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SATO'S PRINCIPLE FOR MICROLOCALIZATION AT THE BOUNDARY OF A CONVEX SET

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INTRODUCTION

Roughly speaking, Sato's fundamental microlocalization principle asserts that Pu=f implies for any partial differential operator P and any solution u, that one has the inclusion $S.S.(u) \subset S.S.(f) \cup \{(x,\eta) : \overset{\circ}{P}(x,\eta) = 0\}$ in the cotangential spherical bundle of the base space. Here, we are going to see that such a formula remains true if one considers a certain type of microlocalization at the boundary that allows to characterize the possible decompositions in sums of holomorphic functions in special imaginary conic domains admitted by a real analytic function or an hyperfunction defined over a convex set Ω of $\mathbb{R}^n$ near a point $x \in \partial\Omega$. To establish this fact, we are basically going to investigate the conditions under which the morphism induced by a linear differential operator with constant coefficients constitutes an isomorphism of the stalks of the microlocalization sheaf C^b.

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GEOMETRICAL BACKGROUND

From now, let Ω denote a convex open set of $\mathbb{R}^n$, F be its closure in the radial compactification $\mathbb{D}^n \simeq \mathbb{R}^n \cup S^\infty_{n-1}$ of $\mathbb{R}^n$ and $\partial\Omega$ its boundary in $\mathbb{D}^n$. We denote by $z=x+iy$ the points of $F+i\mathbb{R}^n$ and by $\zeta=\xi+i\eta$ the directions of $\mathbb{C}^n$, i.e. the elements of $\mathbb{C}^n\setminus\{0\}$. We identify $\mathbb{C}^n$ provided with the hermitian product $<z,\zeta> = \Sigma z_j\overline{\zeta_j}$ with the euclidean space $\mathbb{R}^{2n}$ provided with the scalar product $\mathrm{Re}<z,\zeta>$. We shall denote indifferently by P or P(D) the linear partial differential operator $\sum_{|\alpha|\leq m} c_\alpha D_x^\alpha$ over Ω with constant coefficients c_α or its natural extension to the complex domain $\Sigma c_\alpha D_z^\alpha$ (where $D_{z_j}=(D_{x_j}-iD_{y_j})/2$). The characteristic variety of P will be denoted by

$$\mathrm{Char}(P) = \{\zeta \in \mathbb{C}^n\setminus\{0\} : \overset{\circ}{P}(\zeta) = 0\} ,$$

where $\overset{\circ}{P}$ is the principal part of the operator. Throughout this paper, we will also mean by ω any open subset of F whose intersection with Ω is convex, by S_{n-1} the unit sphere of $\mathbb{R}^n$ and by S^*_{n-1} the unit cosphere of $\mathbb{R}^n$.

Let us now give a brief description of the sheaves of microlocalization we are going to deal with. If $\Gamma \supset \Gamma'$ are open convex cones of $\mathbb{R}^n$ with vertex 0, we denote by $\Lambda(\omega,\Gamma,\Gamma')$ the profile $[\bigcup_{x\in\omega\cap\Omega} \{x\} + i\Gamma] \cup [\bigcup_{x\in\omega\setminus\Omega} \{x\} + i\Gamma']$ and call a tuboid of profile $\Lambda(\omega,\Gamma,\Gamma')$ any intersection of an open convex neighborhood of $\omega\cap\Omega$ in $\Omega+i\mathbb{R}^n$ with an open subset V of $\Lambda(\omega,\Gamma,\Gamma')$ such that, given any compact set $K \subset \Lambda(\omega,\Gamma,\Gamma')$, one can find $\rho_0 > 0$ such that $x+i\rho y$ belongs to V for every $x+iy \in K$ and every ρ in $]0,\rho_0]$. If A and $\mathcal{O}$ denote respectively the sheaf of real analytic functions over Ω and the sheaf of holomorphic functions over $\Omega+i\mathbb{R}^n$, we denote by $\mathcal{O}_0$ [resp. $\mathcal{O}_1$, $\mathcal{O}_2$] the sheaf over SF:= $F \times S_{n-1}$ that associates to any $SF \cap \Lambda(\omega,\Gamma,\Gamma)$ the space $\lim\limits_{\to} \mathcal{O}(V)$, where V runs through the family of tuboids of profile $\Lambda(\omega,\mathbb{R}^n,\mathbb{R}^n)$ [resp. $\Lambda(\omega,\mathbb{R}^n,\Gamma)$, $\Lambda(\omega,\Gamma,\Gamma)$]. The quotient sheaves $\mathcal{O}_{1,0}=\mathcal{O}_1/\mathcal{O}_0$ and $\mathcal{O}_{2,0}=\mathcal{O}_2/\mathcal{O}_0$ allow to define the sheaves $C^{b,k}$ (k=1,2) which vanish over $\Omega \times S^*_{n-1}$ but, whose stalk at any point $(x_0,\eta_0) \in \partial\Omega \times S^*_{n-1}$ is defined by the formula

$$C^{b,k}_{(x_0,\eta_0)} = \lim_{\overrightarrow{m}} \frac{\mathcal{O}_{k,0}[SF \cap \Lambda(\omega_m,\Gamma_m,\Gamma_m)]}{\delta \bigoplus_{j=1}^{n} \mathcal{O}_{k,0}[SF \cap \Lambda(\omega_m,\Gamma_{j,m},\Gamma_{j,m})]} \tag{1}$$

with the following notations. The family ω_m ($m \in \mathbb{N}$) is a decreasing sequence of open neighborhoods of x_0 in F such that $\omega_m \cap \Omega$ is convex for any m, δ is the Čech coboundary operator (i.e. alternate sum of restrictions); if for any $\eta \in S^*_{n-1}$, we denote by E_η the open half-space $\{y \in \mathbb{R}^n : \langle y,\eta\rangle > 0\}$, then Γ_m and $\Gamma_{j,m}$ are defined by $\Gamma_m := \bigcap_{j=1}^{n} E_{\eta_{j,m}}$ and $\Gamma_{j,m} := \bigcap_{k\neq j} E_{\eta_{k,m}}$, where, for each m in $\mathbb{N}$, $\{\eta_{1,m}, \ldots, \eta_{n,m}\}$ is a set of linearly independant points of S^*_{n-1} verifying

$$\begin{cases} \lim\limits_{m\to\infty} \eta_{j,m} = \eta_0 \quad , \quad \forall\ j \in \{1,\ldots,n\} \\ \{\eta_0,\eta_{1,m+1},\ldots,\eta_{n,m+1}\} \subset \gamma_m := \{\sum\limits_{j=1}^{n} r_j\eta_{j,m} : r_j > 0\} \quad , \quad \forall\, m . \end{cases}$$

It is then possible to define the morphisms ρ^k, σ^k and $\tilde{\sigma}^k$ (k=1,2) making exact the rows of the commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \tau_*\mathcal{O}_0 \arrow[r, "\rho^1"] \arrow[d, equal] & \iota_*\mathcal{A} \arrow[r, "\sigma^1"] \arrow[d] & \pi_*C^{b,1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tau_*\mathcal{O}_0 \arrow[r, "\rho^2"] & \iota_*\mathcal{B} \arrow[r, "\sigma^2"] & \pi_*C^{b,2} \arrow[r] & 0 \end{tikzcd} \tag{2}$$

and

$$\begin{array}{ccccccccc} 0 \to \ker \tilde{\sigma}^1 & \longrightarrow & \pi^{-1}\iota_* A & \xrightarrow{\tilde{\sigma}^1} & C^{b,1} & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \to \ker \tilde{\sigma}^2 & \longrightarrow & \pi^{-1}\iota_* B & \xrightarrow{\tilde{\sigma}^2} & C^{b,2} & \to 0 \end{array} \quad , \qquad (3)$$

where ι is the imbedding $\Omega+i\mathbb{R}^n \to F+i\mathbb{R}^n$, τ the projection $SF \to F$, π the projection $S^*F := F \times S^*_{n-1} \to F$, the ρ^k's are restriction morphisms, the σ^k's decomposition morphisms and the $\tilde{\sigma}^k$'s the compositions of the respective $\pi^{-1}\sigma^k$'s with the natural morphisms $\pi^{-1}\pi_* C^{b,k} \longrightarrow C^{b,k}$. We may then define the <u>wave-front sets up to the boundary</u> by $W.F.^b(f) = \mathrm{supp}_{C^{b,k}}(\sigma^k f)$ for $f \in A$ if $k=1$ and $f \in B$ if $k=2$.

In what follows, we are going to develop the complete proofs only in the case of A and $C^{b,1}$ (that we are going to denote now by C^b for short). The case of hyperfunctions is entirely similar and the proofs even contain some simplifications related to the use of O_2 instead of that of O_1.

SOME PRELIMINARY PROPOSITIONS

PROPOSITION 1. *Any operator* P *induces an endomorphism* $\tilde{P}$ *of* C^b *verifying* $\tilde{\sigma}\circ P = \tilde{P}\circ\tilde{\sigma}$ (P *denotes also the trivial extension of the original* P *as an endomorphism of* $\pi^{-1}\iota_* A$). *Moreover,* $\tilde{P}$ *induces a surjective endomorphism of* $C^b_{(x_0,\eta_0)}$ *for any point* (x_0,η_0) *in* S^*F *such that* x_0 *admits a basis of neighborhoods* $\{\omega\}$ *verifying* $PA(\omega\cap\Omega) = A(\omega\cap\Omega)$.

<u>Proof</u>. By the definition of $O_{1,0}$, it is clear that P induces an endomorphism of this sheaf deduced from the usual action of a differential operator over holomorphic functions. The cohomological definition of C^b in terms of derived categories allows to deduce from this morphism the endomorphism $\tilde{P}$: $C^b \to C^b$ by derivation of functors.

The equality $\tilde{P}\circ\tilde{\sigma}=\tilde{\sigma}\circ P$ may then be verified only in the stalks. As the derivation of functors commutes with restrictions, it is clear that P coincides in the stalks with the application of P to elements of the numerator of (1). By commutativity of the diagram

$$\begin{array}{ccc} \pi^{-1}\pi_* C^b & \xrightarrow{\pi^{-1}\pi_*\tilde{P}} & \pi^{-1}\pi_* C^b \\ \downarrow & & \downarrow \\ C^b & \xrightarrow[\tilde{P}]{} & C^b \end{array}$$

and by the stability of the stalks under π^{-1}, we are lead to prove that $\sigma_{x_0}\circ P = (\pi_*\tilde{P})_{x_0}\circ\sigma_{x_0}$ holds for any $x_0 \in F$. By use of Čech cohomology, it is possible to prove that the image under σ of any $f \in (\iota_* A)(\omega) = A(\omega\cap\Omega)$ [where ω is any neighborhood of x_0 such that $\omega\cap\Omega$ is convex] may be represented by a vector $(([f_1], \ldots , [f_{n+1}]))$ with the f_j's sections of $\mathcal{O}_1$ over domains of SF whose projections over S_{n-1} are such that their polarsconstitute a covering of S^*_{n-1} for every x in ω. In the above representation, [ ] means the equivalence class for $\mathcal{O}_0$, (( ))the equivalence class for Čech coboundaries and one has

$$f = \sum_{j=1}^{n+1} f_j \quad \text{over } \omega\cap\Omega . \qquad (*)$$

We get therefore the equality $(\pi_*\tilde{P})\circ\sigma(f) = (([Pf_1],\ldots,[Pf_{n+1}]))$. On the other side, the equality $Pf=\sum_{j=1}^{n+1} Pf_j$ and the uniqueness of the decomposition (*) modulo Čech coboundaries gives also $\sigma\circ P(f) = (([Pf_1],\ldots,[Pf_{n+1}]))$, which is enough.

The surjectivity of $\tilde{P}_{(x_0,\eta_0)}$ when $PA(\omega\cap\Omega) = A(\omega\cap\Omega)$ holds for a basis of neighborhoods of x_0 is then a trivial consequence of the exact sequence of microlocalization (3).□

LEMMA 2. *Let* ω_0 *be an open subset of* F, $\eta_1, \dots, \eta_n \in S^*_{n-1}$ *be linearly independant and* Γ *be the open cone* $\bigcap_{j=1}^{n} E_{\eta_j}$. *For any open convex set* ω *relatively compact in* ω_0 *and any tuboid* V *of profile* $\Lambda(\omega_0,\mathbb{R}^n,\Gamma)$, *there exists a convex tuboid* V' *of profile* $\Lambda(\omega,\Gamma,\Gamma)$ *contained in* V *and which is constituted by the intersection of a convex neighborhood of* $\omega\cap\Omega$ *in* $\Omega+i\mathbb{R}^n$ *with a domain of product type with respect to real and imaginary variables.*

Proof. As convex neighborhood of $\omega\cap\Omega$ that will allow to define V', let us consider the intersection of the convex complex neighborhood of $\omega_0\cap\Omega$ that defines V with $\omega+i\mathbb{R}^n$. From now, let us also denote for short by V and V' the other two open sets whose intersections with the neighborhoods of $\omega_0\cap\Omega$ and $\omega\cap\Omega$ we just mentionned constitute the tuboids of the statement.

Let us denote by η and $\eta_{j,m}$ the traces on the cosphere of $\{\lambda\Sigma\eta_j:\ \lambda<0\}$ and $\{\lambda(\eta_j+m^{-1}\eta):\ \lambda>0\}$;it is direct to verify that for any $m\in\mathbb{N}$ larger than 1, the $\{\eta_{1,m},\dots,\eta_{n,m}\}$ still constitute bases of the dual of $\mathbb{R}^n$ and that the cones $\gamma_m := \bigcap_{j=1}^{n} E_{\eta_{j,m}}$ verify $\cup\gamma_m = \Gamma$ as well as $\gamma_m \subset\subset \gamma_{m+1} \subset\subset \dots \subset\subset \Gamma$ for every $m \ge 2$.

For any $r_m>0$, we have $\bigcap_{j=1}^{n}\{y\in\mathbb{R}^n:\ |y - r_m\eta_{j,m}| < r_m\}\subset\gamma_m$ and therefore, it is straightforward to prove the existence of a sequence of positive numbers r_m that decreases to 0 and such that one has:

$$(\alpha)\qquad B_m := \bar{\omega} + i\bigcap_{j=1}^{n}\{y\in\mathbb{R}^n:\ |y-r_m\eta_{j,m}| < r_m\} \subset V,\quad \forall\, m\ge 2.$$

Let us now also prove the following two relations:

$$\left.\begin{array}{ll}(\beta) & \forall\ K\subset\subset\bar{\omega}+i\gamma_m,\ \exists\rho>0 \text{ s.t. } \{x+i\rho'y: x+iy\in K, 0<\rho'\le\rho\}\subset B_m \\ (\gamma) & \exists\ \delta_m>0 \text{ s.t. } \{x+iy\in B_m:\ \langle y,\eta\rangle > -\delta_m\}\subset B_{m+1}.\end{array}\right\}$$

If (β) does not hold, we can find $x_q+iy_q\in K$ such that $x_q+iq^{-1}y_q$

does not belong to B_m, hence such that

$$|y_q|^2 \geq 2\ r_m\ q\ \inf\ \{<y,\eta_{j_o,m}> \ :\ x+iy \in K\}$$

holds for at least one $j_o \leq n$. As this lower bound is strictly positive, we get a contradiction by making $q\to\infty$. If (γ) does not hold, we can find $x_q+iy_q \in B_m\backslash B_{m+1}$ verifying $<y_q,\eta> \geq -q^{-1}$. As we have $y_q \in \gamma_m \subset \Gamma$ for every q, as up to a positive coefficient independant of q we also have $\sum_{j=1}^{n} <y_q,\eta_j> \leq q^{-1}$ and as up to the choice of a subsequence we may also suppose $x_q+iy_q \to x_o+iy_o \in \overline{\omega}+i\overline{\gamma}_m$, we get $0\leq \lim_{q\to\infty} <y_q,\eta_j> = <y_o,\eta_j> \leq \sum_{j=1}^{n} <y_o,\eta_j> = 0$ for every j which implies $y_o=0$. Considering now the compact set $K:=\{x_q+iy_q|y_q|^{-1} : q \in \mathbb{N}\} \subset \overline{\omega}+i\overline{\gamma}_m \subset \overline{\omega}+i\gamma_{m+1}$, we get by (β) a number $\rho>0$ such that $|y_q| \leq \rho$ implies $x_q+iy_q \in B_{m+1}$. Hence a contradiction because $y_q\to 0$.

We are now going to modify the B_m's in order to get an increasing sequence of convex sets open in $\overline{\omega}+i\mathbb{R}^n$ verifying

$$(\delta) \qquad B'_m \subset \bigcup_{m'=2}^{m} B_{m'} \quad , \quad \forall m \geq 2,$$

and for which there exists a decreasing sequence of positive numbers $\varepsilon_m\to 0$ such that the following equality holds:

$$(\varepsilon) \qquad \{x+iy \in B'_m \ :\ <y,\eta> > -\varepsilon_m\} = \{x+iy \in B_m:\ <y,\eta> > -\varepsilon_m\}.$$

As it is clear that $B'_2=B_2$ and $\varepsilon_2=1$ are suitable, let us proceed by induction and suppose $B'_2, \ldots , B'_{m-1}$ and $\varepsilon_2, \ldots ,\varepsilon_{m-1}$ already determined. Combining (γ) and (ε), we may find $\delta\in]0,\varepsilon_{m-1}[$ such that $\{x+iy \in B'_{m-1}:\ <y,\eta> > -\delta\} \subset B_m$ holds. Let us now prove the existence of $\varepsilon_m \in]0,\delta[$ such that the convex hull of $B'_{m-1} \cup \{x+iy \in B_m:\ <y,\eta> > -\varepsilon_m\}$ is contained in $B_m\cup B'_{m-1}$. If this does not occur, we can find sequences $\theta_q \in [0,1]$, $z_q=x_q+iy_q \in B'_{m-1}$ and $z'_q=x'_q+iy'_q \in B_m$ such that $<y'_q,\eta> > -q^{-1}$ and $\theta_q z_q+(1-\theta_q)z'_q \notin B_m\cup B'_{m-1}$. Let us first remark that we may suppose $\theta_q\to\theta_o\in [0,1]$, $z_q\to x_o+iy_o$ belonging to $\overline{\omega}+i\overline{\gamma}_{m-1}$ and $z'_q\to x'_o\in\overline{\omega}$ because one gets directly $\lim y'_q = 0$ by the same argumentation as above. If the y_q's verify

$<y_q,\eta> > -\delta$ for q large enough, we are lead to a first contradiction by convexity of B_m which implies $\theta_q z_q+(1-\theta_q)z'_q \in B_m$. We may therefore suppose $<y_q,\eta> \le -\delta$ for any q. As we have $<y'_q,\eta>>-\delta/2$ for q large enough, there exist $\mu_q\in[0,1]$ such that $-\delta/2 = <\mu_q y_q+(1-\mu_q)y'_q,\eta>$. If we suppose $\mu_q\to\mu_o\in[0,1]$, we have necessarily $\mu_o\in]0,1[$. As B'_{m-1} is open and convex in $\overline{\omega}+i\mathbb{R}^n$ and admits both $\tilde{x}_o = \theta_o x_o+(1-\theta_o)x'_o$ and $\tilde{x}_o+iy_o$ as points of its closure, it contains $\tilde{x}_o+i\mu_o y_o$ and consequently $\tilde{x}_o+i(\mu_q y_q+(1-\mu_q)y'_q)$ for q large enough. Let us first consider the possibility $\theta_q\in[0,\mu_q]$; as we may write

$$\theta_q y_q+(1-\theta_q)y'_q = (1-\frac{\theta_q}{\mu_q})y'_q + \frac{\theta_q}{\mu_q}(\mu_q y_q +(1-\mu_q)y'_q)$$

with $\tilde{x}_o+iy'_q \in B_m$ and $\tilde{x}_o+i[\mu_q y_q+(1-\mu_q)y'_q] \in \{x+iy \in B'_{m-1}: <y,\eta> > -\delta\} \subset \{z \in B_m: <y,\eta> > -\delta\}$, we get another contradiction by convexity of B_m. The second possibility $\theta_q\in]\mu_q,1]$ provides also a contradiction if one writes

$$\theta_q y_q+(1-\theta_q)y'_q = (\frac{\theta_q - \mu_q}{1 - \mu_q})y_q + (\frac{1 - \theta_q}{1 - \mu_q})[\mu_q y_q + (1-\mu_q)y'_q]$$

because y_q and the factor between brackets belong to B'_{m-1} which is convex.

The conclusion follows then directly by taking as B'_m the convex hull we just considered and as ε_m, the number of which we proved the existence. As a matter of fact, the union of the B'_m 's will provide a convex set which is of required type as one can verify easily by use of (β);it is then straightforward to verify that such a set is of product type with respect to the real and imaginary variables.□

PROPOSITION 3. *If* (x_0,η_0) *is a point of* $\partial\Omega \times S^*_{n-1}$ *such that* $\mathring{P}(\eta_0) \neq 0$, *then* $\tilde{P}$ *induces a bijective endomorphism of* $C^b_{(x_0,\eta_0)}$.

<u>Proof</u>. Let γ_0 be an open convex salient cone of the dual of $\mathbb{R}^n$ containing η_0 and consider the representation of $C^b_{(x_0,\eta_0)}$ given by formula (1) of the introduction. Using the notations of that paragraph, we may of course suppose that the closures in the complement of the origin of the cones $\gamma_m = \{\Sigma r_j \eta_{j,m} : r_j > 0\}$ are contained in γ_0 and that $\mathring{P}$ does not vanish over γ_0.

Any element of $C^b_{(x_0,\eta_0)}$ appears then like the equivalence class of a function f holomorphic over a tuboid V of profile $\Lambda(\omega_m, \mathbb{R}^n, \Gamma_m)$. As it is trivial to prove by using the theory of inductive limits that the restriction map corresponding to the inclusion $\omega_{m+1} + i\Gamma_m \subset \omega_m + i\Gamma_m$ induces in $C^b_{(x_0,\eta_0)}$ the identity operator, lemma 2 allows to suppose that V contains a tuboid V' of profile $\Lambda(\omega_m, \Gamma_m, \Gamma_m)$ which is a convex intersection of a convex neighborhood of $\omega_m \cap \Omega$ with a domain $\tilde{V}'$ of product type. By Malgrange-Ehrenpreis principle, we can solve the equation Pu=f over V'. Let us then remark that $\omega_m \cap \Omega$ is contained in the boundary of V' and that in a neighborhood of any point of $\omega_m \cap \Omega$, V' coincides with $\tilde{V}'$. It is then easy to verify that V' fulfills the condition C(x,I) stated in 4.1 of [1] for $I = -i(\overline{\gamma}_m \setminus \{0\}) \subset -i\gamma_0$. By homogeneity, $\mathring{P}$ does not vanish on I and therefore, theorem 4.1 of [1] asserts that u extends holomorphically on a neighborhood of $\omega_m \cap \Omega$.

The surjectivity of $\tilde{P}_{(x_0,\eta_0)}$ will then follow directly from proposition 1 because u constitutes clearly a section of $\mathcal{O}_1$ over $\omega_m \times (S_{n-1} \cap \Gamma_m)$ such that $\tilde{P}(\tilde{\sigma}u) = \tilde{\sigma}(Pu) = \tilde{\sigma}f$ holds in $C^b[\omega_m \times (S^*_{n-1} \cap \gamma_m)]$ and because the decomposition of f is unique modulo Čech coboundaries.

To prove the injectivity of $\tilde{P}_{(x_0,\eta_0)}$, let us again denote by f a representing function of an element of $C^b_{(x_0,\eta_0)}$, i.e. a holomorphic function defined over a tuboid of profile $\Lambda(\omega_m,\mathbb{R}^n,\Gamma_m)$. By homogeneity of $\overset{\circ}{P}$, we may find $a \in]0,1[$ such that $\overset{\circ}{P}$ does not vanish over $T(\eta_o):=\{\zeta \in \mathbb{C}^n: |\xi|^2+|\eta|^2=1, |\xi|<a, \eta \in \pm\overline{\gamma_{m-1}} \setminus \{0\}\}$.

If $\tilde{P}(\tilde{\sigma}f)=\tilde{\sigma}(Pf)$ vanishes in $C^b_{(x_o,\eta_o)}$, we may suppose that $Pf \in \bigoplus_{j=1}^{n} \mathcal{O}_{1,o}[SF \cap \Lambda(\omega_m,\mathbb{R}^n,\Gamma_{j,m})]$; this means the existence of n tuboids V_j of profile $\Lambda(\omega_m,\mathbb{R}^n,\Gamma_{j,m})$ and of functions $g_j \in \mathcal{O}(V_j)$ and $g_o \in \mathcal{O}_o[SF\cap\Lambda(\omega_m,\Gamma_m,\Gamma_m)]$ such that $Pf=\sum_{j=o}^{n} g_j$ over $\omega_m \cap \Omega$.

By a restriction affecting only the real variables and by a procedure similar to the one we used to prove the surjectivity of $\tilde{P}$ (use of lemma 2), we may suppose that g_o is defined over a convex complex neighborhood $\tilde{V}_o$ of $\omega_m\cap\Omega$ and that the V_j's contain respectively a tuboid $\tilde{V}_j$ of profile $\Lambda(\omega_m,\Gamma_{j,m},\Gamma_{j,m})$ composed by the intersection of a convex complex neighborhood of $\omega_m\cap\Omega$ with a domain of product type. By Malgrange-Ehrenpreis principle, we may again solve the equations $Pf_j=g_j$ over those $\tilde{V}_j$'s; we hence obtain $P(f-\sum_{j=o}^{n} f_j) = 0$ on $V':=V\cap \bigcap_{j=o}^{n} \tilde{V}_j$. According to theorem 2.1 of [1], we may extend $f-\sum_{j=o}^{n} f_j$ to any open convex set V" containing V' such that each hyperplane of $\mathbb{R}^{2n}$ whose normal is characteristic and that intersects V", intersects also V'.

We are going to take

$$V'':= \bigcap_{\zeta\in \mathrm{Char}(P)} \; \bigcup_{z'\in V'} \{z \in \mathbb{C}^n: \mathrm{Re}\langle z-z',\zeta\rangle = 0\}$$

as such a V". As a matter of fact, V" is open because otherwise, we could find a point $z_o \in V''$ and a sequence $z_m \notin V''$ converging to z_o. It should therefore exist some $\zeta_m \in \mathrm{Char}(P)$ such that any $z'\in V'$ verifies $\mathrm{Re}\langle z_m-z',\zeta_m\rangle \neq 0$. By convexity of V' and up to the extraction of a subsequence, we may suppose $\zeta_m \to \zeta_o \in \mathrm{Char}(P)$ and $V' \subset \{z: \mathrm{Re}\langle z_m-z,\zeta_m\rangle > 0\}$ for any m.

By taking the limit and using the fact that V' is open, we obtain $V' \subset \{z : \mathrm{Re}\langle z_0 - z, \zeta_0\rangle > 0\}$, which contradicts $z_0 \in V''$.

The convexity of V'' will follow from the convexity of V' as one may verify directly. More important is the fact that V'' contains $\omega_m \cap \Omega$. As a matter of fact, we shall prove this by distinguishing the two cases $|\xi|<a$ and $|\xi|\geq a$. In the first one, we get necessarily $\eta \notin \pm\overline{\gamma_m}$ and hence, we may find y_0 in Γ_m such that $\langle y_0, \eta\rangle = 0$. Any $x \in \omega_m \cap \Omega$ is then the center of a compact ball b contained in $\omega_m \cap \Omega$; we may then find $\rho_0 > 0$ such that $x' + i\rho y_0$ belongs to V' for any x' in b and any $\rho \in]0, \rho_0]$. As there exists at least one x' in b such that $\langle x-x', \xi\rangle = 0$, the point $z := x' + i\rho y_0 \in V'$ verifies $\mathrm{Re}\langle x-z, \zeta\rangle = 0$. In the second opportunity ($|\xi|\geq a$), there exists certainly $r>0$ such that the function $\langle x-.,\xi\rangle$ takes all the values between $-r$ and r in b. Let us then consider a point y of Γ_m; there exists again $\rho_0 > 0$ such that $x' + i\rho y$ belongs to V' for any x' in b and ρ in $]0, \rho_0]$. Hence we may choose ρ in order to get $|\langle \rho y, \eta\rangle| < r$ and then x' in b to get $\langle x-x', \xi\rangle = \rho\langle y, \eta\rangle$, which is also sufficient.

The injectivity of $\tilde{P}_{(x_0,\eta_0)}$ follows then directly because the function $f - \sum_{j=0}^{n} f_j$ will then constitute in fact a section of $\mathcal{O}_0$. □

SATO'S PRINCIPLE FOR C^b.

THEOREM. *If* u *and* f *are simultaneously real analytic functions (or hyperfunctions) over the intersection* $\omega \cap \Omega$ *and verify* $Pu = f$ *, the following inclusions hold:*

$$W.F.^b(f) \subset W.F.^b(u) \subset W.F.^b(f) \cup \{(x,\eta) \in \partial\Omega \times S^*_{n-1} : \mathring{P}(\eta) = 0\}$$

<u>Proof</u>. The first inclusion is a trivial consequence of

the linearity of P. Let us now consider a point (x_0,η_0) which does not belong neither to $W.F.^b(f)$ nor to $\partial\Omega \times \mathrm{Char}(P)$. If we identify u, f, Pu and Pf with their respective images in $(\pi^{-1}{}_*A)_{(x_0,\eta_0)}$, we get by proposition 1 the relation $\tilde{P}(\tilde{\sigma}u) = \tilde{\sigma}(Pu) = \tilde{\sigma}(f) = 0$ and by proposition 3, this implies that $u=0$, i.e. $(x_0,\eta_0) \notin W.F.^b(u)$.□

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