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3. Self dual connections and lens spaces

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The set of all oriented diffeomorphism classes of $\mathbb{Z}/2$-homology 3-sphere has a structure of additive semigroup by connected sum. Deviding by the subsemigroup of all oriented diffeomorphism classes of boundary of some $\mathbb{Z}/2$-acyclic compact smooth 4-manifolds, we get an abelian group $\Theta_3^{\mathbb{Z}/2}$ called $\mathbb{Z}/2$-homology cobordism group of $\mathbb{Z}/2$-homology 3-spheres. The following facts are known about the structure of $\Theta_3^{\mathbb{Z}/2}$.

(1) The Rochlin invariant induces a surjective homomorphism $\Theta_3^{\mathbb{Z}/2} \longrightarrow \mathbb{Z}/16$.

(2) Two lens spaces $L(p,q)$ and $L(p',q')$ with p and p' odd give a same element of $\Theta_3^{\mathbb{Z}/2}$ iff $p=p'$ and $q=q'$.

(3) $\Theta_3^{\mathbb{Z}/2}$ has an element of infinite order, i.e., $\dim_{\mathbb{Q}}(\Theta_3^{\mathbb{Z}/2}\otimes\mathbb{Q})\geqq 1$. For example the class of Poincaré homology sphere has infinite order.

(1) is now classical. (2) and (3) were obtained by Fintushel-Stern in 1984 in a slightly stronger form. They uses a certain consideration about moduli spaces of self dual connections.

We consider here lens spaces of the form $L(p,1)$ with p odd. Our result is:

Theorem. $L(p,1)$ $p=3,5,\cdots\cdots,2m-1,\cdots$ are all linearly independent over $\mathbb{Z}$. In particular $\dim_{\mathbb{Q}}(\Theta_3^{\mathbb{Z}/2}\otimes\mathbb{Q})=+\infty$.

We also use a moduli space of self dual connections to prove the theorem. The main difference between the argument of Fintushel-Stern and ours lies in the following fact. Their moduli space is compact, but our moduli space has several "ends".

An outline of the proof is as follows. If the theorem does not holds, there is an oriented cobordism between finite lens spaces of the form $L(p,\pm 1)$ with a certain property. Collapcing each component of the boundary, we get a $\mathbb{Z}/2$-acyclic V-manifold, or orbifold. We construct a V-bundle over the V-manifold and consider the moduli space of self dual connections on it.

To control the ends of the moduli space, we need two methods due to Donaldson. One is a description of ends, and the other is a deformation of ends. We show that the deformed moduli space is a smooth one dimensional manifold and the number of its ends is equal to the order of $H^1(\text{V-manifold}, \mathbb{R}/\mathbb{Z})$. Now the V-manifold is $\mathbb{Z}/2$-acyclic, so the number of ends must be odd. But the number of ends of a smooth one dimentional manifold is always even. This is a contradiction.