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Conformally Self-Dual Metrics and Integrability

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0. Introduction

Let $ds^2 = g_{ij}dx^i dx^j$ denote a four-dinensional Riemannian metric and $R_{ijk\ell}$ the components of its Riemann curvature. From the Riemann curvature one can construct the Weyl curvature $C_{ijk\ell}$ and the Ricci curvature R_{ij}, the latter being further decomposed into the tracefree part $R_{ij} - g_{ij}R/4$ and the scalar curvature $R = R_{ij}g^{ij}$. Conversely the Riemann curvature can be reproduced from these three fundamental invariants. These properies are independent of the dimensions.

$$R_{ijk\ell} \quad \left[\begin{array}{ll} C_{ijk\ell} & \text{(Weyl curvature)} \\ R_{ij} - g_{ij}R/4 & \text{(tracefree part of Ricci)} \\ R & \text{(scalar curvature)} \end{array} \right.$$

A feature specific to the four-dimensional case is that the Riemann curvature breakes into the self-dual part and the anti-self-part (written symbolically as R_+ and R_-, respectively). The self-dual part (anti-self-dual part) in itself decomposes into three fundamental components, which are the self-dual part C_+ (anti-self-dual part C_-) of the Weyl curvature, the tracefree part of the Ricci curvature and the scalar curvature. It would be worth noting here that this decomposition (usually formulated by means of "spinor calculus") corresponds to the irreducible decomposition of a class of representations of SL(2): see Atiyah *et al.* [1], Eguchi *et al.* [2], Plebanski [3].

$$R_+ \quad \begin{cases} C_+ & \text{(self-dual part of Weyl)} \\ R_{ij} - g_{ij}R/4 & \\ R & \end{cases}$$

$$R_- \quad \begin{cases} C_- & \text{(anti-self-dual part of Weyl)} \\ R_{ij} - g_{ij}R/4 & \\ R & \end{cases}$$

A four-dimensional metric is said to be <u>conformally self-dual</u> if the anti-self-dual part C_- of the Weyl curvature vanishes. On the other hand it is said to be <u>Einstein</u> (or, more precisely, Einstein <u>without cosmological term</u>) if the Ricci curvature R_{ij} vanishes. These notions have in principle no relation to each other because C_- and R_{ij} are distinct components of the "irreducible decomposition" of the Riemann curvature mentioned above; their combinations however yield various interesting classes of metrics as follows (see Eguchi *et al.* [2]).

CLASSES OF METRICS

a) self-dual Einsten:
$C_- = 0$, $R_{ij} - g_{ij}R/4 = 0$, $R = 0$

b) self-dual Einstein with cosmological constant:
$C_- = 0$, $R_{ij} - g_{ij}R/4 = 0$

c) conformally self-dual: $C_- = 0$

d) Einstein: $R_{ij} = 0$

e) Einstein with cosmological term: $R_{ij} - g_{ij}R/4 = 0$

$$\begin{array}{ccccc} \text{class a)} & \subset & \text{class b)} & \subset & \text{class c)} \\ \cap & & \cap & & \\ \text{class d)} & \subset & \text{class e)} & & \end{array}$$

Metrics in classes a)-c) share a remarkable property that they admit a "twistor-theoretical" description (see Atiyah *et al.* [1], Penrose [4], Ward [5], Hitchin [6], Boyer [7])). This is a sort of "coding" of geometric structures that "encodes" a conformally self-dual metric into a three-dimensional complex manifold (called "twistor space"), from which, in principle, all information on the metric can be "decoded". At present such a satisfactory way of understanding is not known for general Einstein metrics with or without cosmological term, though there is an attempt by LeBrun [8] and Manin and Penkov [9].

The purpose of this article is two-fold: The first one is to introduce basic ideas of twistor theory to be used to describe conformally self-dual metrics (spaces); this occupies most part of this article. The second one is to show how the method presented in my previous work [10] on self-dual Einstein metrics can be extended to the case of general conformally self-dual metrics; this subject is discussed in the last section. Probably some comments should be added here concerning the first point above, because the approach to twistor theory adopted here may look somewhat different from presently more popular ones such as that of Atiyah *et al.* [1]. Until now various methods are known to built up a twistor-theoretical description of geometric structures, but it seems that they can be classified into, roughly speaking, two classes. One may be called "real methods", which is based on the theorem of Auslander and Nirenberg on the integrability of almost complex structures, as adopted by, for example, Atiyah *et al.* [1]. The other one, going back to Penrose's original work [5], could be called "complex methods", whose basis lies in the theorem of Frobenius on the integrability of Pfaffian systems. A concise account of the latter approach is provided in Boyer's paper [7]. In this article we take the latter approach and attempt to make clear, as far as possible, what roles the notion of integrability plays in such a twistor-theoretical description of conformally self-dual metrics. Some basic ideas are also borrowed from the work of Gindikin [11].

1. From metrics to twistor spaces

1.1. COMPLEX METRIC. In "complex method" one starts from a "complex Riemannian metric" (LeBrun [8]), i.e. a nondegenerate complex quadratic form $ds^2 = g_{ij}dz^i dz^j$ on the holomorphic tangent bundle of a four-dimensional complex manifold X, where $z = (z_1, z_2, z_3, z_4)$ denote a set of local coordinates, and $g_{ij} = g_{ij}(z)$ holomorphic functions of z that give metric components. A complex Riemannian metric may be thought of as an analytic continuation of a real analytic metric on a Riemannian manifold into a complexification. The notions of the Levi-Civita connetion, the Riemann curvature, etc. can be extended quite naturally to such a complex metric (see LeBrun [8]). Thus the classification of metrics mentioned in *Introduction* also makes sense for complex metrics. In what follows we consider complex metrics alone, calling them simply "metrics".

1.2. NULL TETRAD. Representing a metric ds^2 by the components g_{ij} is actually not very useful for our purpose. A better way is to use a null tetrad (or null vierbein), i.e. a set of 1-forms (e^1, e^2, e^3, e^4) on X with $e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0$ for which the metric is written

$$ds^2 = 2e^1e^2 + 2e^3e^4 = 2 \det \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix}. \tag{1}$$

A metric can always be written (at least locally) as above as far as the metric in question is nondegenerate. To see this, first take an orthonormal frame $(\omega^1, \omega^2, \omega^3, \omega^4)$ of 1-forms; then define $e^1, \ldots, e^4$ to be: --

$$e^1 = \omega^1 + \sqrt{-1}\omega^2, \quad e^2 = \omega^1 - \sqrt{-1}\omega^2,$$
$$e^3 = \omega^3 + \sqrt{-1}\omega^4, \quad e^4 = \omega^3 - \sqrt{-1}\omega^4,$$

which indeed give a null tetrad that represents the metric as above. Note here that even if one starts from an ordinary real Riemannian metric, the 1-forms $e^1, \ldots, e^4$ do not belong to the cotangent bundle itself, but its complexification. This is a reason why the present formulation inevitably requires the notion of complexification.

1.3. STRUCTURE EQUATIONS AND CONNECTION COEFFICIENTS.

Given a moving frame of 1-forms, the derivation of various geometric quantities (the Levi-Civita connection, the Riemann curvature, etc.) can be reformulated by means of exterior differential forms instead of tensor calculus (see Eguchi *et al* [2]). This often simplifies calculations to a cosiderable extent. In this respect the use of a null tetrad is of particular importance, because it causes a decomposition of the Levi-Civita connection (see Atiyah *et al*. [1], Plebanski [3], Boyer [6]) which corresponds to the decomposion of the Riemann curvature into the self-dual and anti-self-dual parts.

To give a more precise account to this, let us use the so called "spinor notation": Let $\alpha, \beta, \gamma, \ldots$ be a set of indices ("undotted" spinor indices) and $\dot\alpha, \dot\beta, \dot\gamma, \ldots$ another set of indices ("dotted" spinor indices), both of which take values in {1, 2}. A null tetrad is then regarded as a set of 1-forms $e^{\alpha\dot\alpha}$ with two spinor indices: --

$$(2) \qquad (e^{\alpha\dot\alpha}) = \begin{pmatrix} e^{11} & e^{12} \\ e^{21} & e^{22} \end{pmatrix} = \begin{pmatrix} e^{3} & e^{1} \\ -e^{2} & e^{4} \end{pmatrix}.$$

Spinor indices are raised and lowered under the following rule:

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \ \xi^\beta = \xi_\alpha\epsilon^{\alpha\beta}, \ \eta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}, \ \eta^{\dot\beta} = \eta_{\dot\alpha}\epsilon^{\dot\alpha\dot\beta}.$$

where $\epsilon_{\alpha\beta}$ $(= \epsilon^{\alpha\beta})$ and $\epsilon_{\dot\alpha\dot\beta}$ $(= \epsilon^{\dot\alpha\dot\beta})$ denote the totally anti-symmetric spinors to be determined by the contions that $\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = 1$ for $(\alpha.\beta) = (\dot\alpha,\dot\beta) = (1.2)$.

If the null tetrad $(e^{\alpha\dot\alpha})$ is taken as a moving frame. Cartan's first structure equations for the Levi-Civita connection can be written

(3) $$de^{\alpha\dot\alpha} + \Gamma_{+}{}^{\alpha}{}_{\beta} \wedge e^{\beta\dot\alpha} + \Gamma_{-}{}^{\dot\alpha}{}_{\dot\beta} \wedge e^{\alpha\dot\beta} = 0.$$

where $\Gamma_{+}{}^{\alpha}{}_{\beta}$ and $\Gamma_{-}{}^{\dot\alpha}{}_{\dot\beta}$ denote 1-forms with the symmetry

(4) $$\Gamma_{+\alpha\beta} = \Gamma_{+\beta\alpha}. \quad \Gamma_{-\dot\alpha\dot\beta} = \Gamma_{-\dot\beta\dot\alpha} .$$

1-forms $\Gamma_{+}{}^{\alpha}{}_{\beta}$ and $\Gamma_{-}{}^{\dot\alpha}{}_{\dot\beta}$ satisfying (3) and (4) are unique. and can be calculated from the null tetrad. Now let S_+ denote the rank-two vector bundle of undotted spinors μ^{α} on X and S_- the rank-two vector bundle of dotted spinors $\lambda^{\dot\alpha}$ on X. Then taking them as connection coefficients. one can intruduce two connections $\nabla_+ = d + (\Gamma_{+}{}^{\alpha}{}_{\beta})$ on S_+ and $\nabla_- = d + (\Gamma_{-}{}^{\dot\alpha}{}_{\dot\beta})$ on S_-.

These connections give a decomposition of the Levi-Civita connection as mentioned above. Indeed. choosing a null tetrad is eqivalent to fixing an isomorphism $TX \simeq S_+ \otimes S_-$. and under this identification of TX and $S_+ \otimes S_-$ the Levi-Civita connection agrees with the tensor product $\nabla_+ \otimes \nabla_-$. Accordingly the Riemann curvature splits into the direct sum of the curvature forms of ∇_+ and ∇_-. which are nothing but the self-dual part R_+ and the anti- self-dual part R_- of the

Riemann curvature (see the references cited above for details).

1.4. A PFAFFIAN SYSTEM. We now introduce a new variable λ running over a Riemann sphere $\mathbb{P}^1$ and consider the following Pfaffian system on $\mathbb{P}^1 \times X$ (or, to be more precise, the $\mathbb{P}^1$-bundle $\mathbb{P}(S_-)$; see Atiyah *et al.* [1], Boyer [6]): --

(5)
$$\begin{aligned} e^3 + \lambda e^1 &= 0, \\ - e^2 + \lambda e^4 &= 0, \\ d\lambda - \mathcal{D} &= 0, \end{aligned}$$

where $\mathcal{D}$ denotes the 1-form

(6)
$$\mathcal{D} = \lambda^2 \Gamma_{-}{}^{1}{}_{2} + \lambda(\Gamma_{-}{}^{1}{}_{1} - \Gamma_{-}{}^{2}{}_{2}) - \Gamma_{-}{}^{2}{}_{1}.$$

The integrability conditions of system (5) in the sense of Frobenius take the following form:

(7)
$$\begin{aligned} &d(e^3 + \lambda e^1)\wedge(e^3 + \lambda e^1)\wedge(- e^2 + \lambda e^4)\wedge(d\lambda - \mathcal{D}) = 0, \\ &d(- e^2 + \lambda e^4)\wedge(e^3 + \lambda e^1)\wedge(- e^2 + \lambda e^4)\wedge(d\lambda - \mathcal{D}) = 0, \\ &d(d\lambda - \mathcal{D})\wedge(e^3 + \lambda e^1)\wedge(- e^2 + \lambda e^4)\wedge(d\lambda - \mathcal{D}) = 0. \end{aligned}$$

The following proposition is a key to the present construction.

1.5. PROPOSITION [1,4 - 7,11]. Integrability conditions (7) are eqivalent to the conformal self-duality $C_- = 0$.

1.6. REMARKS. i) Eqs. (5) and (7) are form-invariant under "gauge transformations" of the null tetrad,

(8)
$$\begin{pmatrix} e^3 & e^1 \\ - e^2 & e^4 \end{pmatrix} \dashrightarrow \ell_+ \begin{pmatrix} e^3 & e^1 \\ - e^2 & e^4 \end{pmatrix} \ell_-^{-1},$$

if λ is simultaneously transformed as

(9) $\lambda \dashrightarrow (\ell_{-21} + \ell_{-22}\lambda)(\ell_{-11} + \ell_{-12}\lambda)^{-1}$,

where $\ell_+ = (\ell_{+\alpha\beta})$ and $\ell_- = (\ell_{-\dot\alpha\dot\beta})$ are GL(2)-valued functions.

ii) In the above argument we gave an explicit form of the 1-form $\mathscr{D}$ in advance, but this is actually unnesessary because, as noted by Gindikin [11], integrability conditions (7) (to be more precise, the first two equations) determine $\mathscr{D}$ uniquely except for the trivial indeterminancy

(10) $\mathscr{D} \dashrightarrow \mathscr{D}$ + linear combination of $e^3+\lambda e^1$, $-e^2+\lambda e^4$.

From the last equation in (7) one then obtains a system of equations which are indeed equivalent to $C_- = 0$. (It is not hard to see that the above indeterminancy of $\mathscr{D}$ does not affect the final result.) In applications this method of finding an explicit form of the equation $C_- = 0$ is sometimes much simpler than directly executing the calculation of $\Gamma_{-}{}^{\dot\alpha}{}_{\dot\beta}$, etc.

1.7. TWISTOR SPACE. As the proposition stated above shows, a conformally self-dual metric ds^2 (represented by a null tetrad) defines a completely integrable Pfaffian system (5). Let $\mathscr{T}$ denote the set of its maximal (two-dimensional) integral manifolds. Under some assumption on the "convexity" of X with respect to these integral manifolds (see, for example, Ward [5]) $\mathscr{T}$ forms a three-dimensional complex manifold, which is exactly a twistor space corresponding to the given conformally self-dual metric (see Boyer). The space-time manifold X and the twistor space $\mathcal{T}$ are connected by the following triangular diagram of maps: --

(11) $$\begin{array}{ccc} \mathcal{F} \equiv \mathbb{P}^1 \times X & \xrightarrow{p_2} & \mathcal{T} \\ p_1 \downarrow & & \\ X & & \end{array}$$

where p_1 is the projection onto X and p_2 is the map that assigns to each point (λ, z) of $\mathcal{F}$ the integral surface of (5) (i.e. an element of $\mathcal{T}$) passing through (λ, z). This diagram gives a curved space version of the so called "twistor correspondence", which defines a correspondence of subsets of X and of $\mathcal{T}$ such as

"point <---> subset"

or

"subset <---> point".

Note, in particular, that to each point z of X corresponds the subset $p_2 \circ p_1^{-1}(z)$ of $\mathcal{T}$ which is a nonsingular curve isomorphic to $\mathbb{P}^1$ with normal bundle is isomorphic to the pull back of the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over $\mathbb{P}^1$. Thus the spave-time manifold X can be interpreted as a parameter space of this four-parameter family of curves in $\mathcal{T}$. Moreover, this family becomes "complete" in the sense of Kodaira (see Penrose [4], Ward [5], Hitchin [6], Gindikin [11]). Such an interpretation of the role of space-time is a key to the problem of how to (re)produce a conformally self-dual metric from a twistor space (see Penrose [4], Ward [5], Hitchin [6] for details); we shall discuss this in §2.1 and §2.2.

1.8. FIRST INTEGRALS. The complete integrability of Pfaffian system (5) means that there exist (locally) three independent first integrals. We write them $u^\alpha = u^\alpha(\lambda, z)$ ($\alpha = 0, 1, 2$). Each integral manifold belonging to $\mathcal{T}$ can be represented by a set of local equations as: --

(12) $$u^\alpha(\lambda, z) = c^\alpha \quad (\alpha = 0, 1, 2),$$

where c^α's are constants that depend on the integral surface. These constants may be thought of as representing a point (or its coordinates in a local coordinate patch) in $\mathcal{T}$, because the first integrals (u^0, u^1, u^2) give a representation of the map p_2 in terms of local coordinates.

These first integrals can also be characterised as independent solutions of the linear equations

(13) $$\begin{aligned} L_1u &\equiv [-\lambda\partial_3 + \partial_1 + (-\lambda\mathcal{D}_3 + \mathcal{D}_1)\partial/\partial\lambda]u = 0, \\ L_2u &\equiv [\lambda\partial_2 + \partial_4 + (\lambda\mathcal{D}_2 + \mathcal{D}_4)\partial/\partial\lambda]u = 0, \end{aligned}$$

where $\partial_1, \dots, \partial_4$ denote the dual null tetrad, i.e. a unique frame of vector fields obeying the orthogonality relation

(14) $$\langle e^a, \partial_b\rangle = \delta^a{}_b \quad \text{(Kronecker's delta)},$$

and $\mathcal{D}_a$ ($a = 1, \dots, 4$) the tetrad components of $\mathcal{D}$

(15) $$\mathcal{D} = \mathcal{D}_a e^a.$$

As simple calculations show, integrability conditions (7) then become equivalent to the condition that the commutation relation

(16) $$[L_1, L_2] = Q_1L_1 + Q_2L_2$$

is satisfied for some functions Q_1 and Q_2 of (λ, z).

From the above linear equations we can derive another characterisation of first integrals. To see this, let us note the formula $df = (\partial_a f)e^a = (\partial_1 f)e^1 + \ldots + (\partial_4 f)e^4 + (\partial_\lambda f)d\lambda$ that holds for any function f on X. Applying it to u^0, u^1, u^2 and using Eqs. (15) one can compute du^0, du^1, du^2 as:

$$\begin{aligned}
du^0 &= (\lambda\partial_3 u^0 - (-\lambda\mathcal{D}_3 + \mathcal{D}_1)\partial_\lambda u^0)e^1 + (\partial_2 u^0)e^2 + (\partial_3 u^0)e^3 \\
&\quad + (-\lambda\partial_2 u^0 - (\lambda\mathcal{D}_2 + \mathcal{D}_4)\partial_\lambda u^0)e^4 + (\partial_\lambda u^0)d\lambda \\
&= (\partial_\lambda u^0)(d\lambda - (-\lambda\mathcal{D}_3 + \mathcal{D}_1)e^1 - (\lambda\mathcal{D}_2 + \mathcal{D}_4)e^4) \\
&\quad + (\partial_3 u^0)(e^3 + \lambda e^1) - (\partial_2 u^0)(-e^2 + \lambda e^4), \text{ etc.} \ldots
\end{aligned}$$

Eqs. (13) thus turn out to be equivalent to the following equation (cf. [10, section 2]): --

(17) $$\begin{pmatrix} du^0 \\ du^1 \\ du^2 \end{pmatrix} = \begin{pmatrix} \partial_\lambda u^0 & \partial_3 u^0 & -\partial_2 u^0 \\ \partial_\lambda u^1 & \partial_3 u^1 & -\partial_2 u^1 \\ \partial_\lambda u^2 & \partial_3 u^2 & -\partial_2 u^2 \end{pmatrix} \begin{pmatrix} 1 & \mathcal{D}_3 & -\mathcal{D}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\lambda - \mathcal{D} \\ e^1 + \lambda e^3 \\ -e^2 + \lambda e^4 \end{pmatrix}.$$

1.9. <u>AN EXAMPLE</u>. We now consider the case where the null tetrad takes the following form: --

(18) $$\begin{aligned} e^3 &= dx + Adp + Bdq, & e^1 &= dp, \\ -e^2 &= dy + Cdp + Ddq, & e^4 &= dq, \end{aligned}$$

where (p, q, x, y) are coordinates in X and $A, \ldots, D$ are functions of these coordinates. (In fact, this assumption does not cause any loss of generality, because any null tetrad representing a conformally self-dual metric can be re-set into the above form after some gauge and coordinate transformations; see Boyer and Plebanski [12].) The dual null tetrad can be written

$$\text{(19)}\qquad \begin{aligned} \partial_3 &= \partial_x, & \partial_1 &= \partial_p - A\partial_x - C\partial_y, \\ -\partial_2 &= \partial_y, & \partial_4 &= \partial_q - B\partial_x - D\partial_y, \end{aligned}$$

where $\partial_x = \partial/\partial x$, etc. After a long and tedeous calculations it can be shown that the tetrad components of the 1-form $\mathcal{D}$ take the following form: --

$$\text{(20)}\qquad \begin{aligned} \mathcal{D}_3 &= -\partial_2 C - \partial_3 D, & \mathcal{D}_1 &= \partial_4 C - \partial_1 D, \\ -\mathcal{D}_2 &= \partial_2 A + \partial_3 B, & \mathcal{D}_4 &= -\partial_4 A + \partial_1 B. \end{aligned}$$

The conformal self-duality of the metric now reduces to the following system of equations: --

$$\text{(21)}\qquad \begin{aligned} &\partial_1\mathcal{D}_4 - \partial_4\mathcal{D}_1 + \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_3\mathcal{D}_4 = 0, \\ &\partial_1\mathcal{D}_2 - \partial_2\mathcal{D}_1 - \mathcal{D}_3\mathcal{D}_4 + \mathcal{D}_4\mathcal{D}_3 = 0, \\ &\partial_2\mathcal{D}_3 - \partial_3\mathcal{D}_2 = 0. \end{aligned}$$

We next consider first integrals of Pfaffian system (5). It should be note here that the notion of first integrals of an integrable Pfaffian system is of local nature: at any point there indeed exist a maximal number (e.g. for the case of (5), three) of first integrals, but in general they are defined just in a small neighborhood of that point. From the point of view presented in [10], first integrals of particular interest are those defined in a neighborhood of $(\lambda, p, q, x, y) = (\infty, 0, 0, 0, 0)$. A detailed analysis shows that one can choose such first integrals to have the following Laurent expansion in λ: --

$$u^0 = \lambda + u^0_0 + u^0_{-1}\lambda^{-1} + \dots ,$$

(22) $$u^1 = p\lambda + x + pu^0_0 + u^1_{-1}\lambda^{-1} + \dots ,$$

$$u^2 = q\lambda + y + qu^0_0 + u^2_{-1}\lambda^{-1} + \dots ,$$

where the Laurent coefficients of u^α ($\alpha = 0, 1, 2$) are written $u^\alpha_n = u^\alpha_n(p, q, x, y)$ ($n = 1, 0, -1, \dots$). From a geometrical point of view a more appropriate choice of first integrals would be to take ($1/u^0$, u^1/u^0, u^2/u^0) (which also gives a triple of intependent first integrals), because those in (22) carry poles at $\lambda = \infty$ whereas the latter are regular at $\lambda = \infty$ and give a coordinate representation of the map P_2 in the corresponding local chart in $\mathcal{T}$. Nevertheless the choice as shown in (22) is rather useful in order to make clear the relation to the approach presented in [10].

Relations connecting first integrals and null tetrads become particularly simple for the case of first integrals with the Laurent expansion as shown in (22). To see this, let us insert (22) into (16) and consider the infinite number of equations thus obtained for the Laurent coefficients of the first integrals. From the coefficients of λ^1 and λ^0, in particular, the following equations arise: --

(23) $$- A(1 + p\partial_x u^0_0) - Cp\partial_y u^0_0 + u^0_0 - \partial_x u^0_{-1} - \partial_x u^1_{-1} = 0,$$

(24) $$- B(1 + p\partial_x u^0_0) - Dp\partial_y u^0_0 - \partial_y u^0_{-1} - \partial_y u^1_{-1} = 0,$$

(25) $$- Aq\partial_x u^0_0 - C(1 + q\partial_y u^0_0) - \partial_x u^0_{-1} - \partial_x u^2_{-1} = 0,$$

(26) $$- Bq\partial_x u_0^0 - D(1 + q\partial_y u_0^0) + u_0^0 - \partial_y u_{-1}^0 - \partial_y u_{-1}^2 = 0.$$

Thus in generic position (for example, if p and q are sifficiently small) the coefficients $A, \ldots, D$ of the null tetrad can be reproduced from the Laurent coeffcients u_0^0, u_{-1}^0, u_{-1}^1, u_{-1}^2 by soving the above linear algebraic equations.

2. Construction of metrics

2.1. TWISTOR-THEORETICAL CONSTRUCTION. Let $\mathcal{T}$ be a three dimensional complex mnifold with the properties mentioned in §1.7, i.e. $\mathcal{T}$ contains a complete (in the sense of Kodaira) four-parameter analytic family of rational curves (i.e. curves isomorphic to $\mathbb{P}^1$) whose normal bundles are isomorphic to $\mathcal{O}(1) \oplus\mathcal{O}(1)$. Under some condition (convexity etc.) this family of curves forms a four-dimensional complex manifold, which we write X. We now briefly review how a conformally self-dual metric on X can be produced from these data; for details, see references [1, 4-7, 11].

We first re-define basic triangular diagram (11) from the above data. There is no problem in the definition of the first projection p_1. The second projection p_2 can be defined as follows. Take any point (z, λ) of $\mathcal{F} = \mathbb{P}^1 \times X$. To the point z of X there corresponds a rational curve $\hat{z}$ in $\mathcal{T}$, which one may identify with an embedding map

(27) $\hat{z} : \mathbb{P}^1 \dashrightarrow \mathcal{T}$

that parametrises the curve as $\lambda \dashrightarrow \hat{z}(\lambda)$. We then define p_2 as follows:

(28) $p_2(\lambda, z) = \hat{z}(\lambda) \qquad ((\lambda, z) \in \mathcal{F}).$

To see that p_2 is of maximal rank, we have to specify in more detail the structure of the tangent space T_zX and the tangent map $dp_2 : T_{(z,\lambda)}\mathcal{F} \dashrightarrow T_{\hat{z}(\lambda)}\mathcal{T}$. (If $\mathcal{T}$ is the twistor space obtained from a conformally self-dual metric as discussed in §§1.1-7, the above maps indeed agree with those in the original construction.)

From the assumptions we have the following isomorphism:

(29) $T_zX \simeq \Gamma(N(\hat{z}))$,

where $N(\hat{z})$ denotes the normal bundle of the rational curve $\hat{z}$ and $\Gamma(N(\hat{z}))$ the set of all its global sections; this is basically due to the "completenes" of the analytic family of rational curves $\{\hat{z};\ z \in X\}$. Since we also assumed that

(30) $N(\hat{z}) \simeq \hat{z}_*(\mathcal{O}(1)\oplus\mathcal{O}(1))$,

$\Gamma(N(\hat{z}))$ certainly forms a four-dimensional vector space; recall that $\Gamma(\mathcal{O}(1)) \simeq$ {polynomials of first degree in one complex variable } $\simeq \mathbb{C}^2$ (see, for example, Hitchin [6]). Geometrically, the right side of (29) represents the infinitesimal deformations of the curve $\hat{z}$ embedded in $\mathcal{T}$.

The kernel of the tangent map $dp_2 : T_{(\lambda,z)}\mathcal{F} \dashrightarrow T_{\hat{z}(\lambda)}\mathcal{T}$

now can be specified as follows. Since $T_{(\lambda,z)}\mathcal{F}$ decomposes as $T_{(\lambda,z)}\mathcal{F} = T_\lambda\mathbb{P}^1 \oplus T_zX$, any tangent vector at (λ, z) can be represented as $\gamma\partial_\lambda + v$, $\gamma \in \mathbb{C}$, $v \in T_zX$. dp_2 does not vanish on $T_\lambda\mathbb{P}^1$ because $\hat{z} : \mathbb{P}^1 \dashrightarrow \mathcal{T}$ is an embedding, and causes an isomorphism between $dp_2(T_\lambda\mathbb{P}^1)$ and $T_{\hat{z}(\lambda)}(\hat{z})$, the tangent space of the curve $\hat{z}$ at $\hat{z}(\lambda)$. Therefore by factoring out the sourse and target spaces of the tangent map dp_2 by $T_\lambda\mathbb{P}^1$ and $T_{\hat{z}(\lambda)}(\hat{z})$ respectively, one obtains a linear map

$$(31) \qquad T_zX = T_{(\lambda,z)}\mathcal{F}/T_\lambda\mathbb{P}^1 \dashrightarrow N_{\hat{z}(\lambda)}(\hat{z}) = T_{\hat{z}(\lambda)}\mathcal{T}/T_{\hat{z}(\lambda)}(\hat{z})$$

for which the following relation holds:

$$(32) \qquad \mathrm{Ker}((31)) = dp_1(\mathrm{Ker}(dp_2)) \xleftarrow[\sim]{dp_1} \mathrm{Ker}(dp_2).$$

On the other hand it is not hard to see that linear map (31) is just the composition of isomorphism (29) with the evaluation map $\Gamma(N(\hat{z})) \dashrightarrow N_{\hat{z}(\lambda)}(\hat{z})$ that assigns to each section its value at the point $\hat{z}(\lambda)$. Thus under isomorphism (29) we have the isomorphism

$$(33) \quad \mathrm{Ker}((31)) \simeq \{s \in \Gamma(N(\hat{z}));\ s \text{ vanishes at some point of the curve } \hat{z}\}.$$

Recalling that $\Gamma(N(\hat{z})) \simeq \Gamma(\mathcal{O}(1)\oplus\mathcal{O}(1)) = \{(s_1, s_2);\ s_1$ and s_2 are polynomials of first degree in one complex variable $\} \simeq \mathbb{C}^4$ (cf. (30)), the right hand side can further be specified as:

$$(34) \quad \mathrm{Ker}((31)) \simeq \{(s_1, s_2);\ s_1 \text{ and } s_2 \text{ are polynomials in one complex variable that vanish simultaneously for some value}\}.$$

which evidenlty forms a two-dimensional complex vector space. This, in particular, implies that the tangent map dp_2 is of maximal rank. Moreover, not only its dimension, we can also give a more explicit represenation of $\mathrm{Ker}(dp_2)$ as follows. Let $\partial_1, \partial_2, \partial_3, \partial_4$ denote the tangent vectors in $dp_1(\mathrm{Ker}(dp_2))$ that correspond to $(s_1, s_2) = (\lambda, 0), (0, -1), (1, 0), (0, \lambda)$ respectively under isomorphism (34), and e_1, e_2, e_3, e_4 their dual cotangent frame (cf. (14)). Then from (34) one can show without difficulty that

$$
\begin{aligned}
(35)\qquad dp_1(\mathrm{Ker}(dp_2)) &= \mathbb{C}(-\lambda\partial_3 + \partial_1) \oplus \mathbb{C}(\lambda\partial_2 + \partial_4) \\
&= \{e^3 + \lambda e^1 = - e^2 + \lambda e^4 = 0\}.
\end{aligned}
$$

The structure of the "spatial" part of $\mathrm{Ker}(dp_2)$ is thus made fully clear. Because of the isomorphism $dp_1\colon \mathrm{Ker}(dp_2) \tilde{\to} dp_1(\mathrm{Ker}(dp_2))$ the kernel itself can be represented as: --

$$
\begin{aligned}
(36)\qquad \mathrm{Ker}(dp_2) &= \{\langle\mathcal{D}, v\rangle\partial_\lambda + v \;;\; v \in dp_1(\mathrm{Ker}(dp_2))\} \\
&= \{e^3 + \lambda e^1 = - e^2 + e^4\lambda = d\lambda - \mathcal{D} = 0\},
\end{aligned}
$$

where $\mathcal{D}$ is a complex linear form on T_zX whose evaluation is written $v \dashrightarrow \langle\mathcal{D}, v\rangle$.

The linear forms $e^1, \ldots, e^4$ and $\mathcal{D}$ on T_zX found above then define (at least locally) holomorphic 1-forms on X because of the analyticity of the parametrization $X \simeq$ {rational curves $\hat{z}$ in $\mathcal{T}$}, the last formula (36) then being valid at every point of $\mathcal{F}$. We can thus reproduce a Pfaffian system of the same form as (5). Besides, this Pfaffian system is integrable (i.e. Eqs.(7) are satisfied) because the fibers of

p_2, under some convexity condition, give a foliation in $\mathcal{T}$ each of whose leaf is a two-dimensional integral manifold of the Pfaffian systems. In particular, the metric to be constructed from $e^1, \ldots, e^4$ as in (1) is conformally self-dual.

2.2. COORDINATE REPRESENTATION AND ITS GEOMETRICAL MEANING. To describe the above construction more explicitly, we now cover the manifold $\mathcal{T}$ with coordinate patches as $\mathcal{T} = \bigcup_{i \in I} U_{(i)}$ where each $U_{(i)}$ carries a set of local coordinates $u_{(i)} = (u^0_{(i)}, u^1_{(i)}, u^2_{(i)})$. Local coordinates $u_{(i)}$ and $u_{(j)}$ are related on $U_{(i)} \cap U_{(j)}$ as: --

$$(37) \qquad u^\alpha_{(i)} = F^\alpha_{(ij)}(u^0_{(j)}, u^1_{(j)}, u^2_{(j)}), \quad \alpha = 0, 1, 2.$$

The functions $F_{(ij)} = (F^0_{(ij)}, F^1_{(ij)}, F^2_{(ij)})$ $(i \in I)$ arising above are called "patching functions" of the twistor space $\mathcal{T}$. In terms of these local coordinates and patching functions the family of rational curves in $\mathcal{T}$ are represented by a set of functions $u^\alpha_{(i)}(\lambda, z)$ $(\alpha = 0,1,2,\ i \in I)$ which are defined on $p_2^{-1}(U_{(i)})$ respectively and which satisfy Eqs. (37) for all $i, j \in I$. These functions, moreover, have another definite gemetrical meaning, that is, they are first integrals of Pfaffian system (5). This is obvious from the construction because the $u^\alpha_{(i)}(\lambda, z)$'s arise as the coordinate-components of the image of the map p_2 with respect to the above local coordinares (recall that p_2 gives a fibration whose fibers are two-dimensional integral surfaces of Pfaffian system (5)).

Thus we find an alternatiove formulation of the twistor construction of conformally self-dual metrics, which consists of the following steps:

i) To give an appropriate set of holomorphic patching functions $F_{(ij)}$ $(i, j \in I)$ for which the corresponding twistor space includes a complete four-parameter family of rational curves.

ii) To find a set of holomorphic functions $u^{\alpha}_{(i)}(\lambda, z)$ ($\alpha = 0,1,2$, $i \in I$) of five variables $(\lambda, z) = (\lambda, z^1, \dots, z^4)$ that satisfy Eqs. (37), where z runs over a (coordinate patch of a) four-dimensional complex manifold and λ over an open subset $D_{(i)}(z)$ of $\mathbb{P}^1$ that may move as z varries. It is moreover required that the $D_{(i)}(z)$'s cover the whole Riemann sphere

(38) $$\mathbb{P}^1 = \bigcup_{i \in I} D_{(i)}(z).$$

(In the case discussed above, for example, $D_{(i)}(z) = \{\lambda \in \mathbb{P}^1;\ (\lambda, z) \in p_2{}^{-1}(U_{(i)})\}$.)

iii) To construct a null tetrad $(e^1, \dots, e^4)$ and the corresponding metric ds^2 from $u^{\alpha}_{(i)}(\lambda, z)$.

Some examples of self-dual Einstein metrics are indeed constructed along the above process (see Curtis *et al.* [13], Ward [14], Tod and Ward [15], Hitchin [16]). In general, however, the above construction is very hard to execute. The hardest step is ii). The first step is rather easy because, as pointed out by Penrose [4], such a complete family of rational curves do exists as far as the corresponding twistor space $\mathcal{T}$ is a "small deformation" of the flat one $\mathbb{P}^3 \setminus \mathbb{P}^1$. Finding its explicit form, on the other hand, is extremely difficult in general and this forms a main difficulty in this construction. The last step, iii), includes nothing difficult as we observed in previous sections.

2.3. SIMPLEST CASE. In order to illustrate the above construction in more detail, let us now consider the simplest case where $\mathcal{T}$ is covered by just two coordinate patches as $\mathcal{T} = U_{(0)} \cup U_{(\infty)}$ and where the corresponding covering of $\mathbb{P}^1$, $\mathbb{P}^1 = D_{(0)}(z) \cup D_{(\infty)}(z)$, consists of two discs with center at 0 and ∞, respectively; see Penrose [4], Boyer [7], Curtis *et al.* [13], Ward [14], Tod and Ward [15] for details. The family of rational curves parametrised by X is now represented by two triples of functions, $u^\alpha_{(0)}(\lambda, z)$ and $u^\alpha_{(\infty)}(\lambda, z)$ $(\alpha = 0,1,2)$. For the sake of convenience we here further assume that $u^0_{(\infty)}(\lambda, z)$ is normalises as: --

(39) $$u^0_{(\infty)}(\lambda, z) = \lambda^{-1} + O(\lambda^{-2}) \quad \text{as} \quad \lambda \to \infty.$$

(In fact, as a simple argument shows, this does not harm the generality of the present argument). According to what we observed in §1.9, we now consider the following functions: --

(40) $$u^0 = 1/u^0_{(\infty)}, \quad u^1 = u^1_{(\infty)}/u^0_{(\infty)}, \quad u^2 = u^2_{(\infty)}/u^0_{(\infty)}.$$

From their Laurent expansion we can pick out four functions x,

y, p, q as in (22). As far as the manifold $\mathcal{T}$ is "close" enough to the twistor space of the flat space, the functions x, y, p, q are (at least in a small neighborhood of $(p, q, x, y) = (0, 0, 0, 0)$) functionally independent and, therefore, can be adopted as new local coordinates in X. From the construction $u^\alpha(\lambda, z)$ and $u^\alpha_{(0)}(\lambda, z)$, $z = (p, q, x, y)$, now satisfy the functional equations

$$u^\alpha(\lambda, z) = f^\alpha(u^0_{(0)}(\lambda, z), u^1_{(0)}(\lambda, z), u^2_{(0)}(\lambda, z)) \qquad (\alpha = 0, 1, 2), \tag{41}$$

where

$$f^0 = 1/F^0_{(0\infty)}, \quad f^1 = F^1_{(0\infty)}/F^0_{(0\infty)}, \quad f^2 = F^2_{(0\infty)}/F^0_{(0\infty)}, \tag{42}$$

$F^\alpha_{(0\infty)}$, $\alpha = 0, 1, 2$, being patching functions between the coordinate patches $U_{(\infty)}$ and $U_{(0)}$. Thus for the present case the problem of finding a four-parameter family of rational curves in $\mathcal{T}$ reduces to solving Eqs. (41) under the additional requirement that $u^0(\lambda, z)$, $u^1(\lambda, z)/u^0(\lambda, z)$, $u^2(\lambda, z)/u^0(\lambda, z)$ and $u^\alpha_{(0)}(\lambda, z)$ $(z = (p, q, x, y))$ be holomorphic in $U_{(\infty)}$ and in $U_{(0)}$, respectively, where $U_{(\infty)}$ and $U_{(0)}$ are considered open subsets of $\mathbb{C}^3$.

Once the curved twistor construction is re-formulated as above, it is now quite straightforward to incorporate a group-theoretical description of conformally self-dual metrics just the same way as for the case of self-dual Einstein merics discussed in [10]. To see this, we define holomorphic local transformations $u(p,q)$, $u_{(0)}(p,q)$ and f in $\mathbb{C}^3$ (the first two include (p, q) as parameters) to be

$$
\begin{aligned}
&u(p,\ q):\ (\lambda,\ x,\ y) \dashrightarrow (u^{\alpha}(\lambda,\ p,\ q,\ x,\ y))\\
(43)\qquad &u_{(0)}(p,\ q):\ (\lambda,\ x,\ y) \dashrightarrow (u^{\alpha}_{(0)}(\lambda,\ p,\ q,\ x,\ y))\\
&f\ :\ (\lambda,\ x,\ y) \dashrightarrow (f^{\alpha}(\lambda,\ x,\ y)) \qquad (\alpha = 0,\ 1,\ 2),
\end{aligned}
$$

where the variables $(\lambda,\ x,\ y)$ are now considered global coordinates in $\mathbb{C}^3$. Eqs. (41) then can be written

$$(44)\qquad u(p,\ q) = f \circ u_{(0)}(p,\ q).$$

The local transformation $u(p,\ q)$ can further be decomposed as

$$(45)\qquad u(p,\ q) = T(p,q) \circ \varphi(p,q),$$

where

$$
(46)\qquad
\begin{aligned}
&T(p,\ q):\ (\lambda,\ x,\ y) \dashrightarrow (\lambda,\ x + p\lambda,\ y + qy),\\
&\varphi(p,\ q):\ (\lambda,\ x,\ y) \dashrightarrow (u^0,\ u^1 - p\lambda,\ u^2 - q\lambda).
\end{aligned}
$$

Therefore Eq.(44) can rewritten

$$(47)\qquad \varphi(p,\ q) \circ u_{(0)}(p,\ q)^{-1} = T(-p,\ -q) \circ f(p,\ q).$$

Thus we encounter a decomposition problem in the pseudo-group of holomorphic local transformations in $\mathbb{C}^3$ very similar to the one in [10], that is, to decompose the local transformation $T(-p,\ -q) \circ f$ into the composition of such two local transformations that arise on the left side of Eq.(47). Note that $\varphi(p,\ q)$ is now holomorphic in $D_{(\infty)}(p,\ q,\ x,\ y)$.

As mentioned in *Introduction*, conformally self-dual metrics includes self-dual Einstein metrics as a special subclass. It is also straightforward to characterise them in the present formulation. They indeed correspond to the case where the following conditions are satisfied (cf. [10] and references cited therein): --

$$f^0(\lambda, x, y) = \lambda, \quad \frac{\partial(f^1, f^2)}{\partial(x, y)} = 1. \tag{48}$$

2.4. AN APPROACH BY MEANS OF GRASSMANN MANIFOLD. We now conclude this article by showing how the geometry of conformally self-dual metrics can be encoded into a sort of dynamical motion in an infinite dimensional Grassmann manifold (see [10] for the case of self-dual Einstein metrics). We present here basic ideas alone, and omit the full details of the formulation.

The argument employed to derive this result is almost the same as those in [10]. Roughly speaking, it proceeds as follows: --

i) Take an appropriate pseudo-group Γ of holomorphic local transformations in $\mathbb{C}^3$ and a vector space V of holomorphic functions of (λ, x, y) in some domains of $\mathbb{C}^3$ such that the map

$$\begin{aligned} &\rho^* : \Gamma \dashrightarrow GL(V), \\ &\text{where} \quad \rho^*(f)(\xi) = \xi \circ f^{-1} \quad \text{for} \quad f \in \Gamma \quad \text{and} \quad \xi \in V, \end{aligned} \tag{49}$$

defines a linear representation of the pseudo-group Γ.

ii) Let V_ϕ denote the linear subspace of V that consists of elements of V which are also holomorphic in some neighborhoods of $\lambda = 0$, and $\gamma(\varphi(p, q))$ the one defined as: --

(50) $\gamma(\varphi(p, q)) = \rho^*(\varphi(p, q))(V_\phi)$.

iii) The linear subspace $\gamma(\varphi(p, q))$ of V gives a moving point in an infinite dimensional Grassmann manifold GM_V (see [10]) with p and q multi-time parameters.

iv) The dynamics of the above moving point is described by the following simple law: --

(51) $\gamma(\varphi(p, q)) = \exp(p\lambda\partial/\partial x + q\lambda\partial/\partial y)\gamma(\varphi(0, 0))$,

where $\exp(p\lambda\partial/\partial x + q\lambda\partial/\partial y)$ is considered a linear operator that acts on V as:

(52) $\exp(p\lambda\partial/\partial x + q\lambda\partial/\partial y)\xi(\lambda, x, y) = \xi(\lambda, x + p\lambda, y + q\lambda)$

for $\xi = \xi(\lambda, x, y) \in V$.

The choice of the Γ and V is the most delicate part of this construction. In order to avoid cumbersome analytical arguments and to make algebraic features more clear, one may as well take a formal framework as adopted in [10] for the description of "formal metrics"; various algebraic tools developed therein can be applied to the present case almost the same way (with slightest modifications). For example, the vector space $\mathbb{C}\langle\lambda, x, y\rangle$ and its variation $\mathbb{C}[[p, q]]\langle\lambda, x, y\rangle$ defined in [10, section 5], which is formed by formal power series in $\lambda, \lambda^{-1}, x, y$ satisfying some conditions, can play the role of V. The role of Γ is then played by a group of "formal" transformations in $\mathbb{C}^3$ that can act on the above vector spaces of formal Laurent series through the linear representation shown in (49). Such a "formal transformation group" includes various "formal loop groups" introduced in [10]; the latter become subgroups of the former and can be reproduced by imposing constraints as shown in (48). By use of such algebraic tools the above arguments i) - iv) can be justified on a mathematically rigorous basis.

References

1. Atiyah, M.F., Hitchin, N.J. and Singer, I.M.: Self-duality in four dimensional Riemannian geometry. Proc. R. Soc. London A362 (1978), 425-461.
2. Eguchi, T., Gilkey, P.B. and Hanson, A.J.: Gravitation, gauge theories and differential geometry. Phys. Rep. 66 (1980), 213-293.
3. Plebanski, J.F.: Some solutions of complex Einstein equations. J. Math. Phys. 16 (1975), 2395-2402.
4. Penrose, R.: Nonlinear gravitons and twistor theory. Gen. Rel. Grav. 7 (1976), 31-52.
5. Ward, R.S.: Self-dual space times with cosmological constant. comm. Math. Phys. 78 (1980), 1-17.
6. Hitchin, N.J.: Complex manifolds and Einstein's equations. In *Twistor Geometry and Nonlinear Systems*, ed. H.D. Doebner and T.D. Palev, Lect. Notes. Math. #970, pp.73-99, Springer 1982.

7. Boyer, C.P.: The geometry of complex self-dual Einstein spaces. In *Nonlinear Phenomena*, ed. K.B. Wolf, Lect. Notes. Phys. #189, pp.25-45, springer 1983.

8. LeBrun, C.: Spaces of complex null geodesics in complex-Riemannian geometry. Trans. A.M.S. <u>278</u> (1983), 209-231.

9. Manin, Yu.I. and Penkov, I.B.: Null-geodesics of complex Einstein spaces. Func.Anal.Appl. <u>16</u> (1982), 64-66.

10. Takasaki, K.: Aspects of integrability in self-dual Einstein metrics and related equations Preprint, RIMS-527, RIMS, Kyoto University, November 1985.

11. Gindikin, S.G.: Integral geometry and twistors. In *Twistor Geometry and Nonlinear Systems*, ed. H.D. wolf and T.D. Palev, Lect. Notes. Phys. #970, pp.2-42. Springer 1982.

    ______________: Reduction of manifolds of rational curves and related problems of the theory of differential equations. Func. Anal. Appl. <u>18</u> (1984), 278-298.

12. Boyer, C.P. and Plebanski, J.F.: Conformally self-dual spaces and Maxwell's equations. Phys. Lett. <u>106A</u> (1984), 125-129.

13. Curtis, W.D., Lerner, D.E. and Miller, F.R.: Complex pp waves and the nonlinear graviton construction. J. Math. Phys. <u>19</u> (1978), 2024-2027.

14. Ward, R.S.: A class of self-dual solutions of Einstein's equations. Proc. R. soc. London <u>A363</u> (1978), 289-295.

15. Tod, K.P. and Ward, R.S.: Self-dual metrics with self-dual Killing vectors. Proc. R. Soc. London <u>A368</u> (1979), 411-427.

16. Hitchin, N.J.: Polygons and gravitons. Math. Proc. Cambridge Phil. Soc. 85 (1979).

Corrections to "Conformally Self-Dual Metrics and Integrability"

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There are two mistakes in this article that should be corrected as follows.

page 17, line 17. (33) is incorrect. The correct statement is:

$$\mathrm{Ker}((31)) \simeq \{s \in \Gamma(N(\hat{Z}) : s \text{ vanishes at } \hat{Z}(\lambda)\}.$$

page 17, line 21. (34) is incorrect. The correct statement is:

$$\mathrm{Ker}((31)) \simeq \{(s_1, s_2) : s_1 \text{ and } s_2 \text{ are polynimials of degree one in one complex variable that vanish simultaneously at } \lambda\}.$$