Singularity of energy measures on a class of inhomogeneous Sierpinski gaskets

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Abstract. We study energy measures of canonical Dirichlet forms on inhomogeneous Sierpinski gaskets. We prove that the energy measures and suitable reference measures are mutually singular under mild assumptions.

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1 Introduction

Energy measures associated with regular Dirichlet forms are fundamental concepts in stochastic analysis and related fields. For example, the intrinsic metric is defined by using energy measures and appears in Gaussian estimates of the transition probabilities. Energy measures are also crucial for describing the conditions for sub-Gaussian behaviors of transition densities. The energy measures are expected to be singular with respect to (canonical) underlying measures for canonical Dirichlet forms on self-similar fractals, which has been confirmed in many cases [13, 4, 9, 10]. Recently, such a singularity was proved under full off-diagonal sub-Gaussian estimates of the transition densities [11].

In this paper, we study a class of inhomogeneous Sierpinski gaskets as examples that have not yet been covered in the previous studies: they do not necessarily have strict self-similar structures or nice sub-Gaussian estimates. We show that the singularity of the energy measures still holds under mild assumptions. The strategy of our proof is based on quantitative estimates of probability measures on shift spaces, the techniques of which were used in [9, 10]. We expect this study to lead to further progress in stochastic analysis of complicated spaces of this kind.

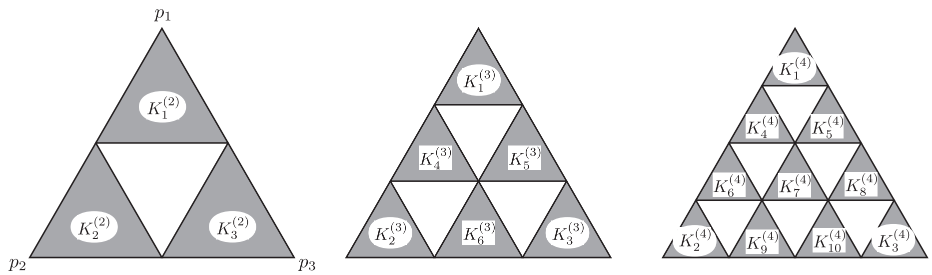

Fig. 1. $K_i^{(\nu)}$, the image of $\tilde{K}$ by the contractive affine map $\psi_i^{(\nu)}$ $(\nu = 2, 3, 4)$.

This paper is organized as follows: In Section 2, we introduce a class of inhomogeneous Sierpinski gaskets and canonical Dirichlet forms defined on them, and state the main results. In Sections 3 and 4, we provide preliminary lemmas and prove the theorems. In Section 5, we make some concluding remarks.

2 Framework and statement of theorems

We begin by recalling 2-dimensional level-ν Sierpinski gaskets SG(ν) for $\nu \geq 2$. Let $N(\nu) = \nu(\nu+1)/2$. Let $\tilde{K}$ be an equilateral triangle in $\mathbb{R}^2$ including the interior. Let $K_i^{(\nu)} \subset \tilde{K}$, $i = 1, 2, \ldots, N(\nu)$, be equilateral triangles including the interior that are obtained by dividing the sides of $\tilde{K}$ in ν, joining these points, and removing all the downward-pointing triangles, as in Figure 1. Let $\psi_i^{(\nu)}$, $i = 1, 2, \ldots, N(\nu)$, be the contractive affine map from $\tilde{K}$ onto $K_i^{(\nu)}$ of type $\psi_i^{(\nu)}(z) = \nu^{-1}z + \alpha_i^{(\nu)}$ for some $\alpha_i^{(\nu)} \in \mathbb{R}^2$. Then, the 2-dimensional level-ν Sierpinski gasket SG(ν) is defined as a unique non-empty compact subset K in $\tilde{K}$ such that

$$K = \bigcup_{i=1}^{N(\nu)} \psi_i^{(\nu)}(K).$$

Let $S_0 = \{1, 2, 3\}$, and let $V_0 = \{p_1, p_2, p_3\}$ be the set of all vertices of $\tilde{K}$. In the definition of SG(ν), the labeling of $K_i^{(\nu)}$ does not matter. For later convenience, we assign $K_i^{(\nu)}$ for $i \in S_0$ to the triangle that contains p_i. As a result, $\psi_i^{(\nu)}$ has a fixed point p_i.

For a general non-empty set X, denote by $l(X)$ the set of all real-valued functions on X. When X is finite, the inner product $(\cdot,\cdot)$ on $l(X)$ is defined by

$$(x, y) = \sum_{p \in X} x(p)y(p), \quad x, y \in l(X).$$

We regard $l(X)$ as the L^2-space on X equipped with the counting measure. Then, the L^2-inner product is identical with $(\cdot,\cdot)$. The induced norm is denoted by $|\cdot|$.

A symmetric linear operator $D=(D_{p,q})_{p,q\in V_0}$ on $l(V_0)$ is defined as

$$D_{p,q}=\begin{cases}-2 & \text{if } p=q,\\ 1 & \text{otherwise.}\end{cases}$$

Let

$$Q(x,y):=(-Dx,y)=-\sum_{p,q\in V_0}D_{p,q}x(q)y(p)$$

for $x,y\in l(V_0)$. More explicitly,

$$\begin{aligned}Q(x,y)&=(x(p_1)-x(p_2))(y(p_1)-y(p_2))+(x(p_2)-x(p_3))(y(p_2)-y(p_3))\\&\quad+(x(p_3)-x(p_1))(y(p_3)-y(p_1)).\end{aligned}$$

This is a Dirichlet form on $l(V_0)$. To simplify the notation, we sometimes write $Q(x)$ for $Q(x,x)$.

Let

$$V_1^{(\nu)}=\bigcup_{i=1}^{N(\nu)}\psi_i^{(\nu)}(V_0).$$

Let $r^{(\nu)}>0$ and $Q^{(\nu)}$ be a symmetric bilinear form on $V_1^{(\nu)}$ that is defined by

$$Q^{(\nu)}(x,y)=\sum_{i=1}^{N(\nu)}\frac{1}{r^{(\nu)}}Q(x\circ\psi_i^{(\nu)}|_{V_0},y\circ\psi_i^{(\nu)}|_{V_0}),\quad x,y\in l(V_1^{(\nu)}).$$

Then, there exists a unique $r^{(\nu)}>0$ such that, for every $x\in l(V_0)$,

$$Q(x,x)=\inf\bigl\{Q^{(\nu)}(y,y)\bigm| y\in l(V_1^{(\nu)})\text{ and }y|_{V_0}=x\bigr\}.\tag{2.1}$$

Hereafter, we fix such $r^{(\nu)}$. For example, $r^{(2)}=3/5$, $r^{(3)}=7/15$, and $r^{(4)}=41/103$, which are confirmed by the concrete calculation.

For each $x\in l(V_0)$, there exists a unique $y\in l(V_1)$ that attains the infimum in (2.1). For $i=1,2,\dots,N(\nu)$, the map $l(V_0)\ni x\mapsto y\circ\psi_i^{(\nu)}|_{V_0}\in l(V_0)$ is linear, which is denoted by $A_i^{(\nu)}$. Then, it holds that

$$Q(x,x)=\sum_{i=1}^{N(\nu)}\frac{1}{r^{(\nu)}}Q(A_i^{(\nu)}x,A_i^{(\nu)}x),\quad x\in l(V_0).\tag{2.2}$$

We can construct a Dirichlet form on $\mathrm{SG}(\nu)$ by using such data, but we omit the explanation because we discuss it in more general situations soon.

For reference, we give a quantitative estimate of $r^{(\nu)}$.

Lemma 2.1. *$1/\nu<r^{(\nu)}<N(\nu)/\nu^2$.*

Proof. This kind of inequality should be well-known (see, e.g., [2, Theorem 1]), and see the proof of [11, Proposition 5.3] (and also [1, Proposition 6.30]) for the second inequality. For the first inequality, let

$$\alpha = \inf\{Q(z,z) \mid z \in l(V_0),\ z(p_1)=1,\ z(p_2)=0\} > 0. \tag{2.3}$$

Then, for general $z \in l(V_0)$,

$$Q(z,z) \geq (z(p_1)-z(p_2))^2\alpha \tag{2.4}$$

by considering $(z-z(p_2))/(z(p_1)-z(p_2))$.

The infimum of (2.3) is attained by $x \in l(V_0)$ given by $x(p_1)=1$, $x(p_2)=0$, $x(p_3)=1/2$ (and $\alpha=3/2$). Take $y \in l(V_1^{(\nu)})$ attaining the infimum of (2.1). Let $I \subset \{1,2,\dots,N(\nu)\}$ be a ν-points set such that, for each $i \in I$, the intersection of $\psi_i^{(\nu)}(V_0)$ and the segment connecting p_1 and p_2 is a two-points set, say $\{\check{p}_i, \hat{p}_i\}$. Note that $3 \notin I$, and y is not constant on $\psi_3^{(\nu)}(V_0)$, which is confirmed by applying the maximum principle (see, e.g., [12, Proposition 2.1.7]) to the graph whose vertices are all points of $V_1^{(\nu)}$ included in the triangle with p_1, p_3 and the middle point of p_1 and p_2 as the three vertices. Therefore,

$$\begin{aligned}
\alpha &= Q^{(\nu)}(y,y)\\
&> \sum_{i\in I} \frac{1}{r^{(\nu)}} Q(y\circ\psi_i^{(\nu)}|_{V_0}, y\circ\psi_i^{(\nu)}|_{V_0})\\
&\geq \frac{1}{r^{(\nu)}} \sum_{i\in I} (y(\check{p}_i)-y(\hat{p}_i))^2\alpha \quad \text{(from (2.4))}\\
&\geq \frac{\alpha}{r^{(\nu)}} \left(\sum_{i\in I}(y(\check{p}_i)-y(\hat{p}_i))\right)^2 \left(\sum_{i\in I} 1\right)^{-1}\\
&= \frac{\alpha}{r^{(\nu)}} \cdot 1 \cdot \nu^{-1}.
\end{aligned}$$

Thus, $1/\nu < r^{(\nu)}$. □

See also [8] for the asymptotic behavior of $r^{(\nu)}$ as $\nu \to \infty$.

We now introduce 2-dimensional inhomogeneous Sierpinski gaskets. We fix a non-empty finite subset T of $\{\nu \in \mathbb{N} \mid \nu \geq 2\}$. For each $\nu \in T$, let $S^{(\nu)}$ denote the set of the letters i^ν for $i = 1,2,\dots,N(\nu)$. We set $S = \bigcup_{\nu\in T} S^{(\nu)}$ and $\Sigma = S^{\mathbb{N}}$. For example, if $T=\{2,3\}$, then

$$S^{(2)} = \{1^2, 2^2, 3^2\}, \quad S^{(3)} = \{1^3, 2^3, 3^3, 4^3, 5^3, 6^3\},$$

and $S = S^{(2)} \cup S^{(3)}$ has nine elements. (Note that i^ν *does not mean* $\underbrace{ii\cdots i}_{\nu}$, the ν-letter word consisting of only i, in this paper.)

For each $v \in S$, a shift operator $\sigma_v\colon \Sigma \to \Sigma$ is defined by $\sigma_v(\omega_1\omega_2\cdots) = v\omega_1\omega_2\cdots$. Let $W_0 = \{\emptyset\}$ and $W_m = S^m$ for $m \in \mathbb{N}$, and define $W_{\leq n} = \bigcup_{m=0}^n W_m$

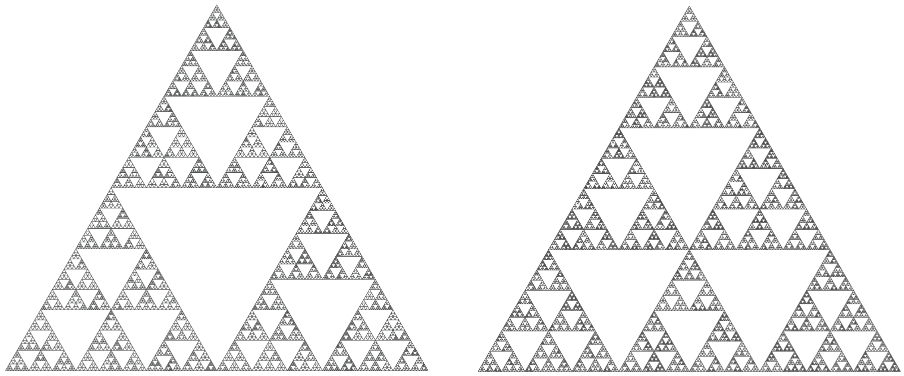

Fig. 2. Examples of inhomogeneous Sierpinski gaskets ($T = \{2, 3\}$).

and $W_* = \bigcup_{m\in\mathbb{Z}_+} W_m$. Here, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. For $w \in W_m$, $|w|$ represents m and is called the length of w. For $w = w_1\cdots w_m \in W_m$ and $w' = w'_1\cdots w'_n \in W_n$, $ww' \in W_{m+n}$ denotes $w_1\cdots w_m w'_1\cdots w'_n$. Also, $\sigma_w\colon \Sigma \to \Sigma$ is defined as $\sigma_w = \sigma_{w_1}\circ\cdots\circ\sigma_{w_m}$, and let $\Sigma_w = \sigma_w(\Sigma)$. For $k \le m$, $[w]_k$ denotes $w_1\cdots w_k \in W_k$. Similarly, for $\omega = \omega_1\omega_2\cdots \in \Sigma$ and $n \in \mathbb{N}$, let $[\omega]_n$ denote $\omega_1\cdots\omega_n \in W_n$. By convention, $\sigma_\emptyset\colon \Sigma \to \Sigma$ is the identity map, $[w]_0 := \emptyset \in W_0$ for $w \in W_*$, and $[\omega]_0 := \emptyset \in W_0$ for $\omega \in \Sigma$.

For $i^\nu \in S$, we define $\psi_{i^\nu} := \psi_i^{(\nu)}$ and $A_{i^\nu} := A_i^{(\nu)}$. For $w = w_1w_2\cdots w_m \in W_m$, ψ_w denotes $\psi_{w_1}\circ\psi_{w_2}\circ\cdots\circ\psi_{w_m}$ and A_w denotes $A_{w_m}\cdots A_{w_2}A_{w_1}$. Here, $\psi_\emptyset$ and $A_\emptyset$ are the identity maps by definition. For $\omega \in \Sigma$, $\bigcap_{m\in\mathbb{Z}_+}\psi_{[\omega]_m}(\tilde{K})$ is a one-point set $\{p\}$. The map $\Sigma \ni \omega \mapsto p \in \tilde{K}$ is denoted by π. The relation $\psi_v\circ\pi = \pi\circ\sigma_v$ holds for $v \in S$.

Now, we fix $L = \{L_w\}_{w\in W_*} \in T^{W_*}$. That is, we assign each $w \in W_*$ to $L_w \in T$. We set $\tilde{W}_0 = \{\emptyset\}$ and

$$\tilde{W}_m = \bigcup_{w\in\tilde{W}_{m-1}} \left\{wv \mid v \in S^{(L_w)}\right\}$$

for $m \in \mathbb{N}$, inductively. Define $\tilde{W}_* = \bigcup_{m\in\mathbb{Z}_+}\tilde{W}_m \subset W_*$, $\tilde{\Sigma} = \{\omega \in \Sigma \mid [\omega]_m \in \tilde{W}_m$ for all $m \in \mathbb{Z}_+\}$ and $G(L) = \pi(\tilde{\Sigma})$. It holds that

$$G(L) = \bigcap_{m\in\mathbb{Z}_+}\bigcup_{w\in\tilde{W}_m}\psi_w(\tilde{K}).$$

We call $G(L)$ an inhomogeneous Sierpinski gasket generated by L. See Figure 2 for a few examples. We equip $G(L)$ with the relative topology of $\mathbb{R}^2$. If $L_w = \nu$ for all $w \in W_*$, then $G(L)$ is nothing but $\mathrm{SG}(\nu)$.

For $m \in \mathbb{N}$, let

$$V_m = \bigcup_{w\in\tilde{W}_m}\psi_w(V_0),$$

and let $V_* = \bigcup_{m\in\mathbb{Z}_+} V_m$. The closure of V_* is equal to $G(L)$.

Next, we define reference measures on $G(L)$. Let

$$\mathcal{A}^{(\nu)} = \left\{ q = \{q_v\}_{v\in S^{(\nu)}} \;\middle|\; q_v > 0 \text{ for all } v \in S^{(\nu)} \text{ and } \sum_{v\in S^{(\nu)}} q_v = 1 \right\}$$

and

$$\mathcal{A} = \left\{ q = \{q_v\}_{v\in S} \;\middle|\; \text{for each } \nu \in T,\ \{q_v\}_{v\in S^{(\nu)}} \in \mathcal{A}^{(\nu)} \right\}.$$

For $q \in \mathcal{A}$, there exists a unique Borel probability measure λ_q on Σ such that

$$\lambda_q(\Sigma_w) = \begin{cases} q_{w_1}\cdots q_{w_m} & \text{if } w = w_1\cdots w_m \in \tilde{W}_m, \\ 0 & \text{if } w \notin \tilde{W}_*. \end{cases}$$

We note that

$$\lambda_q(\Sigma \setminus \tilde{\Sigma}) = \lim_{m\to\infty} \lambda_q\left(\Sigma \setminus \bigcup_{w\in\tilde{W}_m} \Sigma_w \right) = 0.$$

In what follows, q_w denotes $q_{w_1}\cdots q_{w_m}$ for $w = w_1\cdots w_m \in W_m$. By definition, $q_\emptyset = 1$. The Borel probability measure μ_q on $G(L)$ is defined by $\mu_q = (\pi|_{\tilde{\Sigma}})_*\lambda_q$, that is, the image measure of λ_q by $\pi|_{\tilde{\Sigma}}\colon \tilde{\Sigma} \to G(L)$. It is easy to see that μ_q has full support and does not charge any one points. When $T = \{\nu\}$, μ_q is a self-similar measure on $G(L) = \mathrm{SG}(\nu)$.

We next construct a Dirichlet form on $G(L)$. Let $r_{i^\nu} = r^{(\nu)}$ for $i^\nu \in S$, and $r_w = r_{w_1}\cdots r_{w_m}$ for $w = w_1\cdots w_m \in W_m$. By definition, $r_\emptyset = 1$. For $m \in \mathbb{Z}_+$, let

$$\mathcal{E}^{(m)}(x,y) = \sum_{w\in\tilde{W}_m} \frac{1}{r_w} Q(x\circ\psi_w|_{V_0}, y\circ\psi_w|_{V_0}), \quad x,y \in l(V_m).$$

From (2.1) and (2.2), it holds that for every $m \in \mathbb{Z}_+$ and $x \in l(V_m)$,

$$\mathcal{E}^{(m)}(x,x) = \inf\{\mathcal{E}^{(m+1)}(y,y) \mid y \in l(V_{m+1}) \text{ and } y|_{V_m} = x\}.$$

Thus, for any $x \in l(V_*)$, the sequence $\{\mathcal{E}^{(m)}(x|_{V_m}, x|_{V_m})\}_{m=0}^\infty$ is non-decreasing. We define

$$\mathcal{F} = \left\{ f \in C(G(L)) \;\middle|\; \lim_{m\to\infty} \mathcal{E}^{(m)}(f|_{V_m}, f|_{V_m}) < \infty \right\},$$
$$\mathcal{E}(f,g) = \lim_{m\to\infty} \mathcal{E}^{(m)}(f|_{V_m}, g|_{V_m}), \quad f,g \in \mathcal{F},$$

where $C(G(L))$ denotes the set of all real-valued continuous functions on $G(L)$. Then, $(\mathcal{E},\mathcal{F})$ is a resistance form and also a strongly local regular Dirichlet form on $L^2(G(L),\mu_q)$ for any $q \in \mathcal{A}$ (see [7] and [12, Chapter 2]). Here, $C(G(L))$ is regarded as a subspace of $L^2(G(L),\mu_q)$. We equip $\mathcal{F}$ with the inner product $(f,g)_{\mathcal{F}} := \mathcal{E}(f,g) + \int_{G(L)} fg\,d\mu_q$ as usual.

The energy measure $\mu_{\langle f\rangle}$ of $f\in\mathcal{F}$ is a finite Borel measure on $G(L)$, which is characterized by

$$\int_{G(L)} g\,d\mu_{\langle f\rangle}=2\mathcal{E}(f,fg)-\mathcal{E}(f^2,g),\quad g\in\mathcal{F}.$$

By letting $g\equiv 1$, the total mass of $\mu_{\langle f\rangle}$ is $2\mathcal{E}(f,f)$. Another expression of $\mu_{\langle f\rangle}$ is discussed in Section 3.

We introduce the following conditions for $q=\{q_v\}_{v\in S}\in\mathcal{A}$ to describe our main theorem.

(A) $q_{i^\nu}\neq r^{(\nu)}$ for all $i\in S_0$ and $\nu\in T$.
(B) For each $l_0,l_1\in\mathbb{N}$, there exists $l_2\in\mathbb{N}$ such that the following ($\star$) holds for μ_q-a.e. $\omega\in\Sigma$:
   ($\star$) there exist infinitely many $k\in\mathbb{Z}_+$ such that, for every $i,j\in S_0$,

$$\begin{aligned}&[\omega]_k i^{\nu_{k+1}}\cdots i^{\nu_{k+l_0}} j^{\nu_{k+l_0+1}}\cdots j^{\nu_{k+l_0+l_1}} j^{\nu_{k+l_0+l_1+1}}\cdots j^{\nu_{k+l_0+l_1+l_2}}\\ &\in \tilde{W}_{k+l_0+l_1+l_2}\end{aligned}\tag{2.5}$$

implies that

$$\begin{aligned}&\{\nu_m\in T\mid k+l_0+1\le m\le k+l_0+l_1\}\\ &\subset\{\nu_m\in T\mid k+l_0+l_1+1\le m\le k+l_0+l_1+l_2\}.\end{aligned}\tag{2.6}$$

Remark 2.2. (1) Condition ($\star$) is meaningful only for $\omega\in\tilde{\Sigma}$.
(2) For $\omega\in\tilde{\Sigma}$, $k\in\mathbb{Z}_+$, and $i,j\in S_0$, the elements $\nu_{k+1},\nu_{k+2},\dots,\nu_{k+l_0+l_1+l_2}\in T$ so that (2.5) holds are uniquely determined. Indeed, $\nu_{k+1}=L_{[\omega]_k}$, $\nu_{k+2}=L_{[\omega]_k i^{\nu_{k+1}}}$, $\nu_{k+3}=L_{[\omega]_k i^{\nu_{k+1}} i^{\nu_{k+2}}}$, and so on.
(3) A simple sufficient condition for (2.6) is

$$\{\nu_m\mid k+l_0+l_1+1\le m\le k+l_0+l_1+l_2\}=T.\tag{2.7}$$

Theorem 2.3. *Let $q\in\mathcal{A}$. Suppose that Condition* (A) *or* (B) *holds. Then, $\mu_{\langle f\rangle}$ and μ_q are mutually singular for every $f\in\mathcal{F}$.*

We provide some typical examples.

Example 2.4. Let $\nu\in T$ and define $L=\{L_w\}_{w\in W_*}$ by $L_w=\nu$ for all $w\in W_*$. Then, $G(L)$ is equal to $\mathrm{SG}(\nu)$. In this case, Condition ($\star$) is trivially satisfied for all $\omega\in\tilde{\Sigma}$ by letting $l_2=1$ because both sides of (2.6) are equal to $\{\nu\}$. Thus, by Theorem 2.3, $\mu_{\langle f\rangle}\perp\mu_q$ for every $f\in\mathcal{F}$ and $q\in\mathcal{A}$. This singularity has been proved in [10] already.

Example 2.5. Take any sequence $\{\tau_m\}_{m\in\mathbb{Z}_+}\in T^{\mathbb{Z}_+}$ and let $L_w=\tau_{|w|}$ for $w\in W_*$. The set $G(L)$ associated with $L=\{L_w\}_{w\in W_*}$ has been studied in, e.g., [6, 3, 11], and called a scale irregular Sierpinski gasket.

(1) Let $q = \{q_w\}_{w\in S} \in \mathcal{A}$ be given by $q_v = N(\nu)^{-1}$ for $v \in S^{(\nu)}$. The associated measure μ_q is regarded as a uniform measure on $G(L)$. Since $N(\nu)^{-1} < \nu^{-1}$, Condition (A) holds from Lemma 2.1. Therefore, $\mu_{\langle f\rangle} \perp \mu_q$ for any $f \in \mathcal{F}$ from Theorem 2.3. This case was discussed in [11, Section 5].

(2) (a) Suppose that there exists $l_2 \in \mathbb{N}$ such that $\{\tau_{k+1}, \tau_{k+2}, \dots, \tau_{k+l_2}\} = T$ for infinitely many $k \in \mathbb{Z}_+$. Then, Condition ($\star$) is satisfied for all $\omega \in \tilde{\Sigma}$, in view of (2.7).

(b) Suppose that for each $l \in \mathbb{N}$ there exists $k \in \mathbb{Z}_+$ such that $\tau_{k+1} = \tau_{k+2} = \cdots = \tau_{k+l}$. Then, Condition ($\star$) with $l_2 = 1$ is satisfied for all $\omega \in \tilde{\Sigma}$ and $l_0, l_1 \in \mathbb{N}$, but (2.7) may fail to hold for any l_2.

In either case, $\mu_{\langle f\rangle} \perp \mu_q$ for any $f \in \mathcal{F}$ and any $q \in \mathcal{A}$ from Theorem 2.3.

Example 2.6. Let ρ be a probability measure on T with full support. We take a family of T-valued i.i.d. random variables $\{L_w(\cdot)\}_{w\in W_*}$ with distribution ρ that are defined on some probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$. For each $\hat{\omega} \in \hat{\Omega}$, we can define an inhomogeneous Sierpinski gasket $G(L(\hat{\omega}))$ associated with $L(\hat{\omega}) := \{L_w(\hat{\omega})\}_{w\in W_*}$. This is called a random recursive Sierpinski gasket [7]. Then, the following holds.

Theorem 2.7. *For $\hat{P}$-a.s. $\hat{\omega}$, $G(L(\hat{\omega}))$ satisfies Condition* (B) *for all $q \in \mathcal{A}$. That is, for $\hat{P}$-a.s. $\hat{\omega}$, the Dirichlet form on $G(L(\hat{\omega}))$ can apply Theorem 2.3 for all $q \in \mathcal{A}$ to conclude that the energy measures and μ_q are mutually singular for all $q \in \mathcal{A}$.*

Theorems 2.3 and 2.7 are proved in Section 4.

3 Preliminary lemmas

In this section, we provide the necessary concepts and lemmas for proving Theorem 2.3. We fix $L = \{L_w\}_{w\in W_*} \in T^{W_*}$ and $q \in \mathcal{A}$ and retain the notation used in the previous section.

For $w \in \tilde{W}_*$, let K_w denote $\pi(\Sigma_w \cap \tilde{\Sigma})\,(= \psi_w(\tilde{K}) \cap G(L))$.

Let $m \in \mathbb{Z}_+$ and $x \in l(V_m)$. There exists a unique $h \in \mathcal{F}$ that attains

$$\inf\{\mathcal{E}(f,f) \mid f \in \mathcal{F} \text{ and } f|_{V_m} = x\}.$$

We call such h a piecewise harmonic (more precisely, an m-harmonic) function. When $m = 0$, h is called a harmonic function and is denoted by $\iota(x)$.

Lemma 3.1. *For $f \in \mathcal{F}$ and $m \in \mathbb{Z}_+$, let f_m be an m-harmonic function such that $f_m = f$ on V_m. Then, f_m converges to f in $\mathcal{F}$ as $m \to \infty$. In particular, the totality of piecewise harmonic functions is dense in $\mathcal{F}$.*

Proof. The proof is standard. From the maximum principle (see, e.g., [12, Lemma 2.2.3]),

$$\min_{K_w} f \le \min_{\psi_w(V_0)} f = \min_{K_w} f_m \le \max_{K_w} f_m = \max_{\psi_w(V_0)} f \le \max_{K_w} f$$

for any $w \in \tilde{W}_m$. Therefore, f_m converges to f uniformly on $G(L)$, in particular, in $L^2(G(L), \mu_q)$ as $m \to \infty$. Because $\{f_m\}_{m\in\mathbb{Z}_+}$ is bounded in $\mathcal{F}$, it converges to f weakly in $\mathcal{F}$. Because $\lim_{m\to\infty}(f_m, f_m)_{\mathcal{F}} = (f, f)_{\mathcal{F}}$, f_m actually converges to f strongly in $\mathcal{F}$. □

Let $v \in W_*$. We define $L^{[v]} = \{L_w^{[v]}\}_{w\in W_*} \in T^{W_*}$ by $L_w^{[v]} = L_{vw}$. Then, we can define a strongly local regular Dirichlet form $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$ on $L^2(G(L^{[v]}), \mu_q^{[v]})$, where $\mu_q^{[v]}$ is defined in the same way as μ_q with L replaced by $L^{[v]}$. The energy measure of $f \in \mathcal{F}^{[v]}$ is denoted by $\mu_{\langle f\rangle}^{[v]}$. The following lemma is proved in a straightforward manner by going back to the above definition.

Lemma 3.2. (1) *Let $f \in \mathcal{F}$ and $m \in \mathbb{N}$. For each $v \in \tilde{W}_m$, $f^{[v]} := f \circ \psi_v|_{G(L^{[v]})}$ belongs to $\mathcal{F}^{[v]}$. Moreover, it holds that*

$$\mathcal{E}(f, f) = \sum_{v\in\tilde{W}_m} \frac{1}{r_v}\mathcal{E}^{[v]}(f^{[v]}, f^{[v]}) \tag{3.1}$$

and

$$\mu_{\langle f\rangle} = \sum_{v\in\tilde{W}_m} \frac{1}{r_v}(\psi_v|_{G(L^{[v]})})_*\mu_{\langle f^{[v]}\rangle}^{[v]}. \tag{3.2}$$

If f is an m-harmonic function, then $f^{[v]}$ is a harmonic function with respect to $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$.

(2) *It holds that*

$$\mu_q = \sum_{v\in\tilde{W}_m} q_v(\psi_v|_{G(L^{[v]})})_*\mu_q^{[v]}. \tag{3.3}$$

By applying (3.1) with $\mathcal{E}$ replaced by $\mathcal{E}^{[\xi]}$ for $\xi \in \tilde{W}_*$ to $f = \iota(x)$ for $x \in l(V_0)$, we obtain the following identity as a special case:

$$r_\xi^{-1}Q(A_\xi x) = \sum_{\zeta\in W_m;\ \xi\zeta\in\tilde{W}_*} r_{\xi\zeta}^{-1}Q(A_{\xi\zeta}x), \quad m \in \mathbb{Z}_+. \tag{3.4}$$

Let $f \in \mathcal{F}$. For each $m \in \mathbb{Z}_+$, let $\lambda_{\langle f\rangle}^{(m)}$ be a measure on W_m defined as

$$\lambda_{\langle f\rangle}^{(m)}(C) = 2\sum_{v\in C\cap\tilde{W}_m} r_v^{-1}\mathcal{E}^{[v]}(f^{[v]}, f^{[v]}), \quad C \subset W_m.$$

Then, we can verify that $\{\lambda_{\langle f\rangle}^{(m)}\}_{m\in\mathbb{Z}_+}$ are consistent in the sense that $\lambda_{\langle f\rangle}^{(m)}(C) = \lambda_{\langle f\rangle}^{(m+1)}(C \times S)$. By the Kolmogorov extension theorem, there exists a unique Borel measure $\lambda_{\langle f\rangle}$ on Σ such that

$$\lambda_{\langle f\rangle}(\Sigma_C) = \lambda_{\langle f\rangle}^{(m)}(C) \quad \text{for any } m \in \mathbb{Z}_+,\ C \subset W_m,$$

where $\Sigma_C = \bigcup_{v\in C}\Sigma_v$. It is easy to see that $\lambda_{\langle f\rangle}(\Sigma \setminus \tilde{\Sigma}) = 0$.

In particular, if $f=\iota(x)$ for $x\in l(V_0)$, we have

$$\lambda_{\langle\iota(x)\rangle}(\Sigma_C)=2\sum_{v\in C\cap\tilde W_m} r_v^{-1}Q(A_v x),\quad C\subset W_m. \tag{3.5}$$

For simplicity, we write $\lambda_{\langle x\rangle}$ for $\lambda_{\langle\iota(x)\rangle}$.

Lemma 3.3. *For $f\in\mathcal{F}$, $(\pi|_{\tilde\Sigma})_*\lambda_{\langle f\rangle}=\mu_{\langle f\rangle}$.*

Proof. This lemma is proved in [9, Lemma 4.1] when T is a one-point set. In the general case, it suffices to modify the proof line by line by using Lemma 3.2 as a substitution of the self-similar property. We provide a proof here for the reader's convenience.

We define a set function χ_m for $m\in\mathbb{Z}_+$ by

$$\chi_m(A)=\sum_{v\in\tilde W_m}\frac{1}{r_v}\mu^{[v]}_{\langle f^{[v]}\rangle}(\pi(\sigma_v^{-1}(A)))$$

for a σ-compact subset A of $\tilde\Sigma$.

Let B be a closed subset of $G(L)$. For $v\in\tilde W_m$,

$$\begin{aligned}(\psi_v|_{G(L^{[v]})})^{-1}(B)&=\pi\bigl((\pi|_{\tilde\Sigma})^{-1}((\psi_v|_{G(L^{[v]})})^{-1}(B))\bigr)\\&=\pi\bigl(\sigma_v^{-1}((\pi|_{\tilde\Sigma})^{-1}(B))\bigr).\end{aligned}$$

Therefore, $\mu_{\langle f\rangle}(B)=\chi_m\bigl((\pi|_{\tilde\Sigma})^{-1}(B)\bigr)$ from (3.2).

For $C\subset\tilde W_m$,

$$\begin{aligned}\lambda_{\langle f\rangle}(\Sigma_C)&=\lambda^{(m)}_{\langle f\rangle}(C)\\&=2\sum_{v\in C}r_v^{-1}\mathcal{E}^{[v]}(f^{[v]},f^{[v]})\\&=\sum_{v\in\tilde W_m}r_v^{-1}\mu^{[v]}_{\langle f^{[v]}\rangle}\bigl(\pi(\sigma_v^{-1}(\Sigma_C))\bigr)\\&=\chi_m(\Sigma_C).\end{aligned}$$

Here, in the third equality, we used the identity

$$\pi(\sigma_v^{-1}(\Sigma_C))=\begin{cases}G(L^{[v]}) & \text{if } v\in C,\\ \emptyset & \text{otherwise.}\end{cases}$$

Let F be a closed subset of $G(L)$. Then, $(\pi|_{\tilde\Sigma})^{-1}(F)$ is also closed in $\tilde\Sigma$. For $m\in\mathbb{Z}_+$, let $C_m=\bigl\{w\in\tilde W_m \bigm| \Sigma_w\cap(\pi|_{\tilde\Sigma})^{-1}(F)\ne\emptyset\bigr\}$. Then, $\{\Sigma_{C_m}\}_{m=0}^\infty$ is decreasing in m and $\bigcap_{m\in\mathbb{Z}_+}\Sigma_{C_m}=(\pi|_{\tilde\Sigma})^{-1}(F)$. By using the monotonicity of χ_m,

$$\mu_{\langle f\rangle}(F)=\chi_m\bigl((\pi|_{\tilde\Sigma})^{-1}(F)\bigr)\le\chi_m(\Sigma_{C_m})=\lambda_{\langle f\rangle}(\Sigma_{C_m}).$$

Letting $m\to\infty$, we have $\mu_{\langle f\rangle}(F)\le\lambda_{\langle f\rangle}(F)$.

The inner regularity of $\mu_{\langle f\rangle}$ and $\lambda_{\langle f\rangle}$ implies that $\mu_{\langle f\rangle}(B)\le\lambda_{\langle f\rangle}(B)$ for all Borel sets B. Because the total measures of $\mu_{\langle f\rangle}$ and $\lambda_{\langle f\rangle}$ are the same, we also have the reverse inequality by considering $G(L)\setminus B$ in place of B. □

Let $i \in S_0$ and $\nu \in T$. From [12, Proposition A.1.1 and Theorem A.1.2], both 1 and $r^{(\nu)}$ are simple eigenvalues of $A_i^{(\nu)}$, and the modulus of another eigenvalue $s^{(\nu)}$ of $A_i^{(\nu)}$ is less than $r^{(\nu)}$. In our situation, the eigenvectors are explicitly described: the eigenvectors of eigenvalues 1, $r^{(\nu)}$, $s^{(\nu)}$ are constant multiples of

$$\begin{aligned}
&\mathbf{1} := \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad && \tilde{v}_1 := \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad && y_1 := \begin{pmatrix}0\\1\\-1\end{pmatrix} \quad && \text{for } A_1^{(\nu)},\\
&\mathbf{1}, && \tilde{v}_2 := \begin{pmatrix}1\\0\\1\end{pmatrix}, && y_2 := \begin{pmatrix}-1\\0\\1\end{pmatrix} && \text{for } A_2^{(\nu)},\\
&\mathbf{1}, && \tilde{v}_3 := \begin{pmatrix}1\\1\\0\end{pmatrix}, && y_3 := \begin{pmatrix}1\\-1\\0\end{pmatrix} && \text{for } A_3^{(\nu)},
\end{aligned}$$

respectively. Here, we identify $x \in l(V_0)$ with $\begin{pmatrix}x(p_1)\\x(p_2)\\x(p_3)\end{pmatrix}$. It is crucial for subsequent arguments that the eigenvectors of eigenvalue $r^{(\nu)}$ are independent of ν.

Let $\tilde{l}(V_0)$ be the set of all $x \in l(V_0)$ such that $\sum_{p \in V_0} x(p) = 0$. The orthogonal linear space of $\tilde{l}(V_0)$ in $l(V_0)$ is one-dimensional and spanned by $\mathbf{1}$. The function $\tilde{l}(V_0) \ni x \mapsto Q(x,x)^{1/2} \in \mathbb{R}$ defines a norm on $\tilde{l}(V_0)$. Let P denote the orthogonal projection from $l(V_0)$ onto $\tilde{l}(V_0)$. For each $i \in S_0$, $u_i \in l(V_0)$ denotes the column vector $(D_{p,p_i})_{p \in V_0}$.

Lemma 3.4 (see, e.g., [10, Lemma 5] and [12, Lemma A.1.4]). *For each $i \in S_0$ and $\nu \in T$, u_i is an eigenvector of ${}^tA_i^{(\nu)}$ with respect to the eigenvalue $r^{(\nu)}$. Moreover, $u_i \in \tilde{l}(V_0)$.*

We also note that $(u_i, \mathbf{1}) = (u_i, y_i) = 0$. We take $v_i \in l(V_0)$ such that v_i is a constant multiple of $\tilde{v}_i$ and $(u_i, v_i) = 1$.

Lemma 3.5. *Let $i \in S_0$, $x \in l(V_0)$, and $\boldsymbol{\nu} = \{\nu_k\}_{k \in \mathbb{N}} \in T^{\mathbb{N}}$. Then, it holds that*

$$\lim_{n\to\infty} r^{-1}_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} P A_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} x = (u_i, x) P v_i \tag{3.6}$$

and

$$\lim_{n\to\infty} r^{-2}_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} Q(A_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} x) = (u_i, x)^2 Q(v_i). \tag{3.7}$$

Moreover, these convergences are uniform in $i \in S_0$, $x \in \mathcal{C}$, and $\boldsymbol{\nu} \in T^{\mathbb{N}}$, where $\mathcal{C}$ is the inverse image of an arbitrary compact set of $l(V_0)$ by P.

Proof. Note that $P A_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} \mathbf{1} = 0$ and $r^{-1}_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} A_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} v_i = v_i$ for all n. Moreover, $|r^{-1}_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} A_{i^{\nu_1} i^{\nu_2} \dots i^{\nu_n}} y_i| \le \theta^n |y_i|$, where $\theta = \max_{\nu \in T} |s^{(\nu)}/r^{(\nu)}| \in [0,1)$.

For $x \in l(V_0)$ in general, we can decompose x into $x = x_1\mathbf{1} + x_2 v_i + x_3 y_i$. By taking the inner product with u_i on both sides, $(u_i, x) = x_2(u_i, v_i) = x_2$. Therefore, (3.6) holds, and (3.7) follows immediately from (3.6). The uniformity of the convergences is evident from the argument above. □

Although the next lemma can be confirmed by concrete calculation, we provide a proof that is applicable to more general situations.

Lemma 3.6. *The following hold.*

(1) *For every $i, j \in S_0$, $Q(v_i, v_i) = Q(v_j, v_j) > 0$. For $j \in S_0$ and $i, i' \in S_0 \setminus \{j\}$, $(Dv_j)(p_i) = (Dv_j)(p_{i'})$.*
(2) *For every $i, j \in S_0$, $(u_i, v_j) \neq 0$.*
(3) *There exists $\delta_0 > 0$ such that, for each $i \in S_0$, there exists some $i' \in S_0$ satisfying*

$$\big||(Dv_i)(p_i)| - |(Dv_i)(p_{i'})|\big| \geq \delta_0. \tag{3.8}$$

Proof. (1) This is proved in [10, Lemma 10] in more-general situations.
(2) Note that $(u_j, v_j) = 1$. From (1), $(u_i, v_j) = (Dv_j)(p_i)$ is independent of $i \in S_0 \setminus \{j\}$. Moreover, $0 = (Dv_j, \mathbf{1}) = \sum_{i \in S_0}(u_i, v_j)$. Therefore, $(u_i, v_j) = -1/(\#S_0 - 1) = -1/2$ for $i \in S_0 \setminus \{j\}$.
(3) From the proof of (2), we can take $\delta_0 = 1/2$. □

The following are simple estimates used in the proofs of Lemma 4.1 and Theorem 2.3.

Lemma 3.7. *Let $s, t > 0$ and $a > 0$. If $|\log(t/s)| \geq a$, then*

$$|t - s| \geq (1 - e^{-a}) \max\{s, t\}.$$

Proof. We may assume that $s \leq t$. Then, $t/s \geq e^a$, which implies $t - s \geq t - te^{-a} = t(1 - e^{-a})$. □

Lemma 3.8. *Let $d \in \mathbb{N}$ and*

$$\mathcal{P}_d = \left\{ a = (a_1, \ldots, a_d) \in \mathbb{R}^d \;\middle|\; a_k \geq 0 \text{ for all } k = 1, \ldots, d, \text{ and } \sum_{k=1}^d a_k = 1 \right\}.$$

For $a = (a_1, \ldots, a_d),\ b = (b_1, \ldots, b_d) \in \mathcal{P}_d$, it holds that

$$\sum_{k=1}^d \sqrt{a_k b_k} \leq 1 - \frac{|a - b|_{\mathbb{R}^d}^2}{8}.$$

Proof. Since all a_k and b_k are dominated by 1,

$$\begin{aligned}\sqrt{a_k b_k} &= \frac{a_k + b_k}{2} - \frac{(a_k - b_k)^2}{2(\sqrt{a_k} + \sqrt{b_k})^2}\\ &\leq \frac{a_k + b_k}{2} - \frac{(a_k - b_k)^2}{8}.\end{aligned}$$

Taking the sum with respect to k on both sides, we arrive at the conclusion. □

At the end of this section, we introduce a general sufficient condition for singularity of two measures. For $z \in \mathbb{R}$, let

$$z^{\oplus} = \begin{cases} 1/z & (z \neq 0) \\ 0 & (z = 0). \end{cases}$$

Theorem 3.9. *Let $(\Omega, \mathcal{B}, \{\mathcal{B}_n\}_{n \in \mathbb{Z}_+})$ be a measurable space equipped with a filtration such that $\mathcal{B} = \bigvee_{n \in \mathbb{Z}_+} \mathcal{B}_n$. Let P_1 and P_2 be two probability measures on $(\Omega, \mathcal{B})$. Suppose that, for each $n \in \mathbb{Z}_+$, $P_2|_{\mathcal{B}_n}$ is absolutely continuous with respect to $P_1|_{\mathcal{B}_n}$. Let z_n be the Radon–Nikodym derivative $d(P_2|_{\mathcal{B}_n})/d(P_1|_{\mathcal{B}_n})$ for $n \in \mathbb{Z}_+$ and $\alpha_n = z_n z_{n-2}^{\oplus}$ for $n \geq 2$. If*

$$\sum_{n=2}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}]\right) = \infty \quad P_1\text{-a.s.} \tag{3.9}$$

holds, then P_1 and P_2 are mutually singular. Here, $\mathbb{E}^{P_1}[\,\cdot\,|\mathcal{B}_{n-2}]$ denotes the conditional expectation for P_1 given $\mathcal{B}_{n-2}$.

Proof. We modify the proof of [9, Theorem 4.1]. By [14, Theorem VII.6.1], $z_\infty := \lim_{n \to \infty} z_n$ exists $(P_1 + P_2)$-a.e. and

$$P_2(A) = \int_A z_\infty \, dP_1 + P_2(A \cap \{z_\infty = \infty\}), \quad A \in \mathcal{B}. \tag{3.10}$$

Moreover, P_1 and $P_2(\,\cdot\, \cap \{z_\infty = \infty\})$ are mutually singular.

Let

$$Z_1 = \left\{ \sum_{k=1}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_{2k}} \mid \mathcal{B}_{2(k-1)}]\right) = \infty \right\},$$
$$Z_2 = \left\{ \sum_{k=1}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_{2k+1}} \mid \mathcal{B}_{2(k-1)+1}]\right) = \infty \right\}.$$

From (3.9), $P_1(Z_1 \cup Z_2) = 1$. Considering the two filtrations $\{\mathcal{B}_{2k}\}_{k \in \mathbb{Z}_+}$ and $\{\mathcal{B}_{2k+1}\}_{k \in \mathbb{Z}_+}$ and following the proof of [14, Theorem VII.6.4], we have $\{z_\infty = \infty\} = Z_1 = Z_2$ up to P_2-null sets. Therefore, $z_\infty = \infty$ P_2-a.e. on $Z_1 \cup Z_2$. Applying (3.10) to $A = \Omega \setminus (Z_1 \cup Z_2)$, which is a P_1-null set, we have $P_2(A) = P_2(A \cap \{z_\infty = \infty\})$, that is, $z_\infty = \infty$ P_2-a.e. on A. Thus, $P_2(z_\infty = \infty) = 1$ and we conclude that P_1 and P_2 are mutually singular. □

4 Proof of the main results

We introduce some notation. Let $\mathcal{K}$ be a closed set of $l(V_0)$ that is defined as

$$\mathcal{K} = \{x \in l(V_0) \mid 2Q(x, x) = 1\}.$$

For $l_0 \in \mathbb{Z}_+$ and $l_1, l_2 \in \mathbb{N}$, let

$$L(l_0, l_1, l_2) = \left\{ \boldsymbol{\nu} = \{\nu_k\}_{k=1}^\infty \in T^{\mathbb{N}} \;\middle|\; \begin{array}{l} \{\nu_k \mid l_0 + 1 \le k \le l_0 + l_1\} \\ \subset \{\nu_k \mid l_0 + l_1 + 1 \le k \le l_0 + l_1 + l_2\} \end{array} \right\}.$$

We define several constants as follows:

$$\begin{aligned}
\beta_1 &:= \min\{|(u_i, v_j)| \mid i, j \in S_0\} = \min\{|(Dv_i)(p)| \mid i \in S_0,\ p \in V_0\}, \\
\beta_2 &:= \min\{|\log(r_v/q_v)| \mid v \in S,\ r_v \ne q_v\} > 0, \\
\beta_3 &:= \min\{q_v \mid v \in S\} > 0, \\
\beta_4 &:= \min\{r^{(\nu)} \mid \nu \in T\} > 0, \\
\beta_5 &:= 2Q(v_i, v_i) > 0 \quad (i \in S_0).
\end{aligned}$$

By Lemma 3.6(2), $\beta_1 > 0$. In the definition of β_2, $\min \emptyset = 1$ by convention. By Lemma 3.6(1), β_5 is independent of the choice of i.

We fix $q \in \mathcal{A}$. The following is a key lemma for proving Theorem 2.3.

Lemma 4.1. (1) *There exist $N \in \mathbb{N}$ and $N' \in \mathbb{N}$ such that, for any $l \in \mathbb{N}$, there exists $\gamma > 0$ satisfying the following. For all $\boldsymbol{\nu} = \{\nu_k\}_{k=1}^\infty \in L(N, N', l)$ and $x \in \mathcal{K}$, there exist*

$$\begin{aligned}
i &= i(x) \in S_0, \\
j &= j(x, \nu_1, \nu_2, \ldots, \nu_N) \in S_0, \\
m &= m(l, x, \nu_1, \nu_2, \ldots, \nu_{N+N'+l}) \in \{N', N'+1, \ldots, N'+l\}
\end{aligned}$$

such that

$$|2r_\xi^{-1} Q(A_\xi x) - q_\xi| \ge \gamma$$

with $\xi = i^{\nu_1} \cdots i^{\nu_N} j^{\nu_{N+1}} \cdots j^{\nu_{N+m}}$. Here, "$j = j(x, \nu_1, \nu_2, \ldots, \nu_N)$" means that "$j$ depends only on $x, \nu_1, \nu_2, \ldots, \nu_N$," and so on.

(2) *If Condition* (A) *holds, then the claim of item* (1) *holds with "$\boldsymbol{\nu} = \{\nu_k\}_{k=1}^\infty \in L(N, N', l)$" replaced by "$\boldsymbol{\nu} = \{\nu_k\}_{k=1}^\infty \in T^{\mathbb{N}}$."*

Proof. (1) Let φ be a continuous function on $l(V_0)$ that is defined as

$$\varphi(x) = \sum_{i \in S_0} (u_i, x)^2.$$

Since the range of φ on $\mathcal{K}$ is equal to that on a compact set $P(\mathcal{K})$, φ attains a minimum on $\mathcal{K}$, say β_6. Let $x \in \mathcal{K}$. Because

$$0 < Q(x, x) = (-Dx, x) = -\sum_{i \in S_0} (u_i, x) x(p_i),$$

$(u_i, x) \ne 0$ for some $i \in S_0$. This implies that $\varphi(x) > 0$. (In fact, we can confirm that $\varphi(x) \equiv 3/2$.) Thus, $\beta_6 > 0$. Define $\delta' = \beta_6/\#S_0 = \beta_6/3$ and $\mathcal{K}_i = \{x \in \mathcal{K} \mid (u_i, x)^2 \ge \delta'\}$ for $i \in S_0$. It holds that $\mathcal{K} = \bigcup_{i \in S_0} \mathcal{K}_i$.

We fix $x \in \mathcal{K}$. There exists $i \in S_0$ such that $x \in \mathcal{K}_i$. From Lemma 3.6(3), there exists $i' \in S_0$ such that (3.8) holds. By keeping in mind that $(Dv_i)(p_i) = 1$, it follows that

$$\begin{aligned}\bigl|(Dv_i)(p_i)^2 - (Dv_i)(p_{i'})^2\bigr| &= \bigl|1 + |(Dv_i)(p_{i'})|\bigr|\,\bigl||(Dv_i)(p_i)| - |(Dv_i)(p_{i'})|\bigr| \\ &\geq \delta_0. \end{aligned} \tag{4.1}$$

Let $\boldsymbol{\nu} = \{\nu_k\}_{k\in\mathbb{N}} \in T^{\mathbb{N}}$ and define $x_n = r_{i^{\nu_1}\dots i^{\nu_n}}^{-1} A_{i^{\nu_1}\dots i^{\nu_n}} x$ for $n \in \mathbb{N}$. From Lemma 3.4,

$$(u_i, x_n) = (r_{i^{\nu_1}\dots i^{\nu_n}}^{-1}\, {}^t\!A_{i^{\nu_1}\dots i^{\nu_n}} u_i, x) = (u_i, x). \tag{4.2}$$

Let $\delta_1 = \sqrt{\delta'}\beta_1/2$ and $\delta_2 = \delta'\delta_0/3$. By Lemma 3.5, there exists $N \in \mathbb{N}$ independent of the choice of x, i, and $\boldsymbol{\nu}$ such that, for all $p \in V_0$,

$$\bigl||(Dx_N)(p)| - |(u_i, x)(Dv_i)(p)|\bigr| \leq \delta_1 \tag{4.3}$$

and

$$\bigl|(Dx_N)(p)^2 - (u_i, x)^2(Dv_i)(p)^2\bigr| \leq \delta_2. \tag{4.4}$$

From (4.2) and (4.3), for any $j \in S_0$,

$$\begin{aligned}|(u_j, x_N)| &= |(Dx_N)(p_j)| \\ &\geq |(u_i, x)(Dv_i)(p_j)| - \delta_1 \\ &\geq \sqrt{\delta'}\beta_1 - \delta_1 = \delta_1.\end{aligned}$$

By Lemma 3.5,

$$\begin{aligned}\lim_{m\to\infty} r_{j^{\nu_{N+1}}\dots j^{\nu_{N+m}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+m}}} x_N) &= (u_j, x_N)^2 Q(v_j) \\ &\geq \delta_1^2\beta_5/2 > 0.\end{aligned}$$

This convergence is uniform in x, i, j, and $\boldsymbol{\nu}$ because Px_N belongs to some compact set of $\mathcal{K}$ that is independent of them. We take $\delta_3 = \beta_5\delta_2/2$. Then, there exists $N' \in \mathbb{N}$ independent of x, i, j, and $\boldsymbol{\nu}$ such that, for every $n \geq N'$,

$$\bigl|r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}} x_N) - (u_j, x_N)^2 Q(v_j)\bigr| \leq \delta_3/4 \tag{4.5}$$

and

$$\left|\log \frac{r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n-1}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n-1}}} x_N)}{r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}} x_N)}\right| \leq \frac{\beta_2}{2}. \tag{4.6}$$

From (4.1) and (4.4),

$$\begin{aligned}\delta'\delta_0 &\leq (u_i, x)^2\bigl|(Dv_i)(p_i)^2 - (Dv_i)(p_{i'})^2\bigr| \\ &\leq \bigl|(u_i, x)^2(Dv_i)(p_i)^2 - (Dx_N)(p_i)^2\bigr| + \bigl|(Dx_N)(p_i)^2 - (Dx_N)(p_{i'})^2\bigr| \\ &\quad + \bigl|(Dx_N)(p_{i'})^2 - (u_i, x)^2(Dv_i)(p_{i'})^2\bigr| \\ &\leq 2\delta_2 + \bigl|(Dx_N)(p_i)^2 - (Dx_N)(p_{i'})^2\bigr|,\end{aligned}$$

which implies that

$$\big|(Dx_N)(p_i)^2-(Dx_N)(p_{i'})^2\big|\ge \delta'\delta_0-2\delta_2=\delta_2.$$

From the identity $(Dx_N)(p_j)=(u_j,x_N)$ $(j\in S_0)$, we have

$$\begin{aligned}2\delta_3&=\beta_5\delta_2\\&\le \beta_5\big|(u_i,x_N)^2-(u_{i'},x_N)^2\big|\\&\le \big|2Q(v_i)(u_i,x_N)^2-q_w/r_w\big|+\big|2Q(v_{i'})(u_{i'},x_N)^2-q_w/r_w\big|,\end{aligned}$$

where we choose $w=i^{\nu_1}\cdots i^{\nu_N}\in W_N$. Then, for either $j=i$ or i',

$$\big|2Q(v_j)(u_j,x_N)^2-q_w/r_w\big|\ge\delta_3. \tag{4.7}$$

We fix such j. Take any $l\in\mathbb{N}$ and suppose $\boldsymbol{\nu}\in L(N,N',l)$. There are two possibilities:

I) There exists some $k\in\{N'+1,\dots,N'+l\}$ such that $r_{j^{\nu_{N+k}}}\ne q_{j^{\nu_{N+k}}}$.
II) $r_{j^{\nu_{N+k}}}=q_{j^{\nu_{N+k}}}$ for all $k\in\{N'+1,\dots,N'+l\}$.

Suppose Case I). Let $w'=j^{\nu_{N+1}}\cdots j^{\nu_{N+k-1}}\in W_{k-1}$. From (4.6) with $n=k$,

$$\begin{aligned}\frac{\beta_2}{2}&\ge\left|\log\left(r^2_{j^{\nu_{N+k}}}\times\frac{Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+k-1}}}x_N)}{Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+k}}}x_N)}\right)\right|\\&=\left|\log\left(\frac{r_{j^{\nu_{N+k}}}}{q_{j^{\nu_{N+k}}}}\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right)\right|\\&\ge\beta_2-\left|\log\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\right|-\left|\log\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right|.\end{aligned}$$

Therefore, either

$$\left|\log\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\right|\ge\frac{\beta_2}{4}\quad\text{or}\quad\left|\log\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right|\ge\frac{\beta_2}{4}$$

holds. Since $q_{ww'}\ge q_{ww'j^{\nu_{N+k}}}\ge\beta_3^{N+N'+l}$, Lemma 3.7 implies that either

$$|2r^{-1}_{ww'}Q(A_{ww'}x)-q_{ww'}|\ge(1-e^{-\beta_2/4})\beta_3^{N+N'+l} \tag{4.8}$$

or

$$|2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)-q_{ww'j^{\nu_{N+k}}}|\ge(1-e^{-\beta_2/4})\beta_3^{N+N'+l} \tag{4.9}$$

holds.

Next, suppose Case II). Since $\boldsymbol{\nu}\in L(N,N',l)$, $r_{j^{\nu_{N+k}}}=q_{j^{\nu_{N+k}}}$ for all $k\in\{1,\dots,N'\}$. Let $\hat w=j^{\nu_{N+1}}\cdots j^{\nu_{N+N'}}\in W_{N'}$. Note that $q_{\hat w}=r_{\hat w}$. From (4.7) and (4.5),

$$\begin{aligned}\delta_3&\le|2Q(v_j)(u_j,x_N)^2-q_w/r_w|\\&\le|2Q(v_j)(u_j,x_N)^2-2r_{\hat w}^{-2}Q(A_{\hat w}x_N)|+|2r_{\hat w}^{-2}Q(A_{\hat w}x_N)-q_{w\hat w}/r_{w\hat w}|\\&\le\delta_3/2+\beta_4^{-(N+N')}|2r^{-1}_{w\hat w}Q(A_{w\hat w}x)-q_{w\hat w}|.\end{aligned}$$

Therefore,

$$|2r_{w\hat{w}}^{-1}Q(A_{w\hat{w}}x)-q_{w\hat{w}}|\ge\delta_3\beta_4^{N+N'}/2.$$

In conclusion, it suffices to take

$$m=\begin{cases}k-1 & \text{if (4.8) holds in Case I)},\\ k & \text{if (4.8) fails to hold in Case I)},\\ N' & \text{in Case II)}\end{cases}$$

and

$$\gamma=\min\bigl\{(1-e^{-\beta_2/4})\beta_3^{N+N'+l},\ \delta_3\beta_4^{N+N'}/2\bigr\}.$$

(2) In the proof of (1), the condition that $\boldsymbol{\nu}\in L(N,N',l)$ is used only in the discussion of Case II). Under Condition (A), Case II) never happens. Therefore, the arguments are valid for all $\boldsymbol{\nu}\in T^{\mathbb{N}}$. □

Proof (of Theorem 2.3). Let N and N' be natural numbers that are provided in Lemma 4.1. Under Condition (B), take $l_2\in\mathbb{N}$ associated with $l_0=N$ and $l_1=N'$ in (B). Under Condition (A), take $l_2=1$.

Let $M=N+N'+l_2$. For $n\in\mathbb{Z}_+$, let $\mathcal{B}_n$ denote the σ-field on Σ that is generated by $\{\Sigma_w\mid w\in W_{Mn}\}$. Then, $\bigvee_{n=0}^{\infty}\mathcal{B}_n$ is equal to the Borel σ-field on Σ.

Take $x\in\mathcal{K}$. We first prove that $\lambda_{\langle x\rangle}$ and λ_q are mutually singular. For each $n\in\mathbb{Z}_+$, $\lambda_{\langle x\rangle}|_{\mathcal{B}_n}$ is absolutely continuous with respect to $\lambda_q|_{\mathcal{B}_n}$. Indeed, if $\lambda_q(\Sigma_w)=0$ for $w\in W_{Mn}$, then $w\notin\tilde{W}_{Mn}$, which implies $\lambda_{\langle x\rangle}(\Sigma_w)=0$. Let z_n denote the Radon–Nikodym derivative $d(\lambda_{\langle x\rangle}|_{\mathcal{B}_n})/d(\lambda_q|_{\mathcal{B}_n})$.

Under Condition (B), take $\omega=\omega_1\omega_2\cdots\in\tilde{\Sigma}$ such that Condition ($\star$) is satisfied, and let $k\in\mathbb{Z}_+$ in ($\star$). Under Condition (A), take $\omega\in\tilde{\Sigma}$ and $k\in\mathbb{Z}_+$ arbitrarily.

There exists a unique natural number $n\ge 2$ such that $M(n-2)\le k<M(n-1)$. Let $w:=[\omega]_{M(n-2)}\in\tilde{W}_{M(n-2)}$ and $\xi\in W_{2M}$. Using (3.5), we have

$$z_{n-2}=\frac{\lambda_{\langle x\rangle}(\Sigma_w)}{\lambda_q(\Sigma_w)}=\frac{2r_w^{-1}Q(A_wx)}{q_w}\quad\text{on }\Sigma_w$$

and

$$z_n=\begin{cases}\dfrac{2r_{w\xi}^{-1}Q(A_{w\xi}x)}{q_{w\xi}} & \text{if } w\xi\in\tilde{W}_{Mn}\\ 0 & \text{if } w\xi\notin\tilde{W}_{Mn}\end{cases}\quad\text{on }\Sigma_{w\xi}.$$

Then, on $\Sigma_{w\xi}$,

$$\alpha_n:=z_nz_{n-2}^{\oplus}=\begin{cases}\dfrac{Q(A_{w\xi}x)Q(A_wx)^{\oplus}}{q_\xi r_\xi} & \text{if } w\xi\in\tilde{W}_{Mn},\\ 0 & \text{if } w\xi\notin\tilde{W}_{Mn}.\end{cases}$$

If $Q(A_wx)=0$, then $\alpha_n=0$ on Σ_w, which implies that

$$1-\mathbb{E}^{\lambda_q}[\sqrt{\alpha_n}\mid\mathcal{B}_{n-2}](\omega)=1.\tag{4.10}$$

Suppose that $Q(A_w x) \neq 0$. Let $x' = A_w x / \sqrt{2Q(A_w x)} \in \mathcal{K}$. Then,

$$\begin{aligned}\mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}](\omega) &= \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \frac{q_{w\xi}}{q_w} \sqrt{\frac{Q(A_{w\xi}x)}{q_\xi r_\xi Q(A_w x)}} \\ &= \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \sqrt{q_\xi \times 2r_\xi^{-1} Q(A_\xi x')} \\ &\leq 1 - \frac{1}{8} \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \bigl(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\bigr)^2. \qquad (4.11)\end{aligned}$$

Here, the last inequality follows from Lemma 3.8.

Take $\gamma > 0$ in Lemma 4.1 associated with $l = l_2$. Let

$$w' = \omega_{M(n-2)+1}\omega_{M(n-2)+2} \cdots \omega_k \in W_{k-M(n-2)} \quad (w' = \emptyset \text{ if } k = M(n-2))$$

and $\gamma' = \min\{\gamma, \beta_3^M\}$. Note that $q_{w'} \geq \beta_3^M \geq \gamma'$. We consider the following two cases:

i) $|q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')| \geq \gamma\gamma'/3$;
ii) $|q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')| < \gamma\gamma'/3$.

Suppose Case i). Letting $I = \{\zeta \in W_{Mn-k} \mid ww'\zeta \in \tilde{W}_{Mn}\}$, we have

$$\begin{aligned}&\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \bigl(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\bigr)^2 \\ &\quad \geq \sum_{\zeta \in I} \bigl(q_{w'\zeta} - 2r_{w'\zeta}^{-1} Q(A_{w'\zeta}x')\bigr)^2 \\ &\quad \geq \left\{\sum_{\zeta \in I} \bigl(q_{w'\zeta} - 2r_{w'\zeta}^{-1} Q(A_{w'\zeta}x')\bigr)\right\}^2 \left(\sum_{\zeta \in I} 1\right)^{-1} \\ &\quad = \bigl(q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')\bigr)^2 (\#I)^{-1} \qquad \text{(from (3.4))} \\ &\quad \geq (\gamma\gamma'/3)^2 (\#S)^{-2M}.\end{aligned}$$

Next, suppose Case ii). We have

$$\begin{aligned}2r_{w'}^{-1} Q(A_{w'}x') &> q_{w'} - \gamma\gamma'/3 \\ &\geq \gamma' - \gamma'/3 = 2\gamma'/3. \qquad (4.12)\end{aligned}$$

In particular, $Q(A_{w'}x') \neq 0$. Let $x'' = A_{w'}x' / \sqrt{2Q(A_{w'}x')} \in \mathcal{K}$. We make several choices in order as follows:

- Take $i \in S_0$ associated with $x'' \in \mathcal{K}$ in Lemma 4.1.
- Take $\nu_{k+1}, \nu_{k+2}, \ldots, \nu_{k+N} \in T$ such that $ww' i^{\nu_{k+1}} i^{\nu_{k+2}} \cdots i^{\nu_{k+N}} \in \tilde{W}_{k+N}$; these are uniquely determined.
- Take $j \in S_0$ associated with $x'' \in \mathcal{K}$, $i \in S_0$, and $\{\nu_{k+s}\}_{s=1}^N$ in Lemma 4.1.

- Take a unique sequence $\{\nu_s\}_{s=k+N+1}^{\infty} \subset T$ such that

$$ww' i^{\nu_{k+1}} i^{\nu_{k+2}} \cdots i^{\nu_{k+N}} j^{\nu_{k+N+1}} j^{\nu_{k+N+2}} \cdots j^{\nu_{k+N+t}} \in W_{k+N+t}$$

for every $t \in \mathbb{N}$.
- Take $m \in \{N', N'+1, \dots, N'+l_2\}$ associated with $x'' \in \mathcal{K}$, $i \in S_0$, $j \in S_0$, and $\{\nu_{k+s}\}_{s=1}^{\infty}$ in Lemma 4.1.

Note that $\{\nu_{k+s}\}_{s=1}^{\infty} \in L(N, N', l_2)$ under Condition (B).

Let

$$\eta = i^{\nu_{k+1}} i^{\nu_{k+2}} \cdots i^{\nu_{k+N}} j^{\nu_{k+N+1}} j^{\nu_{k+N+2}} \cdots j^{\nu_{k+N+m}} \in W_{N+m}.$$

Then, letting $J = \{\eta' \in W_{Mn-k-N-m} \mid ww'\eta\eta' \in \tilde{W}_{Mn}\}$, we have

$$\begin{aligned}
&\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \left(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\right)^2 \\
&\quad \ge \sum_{\eta' \in J} \left(q_{w'\eta\eta'} - 2r_{w'\eta\eta'}^{-1} Q(A_{w'\eta\eta'} x')\right)^2 \\
&\quad \ge \left\{ \sum_{\eta' \in J} \left(q_{w'\eta\eta'} - 2r_{w'\eta\eta'}^{-1} Q(A_{w'\eta\eta'} x')\right) \right\}^2 \left(\sum_{\eta' \in J} 1 \right)^{-1} \\
&\quad = \left(q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta} x')\right)^2 (\#J)^{-1} \qquad \text{(from (3.4))} \\
&\quad \ge \left(q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta} x')\right)^2 (\#S)^{-2M}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left| q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta} x') \right| \\
&= \left| q_{w'\eta} - 2r_\eta^{-1} Q(A_\eta x'') \cdot 2r_{w'}^{-1} Q(A_{w'} x') \right| \\
&\ge 2r_{w'}^{-1} Q(A_{w'} x') \left| q_\eta - 2r_\eta^{-1} Q(A_\eta x'') \right| - \left| q_{w'} - 2r_{w'}^{-1} Q(A_{w'} x') \right| q_\eta \\
&\ge \frac{2\gamma'}{3} \cdot \gamma - \frac{\gamma\gamma'}{3} \cdot 1 = \gamma\gamma'/3.
\end{aligned}$$

Here, in the last inequality, we used (4.12) and Lemma 4.1.

Therefore, in both Case i) and Case ii),

$$\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \left(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\right)^2 \ge (\gamma\gamma'/3)^2 (\#S)^{-2M}. \tag{4.13}$$

By combining (4.11) with (4.13),

$$1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}](\omega) \ge (\gamma\gamma')^2 (\#S)^{-2M}/72. \tag{4.14}$$

For λ_q-a.s. ω, there are infinitely many n that satisfy (4.10) or (4.14); therefore,

$$\sum_{n=2}^{\infty} \left(1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}]\right) = \infty \quad \lambda_q\text{-a.s.}$$

From Theorem 3.9, we conclude that $\lambda_{\langle x\rangle} \perp \lambda_q$.

Take a σ-compact set B of Σ such that $\lambda_{\langle x\rangle}(B)=1$ and $\lambda_q(B)=0$. Recall that

$$V_* \setminus V_0 = \bigl\{x \in G(L) \bigm| \#(\pi|_{\tilde\Sigma})^{-1}(\{x\}) > 1\bigr\}$$

and $\mu_q(V_*\setminus V_0)=0$. Let $B' = (\pi|_{\tilde\Sigma})^{-1}(V_*\setminus V_0)\cup B$. Because $(\pi|_{\tilde\Sigma})^{-1}(\pi(B'))=B'$, from Lemma 3.3

$$\mu_q(\pi(B')) = \lambda_q\bigl((\pi|_{\tilde\Sigma})^{-1}(\pi(B'))\bigr) = \lambda_q(B') = 0$$

and

$$\mu_{\langle \iota(x)\rangle}(\pi(B')) = \lambda_{\langle x\rangle}\bigl((\pi|_{\tilde\Sigma})^{-1}(\pi(B'))\bigr) = \lambda_{\langle x\rangle}(B') \ge \lambda_{\langle x\rangle}(B) = 1.$$

Therefore, $\mu_{\langle \iota(x)\rangle} \perp \mu_q$. We have now proved that $\mu_{\langle h\rangle} \perp \mu_q$ for all harmonic functions h.

Next, let f be an arbitrary m-piecewise harmonic function. For $v \in \tilde W_m$, we apply the above result to the Dirichlet form $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$ on $L^2(G(L^{[v]}), \mu_q^{[v]})$ and $f^{[v]} := f \circ \psi_v|_{G(L^{[v]})}$ to conclude that $\mu^{[v]}_{\langle f^{[v]}\rangle} \perp \mu_q^{[v]}$. Take a σ-compact subset B_v of $G(L^{[v]})$ such that $\mu^{[v]}_{\langle f^{[v]}\rangle}(G(L^{[v]})\setminus B_v)=0$ and $\mu_q^{[v]}(B_v)=0$. Let

$$B = \bigcup_{v\in\tilde W_m} \psi_v(B_v) \quad\text{and}\quad \hat B = B \setminus (V_*\setminus V_0).$$

From Lemma 3.2 and the property $\mu_q(V_*\setminus V_0)=0$, we have

$$\begin{aligned}\mu_{\langle f\rangle}(B) &\ge \sum_{v\in\tilde W_m} \frac{1}{r_v}\mu^{[v]}_{\langle f^{[v]}\rangle}(B_v) = \sum_{v\in\tilde W_m}\frac{2}{r_v}\mathcal{E}^{[v]}(f^{[v]}, f^{[v]})\\ &= 2\mathcal{E}(f,f) = \mu_{\langle f\rangle}(G(L))\end{aligned}$$

and

$$\mu_q(B) = \mu_q(\hat B) \le \sum_{v\in\tilde W_m} q_v \mu_q^{[v]}(B_v) = 0.$$

Therefore, $\mu_{\langle f\rangle} \perp \mu_q$.

For $f \in \mathcal{F}$ in general, we can take a sequence $\{f_n\}_{n=1}^\infty$ of piecewise harmonic functions that converges to f in $\mathcal{F}$ from Lemma 3.1. For each $n\in\mathbb{N}$, take a Borel set B_n of $G(L)$ such that $\mu_q(B_n)=0$ and $\mu_{\langle f_n\rangle}(G(L)\setminus B_n)=0$. Let $B=\bigcup_{n=1}^\infty B_n$. From a general inequality

$$\left|\sqrt{\mu_{\langle g\rangle}(C)} - \sqrt{\mu_{\langle g'\rangle}(C)}\right| \le \sqrt{\mu_{\langle g-g'\rangle}(C)}$$

for $g, g' \in \mathcal{F}$ and a Borel set C of $G(L)$ (see, e.g., [5, p. 111]), we obtain

$$\mu_{\langle f\rangle}(G(L)\setminus B) = \lim_{n\to\infty}\mu_{\langle f_n\rangle}(G(L)\setminus B) = 0,$$

while $\mu_q(B)=0$. Therefore, $\mu_{\langle f\rangle}\perp\mu_q$. □

Lastly, we prove Theorem 2.7.

Proof (of Theorem 2.7). Since the assertion obviously holds when $\#T = 1$, we may assume that $\#T \geq 2$.

Let $q = \{q_v\}_{v\in S} \in \mathcal{A}$. Take $l_0, l_1 \in \mathbb{N}$ arbitrarily and let $l_2 = \#T$. For $\hat{\omega} \in \hat{\Omega}$, $\tilde{W}_n(\hat{\omega})$ ($n \in \mathbb{Z}_+$) and $\mu_q^{(\hat{\omega})}$ denote the set $\tilde{W}_n$ and the measure μ_q associated with $L(\hat{\omega})$, respectively. We define a probability measure $\mathbb{P}$ on $(\Sigma \times \hat{\Omega}, \mathcal{B}(\Sigma) \otimes \hat{\mathcal{B}})$ by

$$\mathbb{P}(A) = \int_{\hat{\Omega}} \mu_q^{(\hat{\omega})}(A_{\hat{\omega}})\, \hat{P}(d\hat{\omega}), \quad A \in \mathcal{B}(\Sigma) \otimes \hat{\mathcal{B}},$$

where $\mathcal{B}(\Sigma)$ denotes the Borel σ-field on Σ and $A_{\hat{\omega}} = \{\omega \in \Sigma \mid (\omega, \hat{\omega}) \in A\}$. More specifically, if A is expressed as $A = \Sigma_w \times B$ for $w = w_1 w_2 \cdots w_m \in W_m$ and $B = \{\hat{\omega} \in \hat{\Omega} \mid L_v(\hat{\omega}) = \tau_v \text{ for all } v \in W_{\leq n}\}$ for given $m, n \in \mathbb{Z}_+$ with $m - 1 \leq n$ and $\{\tau_v\}_{v \in W_{\leq n}} \in T^{W_{\leq n}}$, then

$$\begin{aligned}\mathbb{P}(A) &= \int_B \mu_q^{(\hat{\omega})}(\Sigma_w)\, \hat{P}(d\hat{\omega}) = \int_B q_w \mathbf{1}_{\tilde{W}_m(\hat{\omega})}(w)\, \hat{P}(d\hat{\omega}) \\ &= \begin{cases} q_w \displaystyle\prod_{v \in W_{\leq n}} \rho(\{\tau_v\}) & \text{if } w_k \in S^{(\tau_{[w]_{k-1}})} \text{ for all } k = 1, 2, \dots, m, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

For $k \in \mathbb{Z}_+$, let $\tilde{U}(k)$ denote the set of all elements $(w, \hat{\omega}) \in W_k \times \hat{\Omega}$ such that the following hold:

(i) $w \in \tilde{W}_k(\hat{\omega})$;

(ii) for any $i, j \in S_0$, if we take $\nu_{k+1}, \nu_{k+2}, \dots, \nu_{k+l_0+l_1+l_2} \in T$ such that

$$w i^{\nu_{k+1}} i^{\nu_{k+2}} \cdots i^{\nu_{k+l_0}} j^{\nu_{k+l_0+1}} j^{\nu_{k+l_0+2}} \cdots j^{\nu_{k+l_0+l_1+l_2}} \in \tilde{W}_{k+l_0+l_1+l_2}(\hat{\omega}),$$

then $\{\nu_{k+l_0+l_1+1}, \nu_{k+l_0+l_1+2}, \dots, \nu_{k+l_0+l_1+l_2}\} = T$.

Define

$$U(k) = \left\{(\omega, \hat{\omega}) \in \Sigma \times \hat{\Omega} \,\middle|\, ([\omega]_k, \hat{\omega}) \in \tilde{U}(k)\right\}$$

and

$$U_{\hat{\omega}}(k) = \{\omega \in \Sigma \mid (\omega, \hat{\omega}) \in U(k)\}, \quad \hat{\omega} \in \hat{\Omega}.$$

Then,

$$\begin{aligned}
\mathbb{P}(U(k)) &= \int_{\hat\Omega} \sum_{w\in W_k} q_w \mathbf{1}_{\tilde U(k)}(w,\hat\omega)\,\hat P(d\hat\omega)\\
&= \sum_{w\in W_k} q_w \hat P\bigl(\{\hat\omega\in\hat\Omega \mid w\in \tilde W_k(\hat\omega)\}\bigr)\left(l_2!\prod_{\nu\in T}\rho(\{\nu\})\right)^{\#(S_0\times S_0)}\\
&= p\sum_{\nu_1,\dots,\nu_k\in T}\ \sum_{\substack{w_j\in S^{(\nu_j)};\\ j=1,\dots,k}}\ \prod_{m=1}^k q_{w_m}\prod_{m=1}^k \rho(\{\nu_m\})\\
&\qquad \Bigl(p:=\bigl(l_2!\textstyle\prod_{\nu\in T}\rho(\{\nu\})\bigr)^9\in(0,1)\Bigr)\\
&= p\left(\sum_{\nu\in T}\sum_{v\in S^{(\nu)}} q_v\rho(\{\nu\})\right)^k\\
&= p.
\end{aligned}$$

In a similar way, we can confirm that $\{U((l_0+l_1+l_2)n)\}_{n\in\mathbb{Z}_+}$ are independent with respect to $\mathbb{P}$.

For $0\le M<N$, we define

$$\begin{aligned}
F_{M,N} &= \bigcap_{n=M+1}^{N}\bigl((\Sigma\times\hat\Omega)\setminus U((l_0+l_1+l_2)n)\bigr),\\
F_{M,N,\hat\omega} &= \{\omega\in\Sigma \mid (\omega,\hat\omega)\in F_{M,N}\},\quad F_{M,\hat\omega}=\bigcap_{N=M+1}^{\infty}F_{M,N,\hat\omega}\quad (\hat\omega\in\hat\Omega),\\
G_{M,N} &= \bigl\{\hat\omega\in\hat\Omega \bigm| \mu_q^{(\hat\omega)}(F_{M,N,\hat\omega})\ge(1-p)^{N/2}\bigr\},\quad G_M=\limsup_{N\to\infty}G_{M,N}.
\end{aligned}$$

Then,

$$\begin{aligned}
\hat P(G_{M,N}) &\le (1-p)^{-N/2}\int_{\hat\Omega}\mu_q^{(\hat\omega)}(F_{M,N,\hat\omega})\,\hat P(d\hat\omega)\\
&= (1-p)^{-N/2}\mathbb{P}(F_{M,N})\\
&= (1-p)^{-N/2}(1-p)^{N-M}\\
&= (1-p)^{(N/2)-M}.
\end{aligned}$$

From the Borel–Cantelli lemma, $\hat P(G_M)=0$. Let

$$\mathcal{U}_q=\bigl\{q'=\{q'_v\}_{v\in S}\in\mathcal{A} \bigm| q'_v/q_v<(1-p)^{-1/(4(l_0+l_1+l_2))}\text{ for all } v\in S\bigr\},$$

which is an open neighborhood of q in $\mathcal{A}$. By letting $\mathcal{F}_n=\sigma(\{\Sigma_w \mid w\in W_{(l_0+l_1+l_2)n}\})$ for $n\in\mathbb{Z}_+$, we have

$$\frac{d\bigl(\mu_{q'}^{(\hat\omega)}|_{\mathcal{F}_n}\bigr)}{d\bigl(\mu_q^{(\hat\omega)}|_{\mathcal{F}_n}\bigr)}\le(1-p)^{-n/4}\quad \mu_q^{(\hat\omega)}\text{-a.e.}$$

for all $q' \in \mathcal{U}_q$ and $\hat{\omega} \in \hat{\Omega}$. Suppose that $q' \in \mathcal{U}_q$ and $\hat{\omega} \notin G_M$. For sufficiently large $N \in \mathbb{N}$, $\hat{\omega} \notin G_{M,N}$. Because $F_{M,N,\hat{\omega}}$ belongs to $\mathcal{F}_N$, we have

$$\mu_{q'}^{(\hat{\omega})}(F_{M,N,\hat{\omega}}) \le (1-p)^{-N/4}\mu_q^{(\hat{\omega})}(F_{M,N,\hat{\omega}}) \le (1-p)^{N/4}$$

for large N, which implies $\mu_{q'}^{(\hat{\omega})}(F_{M,\hat{\omega}}) = 0$. Let $G(q)$ denote $\bigcup_{M\in\mathbb{Z}_+} G_M$. (Here, we specify the dependency of q.) This is a $\hat{P}$-null set. If $\hat{\omega} \notin G(q)$, then $\mu_{q'}^{(\hat{\omega})}\left(\bigcup_{M\in\mathbb{Z}_+} F_{M,\hat{\omega}}\right) = 0$ for $q' \in \mathcal{U}_q$, which means that

$$\mu_{q'}^{(\hat{\omega})}\Bigl(\limsup_{n\to\infty} U_{\hat{\omega}}((l_0+l_1+l_2)n)\Bigr) = 1, \quad q' \in \mathcal{U}_q.$$

Because $\mathcal{A}$ is σ-compact, we can take a countable subset $\{q_\alpha \mid \alpha \in \mathbb{N}\}$ of $\mathcal{A}$ such that $\bigcup_{\alpha\in\mathbb{N}} \mathcal{U}_{q_\alpha} = \mathcal{A}$. Let $\mathcal{N} = \bigcup_{\alpha\in\mathbb{N}} G(q_\alpha)$. Then, $\hat{P}(\mathcal{N}) = 0$ and for $\hat{\omega} \in \hat{\Omega} \setminus \mathcal{N}$,

$$\mu_q^{(\hat{\omega})}\Bigl(\limsup_{k\to\infty} U_{\hat{\omega}}(k)\Bigr) = 1, \quad q \in \mathcal{A}.$$

This implies that, for $\hat{\omega} \in \hat{\Omega} \setminus \mathcal{N}$, $(\star)$ holds with (2.6) replaced by (2.7) for $l_2 = \#T$. $\square$

5 Concluding remarks

We make some remarks about the main results.

(1) The arguments in this paper are valid for some other inhomogeneous fractals. For example, we can obtain similar results for higher-dimensional inhomogeneous Sierpinski gaskets. A crucial property required here is that the eigenfunctions of $A_i^{(\nu)}$ ($i \in S_0$) associated with the eigenvalues $r^{(\nu)}$ do not depend on ν.

(2) Since Condition (B) in Theorem 2.3 is a rather technical constraint, we focus on arguments that are valid more generally and we do not try to make the assumption as weak as possible by relying on concrete structures of fractals under consideration. Indeed, in Lemma 3.6(3), the part "there exists some $i' \in S_0$" can be strengthened to "any $i' \in S_0 \setminus \{i\}$." As a result, in Condition $(\star)$, the part "for every $i, j \in S_0$" can be weakened to "for every $i \in S_0$, for $j = i$ and for some other $j \in S_0$."

(3) We reason that Theorem 2.3 holds true without assuming Condition (A) or (B) in practice.

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