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Category-theoretic Reconstruction of Schemes from Categories of Reduced Schemes

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CATEGORY-THEORETIC RECONSTRUCTION OF SCHEMES FROM CATEGORIES OF REDUCED SCHEMES

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ABSTRACT. Let S be a locally Noetherian normal scheme and $\blacklozenge/S$ a set of properties of S-schemes. Then we shall write $\mathsf{Sch}_{\blacklozenge/S}$ for the full subcategory of the category of S-schemes $\mathsf{Sch}_{/S}$ determined by the objects $X \in \mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$. In the present paper, we shall mainly be concerned with the properties "reduced", "quasi-compact over S", "quasi-separated over S", and "separated over S". We give a functorial category-theoretic algorithm for reconstructing S from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$. This result is analogous to a result of Mochizuki [Mzk04] and may be regarded as a partial generalization of a result of de Bruyn [vDdB19] in the case where S is a locally Noetherian normal scheme.

CONTENTS

INTRODUCTION

Let $\mathbf{U}$ and $\mathbf{V}$ be Grothendieck universes such that $\mathbf{U} \in \mathbf{V}$. Let S be a $\mathbf{U}$-small scheme. In the following, we shall use the term "scheme" to refer to a $\mathbf{U}$-small scheme. Let $\blacklozenge/S$ be a ($\mathbf{V}$-small) set of properties of S-schemes. We shall write

$$\mathsf{Sch}_{\blacklozenge/S}$$

for the full subcategory of the ($\mathbf{V}$-small) category of S-schemes $\mathsf{Sch}_{/S}$ determined by the objects $X \in \mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$. In the present paper, we shall mainly be concerned with the properties

"red", "qcpt", "qsep", "sep"

of S-schemes, i.e., "reduced", "quasi-compact over S", "quasi-separated over S", and "separated over S". If $\blacklozenge/S = \varnothing$, then we simply write $\mathsf{Sch}_{/S}$ for $\mathsf{Sch}_{\blacklozenge/S}$.

In the present paper, we consider the problem of reconstructing the scheme S from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$. In [Mzk04], Mochizuki

gave a solution to this problem in the case where S is locally Noetherian, and $\blacklozenge/S =$ "of finite type over S". In [vDdB19], van Dobben de Bruyn gave a solution to this problem in the case where S is an arbitrary scheme, and $\blacklozenge/S = \varnothing$. The techniques applied in [vDdB19] make essential use of the existence of *non-reduced schemes* in $\mathsf{Sch}_{/S}$. By contrast, in the present paper, we focus on the problem of reconstructing the scheme S from categories of S-schemes that only contain *reduced S-schemes*, hence rely on techniques that differ essentially from the techniques applied in [vDdB19].

If X, Y are objects of a (**V**-small) category $\mathcal{C}$, then we shall write $\mathrm{Isom}(X, Y)$ for the set of isomorphisms from X to Y. By a slight abuse of notation, we shall also regard this set as a discrete category. If $\mathcal{C}, \mathcal{D}$ are (**V**-small) categories, then we shall write $\mathbf{Isom}(\mathcal{C}, \mathcal{D})$ for the (**V**-small) category of equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ and natural isomorphisms. If $\mathcal{C}$ is a (**V**-small) category, and X is an object of $\mathcal{C}$, then we shall write $\mathcal{C}_{/X}$ for the slice category of objects and morphisms equipped with a structure morphism to X. If $f : X \to Y$ is a morphism in a (**V**-small) category $\mathcal{C}$ which is closed under fiber products, then we shall write $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ for the functor induced by the operation of base-change, via f, from X to Y.

Our main result is the following:

Theorem A.

(i) *(Reconstruction 6.22)*
   *Let S be a locally Noetherian normal (**U**-small) scheme,*

   $$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

   a [possibly empty] subset such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$. Then the following may be constructed category-theoretically from $\mathsf{Sch}_{\blacklozenge/S}$ by means of algorithms that are independent of the choice of the subset $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$:
   (a) *for each object T of $\mathsf{Sch}_{\blacklozenge/S}$, a (**V**-small) scheme $T_\mathbf{V}$ and an isomorphism of (**V**-small) schemes $\eta_T : T_\mathbf{V} \xrightarrow{\sim} T$ (where we note that a **U**-small scheme is, in particular, **V**-small), and*
   (b) *for each morphism $f : T_1 \to T_2$ of $\mathsf{Sch}_{\blacklozenge/S}$, a morphism of (**V**-small) schemes $f_\mathbf{V} : T_{1,\mathbf{V}} \to T_{2,\mathbf{V}}$ such that $\eta_{T_2} \circ f_\mathbf{V} = f \circ \eta_{T_1}$.*
(ii) *(Corollary 4.11)*
   *Let S, T be quasi-separated (**U**-small) schemes,*

   $$\blacklozenge, \Diamond \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

   *[possibly empty] subsets such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$, $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \Diamond$. Then if the (**V**-small) categories $\mathsf{Sch}_{\blacklozenge/S}$, $\mathsf{Sch}_{\Diamond/T}$ are equivalent, then $\blacklozenge = \Diamond$.*
(iii) *(Theorem 6.28)*
   *Let S, T be locally Noetherian normal (**U**-small) schemes,*

   $$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$$

   a [possibly empty] subset such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$. Then the natural functor

   $$\mathrm{Isom}(S, T) \to \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S})$$
   $$f \mapsto f^*$$

   *is an equivalence of (**V**-small) categories.*

Our proof of Theorem A proceeds by giving category-theoretic characterizations of various properties of schemes and morphisms of schemes as follows:

- In Section 1, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is isomorphic to the spectrum of a field and use this characterization to give a functorial category-theoretic algorithm for reconstructing the underlying set of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$.
- In Section 2, we consider various scheme-theoretic properties of regular monomorphisms in $\mathsf{Sch}_{\blacklozenge/S}$. These properties will play an important role in Section 3 and Section 4.
- In Section 3, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is isomorphic to the spectrum of a local domain. This characterization is closely related to the category-theoretic characterization of specialization and generization relations between points of a scheme and also yields a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is irreducible, as well as the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is isomorphic to the spectrum of a local ring.
- In Section 4, by using the method of [vDdB19, Lemma 3.7], we give a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is a closed immersion. We then use this characterization to give a functorial category-theoretic algorithm for reconstructing the underlying topological space of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$.
- In Section 5, we give a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is locally of finite presentation. This characterization also yields a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is proper and of finite presentation. These properties form the foundation for the discussion in Section 6.
- In Section 6, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying S-scheme is isomorphic to the projective line $\mathbb{P}^1_S$. We then use this characterization to give a functorial category-theoretic algorithm for reconstructing the underlying schemes of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$. In addition, we discuss some rigidity results related to the various reconstruction algorithms developed in the present paper.

Finally, we remark that, in the above discussion, as well as throughout the present paper, *category-theoretic reconstruction algorithms* are to be understood as algorithms that are *independent* of the choice of the subset $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$.

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Notations and Conventions.

Let $\mathbf{U}, \mathbf{V}$ be Grothendieck universes such that $\mathbf{U} \in \mathbf{V}$. The notation $\mathbb{Z}$ will be used to denote the ring of integers. We shall use the notation

$$\mathsf{Set}_{\mathbf{V}}, \ \mathsf{Top}_{\mathbf{V}}, \ \mathsf{Ring}_{\mathbf{V}}, \ \mathsf{Sch}_{\mathbf{V}}$$

to denote, respectively, the categories of $\mathbf{V}$-small sets, $\mathbf{V}$-small topological spaces, $\mathbf{V}$-small rings, and $\mathbf{V}$-small schemes. We shall use similar notation with the subscript $\mathbf{V}$ omitted to denote the corresponding category of $\mathbf{U}$-small objects. In the following, we shall use the term "scheme" to refer to a $\mathbf{U}$-small scheme.

Let $\mathcal{C}$ be a ($\mathbf{V}$-small) category. We shall write $\mathcal{C}^{\mathrm{op}}$ for the opposite category associated to $\mathcal{C}$. If X is an object of $\mathcal{C}$, then we shall write $\mathcal{C}_{/X}$ for the slice category of objects and morphisms equipped with a structure morphism to X.

Let $\blacklozenge/S$ be a ($\mathbf{V}$-small) set of properties of S-schemes. Then we shall write

$$\mathsf{Sch}_{\blacklozenge/S}$$

for the full subcategory of the ($\mathbf{V}$-small) category of S-schemes $\mathsf{Sch}_{/S}$ determined by the objects X of $\mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$. In the present paper, we shall mainly be concerned with the properties

$$\text{"red", "qcpt", "qsep", "sep"}$$

of S-schemes, i.e., "reduced", "quasi-compact over S", "quasi-separated over S", and "separated over S". Thus, if $\blacklozenge/S = \varnothing$, then $\mathsf{Sch}_{/S} = \mathsf{Sch}_{\blacklozenge/S}$. In the remainder of the present paper,

we *fix* subsets $\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$, $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$.

We shall write $\lim^{\blacklozenge}, \mathrm{colim}^{\blacklozenge}, \times^{\blacklozenge}, \coprod^{\blacklozenge}$ for the (inverse) limit, colimit, fiber product, and push-out in $\mathsf{Sch}_{\blacklozenge/S}$. We shall write $\lim, \mathrm{colim}, \times, \coprod$ for the (inverse) limit, colimit, fiber product, and push-out in Sch. By a slight abuse of notation, we shall use the notation $\varnothing$ to denote the initial object (i.e., the empty scheme) of $\mathsf{Sch}_{\blacklozenge/S}$ or $\mathsf{Sch}_{/S}$.

Let S be a scheme, $Z' \to Z$ a morphism in $\mathsf{Sch}_{\blacklozenge/S}$, and $f : X \to Y$ a morphism over Z in $\mathsf{Sch}_{\blacklozenge/S}$. We shall write $X_{\blacklozenge, Z'}$ for the base-change (if it exists) of X by the morphism $Z' \to Z$ in $\mathsf{Sch}_{\blacklozenge/S}$. We shall write $f_{\blacklozenge, Z'}$ for the base-change (if it exists) of f by the morphism $Z' \to Z$ in $\mathsf{Sch}_{\blacklozenge/S}$. Let $F \in \{X, f\}$. If $Z' = \mathrm{Spec}(A)$, then we shall also write $F_{\blacklozenge, A}$ for $F_{\blacklozenge, Z'}$. If $Z' = \mathrm{Spec}(k(z))$, where $k(z)$ denotes the residue field of $z \in Z$, then we shall also write $F_{\blacklozenge, z}$ for $F_{\blacklozenge, Z'}$. If $\blacklozenge = \varnothing$, then we shall simply write $F_{Z'}$ for $F_{\blacklozenge, Z'}$.

If $f : X \to Y$ is a morphism of schemes, and $Z = \mathrm{Spec}(k(x))$ is the spectrum of the residue field $k(x)$ of X at a point $x \in X$, we shall write $f|_x$ for the composite of the natural morphism $Z \to X$ with f. If $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a morphism of affine schemes, then we shall write $f^{\#} : B \to A$ for the ring homomorphism induced by f. By a slight abuse of notation, if $f^{\#} : B \to A$ is a ring homomorphism, then we shall use the notation f to denote the corresponding morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$.

Let X be a scheme. Then we shall write $\mathcal{O}_X$ for the structure sheaf of X. We shall write $|X|$ for the underlying topological space of X. If the underlying set of $|X|$ is of cardinality 1, then we shall write $*_X$ for the unique element of the underlying set of X. If $F \subset |X|$ is a closed subset, then we shall write F_{red} for the reduced induced closed subscheme determined by F and $X_{\mathrm{red}} :\overset{\mathrm{def}}{=} |X|_{\mathrm{red}}$. If $f : X \to Y$ is a morphism of schemes, then we shall write $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ for the morphism induced by f. For points $x, y \in X$, if x is a specialization of y (where we recall that this includes the case where $x = y$), then we shall write $y \rightsquigarrow x$.

Let S be a scheme; $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ a subset such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge$; I, J (**U**-small) sets; $\{X_i\}_{i \in I}$ a (**U**-small) family of objects of $\mathsf{Sch}_{\blacklozenge/S}$ parametrized by I; $\{f_j\}_{j \in J}$ a (**U**-small) family of morphisms in $\mathsf{Sch}_{\blacklozenge/S}$ parametrized by J; $\mathcal{P}$ a property of schemes; $\mathcal{Q}$ a property of (**U**-small) families of schemes parametrized by I and (**U**-small) families of morphisms of schemes parametrized by J. Assume that S satisfies property $\mathcal{P}$. Then we shall say that

the property that
$$(\{X_i\}_{i \in I}, \{f_j\}_{j \in J})$$
satisfies $\mathcal{Q}$ may be *characterized category-theoretically* from the data
$$(\mathsf{Sch}_{\blacklozenge/S}, \{X_i\}_{i \in I}, \{f_j\}_{j \in J})$$
if for any (**U**-small) family of objects $\{Y_i\}_{i \in I}$ of $\mathsf{Sch}_{\blacklozenge/S}$ parametrized by I, any (**U**-small) family of morphisms $\{g_j\}_{j \in J}$ of $\mathsf{Sch}_{\blacklozenge/S}$ parametrized by J, any scheme T which satisfies property $\mathcal{P}$, any subset $\lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$, and any equivalence $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$,

$(\{Y_i\}_{i \in I}, \{f_j\}_{j \in J})$ satisfies $\mathcal{Q} \iff (\{F(Y_i)\}_{i \in I}, \{F(g_j)\}_{j \in J})$ satisfies $\mathcal{Q}$.

1. The Underlying Set

In this section, we give a functorial category-theoretic algorithm for reconstructing the underlying set of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$.

First, we note the following properties of fiber products in $\mathsf{Sch}_{\blacklozenge/S}$:

Lemma 1.1. *Let S be a scheme. Let $Y \to X$ and $Z \to X$ be morphisms in $\mathsf{Sch}_{\blacklozenge/S}$.*

(i) If $\mathrm{qcpt}, \mathrm{red} \notin \blacklozenge$, then $Y \times_X Z$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the fiber product $Y \times_X^{\blacklozenge} Z$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $Y \times_X Z$.

(ii) If $\mathrm{qcpt} \notin \blacklozenge$ and $\mathrm{red} \in \blacklozenge$, then $(Y \times_X Z)_{\mathrm{red}}$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the fiber product $Y \times_X^{\blacklozenge} Z$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $(Y \times_X Z)_{\mathrm{red}}$.

Assume that either $Y \to X$ or $Z \to X$ is quasi-compact. Then the following hold:

(iii) If $\mathrm{red} \notin \blacklozenge$, then $Y \times_X Z$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the fiber product $Y \times_X^{\blacklozenge} Z$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $Y \times_X Z$.

(iv) If $\mathrm{red} \in \blacklozenge$, then $(Y \times_X Z)_{\mathrm{red}}$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the fiber product $Y \times_X^{\blacklozenge} Z$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $(Y \times_X Z)_{\mathrm{red}}$.

Proof. If $\mathrm{qcpt} \in \blacklozenge$ and either $Y \to X$ or $Z \to X$ is quasi-compact, then it is immediate that $Y \times_X Z$ is quasi-compact over S. Recall that the natural morphism $Y \times_X Z \to Y \times_S Z$ is an immersion, hence separated. Thus, if $\mathrm{sep} \in \blacklozenge$ (respectively, $\mathrm{qsep} \in \blacklozenge$), then since $Y \times_S Z$ is separated (respectively, quasi-separated) over S, $Y \times_X Z$ is separated (respectively, quasi-separated) over S.

Thus, if $\mathrm{red} \notin \blacklozenge$, then the fiber product $Y \times_X Z$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$, which implies that $Y \times_X^{\blacklozenge} Z$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $Y \times_X Z$. This completes the proof of (i) and (iii).

Suppose that $\mathrm{red} \in \blacklozenge$. Then any morphism $f : W \to Y \times_X Z$ from a reduced scheme W factors uniquely through the closed subscheme $(Y \times_X Z)_{\mathrm{red}} \subset Y \times_X Z$. Thus, since $(Y \times_X Z)_{\mathrm{red}}$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$, we conclude that $Y \times_X^{\blacklozenge} Z$ exists in

$\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $(Y \times_X Z)_{\mathrm{red}}$. This completes the proof of (ii) and (iv). $\qquad\square$

Next, we consider a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying set is of cardinality 1.

Lemma 1.2. *Let S be a scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then $|X|$ is of cardinality 1 if and only if the following condition does **not** hold:*

(†) *there exist objects $Y, Z \neq \varnothing$ such that $Y \times_X^{\blacklozenge} Z = \varnothing$.*

In particular, the property that $|X|$ is of cardinality 1 may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

Proof. Assume that there exist two points $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Then $\operatorname{Spec} k(x_1) \times_X^{\blacklozenge} \operatorname{Spec} k(x_2) = \varnothing$. Hence X satisfies the condition (†).

Assume that X satisfies the condition (†), i.e., that there exist $f : Y \to X, g : Z \to X$ such that $Y \times_X^{\blacklozenge} Z = \varnothing$ and $Y, Z \neq \varnothing$. Since $Y, Z \neq \varnothing$, we may assume without loss of generality that $|Y|, |Z|$ are of cardinality 1. To prove that X has two distinct points, it suffices to prove that $f(Y) \cap g(Z) = \varnothing$. Observe that it follows from Lemma 1.1 that $(Y \times_X Z)_{\mathrm{red}} = (Y \times_X^{\blacklozenge} Z)_{\mathrm{red}} = \varnothing$. Hence for any point $x \in X$, the equality

$$(Y_x \times_{\operatorname{Spec} k(x)} Z_x)_{\mathrm{red}} = ((Y \times_X Z)_{\mathrm{red}})_{x, \mathrm{red}} = \varnothing$$

holds. On the other hand, if $f(Y) \cap g(Z) \neq \varnothing$, then for any point $x \in f(Y) \cap g(Z)$, it holds that $(Y_x \times_{\operatorname{Spec} k(x)} Z_x)_{\mathrm{red}} \neq \varnothing$, in contradiction to the displayed equality. $\qquad\square$

Next, we consider a category-theoretic reconstruction of the underlying sets of the objects of $\mathsf{Sch}_{\blacklozenge/S}$.

Reconstruction 1.3. Let S be a scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. We define a (**V**-small) set

$$\mathsf{Pt}_{\blacklozenge/S}(X) \overset{\mathrm{def}}{:=} \{\, p_Z : Z \to X \mid |Z| \text{ is of cardinality } 1 \,\} / \sim,$$

where

$$(p_Z : Z \to X) \sim (p_{Z'} : Z' \to X) \quad :\overset{\mathrm{def}}{\Longleftrightarrow} \quad Z \times_{p_Z, X, p_{Z'}}^{\blacklozenge} Z' \neq \varnothing.$$

Then the set $\mathsf{Pt}_{\blacklozenge/S}(X)$ is defined by means of properties that may be characterized category-theoretically (cf. Lemma 1.2) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$. Since $|Z|$ is of cardinality 1, p_Z defines a point $p_Z(*_Z) \in X$. Furthermore, it follows immediately from Lemma 1.1 that

$$(\dagger) \qquad\qquad p_Z \sim p_{Z'} \iff p_Z(*_Z) = p_{Z'}(*_{Z'}).$$

In particular, the map

$$\eta_X : \mathsf{Pt}_{\blacklozenge/S}(X) \to |X|$$
$$[p_Z] \mapsto p_Z(*_Z)$$

is well-defined and injective. On the other hand, for a point $x \in X$, the equality $\eta_X([\operatorname{Spec} k(x) \to X]) = x$ holds. Hence η_X is a bijection.

Let $(f : X \to Y)$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. We define

$$\mathsf{Pt}_{\blacklozenge/S}(f) : \mathsf{Pt}_{\blacklozenge/S}(X) \to \mathsf{Pt}_{\blacklozenge/S}(Y)$$
$$[p_Z] \mapsto [f \circ p_Z].$$

It follows immediately from (†) that the map $\mathsf{Pt}_{\blacklozenge/S}(f)$ is well-defined. Hence we obtain a functor $\mathsf{Pt}_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Set}_{\mathbf{V}}$.

Let $\bigstar \overset{\text{def}}{:=} \mathsf{Set}$ or $\mathsf{Set}_{\mathbf{V}}$. We define

(‡)
$$U_{\blacklozenge/S}^{\bigstar} : \mathsf{Sch}_{\blacklozenge/S} \to \bigstar$$
$$X \mapsto |X|.$$

Write $i_{\mathbf{U} \in \mathbf{V}}^{\mathsf{Set}} : \mathsf{Set} \to \mathsf{Set}_{\mathbf{V}}$ for the natural inclusion. Then the equality $U_{\blacklozenge/S}^{\mathsf{Set}_{\mathbf{V}}} = i_{\mathbf{U} \in \mathbf{V}}^{\mathsf{Set}} \circ U_{\blacklozenge/S}^{\mathsf{Set}}$ holds. It follows from the equivalence (†), together with the definitions of η_X and $\mathsf{Pt}_{\blacklozenge/S}(f)$, that the following diagram commutes:

(††)
$$
\begin{array}{ccc}
\mathsf{Pt}_{\blacklozenge/S}(X) & \xrightarrow{\mathsf{Pt}_{\blacklozenge/S}(f)} & \mathsf{Pt}_{\blacklozenge/S}(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
|X| & \longrightarrow & |Y|.
\end{array}
$$

Thus we obtain an isomorphism of functors $\eta : \mathsf{Pt}_{\blacklozenge/S} \xrightarrow{\sim} U_{\blacklozenge/S}^{\mathsf{Set}_{\mathbf{V}}} = i_{\mathbf{U} \in \mathbf{V}}^{\mathsf{Set}} \circ U_{\blacklozenge/S}^{\mathsf{Set}}$.

Since the functor $\mathsf{Pt}_{\blacklozenge/S}$ is defined category-theoretically from the data $\mathsf{Sch}_{\blacklozenge/S}$, the following lemma holds:

Lemma 1.4. *Let S, T be schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then S, T, F determine an isomorphism ρ^{Pt} between the two composite functors of the following diagram:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{F} & \mathsf{Sch}_{\lozenge/T} \\
\mathsf{Pt}_{\blacklozenge/S} \downarrow & & \downarrow \mathsf{Pt}_{\lozenge/T} \\
\mathsf{Set}_{\mathbf{V}} & =\!=\!=\!= & \mathsf{Set}_{\mathbf{V}}.
\end{array}
$$

Proof. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. We define a map

$$\rho_X : \mathsf{Pt}_{\blacklozenge/S}(X) \to \mathsf{Pt}_{\lozenge/T}(F(X))$$
$$[p_Z : Z \to X] \mapsto [F(p_Z) : F(Z) \to F(X)].$$

By Lemma 1.2, $|F(Z)|$ is of cardinality 1. Since F is an equivalence,

$$Z \times_X^{\blacklozenge} Z \neq \varnothing \iff F(Z) \times_{F(X)}^{\lozenge} F(Z') \neq \varnothing.$$

Hence ρ_X is well-defined and injective. If Z' is an object of $\mathsf{Sch}_{\lozenge/T}$ whose underlying set is of cardinality 1, $Z' \to F(X)$ is a morphism of $\mathsf{Sch}_{\lozenge/T}$, and F^{-1} is a quasi-inverse of F, then it follows immediately that $\rho_X([F^{-1}(Z') \to X]) = [Z' \to F(X)]$. Hence ρ_X is a bijection.

To complete the proof of Lemma 1.4, it suffices to prove that for any morphism $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathsf{Pt}_{\blacklozenge/S}(X) & \xrightarrow{\rho_X} & \mathsf{Pt}_{\lozenge/T}(F(X)) \\
\mathsf{Pt}_{\blacklozenge/S}(f) \downarrow & & \downarrow \mathsf{Pt}_{\lozenge/T}(F(f)) \\
\mathsf{Pt}_{\blacklozenge/S}(Y) & \xrightarrow{\rho_Y} & \mathsf{Pt}_{\lozenge/T}(F(Y))
\end{array}
$$

Let $p_Z : Z \to X$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$ such that Z is of cardinality 1. Then

$$\rho_Y(\mathsf{Pt}_{\blacklozenge/S}(f)([p_Z])) = \rho_Y([f \circ p_Z]) = [F(f \circ p_Z)] = [F(f) \circ F(p_Z)]$$
$$= \mathsf{Pt}_{\lozenge/T}(F(f))([F(p_Z)]) = \mathsf{Pt}_{\lozenge/T}(F(f))(\rho_X([p_Z])).$$

This completes the proof of Lemma 1.4. $\square$

Corollary 1.5. *Let S, T be schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then the following diagram commutes up to isomorphism:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\ F\ } & \mathsf{Sch}_{\lozenge/T} \\
{\scriptstyle U^{\mathsf{Set}}_{\blacklozenge/S}}\big\downarrow & & \big\downarrow {\scriptstyle U^{\mathsf{Set}}_{\lozenge/T}} \\
\mathsf{Set} & =\!\!=\!\!=\!\!= & \mathsf{Set}\,.
\end{array}
$$

Proof. It follows from Lemma 1.4 and Reconstruction 1.3 that we have natural isomorphisms

$$i^{\mathsf{Set}}_{\mathbf{U} \in \mathbf{V}} \circ U^{\mathsf{Set}}_{\blacklozenge/S} \xleftarrow{\sim} \mathsf{Pt}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Pt}_{\lozenge/T} \circ F \xrightarrow{\sim} i^{\mathsf{Set}}_{\mathbf{U} \in \mathbf{V}} \circ U^{\mathsf{Set}}_{\lozenge/T} \circ F.$$

Since $i^{\mathsf{Set}}_{\mathbf{U} \in \mathbf{V}}$ is fully faithful, the composite isomorphism of the above display determines a natural isomorphism $U^{\mathsf{Set}}_{\blacklozenge/S} \xrightarrow{\sim} U^{\mathsf{Set}}_{\lozenge/T} \circ F$. $\square$

In the following discussion, we shall use the isomorphism of functors η constructed in Reconstruction 1.3 to *identify* (until further notice) $\mathsf{Pt}_{\blacklozenge/S}(-)$ with $|-|$.

Corollary 1.6. *Let S be a scheme. Let $f : Y \to X$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$, $x \in |X|$. Then:*

    (i) *The property that f is surjective may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X)$.*

    (ii) *The property that $x \in \mathrm{Im}(f)$ may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X, x)$.*

Proof. Corollary 1.6 follows immediately from Corollary 1.5. $\square$

In the remainder of this section, we consider a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is isomorphic to the natural morphism to a scheme from the spectrum of the residue field at a point of the scheme.

Lemma 1.7. *Let S be a scheme. Let $X \neq \varnothing$ be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then X is isomorphic to the spectrum of a field if and only if the following conditions hold:*

    (i) $|X|$ *is of cardinality 1;*

    (ii) *every morphism $f : Y \to X$, where $Y \neq \varnothing$, is an epimorphism.*

In particular, the property that X is isomorphic to the spectrum of a field may be characterized category-theoretically (cf. Lemma 1.2) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

Proof. First, we prove necessity. Assume that X is isomorphic to the spectrum of a field. It follows immediately that condition (i) holds. Next, we prove that X satisfies condition (ii). Let $f : Y \to X$ be a morphism, where $Y \neq \varnothing$, and $V = \mathrm{Spec}\, B$ an affine open subscheme of Y. To prove that f is an epimorphism, it suffices to prove that $f|_V$ is an epimorphism. Hence we may assume that $Y = \mathrm{Spec}\, B$ is affine. Since $k = \Gamma(X, \mathcal{O}_X)$ is a field, the ring morphism $f^{\#} : k \to B$ is injective. Let Z be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $g, h : X \rightrightarrows Z$ two morphisms such that $g \circ f = h \circ f$.

Then it suffices to prove that $g = h$. Since $Y \neq \varnothing$, it follows immediately that f is surjective on the underlying sets, hence that g and h induce the same morphism on the underlying sets. Since $|X|$ is of cardinality 1, g and h factor through an affine open subscheme $\operatorname{Spec} C$ of Z. Hence we may assume that $Z = \operatorname{Spec} C$ is affine. Observe that it follows from the equality $g \circ f = h \circ f$ that $f^{\#} \circ g^{\#} = f^{\#} \circ h^{\#}$. Thus, since $f^{\#}$ is injective, we conclude that $g^{\#} = h^{\#}$, i.e., that $g = h$ holds.

Next, we prove sufficiency. Assume that X satisfies conditions (i) and (ii). It follows immediately from condition (i) that X is an affine scheme. Assume that $A = \Gamma(X, \mathcal{O}_X)$ is not a field. Then the natural surjection $f^{\#} : A \to k :\overset{\mathrm{def}}{=} A/\sqrt{0}_A$, where $\sqrt{0}_A$ denotes the nilradical of A, is not injective. Write $q^{\#}, r^{\#} : A' :\overset{\mathrm{def}}{=} A \times_k A \rightrightarrows A$ for the two projections. Since $f^{\#}$ is not injective, $q^{\#} \neq r^{\#}$. Thus, since the equality $f^{\#} \circ q^{\#} = f^{\#} \circ r^{\#}$ holds, we conclude that f is not an epimorphism, in contradiction to our assumption that X satisfies condition (ii). This contradiction implies that A is a field, as desired. $\qquad\square$

Lemma 1.8. *Let S be a scheme. Let $f : Y \to X$ be a morphism of $\mathsf{Sch}_{\spadesuit/S}$. Then f is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism to X from the spectrum of the residue field at a point of X if and only if the following conditions hold:*

(i) Y is isomorphic to the spectrum of a field;

(ii) if $g : Z \to X$ is a morphism of $\mathsf{Sch}_{\spadesuit/S}$ such that Z is isomorphic to the spectrum of a field and $Y \times_X^{\spadesuit} Z \neq \varnothing$, then there exists a unique morphism $h : Z \to Y$ such that $g = f \circ h$.

In particular, the property that $f : Y \to X$ is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism to X from the spectrum of the residue field at a point of X may be characterized category-theoretically (cf. Lemma 1.7) from the data $(\mathsf{Sch}_{\spadesuit/S}, f : Y \to X)$.

Proof. First, we prove necessity. Assume that there exists $x \in X$ such that f is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism $\operatorname{Spec} k(x) \to X$. Then $Y \cong \operatorname{Spec} k(x)$ is isomorphic to the spectrum of a field. Hence condition (i) holds. Next, we prove that f satisfies condition (ii). Let $g : Z \to X$ be a morphism of $\mathsf{Sch}_{\spadesuit/S}$ such that Z is isomorphic to the spectrum of a field and $Y \times_X^{\spadesuit} Z \neq \varnothing$. Write $p_Y : Y \times_X^{\spadesuit} Z \to Y$ and $p_Z : Y \times_X^{\spadesuit} Z \to Z$ for the natural projections. Since $Y \times_X^{\spadesuit} Z \neq \varnothing$, there exists a point $w \in Y \times_X^{\spadesuit} Z$. Then $x = f(*_Y) = f(p_Y(w)) = g(p_Z(w)) = g(*_Z)$. Hence g factors uniquely through Y. Thus f satisfies condition (ii).

Next, we prove sufficiency. Assume that $f : Y \to X$ satisfies conditions (i) and (ii). By condition (i), there exists a field K such that $Y \cong \operatorname{Spec} K$. We set $x :\overset{\mathrm{def}}{=} f(*_Y)$. Since the natural morphism $g : Z = \operatorname{Spec} k(x) \to X$ satisfies condition (ii), there exists $h : Y \to Z$ such that $g \circ h = f$. Since $f : Y \to X$ satisfies condition (ii), there exists $h' : Z \to Y$ such that $g = f \circ h'$. Hence $f = f \circ h' \circ h$ and $g = g \circ h \circ h'$. Thus, it follows from the uniqueness portion of condition (ii) that $h' \circ h = \mathrm{id}_Y$ and $h \circ h' = \mathrm{id}_Z$. Hence f is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism $g : \operatorname{Spec} k(x) \to X$. $\qquad\square$

2. Regular Monomorphisms

In this section, we study various scheme-theoretic properties of regular monomorphisms in $\mathsf{Sch}_{\blacklozenge/S}$ (cf. Lemma 2.8).

Definition 2.1. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. We shall say that $f : X \to Y$ is a *regular monomorphism* if there exist two morphisms $g, h : Y \rightrightarrows Z$ in $\mathcal{C}$ such that f is the equalizer of g and h.

If, for a morphism of schemes $f : X \to Y$, the push-out $Y \coprod_X Y$ exists, then it is often the case that f is the equalizer of the two natural morphisms $Y \rightrightarrows Y \coprod_X Y$. Hence to study various properties of regular monomorphisms in $\mathsf{Sch}_{\blacklozenge/S}$, it is important to study various properties of coproducts in $\mathsf{Sch}_{\blacklozenge/S}$.

First, we note the following properties concerning coproducts in $\mathsf{Sch}_{\blacklozenge/S}$:

Lemma 2.2. *Let S be a scheme. Let X, Y be objects of $\mathsf{Sch}_{\blacklozenge/S}$. Then $X \coprod Y$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the coproduct $X \coprod^{\blacklozenge} Y$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $X \coprod Y$. In the following, we shall simply write $\coprod$ for $\coprod^{\blacklozenge}$.*

Proof. Lemma 2.2 follows from the definition of the disjoint union of schemes. $\square$

By applying the above lemma, we obtain the following category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is connected.

Corollary 2.3. *Let S be a scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then X is connected if and only if, for any objects Y, Z of $\mathsf{Sch}_{\blacklozenge/S}$,*

$$X \cong Y \coprod Z \ \Rightarrow \ Y \cong \varnothing \ \vee \ Z \cong \varnothing.$$

In particular, the property that X is connected may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

Proof. Corollary 2.3 follows from the definition of connectedness. $\square$

Next, we study various properties of push-outs along open immersions.

Lemma 2.4. *Let S be a scheme. Let X be an S-scheme, $U \subset X$ an open subscheme of X. Then the following hold:*
  (i) *The push-out $X \coprod_U X$ exists in $\mathsf{Sch}_{/S}$.*
  (ii) *If X is reduced, then $X \coprod_U X$ is reduced.*
  (iii) *If X is quasi-separated over S, and the inclusion $U \subset X$ is a quasi-compact open immersion, then $X \coprod_U X$ is quasi-separated over S.*
  (iv) *If X is quasi-compact over S, then $X \coprod_U X$ is quasi-compact over S.*

Proof. Since the morphism $U \hookrightarrow X$ is an open immersion, one may construct the push-out $X \coprod_U X$ in $\mathsf{Sch}_{/S}$ by glueing two copies of the scheme X along the open subscheme $U \subset X$. Assertions (i) and (ii) follow immediately.

Next, we prove assertion (iii). Since the property "quasi-separated over S" is Zariski local on S, we may assume that S is an affine scheme. By [Stacks, Tag 01KO], it suffices to prove that if $U_1, U_2 \subset X$ are affine open subsets, then $U_1 \cap U_2$ and $U_1 \cap U_2 \cap U$ are quasi-compact. Since X is quasi-separated, $U_1 \cap U_2$ is quasi-compact. Furthermore, since the inclusion $U \subset X$ is quasi-compact, the intersection $U_1 \cap U_2 \cap U$ is quasi-compact. Hence $X \coprod_U X$ is quasi-separated.

Finally, we prove assertion (iv). Since the property "quasi-compact over S" is Zariski local on S, we may assume that S is an affine scheme. Then X is quasi-compact. Write $i_1, i_2 : X \rightrightarrows X \coprod_U X$ for the two natural open immersions. Then

$X \coprod_U X$ is covered by the two quasi-compact open subschemes $i_1(X)$ and $i_2(X)$. Hence $X \coprod_U X$ is quasi-compact. $\qquad\square$

Next, we note the following property concerning affine schemes of $\mathsf{Sch}_{\blacklozenge/S}$:

Lemma 2.5. *Let S be a quasi-separated scheme. Let X be a scheme over S. Then the following assertions hold:*

    (i) *If X is a quasi-compact scheme, then the structure morphism $f : X \to S$ is quasi-compact. In particular, if X is an affine scheme, then the structure morphism $f : X \to S$ is quasi-compact.*

    (ii) *If X is a separated (respectively, quasi-separated) scheme, then the structure morphism $f : X \to S$ is separated (respectively, quasi-separated). In particular, if X is an affine scheme, then the structure morphism $f : X \to S$ is separated, hence also quasi-separated.*

Proof. Assertion (i) follows immediately from [Stacks, Tag 03GI]. Assertion (ii) follows immediately from [Stacks, Tag 01KV]. $\qquad\square$

By applying Lemma 2.4 and Lemma 2.5, we obtain the following property concerning the existence of push-outs in $\mathsf{Sch}_{\blacklozenge/S}$:

Corollary 2.6. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and U an affine open subscheme of X. Suppose that $\lozenge = \blacklozenge \cup \{\mathrm{qcpt}, \mathrm{sep}\}$. Then the following assertions hold:*

    (i) *U belongs to $\mathsf{Sch}_{\lozenge/S}$, hence, in particular, to $\mathsf{Sch}_{\blacklozenge/S}$.*

    (ii) *Assume that $\mathrm{sep} \notin \blacklozenge$. Then $X \coprod_U X$ (cf. Lemma 2.4 (i)) belongs to $\mathsf{Sch}_{\blacklozenge/S}$. In particular, the push-out $X \coprod_U^{\blacklozenge} X$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ and is naturally isomorphic to $X \coprod_U X$.*

Proof. Assertion (i) follows immediately from Lemma 2.5 (i) and (ii). To prove assertion (ii), it suffices to prove that $X \coprod_U X$ belongs to the category $\mathsf{Sch}_{\blacklozenge/S}$. If $\mathrm{qsep} \notin \blacklozenge$, then assertion (ii) follows from Lemma 2.4 (ii) and (iv). Assume that $\mathrm{qsep} \in \blacklozenge$. Since X is a quasi-separated scheme, and U is an affine scheme, the inclusion $U \hookrightarrow X$ is a quasi-compact open immersion by Lemma 2.5 (i). Hence assertion (ii) follows from Lemma 2.4 (ii), (iii), and (iv). $\qquad\square$

Next, we study the operation "$(-)_{\mathrm{red}}$" in the case of immersions.

Lemma 2.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of S-schemes.*

    (i) *If f is an immersion, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is also an immersion.*

    (ii) *If f is a quasi-compact immersion, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is also a quasi-compact immersion.*

    (iii) *If f is a closed immersion, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is also a closed immersion.*

Proof. It suffices to prove that if f is quasi-compact, an open immersion or a closed immersion, then f_{red} is also quasi-compact, an open immersion or a closed immersion, respectively.

Since f_{red} and f induce same morphism of the underlying topological spaces, if f is quasi-compact, then f_{red} is also quasi-compact.

Assume that f is an open immersion. Then $X_{\mathrm{red}} = (|X|, \mathcal{O}_{Y_{\mathrm{red}}}|_{|X|})$. Hence f_{red} is also an open immersion.

Finally, if a ring homomorphism $A \to B$ is surjective, then $A/\sqrt{0} \to B/\sqrt{0}$ is also surjective. In particular, if f is a closed immersion, then f_{red} is also a closed immersion. $\qquad\square$

Next, we study various scheme-theoretic properties of regular monomorphisms in $\mathsf{Sch}_{\blacklozenge/S}$.

Lemma 2.8. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$.*

   (i) If f is a closed immersion, then f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$.
   (ii) If f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, then f is an immersion.
   (iii) Assume that qsep $\in \blacklozenge$. If f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, then f is a quasi-compact immersion.
   (iv) Assume that sep $\in \blacklozenge$. If f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, then f is a closed immersion.

Proof. First, we prove assertion (i). Assume that f is a closed immersion. Recall that by [Stacks, Tag 0E25], the scheme $Y \coprod_X Y$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$, and f is an equalizer of $Y \rightrightarrows Y \coprod_X Y$ in $\mathsf{Sch}_{\blacklozenge/S}$. Hence f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$. This completes the proof of assertion (i).

Next, we prove assertions (ii), (iii), and (iv). Assume that there exist $g, h : Y \rightrightarrows Z$ such that f is an equalizer of (g, h) in the category $\mathsf{Sch}_{\blacklozenge/S}$. Suppose that $\lozenge = \blacklozenge \setminus \{\mathrm{qcpt}\}$. Then f is an equalizer of (g, h) in the category $\mathsf{Sch}_{\lozenge/S}$. Indeed, suppose that $f' : X' \to Y$ is a morphism in the category $\mathsf{Sch}_{\lozenge/S}$ such that $g \circ f' = h \circ f'$. Since $\blacklozenge \subset \lozenge \cup \{\mathrm{qcpt}, \mathrm{sep}\}$, by Corollary 2.6 (i), any affine open subscheme of X' belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since f is an equalizer of (g, h) in the category $\mathsf{Sch}_{\blacklozenge/S}$, for any open immersion $i' : U' \to X'$ such that U' is affine, there exists a unique morphism $p : U' \to X$ such that $f \circ p = f' \circ i'$. Thus, by allowing U' to vary over the affine open subschemes of X' and gluing together the resulting morphisms $p : U' \to X$, we obtain a unique morphism $q : X' \to X$ such that $f \circ q = f'$. Hence f is an equalizer of (g, h) in the category $\mathsf{Sch}_{\lozenge/S}$.

By Lemma 1.1 (i) and (ii), the fiber product $Z \times_S^{\lozenge} Z$ exists in the category $\mathsf{Sch}_{\lozenge/S}$. Hence we may consider the diagonal morphism $\Delta_Z : Z \to Z \times_S^{\lozenge} Z$. By Lemma 2.7 (i) (cf. also Lemma 1.1 (i) and (ii)), Δ_Z is an immersion. Since f is the equalizer of (g, h) in the category $\mathsf{Sch}_{\lozenge/S}$, f is the pull-back in the category $\mathsf{Sch}_{\lozenge/S}$ of Δ_Z along $(g, h) : Y \to Z \times_S^{\lozenge} Z$. Hence f is an immersion. If, moreover, qsep $\in \blacklozenge$ (respectively, sep $\in \blacklozenge$), then by Lemma 2.7 (ii) and (iii) (cf. also Lemma 1.1 (i) and (ii)), Δ_Z, hence also f, is a quasi-compact (respectively, closed) immersion. This completes the proof of assertions (ii), (iii), and (iv). $\qquad\square$

Finally, we apply the theory developed thus far to conclude a category-theoretic characterization of reduced schemes.

Corollary 2.9. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then X is reduced if and only if every surjective regular monomorphism $Y \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ is an isomorphism. In particular, the property that X is reduced may be characterized category-theoretically (cf. Corollary 1.6 (i)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. First, we prove necessity. Assume that X is reduced, and $f : Y \to X$ is a surjective regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$. By Lemma 2.8 (ii), f is a surjective

closed immersion. Thus, since X is reduced, f is an isomorphism. This completes the proof of necessity.

Next, we prove sufficiency. Assume that X is not reduced. Then the natural morphism $f : X_{\mathrm{red}} \to X$ is a surjective closed immersion, but not an isomorphism in $\mathsf{Sch}_{\blacklozenge/S}$. Thus, by Lemma 2.8 (i), the closed immersion f is a surjective regular monomorphism, but not an isomorphism. This completes the proof of Corollary 2.9.
□

Corollary 2.10. *Let S be a quasi-separated scheme. Let $f : Y \to X$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Then f is isomorphic to the natural morphism $X_{\mathrm{red}} \to X$ in $\mathsf{Sch}_{\blacklozenge/X}$ if and only if for any reduced object Z in $\mathsf{Sch}_{\blacklozenge/S}$ and any morphism $g : Z \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$, there exists a unique morphism $h : Z \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that $f \circ h = g$. In particular, the property that $f : Y \to X$ is isomorphic to the natural morphism $X_{\mathrm{red}} \to X$ in $\mathsf{Sch}_{\blacklozenge/X}$ may be characterized category-theoretically (cf. Corollary 2.9) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X)$.*

Proof. Corollary 2.10 follows immediately from Corollary 2.9. □

3. Local Domains

In this section, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is isomorphic to the spectrum of a local domain and a category-theoretic characterization of specialization and generization relations between points of a scheme.

Definition 3.1. Let X be a scheme and x, y points of X.

(1) We shall say that x is a *generic point* of X if, for any point $x' \in X$ such that $x' \rightsquigarrow x$, it holds that $x' = x$.

(2) We shall say that x, y are *comparable* if either $x \rightsquigarrow y$ or $y \rightsquigarrow x$.

(3) We shall say that X is *local* if there exists a point $v \in X$ such that for any $x' \in X$, $x' \rightsquigarrow v$.

Remark 3.2.

(i) It is immediate that if X is local, then X has a unique closed point.

(ii) If X is local, then every affine neighborhood of the unique closed point of X is equal to X. Thus X is local if and only if X is isomorphic to the spectrum of a local ring.

(iii) It is immediate that a point $x \in X$ is closed if and only if every point $y \in X$ such that $x \rightsquigarrow y$ is equal to x.

Definition 3.3. Let $f : X \to Y$ be a morphism of local schemes. Then we shall say that f is *local* if f maps the unique closed point of X to the unique closed point of Y.

We will use the following property to give a category-theoretic characterization of the property "comparable".

Definition 3.4. Let S be a scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x_1, x_2 \in X$. Write $f_1 : \mathrm{Spec}(k(x_1)) \to X, f_2 : \mathrm{Spec}(k(x_2)) \to X$ for the natural morphisms determined by the points $x_1, x_2 \in X$. We shall say that the triple (X, x_1, x_2) is *strongly local* in $\mathsf{Sch}_{\blacklozenge/S}$ if the following conditions hold:

(i) X is connected.

(ii) For any regular monomorphism $f : Z \to X$, if $x_1, x_2 \in \mathrm{Im}(f)$, then f is an isomorphism.

(iii) The morphism $f_1 \coprod f_2 : \mathrm{Spec}(k(x_1)) \coprod \mathrm{Spec}(k(x_2)) \to X$ determined by f_1 and f_2 is an epimorphism.

(iv) The morphism f_1 is a regular monomorphism.

(v) For any regular monomorphism $f : Z \to X$, if $x_1 \notin \mathrm{Im}(f)$ and $Z \neq \varnothing$, then the morphism $f \coprod f_1 : Z \coprod \mathrm{Spec}(k(x_1)) \to X$ determined by f and f_1 is **not** a regular monomorphism.

Thus the property that (X, x_1, x_2) is strongly local is defined completely in terms of properties that may be characterized category-theoretically (cf. Reconstruction 1.3, Corollary 1.6 (ii), Lemma 1.8, Corollary 2.3) from the data $(\mathsf{Sch}_{\blacklozenge/S}, (X, x_1, x_2))$.

Next, we study basic properties of strongly local triples.

Lemma 3.5. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x_1, x_2 \in X$. Suppose that (X, x_1, x_2) is strongly local in $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

(i) *X is isomorphic to the spectrum of a local domain.*

(ii) *One of x_1, x_2 is the closed point, and the other is the generic point. In particular, x_1, x_2 are comparable.*

Proof. Write $f_1 : \mathrm{Spec}(k(x_1)) \to X, f_2 : \mathrm{Spec}(k(x_2)) \to X$ for the natural morphisms determined by the points $x_1, x_2 \in X$.

Suppose that $x_1 = x_2$. Recall from Definition 3.4 (iv) that the morphism f_1 is a regular monomorphism. Since $x_1 = x_2$, it holds that $x_2 \in \mathrm{Im}(f_1)$. Hence by Definition 3.4 (ii), the morphism f_1 is an isomorphism. Thus X is isomorphic to the spectrum of a field. In particular, assertions (i) and (ii) hold, so in the remainder of the proof of Lemma 3.5, we may suppose that $x_1 \neq x_2$.

By Lemma 2.8 (i), the natural closed immersion $\overline{\{x_1, x_2\}}_{\mathrm{red}} \to X$ is a regular monomorphism. Hence by Definition 3.4 (ii), the regular monomorphism $\overline{\{x_1, x_2\}}_{\mathrm{red}} \to X$ is an isomorphism. Therefore, the following properties hold:

(a) X is reduced.

(b) $|X| = \overline{\{x_1\}} \cup \overline{\{x_2\}}$.

In particular, by property (b), the number of irreducible components of X is at most two.

Suppose that $\mathrm{sep} \in \blacklozenge$. First, we verify that X is local, and x_1 is the unique closed point of X. By Definition 3.4 (iv) and Lemma 2.8 (iv), x_1 is the unique closed point of X. Assume that there exists a point $x \in X$ such that $x_1 \notin \overline{\{x\}}$. Then the morphisms $f : \overline{\{x\}}_{\mathrm{red}} \to X$ and $f \coprod f_1 : \overline{\{x\}}_{\mathrm{red}} \coprod \mathrm{Spec}(k(x_1)) \to X$ are closed immersions. Hence by Lemma 2.8 (i), f and $f \coprod f_1$ are regular monomorphisms. This contradicts the fact that (X, x_1, x_2) satisfies condition (v) of Definition 3.4. Thus $x_1 \in \overline{\{x\}}$ holds for any $x \in X$, i.e., X is local, and x_1 is the unique closed point of X.

Next, we complete the proof of Lemma 3.5 in the case where $\mathrm{sep} \in \blacklozenge$. Since x_1 is the unique closed point of X, it holds that $x_2 \rightsquigarrow x_1$. Hence by property (b), X has a unique generic point x_2. In particular, assertion (ii) holds. By property (a), this implies that X is integral. In particular, assertion (i) holds. This completes the proof of Lemma 3.5 in the case where $\mathrm{sep} \in \blacklozenge$.

Suppose that sep $\notin \blacklozenge$. First, we verify that X is irreducible. Assume that X is not irreducible. By property (b), X has precisely two irreducible components, and x_1, x_2 are the generic points of these two irreducible components. Let U_1 be an affine open neighborhood of x_1 such that $U_1 \subset X \setminus \overline{\{x_2\}}$, and U_2 an affine open neighborhood of x_2 such that $U_2 \subset X \setminus \overline{\{x_1\}}$. By property (b), the equality

$$U_1 \cap U_2 \subset (X \setminus \overline{\{x_2\}}) \cap (X \setminus \overline{\{x_1\}}) = X \setminus (\overline{\{x_1\}} \cup \overline{\{x_2\}}) = \varnothing$$

holds. Since U_1, U_2 are affine, $U :\stackrel{\mathrm{def}}{=} U_1 \cup U_2$ is an affine open subset of X. Hence by Corollary 2.6 (ii), the push-out $X \coprod_U X$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $U_1, U_2 \neq \varnothing$ and $U_1 \cap U_2 = \varnothing$, U is not connected. By Definition 3.4 (i), it holds that $U \neq X$. Hence the two natural open immersions $i_1, i_2 : X \rightrightarrows X \coprod_U X$ are distinct. On the other hand, since x_1, x_2 are contained in U, the equality $i_1 \circ (f_1 \coprod f_2) = i_2 \circ (f_1 \coprod f_2)$ holds, i.e., the morphism $f_1 \coprod f_2 : \mathrm{Spec}(k(x_1)) \coprod \mathrm{Spec}(k(x_2)) \to X$ is not an epimorphism. This contradicts the fact that (X, x_1, x_2) satisfies condition (iii) of Definition 3.4. Thus X is irreducible. Write η for the generic point of X. By property (b), $\eta \in \{x_1, x_2\}$.

Next, we complete the proof of Lemma 3.5 in the case where sep $\notin \blacklozenge$. Let v be the unique point of X such that $\{v, \eta\} = \{x_1, x_2\}$. By properties (a) and (b), to complete the proof of Lemma 3.5, it suffices to prove that the relation $x \rightsquigarrow v$ holds for any $x \in X$. Assume that there exists a point $x \in X$ such that $v \notin \overline{\{x\}}$. Then there exists an affine open neighborhood V of v such that $x \notin V$. By Corollary 2.6 (ii), the push-out $X \coprod_V X$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $V \neq X$, the two natural inclusions $i_1', i_2' : X \rightrightarrows X \coprod_V X$ are distinct. On the other hand, since $\{x_1, x_2\} = \{v, \eta\} \subset V$, the equality $i_1' \circ (f_1 \coprod f_2) = i_2' \circ (f_1 \coprod f_2)$ holds, i.e., the morphism $f_1 \coprod f_2 : \mathrm{Spec}(k(x_1)) \coprod \mathrm{Spec}(k(x_2)) \to X$ is not an epimorphism. This contradicts the fact that (X, x_1, x_2) satisfies condition (iii) of Definition 3.4. Thus for any $x \in X$, the relation $x \rightsquigarrow v$ holds. This completes the proof of Lemma 3.5 in the case where sep $\notin \blacklozenge$. $\qquad\square$

Corollary 3.6. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x_1, x_2, x \in X$. Then the property that (X, x_1, x_2) is strongly local in $\mathsf{Sch}_{\blacklozenge/S}$, and x is either the unique closed point or the unique generic point of X may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2, x)$.*

Proof. Corollary 3.6 follows formally from Reconstruction 1.3, Definition 3.4, and Lemma 3.5. $\qquad\square$

Next, we prove that the triple consisting of the spectrum of a valuation ring, the unique closed point of the spectrum, and the unique generic point of the spectrum is strongly local.

Lemma 3.7. *Let S be a quasi-separated scheme. Let A be a valuation ring equipped with a structure morphism $\mathrm{Spec}(A) \to S$, $v \in \mathrm{Spec}(A)$ the unique closed point, and $\eta \in \mathrm{Spec}(A)$ the unique generic point. Then $(\mathrm{Spec}(A), v, \eta)$ is strongly local in $\mathsf{Sch}_{\blacklozenge/S}$.*

Proof. Since the spectrum of a local ring is connected, $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (i) of Definition 3.4.

Next, we verify that $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (ii) of Definition 3.4. Let $f : Z \to X$ be a regular monomorphism such that $v, \eta \in f(Z)$. Then, by Lemma 2.8 (ii), f is an immersion, i.e., f is the composite of a closed immersion with an open

immersion. Moreover, since $v, \eta \in f(Z)$, the immersion f is a homeomorphism, i.e., f is a surjective closed immersion. Furthermore, since A is reduced, the surjective closed immersion f is an isomorphism. This completes the proof that $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (ii) of Definition 3.4.

Next, we verify that $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (iii) of Definition 3.4. Let $f, g : X \rightrightarrows Y$ be two morphisms such that $f|_v = g|_v, f|_\eta = g|_\eta$. It suffices to prove that $f = g$. Let K be the field of fractions of A and $i^\# : A \to K$ the natural injection. It follows from the definition of v that f and g factor through an affine open neighborhood $V = \mathrm{Spec}(B)$ of $f(v) = g(v)$. Hence, we may assume that $Y = \mathrm{Spec}(B)$ is affine. Since $f|_\eta = g|_\eta$, the equality $i^\# \circ f^\# = i^\# \circ g^\#$ holds. Since $i^\#$ is injective, we conclude that $f^\# = g^\#$, hence that $f = g$. This completes the proof that $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (iii) of Definition 3.4.

Next, by Lemma 2.8 (i), the closed immersion $\mathrm{Spec}(k(v)) \to \mathrm{Spec}(A)$ is a regular monomorphism, i.e., $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (iv) of Definition 3.4.

Finally, we verify that $(\mathrm{Spec}(A), v, \eta)$ satisfies condition (v) of Definition 3.4. Let $f : Z \to X$ be a regular monomorphism such that $v \notin f(Z)$ and $Z \neq \varnothing$. By Lemma 2.8 (ii), it suffices to prove that $g : Z \coprod \mathrm{Spec}(k(v)) \to X$ is not an immersion. Since $Z \neq \varnothing$ and X is local, any open neighborhood of v intersects $f(Z)$. On the other hand, the open neighborhood $\{*_{\mathrm{Spec}(k(v))}\}$ of the point $*_{\mathrm{Spec}(k(v))}$ in $Z \coprod \mathrm{Spec}(k(v))$ does not intersect Z. Thus g is not an immersion. This completes the proof of Lemma 3.7. $\qquad\square$

By applying the theory of strongly local triples developed thus far, we conclude a category-theoretic characterization of comparable pairs of points of a scheme.

Proposition 3.8. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x_1, x_2 \in X$. Then x_1, x_2 are comparable if and only if there exist an object Z, points $z_1, z_2 \in Z$, and a morphism $f : Z \to X$ such that (Z, z_1, z_2) is strongly local in $\mathsf{Sch}_{\blacklozenge/S}$ and $\{f(z_1), f(z_2)\} = \{x_1, x_2\}$. In particular, the property that $x_1, x_2 \in X$ are comparable may be characterized category-theoretically (cf. Reconstruction 1.3, Definition 3.4) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2)$.*

Proof. First, we prove necessity. By Lemma 2.5 (i) (ii), we may assume without loss of generality that X is the spectrum of a local domain A whose unique closed point is x_1, and whose unique generic point is x_2. Write K for the fraction field of A. Let $C \subset K$ be a valuation ring that dominates A (cf. [Stacks, Tag 00IA]), $Z \overset{\mathrm{def}}{:=} \mathrm{Spec}(C)$, $z_1 \in Z$ the unique closed point, $z_2 \in Z$ the unique generic point, and $f : Z \to X$ the morphism induced by $A \subset C$. By Lemma 2.5 (i) (ii), f belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Moreover, by Lemma 3.7, (Z, z_1, z_2) is strongly local. This completes the proof of necessity.

Next, we prove sufficiency. Let (Z, z_1, z_2) be strongly local in $\mathsf{Sch}_{\blacklozenge/S}$ and $f : Z \to X$ a morphism such that $\{f(z_1), f(z_2)\} = \{x_1, x_2\}$. We may assume without loss of generality that $f(z_1) = x_1, f(z_2) = x_2$. Suppose that $z_2 \rightsquigarrow z_1$. Then

$$x_1 = f(z_1) \in f(\overline{\{z_2\}}) \subset \overline{\{f(z_2)\}} = \overline{\{x_2\}}.$$

Thus $x_2 \rightsquigarrow x_1$ holds. Suppose that $z_1 \rightsquigarrow z_2$. Then

$$x_2 = f(z_2) \in f(\overline{\{z_1\}}) \subset \overline{\{f(z_1)\}} = \overline{\{x_1\}}.$$

Thus $x_1 \rightsquigarrow x_2$ holds. This completes the proof of Proposition 3.8. $\qquad\square$

To give a category-theoretic characterization of local objects of $\mathsf{Sch}_{\blacklozenge/S}$, we define the following property:

Definition 3.9. Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x_1, x_2 \in X$. We shall say that the triple (X, x_1, x_2) is *valuative* in $\mathsf{Sch}_{\blacklozenge/S}$ if the following conditions hold:

(i) Any two points of X are comparable.

(ii) (X, x_1, x_2) is strongly local in $\mathsf{Sch}_{\blacklozenge/S}$.

Thus the property that (X, x_1, x_2) is valuative in $\mathsf{Sch}_{\blacklozenge/S}$ is defined completely in terms of properties that may be characterized category-theoretically (cf. Definition 3.4, Proposition 3.8) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2)$.

Remark 3.10.

(i) By Corollary 3.6, if the triple (X, x_1, x_2) is valuative, then for $x \in X$, the property that x is either the closed point of the generic point may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2, x)$.

(ii) It follows immediately from Lemma 3.7 that the triple $(\mathrm{Spec}(A), v, \eta)$ consisting of the spectrum of a valuation ring A, the unique closed point $v \in \mathrm{Spec}(A)$, and the unique generic point $\eta \in \mathrm{Spec}(A)$ is a valuative triple in $\mathsf{Sch}_{\blacklozenge/S}$.

The next lemma will play an important role in our category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is irreducible local (cf. Proposition 3.12).

Lemma 3.11. *Let X be a scheme and $x_1, x_2, x_3 \in X$. Suppose that $x_3 \rightsquigarrow x_2 \rightsquigarrow x_1$. Then there exist a triple $(\mathrm{Spec}(V), v, \eta)$ and a morphism $f : \mathrm{Spec}(V) \to X$ such that $\mathrm{Spec}(V)$ is the spectrum of a valuation ring V, $v \in \mathrm{Spec}(V)$ is the unique closed point, $\eta \in \mathrm{Spec}(V)$ is the unique generic point, $x_2 \in f(\mathrm{Spec}(V)), f(\eta) = x_3,$ and $f(v) = x_1$.*

Proof. To prove the assertion, we may assume without loss of generality that $X = \mathrm{Spec}(A)$ is the spectrum of a local domain A, $x_1 \in X$ is the unique closed point, and $x_3 \in X$ is the unique generic point.

Let $\mathfrak{p}$ be the prime ideal of A which corresponds to $x_2 \in \mathrm{Spec}(A)$, V_1 a valuation ring which dominates $A_{\mathfrak{p}}$ in $k(x_3)$, $\mathfrak{n}_1$ the maximal ideal of V_1, and $p : V_1 \to V_1/\mathfrak{n}_1$ the natural surjection. Since V_1 dominates $A_{\mathfrak{p}}$ in $k(x_3)$, the natural morphism $j : A/\mathfrak{p} \to V_1/\mathfrak{n}_1$ is injective. Let V_2 be a valuation ring which dominates $A/\mathfrak{p}$ in the field $V_1/\mathfrak{n}_1$. Then $V :\overset{\mathrm{def}}{=} p^{-1}(V_2)$ is the desired valuation ring. This completes the proof of Lemma 3.11. $\qquad\square$

Next, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is irreducible local.

Proposition 3.12. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Assume that $|X|$ is not of cardinality 1. Then X is irreducible local if and only if there exist points $x_1 \neq x_2 \in X$ such that the triple (X, x_1, x_2) satisfies the following conditions:*

(i) x_1, x_2 *are comparable.*

(ii) *For any point $x \in X$, x, x_1 and x, x_2 are comparable.*

(iii) *For any valuative triple (Y, y_1, y_2) and any morphism $f : Y \to X$, if $x_1 \in f(Y)$, then $x_1 \in \{f(y_1), f(y_2)\}$.*

(iv) For any valuative triple (Y, y_1, y_2) and any morphism $f : Y \to X$, if $x_2 \in f(Y)$, then $x_2 \in \{f(y_1), f(y_2)\}$.

In particular, the property that X is irreducible local may be characterized category-theoretically (cf. Lemma 1.2, Reconstruction 1.3, Corollary 1.6 (ii), Proposition 3.8) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$. Furthermore, the property that X is irreducible local, and $x \in X$ is either the closed point or the generic point holds if and only if there exist points $x_1, x_2 \in X$ such that $x \in \{x_1, x_2\}$ and the triple (X, x_1, x_2) satisfies the above conditions (i), (ii), (iii), (iv). In particular, the property that X is irreducible local, and $x \in X$ is either the closed point or the generic point may be characterized category-theoretically (cf. Lemma 1.2, Reconstruction 1.3, Corollary 1.6 (ii), Proposition 3.8) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x)$.

Proof. First, we prove necessity. Suppose that X is irreducible local, and $|X|$ is not of cardinality 1. Let $x_1 \in X$ be the unique closed point and $x_2 \in X$ the unique generic point. Then it follows immediately that the triple (X, x_1, x_2) satisfies conditions (i) and (ii).

Next, we prove that X satisfies condition (iii). Let (Y, y_1, y_2) be a valuative triple and $f : Y \to X$ a morphism such that $x_1 \in f(Y)$. Then there exists a point $y \in Y$ such that $f(y) = x_1$. Since either $y_1 \in Y$ or $y_2 \in Y$ is the unique closed point of Y (cf. Lemma 3.5 (ii)), either $y \rightsquigarrow y_1$ or $y \rightsquigarrow y_2$ holds. Hence either $x_1 \rightsquigarrow f(y_1)$ or $x_1 \rightsquigarrow f(y_2)$ holds. Since x_1 is the unique closed point of X, either $f(y_1) = x_1$ or $f(y_2) = x_1$ holds. Thus X satisfies condition (iii).

Next, we prove that X satisfies condition (iv). Let (Y, y_1, y_2) be a valuative triple and $f : Y \to X$ a morphism such that $x_2 \in f(Y)$. Then there exists a point $y \in Y$ such that $f(y) = x_2$. Since either $y_1 \in Y$ or $y_2 \in Y$ is the unique generic point of Y (cf. Lemma 3.5 (ii)), either $y_1 \rightsquigarrow y$ or $y_2 \rightsquigarrow y$ holds. Hence either $f(y_1) \rightsquigarrow x_2$ or $f(y_2) \rightsquigarrow x_2$ holds. Since x_2 is the unique generic point of X, either $f(y_1) = x_2$ or $f(y_2) = x_2$ holds. Thus X satisfies condition (iv). This completes the proof of necessity.

Next, we prove sufficiency. Suppose that $|X|$ is not of cardinality 1, and there exist points $x_1 \neq x_2 \in X$ such that the triple (X, x_1, x_2) satisfies conditions (i), (ii), (iii), (iv). Then since the triple (X, x_2, x_1) satisfies conditions (i), (ii), (iii), (iv), we may assume without loss of generality that $x_2 \rightsquigarrow x_1$.

Since X satisfies condition (ii), if x_1 is a closed point of X, then any point of X is a generalization of x_1. Hence to prove that X is local, it suffices to prove that x_1 is a closed point of X. Suppose that there exists a point $x \in X$ such that $x_2 \rightsquigarrow x_1 \rightsquigarrow x$. By Lemma 3.11, there exists a triple $(\mathrm{Spec}(W), w, \zeta)$ consisting of the spectrum of a valuation ring W, the unique closed point $w \in \mathrm{Spec}(W)$, and the unique generic point $\zeta \in \mathrm{Spec}(W)$, together with a morphism $f : \mathrm{Spec}(W) \to X$, such that $f(\zeta) = x_2, f(w) = x$, and $x_1 \in f(\mathrm{Spec} V)$. By Lemma 2.5 (i) (ii), and Remark 3.10 (ii), the morphism $f : \mathrm{Spec}(W) \to X$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$, and the triple $(\mathrm{Spec}(W), w, \zeta)$ is valuative in $\mathsf{Sch}_{\blacklozenge/S}$. Since X satisfies condition (iii), it holds that $x_1 \in \{f(w), f(\zeta)\} = \{x, x_2\}$. Since $x_1 \neq x_2$, this proves that $x_1 = x$. Thus X is local, and x_1 is the unique closed point of X.

Since X satisfies condition (ii), if x_2 is a generic point of X, then any point of X is a specialization of x_2. Hence to prove that X is irreducible, it suffices to prove that x_2 is a generic point of X. Suppose that there exists a point $\xi \in X$ such that $\xi \rightsquigarrow x_2 \rightsquigarrow x_1$. By Lemma 3.11, there exists a triple $(\mathrm{Spec}(W'), w', \zeta')$ consisting of the spectrum of a valuation ring W', the closed point $w' \in \mathrm{Spec}(W')$, and the

generic point $\zeta' \in \mathrm{Spec}(W')$, together with a morphism $f' : \mathrm{Spec}(W') \to X$, such that $f(\zeta') = \xi, f(w') = x_1$, and $x_2 \in f(\mathrm{Spec}(W'))$. By Lemma 2.5 (i) (ii), and Remark 3.10 (ii), the morphism $f' : \mathrm{Spec}(W') \to X$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$, and the triple $(\mathrm{Spec}(W'), w', \zeta')$ is valuative in $\mathsf{Sch}_{\blacklozenge/S}$. Since X satisfies condition (iv), it holds that $x_2 \in \{f(\zeta'), f(w')\} = \{\xi, x_1\}$. Since $x_1 \neq x_2$, this proves that $x_2 = \xi$. Thus X is irreducible, and x_2 is the unique generic point of X. This completes the proof of sufficiency.

The remaining assertions of Proposition 3.12 are immediate. This completes the proof of Proposition 3.12. $\hfill\square$

To give a category-theoretic characterization of specialization and generization relations between points of a scheme, we first give a category-theoretic characterization of the generic point of an object that appears as the first member of a valuative triple.

Proposition 3.13. *Let S be a quasi-separated scheme. Let (X, x_1, x_2) be a valuative triple in $\mathsf{Sch}_{\blacklozenge/S}$ and $\eta \in X$. Then $\eta \in X$ is the generic point if and only if the following conditions hold:*

(i) $\eta \in \{x_1, x_2\}$.

(ii) For any irreducible local object Y and any morphisms $f, g : X \rightrightarrows Y$, if $f|_\eta = g|_\eta$ then $f = g$.

In particular, the property that (X, x_1, x_2) is valuative in $\mathsf{Sch}_{\blacklozenge/S}$, and $\eta \in X$ is the generic point, may be characterized category-theoretically (cf. Lemma 1.8, Proposition 3.12) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2, \eta)$.

Proof. We may assume without loss of generality that X is not isomorphic to the spectrum of a field. By Lemma 3.5 (i) and Definition 3.9 (ii), X is affine, and $A :\overset{\mathrm{def}}{=} \Gamma(X, \mathcal{O}_X)$ is a local domain. Write K for the field of fractions of A and $i^\# : A \to K$ for the natural inclusion.

First, we prove necessity. Suppose that $\eta \in X$ is the generic point. It follows from Lemma 3.5 (ii) that $\eta \in \{x_1, x_2\}$. Hence condition (i) holds. Let $Y = \mathrm{Spec}(B)$ be the spectrum of a local ring (cf. Remark 3.2 (ii)) and $f, g : X \rightrightarrows Y$ morphisms such that $f|_\eta = g|_\eta$. Since η is the generic point, the equality $i^\# \circ f^\# = i^\# \circ g^\#$ holds. Since $i^\#$ is injective, the equality $f^\# = g^\#$ holds. Hence the equality $f = g$ holds. This completes the proof of necessity.

Next, we prove sufficiency. Suppose that a point $\eta \in X$ satisfies conditions (i) and (ii). By Lemma 3.5 (ii) and condition (i), it suffices to prove that if η is the closed point, then η does **not** satisfy condition (ii). Suppose that $\eta \in X$ is the closed point. Let $\mathfrak{m}_\eta$ be the maximal ideal of A corresponding to η. Since A is not a field, and η is a closed point, there exists a non-zero element $a \in \mathfrak{m}_\eta$. Let $B :\overset{\mathrm{def}}{=} A[t]_{(\mathfrak{m}_\eta, t)}$ be the local domain obtained by localizing the polynomial ring $A[t]$ at the maximal ideal $(\mathfrak{m}_\eta, t)$; $f, g : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ the morphisms determined by the morphisms of A-algebras $f^\#, g^\# : B \rightrightarrows A$ such that $f^\#(t) = 0, g^\#(t) = a$. Then $f|_\eta = g|_\eta$. On the other hand, since $0 \neq a$, it holds that $f \neq g$. Thus η does not satisfy condition (ii). This completes the proof of Proposition 3.13. $\hfill\square$

By a similar argument to the argument given in the proof of Proposition 3.13, one may give a category-theoretic characterization of the closed point of an object that appears as the first member of a valuative triple.

Corollary 3.14. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x, x' \in X$. Then $x' \rightsquigarrow x$ if and only if there exist a valuative triple (V, v_1, v_2), $\eta \in V$, and a morphism $f : V \to X$ such that η is the unique generic point (cf.* Lemma 3.5 (i), Definition 3.9 (ii)) *of V, $f(\eta) = x'$, and $x \in \mathrm{Im}(f)$. In particular, the property that $x' \rightsquigarrow x$ may be characterized category-theoretically (cf.* Definition 3.9, Proposition 3.13) *from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x, x')$.*

Proof. Sufficiency follows immediately. Thus it remains to prove necessity. Suppose that $x' \rightsquigarrow x$. By Lemma 2.5 (i) (ii), we may assume without loss of generality that X is the spectrum of a local domain, $x \in X$ is the unique closed point, and $x' \in X$ is the unique generic point. By Remark 3.10 (ii), Lemma 3.11, there exist a valuative triple (V, v_1, v_2), a point $\eta \in V$, and a morphism $f : V \to X$ such that V is the spectrum of a valuation ring, v_1 is the unique closed point of V, $v_2 = \eta$ is the unique generic point of V, $x' = f(\eta)$, and $x = f(v_1) \in V$. By Lemma 2.5 (i) (ii), f belongs to $\mathsf{Sch}_{\blacklozenge/S}$. This completes the proof of Corollary 3.14. $\qquad \square$

Corollary 3.15. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $x \in X$. Then the following assertions hold:*

(i) *x is a closed point in X if and only if for any point $x' \in X$ such that $x \rightsquigarrow x'$, it holds that $x = x'$. In particular, the property that x is a closed point in X may be characterized category-theoretically (cf.* Corollary 3.14) *from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x)$.*

(ii) *x is a generic point in X if and only if for any point $x' \in X$ such that $x' \rightsquigarrow x$, it holds that $x = x'$. In particular, the property that x is a generic point in X may be characterized category-theoretically (cf.* Corollary 3.14) *from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x)$.*

(iii) *X is local if and only if there exists a point $x_1 \in X$ such that $x' \rightsquigarrow x_1$ holds for all $x' \in X$. In particular, the property that X is local may be characterized category-theoretically (cf.* Corollary 3.14) *from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

(iv) *X is irreducible if and only if there exists a point $x_1 \in X$ such that $x_1 \rightsquigarrow x'$ holds for all $x' \in X$. In particular, the property that X is irreducible may be characterized category-theoretically (cf.* Corollary 3.14) *from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. These assertions follow from the definitions of the various properties under consideration. $\qquad \square$

Corollary 3.16. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

(i) *X is integral if and only if X is irreducible and reduced.*

(ii) *X is isomorphic to the spectrum of a local domain if and only if X is irreducible, reduced and local.*

In particular, the properties of (i) *and* (ii) *may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$ (cf.* Corollary 2.9, Corollary 3.15 (iii) (iv)).

Proof. These assertions follow from the definition of the notion of an integral scheme. $\qquad \square$

Next, we apply the theory developed thus far to conclude a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes

is isomorphic to the natural morphism to a scheme from the spectrum of the local ring at a point of the scheme.

Definition 3.17. Let X be a scheme; $x, \eta \in X$.

(i) We shall say that the pair (X, η) is *irreducible* if X is irreducible, and η is its unique generic point.

(ii) We shall say that the pair (X, x) is *local* if X is local, and x is its unique closed point.

(iii) We shall say that the triple (X, x, η) is *irreducible local* if (X, η) is irreducible, and (X, x) is local.

Let S be a quasi-separated scheme. Suppose that X is an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then these properties may be characterized category-theoretically (cf. Corollary 3.14, Corollary 3.15 (i) (ii) (iii) (iv)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x, \eta)$.

Proposition 3.18. *Let S be a quasi-separated scheme. Let $f : Y \to X$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$ and $x \in X$. Then f is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism to X from the spectrum of the local ring at a point x of X if and only if the following conditions hold:*

(i) *f is a monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$.*

(ii) *There exists a point $y \in Y$ such that (Y, y) is local, and $f(y) = x$.*

(iii) *For any local pair (Z, z) and any morphism $g : Z \to X$ such that $g(z) = x$, there exists a unique local morphism $h : Z \to Y$ such that $g = f \circ h$.*

In particular, the property that f is isomorphic as an object of $\mathsf{Sch}_{/X}$ to the object of $\mathsf{Sch}_{/X}$ that arises from the natural morphism to X from the spectrum of the local ring at a point x of X may be characterized category-theoretically (cf. Definition 3.17 (ii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X, x)$.

Proof. First, we prove necessity. Suppose that $f : Y \to X$ is isomorphic to the natural morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ as X-schemes. Write y for the unique closed point of Y. Then the pair (Y, y) is local, and $f(y) = x$. By the universality of localization, condition (iii) is satisfied. By [Stacks, Tag 01L9], f is a monomorphism of Sch. Hence f satisfies condition (i). This completes the proof of necessity.

Next, we prove sufficiency. Let $g : Z = \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ be the natural localization morphism. Since $f : Y \to X$ satisfies condition (iii), there exists a unique local morphism $h : Z \to Y$ such that $g = f \circ h$. By the necessity portion of Proposition 3.18, g satisfies condition (iii). Hence there exists a unique local morphism $h' : Y \to Z$ such that $f = g \circ h'$. Thus the equalities $f = f \circ h \circ h'$ and $g = g \circ h' \circ h$ hold. Since f satisfies condition (i), i.e., f is a monomorphism, the equality $f = f \circ h \circ h'$ implies $h \circ h' = \mathrm{id}_Y$. Since g satisfies condition (i), i.e., g is a monomorphism, the equality $g = g \circ h' \circ h$ implies $h' \circ h = \mathrm{id}_Z$. Thus the equalities $h \circ h' = \mathrm{id}_Y$ and $h' \circ h = \mathrm{id}_Z$ imply that the morphisms h, h' are isomorphisms between Y, Z. Since $g = f \circ h$, $f : Y \to X$ is isomorphic to $g : Z = \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ as an X-scheme. This completes the proof of Proposition 3.18. $\square$

Finally, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying scheme is isomorphic to the spectrum of a valuation ring.

Proposition 3.19. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then X is isomorphic to the spectrum of a valuation ring if and only if the following conditions hold:*

(i) There exist points $x_1, x_2 \in X$ such that (X, x_1, x_2) is valuative.

(ii) Write η for the unique generic point of X. Then for any integral local object Y and any local morphism $f : Y \to X$, if $Y \times_X^{\blacklozenge} \operatorname{Spec}(k(\eta)) \cong \operatorname{Spec}(k(\eta))$, then f is an isomorphism.

In particular, the property that X is isomorphic to the spectrum of a valuation ring may be characterized category-theoretically (cf. Lemma 1.8, Proposition 3.13, Corollary 3.15 (i), Corollary 3.16 (ii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

Proof. First, we prove necessity. Suppose that $X = \operatorname{Spec}(A)$ is the spectrum of a valuation ring A. Write $\eta \in X$ for the unique generic point and $x \in X$ for the unique closed point. By Remark 3.10 (ii), the triple (X, x, η) is valuative in $\mathsf{Sch}_{\blacklozenge/S}$. Hence X satisfies condition (i). In the remainder of the proof of necessity, we prove that X satisfies condition (ii). Let K be the field of fractions of A, $Y = \operatorname{Spec}(B)$ the spectrum of a local domain, $\xi \in Y$ the unique generic point of Y, $y \in Y$ the unique closed point of Y, and $f : Y \to X$ a local morphism such that $Y \times_X^{\blacklozenge} \operatorname{Spec}(k(\eta)) \cong \operatorname{Spec}(k(\eta))$. Since A, B are local domains, and K is the field of fractions of A, the tensor product $K \otimes_A B$ is integral. Hence $Y \times_X \operatorname{Spec}(k(\eta)) \cong Y \times_X^{\blacklozenge} \operatorname{Spec}(k(\eta)) \cong \operatorname{Spec}(k(\eta))$ (cf. Lemma 1.1 (iii) (iv)), i.e., $B \otimes_A K \cong K$. Since K is a localization of A, the field of fractions of B is isomorphic to K. Since f is a local morphism, the local domain B dominates the valuation ring A in the field K. Since the valuation ring A is maximal for the relation of domination among the set of local subrings in K ([Stacks, Section 00I8]), f is an isomorphism. This completes the proof of necessity.

Next, we prove sufficiency. Suppose that X satisfies conditions (i) and (ii). By Definition 3.9 (ii) and Lemma 3.5 (i), X is isomorphic to the spectrum of a local domain. Write $A :\overset{\text{def}}{=} \Gamma(X, \mathcal{O}_X)$ and K for the field of fractions of A. Let B be a valuation ring which dominates A in K (cf. [Stacks, Tag 00IA]). Since X satisfies condition (ii), the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an isomorphism (cf. Lemma 1.1 (iii) (iv)). Hence $A = B$, i.e., A is a valuation ring. This completes the proof of Proposition 3.19. $\square$

4. THE UNDERLYING TOPOLOGICAL SPACE

In this section, we give a functorial category-theoretic algorithm for reconstructing the underlying topological space of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$. First, we consider the underlying topological space of fiber products in $\mathsf{Sch}_{\blacklozenge/S}$.

Lemma 4.1. *Let S be a quasi-separated scheme. Let $f : X \to Z, g : Y \to Z$ be morphisms of $\mathsf{Sch}_{\blacklozenge/S}$. Suppose that the fiber product $X \times_Z^{\blacklozenge} Y$ exists in $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

(i) *If $\mathrm{red} \notin \blacklozenge$, then the natural morphism of schemes $v : X \times_Z^{\blacklozenge} Y \to X \times_Z Y$ is an isomorphism.*

(ii) *If $\mathrm{red} \in \blacklozenge$, then the natural morphism of schemes $v : X \times_Z^{\blacklozenge} Y \to X \times_Z Y$ is isomorphic to the natural closed immersion $(X \times_Z Y)_{\mathrm{red}} \to X \times_Z Y$ in $\mathsf{Sch}_{/X \times_Z Y}$.*

Proof. First, we prove assertion (i). Let U be an affine scheme and $f : U \to X \times_Z Y$ a morphism of schemes. Then by Lemma 2.5 (i) (ii), together with our assumption

that $\mathrm{red} \notin \blacklozenge$, U belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Hence we obtain the following natural bijections:

$$\mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X \times_Z^{\blacklozenge} Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U, X) \times_{\mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U,Z)} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U, Y)$$

$$= \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X) \times_{\mathrm{Hom}_{\mathsf{Sch}_{/S}}(U,Z)} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, Y)$$

$$\xleftarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X \times_Z Y).$$

Therefore, for any affine scheme U and any morphism $f : U \to X \times_Z Y$, f factors uniquely through the natural morphism $v : X \times_Z^{\blacklozenge} Y \to X \times_Z Y$. This implies that v is an isomorphism. This completes the proof of assertion (i).

Next, we prove assertion (ii). Let U be a reduced affine scheme and $f : U \to X \times_Z Y$ a morphism of schemes. Then by Lemma 2.5 (i) (ii), together with our assumption that U is reduced, U belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Hence we obtain the following natural bijections:

$$\mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X \times_Z^{\blacklozenge} Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U, X) \times_{\mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U,Z)} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(U, Y)$$

$$= \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X) \times_{\mathrm{Hom}_{\mathsf{Sch}_{/S}}(U,Z)} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, Y)$$

$$\xleftarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(U, X \times_Z Y).$$

Therefore, for any reduced affine scheme U and any morphism $f : U \to X \times_Z Y$, f factors uniquely through the natural morphism $v : X \times_Z^{\blacklozenge} Y \to X \times_Z Y$. This implies that v is isomorphic to the natural closed immersion $(X \times_Z Y)_{\mathrm{red}} \to X \times_Z Y$ in $\mathsf{Sch}_{/X \times_Z Y}$. This completes the proof of Lemma 4.1. $\square$

Corollary 4.2. *Let S be a quasi-separated scheme. Let $f : X \to Z, g : Y \to Z$ be morphisms of $\mathsf{Sch}_{\blacklozenge/S}$. Suppose that the fiber product $X \times_Z^{\blacklozenge} Y$ exists in $\mathsf{Sch}_{\blacklozenge/S}$. Then the natural morphism $v : X \times_Z^{\blacklozenge} Y \to X \times_Z Y$ is a surjective closed immersion. In particular, v is a homeomorphism.*

Proof. Corollary 4.2 follows immediately from Lemma 4.1 (i) (ii). $\square$

Next, we give a category-theoretic characterization of the morphisms of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is a closed immersion.

The following result was motivated by [vDdB19, Lemma 3.7].

Proposition 4.3. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then f is a closed immersion if and only if the following conditions hold:*

(i) f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$.

(ii) For any morphism $T \to Y$, the fiber product $X_{\blacklozenge,T} \in \mathsf{Sch}_{\blacklozenge/S}$ exists.

(iii) For any morphism $T \to Y$ and any closed point $t \in T$ such that $t \notin \mathrm{Im}(f_{\blacklozenge,T} : X_{\blacklozenge,T} \to T)$, the morphism $X_{\blacklozenge,T} \coprod \mathrm{Spec}(k(t)) \to T$ is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$.

In particular, the property that f is a closed immersion may be characterized category-theoretically (cf. Lemma 1.8, Corollary 3.15 (i)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X)$.

Proof. First, we prove necessity. Assume that $f : Y \to X$ is a closed immersion. Then, by Lemma 2.8 (i), f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, i.e., f satisfies condition (i). Let $T \to Y$ be a morphism. Since the closed immersion f is quasi-compact, it follows from Lemma 1.1 (iii) (iv) that the fiber product $X_{\blacklozenge,T}$ exists, i.e., f satisfies condition (ii). Let $t \in T \setminus \mathrm{Im}(f_{\blacklozenge,T})$ be a closed point. Since

f_T is a closed immersion, and $t \notin \mathrm{Im}(f_{\blacklozenge,T})$ is a closed point of T, the morphism $X_{\blacklozenge,T} \coprod \mathrm{Spec}(k(t)) \to T$ is a closed immersion. Hence by Lemma 2.8 (i), the morphism $X_{\blacklozenge,T} \coprod \mathrm{Spec}(k(t)) \to T$ is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$. Thus f satisfies condition (iii). This completes the proof of necessity.

Next, we prove sufficiency. Assume that f satisfies conditions (i), (ii), and (iii). Then, by Lemma 2.8 (ii), f is an immersion. Hence it suffices to prove that $\mathrm{Im}(f) \subset Y$ is closed. By condition (ii) and Corollary 4.2, to prove that $\mathrm{Im}(f) \subset Y$ is closed, it suffices to prove that for any affine open subscheme $V \subset Y$, $\mathrm{Im}(f_{\blacklozenge,V}) \subset V$ is closed, i.e., $\overline{\mathrm{Im}(f_{\blacklozenge,V})} = \mathrm{Im}(f_{\blacklozenge,V})$. Let V be an affine open subscheme of Y and $v \in V$ a closed point such that $v \notin \mathrm{Im}(f_{\blacklozenge,V})$. Since f satisfies condition (iii), the morphism $X_{\blacklozenge,V} \coprod \mathrm{Spec}(k(v)) \to V$ is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$. Hence by Lemma 2.8 (ii), the morphism $X_{\blacklozenge,V} \coprod \mathrm{Spec}(k(v)) \to V$ is an immersion, i.e., v is an isolated point of the topological subspace $\mathrm{Im}(f_{\blacklozenge,V}) \cup \{v\} \subset V$. Thus there exists an open neighborhood $v \in V' \subset V$ such that $\mathrm{Im}(f_{\blacklozenge,V}) \cap V' = \varnothing$, i.e., $v \notin \overline{\mathrm{Im}(f_{\blacklozenge,V})}$. This implies that the subset $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V}) \subset V$ does not contain any closed point of V. Since V is an affine scheme, to prove that $\overline{\mathrm{Im}(f_{\blacklozenge,V})} = \mathrm{Im}(f_{\blacklozenge,V})$, it suffices to prove that $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V}) \subset V$ is closed.

By Corollary 4.2, the natural morphism $X_{\blacklozenge,V} \to X_V$ is a surjective closed immersion. Since $f_{\blacklozenge,V} : X_{\blacklozenge,V} \to V$ is the composite of the natural surjective closed immersion $X_{\blacklozenge,V} \to X_V$ and the immersion $f_V : X_V \to V$, $f_{\blacklozenge,V}$ is an immersion. Hence $f_{\blacklozenge,V}$ admits a factorization $X_{\blacklozenge,V} \xrightarrow{g} U \subset V$ such that $g : X_{\blacklozenge,V} \to U$ is a closed immersion, and $U \subset V$ is an open immersion. Let $p \in \overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V})$ be a point. Suppose that $p \in U$. Then, since $p \notin \mathrm{Im}(f_{\blacklozenge,V}) = \mathrm{Im}(g)$, the open neighborhood $p \in U \setminus \mathrm{Im}(g)$ does not intersect $\mathrm{Im}(f_{\blacklozenge,V})$. This contradicts the fact that $p \in \overline{\mathrm{Im}(f_{\blacklozenge,V})}$. Hence $p \notin U$. This implies that $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V}) \subset \overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus U$. Since $\mathrm{Im}(f_{\blacklozenge,V}) \subset U$, the inclusion $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus U \subset \overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V})$ holds. Thus it holds that $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V}) = \overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus U$. Since $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \subset V$ is closed, and $U \subset V$ is open, we thus conclude that the subset $\overline{\mathrm{Im}(f_{\blacklozenge,V})} \setminus \mathrm{Im}(f_{\blacklozenge,V}) \subset V$ is closed. This completes the proof of Proposition 4.3. $\square$

Corollary 4.4. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $f : U \to X$ a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then f is an open immersion if and only if there exists a (necessarily quasi-compact) closed immersion $i : F \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that the following conditions hold:*

(i) $F \times_X^{\blacklozenge} U = \varnothing$ (cf. Lemma 1.1 (iii) (iv)).

(ii) If a morphism $g : Y \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ satisfies $Y \times_X^{\blacklozenge} F = \varnothing$ (cf. Lemma 1.1 (iii) (iv)), then g factors uniquely through $f : U \to X$.

In particular, the property that f is an open immersion may be characterized category-theoretically (cf. Proposition 4.3) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : Y \to X)$.

Proof. First, we prove necessity. Assume that $f : U \to X$ is an open immersion. Write $i : F :\overset{\mathrm{def}}{=} (X \setminus \mathrm{Im}(f))_{\mathrm{red}} \to X$ for the natural closed immersion. Since i is quasi-compact and separated, i belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $F \times_X U = \varnothing$, condition (i) holds. Let $g : Y \to X$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$ such that $Y \times_X^{\blacklozenge} F = \varnothing$. Then the underlying morphism of topological spaces $|g| : |Y| \to |X|$ factors uniquely through the open immersion of topological spaces $|f| : |U| \to |X|$. Thus $g : Y \to X$ factors uniquely through $f : U \to X$. This completes the proof of necessity.

Next, we prove sufficiency. Assume that there exists a closed immersion $i : F \rightarrow X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that $f : U \rightarrow X$ and $i : F \rightarrow X$ satisfy conditions (i) and (ii). Write $j : V = X \setminus \mathrm{Im}(i) \rightarrow X$ for the open immersion determined by the open subset $X \setminus \mathrm{Im}(i) \subset X$. By the necessity portion of Corollary 4.4, the open immersion $j : V \rightarrow X$ satisfies conditions (i) and (ii). Hence there exist morphisms $g : U \rightarrow V, h : V \rightarrow U$ such that $j \circ g = f, f \circ h = j$. By the uniqueness portion of condition (ii), the equalities $h \circ g = \mathrm{id}_U, g \circ h = \mathrm{id}_V$ hold. This proves that g is an isomorphism, i.e., $f = j \circ g$ is an open immersion. This completes the proof of sufficiency. $\qquad\square$

Next, we give a category-theoretic reconstruction of the underlying topological space of an object of $\mathsf{Sch}_{\blacklozenge/S}$.

Reconstruction 4.5. Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. We define a ($\mathbf{V}$-small) topological space

$$\mathsf{Op}_{\blacklozenge/S}(X) :\overset{\mathrm{def}}{=} \left\{ U \subset \mathsf{Pt}_{\blacklozenge/S}(X) \;\middle|\; \begin{array}{l} \text{there exists a collection of open} \\ \text{immersions } \{i_\lambda : U_\lambda \rightarrow X\}_{\lambda \in \Lambda} \text{ in } \mathsf{Sch}_{\blacklozenge/S} \\ \text{such that } U = \bigcup_{\lambda \in \Lambda} \mathrm{Im}(\mathsf{Pt}_{\blacklozenge/S}(i_\lambda)) \end{array} \right\},$$

$$\mathsf{Sp}_{\blacklozenge/S}(X) :\overset{\mathrm{def}}{=} (\mathsf{Pt}_{\blacklozenge/S}(X), \mathsf{Op}_{\blacklozenge/S}(X)).$$

These ($\mathbf{V}$-small) sets are defined completely in terms of properties that may be characterized category-theoretically (cf. Corollary 4.4) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$. Since any open subscheme $U \subset X$ admits a covering by affine open subschemes, it follows from Lemma 2.5 (i) (ii) that the bijection $\eta_X : \mathsf{Sp}_{\blacklozenge/S}(X) \rightarrow |X|$ (cf. Reconstruction 1.3) induces a natural bijection between $\mathsf{Op}_{\blacklozenge/S}(X)$ and the set of open subsets of X. Hence $\mathsf{Sp}_{\blacklozenge/S}(X)$ is a topological space, and the bijection η_X may be regarded (by a slight abuse of notation) as a homeomorphism. Since for any morphism $f : X \rightarrow Y$ of $\mathsf{Sch}_{\blacklozenge/S}$, the diagram

$$(\dagger) \qquad \begin{array}{ccc} \mathsf{Sp}_{\blacklozenge/S}(X) & \xrightarrow[\sim]{\eta_X} & |X| \\ {\scriptstyle \mathsf{Pt}_{\blacklozenge/S}(f)} \downarrow & & \downarrow {\scriptstyle |f|} \\ \mathsf{Sp}_{\blacklozenge/S}(Y) & \xrightarrow[\sim]{\eta_Y} & |Y| \end{array}$$

(cf. Reconstruction 1.3) commutes, the map $\mathsf{Pt}_{\blacklozenge/S}(f) : \mathsf{Sp}_{\blacklozenge/S}(X) \rightarrow \mathsf{Sp}_{\blacklozenge/S}(Y)$ is continuous. We shall write $\mathsf{Sp}_{\blacklozenge/S}(f)$ for $\mathsf{Pt}_{\blacklozenge/S}(f)$. Thus we obtain a functor $\mathsf{Sp}_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \rightarrow \mathsf{Top}_{\mathbf{V}}$.

Let $\bigstar :\overset{\mathrm{def}}{=} \mathsf{Top}$ or $\mathsf{Top}_{\mathbf{V}}$. We define

$$(\ddagger) \qquad \begin{array}{c} U^{\bigstar}_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \rightarrow \bigstar \\ X \mapsto |X|. \end{array}$$

Let $i^{\mathsf{Top}}_{\mathbf{U}\in\mathbf{V}} : \mathsf{Top} \rightarrow \mathsf{Top}_{\mathbf{V}}$ be the inclusion, then the equality $U^{\mathsf{Top}_{\mathbf{V}}}_{\blacklozenge/S} = i^{\mathsf{Top}}_{\mathbf{U}\in\mathbf{V}} \circ U^{\mathsf{Top}}_{\blacklozenge/S}$ holds. Since η_X is a homeomorphism, η may be regarded (by a slight abuse of notation) as a natural isomorphism $\eta : \mathsf{Sp}_{\blacklozenge/S} \xrightarrow{\sim} U^{\mathsf{Top}_{\mathbf{V}}}_{\blacklozenge/S} = i^{\mathsf{Top}}_{\mathbf{U}\in\mathbf{V}} \circ U^{\mathsf{Top}}_{\blacklozenge/S}$.

Since the functor $\mathsf{Sp}_{\blacklozenge/S}$ is defined category-theoretically from the data $\mathsf{Sch}_{\blacklozenge/S}$, the following lemma holds:

Lemma 4.6. *Let S, T be quasi-separated schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then S, T, F determine an isomorphism ρ^{Sp} between the two composite functors of the following diagram:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\ F\ } & \mathsf{Sch}_{\lozenge/T} \\
{\scriptstyle \mathsf{Sp}_{\blacklozenge/S}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{Sp}_{\lozenge/T}} \\
\mathsf{Top_V} & =\!=\!=\!= & \mathsf{Top_V}\,.
\end{array}
$$

Proof. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. By Lemma 1.4, the map $\rho_X : \mathsf{Pt}_{\blacklozenge/S}(X) \to \mathsf{Pt}_{\lozenge/T}(F(X))$ constructed in the proof of Lemma 1.4 is bijective. Hence to prove Lemma 4.6, it suffices to prove that the map $\rho_X : \mathsf{Sp}_{\blacklozenge/S}(X) \to \mathsf{Sp}_{\lozenge/T}(F(X))$ is continuous and open.

By the definitions of the topologies on $\mathsf{Sp}_{\blacklozenge/S}(X)$ and $\mathsf{Sp}_{\lozenge/T}(F(X))$, to prove that ρ_X is continuous, it suffices to prove that for any open immersion $j : V \to F(X)$ in $\mathsf{Sch}_{\lozenge/T}$, $\rho_X^{-1}(\mathrm{Im}(\mathsf{Sp}_{\lozenge/T}(j)))$ is an open subset of $\mathsf{Sp}_{\blacklozenge/S}(X)$. Let $j : V \to F(X)$ be an open immersion. If F^{-1} is a quasi-inverse of F, then by Corollary 4.4, the morphism $j' : F^{-1}(V) \to X$ corresponding to $j : V \to F(X)$ is an open immersion. Moreover, by Lemma 1.4, the equality $\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(j')) = \rho_X^{-1}(\mathrm{Im}(\mathsf{Sp}_{\lozenge/T}(j)))$ holds. Thus ρ_X is continuous.

By the definitions of the topologies on $\mathsf{Sp}_{\blacklozenge/S}(X)$ and $\mathsf{Sp}_{\lozenge/T}(F(X))$, to prove that ρ_X is open, it suffices to prove that for any open immersion $i : V \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$, $\rho_X(\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(i)))$ is an open subset of $\mathsf{Sp}_{\lozenge/T}(X)$. Let $i : U \to X$ be an open immersion. By Corollary 4.4, $F(i) : F(U) \to F(X)$ is an open immersion. Moreover, by Lemma 1.4, the equality $\rho_X(\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(i))) = \mathrm{Im}(\mathsf{Sp}_{\lozenge/T}(F(i)))$ holds. Thus ρ_X is an open map. This completes the proof of Lemma 4.6. $\qquad\square$

Corollary 4.7. *Let S, T be schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then the following diagram commutes up to natural isomorphism:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\ F\ } & \mathsf{Sch}_{\lozenge/T} \\
{\scriptstyle U_{\blacklozenge/S}^{\mathsf{Top}}}\Big\downarrow & & \Big\downarrow{\scriptstyle U_{\lozenge/T}^{\mathsf{Top}}} \\
\mathsf{Top} & =\!=\!=\!= & \mathsf{Top}\,.
\end{array}
$$

Proof. It follows from Lemma 4.6 and Reconstruction 4.5 that we obtain the following natural isomorphisms:

$$
i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Top}} \circ U_{\blacklozenge/S}^{\mathsf{Top}} \xleftarrow{\sim} \mathsf{Sp}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sp}_{\lozenge/T} \circ F \xrightarrow{\sim} i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Top}} \circ U_{\lozenge/T}^{\mathsf{Top}} \circ F.
$$

Since $i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Top}}$ is fully faithful, the composite isomorphism of the above display determines a natural isomorphism $U_{\blacklozenge/S}^{\mathsf{Top}} \xrightarrow{\sim} U_{\lozenge/T}^{\mathsf{Top}} \circ F$. $\qquad\square$

By Reconstruction 4.5, topological properties of schemes or morphisms may be characterized category-theoretically. For example, the following assertion holds:

Corollary 4.8. *Let S be a quasi-separated scheme.*

    (i) Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then the property that X is quasi-compact may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

    (ii) Let $f : Y \to Z$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then the property that f is quasi-compact may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.

Proof. Corollary 4.8 follow immediately from Reconstruction 4.5.   □
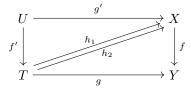
Corollary 4.9. *Let S be a quasi-separated scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$ and $f : Y \to Z$ a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

    (i) *X is quasi-separated if and only if for any two open immersions $U_1 \to X, U_2 \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that U_1, U_2 are quasi-compact, $U_1 \times_X^{\blacklozenge} U_2$ exists and is quasi-compact. In particular, the property that X is quasi-separated may be characterized category-theoretically (cf. Corollary 4.8) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

    (ii) *f is quasi-separated if and only if for any three open immersions $W \to Z, V_1 \to Y, V_2 \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that W, V_1, V_2 are quasi-compact and quasi-separated, and $V_1 \to Y \to Z, V_2 \to Y \to Z$ factor (necessarily uniquely) through $W \to Z$, $V_1 \times_Y^{\blacklozenge} V_2$ exists and is quasi-compact. In particular, the property that f is quasi-separated may be characterized category-theoretically (cf. Corollary 4.8, Corollary 4.9 (i)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.*

    (iii) *X is separated **over** S if and only if the image of the diagonal morphism $\Delta_X^{\blacklozenge} : X \to X \times^{\blacklozenge} X$ (where we note that $X \times^{\blacklozenge} X = X \times_S^{\blacklozenge} X$ exists in $\mathsf{Sch}_{\blacklozenge/S}$ by Lemma 1.1 (i) (ii) (iii) (iv)) is closed. In particular, the property that X is separated over S may be characterized category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. Assertions (i) and (ii) follow formally from Lemma 1.1 (iii) (iv), Lemma 2.5 (i) (ii), Corollary 4.2, and Reconstruction 4.5, together with the definition of the notion of a quasi-separated morphism. Assertion (iii) follows immediately from Corollary 4.2 and Reconstruction 4.5, together with the definition of the notion of a separated morphism.   □

Corollary 4.10. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

    (i) *f is separated if and only if f is quasi-separated, and for any commutative diagram*

in $\mathsf{Sch}_{\blacklozenge/S}$ such that
- *T is isomorphic to the spectrum of a valuation ring, and*
- *$f' : U \to T$ is isomorphic as an object of $\mathsf{Sch}_{/T}$ to the object of $\mathsf{Sch}_{/T}$ that arises from the natural morphism to T from the spectrum of the residue field at the generic point of T,*

it holds that $h_1 = h_2$. In particular, the property that f is separated may be characterized category-theoretically (cf. Lemma 1.8, Corollary 3.15 (ii), Proposition 3.19, Corollary 4.9 (ii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.

(ii) f *is quasi-compact and universally closed if and only if f is quasi-compact,*
and for any diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ g'\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{\ g\ } & Y
\end{array}
$$

in $\mathsf{Sch}_{\blacklozenge/S}$ *such that*
- T *is isomorphic to the spectrum of a valuation ring, and*
- $f' : U \to T$ *is isomorphic as an object of* $\mathsf{Sch}_{/T}$ *to the object of* $\mathsf{Sch}_{/T}$
  that arises from the natural morphism to T from the spectrum of the
  residue field at the generic point of T,

there exists a morphism $h : T \to X$ such that $g' = h \circ f', g = f \circ h$. In
particular, the property that f is quasi-compact and universally closed may
be characterized category-theoretically (cf. Lemma 1.8, Corollary 3.15 (ii),
Proposition 3.19, Corollary 4.8 (ii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.

Proof. These assertions follow from Lemma 2.5 (i) (ii), together with well-known
valuative criteria (cf. [Stacks, Tag 01KD] [Stacks, Tag 01KZ], [Stacks, Tag 01L0],
[Stacks, Tag 01KF]).   □

Finally, we conclude one of the main results of the present paper as follows:

Corollary 4.11. *Let S, T be quasi-separated schemes. If the (**V**-small) cate-*
gories $\mathsf{Sch}_{\blacklozenge/S}$, $\mathsf{Sch}_{\lozenge/T}$ *are equivalent, then the [possibly empty] subsets* $\blacklozenge, \lozenge \subset$
$\{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ *(such that $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge, \lozenge$) coincide, i.e.,* $\blacklozenge = \lozenge$.

Proof. Let $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ be an equivalence. It suffices to prove that $\lozenge \subset \blacklozenge$.
Assume that there exists an element $\star \in \lozenge \setminus \blacklozenge$. Write

$$
Z \overset{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{Z}[\varepsilon, X_1, X_2, \cdots, X_n, \cdots]/(\varepsilon^2)),
$$
$$
W \overset{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{Z}[\varepsilon, X_1, X_2, \cdots, X_n, \cdots]/(\varepsilon^2)) \setminus \{(0, 0, \cdots)\},
$$

where n ranges over the positive integers. Then
- $S \times Z$ satisfies $\mathrm{qcpt}/S, \mathrm{sep}/S$, but not red,
- $(S \times W)_{\mathrm{red}}$ satisfies $\mathrm{red}, \mathrm{sep}/S$, but not qcpt/S,
- $((S \times (Z \coprod_W Z))_{\mathrm{red}}$ satisfies $\mathrm{red}, \mathrm{qcpt}/S$, but not qsep/S, and
- $(\mathbb{A}^1_S \coprod_{\mathbb{A}^1_S \setminus \{0\}} \mathbb{A}^1_S)_{\mathrm{red}}$ satisfies $\mathrm{red}, \mathrm{qcpt}/S, \mathrm{qsep}/S$, but not sep/S.

Since $\star \notin \blacklozenge$, and $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge, \lozenge$, it follows immediately that there exists an
object X of $\mathsf{Sch}_{\blacklozenge/S}$ (which may in fact be taken to be one of the four schemes
in the above display) such that X does not satisfy $\star$. Then by Corollary 2.9,
Corollary 4.8 (ii), and Corollary 4.9 (ii) (iii), $F(X)$ does not satisfy property $\star$.
This contradicts the fact that $\star \in \lozenge$. Thus it holds that $\lozenge \subset \blacklozenge$.   □

5. Locally of Finite Presentation Morphisms

In this section, we give a category-theoretic characterization of the morphisms of
$\mathsf{Sch}_{\blacklozenge/S}$ whose underlying morphism of schemes is locally of finite presentation. To
give a category-theoretic characterization of such morphisms of $\mathsf{Sch}_{\blacklozenge/S}$, we define
the following properties:

Definition 5.1. (i) Let $f : A \to B$ be a local homomorphism between local rings. We shall say that f is *essentially of finite presentation* (respectively, *essentially of finite type*) if f admits a factorization $A \to C \to B$ such that $A \to C$ is a ring homomorphism of finite presentation (respectively, of finite type), and $C \to B$ is the localization morphism **at a prime ideal** of C.

(ii) Let $f : X \to Y$ be a morphism of schemes. We shall say that f is *stalkwise of finite presentation* (respectively, *stalkwise of finite type*) at a point $x \in X$ if the morphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is essentially of finite presentation (respectively, essentially of finite type). We shall say that f is *stalkwise of finite presentation* (respectively, *stalkwise of finite type*) if f is stalkwise of finite presentation at every point of X.

Remark 5.2.

(i) If a morphism of schemes is locally of finite presentation, then it is stalkwise of finite presentation.

(ii) If a morphism of rings is essentially of finite presentation, then it is essentially of finite type.

(iii) For any local ring A, the identity morphism id_A is essentially of finite presentation.

(iv) One verifies immediately that the composite of two morphisms of local rings which are essentially of finite presentation (respectively, essentially of finite type) is essentially of finite presentation (respectively, essentially of finite type).

Lemma 5.3. *Let $p : R \to A, f : A \to B, s : B \to A$ be ring homomorphisms. Then the following assertions hold:*

(i) *If f is of finite type, and $s \circ f = \mathrm{id}_A$, then s is of finite presentation.*

(ii) *If $f \circ p$ is of finite presentation, and $s \circ f = \mathrm{id}_A$, then p is of finite presentation.*

(iii) *Assume that R, A, B are local rings, and p, f are local homomorphisms. If $f \circ p$ is essentially of finite type, then f is also essentially of finite type.*

(iv) *Assume that A, B are local rings, and f, s are local homomorphisms. If f is essentially of finite type, and $s \circ f = \mathrm{id}_A$, then s is essentially of finite presentation.*

(v) *Assume that R, A, B are local rings, and p, f, s are local homomorphisms. If $f \circ p$ is essentially of finite presentation, and $s \circ f = \mathrm{id}_A$, then p is essentially of finite presentation.*

Proof. First, we verify assertion (i). Assume that f is of finite type, and $s \circ f = \mathrm{id}_A$. Then s is surjective. Hence it suffices to prove that $\ker(s)$ is finitely generated. Since f is of finite type, f admits a factorization $A \xrightarrow{i} A[x_1, \cdots, x_r] \xrightarrow{q} B$ such that $i : A \to A[x_1, \cdots, x_r]$ is the natural inclusion, and q is a surjective ring homomorphism. Thus we obtain the equality $\ker(s \circ q) = (x_1 - i(s(q(x_1))), \cdots, x_r - i(s(q(x_r))))$. In particular, $\ker(s \circ q)$ is a finitely generated ideal of $A[x_1, \cdots, x_r]$. Since q is a surjective ring homomorphism, we obtain the equality $\ker(s) = q(\ker(s \circ q))$. Thus $\ker(s)$ is a finitely generated ideal of B. This completes the proof of assertion (i).

Next, we verify assertion (ii). Assume that $f \circ p : R \to B$ is of finite presentation, and $s \circ f = \mathrm{id}_A$. Then f is of finite type. By (i), s is of finite presentation. Hence $p = s \circ f \circ p$ is of finite presentation. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Assume that $f \circ p$ is essentially of finite type. Then $f \circ p$ admits a factorization $R \xrightarrow{g} R' \xrightarrow{i} B$ such that $g : R \to R'$ is a ring homomorphism of finite type, and $i : R' \to B$ is the localization morphism at a prime ideal of R'. Hence the morphism $g' :\stackrel{\mathrm{def}}{=} \mathrm{id}_A \otimes g : A \to A \otimes_R R'$ is a ring homomorphism of finite type. Since $i : R' \to B$ is a localization morphism at a prime ideal of R', the natural morphism $A \otimes_R R' \to B$ determined by the equality $f \circ p = i \circ g$ admits a factorization $A \otimes_R R' \xrightarrow{h} A'' \xrightarrow{j} B$ such that $h : A \otimes_R R' \to A''$ is a surjective ring homomorphism, and $j : A'' \to B$ is the localization morphism at the prime ideal $A'' \cap \mathfrak{m}_B$, where $\mathfrak{m}_B$ is the unique maximal ideal of B. Thus the composite $h \circ g' : A \to A \otimes_R R' \to A''$ is a ring homomorphism of finite type, and we obtain a factorization $A \xrightarrow{h \circ g'} A'' \xrightarrow{j} B$ of f such that $h \circ g' : A \to A''$ is a ring homomorphism of finite type, and $j : A'' \to B$ is the localization morphism at a prime ideal of A''. This proves that f is essentially of finite type.

Next, we verify assertion (iv). Assume that f is essentially of finite type, and $s \circ f = \mathrm{id}_A$. Then f admits a factorization $A \xrightarrow{f'} B' \xrightarrow{q} B$ such that f' is a ring homomorphism of finite type, and q is the localization morphism at a prime ideal of B'. It follows from the equality $s \circ f = \mathrm{id}_A$ that s is a surjective local homomorphism. Hence to prove assertion (iv), it suffices to prove that $\ker(s)$ is a finitely generated ideal. Since f' is a ring homomorphism of finite type, and $s \circ q \circ f' = \mathrm{id}_A$, it follows from (i) that $s \circ q : B' \to A$ is of finite presentation. Hence $\ker(s \circ q)$ is a finitely generated ideal. Since q is the localization morphism at a prime ideal of B', $\ker(s)$ is the localization of $\ker(s \circ q)$ at a prime ideal. Thus $\ker(s)$ is a finitely generated ideal of B. This completes the proof of (iv).

Finally, we verify assertion (v). Assume that $f \circ p$ is essentially of finite presentation, and $s \circ f = \mathrm{id}_A$. By (iii), f is essentially of finite type. Since f is essentially of finite type, and $s \circ f = \mathrm{id}_A$, it follows from (iv) that s is essentially of finite presentation. Thus $p = s \circ (f \circ p)$ is essentially of finite presentation. This completes the proof of Lemma 5.3. $\square$

Corollary 5.4. *Let S, X, Y be schemes and $p : X \to S, f : Y \to X, s : X \to Y$ morphisms. Assume that $f \circ s = \mathrm{id}_X$, and $p \circ f : Y \to S$ is locally of finite presentation (respectively, stalkwise of finite presentation). Then p is locally of finite presentation (respectively, stalkwise of finite presentation).*

Proof. Corollary 5.4 follows immediately from Lemma 5.3 (ii) (v). $\square$

The following lemma will play an important role in the category-theoretic characterization of stalkwise of finite presentation morphisms that we give in Corollary 5.11 below.

Lemma 5.5. *Let S be a quasi-separated scheme. Let $(V_\lambda, \lambda \in \Lambda)$ be a diagram in $\mathsf{Sch}_{\blacklozenge/S}$ indexed by a cofiltered category Λ. Assume that there exists a $\mathbf{U}$-small cofinal (hence cofiltered) subcategory $\Lambda' \subset \Lambda$ such that V_λ is affine for every $\lambda \in \Lambda'$. Then the limits $\lim_{\lambda \in \Lambda} V_\lambda$ and $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ exist in $\mathsf{Sch}_{/S}$ and $\mathsf{Sch}_{\blacklozenge/S}$, respectively, and are naturally isomorphic in $\mathsf{Sch}_{/S}$ ($\supset \mathsf{Sch}_{\blacklozenge/S}$). Moreover, $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ may be taken to be $\mathrm{Spec}(\mathrm{colim}_{\lambda \in \Lambda'^{\mathrm{op}}} \Gamma(V_\lambda, \mathcal{O}_{V_\lambda}))$.*

Proof. It suffices to prove that the limit $\lim_{\lambda \in \Lambda} V_\lambda$ exists in $\mathsf{Sch}_{/S}$ and in fact belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $\Lambda' \subset \Lambda$ is cofinal, we may assume without loss of generality that $\Lambda' = \Lambda$. Then we may take $\lim_{\lambda \in \Lambda} V_\lambda$ to be $\mathrm{Spec}(\mathrm{colim}_{\lambda \in \Lambda} \Gamma(V_\lambda, \mathcal{O}_{V_\lambda}))$. In

particular, $\lim_{\lambda \in \Lambda} V_\lambda$ exists in $\mathsf{Sch}_{/S}$ and is affine. By Lemma 2.5 (i) (ii), $\lim_{\lambda \in \Lambda} V_\lambda$ is separated and quasi-compact over S. If each V_λ is reduced, then $\lim_{\lambda \in \Lambda} V_\lambda$ is also reduced. Thus $\lim_{\lambda \in \Lambda} V_\lambda$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. This completes the proof of Lemma 5.5. $\qquad \square$

Lemma 5.6. *Let* $(C_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ *be a filtered diagram of local rings and local homomorphisms. Then* $\operatorname{colim}_{\lambda \in \Lambda} C_\lambda$ *is local, and the natural morphism* $i_\lambda : C_\lambda \to \operatorname{colim}_{\lambda \in \Lambda} C_\lambda$ *is local.*

Proof. Let $I \subsetneq C :\overset{\mathrm{def}}{=} \operatorname{colim}_{\lambda \in \Lambda} C_\lambda$ be an ideal. Write $\mathfrak{m}_\lambda$ for the maximal ideal of C_λ, $\mathfrak{m} :\overset{\mathrm{def}}{=} \operatorname{colim}_{\lambda \in \Lambda} \mathfrak{m}_\lambda$, and $I_\lambda :\overset{\mathrm{def}}{=} i_\lambda^{-1}(I) \subsetneq C_\lambda$ for the inverse image of the ideal I. Since C_λ is local, $I_\lambda \subset \mathfrak{m}_\lambda$. Since the natural morphism $\operatorname{colim}_{\lambda \in \Lambda} I_\lambda \to I$ is an isomorphism, it holds that $I \subset \mathfrak{m}$. This implies that $\mathfrak{m}$ is a unique maximal ideal of C, i.e., that C is local, and i_λ is local. This completes the proof of Lemma 5.6. $\quad \square$

Corollary 5.7. *Let* S *be a quasi-separated scheme. Then the following assertions hold:*

(i) *Let* $(V_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ *be a* $\mathbf{U}$*-small cofiltered diagram in* $\mathsf{Sch}_{\blacklozenge/S}$ *such that each* V_λ *is local, and each* $p_{\lambda\mu}$ *is local. Then* $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ *exists. Moreover,* $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ *is local, and each projection* $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda \to V_\lambda$ *is local.*

(ii) *Let* $X \in \mathsf{Sch}_{\blacklozenge/S}$ *be an object and* $x \in X$ *a point. Write* $\mathcal{I}$ *for the full subcategory of* $\mathsf{Sch}_{\blacklozenge/X}$ *consisting of all open immersions* $U \to X$ *in* $\mathsf{Sch}_{\blacklozenge/X}$ *such that* x *belongs to the image of* $U \to X$*. (Thus, by a slight abuse of notation,* $\mathcal{I}$ *may also be regarded, by applying the natural functor* $\mathsf{Sch}_{\blacklozenge/X} \to \mathsf{Sch}_{\blacklozenge/S}$*, as a diagram indexed by* $\mathcal{I}$*.) Then* $\lim^{\blacklozenge}_{U \in \mathcal{I}} U$ *exists, and may be taken to be* $\operatorname{Spec}(\mathcal{O}_{X,x})$*.*

Proof. First, we prove assertion (i). Since a local object in $\mathsf{Sch}_{\blacklozenge/S}$ is affine, it follows from Lemma 5.5 that $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ exists, and

$$\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda \cong \operatorname{Spec}(\operatorname{colim}_{\lambda \in \Lambda^{\mathrm{op}}} \Gamma(V_\lambda, \mathcal{O}_{V_\lambda})).$$

By Lemma 5.6, $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda$ is local, and each projection $\lim^{\blacklozenge}_{\lambda \in \Lambda} V_\lambda \to V_\lambda$ is local. Thus assertion (i) holds.

Next, we prove assertion (ii). Since the $\mathbf{U}$-small subcategory

$$\{\, (U \subset X) \in \mathcal{I} \mid U \text{ is an affine open neighborhood of } x \,\} \subset \mathcal{I}$$

is cofinal, assertion (ii) follows from Lemma 5.5. $\qquad \square$

Lemma 5.8. *Let* $f : R \to A$ *be a local homomorphism between* $\mathbf{U}$*-small local rings. Write* LAlg_R *for the category of* $\mathbf{U}$*-small local* R*-algebras and local homomorphisms of* R*-algebras. Then the following assertions hold:*

(i) *If* f *is essentially of finite presentation, then for any* $\mathbf{U}$*-small filtered diagram* $(C_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ *in* LAlg_R*, the natural morphism*
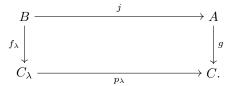
$$\varphi : \operatorname{colim}_{\lambda \in \Lambda} \operatorname{Hom}_{\mathsf{LAlg}_R}(A, C_\lambda) \to \operatorname{Hom}_{\mathsf{LAlg}_R}(A, \operatorname{colim}_{\lambda \in \Lambda} C_\lambda)$$

(where we note that by Lemma 5.6, $\operatorname{colim}_{\lambda \in \Lambda} C_\lambda$ *is local) is surjective.*
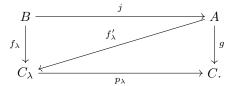
(ii) *There exists a* $\mathbf{U}$*-small filtered diagram* $(C_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ *in* LAlg_R *such that each* R*-algebra* C_λ *is essentially of finite presentation, and* $\operatorname{colim}_{\lambda \in \Lambda} C_\lambda \cong A$*.*

(iii) *Assume that R, A are reduced. Then there exists a $\mathbf{U}$-small filtered diagram $(C_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ in LAlg_R such that each R-algebra C_λ is reduced and essentially of finite type, and $\operatorname{colim}_{\lambda \in \Lambda} C_\lambda \cong A$.*

Proof. First, we prove assertion (i). Let $(C_\lambda, p_{\lambda\mu})_{\lambda \in \Lambda}$ be a $\mathbf{U}$-small filtered diagram in LAlg_R. Write $C :\overset{\mathrm{def}}{=} \operatorname{colim}_{\lambda \in \Lambda} C_\lambda$ and $p_\lambda : C_\lambda \to C$ for the natural morphism. By Lemma 5.6, C is local, and each p_λ is local. Let $g : A \to C$ be a local homomorphism of R-algebras. Since f is essentially of finite presentation, the morphism of local rings $f : R \to A$ admits a factorization $R \overset{h}{\to} B \overset{j}{\to} A$ such that $h : R \to B$ is of finite presentation, and $j : B \to A$ is the localization morphism at a prime ideal of B. Since B is an R-algebra of finite presentation, it follows from [Stacks, Tag 00QO] that there exist an index $\lambda \in \Lambda$ and a morphism $B \overset{f_\lambda}{\longrightarrow} C_\lambda$ of R-algebras such that the following diagram commutes:

Since j is the localization morphism at a prime ideal of B, and p_λ, g are local homomorphisms of local rings, there exists a local morphism $f'_\lambda : A \to C_\lambda$ such that $f_\lambda = f'_\lambda \circ j$:

Since $g \circ j = p_\lambda \circ f_\lambda = p_\lambda \circ f'_\lambda \circ j$, and j is the localization morphism at a prime ideal of B, the equality $g = p_\lambda \circ f'_\lambda$ holds. This implies that

$$\varphi([f'_\lambda]) = p_\lambda \circ f'_\lambda = g,$$

where $[f'_\lambda]$ denotes the element $\in \operatorname{colim}_{\lambda \in \Lambda} \operatorname{Hom}_{\mathsf{LAlg}_R}(A, C_\lambda)$ determined by $f'_\lambda \in \operatorname{Hom}_{\mathsf{LAlg}_R}(A, C_\lambda)$. Thus φ is surjective. This completes the proof of assertion (i).

Next, we prove assertion (ii). By [Stacks, Tag 0BUF], there exists a $\mathbf{U}$-small filtered diagram $(B_\lambda, q_{\lambda\mu})_{\lambda \in \Lambda}$ of R-algebras of finite presentation and homomorphisms of R-algebras such that $\operatorname{colim}_{\lambda \in \Lambda} B_\lambda \cong A$. Write $q_\lambda : B_\lambda \to A$ for the natural morphism, $\mathfrak{m}$ for the maximal ideal of A, $\mathfrak{q}_\lambda :\overset{\mathrm{def}}{=} q_\lambda^{-1}(\mathfrak{m})$, $C_\lambda :\overset{\mathrm{def}}{=} B_{\lambda, \mathfrak{q}_\lambda}$ for the localization of B_λ at $\mathfrak{q}_\lambda$. Then the morphism $q_\lambda : B_\lambda \to A$ induces the local homomorphism of local R-algebras $p_\lambda : C_\lambda \to A$, and any isomorphism $\operatorname{colim}_{\lambda \in \Lambda} B_\lambda \overset{\sim}{\to} A$ induces an isomorphism $\operatorname{colim}_{\lambda \in \Lambda} C_\lambda \overset{\sim}{\to} A$. Since C_λ is essentially of finite presentation over R, this completes the proof of assertion (ii).

Finally, we prove assertion (iii). By assertion (ii), there exists a $\mathbf{U}$-small filtered diagram $(B_\lambda, q_{\lambda\mu})_{\lambda \in \Lambda}$ of essentially of finite type local R-algebras and local homomorphisms such that $\operatorname{colim}_{\lambda \in \Lambda} B_\lambda \cong A$. Write $q_\lambda : B_\lambda \to A$ for the natural morphism. Since A is reduced, $\operatorname{Im}(q_\lambda) \subset A$ is a reduced local R-algebra. Then it holds that $A = \bigcup_{\lambda \in \Lambda} \operatorname{Im}(q_\lambda)$. This implies that $\operatorname{colim}_{\lambda \in \Lambda} \operatorname{Im}(q_\lambda) \cong A$. Write

$C_\lambda :\stackrel{\text{def}}{=} \mathrm{Im}(q_\lambda)$ and $p_{\lambda\mu} : C_\lambda \to C_\mu$ for the morphism induced by $q_{\lambda\mu}$. Since q_λ is local, $(C_\lambda, p_{\lambda\mu})_{\lambda\in\Lambda}$ is a **U**-small filtered diagram in LAlg_R. This completes the proof of Lemma 5.8. $\qquad\square$

Corollary 5.9. *Let S be a quasi-separated scheme. Let X, Y be local objects of $\mathsf{Sch}_{\blacklozenge/S}$ and $f : X \to Y$ a local morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Write $(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}$ for the full subcategory of $\mathsf{Sch}_{\blacklozenge/Y}$ determined by the local objects and the local morphisms. Consider the following condition on f:*

> *(†) For any **U**-small cofiltered diagram $(V_\lambda, p_{\lambda\mu})_{\lambda\in\Lambda}$ of $(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}$, the natural morphism*
>
> $$\varphi : \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Hom}_{(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}}(V_\lambda, X) \to \mathrm{Hom}_{(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}}(\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda, X)$$
>
> *is surjective (where we note that by Corollary 5.7 (i), $\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda$ exists in $(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}$).*

Then the following assertions hold:

> *(i) If the morphism of local rings $f^{\#} : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ is essentially of finite presentation, then condition (†) holds.*
> *(ii) If condition (†) holds, then the morphism of local rings $f^{\#} : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ is essentially of finite type.*

Proof. Assertion (i) follows immediately from Lemma 5.8 (i). In the remainder of the proof of Corollary 5.9, we verify assertion (ii). Assume that f satisfies condition (†). By Lemma 2.5 (i) (ii), Lemma 5.5, and Lemma 5.8 (ii) (iii), there exists a cofiltered diagram $(V_\lambda, p_{\lambda\mu})_{\lambda\in\Lambda}$ of $(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}$ such that $\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda \cong X$, and for any $\lambda \in \Lambda$, $\Gamma(V_\lambda, \mathcal{O}_{V_\lambda})$ is essentially of finite type over $\Gamma(Y, \mathcal{O}_Y)$. Let $g \in \mathrm{Hom}_{(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}}(\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda, X)$ be an isomorphism. Write $p_\lambda : \lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda \to V_\lambda$ for the natural projection. Since f satisfies condition (†), there exist an index $\lambda \in \Lambda$ and a morphism $g' : V_\lambda \to X$ in $(\mathsf{Sch}_{\blacklozenge/Y})_{\mathrm{loc}}$ such that $g' \circ p_\lambda = g$. Since g is an isomorphism, it follows from Lemma 5.3 (iii) that $p_\lambda^{\#}$ is essentially of finite type. Since $g'^{\#} \circ f^{\#} : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(V_\lambda, \mathcal{O}_{V_\lambda})$ is essentially of finite type, it follows from Remark 5.2 (iv) that $f^{\#} = (g^{-1})^{\#} \circ p_\lambda^{\#} \circ (g'^{\#} \circ f^{\#})$ is essentially of finite type. This completes the proof of Corollary 5.9. $\qquad\square$

Definition 5.10. Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. We shall say that f is *category-theoretically stalkwise of finite presentation* if f satisfies following condition:

> For any **U**-small cofiltered diagram $(V_\lambda, p_{\lambda\mu})_{\lambda\in\Lambda}$ of $\mathsf{Sch}_{\blacklozenge/Y}$ such that each V_λ is local, and each $p_{\lambda\mu}$ is local, the natural morphism
>
> $$\varphi : \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(V_\lambda, X) \to \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda, X)$$
>
> is surjective (where we note that by Corollary 5.7 (i), $\lim_{\lambda\in\Lambda}^{\blacklozenge} V_\lambda$ exists in $\mathsf{Sch}_{\blacklozenge/Y}$).

Thus the property that f is category-theoretically stalkwise of finite presentation is defined completely in terms of properties that may be characterized category-theoretically (cf. Corollary 3.15 (i) (iii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.

Corollary 5.11. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

(i) If f is stalkwise of finite presentation, then f is category-theoretically stalk-wise of finite presentation.

(ii) If f is category-theoretically stalkwise of finite presentation, then f is stalk-wise of finite type.

Proof. Assertion (i) follows immediately from Corollary 5.9 (i). Assertion (ii) follows immediately from Corollary 5.9 (ii). □

Definition 5.12. Let X be a scheme. We shall say that X is *stalkwise Noetherian* if for any point $x \in X$, $\mathcal{O}_{X,x}$ is Noetherian.

Corollary 5.13. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Assume that Y is stalkwise Noetherian. Then f is stalkwise of finite presentation if and only if f is category-theoretically stalkwise of finite presentation.*

Proof. Necessity follows immediately from Corollary 5.11 (i). Sufficiency follows immediately from Corollary 5.11 (ii). □

Remark 5.14. Let A be a ring and $f \in A$ an element. Then it holds that $A_f \cong A[x]/(xf - 1)$. Thus A_f is of finite presentation over A.

Lemma 5.15. *Let $f : B \to A$ be a ring homomorphism and $\mathfrak{p} \in \mathrm{Spec}(A)$ a prime ideal. Write $\mathfrak{q} :\stackrel{\mathrm{def}}{=} f^{-1}(\mathfrak{p})$ and $f_\mathfrak{p} : B_\mathfrak{q} \to A_\mathfrak{p}$ for the morphism induced by localizing at $\mathfrak{q}$ and $\mathfrak{p}$, respectively. Assume that $f_\mathfrak{p}$ is essentially of finite presentation. Then there exist a B-algebra C of finite presentation and a prime ideal $\mathfrak{r}$ of C such that $A_\mathfrak{p} \cong C_\mathfrak{r}$ as B-algebras, hence also as $B_\mathfrak{q}$-algebras.*

Proof. Lemma 5.15 follows immediately from Definition 5.1 (i), together with Remark 5.14. □

Definition 5.16. Let X be a scheme, and $x \in X$ a point of X. We shall write $I_X(x)$ for the full subcategory of $\mathsf{Sch}_{\blacklozenge/X}$ consisting of all open immersions $U \to X$ in $\mathsf{Sch}_{\blacklozenge/X}$ such that x belongs to the image of $U \to X$. (Thus, by a slight abuse of notation, if X is an S-scheme, then $I_X(x)$ may also be regarded, by applying the natural functor $\mathsf{Sch}_{\blacklozenge/X} \to \mathsf{Sch}_{\blacklozenge/S}$, as a diagram indexed by $I_X(x)$.)

Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal of A. We shall define $J_A(\mathfrak{p})$ for the full subcategory of A-Alg determined by the set of A-algebras $\{ A_f \mid f \in A \setminus \mathfrak{p} \}$. (Thus, by a slight abuse of notation, if A is a B-algebra, then $J_A(\mathfrak{p})$ may also be regarded, by applying the natural functor A-$\mathsf{Alg} \to B$-Alg, as a diagram indexed by $J_A(\mathfrak{p})$.) Then it follows immediately that $\mathrm{colim}_{A_f \in J_A(\mathfrak{p})} A_f \cong A_\mathfrak{p}$.

Lemma 5.17. *Let $\xi : B \to A$ be a ring homomorphism and $\mathfrak{p} \in \mathrm{Spec}(A)$ a prime ideal. Write $\mathfrak{p}_B :\stackrel{\mathrm{def}}{=} \xi^{-1}(\mathfrak{p})$, $\xi_\mathfrak{p} : B_{\mathfrak{p}_B} \to A_\mathfrak{p}$ for the morphism induced by localizing at $\mathfrak{p}_B$ and $\mathfrak{p}$, respectively. Assume that the following conditions hold:*

(i) *$\xi_\mathfrak{p}$ is essentially of finite presentation.*

(ii) *For any B-algebra R and any prime ideal $\mathfrak{r} \subset R$, the natural map*

$$\mathrm{colim}_{R_r \in J_R(\mathfrak{r})} \mathrm{Hom}_{B\text{-}\mathsf{Alg}}(A, R_r) \to \mathrm{Hom}_{B\text{-}\mathsf{Alg}}(A, \mathrm{colim}_{R_r \in J_R(\mathfrak{r})} R_r)$$

is a bijection.

Then there exists an element $f \in A \setminus \mathfrak{p}$ such that the composite of ξ and the localization morphism $A \to A_f$ is of finite presentation.

Proof. By Lemma 5.15 and condition (i), there exist a B-algebra C of finite presentation, a prime ideal $\mathfrak{q} \subset C$, and a morphism of B-algebras $\psi : A \to C_\mathfrak{q}$ such that $\mathfrak{p} = \psi^{-1}(\mathfrak{q})$, and the morphism $\psi_\mathfrak{p} : A_\mathfrak{p} \xrightarrow{\sim} C_\mathfrak{q}$ induced by localizing at $\mathfrak{p}$ is an isomorphism. If $h \in C \setminus \mathfrak{q}$, then write $\gamma_h : C_h \to C_\mathfrak{q}$ for the localization morphism. By condition (ii), there exist an element $h \in C$ and a morphism of B-algebras $\psi_h : A \to C_h$ such that

$$\psi = \gamma_h \circ \psi_h.$$

By Remark 5.14, C_h is of finite presentation over C, hence over B. Thus, by [Stacks, Tag 00QO], the natural map

$$(\dagger) \qquad \operatorname{colim}_{A_f \in J_A(\mathfrak{p})} \operatorname{Hom}_{B\text{-Alg}}(C_h, A_f) \to \operatorname{Hom}_{B\text{-Alg}}(C_h, \operatorname{colim}_{A_f \in J_A(\mathfrak{p})} A_f)$$

is surjective.

For any elements $f_1, f_2 \in A$ and any prime ideal $\mathfrak{p}_1 \subset A$ such that $f_1 \notin \mathfrak{p}_1$, we shall write

$$\alpha_{f_1} : A \to A_{f_1} \ , \ \alpha_{f_1,f_2} : A_{f_1} \to A_{f_1 f_2} \ , \ \alpha_{f_1,\mathfrak{p}_1} : A_{f_1} \to A_{\mathfrak{p}_1}$$

for the respective localization morphisms. Since the map $(\dagger)$ is surjective, there exist an element $f_1 \in A \setminus \mathfrak{p}$ and a morphism of B-algebras $\phi_{f_1} : C_h \to A_{f_1}$ such that $\psi_\mathfrak{p}^{-1} \circ \gamma_h = \alpha_{f_1,\mathfrak{p}} \circ \phi_{f_1} : C_h \to A_\mathfrak{p}$. Since $\psi_\mathfrak{p}^{-1} \circ \gamma_h \circ \psi_h = \psi_\mathfrak{p}^{-1} \circ \psi = \alpha_{1,\mathfrak{p}} \circ \alpha_1$, it holds that $\alpha_{1,\mathfrak{p}} \circ \alpha_1 = \alpha_{f_1,\mathfrak{p}} \circ \phi_{f_1} \circ \psi_h : A \to A_\mathfrak{p}$:
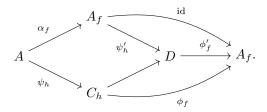
$$
\begin{array}{ccc}
A & \xrightarrow[\sim]{\alpha_1} & A_1 \\
{\scriptstyle \phi_{f_1} \circ \psi_h} \downarrow & & \downarrow {\scriptstyle \alpha_{1,\mathfrak{p}}} \\
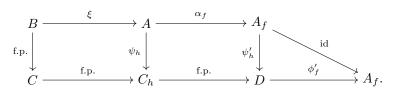A_{f_1} & \xrightarrow{\alpha_{f_1,\mathfrak{p}}} & A_\mathfrak{p}.
\end{array}
$$

Thus, by applying the injectivity portion of condition (ii) in the case where $R = A$, and $\mathfrak{r} = \mathfrak{p}$, we conclude that there exists an element $f_2 \in A$ such that $\alpha_{f_1 f_2} = \alpha_{f_1,f_2} \circ \phi_{f_1} \circ \psi_h : A \to A_{f_1 f_2}$.

Write $f :\overset{\text{def}}{=} f_1 f_2 \in A \setminus \mathfrak{p}$, $D :\overset{\text{def}}{=} C_{h\psi_h(f)} \cong C_h \otimes_A A_f$, $\phi_f :\overset{\text{def}}{=} \alpha_{f_1,f_2} \circ \phi_{f_1}$, and $\psi_h' : A_f \to D$ for the morphism of B-algebras obtained by base changing $\psi_h : A \to C_h$ by $\alpha_f : A \to A_f$. Since $\alpha_f = \alpha_{f_1 f_2} = \alpha_{f_1,f_2} \circ \phi_{f_1} \circ \psi_h = \phi_f \circ \psi_h$, it follows from the universal property of the tensor product that there exists a unique morphism $\phi_f' : D \to A_f$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
 & & A_f & \xrightarrow{\ \text{id}\ } & \\
 & {\scriptstyle \alpha_f}\nearrow & & \searrow{\scriptstyle \psi_h'} & \\
A & & & & D \xrightarrow{\phi_f'} A_f. \\
 & {\scriptstyle \psi_h}\searrow & & \nearrow & \\
 & & C_h & \xrightarrow[\phi_f]{} &
\end{array}
$$

By Remark 5.14, D is of finite presentation over C, hence over B. Thus, it follows from Lemma 5.3 (ii), together with the commutativity of the following diagram,

that the composite $B \xrightarrow{\xi} A \to A_f$ is of finite presentation:

$$
\begin{array}{ccccc}
B & \xrightarrow{\ \ \xi\ \ } & A & \xrightarrow{\ \ \alpha_f\ \ } & A_f \\
\text{f.p.}\big\downarrow & & \psi_h\big\downarrow & & \psi'_h\big\downarrow \quad\quad \text{id} \\
C & \xrightarrow[\text{f.p.}]{} & C_h & \xrightarrow[\text{f.p.}]{} & D \xrightarrow{\ \ \phi'_f\ \ } A_f.
\end{array}
$$

This completes the proof of Lemma 5.17. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 5.18. Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. We shall say that f is *category-theoretically of finite presentation* if f satisfies following conditions:

(i) f is category-theoretically stalkwise of finite presentation.

(ii) For any morphism $Z \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ and any point $z \in Z$, the natural map

$$
\varphi_{z,X} : \mathrm{colim}_{W \in I_Z(z)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(W, X) \to \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(\lim^{\blacklozenge}_{W \in I_Z(z)} W, X)
$$

is a bijection (where we note that by Corollary 5.7 (ii), $\lim^{\blacklozenge}_{W \in I_Z(z)} W$ exists in $\mathsf{Sch}_{\blacklozenge/S}$).

Thus the property that f is category-theoretically of finite presentation is defined completely in terms of properties that may be characterized category-theoretically (cf. Reconstruction 1.3, Corollary 1.6 (ii), Corollary 4.4, Definition 5.10) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f)$.

Corollary 5.19. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then the following assertions hold:*

(i) If f is locally of finite presentation, then f is category-theoretically of finite presentation.

(ii) If f is category-theoretically of finite presentation, and Y is stalkwise Noetherian, then f is locally of finite presentation.

Proof. Assertion (ii) follows immediately from Corollary 5.13 and Lemma 5.17. In the remainder of the proof of Corollary 5.19, we prove assertion (i). Assume that f is locally of finite presentation. By Remark 5.2 (i) and Corollary 5.11 (i), f is category-theoretically stalkwise of finite presentation. Thus f satisfies condition (i) of Definition 5.18. Let $Z \to Y$ be a morphism and $z \in Z$ a point of Z. Since the full subcategory

$$
I_Z^{\mathrm{aff}}(z) :\stackrel{\mathrm{def}}{=} \{\, W \subset Z \mid W \text{ is an affine open neighborhood of } z \,\} \subset I_Z(z)
$$

is cofinal, and $\mathsf{Sch}_{\blacklozenge/Y} \subset \mathsf{Sch}_{/Y}$ is a full subcategory, the natural morphism

$$
\mathrm{colim}_{W \in I_Z^{\mathrm{aff}}(z)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(W, X) \xrightarrow{\ \sim\ } \mathrm{colim}_{W \in I_Z(z)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{/Y}}(W, X)
$$

is an isomorphism. By Lemma 5.5, the natural morphism

$$
\lim^{\blacklozenge}_{W \in I_Z(z)} W \xrightarrow{\ \sim\ } \lim_{W \in I_Z^{\mathrm{aff}}(z)} W
$$

is an isomorphism. Since each $W \in I_Z^{\mathrm{aff}}(z)$ is affine, and f is locally of finite presentation, it follows from [Stacks, Tag 01ZC] that f satisfies condition (ii) of Definition 5.18. This completes the proof of Corollary 5.19. $\qquad\square$

Corollary 5.20. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Assume that Y is stalkwise Noetherian. Then f is locally of finite presentation if and only if f is category-theoretically of finite presentation. In particular, if S is stalkwise Noetherian, then the property that X is locally of finite presentation over S may be characterized category-theoretically (cf. Definition 5.18) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. Corollary 5.20 follows immediately from Corollary 5.19 (i) (ii). $\square$

Corollary 5.21. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Assume that Y is stalkwise Noetherian. Then f is proper and of finite presentation if and only if f is quasi-compact, separated, universally closed, and category-theoretically of finite presentation. In particular, if S is stalkwise Noetherian, then the property that X is proper and of finite presentation over S may be characterized category-theoretically (cf. Corollary 4.10 (i) (ii), Definition 5.18) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. Corollary 5.21 follows immediately from Corollary 5.20. $\square$

6. The Projective Line

In this section, we give a category-theoretic characterization of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ whose underlying S-scheme is isomorphic to the projective line $\mathbb{P}^1_S$. We then use this characterization to give a functorial category-theoretic algorithm for reconstructing the underlying schemes of the objects of $\mathsf{Sch}_{\blacklozenge/S}$ from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$. In addition, we discuss some results related to the various reconstruction algorithms developed in the present paper.

First, we give a category-theoretic characterization of morphisms which are isomorphic to the projection $\mathbb{P}^1_k \to \mathrm{Spec}(k)$, where k is a field.

Lemma 6.1. *Let S be a quasi-separated scheme. Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$ such that X, Y are isomorphic to the spectrum of a field. Write $K :\overset{\mathrm{def}}{=} \Gamma(X, \mathcal{O}_X), k :\overset{\mathrm{def}}{=} \Gamma(Y, \mathcal{O}_Y)$. Then $f^{\#} : k \to K$ is a purely transcendental extension of degree 1 if and only if the following conditions hold:*

  (i) There exists a morphism $g : X \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that g is not an isomorphism, and $f \circ g = f$.
  (ii) If f admits a factorization $X \to Z \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that Z is isomorphic to the spectrum of a field, and $Z \to Y$ is not an isomorphism, then there exists an isomorphism $Z \xrightarrow{\sim} X$ in $\mathsf{Sch}_{\blacklozenge/Y}$.

In particular, the property that

> *X, Y are isomorphic to the spectrum of a field, and $f : X \to Y$ is isomorphic as an object of $\mathsf{Sch}_{/Y}$ to the object of $\mathsf{Sch}_{/Y}$ that arises from a purely transcendental field extension of degree 1*

may be characterized category-theoretically (cf. Lemma 1.7) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : X \to Y)$.

Proof. First, we prove necessity. Assume that $f^{\#} : k \to K$ is a purely transcendental extension of degree 1. Then there exists a transcendental element $t \in K$ such that $K = k(t)$. By considering the k-algebra morphism $k(t) \to k(t), t \mapsto t^2$, we conclude that f satisfies condition (i). Let $k \to L \to k(t)$ be a factorization of $f^{\#}$ such that L is a field, and $k \to L$ is not an isomorphism. Then by Lüroth's

theorem (cf., e.g., [Ha, Chapter IV, Example 2.5.5]), there exists an element $u \in L$ such that $L = k(u)$. Hence we obtain an isomorphism of k-algebras $K \xrightarrow{\sim} L, t \mapsto u$. Thus f satisfies condition (ii). This completes the proof of the necessity.

Next, we prove sufficiency. Assume that $k :\overset{\text{def}}{=} \Gamma(Y, \mathcal{O}_Y)$ and $K :\overset{\text{def}}{=} \Gamma(X, \mathcal{O}_X)$ are fields, and $f : X \to Y$ satisfies conditions (i) and (ii). Since f satisfies condition (i), it follows from [Stacks, Tag 0BMD] that K/k is not an algebraic extension. Hence there exists a transcendental element $t \in K$ over k. Then, by condition (ii), there exists an isomorphism of k-algebras $k(t) \cong K$, which implies that K is a purely transcendental extension of degree 1. This completes the proof of Lemma 6.1. $\square$

Lemma 6.2. *Let S be a quasi-separated scheme. Let X, Y be objects of $\mathsf{Sch}_{\blacklozenge/S}$ such that Y is isomorphic to the spectrum of a field, and $f : X \to Y$ a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. Then $f : X \to Y$ is isomorphic as an object of $\mathsf{Sch}_{\blacklozenge/Y}$ to the object of $\mathsf{Sch}_{\blacklozenge/Y}$ that arises from the natural projection $\mathbb{P}^1_Y \to Y$ if and only if the following conditions hold:*

- *(i) f is proper and of finite presentation.*
- *(ii) X is integral.*
- *(iii) If $\eta \in X$ is the generic point, then the composite $\mathrm{Spec}(k(\eta)) \to X \to Y$ is isomorphic as a Y-scheme to the spectrum of a purely transcendental field extension of degree 1.*
- *(iv) For every closed point $x \in X$, the spectrum of the local ring $\mathrm{Spec}(\mathcal{O}_{X,x})$ is isomorphic to the spectrum of a valuation ring.*

In particular, the property that Y is isomorphic to the spectrum of a field, and $f : X \to Y$ is isomorphic as an object of $\mathsf{Sch}_{\blacklozenge/Y}$ to the object of $\mathsf{Sch}_{\blacklozenge/Y}$ that arises from the natural projection $\mathbb{P}^1_Y \to Y$ may be characterized category-theoretically (cf. Corollary 3.15 (ii), Corollary 3.16 (i), Proposition 3.18, Proposition 3.19, Corollary 5.21, Lemma 6.1) from the data $(\mathsf{Sch}_{\blacklozenge/S}, f : X \to Y)$.

Proof. Lemma 6.2 follows immediately from well-known properties of schemes and valuation rings. $\square$

Lemma 6.3. *Let T be a reduced scheme. Write $0_T, 1_T : T \to \mathbb{A}^1_T$ for the morphisms obtained by base-changing the sections $0, 1 : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{G}_{m,T} :\overset{\text{def}}{=} \mathbb{A}^1_T \setminus \mathrm{Im}(0_T)$. Then the following assertions hold:*

- *(i) Assume that T is the spectrum of a field k. Let $G \subset \mathbb{G}_{m,T}$ be an open subscheme equipped with a group scheme structure over T whose identity section $T \to G$ ($\subset \mathbb{G}_{m,T} \subset \mathbb{A}^1_T$) is 1_T. Then the open immersion $G \subset \mathbb{G}_{m,T}$ is an isomorphism of group schemes over T.*
- *(ii) Assume that T is the spectrum of a field k. Then the set of ring scheme structures on $\mathbb{A}^1_T$ over T whose additive and multiplicative identity sections are 0_T and 1_T, respectively, is of cardinality one.*
- *(iii) Let $G \subset \mathbb{G}_{m,T}$ be an open subscheme equipped with a group scheme structure over T whose identity section $T \to G$ ($\subset \mathbb{G}_{m,T} \subset \mathbb{A}^1_T$) is 1_T. Then the open immersion $G \subset \mathbb{G}_{m,T}$ is an isomorphism of group schemes over T.*
- *(iv) The set of ring scheme structures on $\mathbb{A}^1_T$ over T whose additive and multiplicative identity sections are 0_T and 1_T, respectively, is of cardinality one.*

Proof. First, we prove assertion (i). To prove assertion (i), we may assume without loss of generality that k is algebraically closed. Write $C :\stackrel{\text{def}}{=} \mathbb{P}^1_T \setminus G$ ($\supset \{0, \infty\}$), where we regard G as an open subscheme of $\mathbb{P}^1_T$ by means of the open immersions $G \subset \mathbb{G}_{m,T} \subset \mathbb{P}^1_T$; $m : G \times_T G \to G$ for the multiplication morphism. For any closed point $g \in G$, the automorphism $\rho_g : \mathbb{P}^1_T \xrightarrow{\sim} \mathbb{P}^1_T$ given by $m(g, -) : G \xrightarrow{\sim} G$ satisfies the property $\rho_g(C) = C$. Since the assignment $g \mapsto \rho_g$ is clearly injective, we thus conclude that the group $\{\rho \in \text{Aut}_T(\mathbb{P}^1_T) | \rho(C) = C\}$ is of infinite cardinality. This implies that $C = \{0, \infty\}$, i.e., that the open immersion $G \subset \mathbb{G}_{m,T}$ is an isomorphism of T-schemes.

Next, we prove that the isomorphism $G \xrightarrow{\sim} \mathbb{G}_{m,T}$ is compatible with both group scheme structures over T. Let us identify $\Gamma(G, \mathcal{O}_G)$ with $k[t, 1/t]$ by means of the isomorphism $G \xrightarrow{\sim} \mathbb{G}_{m,T}$. Then the morphism m is determined by the element $m^\#(t) \in k[t, 1/t] \otimes_k k[t, 1/t]$. Moreover, since $t \in k[t, 1/t]$ is invertible, $m^\#(t) \in k[t, 1/t] \otimes_k k[t, 1/t]$ is invertible. Since $m^\#(t)$ is invertible in $k[t, 1/t] \otimes_k k[t, 1/t] \subset k[t, 1/t] \otimes_k k(t)$, there exist an element $f \in k[t, 1/t] \setminus \{0\}$ and an integer a such that $m^\#(t) = t^a \otimes f$. Since $m \circ (1_T \times \text{id}_G) = \text{id}_G$, it holds that $f = t$. Since $m \circ (\text{id}_G \times 1_T) = \text{id}_G$, it holds that $a = 1$. Thus it holds that $m^\#(t) = t \otimes t$. This implies that the isomorphism $G \xrightarrow{\sim} \mathbb{G}_{m,T}$ is compatible with both group scheme structures over T. This completes the proof of assertion (i).

Next, we prove assertion (ii). To prove assertion (ii), we may assume without loss of generality that k is algebraically closed. Write $A :\stackrel{\text{def}}{=} \mathbb{A}^1_T$. Let us identify $\Gamma(A, \mathcal{O}_A)$ with $k[t]$ by means of the isomorphism $A \xrightarrow{\sim} \mathbb{A}^1_T$. Let $(a : A \times_T A \to A, e_a : T \to A, m : A \times_T A \to A, e_m : T \to A)$ be a ring scheme structure on A over T, where a is the addition morphism, $e_a = 0_T$ is the identity section of the additive structure, m is the multiplication morphism, and $e_m = 1_T$ is the identity section of the multiplicative structure. By [AR, Theorem 6.1], the group variety $A^\times$ of units of A is an open subscheme of $A \setminus \{0\} = \mathbb{G}_{m,T}$. Hence, by (i), the open immersion $A^\times \to \mathbb{G}_{m,T}$ is an isomorphism of group schemes over T. This implies that $m^\#(t) = t \otimes t$. Write $\Delta : A \times_T A \times_T A \to A \times_T A \times_T A \times_T A$ for the morphism such that for any T'-valued points $\alpha_1, \alpha_2, \alpha_3 \in A(T')$, $\Delta \circ (\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_1, \alpha_3)$. Then it holds that $m \circ (\text{id} \times a) = a \circ (m \times m) \circ \Delta$. Hence, if we write $a^\#(t) = \sum_{i,j \geq 0} a_{ij} t^i \otimes t^j$, where $a_{ij} \in k$, then it holds that

$$
\begin{aligned}
t \otimes a^\#(t) &= (\text{id} \times a)^\#(m^\#(t)) \\
&= \Delta^\#((m^\# \otimes m^\#)(a^\#(t))) \\
&= \Delta^\# \left((m^\# \otimes m^\#) \left(\sum_{i,j \geq 0} a_{ij} t^i \otimes t^j \right) \right) \\
&= \Delta^\# \left(\sum_{i,j \geq 0} a_{ij} t^i \otimes t^i \otimes t^j \otimes t^j \right) \\
&= \sum_{i,j \geq 0} a_{ij} t^{i+j} \otimes t^i \otimes t^j.
\end{aligned}
$$

This implies that $a^\#(t) = t \otimes a_{10} + a_{01} \otimes t$. Since $a \circ (\text{id}_G \times 0_T) = a \circ (0_T \times \text{id}_G) = \text{id}_G$, it holds that $a^\#(t) = t \otimes 1 + 1 \otimes t$. This implies that A is isomorphic as a ring scheme over T to $\mathbb{A}^1_T$. This completes the proof of assertion (ii).

Since T is reduced, by applying assertions (i) and (ii) to the respective fibers of $G \to T$ and $\mathbb{A}^1_T \to T$, we conclude that assertions (iii) and (iv) hold. This completes the proof of Lemma 6.3. $\square$

The following lemma was motivated by [WW, Proposition 2.3].

Lemma 6.4. *Let V be the spectrum of a DVR. Write $\eta \in V$ for the generic point. Let $f : X \to V$ be a flat separated ring scheme of finite type over V such that the generic fiber X_η of f is isomorphic to the scheme $\mathbb{A}^1_{k(\eta)}$ equipped with its natural ring scheme structure. Then the following assertions hold:*

- (i) *X is affine.*
- (ii) *There exists a unique morphism of ring schemes $h : X \to \mathbb{A}^1_V$ over V such that h_η is an isomorphism of ring schemes over $\mathrm{Spec}(k(\eta))$.*

Proof. Since the generic fiber of f is affine, it follows from [Anan73, Proposition 2.3.1] that X is affine, i.e., assertion (i) holds. In the remainder of the proof of Lemma 6.4, we prove assertion (ii). Write $R :\stackrel{\mathrm{def}}{=} \Gamma(V, \mathcal{O}_V)$, K for the field of fractions of R, and $A :\stackrel{\mathrm{def}}{=} \Gamma(X, \mathcal{O}_X)$. Since A is flat over R, and X_η is isomorphic as a ring scheme over K to $\mathbb{A}^1_K$, we may regard A as a subring of $K[x]$. Thus, to prove assertion (ii), it suffices to prove that $x \in A$.

Write $a^\# : K[x] \to K[x] \otimes_K K[x]$ for the ring homomorphism that defines the *additive* structure of the ring scheme X_η, $m^\# : K[x] \to K[x] \otimes_K K[x]$ for the ring homomorphism that defines the *multiplicative* structure of the ring scheme X_η, $e_a^\# : K[x] \to K$ for the ring homomorphism that defines the *identity section* for the *additive* structure of the ring scheme X_η, and $e_m^\# : K[x] \to K$ for the ring homomorphism that defines the *identity section* for the *multiplicative* structure of the ring scheme X_η. Thus the following equalities hold:

$$a^\#(x) = x \otimes 1 + 1 \otimes x, \quad m^\#(x) = x \otimes x, \quad e_a^\#(x) = 0, \quad e_m^\#(x) = 1.$$

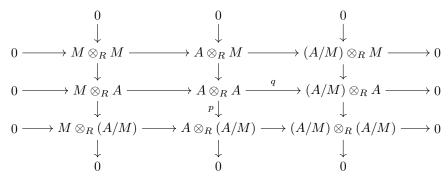Since A is flat over R, we may regard $A \otimes_R A$ as a subring of $K[x] \otimes_K K[x]$. Moreover, since X_η is isomorphic to $\mathbb{A}^1_K$, it follows from Lemma 6.3 that we may regard the ring scheme structure of X is given by the restrictions $a^\#|_A, m^\#|_A, e_a^\#|_A, e_m^\#|_A$. Thus, in particular, the ring homomorphism $m^\#|_A : A \to K[x] \otimes_K K[x]$ factors uniquely through the subring $A \otimes_R A \subset K[x] \otimes_K K[x]$. To prove assertion (ii), it suffices to prove that $x \in A$.

Write $M :\stackrel{\mathrm{def}}{=} (K \cdot 1 + K \cdot x) \cap A \subset A$. Since $e_a^\#|_A$ defines the identity section for the additive structure, it holds that $A \cap (K \cdot 1) = R \cdot 1$, and

$$R \cdot 1 = M \cap (K \cdot 1) \subset M \subset R \cdot 1 + K \cdot x.$$

Since $e_m^\#|_A$ defines the identity section for the multiplicative structure, it holds that $M \subset R \cdot 1 + R \cdot x$. Hence there exists an ideal $I \subset R$ such that $M = R \cdot 1 + I \cdot x$. Next, observe that M and A/M are R-submodules of K-vector spaces, hence, in particular, R-flat. Thus we obtain a commutative diagram of R-flat modules in

which the horizontal and vertical sequences are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R M & \longrightarrow & A \otimes_R M & \longrightarrow & (A/M) \otimes_R M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R A & \longrightarrow & A \otimes_R A & \xrightarrow{q} & (A/M) \otimes_R A & \longrightarrow & 0 \\
& & \downarrow & & {\scriptstyle p}\downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R (A/M) & \longrightarrow & A \otimes_R (A/M) & \longrightarrow & (A/M) \otimes_R (A/M) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

This implies that $M \otimes_R M = (A \otimes_R M) \cap (M \otimes_R A)$ as R-submodules of $A \otimes_R A \subset K[x] \otimes_K K[x]$. Let $t \in M$ be an element. Then, since $M = R \cdot 1 + I \cdot x$, there exist elements $r \in R$ and $s \in I$ such that $t = r + sx$. Moreover, it holds that

$$A \otimes_R A \ni m^{\#}(t) = m^{\#}(r + sx) = r + s \cdot (x \otimes x) = r + (sx) \otimes x = r + x \otimes (sx).$$

Hence it holds that $(p \otimes_R \mathrm{id}_K)(m^{\#}(r + sx)) = 0$, and $(q \otimes_R \mathrm{id}_K)(m^{\#}(r + sx)) = 0$. Thus $m^{\#}|_M : M \to K[x] \otimes_K K[x]$ factors through $(A \otimes_R M) \cap (M \otimes_R A) = M \otimes_R M$. Since $M = R \cdot 1 + I \cdot x$, we conclude that $I \cdot (x \otimes x) \subset I^2 \cdot (x \otimes x)$. This implies that $I \subset I^2$. Hence it holds that $I = R$. Thus it holds that $x \in M \subset A$. This completes the proof of Lemma 6.4. $\qquad\square$

Proposition 6.5. *Let V be the spectrum of a DVR and $f : X \to V$ a flat separated ring scheme of finite type over V. Then f is isomorphic as a ring scheme over V to the projection $\mathbb{A}^1_V \to V$ if and only if the following conditions hold:*

   (i) Each fiber of f is one-dimensional and connected.
   (ii) The generic fiber of f is isomorphic to $\mathbb{A}^1_K$ as a scheme over K.

Proof. Necessity follows immediately. In the remainder of the proof of Proposition 6.5, we prove sufficiency. Assume that $f : X \to V$ satisfies conditions (i) and (ii). Write $v \in V$ for the closed point, $\eta \in V$ for the generic point, $k :\overset{\mathrm{def}}{=} k(v)$, and $K :\overset{\mathrm{def}}{=} k(\eta)$. By Lemma 6.3 (ii), we may assume without loss of generality that the generic fiber of f is isomorphic as a ring scheme over K to $\mathbb{A}^1_K$.

By Lemma 6.4 (ii), there exists a unique morphism of ring schemes $h : X \to \mathbb{A}^1_V$ over V such that h_η is an isomorphism of ring schemes over $\mathrm{Spec}(K)$. Then h_v is a morphism of ring schemes over $\mathrm{Spec}(k)$. Since f satisfies condition (i), the scheme-theoretic image of h_v is a connected closed subscheme of $\mathbb{A}^1_k$. Moreover, since h_v is a morphism of ring schemes over $\mathrm{Spec}(k)$, $0, 1 \in \mathbb{A}^1_k$ are contained in the image of h_v. Hence the scheme theoretic image of h_v is $\mathbb{A}^1_k$. Thus, by [Anan73, Corollary 2.3.3.3], h_v is faithfully flat. Since h_η is an isomorphism, it follows from [Stacks, Tag 039D] that h is faithfully flat. Since X is separated of finite type over V, and $\mathbb{A}^1_V$ is quasi-separated over V, it follows from [Stacks, Tag 01KV], [Stacks, Tag 03GI], and [Stacks, Tag 01T8] that h is separated and of finite type. By Lemma 6.4 (i), X is affine. Write $N :\overset{\mathrm{def}}{=} X \times_{h, \mathbb{A}^1_V, 0_{\mathbb{A}^1_V}} V$. Since the morphism of affine ring schemes h is faithfully flat, separated, and of finite type, N is a flat separated commutative affine group scheme of finite type over V. Since the natural morphism $N_\eta \xrightarrow{\sim} \mathrm{Spec}(K)$ is an isomorphism, and N is flat over V, it holds that $\Gamma(N, \mathcal{O}_N) \subset K$. Since V is

the spectrum of a DVR, and the V-scheme N has a section $V \to N$, it holds that $N \xrightarrow{\sim} V$. Hence h is a monomorphism in Sch. Since h is faithfully flat and of finite type, it follows from [Stacks, Tag 025G] that h is an open immersion. Thus h is an isomorphism. This completes the proof of Proposition 6.5. $\qquad\square$

Remark 6.6. Let R be a mixed characteristic DVR and $\pi \in R$ a uniformizer of R. Write $k :\overset{\text{def}}{=} R/(\pi)$, and p for the characteristic of k. Then the R-scheme $X :\overset{\text{def}}{=} \mathrm{Spec}(R[x,y]/(\pi y - x^{p^2} + x^p)) = \mathrm{Spec}(R[x, (x^{p^2} - x^p)/\pi])$ has the following properties:

- X is a flat affine ring scheme of finite type over R.
- The generic fiber of $X \to \mathrm{Spec}(R)$ is the ring scheme $\mathbb{A}^1_K$.
- The special fiber of $X \to \mathrm{Spec}(R)$ is not connected.

Hence, in Proposition 6.5, if one do not assume the connectedness of the special fiber, then it may not hold that $X \cong \mathbb{A}^1_R$. This example is obtained by forming "Néron's blow-up" (cf. [Stacks, Tag 0BJ1]) of $\mathbb{A}^1_R$ along the closed ring subscheme $\mathrm{Spec}(k[x]/(x^{p^2} - x^p))$ over k of the special fiber $\mathbb{A}^1_k$.

Lemma 6.7. *Let V be an affine scheme; X, Y, Z integral schemes; $f : X \to V$, $g : Y \to V$, and $h : Z \to V$ morphisms. Write $R :\overset{\text{def}}{=} \Gamma(V, \mathcal{O}_V)$. Assume that R is either a DVR or a field, and that the generic fiber of f, g, and h are geometrically reduced and non-empty. Then $X \times_V Y \times_V Z$ is reduced.*

Proof. Write η for the generic point of V. Since X_η, Y_η, Z_η are geometrically reduced over $\mathrm{Spec}(k(\eta))$, it follows from [Stacks, Tag 035Z] that $(X \times_V Y \times_V Z)_\eta \cong X_\eta \times_{\mathrm{Spec}(k(\eta))} Y_\eta \times_{\mathrm{Spec}(k(\eta))} Z_\eta$ is reduced. Since X, Y, Z are integral, and the generic fibers of f, g, h are non-empty, it follows from [Ha, Chapter III, Proposition 9.7] that f, g, h are flat. Hence $X \times_V Y \times_V Z$ is flat over V. Since the scheme-theoretic image of the natural morphism $\mathrm{Spec}(k(\eta)) \to V$ is V, it follows from [Stacks, Tag 081I] that the scheme theoretic image of $(X \times_V Y \times_V Z)_\eta \to X \times_V Y \times_V Z$ is equal to $X \times_V Y \times_V Z$. Since $(X \times_V Y \times_V Z)_\eta$ is reduced, $X \times_V Y \times_V Z$ is also reduced. This completes the proof of Lemma 6.7. $\qquad\square$

Lemma 6.8. *Let S be a quasi-separated scheme. Let V be an object of $\mathsf{Sch}_{\blacklozenge/S}$ which is affine, X an object of $\mathsf{Sch}_{\blacklozenge/S}$ which is quasi-compact (over $\mathbb{Z}$) and integral, and $f : X \to V$ a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Write $R :\overset{\text{def}}{=} \Gamma(V, \mathcal{O}_V)$. Assume that R is either a DVR or a field, and that the generic fiber of f is geometrically reduced. Then there is a natural bijective correspondence between the group scheme structures over V on X and the group object structures on the object $f : X \to V$ of $(\mathsf{Sch}_{\blacklozenge/S})_{/V}$. Similarly, there is a natural bijective correspondence between the ring scheme structures over V on X and the ring object structures on the object $f : X \to V$ of $(\mathsf{Sch}_{\blacklozenge/S})_{/V}$.*

Proof. Lemma 6.8 follows immediately from Lemma 6.7, together with Lemma 1.1 (iii) (iv). $\qquad\square$

Next, to give a category-theoretic characterization of morphisms which are isomorphic to the natural projection $\mathbb{P}^1_S \to S$, we study certain algebraic spaces that parametrize closed immersions of a certain type.

Definition 6.9. Let T be a scheme, X a proper flat T-scheme of finite presentation, Y a T-scheme of finite presentation, $B \subset X$ a closed subscheme which is flat over T,

$f : B \to Y$ a morphism of T-schemes, and U a T-scheme. We define the following sets:

$$\mathrm{Cl}(X, Y)(U) :\overset{\mathrm{def}}{=} \{i : X_U \to Y_U \mid i \text{ is a closed immersion}\},$$

$$\mathrm{Cl}_f(X, Y)(U) :\overset{\mathrm{def}}{=} \{i : X_U \to Y_U \mid i \text{ is a closed immersion such that } i|_{B_U} = f_U\}.$$

Since base-change preserves the required property, we obtain functors

$$\mathrm{Cl}(X, Y) : \mathsf{Sch}^{\mathrm{op}}_{/T} \to \mathsf{Set},$$
$$\mathrm{Cl}_f(X, Y) : \mathsf{Sch}^{\mathrm{op}}_{/T} \to \mathsf{Set}.$$

Lemma 6.10. *Let T be a scheme, X a proper flat T-scheme of finite presentation, Y a separated T-scheme of finite presentation, B a closed subscheme of X which is flat and of finite presentation over T, and $f : B \to Y$ a closed immersion of T-schemes. Then the functors $\mathrm{Cl}(X, Y)$ and $\mathrm{Cl}_f(X, Y)$ are represented by algebraic spaces which are separated and locally of finite presentation over T.*

Proof. First, we prove that the functor $\mathrm{Cl}(X, Y)$ is represented by an algebraic space which is separated and locally of finite presentation over T. Write

$$\mathrm{Hilb}_{Y/T} : \mathsf{Sch}^{\mathrm{op}}_{/T} \to \mathsf{Set}$$

for the Hilbert functor (cf. [Stacks, Tag 0CZY]), i.e., parametrizing closed subschemes of Y that are proper, flat, and of finite presentation over T. Since Y is separated and of finite presentation over T, it follows from [Stacks, Tag 0D01] and [Stacks, Tag 0DM7] that $\mathrm{Hilb}_{Y/T}$ is represented by an algebraic space which is separated and locally of finite presentation over T. Let U_0 be a T-scheme and $i_{U_0} : X_{U_0} \to Y_{U_0}$ a closed immersion. By forming the scheme-theoretic image of X_{U_0} in Y_{U_0} via i_{U_0}, we obtain a map $\varphi(U_0) : \mathrm{Cl}(X, Y)(U_0) \to \mathrm{Hilb}_{Y/T}(U_0)$, which is clearly functorial with respect to U_0. Hence we obtain a morphism of functors $\varphi : \mathrm{Cl}(X, Y) \to \mathrm{Hilb}_{Y/T}$. Thus, by [Stacks, Tag 02YS], [Stacks, Tag 03XQ], and [Stacks, Tag 03KQ], to prove that the functor $\mathrm{Cl}(X, Y)$ is represented by an algebraic space which is separated and locally of finite presentation over T, it suffices to prove that φ is represented by an algebraic space which is separated and locally of finite presentation over T.

Let U_1 be a T-scheme and $U_1 \to \mathrm{Hilb}_{Y/T}$ a morphism of algebraic spaces over T. Then it follows from Yoneda's lemma that the morphism $U_1 \to \mathrm{Hilb}_{Y/T}$ corresponds to a closed subscheme $Z \subset Y_{U_1}$ such that the composite $Z \subset Y_{U_1} \to U_1$ is proper, flat, and of finite presentation. Hence it holds that $U_1 \times_{\mathrm{Hilb}_{Y/T}, \varphi} \mathrm{Cl}(X, Y) \cong \mathrm{Isom}_{U_1}(X_{U_1}, Z)$, where $\mathrm{Isom}_{U_1}(X_{U_1}, Z) \subset \mathrm{Mor}_{U_1}(X_{U_1}, Z)$ is the open sub-algebraic space of the algebraic space $\mathrm{Mor}_{U_1}(X_{U_1}, Z)$ determined by the property of being an isomorphism (cf. [Stacks, Tag 0D1C], [Stacks, Tag 0DPP]). By [Stacks, Tag 0DPN] and [Stacks, Tag 0DPP], $\mathrm{Isom}_{U_1}(X_{U_1}, Z)$ is represented by an algebraic space which is separated and locally of finite presentation over U_1. Thus φ is represented by an algebraic space which is separated and locally of finite presentation over T. This completes the proof of the assertion that the functor $\mathrm{Cl}(X, Y)$ is represented by an algebraic space which is separated and locally of finite presentation over T.

Next, we prove that the functor $\mathrm{Cl}_f(X, Y)$ is represented by an algebraic space which is separated and locally of finite presentation over T. Let U_2 be a T-scheme and $i_{U_2} : X_{U_2} \to Y_{U_2}$ a closed immersion. By composing i_{U_2} with the closed immersion $B_{U_2} \hookrightarrow X_{U_2}$, we obtain a morphism of algebraic spaces $\psi : \mathrm{Cl}(X, Y) \to$

$\text{Cl}(B, Y)$ over T. By [Stacks, Tag 05WT] and [Stacks, Tag 04ZI], ψ is separated and locally of finite presentation. Write $\tilde{f} : T \to \text{Cl}(B, Y)$ for the morphism corresponding to the element $[f : B \to Y] \in \text{Cl}(B, Y)(T)$. Then it holds that $\text{Cl}_f(X, Y) \cong \text{Cl}(X, Y) \times_{\text{Cl}(B,Y), \tilde{f}} T$. This implies that $\text{Cl}_f(X, Y)$ is represented by an algebraic space which is separated and locally of finite presentation over T. This completes the proof of Lemma 6.10. $\square$

Lemma 6.11. *Let X be a stalkwise Noetherian scheme and $x, \xi \in X$ points such that $\xi \rightsquigarrow x$. Then there exist a scheme V and a morphism $f : V \to X$ such that the following conditions hold:*

(i) *V is isomorphic to the spectrum of a discrete valuation ring.*
(ii) *$f(v) = x$, where $v \in V$ is the unique closed point.*
(iii) *$f(\eta) = \xi$, where $\eta \in V$ is the unique generic point.*
(iv) *The natural morphism $k(\xi) \to k(\eta)$ is an isomorphism.*

Proof. Write $\mathfrak{p} :\overset{\text{def}}{=} \ker(\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi})$ and $A :\overset{\text{def}}{=} \mathcal{O}_{X,x}/\mathfrak{p}$. Then A is a subring of $k(\xi)$. By [Stacks, Tag 00PH], there exists a DVR $R \subset k(\xi)$ which dominates A. Then it follows immediately that $V :\overset{\text{def}}{=} \text{Spec}(R)$ and the composite

$$f : \text{Spec}(R) \to \text{Spec}(A) \to X$$

satisfy required properties. This completes the proof of Lemma 6.11. $\square$

Lemma 6.12. *Let Y be a stalkwise Noetherian scheme and $f : X \to Y$ a separated algebraic space of finite presentation over Y. Assume that for any commutative diagram*

$$
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{\eta} & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
\text{Spec}(R) & \xrightarrow{p} & Y
\end{array}
$$

such that R is a DVR, K is the field of fractions of R, and i is the natural morphism, there exists a morphism $q : \text{Spec}(R) \to X$ such that $\eta = q \circ i$, and $p = f \circ q$. Then f is proper.

Proof. Let

$$
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{\eta} & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
\text{Spec}(R) & \xrightarrow{p} & Y
\end{array}
$$

be a commutative diagram such that R is a (not necessary discrete!) valuation ring, K is the field of fractions of R, and i is the natural morphism. Write $r \in \text{Spec}(R)$ for the unique closed point, $y :\overset{\text{def}}{=} p(r)$, $B :\overset{\text{def}}{=} \mathcal{O}_{Y,y}$, $j : \text{Spec}(B) \to Y$ for the natural morphism, $X_B :\overset{\text{def}}{=} X \times_{f,Y,j} B$, $j_X : X_B \to X$ for the natural projection, $p' : \text{Spec}(R) \to \text{Spec}(B)$ for the local morphism induced by the local homomorphism $B \to R$, and $\eta' : \text{Spec}(K) \to X_B$ for the unique morphism such that $j_X \circ \eta' = \eta$

and $f_B \circ \eta' = p' \circ i$:

$$
\begin{array}{ccccc}
\mathrm{Spec}(K) & \xrightarrow{\ \eta'\ } & X_B & \xrightarrow{\ j_X\ } & X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f} \\
\mathrm{Spec}(R) & \xrightarrow{\ p'\ } & \mathrm{Spec}(B) & \xrightarrow{\ j\ } & Y.
\end{array}
$$

Since Y is stalkwise Noetherian, B is Noetherian. Hence we conclude, by considering the case where R is a DVR, from [Stacks, Tag 0CMF] that f_B is proper. This, in turn, implies, in the case where R is *arbitrary* (i.e., not necessarily discrete), by [Stacks, Tag 0A40], that there exists a morphism $q' : \mathrm{Spec}(R) \to X_B$ such that $\eta' = q' \circ i$, and $p' = f_B \circ q'$. Thus, by [Stacks, Tag 0A40], f is proper. This completes the proof of Lemma 6.12. $\qquad\square$

Lemma 6.13. *Let S be a locally Noetherian scheme and $f : X \to S$ an algebraic space over S. Assume that the following conditions hold:*

>   *(i) f is locally of finite presentation.*
>   *(ii) For any morphism $g : \mathrm{Spec}(R) \to S$, where R is a DVR or a field, there exists a unique morphism $h : \mathrm{Spec}(R) \to X$ such that $g = f \circ h$.*

Then the continuous map between underlying topological spaces $|f| : |X| \to |S|$ is a homeomorphism (cf. [Stacks, Tag 03BY], [Stacks, Tag 03BX]). In particular, f is quasi-compact (cf. [Stacks, Tag 03E4]).

Proof. To prove Lemma 6.13, we may assume without loss of generality that S is Noetherian. By condition (ii), the map between underlying sets $|f| : |X| \to |S|$ is bijective. Hence to prove Lemma 6.13, it suffices to prove that for any étale morphism $p : U \to X$ such that U is a quasi-compact scheme, $\mathrm{Im}(|f \circ p|) \subset |S|$ is an open subset.

Let $p : U \to X$ be an étale morphism such that U is a quasi-compact scheme. Then, by condition (i), $f \circ p$ is of finite presentation. Hence, by [Stacks, Tag 054J], $\mathrm{Im}(|f \circ p|) \subset |S|$ is a constructible subset. Thus, by [Stacks, Tag 0542], to prove that $\mathrm{Im}(|f \circ p|) \subset |S|$ is open, it suffices to prove that $\mathrm{Im}(|f \circ p|) \subset |S|$ is stable under generization.

Let $u \in U$ be a point and $\eta \in S$ a point such that $\eta \rightsquigarrow f(p(u))$. Then, by Lemma 6.11, there exists a morphism $g : \mathrm{Spec}(R) \to S$ such that R is a DVR, and $\mathrm{Im}(|g|) = \{\eta, f(p(u))\}$. By condition (ii), there exists a unique morphism $h : \mathrm{Spec}(R) \to X$ such that $g = f \circ h$. Write $v \in \mathrm{Spec}(R)$ for the closed point, $\xi \in \mathrm{Spec}(R)$ for the generic point, U' for the base-change of $p : U \to X$ by $h : \mathrm{Spec}(R) \to X$, $q : U' \to \mathrm{Spec}(R)$ for the natural projection morphism, and $h' : U' \to U$ for the natural projection morphism:

$$
\begin{array}{ccc}
U' & \xrightarrow{\ h'\ } & U \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
\mathrm{Spec}(R) & \xrightarrow{\ h\ } & X.
\end{array}
$$

Since $g = f \circ h$, and $|f| : |X| \to |S|$ is a bijection, it holds that $h(v) = p(u)$, hence that $U' \neq \varnothing$. Since $p : U \to X$ is étale, $q : U' \to \mathrm{Spec}(R)$ is étale. Hence there exists a point $\xi' \in U'$ such that $q(\xi') = \xi$. Thus it holds that

$$
\eta = g(\xi) = f(h(q(\xi'))) = f(p(h'(\xi'))) \in \mathrm{Im}(|f \circ p|).
$$

This implies that $\mathrm{Im}(|f \circ p|) \subset |S|$ is stable under generization, i.e., $\mathrm{Im}(|f \circ p|) \subset |S|$ is open. This completes the proof of Lemma 6.13. $\qquad\square$

Definition 6.14. Let S be a quasi-separated stalkwise Noetherian scheme; $f :$ $X \to S$ an object of $\mathsf{Sch}_{\blacklozenge/S}$; $s_0, s_1, s_\infty : S \to X$ morphisms in $\mathsf{Sch}_{\blacklozenge/S}$. Suppose that $\diamondsuit = \blacklozenge \cup \{\mathrm{red}\}$. Then we shall say that the collection of data (X, s_0, s_1, s_∞) is $\mathbb{P}^1$-*like* in $\mathsf{Sch}_{\blacklozenge/S}$ if the following conditions hold:

(i) X is reduced.

(ii) f is proper and of finite presentation.

(iii) For any object $T \to S$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that T is isomorphic to the spectrum of a field, $X \times_S^\blacklozenge T$ is isomorphic as a T-scheme to $\mathbb{P}^1_T$.

(iv) For any $i, j \in \{0, 1, \infty\}$ such that $i \neq j$, it holds that $S \times_{s_i, X, s_j}^\blacklozenge S = \varnothing$.

(v) For any $i, j, k \in \{0, 1, \infty\}$ such that $\{i, j, k\} = \{0, 1, \infty\}$, there exist an open immersion $\iota : U \hookrightarrow X$ in $\mathsf{Sch}_{\blacklozenge/S}$, morphisms $t_j : S \to U$, $t_k : S \to U$ in $\mathsf{Sch}_{\blacklozenge/S}$, and a ring object structure on U in $\mathsf{Sch}_{\diamondsuit/S}$ such that t_j is the additive identity section, t_k is the multiplicative identity section, $s_j = \iota \circ t_j$, $s_k = \iota \circ t_k$, and $\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(\iota)) = \mathsf{Sp}_{\blacklozenge/S}(X) \setminus \mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(s_i))$ (cf. Reconstruction 4.5).

Thus the property that the collection of data (X, s_0, s_1, s_∞) is $\mathbb{P}^1$-like in $\mathsf{Sch}_{\blacklozenge/S}$ is defined completely in terms of properties that may be characterized category-theoretically (cf. Lemma 1.7, Corollary 2.9, Corollary 4.4, Reconstruction 4.5, Corollary 5.21, Lemma 6.2) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, s_0, s_1, s_\infty)$.

Proposition 6.15. *Let S be a locally Noetherian (hence quasi-separated — cf.* [Stacks, Tag 01OY]*) normal scheme. Let X be an object of $\mathsf{Sch}_{\blacklozenge/S}$. Then X is isomorphic to $\mathbb{P}^1_S$ as an S-scheme if and only if the following conditions hold:*

(i) There exist morphisms $s_0, s_1, s_\infty : S \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that the collection of data (X, s_0, s_1, s_∞) is $\mathbb{P}^1$-like in $\mathsf{Sch}_{\blacklozenge/S}$.

(ii) For any $\mathbb{P}^1$-like collection of data (Y, t_0, t_1, t_∞) in $\mathsf{Sch}_{\blacklozenge/S}$, there exists a unique closed immersion $h : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that for each $i \in \{0, 1, \infty\}$, $h \circ s_i = t_i$.

In particular, the property that X is isomorphic to $\mathbb{P}^1_S$ as an S-scheme may be characterized category-theoretically (cf. Proposition 4.3, Definition 6.14) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

Proof. First, we prove necessity. Assume that $X \cong \mathbb{P}^1_S$. For any $i \in \{0, 1, \infty\}$, write $s_i : S \to X$ for the morphism in $\mathsf{Sch}_{\blacklozenge/S}$ obtained by base-changing the sections $0, 1, \infty : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_\mathbb{Z}$. Then it follows immediately from Lemma 1.1 (iii) (iv) that the collection of data (X, s_0, s_1, s_∞) satisfies conditions (i), (ii), (iii), and (iv) of Definition 6.14. Moreover, since for any $i \in \{0, 1, \infty\}$, $\mathbb{A}^1_S \cong X \setminus s_i$, it follows immediately that the collection of data (X, s_0, s_1, s_∞) satisfies condition (v) of Definition 6.14. Hence X satisfies condition (i) of Proposition 6.15.

Next, we verify that X satisfies condition (ii) of Proposition 6.15. Let (Y, t_0, t_1, t_∞) be a $\mathbb{P}^1$-like collection of data in $\mathsf{Sch}_{\blacklozenge/S}$. Write

- $B := S \coprod S \coprod S$,
- $s := s_0 \coprod s_1 \coprod s_\infty : B \to X$ for the closed immersion,
- $t := t_0 \coprod t_1 \coprod t_\infty : B \to Y$ for the closed immersion,

- $C :\overset{\text{def}}{=} \mathrm{Cl}_t(X,Y)$, where we regard B as a closed subscheme of X by means of s, and we take "T" to be S (cf. Definition 6.9, Lemma 6.10), and
- $p : C \to S$ for the structure morphism.

Then, to verify that X satisfies condition (ii) of Proposition 6.15, it suffices to prove that the set of S-valued points $C(S)$ is of cardinality 1. Hence, in particular, since S is normal, it suffices to prove that the composite $C_{\mathrm{red}} \to C \xrightarrow{p} S$ is an isomorphism. By Lemma 6.10, p is a morphism of algebraic spaces which is separated and locally of finite presentation. Let $T \to S$ be a morphism such that T is isomorphic to the spectrum of a field. Then, by Lemma 2.5 (i) (ii), $T \to S$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Moreover, by Lemma 1.1 (iii) (iv) and Definition 6.14 (iii), $C(T)$ is of cardinality 1. In particular, by allowing T to vary, we conclude that $\dim C_T = 0$, where we write $C_T :\overset{\text{def}}{=} C \times_S T$. Since the natural projection $p_T : C_T \to T$ is locally of finite presentation, it follows from [Stacks, Tag 06LZ] that C_T is a scheme. Hence, since $C(T)$ is of cardinality 1 for arbitrary T, we conclude that for arbitrary T, the composite $C_{T,\mathrm{red}} \to C_T \xrightarrow{p_T} T$ is an isomorphism.

Next, we prove the following assertion:

(†) For any morphism $\mathrm{Spec}(R) \to S$ such that R is a DVR, the set of R-valued points $C(R)$ is of cardinality 1.

(Here, we observe that it follows formally, by considering the ring of formal power series in one variable over a field, from (†) that (†) continues to hold even if the condition "R is a DVR" is replaced by the condition "R is a field".) Let $\mathrm{Spec}(R) \to S$ be a morphism of schemes such that R is a DVR. Write $V :\overset{\text{def}}{=} \mathrm{Spec}(R)$. Then, by Lemma 2.5 (i) (ii), $V \to S$ belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $X \cong \mathbb{P}^1_S$, to prove assertion (†), it suffices to prove that $Y_{V,\mathrm{red}} \cong \mathbb{P}^1_V$. Suppose that $\lozenge = \blacklozenge \cup \{\mathrm{red}\}$. Write η for the generic point of V and v for the closed point of V. By Lemma 1.1 (iv), the fiber product $Z :\overset{\text{def}}{=} Y \times^{\lozenge}_S V$ exists in $\mathsf{Sch}_{\lozenge/S}$ and is naturally isomorphic to $Y_{V,\mathrm{red}}$. By Definition 6.14 (ii), Y is proper over S. Hence the projection morphism $Z \to V$ is proper. By Definition 6.14 (iii), it holds that $Z \times^{\lozenge}_V \mathrm{Spec}(k(\eta)) \cong \mathbb{P}^1_{k(\eta)}$, and $Z \times^{\lozenge}_V \mathrm{Spec}(k(v)) \cong \mathbb{P}^1_{k(v)}$. Write $\overline{Z}_\eta \subset Z$ for the schematic closure of the generic fiber of the natural morphism $Z \to V$. Thus, since V is the spectrum of a DVR, the natural morphism $\overline{Z}_\eta \to V$ is proper and flat. Hence, by [Stacks, Tag 0D4J], the dimension of the special fiber of the natural morphism $\overline{Z}_\eta \to V$ is one. Since $Z \times^{\lozenge}_V \mathrm{Spec}(k(v)) \cong \mathbb{P}^1_{k(v)}$, the morphism of underlying topological spaces $|\overline{Z}_\eta| \to |Z|$ is a homeomorphism. Thus, since Z is reduced, the closed immersion $\overline{Z}_\eta \xrightarrow{\sim} Z$ is an isomorphism. In particular, Z is integral.

Since V is reduced, and $Z \cong Y_{V,\mathrm{red}}$, for each $i \in \{0, 1, \infty\}$, there exists a unique section $s'_{i,V} : V \to Z$ of the projection $Z \to V$ such that the composite of $s'_{i,V}$ with the closed immersion $Z \subset Y_V$ is $s_{i,V} : V \to Y_V$. Next, observe that by Lemma 1.1 (iv), for each $i \in \{0, 1, \infty\}$, the natural morphism $Z \setminus \mathrm{Im}(s'_{i,V}) \xrightarrow{\sim} (Y \setminus \mathrm{Im}(s_i)) \times^{\lozenge}_S V$ is an isomorphism. Hence, by Definition 6.14 (iv) (v), for any $i, j, k \in \{0, 1, \infty\}$ such that $\{i, j, k\} = \{0, 1, \infty\}$, $Z \setminus \mathrm{Im}(s'_{i,V})$ has a ring object structure in $(\mathsf{Sch}_{\lozenge/S})_{/V}$ such that (the morphism $V \to Z \setminus \mathrm{Im}(s'_{i,V})$ determined by) $s'_{j,V}$ is the multiplicative identity section, and (the morphism $V \to Z \setminus \mathrm{Im}(s'_{i,V})$ determined by) $s'_{k,V}$ is the additive identity section. Since $Z \times^{\lozenge}_V k(\eta) \cong \mathbb{P}^1_{k(\eta)}$, for each $i \in \{0, 1, \infty\}$, it holds that $(Z \setminus \mathrm{Im}(s'_{i,V}))_\eta \cong \mathbb{A}^1_{k(\eta)}$ (as $k(\eta)$-schemes). In particular, for each $i \in \{0, 1, \infty\}$,

$(Z \setminus \mathrm{Im}(s'_{i,V}))_\eta$ is geometrically reduced over $k(\eta)$. Moreover, since $Z \to V$ is proper, Z is Noetherian. In particular, for any $i \in \{0, 1, \infty\}$, $Z \setminus \mathrm{Im}(s'_{i,V})$ is quasi-compact over $\mathbb{Z}$. Hence, since Z is integral, it follows from Lemma 6.8 that for any $i, j, k \in \{0, 1, \infty\}$ such that $\{i, j, k\} = \{0, 1, \infty\}$, the ring object structure on $Z \setminus \mathrm{Im}(s'_{i,V})$ in $(\mathsf{Sch}_{\Diamond/S})_{/V}$ induces a ring scheme structure over V on $Z \setminus \mathrm{Im}(s'_{i,V})$ such that (the morphism $V \to Z \setminus \mathrm{Im}(s'_{i,V})$ determined by) $s'_{j,V}$ is the multiplicative identity section, and (the morphism $V \to Z \setminus \mathrm{Im}(s'_{i,V})$ determined by) $s'_{k,V}$ is the additive identity section.

For each $i \in \{0, 1, \infty\}$, since $Z \setminus \mathrm{Im}(s'_{i,V})$ is a flat separated ring scheme of finite type over V, $(Z \setminus \mathrm{Im}(s'_{i,V}))_{v,\mathrm{red}} \cong \mathbb{A}^1_{k(v)}$ (as $k(v)$-schemes) is one-dimensional and connected, and $(Z \setminus \mathrm{Im}(s'_{i,V}))_\eta \cong \mathbb{A}^1_{k(\eta)}$ (as ring schemes over $k(\eta)$), it follows from Proposition 6.5 that the ring scheme $Z \setminus \mathrm{Im}(s'_{i,V})$ over V is isomorphic as a ring scheme over V to the projection $\mathbb{A}^1_V \to V$. Moreover, for any $i, j, k \in \{0, 1, \infty\}$ such that $\{i, j, k\} = \{0, 1, \infty\}$, since $s'_{j,V}$ is the multiplicative identity section of the ring scheme $Z \setminus \mathrm{Im}(s'_{i,V})$ over V, and $s'_{k,V}$ is the additive identity section of the ring scheme $Z \setminus \mathrm{Im}(s'_{i,V})$ over V, by observing that Z is naturally isomorphic to the V-scheme obtained by gluing together $Z \setminus \mathrm{Im}(s'_{i,V})$ and $Z \setminus \mathrm{Im}(s'_{k,V})$ along $Z \setminus (\mathrm{Im}(s'_{i,V}) \cup \mathrm{Im}(s'_{k,V}))$, we conclude that $Z \cong \mathbb{P}^1_V$. Thus, in particular, $C(V)$ is of cardinality 1, i.e., assertion (†) holds.

By Lemma 6.13 and assertion (†), $p : C \to S$ is a (necessarily quasi-compact) homeomorphism. Hence, by Lemma 6.10, Lemma 6.12, and assertion (†), $p : C \to S$ is proper. Moreover, by [Stacks, Tag 0A4X] and assertion (†), $p : C \to S$ is finite. In particular, since S is a scheme, C is a finite S-scheme, hence may be written in the form $\mathrm{Spec}_S(p_*\mathcal{O}_C)$. By Definition 6.14 (iii), for any generic point $\eta \in S$, it holds that $C_\eta \cong \mathrm{Spec}(k(\eta))$. Since

- S is normal,
- $\mathcal{O}_S, p_*\mathcal{O}_{C_{\mathrm{red}}}$ are subsheaves of the sheaf of rational functions $\mathcal{K}_S$ on S, and
- the morphism of sheaves of rings $\mathcal{O}_S \to p_*\mathcal{O}_{C_{\mathrm{red}}}$ is finite and compatible with the inclusions $\mathcal{O}_S, p_*\mathcal{O}_{C_{\mathrm{red}}} \subset \mathcal{K}_S$,

the composite $C_{\mathrm{red}} \to C \xrightarrow{p} S$ is an isomorphism. Thus $C(S)$ is of cardinality 1, i.e., X satisfies condition (ii) of Proposition 6.15. This completes the proof of necessity.

Next, we prove sufficiency. Assume that X satisfies conditions (i) and (ii) of Proposition 6.15. Write $Y := \mathbb{P}^1_S$ and for each $i \in \{0, 1, \infty\}$, $t_i : S \to Y$ for the morphism in $\mathsf{Sch}_{\blacklozenge/S}$ obtained by base-change of the sections $0, 1, \infty : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$. By the necessity portion of Proposition 6.15, (Y, t_0, t_1, t_∞) satisfies condition (ii) of Proposition 6.15. In particular, since X satisfies condition (i) of Proposition 6.15, there exists a unique closed immersion $h_1 : Y \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that for any $i \in \{0, 1, \infty\}$, $h_1 \circ t_i = s_i$. By the necessity portion of Proposition 6.15, (Y, t_0, t_1, t_∞) satisfies condition (i) of Proposition 6.15. In particular, since X satisfies condition (ii) of Proposition 6.15, there exists a unique closed immersion $h_2 : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that for any $i \in \{0, 1, \infty\}$, $h_2 \circ s_i = t_i$. Then it holds that $h_1 \circ h_2 \circ s_i = s_i$, and $h_2 \circ h_1 \circ t_i = t_i$. Since (X, s_0, s_1, s_∞) satisfies conditions (i) (ii) of Proposition 6.15, it follows from the uniqueness portion of condition (ii) of Proposition 6.15 that $h_1 \circ h_2 = \mathrm{id}_X$. Since (Y, t_0, t_1, t_∞) satisfies conditions (i) (ii) of Proposition 6.15, it follows from the uniqueness portion of condition (ii) of Proposition 6.15 that $h_2 \circ h_1 = \mathrm{id}_Y$. This implies that $X \cong Y$. This completes the proof of Proposition 6.15. $\square$

Corollary 6.16. *Let S be a locally Noetherian normal scheme. Let X be a ring object of $\mathsf{Sch}_{\blacklozenge/S}$. Write $0_X : S \to X, 1_X : S \to X$ for the additive and multiplicative identity sections. Then X is isomorphic as a ring scheme over S to $\mathbb{A}^1_S$ if and only if $S \times^{\blacklozenge}_{0_X, X, 1_X} S = \varnothing$, and there exist an open immersion $\iota : X \to \mathbb{P}^1_S$ in $\mathsf{Sch}_{\blacklozenge/S}$ and sections $s_0, s_1, s_\infty : S \to \mathbb{P}^1_S$ such that the collection of data $(\mathbb{P}^1_S, s_0, s_1, s_\infty)$ is $\mathbb{P}^1$-like, and $\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(\iota)) = \mathsf{Sp}_{\blacklozenge/S}(\mathbb{P}^1_S) \setminus \mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(s_\infty))$. In particular, the property that X is isomorphic as a ring scheme over S to $\mathbb{A}^1_S$ may be characterized category-theoretically (cf. Corollary 4.4, Reconstruction 4.5, Proposition 6.15) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.*

Proof. Necessity follows immediately from Lemma 1.1 (iii) (iv), Reconstruction 4.5, and Proposition 6.15. Next, we prove sufficiency. Assume that $S \times^{\blacklozenge}_{0_X, X, 1_X} S = \varnothing$, and, moreover, that there exist an open immersion $\iota : X \to \mathbb{P}^1_S$ in $\mathsf{Sch}_{\blacklozenge/S}$ and sections $s_0, s_1, s_\infty : S \to \mathbb{P}^1_S$ such that the collection of data $(\mathbb{P}^1_S, s_0, s_1, s_\infty)$ is $\mathbb{P}^1$-like, and $\mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(\iota)) = \mathsf{Sp}_{\blacklozenge/S}(\mathbb{P}^1_S) \setminus \mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(s_\infty))$. Then $X \cong \mathbb{A}^1_S$. Hence, by Lemma 1.1 (iii) (iv), the ring object structure on X determines a ring scheme structure over S on $X \cong \mathbb{A}^1_S$. Since $S \times^{\blacklozenge}_{0_X, X, 1_X} S = \varnothing$, it follows from Lemma 1.1 (iii) (iv) that $\mathrm{Im}(0_X) \cap \mathrm{Im}(1_X) = \varnothing$. Hence there exists an isomorphism of S-schemes $f : X \xrightarrow{\sim} \mathbb{A}^1_S$ such that $0_S = f \circ 0_X, 1_S = f \circ 1_X$. Thus, by Lemma 6.3 (iv), f is an isomorphism of ring schemes over S. This completes the proof of Corollary 6.16. $\square$

Next, we consider a category-theoretic reconstruction of the underlying schemes of objects of $\mathsf{Sch}_{\blacklozenge/S}$.

Definition 6.17. Let S be a locally Noetherian normal scheme.

(i) We shall write $\mathcal{A}^1_{\blacklozenge/S}$ for the (**V**-small) connected groupoid of ring objects and isomorphisms of ring objects of $\mathsf{Sch}_{\blacklozenge/S}$ such that $\mathbb{A}^1_S \in \mathcal{A}^1_{\blacklozenge/S}$. Note that, by Lemma 6.3 (iv), for any objects $A, B \in \mathcal{A}^1_{\blacklozenge/S}$, $\mathrm{Hom}_{\mathcal{A}^1_{\blacklozenge/S}}(A, B)$ is of cardinality one.

(ii) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. Write $\mathsf{OpIm}_{\blacklozenge/S}(X) \subset (\mathsf{Sch}_{\blacklozenge/S})_{/X}$ for the full subcategory determined by quasi-compact open immersions $U \to X$. Then the category $\mathsf{OpIm}_{\blacklozenge/S}(X)$ is defined completely in terms of properties that may be characterized category-theoretically (cf. Corollary 4.4, Corollary 4.8 (ii)) from the data $(\mathsf{Sch}_{\blacklozenge/S}, X)$.

(iii) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. Write $\mathsf{Op}^{\mathsf{cat}}_{\blacklozenge/S}(X)$ for the category whose objects are elements of $\mathsf{Op}_{\blacklozenge/S}(X)$ (cf. Reconstruction 4.5), and whose morphisms are the inclusion morphisms. We define the following functor:

$$\varphi_X : \mathsf{OpIm}_{\blacklozenge/S}(X) \to \mathsf{Op}^{\mathsf{cat}}_{\blacklozenge/S}(X)$$
$$[i : U \to X] \mapsto \mathrm{Im}(\mathsf{Sp}_{\blacklozenge/S}(i)).$$

Note that, by the definition of the homeomorphism $\eta_X : \mathsf{Sp}_{\blacklozenge/S}(X) \xrightarrow{\sim} |X|$ (cf. Reconstruction 4.5), for any object $[i : U \to X] \in \mathsf{OpIm}_{\blacklozenge/S}(X)$, $\varphi_X(i : U \to X) = \eta_X^{-1}(\mathrm{Im}(|i|))$.

(iv) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object and $U_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(X)$ an open subset. We shall write $\mathcal{I}_{\blacklozenge/S}(X; U_\blacklozenge^{\mathsf{Sp}}) \subset \mathsf{OpIm}_{\blacklozenge/S}(X)^{\mathrm{op}}$ for the full subcategory determined by the objects $[i : U' \to X] \in \mathcal{I}_{\blacklozenge/S}(X; U_\blacklozenge^{\mathsf{Sp}})$ such that $\varphi_X(i : U' \to X) \subset U_\blacklozenge^{\mathsf{Sp}}$.

(v) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. We shall write $\iota_X : \mathsf{OpIm}_{\blacklozenge/S}(X) \to \mathsf{Sch}_{\blacklozenge/S}$ for the forgetful functor.

(vi) Let $f : X \to Y$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. By Lemma 1.1 (iii) (iv), for any object $[i : V \to Y] \in \mathsf{OpIm}_{\blacklozenge/S}(Y)$, the natural projection $i_{\blacklozenge,X} : V \times_Y^\blacklozenge X \to X$ is an object of $\mathsf{OpIm}_{\blacklozenge/S}(X)$. Hence we obtain a functor $f^{-1} : \mathsf{OpIm}_{\blacklozenge/S}(Y) \to \mathsf{OpIm}_{\blacklozenge/S}(X)$.

Definition 6.18. Let S be a scheme.

(i) Let $\bigstar :\overset{\mathrm{def}}{=} \mathsf{Sch}$ or $\mathsf{Sch}_{\mathbf{V}}$. We define
$$U_{\blacklozenge/S}^\bigstar : \mathsf{Sch}_{\blacklozenge/S} \to \bigstar$$
$$X \mapsto X.$$

(ii) Write $i_{\mathbf{U} \in \mathbf{V}}^{\mathsf{Sch}} : \mathsf{Sch} \to \mathsf{Sch}_{\mathbf{V}}$ for the natural inclusion functor. Then the equality $U_{\blacklozenge/S}^{\mathsf{Sch}_{\mathbf{V}}} = i_{\mathbf{U} \in \mathbf{V}}^{\mathsf{Sch}} \circ U_{\blacklozenge/S}^{\mathsf{Sch}}$ holds.

(iii) For any morphism of schemes $f : X \to Y$, any open subset $V^{|\cdot|} \subset |Y|$, and any open subset $U^{|\cdot|} \subset |f|^{-1}(V^{|\cdot|})$, we shall write
$$f^\#(V^{|\cdot|}, U^{|\cdot|}) : \mathcal{O}_Y(V^{|\cdot|}) \to \mathcal{O}_X(U^{|\cdot|})$$
for the morphism of rings determined by the composite of the morphism of rings $\mathcal{O}_Y(V^{|\cdot|}) \to \mathcal{O}_X(|f|^{-1}(V^{|\cdot|}))$ induced by f and the restriction morphism $\mathcal{O}_X(|f|^{-1}(V^{|\cdot|})) \to \mathcal{O}_X(U^{|\cdot|})$.

Reconstruction 6.19 ($\mathsf{RSp}_{\blacklozenge/S}^A(X)$). Let S be a locally Noetherian normal scheme. Let $A \in \mathcal{A}_{\blacklozenge/S}^1$ a ring object in $\mathsf{Sch}_{\blacklozenge/S}$ such that $A \cong \mathbb{A}_S^1$ as ring objects in $\mathsf{Sch}_{\blacklozenge/S}$.

(i) For any object $X \in \mathsf{Sch}_{\blacklozenge/S}$, we define a functor
$$\breve{\mathcal{O}}_{\blacklozenge/S,X}^A :\overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(\iota_X(-), A) : \mathsf{OpIm}_{\blacklozenge/S}^{\mathrm{op}}(X) \to \mathsf{Ring}.$$

Then, since A is isomorphic as a ring object in $\mathsf{Sch}_{\blacklozenge/S}$ to $\mathbb{A}_S^1$, and $\mathbb{A}_S^1$ represents the global section functor $\mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Ring}$, there exists a unique assignment
$$(X, i) \mapsto [\breve{\eta}_X^{A,\#}(i) : \mathcal{O}_X(\mathrm{Im}(|i|)) \xrightarrow{\sim} \breve{\mathcal{O}}_{\blacklozenge/S,X}^A(i)],$$
where X ranges over the objects of $\mathsf{Sch}_{\blacklozenge/S}$, and i ranges over the objects of $\mathsf{OpIm}_{\blacklozenge/S}(X)$, such that $\breve{\eta}_X^{A,\#}(i)$ is an isomorphism of rings, and for any diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ i\ } & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
V & \xrightarrow{\ j\ } & Y
\end{array}
$$

in $\mathsf{Sch}_{\blacklozenge/S}$, if i, j are quasi-compact open immersions, then the diagram of rings

$$(\dagger) \qquad \begin{array}{ccc} \mathcal{O}_Y(\mathrm{Im}(|j|)) & \xrightarrow[\sim]{\check{\eta}_Y^{A,\#}(j)} & \check{\mathcal{O}}_{\blacklozenge/S,Y}^A(j) \\ {\scriptstyle f^{\#}(\mathrm{Im}(|j|),\mathrm{Im}(|i|))}\downarrow & & \downarrow{\scriptstyle (-)\circ g} \\ \mathcal{O}_X(\mathrm{Im}(|i|)) & \xrightarrow[\sim]{\check{\eta}_X^{A,\#}(i)} & \check{\mathcal{O}}_{\blacklozenge/S,X}^A(i) \end{array}$$

commutes.

(ii) Let $f : X \to Y$ be a morphism of $\mathsf{Sch}_{\blacklozenge/S}$. For any open subset $U_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(X)$, we define

$$\mathcal{O}_{\blacklozenge/S,X}^A(U_\blacklozenge^{\mathsf{Sp}}) \overset{\mathrm{def}}{:=} \lim_{i \in \mathcal{I}_{\blacklozenge/S}(X;U_\blacklozenge^{\mathsf{Sp}})} \check{\mathcal{O}}_{\blacklozenge/S,X}^A(i).$$

For any open subset $U^{|\cdot|} \subset |X|$, we define

$$\eta_X^{A,\#}(U^{|\cdot|}) : \mathcal{O}_X(U^{|\cdot|}) \xrightarrow{\sim} \mathcal{O}_{\blacklozenge/S,X}^A(\eta_X^{-1}(U^{|\cdot|}))$$

(cf. Reconstruction 4.5) to be the composite of the following two isomorphisms

$$\mathcal{O}_X(U^{|\cdot|}) \xrightarrow{\sim} \lim_{i \in I} \mathcal{O}_X(\mathrm{Im}(|i|)) \xrightarrow[\sim]{\lim_{i \in I} \check{\eta}_X^{A,\#}(i)} \lim_{i \in I} \check{\mathcal{O}}_{\blacklozenge/S,X}^A(i) = \mathcal{O}_{\blacklozenge/S,X}^A(\eta_X^{-1}(U^{|\cdot|})),$$

where $I \overset{\mathrm{def}}{:=} \mathcal{I}_{\blacklozenge/S}(X, \eta_X^{-1}(U^{|\cdot|}))$. For any open subsets $V_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(Y)$ and $U_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(f)^{-1}(V_\blacklozenge^{\mathsf{Sp}})$, we define

$$\mathcal{O}_{\blacklozenge/S,f}^A(V_\blacklozenge^{\mathsf{Sp}}, U_\blacklozenge^{\mathsf{Sp}}) \overset{\mathrm{def}}{:=} \lim[\check{\mathcal{O}}_{\blacklozenge/S,Y}^A(j) \to \check{\mathcal{O}}_{\blacklozenge/S,X}^A(i)]$$
$$: \mathcal{O}_{\blacklozenge/S,Y}^A(V_\blacklozenge^{\mathsf{Sp}}) \to \mathcal{O}_{\blacklozenge/S,X}^A(U_\blacklozenge^{\mathsf{Sp}}),$$

where the limit is indexed by the opposite category associated to the category whose objects are commutative diagrams

$$(i_1 : U_1 \to X, j_1 : V_1 \to Y, g_1) \qquad\qquad \begin{array}{ccc} U_1 & \xrightarrow{i_1} & X \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f} \\ V_1 & \xrightarrow{j_1} & Y \end{array}$$

such that $\varphi_X(i_1) \subset U_\blacklozenge^{\mathsf{Sp}}$ and $\varphi_Y(j_1) \subset V_\blacklozenge^{\mathsf{Sp}}$, and whose morphisms $(i_1 : U_1 \to X, j_1 : V_1 \to Y, g_1) \to (i_2 : U_2 \to X, j_2 : V_1 \to Y, g_2)$ are pairs of morphisms $(s : U_1 \to U_2, t : V_1 \to V_2)$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that $i_1 = i_2 \circ s, j_1 = j_2 \circ t$, and the diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{s} & U_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ V_1 & \xrightarrow{t} & V_2 \end{array}$$

commutes. Then, since the diagram [†] commutes, for any subset $V^{|\cdot|} \subset |Y|$ and any open subset $U^{|\cdot|} \subset |f|^{-1}(V^{|\cdot|})$, the diagram of rings

$$
(\ddagger) \qquad
\begin{array}{ccc}
\mathcal{O}_Y(V^{|\cdot|}) & \xrightarrow[\sim]{\eta_Y^{A,\#}(V^{|\cdot|})} & \mathcal{O}_{\blacklozenge/S,Y}^A(\eta_Y^{-1}(V^{|\cdot|})) \\[2pt]
{\scriptstyle f^\#(V^{|\cdot|},U^{|\cdot|})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{O}_{\blacklozenge/S,f}^A(\eta_Y^{-1}(V^{|\cdot|}),\eta_Y^{-1}(U^{|\cdot|}))} \\[2pt]
\mathcal{O}_X(U^{|\cdot|}) & \xrightarrow[\sim]{\eta_X^{A,\#}(U^{|\cdot|})} & \mathcal{O}_{\blacklozenge/S,X}^A(\eta_X^{-1}(U^{|\cdot|}))
\end{array}
$$

commutes. For any open subsets $U_\blacklozenge^{\mathsf{Sp}} \subset U'^{\mathsf{Sp}}_\blacklozenge \subset \mathsf{Sp}_{\blacklozenge/S}(X)$, we define

$$
(-)|_{U_\blacklozenge^{\mathsf{Sp}}} \overset{\mathrm{def}}{:=} \mathcal{O}_{\blacklozenge/S,\mathrm{id}_X}^A(U'^{\mathsf{Sp}}_\blacklozenge, U_\blacklozenge^{\mathsf{Sp}}) : \mathcal{O}_{\blacklozenge/S,X}^A(U'^{\mathsf{Sp}}_\blacklozenge) \to \mathcal{O}_{\blacklozenge/S,X}^A(U_\blacklozenge^{\mathsf{Sp}}).
$$

(iii) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. Since the diagram [‡] commutes, it follows from the definitions of $\mathcal{O}_{\blacklozenge/S,X}^A(U_\blacklozenge^{\mathsf{Sp}})$ and $(-)|_{U_\blacklozenge^{\mathsf{Sp}}}$ that the rings $\mathcal{O}_{\blacklozenge/S,X}^A(U_\blacklozenge^{\mathsf{Sp}})$ and the ring homomorphisms $(-)|_{U_\blacklozenge^{\mathsf{Sp}}}$ form a presheaf $\mathcal{O}_{\blacklozenge/S,X}^A : \mathsf{Op}_{\blacklozenge/S}^{\mathsf{cat}}(X)^{\mathrm{op}} \to \mathsf{Ring}$ on $\mathsf{Sp}_{\blacklozenge/S}(X)$, and the family of isomorphisms of rings

$$
(\eta_X^{A,\#}(U^{|\cdot|}) : \mathcal{O}_X(U^{|\cdot|}) \xrightarrow{\sim} \mathcal{O}_{\blacklozenge/S,X}^A(\eta_X^{-1}(U^{|\cdot|})))_{U^{|\cdot|} \in \mathsf{Open}(|X|)},
$$

where $\mathsf{Open}(|X|)$ is the set of the open sets of $|X|$, determines an isomorphism of presheaves $\eta_X^{A,\#} : \mathcal{O}_X \xrightarrow{\sim} \eta_{X,*}\mathcal{O}_{\blacklozenge/S,X}^A$ on $|X|$. Since $\mathcal{O}_X$ is a sheaf on $|X|$, and $\eta_X : \mathsf{Sp}_{\blacklozenge/S}(X) \xrightarrow{\sim} |X|$ is a homeomorphism, the presheaf $\mathcal{O}_{\blacklozenge/S,X}^A$ on $\mathsf{Sp}_{\blacklozenge/S}(X)$ is a sheaf. Hence, the pair $\mathsf{RSp}_{\blacklozenge/S}^A(X) \overset{\mathrm{def}}{:=} (\mathsf{Sp}_{\blacklozenge/S}(X), \mathcal{O}_{\blacklozenge/S,X}^A)$ is a (**V**-small) ringed space, and the pair $\eta_X^A \overset{\mathrm{def}}{:=} (\eta_X, \eta_X^{A,\#})$ may be regarded as an isomorphism of ringed spaces $\eta_X^A : \mathsf{RSp}_{\blacklozenge/S}^A(X) \xrightarrow{\sim} X$. In particular, $\mathsf{RSp}_{\blacklozenge/S}^A(X)$ is a (**V**-small) scheme.

Reconstruction 6.20 ($\mathsf{RSp}_{\blacklozenge/S}^A(f)$)**.** Let S be a locally Noetherian normal scheme. Let $f : X \to Y$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$, and $A \in \mathcal{A}_{\blacklozenge/S}^1$ a ring object in $\mathsf{Sch}_{\blacklozenge/S}$ such that $A \cong \mathbb{A}_S^1$ as ring objects in $\mathsf{Sch}_{\blacklozenge/S}$. For any open subset $V_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(Y)$, we define

$$
\mathcal{O}_{\blacklozenge/S,f}^{A,\#}(V_\blacklozenge^{\mathsf{Sp}}) \overset{\mathrm{def}}{:=} \mathcal{O}_{\blacklozenge/S,f}^A(V_\blacklozenge^{\mathsf{Sp}}, \mathsf{Sp}_{\blacklozenge/S}(f)^{-1}(V_\blacklozenge^{\mathsf{Sp}}))
$$
$$
: \mathcal{O}_{\blacklozenge/S,Y}^A(V_\blacklozenge^{\mathsf{Sp}}) \to \mathcal{O}_{\blacklozenge/S,X}^A(\mathsf{Sp}_{\blacklozenge/S}(f)^{-1}(V_\blacklozenge^{\mathsf{Sp}})).
$$

Since the diagram Reconstruction 6.19 [‡] commutes, the family of morphisms

$$
\mathcal{O}_{\blacklozenge/S,f}^{A,\#} \overset{\mathrm{def}}{:=} \left(\mathcal{O}_{\blacklozenge/S,f}^{A,\#}(V_\blacklozenge^{\mathsf{Sp}}) : \mathcal{O}_{\blacklozenge/S,Y}^A(V_\blacklozenge^{\mathsf{Sp}}) \to \mathcal{O}_{\blacklozenge/S,X}^A(\mathsf{Sp}_{\blacklozenge/S}(f)^{-1}(V_\blacklozenge^{\mathsf{Sp}})) \right)_{V_\blacklozenge^{\mathsf{Sp}} \in \mathsf{Op}_{\blacklozenge/S}(Y)}
$$

determines a morphism of sheaves

$$
\mathcal{O}_{\blacklozenge/S,f}^{A,\#} : \mathcal{O}_{\blacklozenge/S,Y}^A \to \mathsf{Sp}_{\blacklozenge/S}(f)_* \mathcal{O}_{\blacklozenge/S,X}^A
$$

on $\mathsf{Sp}_{\blacklozenge/S}(Y)$. Hence the pair

$$
\mathsf{RSp}_{\blacklozenge/S}^A(f) \overset{\mathrm{def}}{:=} (\mathsf{Sp}_{\blacklozenge/S}(f), \mathcal{O}_{\blacklozenge/S,f}^{A,\#})
$$

determines a morphism of (**V**-small) schemes

$$
\mathsf{RSp}_{\blacklozenge/S}^A(f) : \mathsf{RSp}_{\blacklozenge/S}^A(X) \to \mathsf{RSp}_{\blacklozenge/S}^A(Y).
$$

Moreover, since the diagram Reconstruction 6.19 (‡) commutes, for any object $A \in \mathcal{A}^1_{\blacklozenge/S}$, the diagram of (**V**-small) schemes

$$
(\ddagger) \qquad
\begin{array}{ccc}
\mathsf{RSp}^A_{\blacklozenge/S}(X) & \xrightarrow[\sim]{\eta^A_X} & X \\
{\scriptstyle \mathsf{RSp}^A_{\blacklozenge/S}(f)} \downarrow & & \downarrow {\scriptstyle f} \\
\mathsf{RSp}^A_{\blacklozenge/S}(Y) & \xrightarrow[\sim]{\eta^A_Y} & Y
\end{array}
$$

commutes. This implies that $\mathsf{RSp}^A_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}_{\mathbf{V}}$ is a functor, and the family of isomorphisms of (**V**-small) schemes $\eta^A \overset{\mathrm{def}}{:=} (\eta^A_X : \mathsf{RSp}^A_{\blacklozenge/S}(X) \xrightarrow{\sim} X)_{X \in \mathsf{Sch}_{\blacklozenge/S}}$ determines an isomorphism of functors $\eta^A : \mathsf{RSp}^A_{\blacklozenge/S} \xrightarrow{\sim} U^{\mathsf{Sch}_{\mathbf{V}}}_{\blacklozenge/S} = i^{\mathsf{Sch}}_{\mathbf{U} \in \mathbf{V}} \circ U^{\mathsf{Sch}}_{\blacklozenge/S}$.

Reconstruction 6.21 $(\eta^{A,B})$. Let S be a locally Noetherian normal scheme.

(i) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object and $[A \xrightarrow{\sim} B] \in \mathcal{A}^1_{\blacklozenge/S}$ be an isomorphism of ring objects in $\mathsf{Sch}_{\blacklozenge/S}$. For any object $[i : U \to X] \in \mathsf{OpIm}_{\blacklozenge/S}(X)$, write

$$
\check{\eta}^{A,B,\#}_X(i) : \check{\mathcal{O}}^A_{\blacklozenge/S,X}(i) \xrightarrow{\sim} \check{\mathcal{O}}^B_{\blacklozenge/S,X}(i)
$$

(cf. Reconstruction 6.19 (i)) for the isomorphism of rings obtained by composing with the isomorphism of ring objects $A \xrightarrow{\sim} B$. Then it follows from the definitions of $\check{\mathcal{O}}^*_{\blacklozenge/S,X}(i)$ and $\check{\eta}^{A,B,\#}_X(i)$ that for any diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ i\ } & X \\
{\scriptstyle g} \downarrow & & \downarrow {\scriptstyle f} \\
V & \xrightarrow{\ j\ } & Y,
\end{array}
$$

if i, j are quasi-compact immersions, then the diagrams of rings

$$
(\dagger) \qquad
\begin{array}{ccc}
\check{\mathcal{O}}^A_{\blacklozenge/S,Y}(j) & \xrightarrow[\sim]{\check{\eta}^{A,B,\#}_Y(j)} & \check{\mathcal{O}}^B_{\blacklozenge/S,Y}(j) \\
{\scriptstyle (-)\circ g} \downarrow & & \downarrow {\scriptstyle (-)\circ g} \\
\check{\mathcal{O}}^A_{\blacklozenge/S,X}(i) & \xrightarrow[\sim]{\check{\eta}^{A,B,\#}_X(i)} & \check{\mathcal{O}}^B_{\blacklozenge/S,X}(i).
\end{array}
$$

commutes.

(ii) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object, $U^{\mathsf{Sp}}_{\blacklozenge} \subset \mathsf{Sp}_{\blacklozenge/S}(X)$ an open subset, and $[A \xrightarrow{\sim} B] \in \mathcal{A}^1_{\blacklozenge/S}$ an isomorphism of ring objects in $\mathsf{Sch}_{\blacklozenge/S}$. Write

$$
\eta^{A,B,\#}_X(U^{\mathsf{Sp}}_{\blacklozenge}) : \mathcal{O}^A_{\blacklozenge/S,X}(U^{\mathsf{Sp}}_{\blacklozenge}) \xrightarrow{\sim} \mathcal{O}^B_{\blacklozenge/S,X}(U^{\mathsf{Sp}}_{\blacklozenge})
$$

for the isomorphism defined by

$$
\eta^{A,B,\#}_X(U^{\mathsf{Sp}}_{\blacklozenge}) \overset{\mathrm{def}}{:=} \lim_{i \in \mathcal{I}_{\blacklozenge/S}(X;U^{\mathsf{Sp}}_{\blacklozenge})} \check{\eta}^{A,B,\#}_X(i).
$$

Since the diagram (†) commutes, for any morphism $f : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$, any open subset $V^{\mathsf{Sp}}_{\blacklozenge} \subset \mathsf{Sp}_{\blacklozenge/S}(Y)$, and any open subset $U^{\mathsf{Sp}}_{\blacklozenge} \subset \mathsf{Sp}_{\blacklozenge/S}(X)$ such that

$U_\blacklozenge^{\mathsf{Sp}} \subset \mathsf{Sp}_{\blacklozenge/S}(f)^{-1}(V_\blacklozenge^{\mathsf{Sp}})$, the diagram of rings

$$(\ddagger) \qquad \begin{array}{ccc} \mathcal{O}^A_{\blacklozenge/S,Y}(V_\blacklozenge^{\mathsf{Sp}}) & \xrightarrow[\sim]{\eta_Y^{A,B,\#}(V_\blacklozenge^{\mathsf{Sp}})} & \mathcal{O}^B_{\blacklozenge/S,Y}(V_\blacklozenge^{\mathsf{Sp}}) \\ {\scriptstyle \mathcal{O}^A_{\blacklozenge/S,f}(V_\blacklozenge^{\mathsf{Sp}},U_\blacklozenge^{\mathsf{Sp}})} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathcal{O}^B_{\blacklozenge/S,f}(V_\blacklozenge^{\mathsf{Sp}},U_\blacklozenge^{\mathsf{Sp}})} \\ \mathcal{O}^A_{\blacklozenge/S,X}(U_\blacklozenge^{\mathsf{Sp}}) & \xrightarrow[\sim]{\eta_X^{A,B,\#}(U_\blacklozenge^{\mathsf{Sp}})} & \mathcal{O}^B_{\blacklozenge/S,X}(U_\blacklozenge^{\mathsf{Sp}}) \end{array}$$

commutes.

(iii) Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. Then the commutativity of the diagram $(\ddagger)$ implies that the family of isomorphisms

$$\eta_X^{A,B,\#} := \left(\eta_X^{A,B,\#}(U_\blacklozenge^{\mathsf{Sp}}) : \mathcal{O}^A_{\blacklozenge/S,X}(U_\blacklozenge^{\mathsf{Sp}}) \xrightarrow{\sim} \mathcal{O}^B_{\blacklozenge/S,X}(U_\blacklozenge^{\mathsf{Sp}}) \right)_{U_\blacklozenge^{\mathsf{Sp}} \in \mathsf{Op}_{\blacklozenge/S}(X)}$$

determines an isomorphism $\eta_X^{A,B,\#} : \mathcal{O}^A_{\blacklozenge/S,X} \xrightarrow{\sim} \mathcal{O}^B_{\blacklozenge/S,X}$ of sheaves of rings on $\mathsf{Sp}_{\blacklozenge/S}(X)$. In particular, the pair $\eta_X^{A,B} := (\mathrm{id}_{\mathsf{Sp}_{\blacklozenge/S}(X)}, \eta_X^{A,B,\#})$ determines an isomorphism of schemes $\eta_X^{A,B} : \mathsf{RSp}^A_{\blacklozenge/S}(X) \xrightarrow{\sim} \mathsf{RSp}^B_{\blacklozenge/S}(X)$. Moreover, the commutativity of the diagram $(\ddagger)$ implies that for any morphism $f : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$, the diagram of ($\mathbf{V}$-small) schemes

$$(\dagger\dagger) \qquad \begin{array}{ccc} \mathsf{RSp}^A_{\blacklozenge/S}(X) & \xrightarrow[\sim]{\eta_X^{A,B}} & \mathsf{RSp}^B_{\blacklozenge/S}(X) \\ {\scriptstyle \mathsf{RSp}^A_{\blacklozenge/S}(f)} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathsf{RSp}^B_{\blacklozenge/S}(f)} \\ \mathsf{RSp}^A_{\blacklozenge/S}(Y) & \xrightarrow[\sim]{\eta_Y^{A,B}} & \mathsf{RSp}^B_{\blacklozenge/S}(Y) \end{array}$$

commutes. Thus, the family of isomorphisms

$$\eta^{A,B} := \left(\eta_X^{A,B} : \mathsf{RSp}^A_{\blacklozenge/S}(X) \xrightarrow{\sim} \mathsf{RSp}^B_{\blacklozenge/S}(X) \right)_{X \in \mathsf{Sch}_{\blacklozenge/S}}$$

determines an isomorphism of functors $\eta^{A,B} : \mathsf{RSp}^A_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{RSp}^B_{\blacklozenge/S}$. Moreover, it follows immediately from the definition of $\eta_X^{A,B,\#}$ that for any (iso)morphisms $A \xrightarrow{\sim} B, B \xrightarrow{\sim} C$ in $\mathcal{A}^1_{\blacklozenge/S}$, $\eta^A = \eta^B \circ \eta^{A,B}$, and $\eta^{A,C} = \eta^{B,C} \circ \eta^{A,B}$.

Reconstruction 6.22 ($\mathsf{RSp}_{\blacklozenge/S}$). Let S be a locally Noetherian normal scheme. By Reconstruction 6.20 and Reconstruction 6.21, we obtain a functor

$$\mathcal{A}^1_{\blacklozenge/S} \to \mathrm{Hom}(\mathsf{Sch}_{\blacklozenge/S}, \mathsf{Sch}_{\mathbf{V}}),$$
$$A \mapsto \mathsf{RSp}^A_{\blacklozenge/S},$$
$$[A \xrightarrow{\sim} B] \mapsto \eta^{A,B},$$

where $\mathrm{Hom}(\mathsf{Sch}_{\blacklozenge/S}, \mathsf{Sch}_{\mathbf{V}})$ denotes the ($\mathbf{V}$-small) category of functors $\mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}_{\mathbf{V}}$ and morphisms of functors. Define

$$\mathsf{RSp}_{\blacklozenge/S} := \mathrm{colim}_{A \in \mathcal{A}^1_{\blacklozenge/S}} \mathsf{RSp}^A_{\blacklozenge/S} \ : \ \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}_{\mathbf{V}} \, .$$

Since $\mathcal{A}^1_{\blacklozenge/S}$ is a connected ($\mathbf{V}$-small) groupoid, the colimit $\mathsf{RSp}_{\blacklozenge/S}$ exists, and the natural morphism $\mathsf{RSp}^A_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{RSp}_{\blacklozenge/S}$ is an isomorphism. By Reconstruction 6.21, the family of isomorphisms of functors $\eta^A : \mathsf{RSp}^A_{\blacklozenge/S} \xrightarrow{\sim} U^{\mathsf{Sch}_{\mathbf{V}}}_{\blacklozenge/S}$ determines an isomorphism of functors $\eta : \mathsf{RSp}_{\blacklozenge/S} \xrightarrow{\sim} U^{\mathsf{Sch}_{\mathbf{V}}}_{\blacklozenge/S} = i^{\mathsf{Sch}}_{\mathbf{U} \in \mathbf{V}} \circ U^{\mathsf{Sch}}_{\blacklozenge/S}$.

Lemma 6.23. *Let S, T be locally Noetherian normal schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then the following assertions hold:*

(i) *Let $A_\blacklozenge \in \mathcal{A}^1_{\blacklozenge/S}$. Write $A_\lozenge \overset{\mathrm{def}}{:=} F(A_\blacklozenge)$ for the ring object of $\mathsf{Sch}_{\lozenge/T}$ determined by $A_\blacklozenge$. Then $A_\lozenge \in \mathcal{A}^1_{\lozenge/T}$.*

(ii) *Let $A_\blacklozenge \in \mathcal{A}^1_{\blacklozenge/S}$. Write $A_\lozenge \overset{\mathrm{def}}{:=} F(A)$ for the ring object of $\mathsf{Sch}_{\lozenge/T}$ determined by $A_\blacklozenge$. Then $S, T, F, A_\blacklozenge$ determine an isomorphism ρ^A between the two composite functors of the following diagram:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\ F\ } & \mathsf{Sch}_{\lozenge/T} \\[2pt]
{\scriptstyle \mathsf{RSp}^{A_\blacklozenge}_{\blacklozenge/S}} \downarrow & & \downarrow {\scriptstyle \mathsf{RSp}^{A_\lozenge}_{\lozenge/T}} \\[2pt]
\mathsf{Sch}_{\mathbf{V}} & =\!=\!= & \mathsf{Sch}_{\mathbf{V}}\, .
\end{array}
$$

(iii) *S, T, F determine an isomorphism ρ^{Rsp} between the two composite functors of the following diagram:*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\ F\ } & \mathsf{Sch}_{\lozenge/T} \\[2pt]
{\scriptstyle \mathsf{RSp}_{\blacklozenge/S}} \downarrow & & \downarrow {\scriptstyle \mathsf{RSp}_{\lozenge/T}} \\[2pt]
\mathsf{Sch}_{\mathbf{V}} & =\!=\!= & \mathsf{Sch}_{\mathbf{V}}\, .
\end{array}
$$

Proof. Assertion (i) follows immediately from Corollary 6.16.

Next, we prove assertion (ii). Let F^{-1} be a quasi-inverse of F. Since F^{-1} is a left adjoint of F, and $A_\lozenge = F(A_\blacklozenge)$, there exists an isomorphism of functors

$(\bigstar) \qquad \check{A}_* : \mathrm{Hom}_{\mathsf{Sch}_{\lozenge/T}}(*, A_\lozenge) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(F^{-1}(*), A_\blacklozenge) : \mathsf{Sch}_{\lozenge/T} \to \mathsf{Ring}.$

For any object $X \in \mathsf{Sch}_{\blacklozenge/S}$ and any object $[i_\lozenge : U_\lozenge \to F(X)] \in \mathsf{OpIm}_{\lozenge/T}(F(X))$, we write $U_\blacklozenge \overset{\mathrm{def}}{:=} F^{-1}(U_\lozenge)$, $i_\blacklozenge : U_\blacklozenge \to X$ for the morphism corresponding to $i_\lozenge : U_\lozenge \to F(X)$ (where we note that by Corollary 4.4 and Corollary 4.8 (ii), for any quasi-compact open immersion $U_\lozenge \to F(X)$ in $\mathsf{Sch}_{\lozenge/T}$, the corresponding morphism $F^{-1}(U_\lozenge) \to X$ in $\mathsf{Sch}_{\blacklozenge/S}$ is also a quasi-compact open immersion), and

$$
\check{A}(i_\lozenge) \overset{\mathrm{def}}{:=} \check{A}_{U_\lozenge} : \check{\mathcal{O}}^{A_\lozenge}_{\lozenge/T, F(X)}(i_\lozenge) \to \check{\mathcal{O}}^{A_\blacklozenge}_{\blacklozenge/S, X}(i_\blacklozenge)
$$

(cf. Reconstruction 6.19 (i)). Then, for any morphism $f : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$ and any diagram

$$
\begin{array}{ccc}
U_\lozenge & \xrightarrow{\ i_\lozenge\ } & F(X) \\[2pt]
{\scriptstyle g} \downarrow & & \downarrow {\scriptstyle F(f)} \\[2pt]
V_\lozenge & \xrightarrow{\ j_\lozenge\ } & F(Y)
\end{array}
$$

such that $i_\lozenge, j_\lozenge$ are quasi-compact open immersions, the diagram of rings

$$
(\dagger) \qquad
\begin{array}{ccc}
\check{\mathcal{O}}^{A_\lozenge}_{\lozenge/T, F(Y)}(j_\lozenge) & \xrightarrow[\sim]{\check{A}(j_\lozenge)} & \check{\mathcal{O}}^{A_\blacklozenge}_{\blacklozenge/S, Y}(j_\blacklozenge) \\[6pt]
{\scriptstyle (-)\circ g} \downarrow & & \downarrow {\scriptstyle (-)\circ F^{-1}(g)} \\[6pt]
\check{\mathcal{O}}^{A_\lozenge}_{\lozenge/T, F(X)}(i_\lozenge) & \xrightarrow[\sim]{\check{A}(i_\lozenge)} & \check{\mathcal{O}}^{A_\blacklozenge}_{\blacklozenge/S, X}(i_\blacklozenge)
\end{array}
$$

commutes.

Let $X \in \mathsf{Sch}_{\blacklozenge/S}$ be an object. Since F^{-1} is a left adjoint of F, F^{-1} induces an equivalence of categories

$$\mathsf{OpIm}_{\lozenge/T}(F(X)) \xrightarrow{\sim} \mathsf{OpIm}_{\blacklozenge/S}(X).$$

By the definition of the homeomorphism ρ_X (cf. the proof of Lemma 4.6) associated to the S, T, F under consideration, for any open subset $U_{\lozenge}^{\mathsf{Sp}} \subset \mathsf{Sp}_{\lozenge/T}(F(X))$, F^{-1} induces an equivalence of categories

$$F_{X;U_{\lozenge}^{\mathsf{Sp}}}^{\lozenge \to \blacklozenge} : \mathcal{I}_{\lozenge/T}(F(X); U_{\lozenge}^{\mathsf{Sp}}) \xrightarrow{\sim} \mathcal{I}_{\blacklozenge/S}(X; \rho_X^{-1}(U_{\lozenge}^{\mathsf{Sp}})).$$

For any open subset $U_{\lozenge}^{\mathsf{Sp}} \subset \mathsf{Sp}_{\lozenge/T}(F(X))$, if we write $U_{\blacklozenge}^{\mathsf{Sp}} \overset{\mathrm{def}}{:=} \rho_X^{-1}(U_{\lozenge}^{\mathsf{Sp}})$, $I_{\lozenge} \overset{\mathrm{def}}{:=} \mathcal{I}_{\lozenge/T}(F(X); U_{\lozenge}^{\mathsf{Sp}})$, and $I_{\blacklozenge} \overset{\mathrm{def}}{:=} \mathcal{I}_{\blacklozenge/S}(X; U_{\blacklozenge}^{\mathsf{Sp}})$, then we define an isomorphism of rings

$$\rho_X^{A,\#}(U_{\lozenge}^{\mathsf{Sp}}) : \mathcal{O}_{\lozenge/T,F(X)}^{A\lozenge}(U_{\lozenge}^{\mathsf{Sp}}) \xrightarrow{\sim} \mathcal{O}_{\blacklozenge/S,X}^{A\blacklozenge}(U_{\blacklozenge}^{\mathsf{Sp}})$$

(cf. Reconstruction 6.19 (ii)) as the composite of

$$\lim_{i_{\lozenge} \in I_{\lozenge}} \check{A}(i_{\lozenge}) : \mathcal{O}_{\lozenge/T,F(X)}^{A\lozenge}(U_{\lozenge}^{\mathsf{Sp}}) = \lim_{i_{\lozenge} \in I_{\lozenge}} \check{\mathcal{O}}_{\lozenge/T,F(X)}^{A\lozenge}(i_{\lozenge}) \xrightarrow{\sim} \lim_{i_{\lozenge} \in I_{\lozenge}} \check{\mathcal{O}}_{\blacklozenge/S,X}^{A\blacklozenge}(F_{X;U_{\lozenge}^{\mathsf{Sp}}}^{\lozenge \to \blacklozenge}(i_{\lozenge}))$$

and the inverse of the natural isomorphism

$$\mathcal{O}_{\blacklozenge/S,X}^{A\blacklozenge}(U_{\blacklozenge}^{\mathsf{Sp}}) = \lim_{i_{\blacklozenge} \in I_{\blacklozenge}} \check{\mathcal{O}}_{\blacklozenge/S,X}^{A\blacklozenge}(i_{\blacklozenge}) \xrightarrow{\sim} \lim_{i_{\lozenge} \in I_{\lozenge}} \check{\mathcal{O}}_{\blacklozenge/S,X}^{A\blacklozenge}(F_{X;U_{\lozenge}^{\mathsf{Sp}}}^{\lozenge \to \blacklozenge}(i_{\lozenge}))$$

induced by $F_{X;U_{\lozenge}^{\mathsf{Sp}}}^{\lozenge \to \blacklozenge}$. Since the diagram (†) commutes, for any morphism $f : X \to Y$ in $\mathsf{Sch}_{\blacklozenge/S}$, any open subset $V_{\lozenge}^{\mathsf{Sp}} \subset \mathsf{Sp}_{\lozenge/T}(F(Y))$, and any open subset $U_{\lozenge}^{\mathsf{Sp}} \subset \mathsf{Sp}_{\lozenge/T}(F(f))^{-1}(V_{\lozenge}^{\mathsf{Sp}})$, if we write $U_{\blacklozenge}^{\mathsf{Sp}} \overset{\mathrm{def}}{:=} \rho_X^{-1}(U_{\lozenge}^{\mathsf{Sp}})$ and $V_{\blacklozenge}^{\mathsf{Sp}} \overset{\mathrm{def}}{:=} \rho_Y^{-1}(V_{\lozenge}^{\mathsf{Sp}})$, then the diagram of rings

$$(\ddagger) \qquad \begin{array}{ccc} \mathcal{O}_{\lozenge/T,F(Y)}^{A\lozenge}(V_{\lozenge}^{\mathsf{Sp}}) & \xrightarrow[\sim]{\rho_Y^{A,\#}(V_{\lozenge}^{\mathsf{Sp}})} & \mathcal{O}_{\blacklozenge/S,Y}^{A\blacklozenge}(V_{\blacklozenge}^{\mathsf{Sp}}) \\ {\scriptstyle \mathcal{O}_{\lozenge/T,F(f)}^{A\lozenge}(V_{\lozenge}^{\mathsf{Sp}}, U_{\lozenge}^{\mathsf{Sp}})} \downarrow & & \downarrow {\scriptstyle \mathcal{O}_{\blacklozenge/S,f}^{A\blacklozenge}(V_{\blacklozenge}^{\mathsf{Sp}}, U_{\blacklozenge}^{\mathsf{Sp}})} \\ \mathcal{O}_{\lozenge/T,F(X)}^{A\lozenge}(U_{\lozenge}^{\mathsf{Sp}}) & \xrightarrow[\sim]{\rho_X^{A,\#}(U_{\lozenge}^{\mathsf{Sp}})} & \mathcal{O}_{\blacklozenge/S,X}^{A\blacklozenge}(U_{\blacklozenge}^{\mathsf{Sp}}) \end{array}$$

(cf. Reconstruction 6.19 (ii)) commutes. Hence the family of isomorphisms of rings

$$\rho_X^{A,\#} \overset{\mathrm{def}}{:=} (\rho_X^{A,\#}(U_{\lozenge}^{\mathsf{Sp}}))_{U_{\lozenge}^{\mathsf{Sp}} \in \mathsf{Op}_{\lozenge/T}(F(X))}$$

(cf. Reconstruction 4.5) determines an isomorphism of sheaves of rings $\rho_X^{A,\#} : \mathcal{O}_{\lozenge/T,F(X)}^{A\lozenge} \xrightarrow{\sim} \rho_{X,*} \mathcal{O}_{\blacklozenge/S,X}^{A\blacklozenge}$ (cf. Reconstruction 6.19 (iii)). In particular, the pair $\rho_X^A \overset{\mathrm{def}}{:=} (\rho_X, \rho_X^{A,\#})$ determines an isomorphism of schemes $\rho_X^A : \mathsf{RSp}_{\blacklozenge/S}^{A\blacklozenge}(X) \xrightarrow{\sim} \mathsf{RSp}_{\lozenge/T}^{A\lozenge}(F(X))$ (cf. Reconstruction 6.19 (iii)).

Let $f : X \to Y$ be a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Since the diagram ($\ddagger$) commutes, the following diagram of sheaves of rings on $\mathsf{RSp}_{\lozenge/T}^{A_\lozenge}(F(Y))$ commutes:

$$
\begin{array}{ccc}
\mathcal{O}_{\lozenge/T, F(Y)}^{A_\lozenge} & \xrightarrow{\quad \rho_Y^{A,\#} \quad} & \rho_{Y,*}\mathcal{O}_{\blacklozenge/S,Y}^{A_\blacklozenge} \\
\Big\downarrow{\scriptstyle \mathcal{O}_{\lozenge/T, F(f)}^{A_\lozenge,\#}} & & \Big\downarrow{\scriptstyle \rho_{Y,*}\mathcal{O}_{\blacklozenge/S,f}^{A_\blacklozenge,\#}} \\
 & & \rho_{Y,*}\mathsf{Sp}_{\blacklozenge/S}(f)_*\mathcal{O}_{\blacklozenge/S,X}^{A_\blacklozenge} \\
 & & \| \\
\mathsf{Sp}_{\lozenge/T}(F(f))_*\mathcal{O}_{\lozenge/T, F(X)}^{A_\lozenge} & \xrightarrow{\mathsf{Sp}_{\lozenge/T}(F(f))_* \rho_X^{A,\#}} & \mathsf{Sp}_{\lozenge/T}(F(f))_*\rho_{X,*}\mathcal{O}_{\blacklozenge/S,X}^{A_\blacklozenge}.
\end{array}
$$

(cf. Lemma 4.6, Reconstruction 6.20). Hence the diagram of schemes

$$
\begin{array}{ccc}
\mathsf{RSp}_{\blacklozenge/S}^{A_\blacklozenge}(X) & \xrightarrow[\sim]{\rho_X^A} & \mathsf{RSp}_{\lozenge/T}^{A_\lozenge}(F(X)) \\
\Big\downarrow{\scriptstyle \mathsf{RSp}_{\blacklozenge/S}^{A_\blacklozenge}(f)} & & \Big\downarrow{\scriptstyle \mathsf{RSp}_{\lozenge/T}^{A_\lozenge}(F(f))} \\
\mathsf{RSp}_{\blacklozenge/S}^{A_\blacklozenge}(Y) & \xrightarrow[\sim]{\rho_Y^A} & \mathsf{RSp}_{\lozenge/T}^{A_\lozenge}(F(Y))
\end{array}
$$

commutes. Thus the family of isomorphisms

$$
\rho^A :\overset{\mathrm{def}}{=} \left(\rho_X^A : \mathsf{RSp}_{\blacklozenge/S}^{A_\blacklozenge}(X) \xrightarrow{\sim} \mathsf{RSp}_{\lozenge/T}^{A_\lozenge}(F(X)) \right)_{X \in \mathsf{Sch}_{\blacklozenge/S}}
$$

determines an isomorphism of functors $\rho^A : \mathsf{RSp}_{\blacklozenge/S}^{A_\blacklozenge} \xrightarrow{\sim} \mathsf{RSp}_{\lozenge/T}^{A_\lozenge} \circ F$. This completes the proof of assertion (ii).

Finally, we prove assertion (iii). Let $p : A_\blacklozenge \to B_\blacklozenge$ be an isomorphism in $\mathcal{A}_{\blacklozenge/S}^1$. Write $A_\lozenge :\overset{\mathrm{def}}{=} F(A_\blacklozenge)$ and $B_\lozenge :\overset{\mathrm{def}}{=} F(B_\blacklozenge)$. Then, by (i), $A_\lozenge, B_\lozenge \in \mathcal{A}_{\lozenge/T}^1$. Moreover, for any quasi-inverse F^{-1} of F and any object $* \in \mathsf{Sch}_{\lozenge/T}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Sch}_{\lozenge/T}}(*, A_\lozenge) & \xrightarrow[\sim]{\check{A}_*} & \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(F^{-1}(*), A_\blacklozenge) \\
\Big\downarrow{\scriptstyle F(p)\circ(-)} & & \Big\downarrow{\scriptstyle p\circ(-)} \\
\mathrm{Hom}_{\mathsf{Sch}_{\lozenge/T}}(*, B_\lozenge) & \xrightarrow[\sim]{\check{B}_*} & \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(F^{-1}(*), B_\blacklozenge),
\end{array}
$$

(cf. ($\bigstar$)). This commutativity implies that the functors "$\eta^{A,B}$" of Reconstruction 6.21 are compatible with passage from "$\blacklozenge$" to "$\lozenge$". Thus, by applying Reconstruction 6.22, we obtain an isomorphism

$$
\rho^{\mathsf{Rsp}} :\overset{\mathrm{def}}{=} \mathrm{colim}_{A_\blacklozenge \in \mathcal{A}_{\blacklozenge/S}^1} \rho^A : \mathsf{RSp}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{RSp}_{\lozenge/T} \circ F.
$$

This completes the proof of Lemma 6.23. □

Corollary 6.24. *Let S, T be locally Noetherian normal schemes and $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$ an equivalence. Then the following diagram commutes (up to natural isomorphism):*

$$
\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\quad F \quad} & \mathsf{Sch}_{\lozenge/T} \\
\Big\downarrow{\scriptstyle U_{\blacklozenge/S}^{\mathsf{Sch}}} & & \Big\downarrow{\scriptstyle U_{\lozenge/T}^{\mathsf{Sch}}} \\
\mathsf{Sch} & =\!\!=\!\!=\!\!=\!\!= & \mathsf{Sch}\,.
\end{array}
$$

Proof. By Reconstruction 6.22 and Lemma 6.23 (iii), we obtain the following natural isomorphisms:

$$i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Sch}} \circ U_{\blacklozenge/S}^{\mathsf{Sch}} \xleftarrow{\ \sim\ } \mathsf{RSp}_{\blacklozenge/S} \xrightarrow[\sim]{\ \rho^{\mathsf{Rsp}}\ } \mathsf{RSp}_{\lozenge/T} \circ F \xrightarrow{\ \sim\ } i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Sch}} \circ U_{\lozenge/T}^{\mathsf{Sch}} \circ F,$$

where the first and the third isomorphisms are the "$\blacklozenge$" and "$\lozenge$" versions of the isomorphism "η" of Reconstruction 6.22. Since $i_{\mathbf{U}\in\mathbf{V}}^{\mathsf{Sch}}$ is fully faithful, the composite isomorphism of the above display determines a natural isomorphism $U_{\blacklozenge/S}^{\mathsf{Sch}} \xrightarrow{\sim} U_{\lozenge/T}^{\mathsf{Sch}} \circ F$. $\qquad\square$

In the remainder of this section, we discuss some properties of $\mathsf{Sch}_{\blacklozenge/S}$ related to our reconstructions.

The following proposition concerns the rigidity of various forgetful functors.

Proposition 6.25. *Let S be a quasi-separated scheme. Then the following assertions hold:*

(i) $\mathrm{Aut}(U_{\blacklozenge/S}^{\mathsf{Set}}) = \left\{ \mathrm{id}_{U_{\blacklozenge/S}^{\mathsf{Set}}} \right\}$ *(cf. Reconstruction 1.3 (‡)).*

(ii) $\mathrm{Aut}(U_{\blacklozenge/S}^{\mathsf{Top}}) = \left\{ \mathrm{id}_{U_{\blacklozenge/S}^{\mathsf{Top}}} \right\}$ *(cf. Reconstruction 4.5 (‡)).*

(iii) $\mathrm{Aut}(U_{\blacklozenge/S}^{\mathsf{Sch}}) = \left\{ \mathrm{id}_{U_{\blacklozenge/S}^{\mathsf{Sch}}} \right\}$ *(cf. Definition 6.18 (i)).*

Proof. First, we prove assertion (i). Let $\alpha : U_{\blacklozenge/S}^{\mathsf{Set}} \xrightarrow{\sim} U_{\blacklozenge/S}^{\mathsf{Set}}$ be an isomorphism. Then, for any object $X \in \mathsf{Sch}_{\blacklozenge/S}$ and any point $x \in X$, if we write $f : \mathrm{Spec}(k(x)) \to X$ for the morphism of $\mathsf{Sch}_{\blacklozenge/S}$ determined by the point $x \in X$, then the diagram

$$
\begin{array}{ccc}
\{x\} = U_{\blacklozenge/S}^{\mathsf{Set}}(\mathrm{Spec}(k(x))) & \xrightarrow{\ U_{\blacklozenge/S}^{\mathsf{Set}}(f)\ } & U_{\blacklozenge/S}^{\mathsf{Set}}(X) = |X| \\
{\scriptstyle \alpha_{\mathrm{Spec}(k(x))}}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\
\{x\} = U_{\blacklozenge/S}^{\mathsf{Set}}(\mathrm{Spec}(k(x))) & \xrightarrow{\ U_{\blacklozenge/S}^{\mathsf{Set}}(f)\ } & U_{\blacklozenge/S}^{\mathsf{Set}}(X) = |X|
\end{array}
$$

commutes. Since $U_{\blacklozenge/S}^{\mathsf{Set}}(f)(x) = x$, it holds that $\alpha_X(x) = x$. Hence for any object $X \in \mathsf{Sch}_{\blacklozenge/S}$, it holds that $\alpha_X = \mathrm{id}_{U_{\blacklozenge/S}^{\mathsf{Set}}(X)}$. This completes the proof of assertion (i).
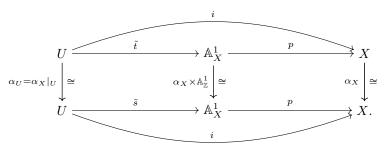
Assertion (ii) follows immediately from assertion (i).

Finally, we prove (iii). Let $\alpha : U_{\blacklozenge/S}^{\mathsf{Sch}} \xrightarrow{\sim} U_{\blacklozenge/S}^{\mathsf{Sch}}$ be an isomorphism. Then α is a family of isomorphisms of schemes $(\alpha_X : X \xrightarrow{\sim} X)_{X \in \mathsf{Sch}_{\blacklozenge/S}}$ such that for any morphism $f : X \to Y$ of $\mathsf{Sch}_{\blacklozenge/S}$, the following diagram of schemes commutes:
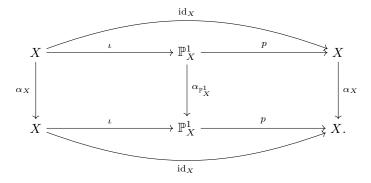
$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\
X & \xrightarrow{\ f\ } & Y.
\end{array}
$$

By assertion (ii), each morphism α_X induces the identity morphism on the underlying topological space $U_{\blacklozenge/S}^{\mathsf{Top}}(X)$ of X. Hence to prove (iii), it suffices to prove that for any $X \in \mathsf{Sch}_{\blacklozenge/S}$, the isomorphism of sheaves of rings $\alpha_X^{\#} : \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$ is the identity morphism. Let U be an affine open subset of X and $s \in \mathcal{O}_X(U)$ a section.

Write $i : U \to X$ for the open immersion. To prove that $\alpha_X^{\#} = \mathrm{id}_{\mathcal{O}_X}$, it suffices to prove that $\alpha_X^{\#}(U)(s) = s$. By Corollary 2.6 (i), U belongs to $\mathsf{Sch}_{\blacklozenge/S}$. Since $i \circ \alpha_U = \alpha_X \circ i$, it holds that $\alpha_U = \alpha_X|_U$. Write $p : \mathbb{A}_X^1 \to X$ for the natural projection and $\tilde{s} : U \to \mathbb{A}_X^1$ for the morphism over X in $\mathsf{Sch}_{\blacklozenge/S}$ corresponding to s such that $p \circ \tilde{s} = i$. If $\tilde{t} : U \to \mathbb{A}_X^1$ is the morphism corresponding to the section $t = \alpha_X^{\#}(U)(s) \in \mathcal{O}_X(U)$, then the following diagram of schemes commutes:

Since $\tilde{s} \circ \alpha_U = \alpha_{\mathbb{A}_X^1} \circ \tilde{s}$, to prove that $t = s$, it suffices to prove that $\alpha_{\mathbb{A}_X^1} = \mathrm{id}_{\mathbb{A}_{\mathbb{Z}}^1} \times \alpha_X$. Moreover, since $\mathbb{A}_X^1 \subset \mathbb{P}_X^1$ is an open subscheme, it suffices to prove that $\alpha_{\mathbb{P}_X^1} = \mathrm{id}_{\mathbb{P}_{\mathbb{Z}}^1} \times \alpha_X$. Write $0_X, 1_X, \infty_X : X \to \mathbb{P}_X^1$ for the morphisms obtained by base-changing the sections $0, 1, \infty : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}_{\mathbb{Z}}^1$. Then for any $\iota \in \{0_X, 1_X, \infty_X\}$, the following diagram commutes:

This commutativity implies that $\alpha_{\mathbb{P}_X^1}^{-1} \circ (\mathrm{id}_{\mathbb{P}_{\mathbb{Z}}^1} \times \alpha_X) : \mathbb{P}_X^1 \xrightarrow{\sim} \mathbb{P}_X^1$ is an automorphism of $\mathbb{P}_X^1$ over X which preserves $0_X, 1_X, \infty_X$, hence that $\alpha_{\mathbb{P}_X^1}^{-1} \circ (\mathrm{id}_{\mathbb{P}_{\mathbb{Z}}^1} \times \alpha_X) = \mathrm{id}_{\mathbb{P}_X^1}$. Thus $\alpha_{\mathbb{P}_X^1} = \mathrm{id}_{\mathbb{P}_{\mathbb{Z}}^1} \times \alpha_X$. This completes the proof of Proposition 6.25. $\square$

The following proposition concerns the rigidity of the composite of $U_{\blacklozenge/S}^{\mathsf{Sch}}$ (cf. Definition 6.18 (i)) with the functor f^* determined by base-change by f. For a morphism $f : S \to T$ of $\mathsf{Sch}_{\blacklozenge/\mathbb{Z}}$, we shall write

$$\mathrm{Aut}_T(f) :\overset{\mathrm{def}}{=} \left\{ \psi : S \xrightarrow{\sim} S \mid f = f \circ \psi \right\}$$

for the group of automorphisms of S over T.

Proposition 6.26. *Let T be a quasi-separated scheme and $f : S \to T$ a morphism in $\mathsf{Sch}_{\blacklozenge/T}$. Assume that $\mathrm{id}_T : T \to T$ belongs to $\mathsf{Sch}_{\blacklozenge/T}$. Write $f^* :\overset{\mathrm{def}}{=} (-) \times_T^{\blacklozenge} S :$*

$\mathsf{Sch}_{\blacklozenge/T} \to \mathsf{Sch}_{\blacklozenge/S}$ *for the functor determined by the operation of base-change, via* f, *from* T *to* S *(cf.* [Stacks, Tag 03GI], *Lemma 1.1 (i) (ii) (iii) (iv)).*

(i) *For any automorphism* $\psi \in \mathrm{Aut}_T(f)$, *the family of automorphisms*

$$\alpha^{\psi} :\overset{\mathrm{def}}{=} (\mathrm{id}_Y \times^{\blacklozenge}_T \psi : Y \times^{\blacklozenge}_T S \xrightarrow{\sim} Y \times^{\blacklozenge}_T S)_{Y \in \mathsf{Sch}_{\blacklozenge/T}}$$

determines an automorphism $\alpha^{\psi} : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$. *Moreover,* $(\alpha^{\psi})_T = \psi$.

(ii) *For any automorphism* $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$, $\alpha_{\mathbb{A}^1_T} = \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times \alpha_T$.

(iii) *For any automorphism* $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$, $\alpha_T : S \xrightarrow{\sim} S$ *belongs to* $\mathrm{Aut}_T(f)$.

(iv) *For any automorphism* $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$ *and any object* $Y \in \mathsf{Sch}_{\blacklozenge/T}$, $\alpha_Y = \mathrm{id}_Y \times^{\blacklozenge}_T \alpha_T$.

(v) *The morphism of groups*

$$(-)_T : \mathrm{Aut}(U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*) \to \mathrm{Aut}(S)$$
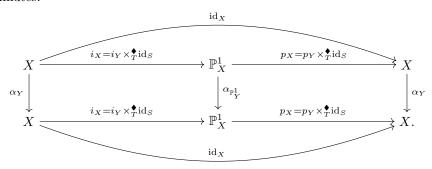$$\alpha \mapsto \alpha_T$$

is injective, and its image is equal to $\mathrm{Aut}_T(f) \subset \mathrm{Aut}(S)$. *In particular,* $\mathrm{Aut}(U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*) \cong \mathrm{Aut}_T(f)$.

Proof. Assertion (i) follows immediately from the definitions.

Next, we prove assertion (ii). Let $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$ be an automorphism. Thus α is a family of automorphisms $(\alpha_Y : Y \times^{\blacklozenge}_T S \xrightarrow{\sim} Y \times^{\blacklozenge}_T S)_{Y \in \mathsf{Sch}_{\blacklozenge/T}}$ such that for any morphism $g : Y \to Y'$ in $\mathsf{Sch}_{\blacklozenge/T}$ the following diagram in Sch commutes:

$$(\dagger) \quad \begin{array}{ccc} Y \times^{\blacklozenge}_T S & \xrightarrow{g \times^{\blacklozenge}_T \mathrm{id}_S} & Y' \times^{\blacklozenge}_T S \\ \alpha_Y \downarrow \cong & & \alpha_{Y'} \downarrow \cong \\ Y \times^{\blacklozenge}_T S & \xrightarrow{g \times^{\blacklozenge}_T \mathrm{id}_S} & Y' \times^{\blacklozenge}_T S. \end{array}$$

Let $Y \in \mathsf{Sch}_{\blacklozenge/T}$ be an object. Write $X :\overset{\mathrm{def}}{=} Y \times^{\blacklozenge}_T S$; $0_X, 1_X, \infty_X : X \to \mathbb{P}^1_X$ for the morphisms obtained by base-changing the sections $0, 1, \infty : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$; $0_Y, 1_Y, \infty_Y : Y \to \mathbb{P}^1_Y$ for the morphisms obtained by base-changing the sections $0, 1, \infty : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}$; $p_X : \mathbb{P}^1_X \to X$ and $p_Y : \mathbb{P}^1_Y \to Y$ for the natural projections. Then, since $\mathbb{P}^1_X$ is naturally isomorphic to $\mathbb{P}^1_Y \times^{\blacklozenge}_T S$, for each $i \in \{0, 1, \infty\}$, it holds that $i_X = i_Y \times^{\blacklozenge}_T \mathrm{id}_S$ and $p_X = p_Y \times^{\blacklozenge}_T \mathrm{id}_S$, and the following diagram in Sch commutes:
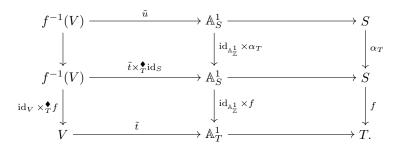
Hence $\alpha_{\mathbb{P}^1_Y}^{-1} \circ (\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \alpha_Y) : \mathbb{P}^1_X \xrightarrow{\sim} \mathbb{P}^1_X$ is an automorphism of $\mathbb{P}^1_X$ over X which preserves $0_X, 1_X, \infty_X$. This implies that $\alpha_{\mathbb{P}^1_Y}^{-1} \circ (\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \alpha_Y) = \mathrm{id}_{\mathbb{P}^1_X}$. Thus $\alpha_{\mathbb{P}^1_Y} = \mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \alpha_Y$. In particular, $\alpha_{\mathbb{P}^1_T} = \mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} \times \alpha_T$. Since $\mathbb{A}^1_T \subset \mathbb{P}^1_T$ is an open subscheme, the commutativity of (†) implies that $\alpha_{\mathbb{A}^1_T} = \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times \alpha_T$. This completes the proof of assertion (ii).

Next, we prove assertion (iii). Let $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^* \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/S} \circ f^*$ be an automorphism and $q \in T$ a point. Write $Y :\overset{\mathrm{def}}{=} \mathrm{Spec}(k(q))$ and $i : Y \to T$ for the natual morphism. Then, by the commutativity of (†), $\alpha_T \circ (i \times^{\blacklozenge}_T \mathrm{id}_S) = (i \times^{\blacklozenge}_T \mathrm{id}_S) \circ \alpha_Y$. Hence, since $i \times^{\blacklozenge}_T \mathrm{id}_S : Y \times^{\blacklozenge}_T S \to S$ and $\mathrm{id}_Y \times^{\blacklozenge}_T f : Y \times^{\blacklozenge}_T S \to Y$ are the natural projections, for any $p \in f^{-1}(q)$,

$$
\begin{aligned}
|f|(|\alpha_T|(|i \times^{\blacklozenge}_T \mathrm{id}_S|(p))) &= |f|(|i \times^{\blacklozenge}_T \mathrm{id}_S|(|\alpha_Y|(p))) \\
&= |i|(|\mathrm{id}_Y \times^{\blacklozenge}_T f|(|\alpha_Y|(p))) \\
&\in \mathrm{Im}(|i|) = \{q\}.
\end{aligned}
$$

This implies that $|f \circ \alpha_T| = |f|$. Hence, to prove assertion (iii), it suffices to prove that $f^{\#} = f^{\#} \circ f_*(\alpha_T^{\#}) : \mathcal{O}_T \to f_* \mathcal{O}_S$.

Let $V \subset T$ be an affine open subscheme and $t \in \mathcal{O}_T(V)$ a section. Then, by Corollary 2.6 (i), $V \in \mathsf{Sch}_{\blacklozenge/T}$. Write $f^{\#}_V : \mathcal{O}_T(V) \to \mathcal{O}_S(f^{-1}(V))$ for the morphism of rings of sections over $V \subset T$ and $\alpha^{\#}_{T,f^{-1}(V)} : \mathcal{O}_S(f^{-1}(V)) \xrightarrow{\sim} \mathcal{O}_S(f^{-1}(V))$ for the morphism of rings of sections over $f^{-1}(V) = V \times^{\blacklozenge}_T S \subset S$. Then $t \in \mathcal{O}_T(V)$ corresponds to a morphism of T-schemes $\tilde{t} : V \to \mathbb{A}^1_T$ in $\mathsf{Sch}_{\blacklozenge/T}$, $f^{\#}_V(t) \in \mathcal{O}_S(f^{-1}(V))$ corresponds to the morphism of S-schemes $\tilde{t} \times^{\blacklozenge}_T \mathrm{id}_S : f^{-1}(V) \to \mathbb{A}^1_S$ in $\mathsf{Sch}_{\blacklozenge/S}$, and $\alpha^{\#}_{T,f^{-1}(V)}(f^{\#}_V(t)) \in \mathcal{O}_S(f^{-1}(V))$ corresponds to the morphism of S-schemes $\tilde{u} : f^{-1}(V) \to \mathbb{A}^1_S$ in $\mathsf{Sch}_{\blacklozenge/S}$ such that all small square in the following diagram in Sch are Cartesian:

$$
\begin{array}{ccccc}
f^{-1}(V) & \xrightarrow{\tilde{u}} & \mathbb{A}^1_S & \longrightarrow & S \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times \alpha_T} & & \downarrow{\scriptstyle \alpha_T} \\
f^{-1}(V) & \xrightarrow{\tilde{t} \times^{\blacklozenge}_T \mathrm{id}_S} & \mathbb{A}^1_S & \longrightarrow & S \\
{\scriptstyle \mathrm{id}_V \times^{\blacklozenge}_T f} \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times f} & & \downarrow{\scriptstyle f} \\
V & \xrightarrow{\tilde{t}} & \mathbb{A}^1_T & \longrightarrow & T.
\end{array}
$$

By assertion (ii), $\mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times \alpha_T = \alpha_{\mathbb{A}^1_T}$. Moreover, since the diagram the diagram (†) and the upper left-hand square of the above diagram are Cartesian, it holds that $\tilde{u} = \tilde{t} \times^{\blacklozenge}_T \mathrm{id}_S$. This implies that $\alpha^{\#}_{T,f^{-1}(V)}(f^{\#}_V(t)) = f^{\#}_V(t)$. This completes the proof of assertion (iii).

Next, we prove assertion (iv). By assertion (ii), $\mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times \alpha_T = \alpha_{\mathbb{A}^1_T}$. Hence, by the commutativity of (†) and Corollary 2.6 (i), for any affine open subscheme $V \subset T$,

$$(\ddagger) \qquad\qquad \alpha_{\mathbb{A}^1_V} = \mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}} \times (\alpha_T|_{f^{-1}(V)}).$$

Moreover, by the commutativity of (†) and equation (‡), for any $n \geq 1$ and any affine open subscheme $V \subset T$,

$$(\dagger\dagger) \qquad\qquad \alpha_{\mathbb{A}_V^n} = \mathrm{id}_{\mathbb{A}_{\mathbb{Z}}^n} \times (\alpha_T|_{f^{-1}(V)}).$$

Thus, by the commutativity of (†), equation (††), and assertion (iii), for any affine open subscheme $\mathrm{Spec}(B) \cong V \subset T$ and any (**U**-small) set I, if we write $B :\overset{\mathrm{def}}{=} \Gamma(V, \mathcal{O}_V)$ and $N :\overset{\mathrm{def}}{=} B^{\oplus I}$, then

$$(\ddagger\ddagger) \quad \alpha_{\mathrm{Spec}(\mathrm{Sym}_B(N))} = \mathrm{id}_{\mathrm{Spec}(\mathrm{Sym}_{\mathbb{Z}}(\mathbb{Z}^{\oplus I}))} \times (\alpha_T|_{f^{-1}(V)}) = \mathrm{id}_{\mathrm{Spec}(\mathrm{Sym}_B(N))} \times_T^{\blacklozenge} \alpha_T.$$

Note that for any affine open subscheme $V \subset T$ and any affine scheme $\mathrm{Spec}(A) \to V$, if we write $B :\overset{\mathrm{def}}{=} \Gamma(V, \mathcal{O}_V)$, then $\mathrm{Spec}(A)$ may be regarded as a closed subscheme of $\mathrm{Spec}(\mathrm{Sym}_B(B^{\oplus A}))$. Hence, by the commutativity of (†), equation (‡‡), and Corollary 2.6 (i), for any affine open subscheme $V \subset T$ and any affine scheme W over V (in $\mathsf{Sch}_{\blacklozenge/T}$), $\alpha_W = \mathrm{id}_W \times_T^{\blacklozenge} \alpha_T$. Thus, by the commutativity of (†), for any object $Y \in \mathsf{Sch}_{\blacklozenge/T}$, $\alpha_Y = \mathrm{id}_Y \times_T^{\blacklozenge} \alpha_T$. This completes the proof of assertion (iv).

Assertion (v) follows immediately from assertions (i), (iv), and (iii). This completes the proof of Proposition 6.26. $\square$

Corollary 6.27. *Let T be a quasi-separated scheme and $f : S \to T$ a morphism in $\mathsf{Sch}_{\blacklozenge/T}$. Assume that $\mathrm{id}_T : T \to T$ belongs to $\mathsf{Sch}_{\blacklozenge/T}$. Write $f^* :\overset{\mathrm{def}}{=} (-) \times_T^{\blacklozenge} S : \mathsf{Sch}_{\blacklozenge/T} \to \mathsf{Sch}_{\blacklozenge/S}$ for the functor determined by the operation of base-change, via f, from T to S (cf. [Stacks, Tag 03GI], Lemma 1.1 (i) (ii) (iii) (iv)). Then $\mathrm{Aut}(f^*) = \{\mathrm{id}_{f^*}\}$.*

Proof. Let $\alpha : f^* \xrightarrow{\sim} f^*$ be an isomorphism. Then, by Proposition 6.26 (iv), for any object $Y \in \mathsf{Sch}_{\blacklozenge/T}$, it holds that $\alpha_Y = \mathrm{id}_Y \times_T^{\blacklozenge} \alpha_T$. Since $\alpha_T : S \xrightarrow{\sim} S$ is an isomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, it holds that $\alpha_T = \mathrm{id}_S$. This implies that $\alpha = \mathrm{id}_{f^*}$. This completes the proof of Corollary 6.27. $\square$

Finally, we prove the main result of the present paper.

Theorem 6.28. *Let S and T be locally Noetherian normal schemes and $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ a subset. Then the natural functor*
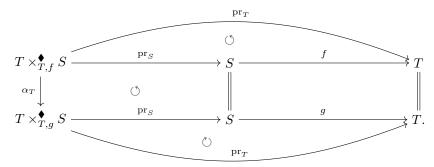
$$\mathrm{Isom}(S, T) \to \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S})$$
$$f \mapsto f^*$$

*is an equivalence of (**V**-small) categories.*

Proof. First, we observe that since $\mathrm{Isom}(S, T)$ is a discrete category, the functor $f \mapsto f^*$ is faithful.
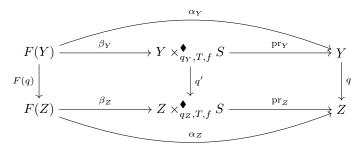
Next, we verify that the functor $f \mapsto f^*$ is full. Let $f, g : S \xrightarrow{\sim} T$ be isomorphisms. Assume that there exists an isomorphism of functors $\alpha : f^* \xrightarrow{\sim} g^*$. Write $h :\overset{\mathrm{def}}{=} g^{-1} \circ f : S \xrightarrow{\sim} S$ for the isomorphism of T-schemes from $f : S \to T$ to $g : S \to T$, $\tilde{\alpha} : U_{\blacklozenge/S}^{\mathsf{Sch}} \circ f^* \xrightarrow{\sim} U_{\blacklozenge/S}^{\mathsf{Sch}} \circ g^*$ (cf. Definition 6.18 (i)) for the isomorphism of functors induced by $\alpha : f^* \xrightarrow{\sim} g^*$, and $\beta :\overset{\mathrm{def}}{=} (\mathrm{id}_Y \times_T^{\blacklozenge} h : Y \times_{T,f}^{\blacklozenge} S \xrightarrow{\sim} Y \times_{T,g}^{\blacklozenge} S)_{Y \in \mathsf{Sch}_{\blacklozenge/T}}$ for the isomorphism of functors $\beta : U_{\blacklozenge/S}^{\mathsf{Sch}} \circ f^* \xrightarrow{\sim} U_{\blacklozenge/S}^{\mathsf{Sch}} \circ g^*$. Then $\tilde{\alpha}^{-1} \circ \beta \in \mathrm{Aut}(U_{\blacklozenge/S}^{\mathsf{Sch}} \circ f^*)$. Hence, by Proposition 6.26 (iii), $\alpha_T^{-1} \circ \beta_T \in \mathrm{Aut}_T(f)$. Since $\beta_T = \mathrm{id}_T \times_T^{\blacklozenge} (g^{-1} \circ f)$, the isomorphism of S-schemes $\alpha_T : T \times_{T,f}^{\blacklozenge} S \xrightarrow{\sim} T \times_{T,g}^{\blacklozenge} S$

is an isomorphism over T. Hence the outer square in the following diagram commutes:

$$
\begin{array}{ccccc}
& & \mathrm{pr}_T & & \\
& & \circlearrowright & & \\
T \times^{\blacklozenge}_{T,f} S & \xrightarrow{\mathrm{pr}_S} & S & \xrightarrow{f} & T \\
\alpha_T \downarrow & \circlearrowright & \| & & \| \\
T \times^{\blacklozenge}_{T,g} S & \xrightarrow{\mathrm{pr}_S} & S & \xrightarrow{g} & T. \\
& & \circlearrowright & & \\
& & \mathrm{pr}_T & &
\end{array}
$$

This implies that $f = g$, hence that $\alpha \in \mathrm{Aut}(f^*)$. Thus, by Corollary 6.27, $\alpha = \mathrm{id}_{f^*}$, which implies that the functor $f \mapsto f^*$ is full.

Finally, we verify that the functor $f \mapsto f^*$ is essentially surjective. Let $F : \mathsf{Sch}_{\blacklozenge/T} \xrightarrow{\sim} \mathsf{Sch}_{\blacklozenge/S}$ be an equivalence. By Corollary 6.24, there exists an isomorphism of functors $\alpha : U^{\mathsf{Sch}}_{\blacklozenge/S} \circ F \xrightarrow{\sim} U^{\mathsf{Sch}}_{\blacklozenge/T}$. Let $q_Y : Y \to T$ and $q_Z : Z \to T$ be objects of $\mathsf{Sch}_{\blacklozenge/T}$ and $q : Y \to Z$ a morphism in $\mathsf{Sch}_{\blacklozenge/T}$. Write $\beta_T : F(T) \to S$ for the structure morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Since $F(T)$ is a terminal object of $\mathsf{Sch}_{\blacklozenge/S}$, the morphism β_T is an isomorphism. Write $f :\overset{\mathrm{def}}{=} \alpha_T \circ \beta_T^{-1} : S \xrightarrow{\sim} T$, $\mathrm{pr}_Y : Y \times^{\blacklozenge}_{q_Y,T,f} S \xrightarrow{\sim} Y$ and $\mathrm{pr}_Z : Z \times^{\blacklozenge}_{q_Z,T,f} S \xrightarrow{\sim} Z$ for the first projections, $p_Y : Y \times^{\blacklozenge}_{q_Y,T,f} S \to S$ and $p_Z : Z \times^{\blacklozenge}_{q_Z,T,f} S \to S$ for the second projections, $\beta_Y :\overset{\mathrm{def}}{=} \mathrm{pr}_Y^{-1} \circ \alpha_Y : F(Y) \xrightarrow{\sim} Y \times^{\blacklozenge}_{q_Y,T,f} S$, $\beta_Z :\overset{\mathrm{def}}{=} \mathrm{pr}_Z^{-1} \circ \alpha_Z : F(Z) \xrightarrow{\sim} Z \times^{\blacklozenge}_{q_Z,T,f} S$, and $q' : Y \times^{\blacklozenge}_{q_Y,T,f} S \to Z \times^{\blacklozenge}_{q_Z,T,f} S$ for the unique morphism such that $p_Y = p_Z \circ q'$ and $q \circ \mathrm{pr}_Y = \mathrm{pr}_Z \circ q'$. Then it follows from the definition of α that the following diagram commutes:

$$
\begin{array}{ccccc}
& \alpha_Y & & & \\
F(Y) & \xrightarrow{\beta_Y} & Y \times^{\blacklozenge}_{q_Y,T,f} S & \xrightarrow{\mathrm{pr}_Y} & Y \\
F(q) \downarrow & & q' \downarrow & & q \downarrow \\
F(Z) & \xrightarrow{\beta_Z} & Z \times^{\blacklozenge}_{q_Z,T,f} S & \xrightarrow{\mathrm{pr}_Z} & Z \\
& & \alpha_Z & &
\end{array}
$$

Thus the family of isomorphisms $(\beta_Y : F(Y) \xrightarrow{\sim} Y \times^{\blacklozenge}_{q_Y,T,f} S)_{Y \in \mathsf{Sch}_{\blacklozenge/T}}$ determines an isomorphism of functors $\beta : F \xrightarrow{\sim} f^*$. This completes the proof of Theorem 6.28.
□

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