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Anabelian Group-theoretic Properties of the Pro-p Absolute Galois Groups of Henselian Discrete Valuation Fields

By

Arata MINAMIDE and Shota TSUJIMURA

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract

Let p be a prime number; K a Henselian discrete valuation field of characteristic 0 such that the residue field is an infinite field of characteristic p. Write G_K for the absolute Galois group of K. In our previous papers, under the assumption that K contains a primitive p-th root of unity ζ_p, we proved that any almost pro-p-maximal quotient of G_K satisfies certain "anabelian" group-theoretic properties called very elasticity and strong internal indecomposability. In the present paper, we generalize this result to the case where K does not necessarily contain ζ_p. Then, by applying this generalization, together with some facts concerning Hilbertian fields, we prove the semi-absoluteness of isomorphisms between the pro-p étale fundamental groups of smooth varieties over certain classes of fields of characteristic 0. Moreover, we observe that there are various similarities between the maximal pro-p quotient G_K^p of G_K and nonabelian free pro-p groups. For instance, we verify that every topologically finitely generated closed subgroup of G_K^p is a free pro-p group. One of the key ingredients of our proofs is "Artin-Schreier theory in characteristic zero" introduced by MacKenzie and Whaples.

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Contents

Introduction

Let p be a prime number. For any field F of characteristic 0, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F;

$$G_F \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F).$$

We shall fix a primitive p-th root of unity $\zeta_p \in \overline{F}$. For any profinite group G, we shall write G^p for the maximal pro-p quotient of G.

In [8], [9], we proved/observed that various profinite groups related to *anabelian geometry* satisfy the following distinctive group-theoretic properties:

- *elasticity* — i.e., the property that every nontrivial topologically finitely generated normal closed subgroup of an open subgroup is open;
- *internal indecomposability* — i.e., the property that the centralizer [in the given group] of every nontrivial normal closed subgroup is trivial

[cf. Definition 1.1, (iii), (iv)]. For instance, we proved the following result [cf. [8], Theorem C; [9], Theorem A, (i)]:

Theorem. *Let K be a Henselian discrete valuation field of characteristic* 0 *such that the residue field is an infinite field of characteristic p. Suppose that*

$$\zeta_p \in K.$$

Then any almost pro-p-maximal quotient of G_K — i.e., the quotient of G_K by the kernel of the natural surjection $N \twoheadrightarrow N^p$ associated to a normal open subgroup $N \subseteq G_K$ — is very elastic — i.e., elastic and not topologically finitely generated — and strongly internally indecomposable — i.e., every open subgroup is internally indecomposable.

Here, we note that *Kummer theory* [together with some arguments concerning cyclotomic characters] played an essential role in our proof of Theorem. This is precisely the reason why we needed the assumption concerning ζ_p. In the present paper, by applying "*Artin-Schreier theory in characteristic zero*", introduced by MacKenzie and Whaples, instead of Kummer theory, we prove the following result [cf. Theorem 4.4]:

Theorem A. *Let K be a Henselian discrete valuation field of characteristic 0 such that the residue field is an infinite field of characteristic p. Then any almost pro-p-maximal quotient of G_K is very elastic and strongly internally indecomposable.*

Moreover, we also verify that, in the maximal pro-p quotient case, the following strong "rigidity" properties hold [cf. Proposition 3.3; Corollary 3.6; Theorem 3.8]:

Theorem B. *In the notation of Theorem A, let H be a nontrivial closed subgroup of G_K^p. Then the following hold:*

(i) Suppose that H is topologically finitely generated. Then H is a free pro-p group. [In particular, G_K^p is torsion-free.]

(ii) Suppose that the $i(>0)$-th [topological] derived subgroup of H is trivial. Then H is isomorphic to $\mathbb{Z}_p$. [In particular, every abelian closed subgroup of G_K^p is pro-cyclic.]

(iii) Suppose that H is isomorphic to $\mathbb{Z}_p$. Then the commensurator

$$C_{G_K^p}(H) \overset{\text{def}}{=} \{g \in G_K^p \mid H \cap gHg^{-1} \text{ is open in } H \text{ and } gHg^{-1}\}$$

of H in G_K^p is closed, and, moreover, isomorphic to $\mathbb{Z}_p$.

Finally, by applying Theorem A, together with some facts concerning Hilbertian fields [i.e., fields for which "Hilbert's irreducibility theorem" holds], we prove the *semi-absoluteness* of isomorphisms between the pro-p étale fundamental groups of smooth varieties [i.e., smooth, of finite type, separated, and geometrically integral schemes] over certain classes of fields of characteristic 0 [cf. Theorem 5.2; [11], Definition 2.4, (ii)]:

Theorem C. *Let K, K' be fields of characteristic 0; X, X' smooth varieties over K, K', respectively;*

$$\alpha : \Pi_X^p \overset{\sim}{\to} \Pi_{X'}^p,$$

— where we denote by $\Pi_{(-)}$ the étale fundamental group of $(-)$ [relative to a suitable choice of basepoint] — an isomorphism of profinite groups. Suppose that

- *K is either a Henselian discrete valuation field of characteristic 0 such that the residue field is an infinite field of characteristic p or a Hilbertian field;*
- *K' is either a Henselian discrete valuation field of characteristic 0 such that the residue field is a field of characteristic p or a Hilbertian field.*

Then α induces an isomorphism $G_K^p \xrightarrow{\sim} G_{K'}^p$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \downarrow & & \downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p \end{array}$$

— where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the smooth varieties X, X'.

Theorem C may be regarded as a generalization of [15], Theorem A, (i) [cf. also Remark 5.2.1].

The present paper is organized as follows. In §1, we recall basic notions on profinite groups, and verify some auxiliary results which will be used later. In §2, we discuss "Artin-Schreier theory in characteristic zero" which plays an essential role in the present paper. In §3, we study the maximal pro-p quotient case. In particular, by applying results in §1, §2, we verify Theorem B and the maximal pro-p quotient case of Theorem A. In §4, by applying results of §1, §2, §3, we complete our proof of Theorem A. In §4, we also discuss the case of Henselian discrete valuation fields of characteristic p. In §5, by applying [a special case of] Theorem A, together with some facts concerning Hilbertian fields, we prove Theorem C.

Notations and Conventions

Numbers: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\mathbb{Z}_{\geq 1}$ will be used to denote the set of positive integers. If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; the notation $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p. If A is a commutative ring, then the notation $A^\times$ will be used to denote the group of units of A.

Fields: Let F be a field; F^{sep} a separable closure of F; p a prime number. Then we shall write $\mathrm{char}(F)$ for the characteristic of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$. If $\mathrm{char}(F) \neq p$, then we shall fix a primitive p-th root of unity $\zeta_p \in F^{\mathrm{sep}}$. If $\mathrm{char}(F) = p$, then we shall write $k^p \stackrel{\mathrm{def}}{=} \{a^p \mid a \in k\}$.

Profinite groups: Let G be a profinite group. If p is a prime number, then we shall write G^p for the maximal pro-p quotient of G. If $i \geq 0$ is an integer, then we shall write $G[i+1]$ for the $(i+1)$-th derived subgroup $\overline{[G[i], G[i]]}$ of G, where $G[0] \stackrel{\mathrm{def}}{=} G$.

Fundamental groups: Let S be a connected locally Noetherian scheme. Then we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. [Note that, for any field F, $\Pi_{\mathrm{Spec}(F)} \cong G_F$.]

1 Some profinite group theory

In the present section, let p be a prime number.

In the present section, we first recall various notions — such as *slimness* and *elasticity* — associated to profinite groups. Then we observe that, in many situations, elasticity is a *stronger* property than slimness [cf. Proposition 1.2]. Next, we verify some technical results [Proposition 1.4; Lemmas 1.7, 1.8] which will be used to prove [for instance] very elasticity of almost pro-p-maximal quotients of the absolute Galois groups of Henselian discrete valuation fields of characteristic 0 such that the residue fields are infinite fields of characteristic p [cf. §3, §4].

Definition 1.1 ([11], Notations and Conventions; [11], Definition 1.1, (ii); [9], Definition 1.1, (vi); [9], Proposition 1.2)**.** Let G be a profinite group; $H \subseteq G$ a closed subgroup of G.

(i) We shall write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$$

for the *centralizer* of H in G; $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid gHg^{-1} = H\}$$

for the *normalizer* of H in G;

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid H \cap gHg^{-1} \text{ is open in } H \text{ and } gHg^{-1}\}.$$

for the *commensurator* of H in G. Note that although $Z_G(H)$ and $N_G(H)$ are closed in G, $C_G(H)$ is not necessarily closed in G [cf. [10], the discussion entitled "Topological Groups" in §0].

(ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G.

(iii) We shall say that G is *elastic* if every nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G is open. If G is elastic, but not topologically finitely generated, then we shall say that G is *very elastic.*

(iv) We shall say that G is *internally indecomposable* if $Z_G(H) = \{1\}$ for every nontrivial normal closed subgroup $H \subseteq G$. We shall say that G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

(v) We shall say that G is *almost pro-cyclic* if there exists an open subgroup [of G] that is pro-cyclic.

Remark 1.1.1. Let G be a profinite group. Then the following hold [cf. [8], Proposition 1.2]:

(i) G is slim if and only if, for every open subgroup $U \subseteq G$, $Z(U) = \{1\}$.

(ii) Suppose that G is nontrivial. Then G is very elastic if and only if every topologically finitely generated normal closed subgroup of G is trivial.

Remark 1.1.2. Let G be a strongly internally indecomposable profinite group. Then it follows immediately from Remark 1.1.1, (i), that G is slim.

Proposition 1.2. *Let G be an elastic profinite group. Suppose that G is not almost pro-cyclic. Then G is slim.*

Proof. Since every open subgroup of G is elastic and not almost pro-cyclic, to verify Proposition 1.2, it suffices to show that G is center-free. Let $g \in Z(G)$ be an element. Suppose that g is nontrivial. Write $H \subseteq G$ for the closed subgroup of G topologically generated by g. In particular, H is a nontrivial topologically finitely generated normal closed subgroup of G [cf. the inclusion $H \subseteq Z(G)$]. Thus, since G is elastic, we conclude that H is open in G — in contradiction to our assumption that G is not almost pro-cyclic. This completes the proof of Proposition 1.2. □

Remark 1.2.1. Proposition 1.2 implies that, in many situations, elasticity is a *stronger* property than slimness.

Definition 1.3 ([11], Definition 1.1, (iii))**.** Let G, Q be profinite groups; $q : G \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups]. Then we shall say that Q is an *almost pro-p-maximal quotient* of G if there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$.

Remark 1.3.1. It follows from the various definitions involved that, in the notation of Definition 1.3, the following hold:

(i) Let Q' be an open subgroup of Q. Then Q' is an almost pro-p-maximal quotient of $q^{-1}(Q')$.

(ii) The natural surjection $G^p \twoheadrightarrow Q^p$ is an isomorphism. In particular, if Q is a pro-p group, then Q may be [naturally] identified with the maximal pro-p quotient of G.

Proposition 1.4. *Let G be a profinite group. Suppose that every open subgroup H of G satisfies the following conditions:*

(a) H^p is center-free.

(b) Let $l \neq p$ be a prime number; $N \subseteq H$ a normal open subgroup of H of index l. Then the natural surjection $N^p \twoheadrightarrow H^p$ is not bijective.

Then any almost pro-p-maximal quotient Q of G is slim.

Proof. Write $q : G \twoheadrightarrow Q$ for the natural surjection. Note that there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$. To verify Proposition 1.4, it suffices to show that $Z(Q) = \{1\}$ [cf. Remark 1.3.1, (i)]. First, we claim the following:

> Claim 1.4.A: If an element $y \in Z(Q)$ satisfies $y^p = 1$, then we have $y = 1$.

Indeed, write $I \subseteq Q$ for the [finite] closed subgroup of Q generated by y. Then since $IN^p = I \times N^p$ is a pro-p open subgroup of Q, it follows from Remark 1.3.1, (i), (ii), that IN^p may be identified with the maximal pro-p quotient of $q^{-1}(IN^p)$. In particular, since IN^p is slim [cf. condition (a)], we conclude that $I = \{1\}$ [cf. [8], Lemma 1.3]. This completes the proof of Claim 1.4.A.

Next, we claim the following:

> Claim 1.4.B: Let $l \neq p$ be a prime number. If an element $y \in Z(Q)$ satisfies $y^l = 1$, then we have $y = 1$.

Indeed, write $I \subseteq Q$ for the [finite] closed subgroup of Q generated by y. Then since $IN^p = I \times N^p$ is an open subgroup of Q, it follows from Remark 1.3.1, (i), that IN^p may be identified with an almost pro-p-maximal quotient of $q^{-1}(IN^p)$. In particular, we have

$$N^p \overset{\sim}{\to} (I \times N^p)^p = (IN^p)^p \overset{\sim}{\to} (q^{-1}(IN^p))^p$$

[cf. Remark 1.3.1, (ii)]. Now suppose that $I \neq \{1\}$. Then since $N \subseteq q^{-1}(IN^p)$ is a normal open subgroup of $q^{-1}(IN^p)$ of index l, we obtain a contradiction [cf. condition (b)]. Therefore, we conclude that $I = \{1\}$. This completes the proof of Claim 1.4.B.

Finally, let us complete our proof of Proposition 1.4. Let $x \in Z(Q)$ be an element. In particular, there exists an integer $m \in \mathbb{Z}_{\geq 1}$ such that

$$x^m \in Z(Q) \cap N^p \subseteq Z(N^p) = \{1\}$$

[cf. condition (a)]. Thus, in light of Claims 1.4 A and 1.4 B, we conclude that $x = 1$. This completes the proof of Proposition 1.4. □

Lemma 1.5. *Let G be a nontrivial free pro-p group; $I \subseteq G$ a closed subgroup that is isomorphic to $\mathbb{Z}_p$. Then the following hold:*

(i) G is elastic.

(ii) The closed subgroup $N_G(I) \subseteq G$ is isomorphic to $\mathbb{Z}_p$.

Proof. Assertion (i) follows immediately from [13], Theorem 8.6.6. Next, we consider assertion (ii). Note that $N_G(I)$ is a [nontrivial] free pro-p group [cf. [13], Corollary 7.7.5]. Then since I is a nontrivial topologically finitely generated normal closed subgroup of $N_G(I)$, it follows from assertion (i) that I is open in $N_G(I)$, hence that $N_G(I)$ is almost pro-cyclic. This implies that $N_G(I)$ is isomorphic to $\mathbb{Z}_p$. This completes the proof of assertion (ii). $\square$

Lemma 1.6. *Let A be an integral domain; M an A-module such that every nontrivial finitely generated A-submodule is isomorphic to A. Write K for the quotient field of A. Then there exists an injective A-homomorphism*

$$\iota : M \hookrightarrow K.$$

In particular, if $A = \mathbb{Z}_p$, and, moreover, M is nontrivial, then M is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Q}_p$ [cf. the fact that every nontrivial proper $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p$ can be written as $p^n\mathbb{Z}_p$ ($n \in \mathbb{Z}$)].

Proof. Lemma 1.6 is immediate in the case where $M = \{0\}$. Thus, we may assume without loss of generality that $M \neq \{0\}$. Let $m \in M$ be a nontrivial element. We note that for any $x \in M$, the A-submodule M_x of M generated by m and x is isomorphic to A. Let $y \in M_x$ be a generator. Then there exist [unique] elements a, $b \in A$ such that $m = ay$ and $x = by$, where $a \neq 0$. Now we define a map ι as follows:

$$\iota : M \to K; \quad x \mapsto b/a.$$

[Note that this correspondence does not depend on the choice of the generator y.] One then verifies easily that ι is, in fact, an injective A-homomorphism. This completes the proof of Lemma 1.6. $\square$

Lemma 1.7. *Let T be a nontrivial pro-p group such that every nontrivial topologically finitely generated closed subgroup of T is isomorphic to $\mathbb{Z}_p$ [in the category of profinite groups]. Then T is isomorphic to $\mathbb{Z}_p$ [in the category of profinite groups].*

Proof. First, we claim the following:

T is an abelian group.

Indeed, let x, $y \in T$ be nontrivial elements. Write $T_{\{x,y\}}$ for the closed subgroup of T topologically generated by x and y. Then since $T_{\{x,y\}}$ is isomorphic to $\mathbb{Z}_p$, we conclude that x and y commute. This completes the proof of the claim.

In light of this claim, T admits a natural structure of $\mathbb{Z}_p$-module. Then we note that every nontrivial finitely generated $\mathbb{Z}_p$-submodule of T is a nontrivial topologically finitely generated closed subgroup of T, hence that it is isomorphic to $\mathbb{Z}_p$ [in the category of profinite groups]. In particular, since $T \neq \{0\}$, it follows from Lemma 1.6 that there exists a bijective $\mathbb{Z}_p$-homomorphism

$$\mathbb{Z}_p \ \stackrel{\sim}{\to}\ T.$$

[Here, note that the additive group of any field of characteristic zero — such as $\mathbb{Q}_p$ — does not admit a structure of profinite group. Indeed, such an additive group does not have a proper subgroup of finite index.] Therefore, since every continuous bijection between compact Hausdorff spaces is bi-continuous, we conclude that T is isomorphic to $\mathbb{Z}_p$ in the category of profinite groups. This completes the proof of Lemma 1.7. □

Lemma 1.8. *Let M be a field of characteristic $\neq p$. Then the natural homomorphism*

$$H^2(G_M, \mathbb{F}_p) \to H^2(G_{M(\zeta_p)}, \mathbb{F}_p)$$

is injective.

Proof. In light of the Hochschild-Serre spectral sequence associated to the natural exact sequence

$$1 \longrightarrow G_{M(\zeta_p)} \longrightarrow G_M \longrightarrow \mathrm{Gal}(M(\zeta_p)/M) \longrightarrow 1,$$

to verify Lemma 1.8, it suffices to show that

$$H^1(\mathrm{Gal}(M(\zeta_p)/M), H^1(G_{M(\zeta_p)}, \mathbb{F}_p)) = \{0\}, \qquad H^2(\mathrm{Gal}(M(\zeta_p)/M), \mathbb{F}_p) = \{0\}.$$

However, these equalities follow immediately from the fact that $[M(\zeta_p) : M]$ is coprime to p. This completes the proof of Lemma 1.8. □

2 Artin-Schreier equations in characteristic zero

In the present section, let p be a prime number; A a Henselian discrete valuation ring of residue characteristic p; π a uniformizer of A. Write K (respectively, k) for the quotient (respectively, residue) field of A; v_K for the discrete valuation on K such that $v_K(\pi) = 1$. For any $x \in A$, write $\overline{x}$ for the image of x in k. For every finite extension M of K, write $e_{M/K}$ for the ramification index of the extension $K \subseteq M$;

$$\underline{v}_M \ \stackrel{\mathrm{def}}{=}\ \tfrac{1}{[M:K]} \cdot v_K \circ N_{M/K} : M^\times \ \twoheadrightarrow\ \tfrac{1}{e_{M/K}}\mathbb{Z}\ \ (\subseteq\ \mathbb{Q})$$

— where we denote by $N_{M/K}$ the norm map of the extension $K \subseteq M$. [Note that we have $\underline{v}_M|_{K^\times} = v_K$.] Moreover, we suppose that

$$\mathrm{char}(K) \ = \ 0.$$

Write e for the absolute ramification index [i.e., $v_K(p)$] of K.

Our goal of the present section is to prove the following results:

- Suppose that $e \geq p+1$, and that k is an imperfect field of characteristic p. Let $F \subseteq G_K^p$ be a topologically finitely generated closed subgroup. Write $K \subseteq K_F$ for the pro-p extension of K associated to F. Then there exists a weakly unramified [pro-p] extension $K \subseteq L \subseteq K_F$ such that the residue field of L is *perfect* [cf. Lemma 2.5].
- If k is infinite, then G_K^p is *not topologically finitely generated* [cf. Lemma 2.6].

To do this, we begin by reviewing "*Artin-Schreier theory in characteristic zero*" [cf. Lemma 2.1; [2], Chapter III, §2, (2.5); [2], Chapter III, §2, Exercise 1; [6]].

Lemma 2.1. *Suppose that k is an imperfect (respectively, arbitrary) field of characteristic p. Let $\beta_1 \in A^\times$ be a unit such that $\overline{\beta}_1 \notin k^p$ (respectively, $\overline{\beta}_1 \neq 0$); $\beta_2 \in K$ an element such that $v_K(\beta_2) = -1$; λ a root of the equation*

$$X^p - X - \beta_1\beta_2^p \ = \ 0 \quad (\textit{respectively,} \ \ X^p - X - \beta_1\beta_2 \ = \ 0).$$

Then the extension $K \subseteq K(\lambda)$ satisfies the following:

- *$K \subseteq K(\lambda)$ is a weakly unramified (respectively, totally ramified) extension of degree p.*
- *The residue field of $K(\lambda)$ is $k(\overline{\beta}_1^{\frac{1}{p}})$ (respectively, k).*

Suppose further that $e \geq p$ (respectively, $e \geq 1$). Then $K \subseteq K(\lambda)$ is a Galois extension. In this case, the roots [in $K(\lambda)$] of the above equation can be written as

$$\lambda, \quad \lambda + z_1, \quad \lambda + z_2, \quad \ldots, \quad \lambda + z_{p-1}$$

— where z_i is an element of the ring of integers in $K(\lambda)$ such that the image of z_i in the residue field is i ($\in k$). Moreover, there exists an element $\rho \in \mathrm{Gal}(K(\lambda)/K)$ such that

$$\underline{v}_{K(\lambda)}(\rho(\lambda) - \lambda - 1) \ > \ 0.$$

Proof. We begin by considering the non-resp'd case of the first assertion. We claim the following:

The residue field of $K(\lambda)$ contains the p-th root of $\overline{\beta}_1$.

Indeed, observe that the equality $\lambda^p - \lambda = \beta_1\beta_2^p$ implies that $\underline{v}_{K(\lambda)}(\lambda) = -1$ [cf. our assumption that $v_K(\beta_1) = 0$, and $v_K(\beta_2) = -1$]. In particular, we conclude that $\lambda\beta_2^{-1}$ is a unit of the ring of integers in $K(\lambda)$. Then it follows from the equality

$$(\lambda\beta_2^{-1})^p - \beta_1 \;=\; (\lambda\beta_2^{-1}) \cdot \beta_2^{1-p},$$

together with the fact that $\underline{v}_{K(\lambda)}(\beta_2^{1-p}) = p - 1 > 0$, that the residue field of $K(\lambda)$ contains the p-th root of $\overline{\beta}_1$. This completes the proof of the claim. Now this claim implies that p divides $[K(\lambda) : K]$, hence that

- $K \subseteq K(\lambda)$ is a weakly unramified extension of degree p;
- the residue field of $K(\lambda)$ is $k(\overline{\beta}_1^{\frac{1}{p}})$.

This completes the proof of the non-resp'd case of the first assertion.

Next, let us consider the resp'd case of the first assertion. In this case, the equality $\lambda^p - \lambda = \beta_1\beta_2$ implies that $p \cdot \underline{v}_{K(\lambda)}(\lambda) = -1$. In particular, since p divides $e_{K(\lambda)/K}$ [cf. the fact that $\underline{v}_{K(\lambda)}(\lambda) \in e_{K(\lambda)K}^{-1} \cdot \mathbb{Z}$], we conclude that

- $K \subseteq K(\lambda)$ is a totally ramified extension of degree p.

This completes the proof of the resp'd case of the first assertion.

In the following, suppose that $e \geq p$ (respectively, $e \geq 1$). Let us consider the polynomial

$$g(Y) \;=\; (\lambda + Y)^p - (\lambda + Y) - \beta_1\beta_2^p$$
$$(\text{respectively, } g(Y) \;=\; (\lambda + Y)^p - (\lambda + Y) - \beta_1\beta_2).$$

Here, we observe that

$$g(Y) \;=\; Y^p + \binom{p}{1}\lambda Y^{p-1} + \cdots + \binom{p}{p-1}\lambda^{p-1}Y - Y.$$

Write $\Box$ for -1 (respectively, $-1/p$). Then since

$$\underline{v}_{K(\lambda)}\Big(\binom{p}{j}\lambda^j\Big) \;=\; \underline{v}_{K(\lambda)}(p) + j \cdot \underline{v}_{K(\lambda)}(\lambda) \;=\; e + j \cdot \Box \;>\; 0$$

— where $j \in \{1, \ldots, p-1\}$ — it follows from our assumption that K [hence, in particular $K(\lambda)$] is Henselian that $g(Y)$ splits completely in $K(\lambda)$ as follows:

$$g(Y) \;=\; Y(Y - z_1)(Y - z_2)\cdots(Y - z_{p-1})$$

— where z_i is an element of the ring of integers in $K(\lambda)$ such that the image of z_i in the residue field is i ($\in k$). In particular, λ, $\lambda + z_1$, $\lambda + z_2$, $\ldots$, $\lambda + z_{p-1}$ are the roots [in $K(\lambda)$] of the equation $X^p - X - \beta_1\beta_2^p = 0$ (respectively, $X^p - X - \beta_1\beta_2 = 0$).

Finally, we note that for any $\sigma \in \mathrm{Gal}(K(\lambda)/K)$, $\sigma(\lambda) - \lambda$ is a root of the equation $g(Y) = 0$. Thus, there exists an element $\rho \in \mathrm{Gal}(K(\lambda)/K)$ such that $\rho(\lambda) - \lambda = z_1$. Therefore, we conclude that

$$\underline{v}_{K(\lambda)}(\rho(\lambda) - \lambda - 1) \;=\; \underline{v}_{K(\lambda)}(z_1 - 1) \;>\; 0.$$

This completes the proof of Lemma 2.1. $\Box$

Remark 2.1.1. In the case where $\mathrm{char}(K) = p$, Lemma 2.1 also holds without the assumptions concerning "e" [i.e., the finite extension "$K \subseteq K(\lambda)$" is automatically Galois]. Indeed, in the notation of the proof of Lemma 2.1, since $\mathrm{char}(K) = p$, the polynomial $g(Y)$ splits completely in $K(\lambda)$ as follows:

$$g(Y) \;=\; Y(Y-1)(Y-2)\cdots(Y-p+1).$$

In particular, in this case, we can take z_i to be i $(\in K)$.

Definition 2.2. We shall write

$$\mathcal{A} \;\stackrel{\text{def}}{=}\; \{x \in A \mid \overline{x} \in (k^p)^{\times}\} \;\; (\subseteq\; A^{\times}).$$

In the case where k is imperfect, we fix a unit

$$\gamma \;\in\; A^{\times}$$

such that $\overline{\gamma} \notin k^p$. Suppose that $e \geq p$ (respectively, $e \geq 1$), and that k is an imperfect (respectively, arbitrary) field of characteristic p.

For any $x \in \mathcal{A}$ (respectively, $x \in A^{\times}$), let λ_x be a root of the equation

$$X^p - X - (\gamma x)\cdot(\pi^{-1})^p \;=\; 0 \quad (\text{respectively, } \; X^p - X - x\cdot\pi^{-1} \;=\; 0).$$

Then we shall write $\phi_x \in \mathrm{Hom}(G_K^p, \mathbb{F}_p)$ (respectively, $\psi_x \in \mathrm{Hom}(G_K^p, \mathbb{F}_p)$) for the surjective homomorphism

$$G_K^p \;\to\; \mathbb{F}_p; \quad \sigma \;\mapsto\; (\sigma(\lambda_x) - \lambda_x) \bmod \mathfrak{m}_x$$

— where we denote by $\mathfrak{m}_x$ the maximal ideal of the ring of integers of $K(\lambda_x)$; by abuse of notation, we denote by "$\mathbb{F}_p$" the additive group of the prime field of k [cf. Lemma 2.1]. [We note that the finite extension of K associated to $\mathrm{Ker}(\phi_x) \subseteq G_K^p$ (respectively, $\mathrm{Ker}(\psi_x) \subseteq G_K^p$) coincides with $K(\lambda_x)$, and that the residue field of $K(\lambda_x)$ is $k(\overline{\gamma}^{\frac{1}{p}}\overline{x}^{\frac{1}{p}}) = k(\overline{\gamma}^{\frac{1}{p}})$ (respectively k).] In particular, we obtain a map

$$\phi : \mathcal{A} \;\to\; \mathrm{Hom}(G_K^p, \mathbb{F}_p); \quad x \;\mapsto\; \phi_x$$

$$(\text{respectively, } \; \psi : A^{\times} \;\to\; \mathrm{Hom}(G_K^p, \mathbb{F}_p); \quad x \;\mapsto\; \psi_x).$$

[Note that the construction of ϕ depends on the choice of $\gamma \in A^{\times}$.]

Remark 2.2.1. In the case where $\mathrm{char}(K) = p$, we can define positive characteristic versions of "ϕ_x" and "ψ_x" [hence, in particular, "ϕ" and "ψ"] — for which, by abuse of notation, we shall write ϕ_x and ψ_x — of Definition 2.2 without the assumptions concerning "e" as follows: In the case where k is imperfect, we fix a unit

$$\gamma \;\in\; A^{\times}$$

such that $\overline{\gamma} \notin k^p$. If k is an imperfect (respectively, arbitrary) field of characteristic p, then for any $x \in \mathcal{A}$ [cf. Definition 2.2] (respectively, $x \in A^\times$), let λ_x be a root of the equation

$$X^p - X - (\gamma x) \cdot (\pi^{-1})^p \ = \ 0 \quad \text{(respectively, } \ X^p - X - x \cdot \pi^{-1} \ = \ 0).$$

Then we shall write $\phi_x \in \mathrm{Hom}(G_K^p, \mathbb{F}_p)$ (respectively, $\psi_x \in \mathrm{Hom}(G_K^p, \mathbb{F}_p)$) for the surjective homomorphism

$$G_K^p \ \to \ \mathbb{F}_p; \ \ \sigma \ \mapsto \ \sigma(\lambda_x) - \lambda_x$$

— where, by abuse of notation, we denote by "$\mathbb{F}_p$" the additive group of the prime field of K [cf. Remark 2.1.1].

Lemma 2.3. *Suppose that $e \geq p+1$ (respectively, $e \geq 2$), and that k is an imperfect (respectively, arbitrary) field of characteristic p. In the notation of Definition 2.2, let x, $y \in \mathcal{A}$ (respectively, x, $y \in A^\times$) be elements such that $\overline{x} \neq \overline{y}$. [In particular, $x - y \in \mathcal{A}$ (respectively, $x - y \in A^\times$).] Then it holds that $\phi_x - \phi_y = \phi_{x-y}$ (respectively, $\psi_x - \psi_y = \psi_{x-y}$).*

Proof. In the notation of Definition 2.2, for any $x \in \mathcal{A}$ (respectively, $x \in A^\times$), we set

$$\delta_x \ \stackrel{\text{def}}{=} \ (\gamma x) \cdot (\pi^{-1})^p \quad \text{(respectively, } \delta_x \ \stackrel{\text{def}}{=} \ x \cdot \pi^{-1}).$$

In particular, it holds that $\lambda_x^p - \lambda_x = \delta_x$. Thus, for any x, $y \in \mathcal{A}$ (respectively, x, $y \in A^\times$) such that $\overline{x} \neq \overline{y}$, we have

$$\lambda_x^p - \lambda_y^p - \lambda_{x-y}^p \ = \ \lambda_x - \lambda_y - \lambda_{x-y}. \tag{1}$$

Write $L \stackrel{\text{def}}{=} K(\lambda_x, \lambda_y, \lambda_{x-y})$; $\Box$ for -1 (respectively, $-1/p$). Here, we note that

$$\underline{v}_L(\lambda_x) \ = \ \underline{v}_L(\lambda_y) \ = \ \underline{v}_L(\lambda_{x-y}) \ = \ \Box \tag{2}$$

[cf. the equality $\lambda_x^p - \lambda_x = \delta_x$]. Now we claim the following:

$$\underline{v}_L((\lambda_x - \lambda_y - \lambda_{x-y})^p - (\lambda_x^p - \lambda_y^p - \lambda_{x-y}^p)) \ > \ 0.$$

Indeed, observe that each term of $(\lambda_x - \lambda_y - \lambda_{x-y})^p - (\lambda_x^p - \lambda_y^p - \lambda_{x-y}^p)$ can be written as

$$p \cdot s \cdot \lambda_x^A (-\lambda_y)^B (-\lambda_{x-y})^C$$

— where s is a positive integer; A, B, C are nonnegative integers such that $A + B + C = p$. Thus, it follows from the equality (2), together with our assumption that $e \geq p+1$ (respectively, $e \geq 2$), that

$$\underline{v}_L(p \cdot s \cdot \lambda_x^A (-\lambda_y)^B (-\lambda_{x-y})^C) \ = \ (e + p \cdot \Box) + \underline{v}_L(s) \ > \ 0.$$

This completes the proof of the claim. In light of this claim and the equality (1), we obtain that

$$\underline{v}_L((\lambda_x - \lambda_y - \lambda_{x-y})^p - (\lambda_x - \lambda_y - \lambda_{x-y})) \ > \ 0.$$

This inequality implies that $\lambda_x - \lambda_y - \lambda_{x-y}$ is an integer in L, and that

$$\lambda_x - \lambda_y - \lambda_{x-y} \bmod \mathfrak{n} \ \in \ \mathbb{F}_p$$

— where we denote by $\mathfrak{n}$ the maximal ideal of the ring of integers of L. Thus, we conclude that for any $\sigma \in G_K^p$, we have

$$\begin{aligned} & (\sigma(\lambda_x) - \lambda_x) - (\sigma(\lambda_y) - \lambda_y) - (\sigma(\lambda_{x-y}) - \lambda_{x-y}) \bmod \mathfrak{n} \\ = \ & \sigma(\lambda_x - \lambda_y - \lambda_{x-y}) - (\lambda_x - \lambda_y - \lambda_{x-y}) \bmod \mathfrak{n} \\ = \ & 0 \bmod \mathfrak{n}. \end{aligned}$$

This completes the proof of Lemma 2.3. □

Remark 2.3.1. In the case where $\mathrm{char}(K) = p$, suppose that k is an imperfect (respectively, arbitrary) field of characteristic p. Let x, $y \in \mathcal{A}$ (respectively, x, $y \in A^\times$) be elements such that $\overline{x} \neq \overline{y}$. Then, in the notation of Remark 2.2.1, it holds that $\phi_x - \phi_y = \phi_{x-y}$ (respectively, $\psi_x - \psi_y = \psi_{x-y}$). Indeed, in this case, we have

$$(\lambda_x - \lambda_y - \lambda_{x-y})^p \ = \ \lambda_x^p - \lambda_y^p - \lambda_{x-y}^p \ = \ \lambda_x - \lambda_y - \lambda_{x-y}.$$

Therefore, we conclude that $\lambda_x - \lambda_y - \lambda_{x-y}$ is contained in $\mathbb{F}_p$ [i.e, the additive group of the prime field of K] — cf. also the proof of Lemma 2.3.

Lemma 2.4. *Suppose that $e \geq p+1$, and that k is an imperfect field of characteristic p. Let $F \subseteq G_K^p$ be a topologically finitely generated closed subgroup; $k \subseteq k_1$ a purely inseparable extension of degree p. Write $K \subseteq K_F$ for the pro-p extension of K associated to F. Then there exists a Galois extension $K \subseteq K_1$ of degree p such that the residue field of K_1 is k_1, and $K_1 \subseteq K_F$. [Note that the extension $K \subseteq K_1$ is weakly unramified.]*

Proof. Let $T_1 \in k_1 \setminus k$ be an element. Write $T \stackrel{\mathrm{def}}{=} T_1^p \in k \setminus k^p$. Let $\gamma \in A^\times$ be a lifting of T. In particular, we have a map

$$\phi : \mathcal{A} \ \to \ \mathrm{Hom}(G_K^p, \mathbb{F}_p)$$

[cf. Definition 2.2]. Let $\tau : (k^p)^\times \to \mathcal{A}$ be a [set-theoretic] section of the natural surjection $\mathcal{A} \twoheadrightarrow (k^p)^\times$. Now we consider the following composite of maps:

$$(k^p)^\times \ \xrightarrow{\tau} \ \mathcal{A} \ \xrightarrow{\phi} \ \mathrm{Hom}(G_K^p, \mathbb{F}_p) \ \longrightarrow \ \mathrm{Hom}(F, \mathbb{F}_p)$$

— where the third arrow is the homomorphism induced by the inclusion $F \subseteq G_K^p$. Here, note that $(k^p)^\times$ (respectively, $\mathrm{Hom}(F, \mathbb{F}_p)$) is a(n) infinite (respectively, finite) set [cf. our assumption that F is topologically finitely generated].

In particular, the above composite is not injective. Thus, there exist distinct elements a, $b \in (k^p)^\times$ such that the image of the element

$$\phi_{\tau(a)-\tau(b)} \ = \ \phi_{\tau(a)} - \phi_{\tau(b)} \ \in \ \mathrm{Hom}(G_K^p, \mathbb{F}_p)$$

[cf. Lemma 2.3] via the natural homomorphism $\mathrm{Hom}(G_K^p, \mathbb{F}_p) \to \mathrm{Hom}(F, \mathbb{F}_p)$ is 0. Then it follows from the various definitions involved that the finite Galois extension $K \subseteq K_1$ of degree p associated to $\mathrm{Ker}(\phi_{\tau(a)-\tau(b)}) \subseteq G_K^p$ satisfies the following:

- $K_1 \subseteq K_F$.
- The residue field of K_1 is $k(T_1) = k_1$.

This completes the proof of Lemma 2.4. □

Lemma 2.5. *In the notation of Lemma 2.4, there exists a weakly unramified [pro-p] extension $K \subseteq L \subseteq K_F$ [hence, in particular, L is a Henselian discrete valuation field] such that the residue field of L is perfect.*

Proof. Let $\{t_i \ (i \in I)\}$ be a p-basis of k; for each $(i, j) \in I \times \mathbb{Z}_{\geq 1}$,

$$K_{i,j-1} \ \subseteq \ K_{i,j} \ \ (\subseteq \ K_F)$$

— where $K_{i,0} \stackrel{\mathrm{def}}{=} K$ — a weakly unramified extension of degree p such that the residue field of $K_{i,j}$ is generated by the p^j-th root of t_i over k [cf. Lemma 2.4]. Write

$$L \ \ (\subseteq \ K_F)$$

for the composite field of the fields $\{K_{i,j} \mid (i, j) \in I \times \mathbb{Z}_{\geq 1}\}$. Then one verifies immediately that L is a desired extension of K. This completes the proof of Lemma 2.5. □

Lemma 2.6. *If k is infinite, then G_K^p is not topologically finitely generated.*

Proof. By replacing K by the field obtained by adjoining a root of the equation $X^p - X - \pi^{-1} = 0$ to K, we may assume without loss of generality that $e \geq p$ [cf. the resp'd case of Lemma 2.1]. Let $\tau : k^\times \to A^\times$ be a [set-theoretic] section of the natural surjection $A^\times \twoheadrightarrow k^\times$. Now we consider the following composite of maps:

$$k^\times \ \stackrel{\tau}{\longrightarrow} \ A^\times \ \stackrel{\psi}{\longrightarrow} \ \mathrm{Hom}(G_K^p, \mathbb{F}_p)$$

[cf. Definition 2.2]. Then it follows immediately from Lemma 2.3 that this composite is injective, hence that $\mathrm{Hom}(G_K^p, \mathbb{F}_p)$ is infinite. Therefore, we conclude that G_K^p is not topologically finitely generated. This completes the proof of Lemma 2.6. □

Remark 2.6.1. An alternative proof of Lemma 2.6 is given in [the proof of] [1], Lemma 2.3, (i) from the point of view of *Kummer theory.*

3 Very elasticity and strong internal indecomposability — the maximal pro-p quotient case

In the present section, we shall continue to use the notation of §2. On the other hand, in the present section, we suppose that

$$k \text{ is an infinite field.}$$

In the present section, we prove that G_K^p is *very elastic* [cf. Theorem 3.4] and *strongly internally indecomposable* [cf. Corollary 3.7]. We also compute the *normalizers* of infinite pro-cyclic closed subgroups of G_K^p [cf. Proposition 3.5]. As an application of this computation, we prove that any nontrivial closed subgroup $H \subseteq G_K^p$ such that $H[i] = \{1\}$ $(i > 0)$ is isomorphic to $\mathbb{Z}_p$ [cf. Corollary 3.6]. Finally, we compute the *commensurators* of infinite pro-cyclic closed subgroups of G_K^p [cf. Theorem 3.8].

Lemma 3.1. *Suppose that k is perfect. Write $B_K \stackrel{\text{def}}{=} H^2(G_K, (K^{\text{sep}})^\times)$; $B_k \stackrel{\text{def}}{=} H^2(G_k, (k^{\text{sep}})^\times)$. Then the following hold:*

(i) We have a natural exact sequence

$$0 \longrightarrow B_k \longrightarrow B_K \longrightarrow \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

(ii) Let L be a finite extension of K. Write k_L for the residue field of L; $B_L \stackrel{\text{def}}{=} H^2(G_L, (K^{\text{sep}})^\times)$; $B_{k_L} \stackrel{\text{def}}{=} H^2(G_{k_L}, (k^{\text{sep}})^\times)$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_k & \longrightarrow & B_K & \longrightarrow & \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_{k_L} & \longrightarrow & B_L & \longrightarrow & \mathrm{Hom}(G_{k_L}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

— where the horizontal sequences are the exact sequences of assertion (i); the left-hand and middle vertical arrows are the natural homomorphisms; the right-hand vertical arrow is the homomorphism induced by the natural inclusion $G_{k_L} \subseteq G_k$ and the homomorphism $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ determined by the multiplication by the ramification index $e_{L/K}$.

Proof. First, we note that the natural homomorphism

$$B_K \to B_{\widehat{K}}$$

— where we denote by $\widehat{K}$ the completion of K — is an isomorphism [cf., e.g., [7], Chapter IV, Exercise 2.20, (c)]. Thus, by replacing K by $\widehat{K}$, we may assume without loss of generality that K is complete. Then Lemma 3.1 follows from [14], Chapter XII, §3, Theorem 2; [14], Chapter XII, §3, Exercise 2. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Suppose that k is perfect. Let $M \subseteq K^{\mathrm{sep}}$ be an algebraic extension of K such that the ramification index is divisible by p^{∞}. Then $H^2(G_M, \mathbb{F}_p) = \{0\}$. In particular, G_M^p is a free pro-p group.*

Proof. By replacing K (respectively, M) by $K(\zeta_p)$ (respectively, $M(\zeta_p)$), we may assume without loss of generality that $\zeta_p \in K$ [cf. Lemma 1.8]. Let L be a finite extension of K such that $L \subseteq M$. We observe that since k [hence, in particular, k^{sep}] is perfect, the p-th power map on $(k^{\mathrm{sep}})^{\times}$ induces an isomorphism

$$H^2(G_{k_L}, (k^{\mathrm{sep}})^{\times}) \xrightarrow{\sim} H^2(G_{k_L}, (k^{\mathrm{sep}})^{\times}).$$

Thus, it follows immediately from Lemma 3.1, (i), together with Hilbert's theorem 90, that

$$H^2(G_L, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Hom}(G_{k_L}, \mathbb{F}_p).$$

Therefore, since the ramification index of the extension $K \subseteq M$ is divisible by p^{∞}, we conclude from Lemma 3.1, (ii), that

$$H^2(G_M, \mathbb{F}_p) \xrightarrow{\sim} \varinjlim_{K \subseteq L \subseteq M} H^2(G_L, \mathbb{F}_p) = \{0\}$$

— where $K \subseteq L$ $(\subseteq M)$ ranges over the finite extensions of K. This completes the proof of Lemma 3.2. □

Proposition 3.3. *Let $F \subseteq G_K^p$ be a topologically finitely generated closed subgroup. Then there exists a closed subgroup $Q \subseteq G_K^p$ of infinite index such that the following conditions hold:*

- *F is a closed subgroup of Q of infinite index.*
- *Q is a free pro-p group.*

In particular, F is a free pro-p group, and G_K^p is torsion-free.

Proof. Let us first consider the collection

$$\{F_i \mid i \in I\}$$

of all open subgroups of G_K^p such that $F \subseteq F_i$. Here, we note that it holds that

$$F = \bigcap_{i \in I} F_i.$$

Write K_i for the finite extension of K associated to F_i. Now we claim that there exists an element $i \in I$ such that the ramification index of the extension $K \subseteq K_i$ is $\geq p$. Indeed, suppose that for any $i \in I$, the ramification index of the extension $K \subseteq K_i$ is 1. Then since the composite field of the fields $\{K_i \mid i \in I\}$ is a Henselian discrete valuation field of characteristic zero whose residue field is an infinite field of characteristic p, it follows from Lemma 2.6 that F is not topologically finitely generated — in contradiction to our assumption that F is topologically finitely generated. This completes the proof of the claim.

In light of this claim, by replacing K by a suitable finite extension, we may assume without loss of generality that $e \geq p$. Moreover, by applying the same argument once more, we may assume without loss of generality that $e \geq p^2$. Thus, since F is topologically finitely generated, by replacing K by a suitable weakly unramified [pro-p] extension, we may assume without loss of generality that k is perfect [cf. Lemma 2.5].

Write φ for the natural surjection $G_K^p \twoheadrightarrow G_k^p$. Then, by replacing K by the unramified [pro-p] extension associated to the closed subgroup $\varphi^{-1}(\varphi(F))$ [of G_K^p], we may assume without loss of generality that $\varphi(F) = G_k^p$. Since F is topologically finitely generated, and, moreover, G_K^p is not topologically finitely generated [cf. Lemma 2.6], there exists a normal closed subgroup $Q \subseteq G_K^p$ of infinite index such that

- F is a closed subgroup of Q of infinite index.

Write

$$K \subseteq K_Q \quad (\subseteq K^{\text{sep}})$$

for the pro-p extension of K associated to Q. Note that since $\varphi(Q) = G_k^p$, the ramification index of the extension $K \subseteq K_Q$ is divisible by p^∞. Then it follows immediately from Lemma 3.2 that

- Q is a free pro-p group.

This completes the proof of Proposition 3.3. □

Remark 3.3.1. Suppose that k is finite. Then, since G_K^p is topologically finitely generated [cf. [12], Theorem 7.4.1], the first assertion of Proposition 3.3 does not hold. However, since the kernel of the natural surjection $G_K^p \twoheadrightarrow G_k^p$ $(\cong \mathbb{Z}_p)$ is torsion-free [cf. Proposition 3.3], we conclude that G_K^p is torsion-free.

Remark 3.3.2. As we have seen in Proposition 3.3, every topologically finitely generated closed subgroup of G_K^p is a free pro-p group. Note that the absolute Galois groups of Hilbertian fields satisfy a similar property [cf. [4], Theorem A].

Theorem 3.4. *G_K^p is very elastic.*

Proof. To verify Theorem 3.4, it suffices to show that every topologically finitely generated normal closed subgroup of G_K^p is trivial [cf. Remark 1.1.1, (ii)]. This follows immediately from Lemma 1.5, (i); Proposition 3.3. This completes the proof of Theorem 3.4. □

Remark 3.4.1. Suppose that k is finite. Then G_K^p is topologically finitely generated [cf. Remark 3.3.1] and elastic [cf. [11], Theorem 1.7, (ii)].

Proposition 3.5. *Let $I \subseteq G_K^p$ be a closed subgroup that is isomorphic to $\mathbb{Z}_p$. Then the closed subgroup $N_{G_K^p}(I) \subseteq G_K^p$ is isomorphic to $\mathbb{Z}_p$.*

Proof. To verify Proposition 3.5, it suffices to show that every nontrivial topologically finitely generated closed subgroup F of $N_{G_K^p}(I)$ is isomorphic to $\mathbb{Z}_p$ [cf. Lemma 1.7]. Note that, by replacing F by IF [which is topologically finitely generated], we may assume without loss of generality that $I \subseteq F$. Then the fact that $F \cong \mathbb{Z}_p$ follows immediately from Lemma 1.5, (ii); Proposition 3.3. This completes the proof of Proposition 3.5. □

Remark 3.5.1. Proposition 3.5 also holds in the case where k is finite. To verify this fact, in the notation of Proposition 3.5, we claim the following:

$$N_{G_K^p}(I) \text{ is a closed subgroup of } G_K^p \text{ of infinite index.}$$

Indeed, suppose that $N_{G_K^p}(I)$ is an open subgroup of G_K^p. Then, by replacing K by its finite extension, we may assume without loss of generality that I is normal in G_K^p. In particular, it follows from the elasticity of G_K^p [cf. Remark 3.4.1] that G_K^p is almost pro-cyclic — a contradiction [cf. [12], Theorem 7.5.11]. This completes the proof of the claim.

In light of this claim, together with [11], Proposition 1.6, (iii), we conclude that $N_{G_K^p}(I)$ is a free pro-p group. Then the fact that $N_{G_K^p}(I) \cong \mathbb{Z}_p$ follows immediately from Lemma 1.5, (ii).

Corollary 3.6. *Let i be a positive integer; $H \subseteq G_K^p$ a nontrivial closed subgroup such that $H[i] = \{1\}$. Then H is isomorphic to $\mathbb{Z}_p$.*

Proof. We verify Corollary 3.6 by induction on i. First, we consider the case where $i = 1$ [i.e., the case where H is abelian]. Let $I \subseteq H$ be a closed subgroup that is isomorphic to $\mathbb{Z}_p$ [cf. Proposition 3.3]. Then we have

$$H \ \subseteq\ Z_{G_K^p}(I) \ \subseteq\ N_{G_K^p}(I) \ \cong\ \mathbb{Z}_p$$

[cf. Proposition 3.5]. Thus, we conclude that $H \cong \mathbb{Z}_p$. This completes the proof of the case where $i = 1$.

Now suppose that $i \geq 2$, and that the induction hypothesis is in force. In particular, if $H[i-1] = \{1\}$, then it follows from our induction hypothesis that $H \cong \mathbb{Z}_p$. Thus, we may assume without loss of generality that $H[i-1] \neq \{1\}$. Then since $H[i-1] \cong \mathbb{Z}_p$ [cf. the first paragraph], it follows from Proposition 3.5 that

$$H \ \subseteq\ N_{G_K^p}(H[i-1]) \ \cong\ \mathbb{Z}_p,$$

hence that $H \cong \mathbb{Z}_p$. This completes the proof of Corollary 3.6. □

Remark 3.6.1. It follows from Corollary 3.6 that every abelian closed subgroup of G_K^p is pro-cyclic. It is well-known that $G_{\mathbb{Q}}$ also satisfies this property [cf. [4], Theorem C]. However, there exists a Hilbertian field [e.g., $\mathbb{Q}(t)$] whose absolute Galois group does not satisfy this property [cf. [4], Theorem H].

Corollary 3.7. *G_K^p is strongly internally indecomposable. [Note that this implies the slimness of G_K^p — cf. Remark 1.1.2].*

Proof. To verify Corollary 3.7, it suffices to show that G_K^p is internally indecomposable. Let N be a nontrivial normal closed subgroup of G_K^p. Write C for the centralizer of N in G_K^p. Then since $N \cap C$ is an abelian [normal] closed subgroup of G_K^p, it follows from Corollary 3.6 that

$$N \cap C \ = \ \{1\} \quad \text{or} \quad N \cap C \ \cong \ \mathbb{Z}_p.$$

Here, we observe that if $N \cap C \cong \mathbb{Z}_p$, then this contradicts the very elasticity of G_K^p [cf. Theorem 3.4]. Thus, we conclude that $N \cap C = \{1\}$.

Now suppose that $C \neq \{1\}$. Let $x \in N$, $y \in C$ be nontrivial elements. Write $T_{\{x,y\}}$ (respectively, T_x; T_y) for the closed subgroup of G_K^p topologically generated by x and y (respectively, x; y). Then it follows from the various definitions involved, together with the equality $N \cap C = \{1\}$, that

$$T_{\{x,y\}} \ = \ T_x \times T_y.$$

On the other hand, since $T_{\{x,y\}} \cong \mathbb{Z}_p$ [cf. Corollary 3.6], this equality contradicts the [easily verified] fact that $\mathbb{Z}_p$ does not have nontrivial direct product decompositions. Thus, we conclude that $C = \{1\}$. This completes the proof of Corollary 3.7. □

Theorem 3.8. *Let $I \subseteq G_K^p$ be a closed subgroup that is isomorphic to $\mathbb{Z}_p$. Then the subgroup $C_{G_K^p}(I) \subseteq G_K^p$ is closed, and, moreover, isomorphic to $\mathbb{Z}_p$.*

Proof. First, we claim the following:

$$(I \ \subseteq) \ \ C_{G_K^p}(I) \ \subseteq \ \bigcup_{n \geq 0} N_{G_K^p}(p^n I).$$

Indeed, let $g \in C_{G_K^p}(I)$ be an element. Write φ for the inner automorphism of G_K^p determined by $* \mapsto g \cdot * \cdot g^{-1}$;

$$I_1 \ \stackrel{\text{def}}{=} \ I \cap \varphi(I) \ \neq \ \{1\}; \quad I_2 \ \stackrel{\text{def}}{=} \ I \cap \varphi^{-1}(I) \ \neq \ \{1\}.$$

Here, we note that since $I \cong \mathbb{Z}_p$, it holds that

$$I_1 \ \subseteq \ I_2 \quad \text{or} \quad I_2 \ \subseteq \ I_1.$$

Thus, we may assume without loss of generality that $I_2 \subseteq I_1$. In particular, since $I_1 = \varphi(I_2)$, we obtain a sequence of [nontrivial] abelian closed subgroups of G_K^p as follows:

$$(\{1\} \neq) \ I_2 \ \subseteq \ \varphi(I_2) \ \subseteq \ \varphi^2(I_2) \ \subseteq \ \cdots \ (\subseteq \ G_K^p).$$

Now write

$$J \stackrel{\text{def}}{=} \bigcup_{m \geq 0} \varphi^m(I_2) \ \ (\subseteq \ G_K^p).$$

Then since $J \neq \{1\}$ is an abelian subgroup of G_K^p, it follows from Corollary 3.6 that its closure $\overline{J}$ is isomorphic to $\mathbb{Z}_p$. Thus, since $[\overline{J} : I_2] < +\infty$, there exists a nonnegative integer m such that $\varphi^m(I_2) = \varphi^{m+1}(I_2)$. Therefore, we conclude that $\varphi(I_2) = I_2$, hence that

$$g \ \in \ N_{G_K^p}(I_2).$$

This completes the proof of the claim.

Next, we observe that we have a sequence of [nontrivial] abelian closed subgroups of G_K^p as follows:

$$(I \ \subseteq) \ N_{G_K^p}(I) \ \subseteq \ N_{G_K^p}(pI) \ \subseteq \ N_{G_K^p}(p^2 I) \ \subseteq \ \cdots \ (\subseteq \ G_K^p)$$

[cf. Proposition 3.5]. Now write

$$U \stackrel{\text{def}}{=} \bigcup_{n \geq 0} N_{G_K^p}(p^n I) \ \ (\subseteq \ G_K^p).$$

Then since $U \neq \{1\}$ is an abelian subgroup of G_K^p, it follows from Corollary 3.6 that its closure $\overline{U}$ is isomorphic to $\mathbb{Z}_p$. Thus, since $[\overline{U} : I] < +\infty$, in light of the above claim, we conclude that $C_{G_K^p}(I)$ is a closed subgroup of G_K^p, and, moreover, isomorphic to $\mathbb{Z}_p$. This completes the proof of Theorem 3.8. □

Remark 3.8.1. Corollaries 3.6, 3.7, Theorem 3.8 also hold in the case where k is finite [cf. Remarks 3.3.1, 3.4.1, 3.5.1; the proofs of Corollaries 3.6, 3.7, Theorem 3.8]. We leave the routine details to the reader.

Remark 3.8.2. In the notation of Theorem 3.8, suppose further that I is saturated in G_K^p, i.e., I satisfies the following condition:

> If an element $g \in G_K^p$ satisfies $g^n \in I$, where $n \in \mathbb{Z}_{\geq 1}$ is an integer, then it holds that $g \in I$.

Then, as is easily verified, we have $C_{G_K^p}(I) = I$.

Remark 3.8.3. As we have seen in the present section, there are various similarities between G_K^p and nonabelian free pro-p groups [from the point of view of group-theoretic properties].

4 Very elasticity and strong internal indecomposability — the almost pro-p-maximal quotient case

In the present section, we shall continue to use the notation of §2. On the other hand, in the present section, we also consider the case where

$$\mathrm{char}(K) \;=\; p \;> 0.$$

However, if $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) = p$), then we suppose that

k is an infinite (respectively, arbitrary) field of characteristic p.

In the present section, we first verify a technical lemma [cf. Lemma 4.1] as an application of the theory of §2. Then, by applying this lemma, we prove that any almost pro-p-maximal quotient of G_K is *slim* [cf. Theorem 4.3]. This allows one to generalize [8], Theorem C; [9], Theorem A, (i), to the case where $\zeta_p \notin K$ [cf. Theorem 4.4]. In the case where $\mathrm{char}(K) = p$, our proof of the slimness portion may be regarded as an *easier alternative proof* — in which we do not apply the [highly nontrivial] theory of *fields of norms* — of [8], Theorem 2.10.

Lemma 4.1. *Let $l \neq p$ be a prime number; L a cyclic extension of K of degree l. Then the natural surjection $G_L^p \twoheadrightarrow G_K^p$ is not bijective.*

Proof. In the case where $\mathrm{char}(K) = 0$, by replacing K and L by the field K' obtained by adjoining a root of the equation $X^p - X - \pi^{-1} = 0$ to K and $K'L$, respectively, we may assume without loss of generality that $e \geq p$ [cf. the resp'd case of Lemma 2.1]. Write Q for the profinite group obtained by forming the quotient of G_K by the kernel of the natural surjection $G_L \twoheadrightarrow G_L^p$. In the following, if $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) = p$), then, by abuse of notation, we denote by $\mathbb{F}_p$ [the additive group of] the prime field of k (respectively, K). To verify Lemma 4.1, it suffices to show that the natural injection

$$(H^1(Q^p, \mathbb{F}_p) \;\overset{\sim}{\to})\;\; H^1(Q, \mathbb{F}_p) \;\overset{\sim}{\to}\; H^1(G_L^p, \mathbb{F}_p)^{\mathrm{Gal}(L/K)} \;\hookrightarrow\; H^1(G_L^p, \mathbb{F}_p)$$

is not bijective. In particular, to verify Lemma 4.1, in the notation of Definition 2.2 (respectively, Remark 2.2.1), it suffices to show that there exists $x \in B^\times$ — where we denote by B the ring of integers of L — such that

$$\tau \cdot \psi_x \;\neq\; \psi_x \;\; (\in\; \mathrm{Hom}(G_L^p, \mathbb{F}_p) \;=\; H^1(G_L^p, \mathbb{F}_p))$$

— where τ is a generator of $\mathrm{Gal}(L/K)$.

First, we consider the case where L is a totally ramified extension of K. Let π_L be a uniformizer of L. In particular, we have

$$\pi \;=\; \pi_L^l \cdot u$$

— where $u \in B$ is a unit. Here, we note that [since the residue field extension is trivial], by replacing π and u by $\pi \cdot u'$ and $u \cdot u'$, where $u' \in A^\times$ is a suitable unit, respectively, we may assume without loss of generality that the image of u in the residue field of B is $\overline{1}$. Then, by applying "Hensel's Lemma" to the polynomial $g(Y) = Y^l - u$ [cf. our assumption that $l \neq p$], we obtain a unit $v \in B^\times$ such that $u = v^l$. Thus, by replacing π_L by $\pi_L \cdot v$, we may assume without loss of generality that

$$\pi \;=\; \pi_L^l.$$

In particular,

$$\omega_l \;\stackrel{\text{def}}{=}\; \tau(\pi_L^{-1}) \cdot \pi_L \;\;(\in\; B^\times)$$

is a primitive l-th root of unity. Now we set

$$x \;\stackrel{\text{def}}{=}\; 1.$$

Then we claim the following:

Claim 4.1.A: It holds that $\tau \cdot \psi_x = \psi_{\omega_l}$.

Indeed, let $\overline{\tau} \in Q$ be a lifting of $\tau \in \mathrm{Gal}(L/K)$; λ_x a root of the equation

$$X^p - X - \pi_L^{-1} \;=\; 0.$$

Write $\mathcal{O}_x$ for the ring of integers of $L(\lambda_x)$; $\mathfrak{m}_x$ for the maximal ideal of $\mathcal{O}_x$. Then, in the case where $\mathrm{char}(K) = 0$ (respectively, $\mathrm{char}(K) = p$), we have

$$\tau \cdot \psi_x : G_L^p \;\to\; \mathbb{F}_p; \;\; \sigma \;\mapsto\; (\overline{\tau}^{-1} \circ \sigma \circ \overline{\tau}(\lambda_x) - \lambda_x) \bmod \mathfrak{m}_x$$

$$(\text{respectively,} \;\; \tau \cdot \psi_x : G_L^p \;\to\; \mathbb{F}_p; \;\; \sigma \;\mapsto\; \overline{\tau}^{-1} \circ \sigma \circ \overline{\tau}(\lambda_x) - \lambda_x).$$

On the other hand, since $\overline{\tau}(\lambda_x)$ is a root of the equation

$$X^p - X - \omega_l \cdot \pi_L^{-1} \;=\; 0,$$

we have

$$\psi_{\omega_l} : G_L^p \;\to\; \mathbb{F}_p; \;\; \sigma \;\mapsto\; \overline{\tau}(\overline{\tau}^{-1} \circ \sigma \circ \overline{\tau}(\lambda_x) - \lambda_x) \bmod \overline{\tau}(\mathfrak{m}_x)$$

$$(\text{respectively,} \;\; \psi_{\omega_l} : G_L^p \;\to\; \mathbb{F}_p; \;\; \sigma \;\mapsto\; \overline{\tau}(\overline{\tau}^{-1} \circ \sigma \circ \overline{\tau}(\lambda_x) - \lambda_x))$$

— where we note that "$\overline{\tau}(\mathfrak{m}_x)$" is the maximal ideal of the Henselian discrete valuation ring $\overline{\tau}(\mathcal{O}_x)$ [which is the ring of integers of $\overline{\tau}(L(\lambda_x)) = L(\overline{\tau}(\lambda_x))$]. Thus, we conclude that $\tau \cdot \psi_x = \psi_{\omega_l}$. This completes the proof of Claim 4.1.A.

In light of Claim 4.1.A, the fact that $\tau \cdot \psi_x \neq \psi_x$ follows immediately from Lemma 2.3 (respectively, Remark 2.3.1). This completes the proof in the case where L is a totally ramified extension of K.

Next, we consider the case where L is an unramified extension of K. [In this case, $\pi \in K$ is a uniformizer of L.] Let $u \in B^\times$ be a unit such that

$$\tau(u) - u \;\in\; B^\times.$$

[Indeed, let r be an element of the residue field of B that is not contained in k. Then any lifting $\in B^\times$ of r is such an element.] Now we set

$$x \stackrel{\text{def}}{=} u.$$

Then we claim the following:

Claim 4.1.B: It holds that $\tau \cdot \psi_x = \psi_{\tau(u)}$.

Indeed, Claim 4.1.B follows from a similar argument to the argument applied in the proof of Claim 4.1.A. [On the other hand, in this proof, instead of the above two equations

$$X^p - X - \pi_L^{-1} \ = \ 0 \quad \text{and} \quad X^p - X - \omega_l \cdot \pi_L^{-1} \ = \ 0,$$

we consider the equations

$$X^p - X - u \cdot \pi^{-1} \ = \ 0 \quad \text{and} \quad X^p - X - \tau(u) \cdot \pi^{-1} \ = \ 0,$$

respectively.]

In light of Claim 4.1.B, the fact that $\tau \cdot \psi_x \neq \psi_x$ follows immediately from Lemma 2.3 (respectively, Remark 2.3.1). This completes the proof in the case where L is an unramified extension of K, hence also of Lemma 4.1. □

Proposition 4.2. *If* $\mathrm{char}(K) = p$, *then* G_K^p *is a free pro-p group of infinite rank. In particular,* G_K^p *is very elastic [cf. Lemma 1.5, (i)] and strongly internally indecomposable [cf. [9], Proposition 1.5]. [Note that this implies the slimness of* G_K^p *— cf. Remark 1.1.2].*

Proof. By replacing K by its completion, we may assume without loss of generality that K is complete [cf. [8], Lemma 3.1]. In particular, it follows from Cohen's structure theorem [cf. [5], Chapter I, Theorem 5.5A] that K is isomorphic to $k((t))$. Then Proposition 4.2 follows immediately from [12], Proposition 6.1.7. □

Theorem 4.3. *Any almost pro-p-maximal quotient of* G_K *is slim.*

Proof. Theorem 4.3 follows immediately from Proposition 1.4; Corollary 3.7; Lemma 4.1; Proposition 4.2. □

Remark 4.3.1. In the case where $\mathrm{char}(K) = p$, our proof of Theorem 4.3 may be regarded as an easier alternative proof — in which we do not apply the [highly nontrivial] theory of fields of norms — of [8], Theorem 2.10.

Theorem 4.4. *Any almost pro-p-maximal quotient of G_K is very elastic and strongly internally indecomposable.*

Proof. Write $q : G_K \twoheadrightarrow Q$ for the natural surjection. Note that there exists a normal open subgroup $N \subseteq G_K$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$. Then since N^p is very elastic [cf. Theorem 3.4; Proposition 4.2] and strongly internally indecomposable [cf. Corollary 3.7; Proposition 4.2], Theorem 4.4 follows immediately from Theorem 4.3; [8], Lemma 1.4; [9], Proposition 1.6. □

Remark 4.4.1. Suppose that $\mathrm{char}(K) = 0$ and k is finite. Then any almost pro-p-maximal quotient of G_K is topologically finitely generated [cf. [12], Theorem 7.4.1], elastic [cf. [11], Theorem 1.7, (ii)], and strongly internally indecomposable [cf. Remark 3.8.1; the proof of Theorem 4.4; [11], Proposition 1.6, (i)]. We leave the routine details to the reader.

Remark 4.4.2. Theorem 4.4 may be regarded as a generalization of [8], Theorem C; [9], Theorem A, (i).

5 Application to pro-*p* absolute anabelian geometry over mixed characteristic Henselian discrete valuation fields

In the present section, let p be a prime number.

In the present section, we verify the *semi-absoluteness* of isomorphisms between the pro-p étale fundamental groups of smooth varieties over certain classes of fields of characteristic 0 [cf. Theorem 5.2; [11], Definition 2.4, (ii)]. Our result may be regarded as a generalization of [15], Theorem A, (i).

Proposition 5.1. *Let K be a Hilbertian field. Then G_K^p is very elastic.*

Proof. Since K is Hilbertian, for any integer $n \in \mathbb{Z}_{\geq 1}$, there exists an epimorphism [in the category of profinite groups]

$$G_K^p \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^n$$

[cf. [3], Corollary 16.3.6]. In particular, we conclude that G_K^p is not topologically finitely generated. Then the [very] elasticity of G_K^p follows from a similar argument to the argument applied in the proof of [15], Proposition 4.3. □

Theorem 5.2. *Let K, K' be fields of characteristic 0; X, X' smooth varieties [i.e., smooth, of finite type, separated, and geometrically integral schemes] over K, K', respectively;*

$$\alpha : \Pi_X^p \overset{\sim}{\to} \Pi_{X'}^p$$

an isomorphism of profinite groups. Suppose that

- *K is either a Henselian discrete valuation field of characteristic 0 such that the residue field is an infinite field of characteristic p or a Hilbertian field;*
- *K' is either a Henselian discrete valuation field of characteristic 0 such that the residue field is a field of characteristic p or a Hilbertian field.*

Then α induces an isomorphism $G_K^p \overset{\sim}{\to} G_{K'}^p$ that fits into a commutative diagram

$$\begin{array}{ccc} \Pi_X^p & \xrightarrow[\alpha]{\sim} & \Pi_{X'}^p \\ \downarrow & & \downarrow \\ G_K^p & \xrightarrow{\sim} & G_{K'}^p \end{array}$$

— where the vertical arrows denote the natural surjections [determined up to composition with an inner automorphism] induced by the structure morphisms of the smooth varieties X, X'.

Proof. In light of Theorem 3.4 and Proposition 5.1, Theorem 5.2 follows from a similar argument to the argument applied in the proof of [15], Theorem 4.5. □

Remark 5.2.1. It is natural to pose the following question:

> Question: In Theorem 5.2, when K is a Henselian discrete valuation field, can the assumption that the residue field of K is infinite be dropped?

However, at the time of writing, the authors do not know whether this question is affirmative or not [cf. [15], Theorem A, (ii)].

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Updated versions of [8], [9], [15] may be found at the following URL:
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(Arata Minamide) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: minamide@kurims.kyoto-u.ac.jp

(Shota Tsujimura) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: stsuji@kurims.kyoto-u.ac.jp