>

Thus the above observation gives rise to the following problem naturally :

Can we find certain conditions on H which control the behavior of w in the sector of $\mathbb{R}^2$ containing the tangent cone ?

We shall formulate such conditions for more general cases in §2.

<u>Observation 2.</u> Let $w(x,y) = x^2 - 2\,x\,y^{50}$. Then $k = 2$, and $w \in J^{100}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[100]}(2,1)$. We put

$$w(XY,Y) = Y^2\, H_1(X,Y) \ ,$$

$$H_k(XY,Y) = Y^2\, H_{k+1}(X,Y) \quad (1 \leqq k \leqq 48) \ .$$

After 49 times blowing-ups, we have $H_{49}(X,Y) = X^2 - 2\,X\,Y$, and $j^2H_{49} \in J^2(2,1)$ is C^0-sufficient in $\mathcal{E}_{[2]}(2,1)$. Then we notice the following relation at once :

$$\underline{100 = 2 + 2 \times 49} \ .$$

This observation leads us to the following question :

If j^rH is C^0-sufficient in $\mathcal{E}_{[r]}(2,1)$, then is $w \in J^{k+r}(2,1)$ C^0-sufficient in $\mathcal{E}_{[k+r]}(2,1)$?

This quetion will be answered in Corollary 1' in §2. By applying Corollary 1' 49 times, we can explain the above example.

§2. Results.

Let $w : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be a polynomial of degree $k + r$, and w be written as follows :

$$w(x) = Z_k(x) + G(x) \quad \text{with} \quad j^kG = 0 \ .$$

We put $A = S^{n-1} \cap Z_k^{-1}(0)$ and

$B[a] = \{ \sigma_a \in O(n) \mid \sigma_a(a) = e_n \ (= (0, \cdots, 0, 1) \}$ where $a \in A$. For $\sigma_a \in B[a]$, we write $w_{\sigma(a)} = w \circ \sigma_a^{-1}$. Here we put

$$w_{\sigma(a)}(X_1X_n, \cdots, X_{n-1}X_n, X_n) = X_n^k H_{\sigma(a)}(X_1, \cdots, X_n) .$$

Then $Z_k \circ \sigma_a^{-1}$ is a homogeneous polynomial of degree k and $Z_k \circ \sigma_a^{-1}(e_n) = 0$. Therefore $Z_k \circ \sigma_a^{-1}$ does not contain the term $a\, x_n^k$ ($a \neq 0$) . Thus $H_{\sigma(a)}$ is a polynomial with $H_{\sigma(a)}(0) = 0$. For $r \in \mathbb{N}$ and $\varepsilon_a > 0$, let $\mathcal{R}_r^n(H_{\sigma(a)};\varepsilon_a)$ denote the set $\{ X \in \mathbb{R}^n \mid |H_{\sigma(a)}(X)| \leq \varepsilon_a |X_n|^r \}$. Then we have the following characterization of C^0-sufficiency of (k + r)-jets by using the "after blowing-up" functions $H_{\sigma(a)}$.

<u>Theorem 1.</u> For $w \in J^{k+r}(n,1)$, the following conditions are equivalent.

(1) w is C^0-sufficient in $\mathcal{E}_{[k+r]}(n,1)$.

(2) For any $a \in A$, there exist $\sigma_a \in B[a]$ and C_a , $\varepsilon_a > 0$ such that

$$(**) \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, X_n \frac{\partial H_{\sigma(a)}}{\partial X_n}\right)\right| \geq C_a |X_n|^r$$

in $\mathcal{R}_r^n(H_{\sigma(a)};\varepsilon_a)$ around $0 \in \mathbb{R}^n$.

In the special case $n = 2$, we have

<u>Theorem 2.</u> For $w \in J^{k+r}(2,1)$, the following conditions are equivalent.

(1) w is C^0-sufficient in $\mathcal{E}_{[k+r]}(2,1)$.

(2) For any $a \in A$, there exist $\sigma_a \in B[a]$ and $C_a > 0$ such that

$$(**) \qquad \left|\left(\frac{\partial H_{\sigma(a)}}{\partial X}, Y\frac{\partial H_{\sigma(a)}}{\partial Y}\right)\right| \geq C_a \, |Y|^r \quad \text{around } 0 \in \mathbb{R}^2 .$$

<u>Remark 1.</u> If for some $\sigma_a \in B[a]$, there exists $C_a > 0$ such that (**) holds, then for any $\sigma_a \in B[a]$, there exists $C_a > 0$ such that (**) holds. In other words, the property (**) does not depend on σ_a ; We merely use σ_a for the formulation.

As a corollary of Theorem 1, we have

<u>Corollary 1.</u> If for any $a \in A$, there exists $\sigma_a \in B[a]$ such that $j^r H_{\sigma(a)}$ is C^0-sufficient in $\mathcal{E}_{[r]}(n,1)$, then $w \in J^{k+r}(n,1)$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(n,1)$.

<u>Corollary 2.</u> For the Weierstrass jet (*) $w \in J^{k+r}(2,1)$ in §1, the following conditions are equivalent,

(1) w is C^0-sufficient in $\mathcal{E}_{[k+r]}(2,1)$.

(2) There exists $C > 0$ such that

$$\left|\left(\frac{\partial H}{\partial X}, Y\frac{\partial H}{\partial Y}\right)\right| \geq C \, |Y|^r \quad \text{around } 0 \in \mathbb{R}^2 .$$

<u>Corollary 1'.</u> Let w be in the Weierstrass form (*) .

If $j^r H$ is C^0-sufficient in $\mathcal{E}_{[r]}(2,1)$, then $w \in J^{k+r}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(2,1)$.

§3. Examples.

We shall give first examples to show that it is easier to check the hypothesis of Corollary 2 than that of the Kuiper-Kuo Theorem. Example 4 (which will be stated in §4) is one of such examples too.

<u>Example 1.</u> Let $w(x,y) = x^{10} - 10\,x\,y^{11}$. Then we have $k = 10$ and $H(X,Y) = X^{10} - 10\,X\,Y^2$. It is easy to see that the condition (2) in Corollary 2 is satisfied for $r = 3$. Thus, from Corollary 2, it follows that $w \in J^{13}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[13]}(2,1)$.

<u>Example 2.</u> Let $w(x,y) = x^{10} - 10\,x\,y^{11} + y^{12}$. Then we have $k = 10$ and $H(X,Y) = X^{10} - 10\,X\,Y^2 + Y^2$. Similarly the condition (2) in Corollary 2 is satisfied for $r = 2$. Thus $w \in J^{12}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[12]}(2,1)$.

Next, we give an example which is related to the results in this note and some other facts in "singularity theory" .

<u>Example 3.</u> Let $P_t(x,y) = x^3 + 3\,x\,y^8 + t\,y^{10}$. We put $P_t(XY,Y) = Y^3\,H_t(X,Y)$ and $H_t(XY,Y) = Y^3\,G_t(X,Y)$. Then we have $H_t(X,Y) = X^3 + 3\,X\,Y^6 + t\,Y^7$ and $G_t(X,Y) = X^3 + 3\,X\,Y^4 + t\,Y^4$.

For $t \neq 0$, $j^4 G_t(X,Y) = X^3 + t\,Y^4$ is C^0-sufficient in $\mathcal{E}_{[4]}(2,1)$. Therefore $X^3 + t\,Y^4$ is $\mathcal{R}$-C^0 equivalent to $X^3 + 3\,X\,Y^4 + t\,Y^4$ for $t \neq 0$. Hence it follows from the proof of the theorems in this note that

$$x^3 + t\,y^{10} \text{ is } \mathcal{R}\text{-}C^0 \text{ equivalent to } x^3 + 3\,x\,y^8 + t\,y^{10} \quad \text{for } t \neq 0 \qquad \cdots (1).$$

Thus <u>we can omit the lower ordered term</u> $3\,x\,y^8$ from $x^3 + 3\,x\,y^8 + t\,y^{10}$, <u>using the notion of sufficiency of jets</u>.

On the other hand, by the Kuiper-Kuo theorem, $j^9 P_t = x^3 + 3\,x\,y^8$ is C^0-sufficient in $\mathcal{E}_{[9]}(2,1)$. Thus

$$x^3 + 3\,x\,y^8 \text{ is } \mathcal{R}\text{-}C^0 \text{ equivalent to } x^3 + 3\,x\,y^8 + t\,y^{10} \quad \text{for any } t \qquad \cdots (2).$$

(Recently T. C. Kuo has pointed out to me that $x^3 + 3\,x\,y^8$ is <u>not</u> "blow-analytic equivalent" (in his sense) to $x^3 + 3\,x\,y^8 + y^{10}$.) Note that $x^3 + 3\,x\,y^8$ is a weighted homogeneous polynomial of type $(\frac{1}{3}, \frac{1}{12})$ with an isolated singularity. The weight of y^{10} equals $\frac{5}{6} < 1$. In this case, $x^3 + 3\,x\,y^8$ <u>controls the behavior of the lower weight term</u> $t\,y^{10}$.

By (1) and (2), $x^3 + 3\,x\,y^8$ is $\mathcal{R}$-C^0 equivalent to $x^3 + t\,y^{10}$ for $t \neq 0$. This fact also follows from Henry King's results on analytic functions with an isolated singularity.

§4. The converse problem.

When we consider the problem of omitting lower ordered terms

by using the notion of sufficiency of jets, the converse problem of Corollary 1' is important too. Consider a polynomial $w(x,y) = x^k + G(x,y)$ with an isolated singularity and $j^k G = 0$. Then there exists $r \geq 1$ such that $j^{k+r}w$ is C^0-sufficient in $\mathcal{E}_{[k+r]}(2,1)$. Let $r(w)$ denote "the degree of C^0-sufficiency of w", namely the smallest integer having the above property. Then we put $w(XY,Y) = Y^k H(X,Y)$. It follows that H also has an isolated singularity at 0. Similarly there exists $\ell \geq 1$ such that $j^\ell H$ is C^0-sufficient in $\mathcal{E}_{[\ell]}(2,1)$. Let $\ell(w)$ be the degree of C^0-sufficiency of H. Here we put

$$A_r = \left\{ x^k + G \;\middle|\; \begin{array}{l} G : \text{polynomial of deg.} \leq k + r \,,\; j^k G = 0 \\ j^{k+r}(x^k+G) : C^0\text{-sufficient in } \mathcal{E}_{[k+r]}(2,1) \end{array} \right\}.$$

Then the following fact follows easily from Corollary 1' :

$$\ell(w) \geq r \quad \text{for any} \quad w \in A_r - A_{r-1} .$$

We now make some remarks on the converse problem.

<u>Proposition.</u> For $w \in A_r$,

$$\ell(w) \leq \max \{ k,\ r + 1,\ r(k - 2) + 1 \} .$$

<u>Remark 2.</u> In the above proposition,

(1) We can replace r by $r(w)$;

(2) If $k \leq 3$, then $\ell(w) \leq \max (k,\ r(w) + 1)$.

<u>Problem.</u> More generally, is it true that

$$\ell(w) \leqq \max\ (\ k,\ r(w) + 1\ )\ ?$$

In fact, $\ell(w) = r(w)$ or k for almost all w. The following is an example satisfying $r(w) < r(w) + 1 < \ell(w) < k$.

<u>Example 4.</u> Let $w(x,y) = x^8 + x^4 y^5 + x y^9$. Then we have $k = 8$ and $r(w) = 3$. On the other hand, $H(X,Y) = X^8 + X^4 Y + X Y^2$. Then we have $\ell(w) = 7$.

References

[1] J. Bochnak and W. Kucharz : Sur les germs d'applications différentiables à singularités isolées, Trans. Amer. Math. Soc., 252 (1979), 115-131.

[2] J. Bochnak and T. C. Kuo : Different realizations of a non-sufficient jet, Indag. Math., 34 (1972), 24-31.

[3] J. Bochnak and S. Łojasiewicz : A converse of the Kuiper-Kuo theorem, Proc. of Liverpool Singularities Sympo. I, Lecture Notes in Math., 192, Springer, 1971, 254-261.

[4] S. H. Chang and Y. C. Lu : On C^0-sufficiency of complex jets, Canad. J. Math., 25 (1973), 877-880.

[5] M. Kirshenbaum and Y. C. Lu : Sufficiency of Weierstrass jets, Canad. J. Math., 35 (1983), 167-176.

[6] T. C. Kuo : Characterizations of v-sufficiency of jets, Topology, 11 (1972), 115-131.

[7] T. C. Kuo : On C^0-sufficiency of jets of potential functions, Topology, 8 (1969), 167-171.

[8] T. C. Kuo : Modified trivializations of singularities, J. Math. Soc. Japan, 32(1980), 605-614.

[9] T. C. Kuo and J. N. Ward : A theorem on almost analytic equisingularities, J. Math. Soc. Japan, 33 (1981), 471-484.

[10] Y. C. Lu : Sufficiency of jets in $J^r(2,1)$ via Decomposition, Invent. Math., 10 (1970), 119-127.

Hyogo University of Teacher Education