数理解析研究所講究録
579巻 1985年 41-54

On semigroups generated by "regularly" m-accretive operators

Michiaki Watanabe 新潟大教養 渡辺道昭
Faculty of General Education
Niigata University

INTRODUCTION

Let's denote by $\{S(t): t \geqq 0\}$ the semigroup generated by an m-accretive operator A in a real Banach space $(X,|\cdot|)$ in the sense of Crandall-Liggett[3] . As is well known,

$$S(t)x = \lim_{\lambda \downarrow 0} (I + \lambda A)^{-[t/\lambda]}x \quad \text{for} \quad x \in \overline{D(A)}$$

uniformly on every bounded subinterval of $[0,\infty)$, and satisfies

$$S(\cdot)x \in C([0,\infty);X) \quad ;$$

$$|S(t)x - S(t)y| \leqq |x - y| \quad \text{for} \quad x,y \in \overline{D(A)}.$$

In this article, I shall try to strengthen the accretiveness condition of A as follows :

$$
\text{(A)} \quad \begin{cases} D(A) \subset (V, \|\cdot\|), \\ \text{which is a Banach space continuously included in } X\ ; \\ |u - v|^p + C\lambda\|u - v\|^p \leqq |u + \lambda Au - v - \lambda Av|^p \\ \text{for } \lambda > 0 \text{ and } u,v \in D(A), \text{ where } C > 0 \text{ and } p > 1 \\ \text{are constants.} \end{cases}
$$

Here, I am interested in the problem :

1. What is the property of $\{S(t): t \geqq 0\}$ like ?
2. How about the condition on B under which $A + B$ becomes m-accretive ?

The condition (A), which appears somewhat curious, is equivalent to each of the following :

(A)' $\quad \tau(u - v, Au - Av)|u - v|^{p-1} \geqq (C/p)\|u - v\|^p$

where $\tau(x,y) = \inf_{\lambda>0} \lambda^{-1}(|x + \lambda y| - |x|)$;

(A)" $|u - v|^p + (C/p)\lambda\|u - v\|^p \leqq |u + \lambda Au - v - \lambda Av||u - v|^{p-1}$.

So in the case of a Hilbert space, the condition (A)' with $p = 2$ is nothing but a Gårding-type inequality. Minty[5] once used this kind of inequality to obtain a generalization of the Lax-Milgram lemma. Moreover, if A satisfies

$$|u - v| + K\lambda^{\alpha}\|u - v\| \leqq |u + \lambda Au - v - \lambda Av|$$

for constants $K > 0$ and $0 < \alpha < 1$, then (A) holds with $p = \alpha^{-1}$ and $C = K^p$.

PROPERTIES OF $S(t)$

First, I shall find out some properties, "smoothing effect" for example, of the semigroup generated by a possibly "nonlinear and multivalued" operator A satisfying (A) and

$$R(I + \lambda A) = X \quad \text{for} \quad \lambda > 0.$$

THEOREM 1. Let $\{S(t): t \geqq 0\}$ be the semigroup generated in X by an m-accretive operator A satisfying (A). Then, the following hold :

(S)
- (i) $S(\cdot)u \in C([0,\infty);V)$ for $u \in D(A)$;
- (ii) $S(\cdot)x \in L^p(0,T;V)\ (T > 0)$ for $x \in \overline{D(A)}$ and satisfies

$$|S(t)x - S(t)y|^p + C\int_0^t \|S(r)x - S(r)y\|^p\, dx \leqq |x - y|^p$$

for $t \geqq 0$ and $x,y \in \overline{D(A)}$.

REMARK ON "INTEGRAL SOLUTION". As a by-product, we obtain that for each $x \in \overline{D(A)}$, $u(t) = S(t)x$ is a unique "integral solution" of the Cauchy problem for

$$du(t)/dt + Au(t) = 0, \quad 0 \leqq t < T$$

in the following sense :

$$\begin{cases} u(0) = x \;; \\ u \text{ belongs to } C([0,T);X) \cap L^p(0,T;V) \text{ satisfying} \\ \quad |u(t) - v|^p - |x - v|^p + c\displaystyle\int_0^t \|u(r) - v\|^p \, dr \\ \quad \leqq \displaystyle\int_0^t \tau(u(r) - v, -Av)\cdot p|u(r) - v|^{p-1} \, dr \\ \quad \text{for } v \in D(A) \text{ and } 0 \leqq t < T. \end{cases}$$

By a similar method, the unique integral solution of the problem

$$du(t)/dt + Au(t) = f(t), \; 0 \leqq t < T;$$
$$u(0) = x$$

could be constructed for given $f \in L^p(0,T;X)$ and $x \in \overline{D(A)}$.

REMARK ON "CONVERSE PROBLEM". It is well known(see [1,8]) that, if X is a "nice" Banach space, then there is a biunique correspondence between m-accretive operators in X and nonlinear semigroups on "nonexpansive retracts" of X. I feel it would be interesting to determine if there is an analogous correspondence between m-accretive operators satisfying (A) and nonlinear semigroups which satisfy (S) of Theorem 1.

In the case, at least, that A is linear, we can prove the following.

PROPOSITION 2. Let $\{S(t): t \geqq 0\}$ be the strongly continuous semigroup in X generated by a linear m-accretive operator A with domain dense in X. Then, the conditions (A) and (S) are equivalent.

PROOF. We shall show here the implication (S) $\to$ (A). Recalling the property of $\tau(x,y)$, we see that

$$\lambda^{-1}(|x + \lambda t^{-1}(I - S(t))x|^p - |x|^p)$$

$$\geqq \tau(x, t^{-1}(I - S(t))x) \cdot p|x|^{p-1}$$

$$\geqq t^{-1}(|x|^p - |S(t)x|^p)$$

for $x \in X$ and $\lambda > 0$. So, using (ii) of (S), we get

$$|x|^p + C\lambda \cdot t^{-1} \int_0^t \|S(r)x\|^p \, dr \leqq |x + \lambda t^{-1}(I - S(t))x|^p.$$

Putting $x = u \in D(A)$, noting (i) of (S) and going to the limit as $t \downarrow 0$, we obtain (A). Q.E.D.

In this linear case, replace $A + bI$ for a real number b by A if necessary, and denote by $D(A^\alpha)$ the domain of fractional power A^α, $0 < \alpha < 1$ with graph norm $\|\cdot\|_\alpha$. Let's recall here that A^α is given by $A^\alpha = (A^{-\alpha})^{-1}$ and

$$A^{-\alpha}x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1}x \, d\lambda, \; x \in X,$$

which is absolutely convergent in X.

Proposition 2 with $V = D(A^\alpha)$, $0 < \alpha < 1$ gives a new characterization of generators of "linear differentiable semi-groups" in X. I think this is meaningful , together with the proposition itself, in that our method does not require any use of complex numbers.

Let's note the fact that the estimate

$$\|(I + \lambda A)^{-1}x\|_\alpha \leqq K\lambda^{-\alpha}|x| \quad \text{for} \quad x \in X \quad \text{and} \quad \lambda \in (0,1]$$

always holds, and go into the following assertion.

<u>COROLLARY 3.</u> Let A be a linear m-accretive operator with domain dense in X satisfying, for $x \in X$ and $\lambda > 0$,

$$\|(I + \lambda A)^{-1}x\|_\alpha \leqq K\lambda^{-\alpha}(|x| - |(I + \lambda A)^{-1}x|)$$

with a constant $K > 0$. Then, for each $x \in X$, $S(t)x$ belongs to $D(A)$ for a.a. $t > 0$.

<u>PROOF.</u> Recalling the remark given at the end of the introduction, we see that A satisfies (A) with $p = \alpha^{-1}$. So, for each $x \in X$, $S(\cdot)x$ belongs to $L^p(0,T;D(A^\alpha))$ $(T > 0)$ and hence $S(t)x$ belongs to $D(A^\alpha)$ for a.a. $t > 0$.

Thus, writing as $N\alpha \leqq 1 < (N + 1)\alpha$ for some positive integer N , we have for a.a. $t > 0$

$$AS(t)x = A^{1-N\alpha}S(t - t_1 - \cdots - t_N)A^\alpha S(t_N) \cdots A^\alpha S(t_1)x$$

by $N + 1$ applications of the above inclusion. Q.E.D.

I shall refer the reader to the paper[4], where characterizations of generators of linear analytic semigroups in X are given without use of complex numbers.

Let's turn now to the proof of Thoerem 1. To this end, the following three lemmas are necessary to establish under the hypothesis of the theorem.

LEMMA 4. Set $J_\lambda = (I + \lambda A)^{-1}$ with $J_0 = I$. Then, for each $u \in D(A)$, $J_\lambda u$ is continuous in $\lambda \in [0,1]$ in the topology of V.

PROOF. Note first that for $x, y \in X$ and $\lambda > 0$,

$$\|J_\lambda x - J_\lambda y\| \leqq (C\lambda)^{-1/p}|x - y|,$$

which is derived at once from (A). Then, using the equality

$$J_\lambda x = J_\mu(\mu\lambda^{-1}x + (\lambda - \mu)\lambda^{-1}J_\lambda x), \quad 0 < \mu < \lambda,$$

we have

$$\|J_\lambda x - J_\mu x\| \leqq (C\mu)^{-1/p}\lambda^{-1}(\lambda - \mu)|J_\lambda x - x| .$$

Moreover, we have for $u \in D(A)$,

$$\|J_\lambda u - u\| \leqq (C\lambda)^{-1/p}|u - (I + \lambda A)u| = C^{-1/p}\lambda^{1-1/p}|Au|,$$

which implies the continuity in V at $\lambda = 0$ of $J_\lambda u$. Q.E.D.

LEMMA 5. Let $\{S(t): t \geqq 0\}$ be the semigroup generated in X by A. Then, for each $u \in D(A)$, $S(\cdot)u$ belongs to $C([0,\infty);V)$ and

$$J_\lambda^{[t/\lambda]}u \quad \to \quad S(t)u \quad \text{in} \quad V \quad \text{as} \quad \lambda \downarrow 0$$

uniformly on every bounded subinterval of $[0,\infty)$.

PROOF. By (A)' we have

$$\|J_\lambda^m u - J_\mu^n u\|^p \leqq (p/C)(|AJ_\lambda^m u| + |AJ_\mu^n u|)|J_\lambda^m u - J_\mu^n u|^{p-1}.$$

Since $|AJ_\lambda^n u| \leqq |Au|$, this estimate becomes

$$\|J_\lambda^m u - J_\mu^n u\| \leqq (2p/C)^{1/p}|Au|^{1/p}|J_\lambda^m u - J_\mu^n u|^{1-1/p}.$$

Here, let's recall the results due to Crandall-Liggett[3], described in the beginning of the present article. First, the above estimate implies that $J_\lambda^{[t/\lambda]}u$ converges in V to some $v(t)$ as $\lambda \downarrow 0$ and the convergence is uniform on every bounded subinterval of $[0,\infty)$. But, since $V \subset X$ continuously and $J_\lambda^{[t/\lambda]}u$ converges in X to $S(t)u$, $v(t)$ has to coincide with $S(t)u$ for all $t \geqq 0$. Next, again making use of the above estimate, we have, for $t,s \geqq 0$,

$$\|S(t)u - S(s)u\| \leqq (2p/C)^{1/p}|Au|^{1/p}|S(t)u - S(s)u|^{1-1/p}.$$

This implies the continuity in V of $t \to S(t)u$. Q.E.D.

LEMMA 6. For each $u,v \in D(A)$ and $t \geqq 0$,

$$|S(t)u - S(t)v|^p + C\int_0^t \|S(r)u - S(r)v\|^p \, dr \leqq |u - v|^p.$$

PROOF. Dealing with (A), we have for $x,y \in X$,

$$C\lambda\|J_\lambda^i x - J_\lambda^i y\|^p \leqq |J_\lambda^{i-1}x - J_\lambda^{i-1}y|^p - |J_\lambda^i x - J_\lambda^i y|^p.$$

Summing up these estimates for $i = 1,\ldots,n$, we get

$$|J_\lambda^n x - J_\lambda^n y|^p + C\sum_{i=1}^n \lambda\|J_\lambda^i x - J_\lambda^i y\|^p \leqq |x - y|^p.$$

Noting Lemma 4, we have for $u,v \in D(A)$,

$$|J_\lambda^n u - J_\lambda^n v|^p + C\int_\lambda^{(n+1)\lambda} \|J_\lambda^{[r/\lambda]}u - J_\lambda^{[r/\lambda]}v\|^p \, dr \leqq |u - v|^p.$$

Finally, noting Lemma 5 and taking the limit as $\lambda \downarrow 0$ with $t \geqq \lambda$ and $n = [t/\lambda]$, we obtain the desired estimate. Q.E.D.

PROOF OF THEOREM 1. The property (i) of (S) has been shown in Lemma 5.

Let's go into the proof of (ii). Take $x \in \overline{D(A)}$ and let $\{u_n\}_{n=1}^\infty$ be a sequence in $D(A)$ such that $|u_n - x| \to 0$ as $n \to \infty$. Using Lemma 6, we have for $T > 0$,

$$C \int_0^T \| S(r)u_m - S(r)u_n \|^p \, dr \leqq |u_m - u_n|^p,$$

which implies that $S(\cdot)u_n$ converges in the topology of the Banach space $L^p(0,T;V)$ to some w as $n \to \infty$. But, V is included in X continuously and $S(\cdot)u_n$ tends to $S(\cdot)x$ in $L^p(0,T;X)$ as $n \to \infty$. Hence, $w(t)$ has to coincides with $S(t)x$ for a.a. $t \in (0,T)$.

Next, let $x,y \in \overline{D(A)}$, and $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in $D(A)$ which converge in X to x and y respectively. Then, we obtain the desired inequality by taking the limit as $n \to \infty$ in

$$|S(t)u_n - S(t)v_n|^p + C \int_0^t \| S(r)u_n - S(r)v_n \|^p \, dr \leqq |u_n - v_n|^p.$$

PERTURBATION FOR $S(t)$

Next, I shall give a condition on an operator B in X under which, if A is m-accretive and satisfies (A), then $A + B$ is again m-accretive. In perturbation theory in this direction for nonlinear semigroups, A was assumed to be m-accretive simply, but B was sometimes forced to satisfy the condition instead(see [2,7,10]) :

$$D(B) \supset \overline{D(A)},$$

which is too restrictive for B to become a "differential" operator.

The stronger is our condition on A, the weaker is the restriction of B expected to be. Indeed, we shall impose on B the following condition :

$$(B) \quad \begin{cases} D(B) \supset V ; \\ \text{There is a constant } L \geqq 1 \text{ such that} \\ |Bu - Bv| \leqq L\|u - v\| \quad \text{for all } u,v \in V. \end{cases}$$

The Lipschitz continuity condition of $B : V \to X$ is simple but new. In fact, B can replace a differential operator. Our method can be used to treat the same problem in the case that B satisfies a "local" Lipschitz continuity condition from V to X. However, to clarify our idea, we confine ourselves here to the condition (B), and then explain below that the condition (A) is nice enough for A to "absorb" B.

<u>PROPOSITION 7.</u> Let A be an m-accretive operator satisfying (A), and B be an operator satisfying (B). Then, $A + B$ is m-accretive and satisfies a condition similar to (A).

<u>PROOF.</u> Using (A)" and (B), we have

$$|u - v|^p + (C/p)\lambda\|u - v\|^p$$

$$\leqq (|u + \lambda(A + B)u - v - \lambda(A + B)v| + L\lambda\|u - v\|)|u - v|^{p-1}.$$

Applying to the right-hand side the inequality :

$$L \cdot YZ^{p-1} \leqq (C/(2p))Y^p + \omega Z^p \quad \text{for} \quad Y,Z \geqq 0$$

$$\text{with} \quad \omega = (1 - 1/p)L^{p/(p-1)}(2/C)^{1/(p-1)},$$

which is a simple consequence of Young's inequality, we get

$$(1 - \omega\lambda)|u - v|^p + (C/(2p))\lambda\|u - v\|^p$$

$$\leqq |u + \lambda(A + B)u - v - \lambda(A + B)v|\,|u - v|^{p-1}$$

for $\lambda > 0$ and $u,v \in D(A + B)(= D(A))$. This is similar to (A)".

Finally, let's prove :

$$R(I + \lambda(A + B)) = X \quad \text{for} \quad 0 < \lambda < \omega^{-1}.$$

take $x \in X$ and set $Tu = J_\lambda(x - \lambda Bu)$ for $\lambda > 0$. Then, the operator T maps V into $D(A)$ and hence into V. In view of (A)" , we see

$$|Tu - Tv|^p + (C/p)\lambda\|Tu - Tv\|^p \leqq \lambda|Bu - Bv|\,|Tu - Tv|^{p-1}$$

$$\leqq L\lambda\|u - v\|\,|Tu - Tv|^{p-1}$$

for $\lambda > 0$ and $u,v \in V$. Again using the above consequence of Young's inequality, we have

$$(1 - \omega\lambda)|Tu - Tv|^p + (C/p)\lambda\|Tu - Tv\|^p$$
$$\leqq (C/(2p))\lambda\|u - v\|^p .$$

Hence, if $0 < \lambda < \omega^{-1}$, then

$$\|Tu - Tv\| \leqq 2^{-1/p}\|u - v\| \quad \text{with} \quad 0 < 2^{-1/p} < 1$$

and a well-known fixed point theorem applies. Let u be the unique fixed point of T in V. Then, $u = J_\lambda(x - \lambda Bu)$ and hence u belongs to $D(A + B)$ and $u + \lambda(A + B)u = x$.

Thus, we have proved that $A + B$ has properties quite similar to those of A. Q.E.D.

NOTE. This summer, I had an opportunity to talk with A. Pazy about Theorem 1 and Proposition 7. I feel that our theory partially resembles that in Section 3 of his paper[6]. Here, the semigroup generated by A satisfying , instead of (A) essentially,

$$\phi(u) + \lambda\psi(u) \leqq \phi(u + \lambda Au) \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad u \in D(A),$$

where ϕ and ψ are lower semicontinuous functions on X, is discussed together with remarkable applications.

I am grateful to Professor A. Pazy for his kind advices.

REFERENCES

[1] J.- B. Baillon, Générateurs et semi-groupes dans les espaces de Banach uniformément lisses, J. Funct. Anal. 29 (1978), 199-213.

[2] V. Barbu, Continuous perturbations of nonlinear m-accretive operators in Banach spaces, Boll. Un. Mat. Ital. 6 (1972), 270-278.

[3] M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, Amer. J. Math. 93 (1971), 265-298.

[4] M. G. Crandall, A. Pazy and L. Tartar, Remarks on generators of analytic semigroups, Israel J. Math. 32 (1979), 363-374.

[5] G. J. Minty, Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29 (1962), 341-346.

[6] A. Pazy, The Lyapunov method for semigroups of nonlinear contractions in Banach spaces, J. d'Anal. Math. 40 (1981), 239-262.

[7] M. Pierre, Perturbations localement Lipschitziennes et continues d'opérateurs m-accrétifs, Proc. Amer. Math. Soc. 58 (1976), 124-128.

[8] S. Reich, Product formulas, nonlinear semigroups, and accretive operators, J. Funct. Anal. 36 (1980), 147-168.

[9] M. Watanabe, On semigroups generated by m-accretive operators in a strict sense, Proc. Amer. Math. Soc., to appear.

[10] G. Webb, Continuous nonlinear perturbations of linear accretive operators in Banach spaces, J. Funct. Anal. 10 (1972), 191-203.

新潟市　五十嵐 2の町 8050

新潟大学　教養部　渡辺道昭