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Curves and symmetric spaces III: BN-special vs. 1-PS degeneration

To the memory of Professor C.S. Seshadri

By

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Abstract

A linear section theorem for Brill-Noether general curves of genus $g = 7, 8, 9$ is extended to Brill-Noether special ones by replacing the three symmetric spaces $OG(5, 10)^+ \subset \mathbb{P}^{15}$, $G(2, 6) \subset \mathbb{P}^{14}$ and the 6-dimensional Lagrangian Grassmannian $G(3, 6, \sigma) \subset \mathbb{P}^{13}$ with their suitable 1-PS limits $\Sigma'_{2g-2} \subset \mathbb{P}^{22-g}$.

Keywords[1] — canonical curve, symmetric space, Brill-Noether theory

In [2] and [5], it was found that the basic projective model $\Sigma_{2g-2} \subset \mathbb{P}_*(V)$ of a homogeneous variety $\Sigma_{2g-2} = G/(\text{parabolic subgp.})$ has a canonical curve $C_{2g-2} \subset \mathbb{P}^{g-1}$ of genus g as linear section for $g = 7, 8, 9, 10$. Except the last one, three are symmetric spaces of dimension $24 - 2g$. The following is proved:

Theorem 1 ([6], [7], [8], [9]) *A Brill-Noether general curve of genus g is isomorphic to a (transversal) linear section $\Sigma_{2g-2} \cap H_1 \cap \cdots \cap H_{23-2g}$ of of the basic projective model $\Sigma_{2g-2} \subset \mathbb{P}_*(V)$ for $g = 7, 8, 9$.*

Here a curve C of genus g is *Brill-Noether general* if $h^0(\xi)h^0(K_C\xi^{-1}) \leq g$ holds for every line bundle ξ on C with $h^0(\xi) \geq 2$ and$h^0(K_C\xi^{-1}) \geq 2$.

The Lie algebra of G and its $(23 - g)$-dimensional representation V is given in Table 1.

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Table 1: Symmetric spaces with a canonical curve section

g	7	8	9
$\Sigma_{2g-2} \subset \mathbb{P}^{22-g}$	$OG(5,10)^+ \subset \mathbb{P}^{15}$	$G(2,6) \subset \mathbb{P}^{14}$	$SpG(3,6) \subset \mathbb{P}^{13}$
Lie algebra	$so(10)$	$sl(6)$	$sp(6)$
V	16-dim'l spin	$\bigwedge^2 \mathbb{C}^6$	(14-dim'l) $\subset \bigwedge^3 \mathbb{C}^6$

In the moduli space $\mathcal{M}_g$ of curves of genus g, the curves which are not Brill-Noether general form a proper (Zariski) closed subset ([1, Chap. 5]), which we denote by BNS_g. For $g = 7, 8, 9$, BNS_g is an irreducible divisor. Our purpose of this article is to show Theorem 1 extends to a non-empty open set of $BNS_g \subset \mathcal{M}_g$. The main result is the following:

Theorem 2 *For each $g = 7, 8, 9$, there exists a one-parameter subgroup $\lambda \in \mathbb{G}_m \subset SL(V)$ such that a curve C of genus g corresponding to a general point of BNS_g is isomorphic to a transversal linear section of the 1-PS degeneration*

$$\left[\Sigma'_{2g-2} \subset \mathbb{P}^{22-g}\right] := \lim_{\lambda \to 0} \left[\Sigma^{\lambda}_{2g-2} \subset \mathbb{P}^{22-g}\right].$$

Table 2: Brill-Noether special vs. 1-PS degeneration

g	7	8	9
(r, s)	$(2, 4)$	$(3, 3)$	$(2, 5)$
BNS_g	G^1_4	G^2_7	G^1_5
Levi part	$so(4) \oplus so(6)$	$sl(3) \oplus sl(3) \oplus \mathbb{C}$	$sl(2) \oplus sp(4)$

The degenerations $\Sigma'_{2g-2} \subset \mathbb{P}^{22-g}$ will be constructed section by section. They are singular along the linear subspace P of dimension $21 - 2g$, and contained in the cone over the Segre variety $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \subset \mathbb{P}^g$

$$P \vee \left[\mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \subset \mathbb{P}^g\right] := \bigcup_{p \in P, q \in \mathbb{P} \times \mathbb{P}} \overline{pq} \subset \mathbb{P}^{22-g} \tag{1}$$

with vertex P, where the pair (r, s) of positive integers with $rs = g + 1$ is given in Table 2, whose last line gives the Levi part of the centralizer of the 1-PS in

Theorem 2. More precisely, let

$$(x_1 : \cdots : x_{22-2g}), \quad (u_1 : \cdots : u_r), \quad (v_1 : \cdots : v_s) \tag{2}$$

be the homogeneous coordinates of $P, \mathbb{P}^{r-1}, \mathbb{P}^{s-1}$, and we assign them bi-degree $(1,1),(1,0),(0,1)$, respectively. Then we have

(1) $g=7$**:** $\Sigma'_{12} \subset \mathbb{P}^7 \vee [\mathbb{P}^1 \times \mathbb{P}^3]$ is a complete intersection $f_1(x,u,v) = f_2(x,u,v) = 0$ of two divisors of bi-degree $(1,2)$.

(2) $g=8$**:** $\Sigma'_{14} \subset \mathbb{P}^5 \vee [\mathbb{P}^2 \times \mathbb{P}^2]$ is a complete intersection $f_1(x,u,v) = f_2(x,u,v) = 0$ of two divisors of bi-degree $(1,2)$ and $(2,1)$.

(3) $g=9$**:** $\Sigma'_{16} \subset \mathbb{P}^3 \vee [\mathbb{P}^1 \times \mathbb{P}^4]$ is the common zero locus of principal 4×4 minors of the skew-symmetric matrix

$$\begin{pmatrix} 0 & (1,1) & (1,1) & (1,1) & (1,1) \\ & 0 & (0,1) & (0,1) & (0,1) \\ & & 0 & (0,1) & (0,1) \\ & \ominus & & 0 & (0,1) \\ & & & & 0 \end{pmatrix} \tag{3}$$

whose (i,j)-entries are bi-homogenious polynomials $f_{ij}(x,u,v)$ of prescribed bi-degree.

Notation G^r_d denotes the (Zariski closure of) locus of curves with a g^r_d, that is, an r-dimensional linear system of degree d, in the moduli space $\mathcal{M}_g$.

1 Degeneration of orthogonal Grassmannian

Let $(\mathbb{C}^{10}, \langle\,,\rangle)$ be a 10-dimensional inner product space. The totally isotropic 5-dimensional spaces are parametrized by the disjoint union of two smooth subvarieties $OG(5,10)^{\pm}$ in the Grassmannian variety $G(5,10)$. Both $OG(5,10)^+$ and $OG(5,10)^-$ are 10-dimensional and embedded into $\mathbb{P}^{15}$ by spinor coordinates. The projective varieties $OG(5,10)^{\pm} \subset \mathbb{P}^{15}$ have a Brill-Norther general canonical curve of genus 7 as (complete) linear section. In this section we construct a 1-PS degeneration $OG(5,10)^+ \subset \mathbb{P}^{15}$ which has a Brill-Norther special curve of genus 7 as linear section.

Let V be a (2^{n-1}-dimensional) half spinor representation of the orthogonal Lie algebra $so(2n)$. The restriction of V to a Lie subalgebra $so(2n-2)$ decomposes in to the direct sum of two half spinor representations. The further restriction to

$so(2n-4)$ is the direct sum of two copies of half spinor representations $U^{\pm}$. More precisely, $so(2n)$ contains

$$\mathfrak{g}_0 := so(4) \oplus so(2n-4) \simeq sl(2) \oplus sl(2) \oplus so(2n-4)$$

as Lie subalgebra, and we have the decomposition

$$V = (\mathbb{C}^2 \otimes U^+) \oplus (\mathbb{C}^2 \otimes U^-)$$

as representation of $\mathfrak{g}_0$.

Returning to our situation we put $n = 5$. Then $U^{\pm}$ are dual to each other as representation of $so(6) \simeq sl(4)$. Hence the 16-dimensional representation V of $so(10)$ decomposes

$$V = (\mathbb{C}^2 \otimes \mathbb{C}^4) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{4,*}) \tag{4}$$

as representation of $\mathfrak{g}_0 \simeq sl(2) \oplus sl(2) \oplus sl(4)$.

The orthogonal Grassmannian $\Sigma_{12} = OG(5,10)^+ \subset \mathbb{P}^{15}$ is defined by 10 quadratic equations (see e.g. [8]). In terms of a system of homogeneous coordinates

$$(x_{11} : \cdots : x_{14} : x_{21} : \cdots : x_{24} : z_{11} : \cdots : z_{14} : z_{21} : \cdots : z_{24}) \tag{5}$$

compatible with (4), the 10 defining equations consists of four equations

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix} \begin{pmatrix} z_{11} & z_{21} \\ z_{12} & z_{22} \\ z_{13} & z_{23} \\ z_{14} & z_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

and six equations

$$\begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix} \pm \begin{vmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l} \end{vmatrix} = 0, \quad 1 \le i < j \le 4, \tag{7}$$

where $\{k, l\}$ is the complement of $\{i, j\}$ in $\{1, 2, 3, 4\}$ and the sign $\pm$ is chosen suitably.

We define a one-parameter subgroup $\lambda \in \mathbb{G}_m$ of $SL(16)$ by $(x, z) \mapsto (\lambda x, \lambda^{-1} z)$. The centralizer of $\mathbb{G}_m$ in $\mathfrak{g} = so(10)$ is $\mathfrak{g}_0$. The four equations (6) are invariant under this $\mathbb{G}_m$-action. The six equations (7) converge to

$$\begin{vmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l} \end{vmatrix} = 0 \tag{8}$$

as $\lambda \to 0$. Therefore, the limit Σ'_{12} of $\Sigma^{\lambda}_{12} \subset \mathbb{P}^{15}$ is contained in the cone

$$\mathbb{P}^7 \vee [\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7] \subset \mathbb{P}^{15}$$

over the Segre variety with vertex $\mathbb{P}^7$. Furthermore, in terms of the coordinates

$$((x_{11} : \cdots : x_{24}) : (u_1 : u_2) \times (v_1 : v_2 : v_3 : v_4)),$$

the limit Σ'_{12} is defined by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

in the cone $\mathbb{P}^7 \vee [\mathbb{P}^1 \times \mathbb{P}^3]$. In particular, the limit is a complete intersection of two divisors of bi-degree $(1,2)$.

More geometrically, the limit Σ'_{12} is the incident join

$$\bigcup_{p,q,<b_1,d>=<b_2,d>=0} \overline{pq} \subset \mathbb{P}^7 \vee [\mathbb{P}^1 \times \mathbb{P}^3] \subset \mathbb{P}^{15}, \tag{10}$$

where we put $p = (a_1 \otimes b_1 + a_2 \otimes b_2) \in \mathbb{P}^7, q = (c \otimes d), (c) \in \mathbb{P}^1, (d) \in \mathbb{P}^3$.

Now we are ready to consider a tetragonal curve C of genus 7 and recall the following:

Proposition 3 ([8, §6]) *Assume that a genus 7 curve C with a g^1_4 has no g^1_3 or g^2_6 and is not bi-elliptic. Then C is a complete intersection $D_1 \cap D_2 \cap D_3$ of three divisors of bi-degree $(1,1), (1,2)$ and $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$.*

Proof of Theorem 2 ($g = 7$) Let $\tilde{C} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be the intersection $D_2 \cap D_3$ of two divisors of bi-degree $(1,2)$ in Proposition 3, that is,

$$\tilde{C} : \sum_{i,j,k} a_{ijk} u_i v_j v_k = \sum_{i,j,k} a'_{ijk} u_i v_j v_k = 0$$

in $\mathbb{P}^1 \times \mathbb{P}^3$. Then $\tilde{C}$ is cut out from Σ'_{12} by the 8 hyperplanes,

$$H_k : x_{1k} = \sum_{i,j} a_{ijk} z_{ij}, \quad H'_k : x_{2i} = \sum_{i,j} a'_{ijk} z_{ij}, \quad k = 1,2,3,4 \tag{11}$$

that is, we have

$$\tilde{C} = H_1 \cap \cdots \cap H_4 \cap H'_1 \cap \cdots \cap H'_4 \cap \Sigma'_{12}.$$

Hence C is a linear section of Σ'_{12}. A general member of $BNS_7 \subset \mathcal{M}_7$ has a g^1_4, but has no g^1_3 or g^2_6. Hence we have Theorem 2. □

2 Degeneration of Grassmannian

Let $G(2,6)$ be the Grassmannian of 2-dimensional subspaces of a fixed 6-dimensional vector space. The projective variety $G(2,6) \subset \mathbb{P}^{14}$, embedded by Plücker coordinates, has a Brill-Noether general curve of genus 8 as transversal linear section. In this section we construct a 1-PS degeneration of this symmetric space corresponding to Brill-Noether specialization.

The second wedge representation $V = \bigwedge^2 \mathbb{C}^6$ of the Lie algebra $sl(6)$ decomposes

$$V = (\mathbb{C}^{3,*} \oplus \mathbb{C}^{3.*}) \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3) \tag{12}$$

as representation of the Lie subalgebra $sl(3) \oplus sl(3)$. We take

$$\begin{pmatrix} 0 & y_3 & -y_2 & z_{11} & z_{12} & z_{13} \\ & 0 & y_1 & z_{21} & z_{22} & z_{23} \\ & & 0 & z_{31} & z_{32} & z_{33} \\ & & & 0 & x_3 & -x_2 \\ & \ominus & & & 0 & x_1 \\ & & & & & 0 \end{pmatrix}$$

as a system of homogeneous coordinates of the 8-dimensional Grassmannian $G(2,6) \subset \mathbb{P}^{14}$. Then the Plücker relation decomposes into 9 relations

$$\begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix} + adj \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = 0 \tag{13}$$

and 6 relations

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

$$(y_1, y_2, y_3) \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = (0,0,0). \tag{15}$$

We define a one-parameter subgroup $\lambda \in \mathbb{G}_m$ of $SL(15)$ by

$$(x, y, z) \mapsto (\lambda^3 x, \lambda^3 y, \lambda^{-2} z).$$

Then, while both (14) and (15) are (semi-)invariant under this 1-PS, the 9-equations (13) converge to

$$adj \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = 0 \tag{16}$$

as $\lambda \to 0$. Hence the limit Σ'_{14} of $\Sigma^{\lambda}_{14} \subset \mathbb{P}^{14}$ as $\lambda \to 0$ is contained in the cone

$$\mathbb{P}^5 \vee [\mathbb{P}^2 \times \mathbb{P}^2] \subset \mathbb{P}^{14}$$

over the Segre variety

$$\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, ((u_1 : u_2 : u_3), (v_1 : v_2 : v_3)) \mapsto (z_{ij})_{i,j=1,2,3}, \quad z_{ij} = u_i v_j$$

with vertex $\mathbb{P}^5$. By (14) and (15), we have

$$\sum_{i=1}^{3} x_i v_i = 0, \quad \sum_{i=1}^{3} y_i u_i = 0 \tag{17}$$

under the coordinate system $(x_i : y_j : u_i v_j)$ of (2), that is, the limit Σ'_{14} is a complete intersection of two divisors of bi-degree $(1,2)$ and $(2,1)$ in $\mathbb{P}^5 \vee \mathbb{P}^2 \times \mathbb{P}^2$. Geometrically Σ'_{14} is the incident join

$$\bigcup_{p,q,<a,c>=<b,d>=0} \overline{pq} \subset \mathbb{P}^5 \vee [\mathbb{P}^2 \times \mathbb{P}^2] \subset \mathbb{P}^{14}, \tag{18}$$

where we put $p = (a,b) \in \mathbb{P}^5, q = (c \otimes d), (c) \in \mathbb{P}^2, (d) \in \mathbb{P}^2$.

Now we consider a curve C of genus 8 with a g^2_7 and recall the following:

Proposition 4 ([4, §1]) *Assume that a curve C of genus 8 with a g^2_7 has no g^1_3 or g^2_6. Then C is a complete intersection $D_1 \cap D_2 \cap D_3$ of three divisors of bi-degree $(1,1), (1,2)$ and $(2,1)$ in the product $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof of Theorem 2 ($g = 8$) Let $\tilde{C} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the intersection $D_2 \cap D_3$ of two divisors of bi-degree $(1,2)$ and $(2,1)$ in Proposition 4, that is,

$$\tilde{C} : \sum_{j,k,l} a_{jkl} u_j v_k v_l = \sum_{i,j,k} a'_{ijk} u_i u_j v_k = 0.$$

Then $\tilde{C}$ is cut out from Σ'_{14} by the 6 hyperplanes,

$$H_l : x_l = \sum_{j,k} a_{jkl} z_{jk}, \quad \text{and} \quad H'_i : y_i = \sum_{j,k} a'_{ijk} z_{jk}, \quad l, i = 1,2,3, \tag{19}$$

that is, we have

$$\tilde{C} = H_1 \cap H_2 \cap H_3 \cap H'_1 \cap H'_2 \cap H'_3 \cap \Sigma'_{14}.$$

Hence C is a linear section of Σ'_{14}. A general member of $BNS_8 \subset \mathcal{M}_8$ has a g^2_7, but has no g^1_3 or g^2_6. Hence we have Theorem 2. □

3 Degenerated Lagrangian Grassmannian

Let $(\mathbb{C}^6, \sigma)$, $\sigma : \mathbb{C}^6 \times \mathbb{C}^6 \to \mathbb{C}$, be a 6-dimensional skew inner product space. The Lagrangian subspaces U form a smooth 6-dimensional subvariety

$$G(3,6,\sigma) := \{[U] | \sigma|_{U\times U} = 0\} \tag{20}$$

in the 9-dimensional Grassmannian $G(3,6)$. $G(3,6,\sigma)$ is a symmetric space of the symmetric group $Sp(6)$, and embedded into the projective space $\mathbb{P}^{13}$ associated with a 14-dimensional irreducible representation V. This is nothing but a Plücker embedding.

When restricting to the Lie subalgebra $sl(2) \oplus sp(4) \subset sp(6)$, the representation V decomposes as

$$V = \mathbb{C}^4 \oplus (\mathbb{C}^2 \otimes W), \tag{21}$$

where $\mathbb{C}^2, \mathbb{C}^4$ are vector representations of $sl(2), sp(4)$, respectively, and W the 5-dimensional irreducible one of $sp(4)$. For a suitable one-parameter subgroup $\lambda \in \mathbb{G}_m \subset SL(14)$ compatible with (21), the limit of $G(3,6,\sigma)(=\Sigma_{16})$ as $\lambda \to 0$ is $G(3,6,\sigma')(=\Sigma'_{16})$ for a skew-symmetric bilinear form $\sigma' : \mathbb{C}^6 \times \mathbb{C}^6 \to \mathbb{C}$ of rank 4.

We describe the quadratic equations of $G(3,6,\sigma')$ in $\mathbb{P}^{13}$, restricting those of $G(3,6) \subset \mathbb{P}^{19}$. For our purpose it is convenient to regard $G(3,6)$ as the closure of the image of the Veronese-like map

$$\mathrm{Mat}_3(\mathbb{C}) \to \mathbb{P}(\mathbb{C} \oplus \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C}) \oplus \mathbb{C}), \quad A \mapsto (1 : A : adj(A) : \det A)$$

of the (Jordan) algebra $\mathrm{Mat}_3(\mathbb{C})$ of 3×3 matrices. The Lagrangian Grassmannian $G(3,6,\sigma)$ and its degeneration $G(3,6,\sigma')$ are obtained when restricting to symmetric matrices and partly symmetric matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},$$

respectively. In both cases, they are defined by the 21(=6+6+9) quadratic equations

$$adj(A) = bB, \quad aA = adj(B), \quad AB = ab \cdot I_3 \tag{22}$$

in the matrix coordinate $(b : A : B : a)$ of $\mathbb{P}^{13}$, where I_3 is the unit matrix.

Following the decomposition (21), we take

$$\left(z_1 : \begin{pmatrix} z_2 & z_3 & x_1 \\ z_3 & z_4 & x_2 \\ 0 & 0 & t_5 \end{pmatrix} : \begin{pmatrix} t_4 & -t_3 & x_3 \\ -t_3 & t_2 & x_4 \\ 0 & 0 & z_5 \end{pmatrix} : t_5 \right)$$

as coordinate of $G(3,6,\sigma') \subset \mathbb{P}^{13}$. Then 10 of the 21 equations (22) coincide with the vanishing of 2×2 minors of

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{pmatrix}.$$

Therefore, $G(3,6,\sigma')$ is contained in the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^4$ with vertex $\mathbb{P}^3 = \mathbb{P}^3_{(x_1:x_2:x_3:x_4)}$. Putting $z_i = u_1 v_i, t_i = u_2 v_i, 1 \le i \le 5$, the remaining equations are reduced to the defining equation

$$v_1 v_5 + v_2 v_4 + v_3^2 = 0 \tag{23}$$

of the 3-dimensional symplectic Grassmannian $G(2,4,\bar{\sigma}') \simeq Q^3 \subset \mathbb{P}^4$, and the 4 equations

$$\begin{pmatrix} 0 & v_5 & v_3 & v_4 \\ & 0 & -v_2 & -v_3 \\ & & 0 & v_1 \\ & \ominus & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{24}$$

Combining (23) and (24), we have the following

Proposition 5 *The degenerated Lagrangian Grassmannian $G(3,6,\sigma')$ is the common zero locus of the principal 4×4-Pfaffians of the skew-symmetric matrix*

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ & 0 & v_5 & v_3 & v_4 \\ & & 0 & -v_2 & -v_3 \\ & \ominus & & 0 & v_1 \\ & & & & 0 \end{pmatrix}$$

in the system of coordinates (2).

Now we are ready to consider a pentagonal curve of genus 9.

Proposition 6 (Sagraloff [10, Theorem 4.5.4]) *Assume that a curve C of genus 9 has a g^1_5 ξ and also that ξ is* regular*, that is, $h^0(\xi^2) = 3$. Assume further that C has no g^1_4, g^2_6, or g^1_5 other than ξ. Then, by Buchsbaum-Eisenbud [3], C is defined by Pfaffian of 4×4 principal minors of a 5×5 alternating matrix in a 4-dimensional scroll S. Moreover, S is isomorphic to the $\mathbb{P}^3$-bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 3})$ over $\mathbb{P}^1$, and the 5×5 skew-symmetric matrix is of the form*

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ & 0 & b_{12} & b_{13} & b_{14} \\ & & 0 & b_{23} & b_{24} \\ & \ominus & & 0 & b_{34} \\ & & & & 0 \end{pmatrix}, \quad a_i \in H^0(S, \mathcal{L}(1)), b_{ij} \in H^0(S, \mathcal{L}), \tag{25}$$

where $\mathcal{L}$ is the tautological line bundle of the $\mathbb{P}^3$-bundle $S/\mathbb{P}^1$.

Proof of Theorem 2 ($g = 9$) A general member of $BNS_9 \subset \mathcal{M}_9$ has a g^1_5, but has no g^1_4 or g^2_6. Furthermore, C satisfies also the remaining assumption in the proposition by [10]. A general 4-dimensional linear section $\mathbb{P}^3 \vee [\mathbb{P}^1 \times \mathbb{P}^4] \cap H_1 \cap \cdots \cap H_5$ is the scroll $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}^{\oplus 3}) \subset \mathbb{P}^8$ (over $\mathbb{P}^1$). Since $H^0(S, \mathcal{L})$ is of 5-dimensional, we can normalize b_{ij}'s so that $b_{13} + b_{24} = 0$ in Proposition 6. Hence C is a linear section of the degenerated Lagrangian Grassmannian $G(3, 6, \sigma')$ by Proposition 5. □

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