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How to change the sign of energy?
- Report on my three months visit to YITP

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Abstract. In this report I review some of my research activities during my visit to the Yukawa Institute for Theoretical Physics in Kyoto. In particular, during these three months, I was interested in explicitly time-dependent canonical transformations. These transformations map Hamiltonian equations into other Hamiltonian equations and are especially interesting in the context of cosmology. Indeed, there the focus is on the perturbations around time-dependent cosmological backgrounds, so that the Hamiltonian for perturbations is explicitly time-dependent. Especially, the time-dependent canonical transformations are helpful for a reformulation of theories with ghosts–cosmological perturbations with the negative kinetic energies. During my stay at YITP, I have proven that one can always perform such a canonical transformation that a new description of the system does not involve ghost degrees of freedom.

Keywords: canonical transformations, ghost degrees of freedom, cosmological perturbations, Hamilton-Jacobi equation

1. Classical Time-Dependent Canonical Transformations

Most of the time during my visit to YITP I was occupied with different thoughts about canonical transformations and their applications to cosmology. In particular, I was interested to apply explicitly time-dependent transformations to cosmological perturbations. Cosmological perturbations are described by actions with explicitly time-dependent coefficients, hence a change from one time-dependent description to another one is a rather natural procedure. Here I will review some of the results on this topic obtained during my stay at YITP. A by far more detailed discussion including quantum canonical transformations and particularly useful examples will be published in (De Felice, 2016).

Let us first refresh the general formalism of canonical transformations.

Consider a Hamiltonian system with canonical variables (p_i, q_i), (with i taking values from 1 to N, where N is the number of degrees of freedom) and with the Hamiltonian $H = H(p_i, q_i, t)$. Further, for simplicity of notation we will suppress the indices.

It is well known, (see e.g. (Arnold, 1989) that univalent canonical transformations

$$(p, q, H) \to (\pi, \theta, \mathcal{H}) ,$$

preserve the Poincaré–Cartan integral invariant

$$I = \oint pdq - Hdt = \oint \pi d\theta - \mathcal{H}dt , \quad (1)$$

from where it follows that

$$pdq - Hdt - (\pi d\theta - \mathcal{H}dt) = dF , \quad (2)$$

where $F = F(q, \theta, t)$ is the so-called generating function. In this way one obtains

$$p = \frac{\partial F}{\partial q} , \qquad \pi = -\frac{\partial F}{\partial \theta} ,$$

while the explicit-time dependence of the generating function changes the value of the Hamiltonian as

$$\frac{\partial F}{\partial t} = \mathcal{H}(\pi, \theta, t) - H\,(p, q, t) .$$

Thus one can find the corresponding generating function by solving the following nonlinear first-order partial differential equation (PDE)

$$\frac{\partial F}{\partial t} = \mathcal{H}\left(-\frac{\partial F}{\partial \theta}, \theta, t\right) - H\left(\frac{\partial F}{\partial q}, q, t\right) . \quad (3)$$

In particular, if $\mathcal{H} = 0$, the last equation is called the Hamilton-Jacobi equation. The general solution of this equation – the so-called *general integral* contains a free function. But to find a generating function one only needs the so-called *complete integral* of this equation, which only contains as many free parameters as independent variables, (see discussion in e.g. (Landau, 1976)).

This PDE for the generating function can be rewritten as the Hamilton-Jacobi equation for a larger system. To achieve this it is convenient to enlarge the configuration space of the system and introduce coordinates $Q = (q, \theta)$ with momenta corresponding canonical momenta defined as

$$P = \frac{\partial F}{\partial Q} \ ,$$

so that $Q_1 = q$, $Q_2 = \theta$, $P_1 = p$, $P_2 = -\pi$. Note that $(\pi, \theta) \rightarrow (P_2, Q_2)$ is not an univalent canonical transformation, rather it is an *extended canonical transformation* with valence $c = -1$ (for details see e.g. (Gantmacher, 1975)) so that the corresponding Hamiltonian

$$H_2(P_2, Q_2, t) = c\mathcal{H} = -\mathcal{H}.$$

The total Hamiltonian is then

$$H_{\pm}(P, Q, t) = H_1(P_1, Q_1, t) + H_2(P_2, Q_2, t) = H(P_1, Q_1, t) - \mathcal{H}(-P_2, Q_2, t)\,. \tag{4}$$

In this case the equation defining the generating function (3) takes exactly the form of the Hamilton-Jacobi equation for a system with the twice as many coordinates in the configuration space

$$\frac{\partial F(Q,t)}{\partial t} + H_{\pm}\left(\frac{\partial F}{\partial Q}, Q, t\right) = 0\,. \tag{5}$$

Clearly the *complete integral* of this equation (5) is the on-shell classical action (also called Hamilton's principal function) $S_{\pm}(Q, t)$ for the motion of the enlarged system with the Hamiltonian $H_{\pm}$ and final coordinates Q. The initial coordinates are just free parameters in the *complete integral.*

The generating function can be differentiated with respect to θ and q to obtain π and p. Further, for a non-degenerate generating function the system of equations $\pi = -F_{,\theta}(q, \theta, t)$ and $p = F_{,q}(q, \theta, t)$ can be solved to find $\pi(p, q, t)$ and $\theta(p, q, t)$. It is important to note that the resulting canonical transformation generated by $F = S_{\pm}(Q, t)$ is univalent.

2. Changing the sign of the Hamiltonian

In particular, one can apply these results from above to obtain a generating function for canonical transformations to system with the Hamiltonian with an opposite (in particular negative) sign. In this case $\mathcal{H} = -H$ and the auxiliary Hamiltonian (4) is

$$H_{\pm}(P, Q, t) = H(P_1, Q_1, t) + H(-P_2, Q_2, t)\,. \tag{6}$$

Another way to understand the transformation changing the sign of the Hamiltonian and to find explicit formulas for the new canonical coordinates and momenta is based on the facts that i) canonical transformations build a group ii) motion is a canonical transformation. This way does not require a search for the generating function.

Indeed, the classical motion is a canonical transformation to the initial data

$$(p, q, H(p, q, t)) \rightarrow (P_0, Q_0, 0)\,.$$

So that there are trivially exist functions $p(P_0, Q_0, t)$ and $q(P_0, Q_0, t)$ along with inverse $P_0(p, q, t)$ and $Q_0(p, q, t)$. The inverse transformation given by the latter formulas is also canonical. On the other hand one can always consider another system with coordinates θ momenta π and the Hamiltonian $\mathcal{H} = -H(\pi, \theta, t)$. For this system one can consider a motion from *the same initial data*

$$(\pi, \theta, -H(\pi, \theta, t)) \rightarrow (P_0, Q_0, 0)\,.$$

So that there are functions $\pi(P_0, Q_0, t)$ and $\theta(P_0, Q_0, t)$ along with inverse $P_0(\pi, \theta, t)$ and $Q_0(\pi, \theta, t)$. Hence, there exist functions

$$\pi(p, q, t) = \pi(P_0(p, q, t), Q_0(p, q, t), t) ,$$

and

$$\theta(p, q, t) = \theta(P_0(p, q, t), Q_0(p, q, t), t) .$$

These functions define a canonical transformation, because of the group properties of canonical transformations. In particular, it is not hard to find explicit form of these transformations for a lineal system.

3. Cosmological perturbations

Here we shortly comment on the application to the cosmological perturbations. In the linear theory of scalar cosmological perturbations one is interested in the dynamics of a particular combination $\mathcal{R}$ of the scalar perturbations of the metric and matter (see (Mukhanov, 1986, 1992) and (Sasaki, 1986)). The dynamics are described by the following quadratic action for $\mathcal{R}$

$$S = \frac{1}{2}\int d\eta\, d^3\boldsymbol{x}\, Z\, ((\mathcal{R}')^2 - c_s^2 (\partial_i \mathcal{R})^2) ,$$

where both coefficients $Z = Z(\eta)$ and the sound speed $c_s^2 = c_s^2(\eta)$ are constructed out of the background quantities like energy density, pressure, Hubble parameter etc. Thus these coefficients are explicit functions of the conformal time η. In particular, for some theories, it happens that $Z < 0$ so that the perturbations are *ghosts*. It is convenient to go to the Fourier space and rewrite the action as

$$S = \frac{1}{2}\int d\eta\, d^3\boldsymbol{k}\, Z\, (|\mathcal{R}_{\boldsymbol{k}}'|^2 - c_s^2 k^2 |\mathcal{R}_{\boldsymbol{k}}|^2) = \int d^3k\, S_k \ .$$

Now each mode one can treat separately as a dynamical system. In particular, the corresponding Hamiltonian is

$$\mathcal{H}_k = \frac{|P_{\boldsymbol{k}}|^2}{2Z} + \frac{Z}{2} c_s^2 k^2 |\mathcal{R}_{\boldsymbol{k}}|^2 . \tag{7}$$

In that case one can always perform a canonical transformation (5) with the Hamiltonian (6) constructed out of (7) so that the new Hamiltonian for the mode is

$$\mathrm{H}_k = -\frac{|\pi_{\boldsymbol{k}}|^2}{2Z} - \frac{Z}{2} c_s^2 k^2 |r_{\boldsymbol{k}}|^2 .$$

In this way, if the original modes were ghosts the new modes are normal and vice versa. Thus simple time-dependent canonical transformations can extinguish ghosts. The price to pay is a structure of interactions different from those for $\mathcal{R}$ and even stronger time-dependent and $\boldsymbol{k}$-dependent couplings.

4. Other research activity while at the YITP

During my stay at YITP I also finished work on gauge issues and Weyl-invariance in the *Mimetic Gravity* (Hammer, 2015). Further I have been working on Null Energy Condition and speed of propagation for the gravitational waves in the scalar-tensor theories. Moreover, in collaboration with

Damien Easson, I was investigating the plausibility of the cosmological oscillatory attractors recently introduced by Wilczek and collaborators in (Bains, 2015). This study will be published in May 2016. Thanks to the generous support from the International Research Unit of Advanced Future Studies, Kyoto University Research Coordination Alliance, I have visited and have given seminars at:

1. Theoretical High-Energy Physics Lab at the Department of Physics, Tokyo Metropolitan University,
2. Leung Center for Cosmology and Particle Astrophysics of the National Taiwan University
3. Jockey Club Institute for Advanced Study of the Hong Kong University of Science and Technology,
4. Department of Physics, Tokyo Institute of Technology

Moreover, I have attended and have given a talk at a very nice and intensive workshop "New perspectives on cosmology" at the Asia Pacific Center for Theoretical Physics, Pohang, South Korea. During some of these visits, as a result of the intensive exchange of ideas, I started new interesting projects and found new collaborators.

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