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Infinite geometric groups and sets

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We investigate geometric groups and sets of permutations of an infinite set. (These are a generalisation of sharply t-transitive groups and sets.) We prove non-existence of groups, and give constructions of sets, for certain parameters.

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1. Introduction.

It is known that sharply t-transitive groups of permutations of an infinite set exist only for $t \leq 3$ [4], while sharply t-transitive sets exist for all t [1].

Geometric groups and sets of permutations have been proposed as a natural generalisation of sharply t-transitive groups and sets [2]. Our purpose is to investigate such objects on infinite sets. Not surprisingly, we give nonexistence results for groups, and constructions for sets.

Let $L = \{\ell_0, \ell_1, \ldots, \ell_{s-1}\}$ be a finite set of natural numbers, with $\ell_0 < \ldots < \ell_{s-1}$. The permutation group G on the set X is a <u>geometric group of type</u> L if there exist points $x_1, \ldots, x_s \in X$ such that

(i) the stabiliser of $x_1, \ldots, x_s$ is the identity;

(ii) for $i < s$, the stabiliser of $x_1, \ldots, x_i$ fixes ℓ_i points and acts transitively on its non-fixed points.

<u>Theorem 1</u>. There is no infinite geometric group of type $\{0, m, 2m, \ldots, (p-1)m\}$, where p is prime and p divides m.

This theorem is proved in Section 2. The case $p = m = 2$ is an infinite analogue of a theorem of Tsuzuku [5]. The theorem is new only for $p \leq 3$, since the non-existence of infinite geometric groups of type $\{0, m, \ldots, (t-1)m\}$ for any $t \geq 4$ follows from a theorem of Yoshizawa [6] (see [3]). However, the case $p = 2$ is the most interesting. It has the following consequence, also derived in Section 2:

Corollary 2. Let $L = \{\ell_0, \ldots, \ell_{s-1}\}$, with $\ell_0 < \ldots < \ell_{s-1}$. If $\ell_{i+1} - \ell_i$ is even for some $i \in \{0, \ldots, s-2\}$, then no geometric group of type L exists.

Geometric sets are harder to define; but our examples (in common with all known finite examples) satisfy an additional, simplifying condition. Let $L = \{\ell_0, \ldots, \ell_{s-1}\}$ with $\ell_0 < \ldots < \ell_{s-1}$. The set S of permutations of X is a special geometric set of type L if there is a matroid $\mathcal{M}$ of rank s on X satisfying

(i) any element of S is an automorphism of $\mathcal{M}$;

(ii) for $i < s$, a flat of $\mathcal{M}$ of rank i has cardinality ℓ_i;

(iii) for any $g, h \in S$, $\{x \in X \mid g(x) = h(x)\}$ is a flat of $\mathcal{M}$;

(iv) S is sharply basis-transitive, that is, if $(x_1, \ldots, x_s)$ and $(y_1, \ldots, y_s)$ are bases of $\mathcal{M}$, there is a unique $g \in S$ with $g(x_i) = y_i$ for $i = 1, \ldots, s$.

It is easily seen that any geometric group is a special geometric set; the bases of $\mathcal{M}$ are the s-tuples with the property of the definition. A geometric set in the sense of [2] is special if and only if it is unisupported (all the matroids $\hat{g}$ coincide) and consists of automorphisms of this common matroid.

In Section 3, we prove:

Theorem 3. There exist special geometric sets of permutations of a countable set, of each of the following types:

(i) $L = \{0, m, 2m, \ldots, (t-1)m\}$ for any $t, m > 1$;

(ii) $L = \{0, 1, q, \ldots, q^{t-1}\}$ for any $t > 1$ and any prime power q.

2. Geometric groups.

We begin with a lemma. This lemma, in more general form, is due to Yoshizawa [6]; we repeat the proof for completeness. However, by Yoshizawa's theorem [6], the lemma is vacuous unless $p \leq 3$.

Lemma 2.1. Let p be a prime. Suppose that G is p-transitive on an infinite set, and that the stabiliser of p points is finite with order divisible by p. Then some element of order p fixes infinitely many points.

Proof. Suppose that G is a counterexample to the lemma. Let $g \in G$ be an element of order p with (finite) fixed point set F (with $|F| \geq p$) and (infinitely many) cycles $C_1, C_2, \ldots$. For each $i \in \mathbb{N}$, g normalises the pointwise stabiliser H_i of C_i; so there is an element $h_i \in H_i$ of order p which commutes with g. Thus h_i maps F to F. But infinitely many of the elements h_i are different, since by assumption an element of order p can belong to only finitely many subgroups H_i. Thus, the setwise stabiliser of F is infinite, and so is its pointwise stabiliser, contrary to assumption.

If L is finite and $0 \in L$, a geometric group G of type L is transitive, and the stabiliser of a point fixes ℓ_1 points, where $\ell_1 = \min(L \setminus \{0\})$. These fixed point sets are blocks of imprimitivity for G; for brevity, we call them blocks. Note that the stabiliser of a point acts on the set of its non-fixed points as a geometric group H of type $L' = \{\ell - \ell_1 \mid \ell \in L, \ell \neq \ell_0\}$. We require a lemma about the kernel of the action of G on the set of blocks.

Lemma 2.2. Let G be geometric of type L, where L is finite and $0 \in L$. Then the kernel K of the action of G on its set of blocks is semiregular; that is, no non-identity element fixes every block and a point in one of them.

Proof. Observe first that K is finite; indeed, $|K| \leq \ell_1^s$, where $s = |L|$, because the stabiliser of s independent points is trivial. The lemma is proved by induction on s, being clear when s = 1. We suppose the result true for L'-geometric groups with $|L'| = s-1$.

Let H be the stabiliser of a point x, acting on its non-fixed points. By the induction hypothesis, the stabiliser of all H-blocks is semiregular. Each H-block is a union of G-blocks; so, if K is the kernel of the action on G-blocks, then $K_{xy} = 1$ whenever x and y lie in different G-blocks. Thus, as x runs over a set of representatives for the G-blocks, the subgroups K_x all have the same order and intersect pairwise in the identity. Because there are infinitely many G-blocks but K is finite, this requires $K_x = 1$.

We turn now to the proof of Theorem 1. Suppose that G is a geometric group of type $\{0, m, \ldots, (p-1)m\}$, where p is prime and $p|m$, and let K be the kernel of the action of G on blocks.

Any two elements of order p in $G \setminus K$ are conjugate. For, let g and h be two such elements; let $(x_1, \ldots, x_p)$ be a cycle of g with its points in distinct blocks, and $(y_1, \ldots, y_p)$ a cycle of h with the same property. Then $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_p\}$ are bases of the matroid $\mathfrak{M}$. So h is the unique element of G having a cycle $(y_1, \ldots, y_p)$; and there exists $k \in G$ with $k(x_i) = y_i$ for $i = 1, \ldots, p$. Then kgk^{-1} has a cycle $(y_1, \ldots, y_p)$, and so $kgk^{-1} = h$.

By Lemma 2.2, it follows that any two elements of order p which fix a point are conjugate.

Now G acts p-transitively on the set of blocks, and the stabiliser of p blocks is finite with order divisible by p (its order is a multiple of m^{p-1} and a divisor of m^p). Let g be an element of order p fixing p-1 blocks $B_1, \ldots, B_{p-1}$ pointwise and anothe block setwise. By the above remarks and Lemma 2.1, g fixes infinitely many blocks setwise, say B_1', B_2', ...

For any $i \geq p$, g normalises the subgroup H_i fixing B_i, B_2', ..., B_{p-1}' pointwise and B_1' setwise; so H_i contains an element h_i of order p commuting with g. Then h_i fixes setwise the fixed point set $B_1 \cup \ldots \cup B_{p-1}$ of g, and preserves the block system; so it fixes $B_1, \ldots, B_{p-1}$ setwise. Also, the elements h_i are all distinct, since if $h_i = h_j$ then h_i fixes $B_i', B_j', B_2', \ldots, B_p'$ (a total of p blocks) pointwise. Thus the setwise stabiliser of $B_1, \ldots, B_{p-1}, B_1'$ is infinite, a contradiction.

Proof of Corollary 2. Well-known necessary conditions for the existence of perfect matroid designs (see [2]) show that $\ell_{i+1}-\ell_i$ divides $\ell_{s-1}-\ell_{s-2}$; so we may assume that $\ell_{s-1}-\ell_{s-2}$ is even. Then the stabiliser of s-2 independent points, acting on its set of non-fixed points, is geometric of type $\{0, \ell_{s-1}-\ell_{s-2}\}$, contradicting Theorem 1.

3. Geometric sets

We begin with some general remarks.

1. The property of being a special geometric set of type L is unaffected by left or right multiplication by a fixed permutation. So we may assume, without loss, that a special geometric set contains the identity.

2. If $\min(L) = \ell_0 > 0$, then the existence of special geometric sets of type L and of type $L - \ell_0 = \{\ell - \ell_0 \mid \ell \in L\}$ are equivalent. For we may add points fixed by every permutation; conversely, if S has type L and contains the identity, then the members of S have ℓ_0 common fixed points, which may be deleted.

3. If $0 \in L$, then the existence of a special geometric set of type L implies the existence of one of type $L \setminus \{0\}$ (for example, all permutations in L which fix some given point).

As an illustration, Theorem 3(ii) implies the existence of special geometric sets of type $\{0, 1, 3, 7, \ldots, 2^t-1\}$.

We turn now to the proof of Theorem 3. The strategy is to prescribe in advance the matroid $\mathcal{M}$, and to construct freely the required set of permutations.

Consider the case $L = \{0, m, \ldots, (t-1)m\}$. Let X be the disjoint union of countably many m-sets $X_1, X_2, \ldots$, called blocks, where $X_i = \{x_{i0}, x_{i1}, \ldots, x_{im-1}\}$. We construct a sequence of pairs $(\mathcal{G}_n, m_n)$, where $\mathcal{G}_n$ is a set of bijections between subsets of X, and m_n is a positive integer, so that the following conditions are satisfied, where $A_n = X_1 \ldots X_{m_n}$:

(i) Each member of $\mathcal{G}_n$ is contained in a unique member of $\mathcal{G}_{n+1}$, and $m_n < m_{n+1}$.

(ii) The domain and range of each member of $\mathcal{G}_n$ are unions of blocks and contain A_n.

(iii) If $(y_1, \ldots, y_t)$ and $(z_1, \ldots, z_t)$ are t-tuples of elements of A_n, the members of each tuple lying in distinct blocks, then a unique member of $\mathcal{G}_n$ carries y_i to z_i for $i = 1, \ldots, t$.

(iv) If $g \in \mathcal{G}_n$ and $g(x_{ij}) = x_{k\ell}$, then $g(x_{ij+s}) = x_{k\ell+s}$ for $s = 1, \ldots, m-1$, where the second subscript is taken mod m.

It is clear that a starting pair $(\mathcal{G}_0, m_0)$ satisfying (ii)-(iv) exists.

Suppose that $(\mathcal{G}_n, m_n)$ has been constructed. The construction of $(\mathcal{G}_{n+1}, m_{n+1})$ involves two steps.

(a) First, let m_{n+1} be the largest index of a block contained in the domain or range of an element of $\mathcal{G}_n$, and $A_{n+1} = X_1 \cup \ldots \cup X_{m_{n+1}}$. For each pair of t-tuples from A_{n+1} as in (iii), if there is not already a member of $\mathcal{G}_n$ carrying the first to the second, adjoin such a bijection between the unions of the blocks containing the two t-tuples, in such a way that (iv) is satisfied.

(b) Now extend each bijection so that its domain and range contain A_{n+1}. For example, if g is not defined on the block $B_i \in A_{n+1}$, select the first block not already used in the construction, say B_j, and let g map B_i to B_j so that (iv) is satisfied.

It is now clear that all the conditions hold for $(\mathcal{G}_{n+1}, m_{n+1})$.

Now let $\mathcal{G}$ be the set of permutations of X obtained as direct limits (or unions) of sequences (g_n) of bijections, where $g_n \in \mathcal{G}_n$ and $g_n \subseteq g_{n+1}$. (all such direct limits are permutations, by (ii).) Clearly any element of $\mathcal{G}$ permutes the blocks among themselves; so $\mathcal{G}$ consists of automorphisms of the matroid $\mathcal{M}$ whose i-flats are unions of i distinct blocks for $i \leq t-1$. We must show that, if $g, h \in \mathcal{G}$ with $g \neq h$, then $\{x \mid g(x) = h(x)\}$ is the union of fewer than t blocks. By (iv), this set is a union of blocks. At the first stage at which both g_n and h_n are defined, this set contains fewer than t blocks, by (iii) (or by assumption if $n = 0$); and the prescription of (b) guarantees that no further agreement occurs. Finally, the transitivity of $\mathcal{G}$ on bases of $\mathcal{M}$ (t-tuples from distinct blocks) is clear from (ii).

Now consider the case $L = \{0, 1, q, \ldots, q^{t-1}\}$, where q is a prime power. This time, let X be the point set of an affine space of countable dimension over GF(q), with affine basis $\{x_0, x_1, x_2, \ldots\}$. (We may take X to be a GF(q)-vector space of countable dimension, with $x_0 = 0$ and $\{x_1, x_2, \ldots\}$ a vector space basis.) Again we construct pairs $(\mathcal{G}_n, m_n)$, where A_n is the affine span of $\{x_0, \ldots, x_{m_n}\}$. The conditions are:

(i) As before.

(ii) Each member of $\mathcal{G}_n$ is an affine bijection between affine subspaces of X; its domain and range contain A_n.

(iii) If $(y_0, \ldots, y_t)$ and $(z_0, \ldots, z_t)$ are affine independent (y+1)-tuples of elements of A_n, then a unique element of $\mathcal{G}_n$ carries y_i to z_i for $i = 0, \ldots, t$.

The construction iis as before; the analogue of (iv) is the requirement that the transformations are affine (so that domains and ranges are affine subspaces). In (a), for each pair of tuples as in (iii) for which no transformation yet carries the first to the second, adjoin the unique affine transformation from $\langle y_0, \ldots, y_t \rangle$ to $\langle z_0, \ldots, z_t \rangle$ carrying y_i to z_i for $i = 0, \ldots, t$. The extension process in (b) requires comment. Suppose that $g: U \rightarrow V$, where $g \in \mathcal{G}_n$. Let $\dim(A_{n+1}) - \dim(A_{n+1} \cap U) = r$. Let $x_{j+1}, \ldots, x_{j+r}$ be the first r basis vectors not previously used, and extend g to an affine transformation from $\langle U, A_{n+1} \rangle$ to $\langle V, x_{j+1}, \ldots, x_{j+r} \rangle$. The extension of the range is done similarly. The proof that the construction works is as before, noting that $\{x \mid g(x) = h(x)\}$ is an affine subspace.

References

1. A. Barlotti and K. Strambach, k-transitive permutation groups and k-planes, Math. Z. 185 (1984), 465-485.
2. P. J. Cameron and M. Deza, On permutation geometries, J. London Math. Soc. (2), 20 (1979), 373-386.
3. P. J. Cameron, M. Deza and P. Frankl, Sharp sets of permutations, J. Algebra, to appear.
4. J. Tits, Généralisation des groupes projectifs basée sur leurs propriétés de transitivité, Acad. Roy. Belgique Cl. Sci. Mem. 27 (1952).
5. T. Tsuzuku, Transitive extensions of certain permutation groups of rank 3, Nagoya J. Math. 31 (1968), 31-36.
6. M. Yoshizawa, On infinite four-transitive permutation groups, J. London Math. Soc. (2), 19 (1979), 437-438.