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THREE THEOREMS ON THE COMPUTABILITY OF LINEAR OPERATORS THEIR EIGENVALUES AND EIGENVECTORS

by

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In this paper, three questions are considered. Associated with each, a theorem is presented. Let us begin by discussing the first question. Which processes in analysis and physics preserve computability and which do not?

In order to answer this question, we need a definition of "computable function of a real variable". Luckily, Grzegorczyk and Lacombe have provided this (cf. [3], [4], [6]). The following is equivalent to the classical Grzegorczyk-Lacombe definitions [11].

Definition 1. Let a and b be recursive reals. Then $\varphi(x)$ is a recursive function of a real variable on $[a,b]$ if there exist five recursive functions b,c,s,d,h such that

$$|\varphi(x) - \sum_{j=0}^{d(n)} (-1)^{s(n,j)} \frac{b(n,j)}{c(n,j)} x^j| < \frac{1}{k+1}$$

if $n \geq h(k)$.

Note that the above definition is really an effective version of the well-known Weierstrass approximation theorem. It states that there exists a recursively enumerable sequence of polynomials which converges recursively to $\varphi(x)$. The convergence is in the uniform or L^∞ norm. For our purposes

it will be useful to view the definition in a slightly different way. Consider the sequence of monomials $1, x, x^2, \ldots$. Consider further the linear span of the monomials over the rationals--i.e., finite linear combinations of the monomials with rational coefficients. The computable functions are precisely those functions which are in the effective closure of the (linear span of the) monomials in L^∞ norm. The monomials provide an example of an effective generating set. We shall have more to say about this concept later.

Definition 1 may be extended to computable sequences $\{\varphi_m\}$ in an obvious way.

We now turn to Theorem I. It delineates precisely between those linear operators which preserve computability and those which do not. In order to obtain a variety of applications to physics and analysis, we present this theorem in a very general form in terms of a "computability structure" on a Banach space. Note that we do not define a computable Banach space. We reason about Banach space theory classically--much as physicists and analysts do. We merely put a computability structure on it. This is done axiomatically. The precise concept axiomatized is "computable sequence of points" of the Banach space.

Let us assume for the moment that we have defined the concept "Banach space with a computability structure". In order to state Theorem I, we need the following definition.

Definition 2. Let X be a Banach space with a computability structure. Then X is effectively separable if there is a computable sequence $\{e_i\}$

whose linear span is dense in X. Such an $\{e_i\}$ is called an <u>effective generating set</u>.

<u>Note</u>. In a previous paper [15] we assumed that the effective generating set was not merely dense, but "effectively dense". We have since learned that this assumption is redundant.

And now for Theorem I. As is usual in functional analysis, a linear operator $T: X \to Y$ is only assumed to be defined on a dense subset of X. Roughly Theorem I states that, under some very mild conditions which are always satisfied in practical situations, a linear operator preserves computability if and only if it is bounded (cf. [15]). More precisely

<u>Theorem I. Let X and Y be Banach spaces with computability structures, and let $\{e_n\}$ be an effective generating set for X. Let $T: X \to Y$ be a closed linear operator whose domain includes $\{e_n\}$ and such that $\{Te_n\}$ is a computable sequence of Y. Then T maps every computable element of its domain into a computable element if and only if T is bounded.</u>

The Computability Structure

We now turn to a description of a computability structure. As stated above, this structure is defined axiomatically. The undefined notion is <u>computable sequence</u> $\{x_n\}$ of points of the Banach space. A point x is computable if the sequence $x,x,x,\ldots,x\ldots$ is computable. A double sequence $\{x_{nk}\}$ is computable if its elements can be arranged in a computable sequence $\{y_m\}$ by one of the usual pairing functions.

In order to give the axioms, we need the following simple definition.

Definition 3. The double sequence $\{x_{nk}\}$ converges to the sequence $\{x_n\}$ as $k \to \infty$ effectively in k and n if there exists a recursive function e such that

$$\|x_{nk} - x_n\| \leq \frac{1}{2^N}$$

for all $k \geq e(n,N)$.

In our published papers, five axioms are given. We have since found that these five can be replaced by three--one axiom for each of the basic concepts of Banach space theory (linearity, limit and norm).

Axiom 1 (Linear Forms). Let $\{x_n\}$ and $\{y_n\}$ be computable sequences in X , let $\{\alpha_{nk}\}$ and $\{\beta_{nk}\}$ be computable double sequences of real or complex numbers, and let $d: \mathbb{N} \to \mathbb{N}$ be a recursive function. Then the sequence

$$s_n = \sum_{k=0}^{d(n)} (\alpha_{nk} x_k + \beta_{nk} y_k)$$

is computable in X .

Axiom 2 (Limit). Let $\{x_{nk}\}$ be a computable double sequence in X such that $\{x_{nk}\}$ converges to $\{x_n\}$ as $k \to \infty$, effectively in k and n . Then $\{x_n\}$ is a computable sequence in X .

Axiom 3 (Norm). If $\{x_n\}$ is a computable sequence in X , then the norms $\{\|x_n\|\}$ form a computable sequence of reals.

It is clear that these axioms are "minimal" (i.e., they are just sufficient to cover the three basic notions of Banach space theory --

linearity, limit and norm). Recently we have found that these axioms are in a sense "maximal". Namely, for effectively separable spaces, they determine the computability structure uniquely. More precisely

Stability Lemma. Let $\{e_n\}$ be a sequence whose linear span is dense in the Banach space X. Let $\mathcal{S}'$ and $\mathcal{S}''$ be computability structures on X such that $\{e_n\} \in \mathcal{S}'$ and $\{e_n\} \in \mathcal{S}''$. Then $\mathcal{S}' = \mathcal{S}''$.

Thus it is not possible to add axioms to an effectively separable Banach space to obtain a more intuitive notion of computability.

Examples of Computability Structures

An obvious example is given by definition 1, and its extension to computable sequences $\{\varphi_m\}$. The space is $C[a,b]$ with the L^∞ (i.e., uniform) norm. More generally we consider L^p-computability for p, a recursive real with $p \geq 1$. We need this definition because so much of the work in theoretical physics and analysis is concerned with the properties of L^p spaces and of linear operators on these spaces.

In order to present the definition of L^p-computability let us return for a moment to Definition 1, the classical definition of a recursive function of a real variable. This was defined as the effective closure of the (linear span of the) monomials $1, x, x^2, \ldots$ in the L^∞ norm. Now the L^∞ norm is more stringent than the L^p norm for $p < \infty$. Thus one possible definition for L^p computability for $p < \infty$ is to take the effective closure (of the span) of $1, x, x^2 \ldots$ in the L^p norm. Call this class $\mathrm{Comp}^p[a,b]$.

There are several other equivalent definitions. Suppose, for example, the approach is via measure and integration. It is natural to consider "computable step functions"--e.g. those with rational values and jump points. Now take the effective closure of these functions under the L^p norm. We get the same class, $\mathrm{Comp}^p[a,b]$. Suppose, instead, one is dealing with Fourier series. Then it is natural to take the effective closure of the "trigonometric polynomials" in the L^p norm. Again we obtain the class $\mathrm{Comp}^p[a,b]$. In general, if we take the effective closure of the span of any effective generating set under the L^p norm, we obtain $\mathrm{Comp}^p[a,b]$. This is the same class of functions as can be obtained by taking the effective closure of all "definition 1 - computable" functions in L^p norm. Note that if we apply these procedures in the case that $p = \infty$ we get back what we started with--the "definition 1 - computable" functions. However some care must be exercised. A moment's thought will convince the reader that $C[a,b]$ is the only space where there are no computable step functions.

The above definitions may be extended to $L^p(R)$, $L^p[\vec{a},\vec{b}]$, $L^p(\mathbb{R}^n)$ - where $[\vec{a},\vec{b}]$ is the n-dimensional rectangle with recursive coordinates (a_i,b_i) . More generally they may be extended to any Banach space in which the polynomials are dense.

Applications of Theorem I

In practical situations the hypothesis of this theorem is always satisfied. Thus to determine whether computability is preserved, we merely determine whether or not the operator is of bounded norm. We have found it

necessary to use a variety of Banach space norms, sometimes even considering several different norms in connection with the same problem.

Let us first consider the wave equation. We note that the Kirchhoff solution operator is unbounded in the uniform norm. Hence computability in the sense of definition 1 is not preserved. Thus there exists an example of the wave equation with computable initial data, such that the unique solution, although continuous, is not computable. This specific result was obtained directly by the authors in a previous paper [13]. On the other hand, if we use a norm which is better adapted to the wave equation--the so-called "energy norm"--then computability is preserved.

By contrast, the solution operator for the heat equation is bounded in uniform norm. Hence computability in the sense of definition 1 is preserved. A similar result is obtained for Laplace's equation on regions of suitable shape. These two results also hold if "definition 1 - computability" is replaced by L^p -- computability for $p \neq \infty$.

In the case of Fourier series and transforms, we know precisely for which values of p and r the mappings from L^p (or ℓ^p) to L^r (or ℓ^r) are bounded. Hence we know precisely for which p and r computability is preserved. We note that for some p and r , computability is preserved; for others it is not.

Even trivial operators yield information. Since integration is a bounded operator, the integral of a computable function of a real variable is computable in the sense of definition 1. Since differentiation is not a bounded operator, we obtain an example of a "definition 1 - computable"

function whose derivative, although continuous, is not computable. This last result was obtained by Myhill in 1971, [9]. Even the identity operator provides information. For, applying the identity operator $I: L^p$ (or ℓ^p) $\to$ L^r (or ℓ^r), we obtain a complete description of the relation between L^p (or ℓ^p) computability and L^r (or ℓ^r) computability. For example, L^p and L^r computability are different if $p \neq r$. Similarly for ℓ^p and ℓ^r.

We now turn to Theorem II. It is associated with the following question. Determine the "computability relationships" between an operator and its eigenvalues in a general setting.

Theorem II also uses the notion of a computability structure--this time on a Hilbert space. In order to present this theorem we need the definition of an "effectively determined" operator. This involves an extension of the notion of effective generating set given earlier.

<u>Definition 4</u>. Let X and Y be Banach spaces with computability structures, and let $T: X \to Y$ be a (bounded or unbounded) closed linear operator. Then T is <u>effectively determined</u> if there is an effective generating set for the graph of T.

(More precisely, this means that there is a computable sequence $\{e_n\}$ in X such that $\{Te_n\}$ is computable in Y, and the set of pairs (e_n, Te_n) spans a dense linear subspace of the graph of T -- i.e. a dense subspace of $\{(x,y): y = Tx\}$ in $X \times Y$).

It is easy to check that the standard operators of analysis and physics are effectively determined. We now answer the question about eigenvalues/spectrum which was posed above (cf. [16]).

Theorem II. Let H be a Hilbert space with a computability structure. Let $T: H \to H$ be an effectively determined (bounded or unbounded) self-adjoint operator. Then there exists a computable sequence of real numbers $\{\lambda_n\}$ and a recursively enumerable set of integers A such that:

(a) The set of eigenvalues of T coincides with $\{\lambda_n : n \in \mathbb{N} - A\}$. In particular, each eigenvalue is computable.

(b) Each $\lambda_n \in$ spectrum(T), and the spectrum of T coincides with the closure of $\{\lambda_n\}$ in $\mathbb{R}$.

(c) Conversely, for any sequence $\{\lambda_n\}$ and set A as in (a) above, there exists an effectively determined self-adjoint operator T whose set of eigenvalues is $\{\lambda_n : n \in \mathbb{N} - A\}$. Likewise for (b), the closure of any computable real sequence $\{\lambda_n\}$ occurs as the spectrum of an effectively determined self-adjoint operator.

Note that in (b) and (c) above the spectrum of T is not the effective closure of $\{\lambda_n\}$; it is merely the closure of this set.

For operators which are compact--e.g. integral operators of the form $T(f)(x) = \int_a^b K(x,y) \cdot f(y)dy$ with continuous kernel K --we have the following stronger result.

Corollary. Let $T: H \to H$ be compact, self-adjoint, and effectively determined. Then the eigenvalues of T form a computable sequence of real numbers.

(Thus the set A in part (a) above can be taken to be empty.)

In general, the behavior of eigenvalues can be highly discontinuous. Thus arbitrarily small perturbations of a self-adjoint operator can cause an

eigenvalue to disappear while new eigenvalues in quite different locations are being suddenly created. Examples of this behavior, involving bounded effectively determined operators, are given in [16]. Such discontinuities frequently indicate noncomputability. Nevertheless the eigenvalues are computable.

Theorem II can be generalized to the case of bounded normal operators. However, for operators which are not normal, our results fail-- as the following example shows. (For details see [16].)

Example. There exists an effectively determined, bounded (but not normal) operator $T: H \to H$ which has a noncomputable eigenvalue.

For the sake of brevity, we do not include any specific applications of Theorem II. We recall the two hypotheses of this theorem: that the operator be self-adjoint and effectively determined. In physics and classical analysis one knows which operators are self-adjoint. In addition, in practice, it is easy to verify that an operator is effectively determined. All one needs is a dense sequence of very smooth functions, suitably adjusted to the boundary conditions, on which the operator acts effectively. Note that the operators of nonrelativistic quantum mechanics are self-adjoint and effectively determined. These and other well known operators have been studied intensively.

It should be remarked that the proof of Theorem II gives an algorithm for computing the λ_n

We turn now to Theorem III. It answers a question concerning the computability of eigenvectors. Theorem III has not yet appeared in any published paper.

Theorem III. Let $H = L^2[0,1]$ with the natural L^2-computability structure described above under Examples of Computability Structures. There exists an effectively determined, compact, self-adjoint operator $T: H \to H$ with the following properties

(1) The number $\lambda = 0$ is an eigenvalue of T of multiplicity one (i.e. the space of eigenvectors corresponding to $\lambda = 0$ is one dimensional).

(2) None of the eigenvectors corresponding to $\lambda = 0$ is computable.

Open Problems

Open problems abound. Let us mention four areas.

Recall that the reasoning in this paper is blatantly classical. What are the constructive analogs--for various notions of constructivity--of these results?

For our second area we recall that the original formulation of definition 1 was given in terms of Kleene functionals on functions from non-negative integers into themselves. Many equivalent definitions--more amenable to work in analysis--were then given. As we have seen, the definition of L^p-computability is a generalization of definition 1, and the concept of a computability structure is even more general. Perhaps it is time to go back and investigate the relation of work in this paper to functionals of higher type, E-recursion theory, as well as various aspects of descriptive set theory, etc. (see, for example, [8] and [10]-including the bibliographies).

A third area arises naturally by considering our proofs--see also

[12, 13, 14, 15, 16]. As expected these proofs make use of recursively enumerable nonrecursive sets and recursively inseparable pairs of sets. In fact, for most of the results, any recursively enumerable nonrecursive set--of any degree of unsolvability--will do. These results can most certainly be refined by combining techniques (including priority arguments) and results from degrees of unsolvability with our results.

A fourth area is concerned with more restricted classes of computability. The work in this paper is based on the general notion of recursive/partial recursive function. It is of interest to investigate this work from the viewpoint of polynomial time/polynomial space computability [1], [5], Grzegorczyk's hierarchy of primitive recursive functions [2], the Löb-Wainer hierarchy [7], etc.

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