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Long Cycles through Specified Vertices in a Graph

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ABSTRACT

In this paper, we consider the length of the longest cycle through specified vertices. We show the following two results. (1) Let $G$ be a $k$-connected graph of order at least $2k$ and circumference $l$. Suppose $m < k$. Then for any $m$ vertices of $G$, $G$ has a cycle which contains all of them and has length at least $\frac{k-m}{k}l + 2m$. (2) Let $G$ be a 3-connected planar graph with circumference $l$. Then for any three vertices of $G$, there exists a cycle which contains all of them and has length at least $\frac{1}{4}l + 3$.

Here, we consider finite simple graphs. Let $G$ be a graph. By Dirac's theorem[3] $G$ has a cycle through specified $k$ vertices. In [2] Dirac also showed that a 2-connected graph of order $n$ and minimum degree at least $d$ has a cycle of length at least $\min\{n, 2d\}$. Locke[4] and Voss[7] generalized his result by showing that under the same conditions the graph has a cycle of length at least $\min\{n, 2d\}$ which contains specified two vertices.

These results lead us to the following question: Does a $k$-connected graph have a long cycle through specified $m$ vertices ($m \leq k$)? In this paper we investigate this question.

For basic graph-theoretic terminology, we refer the reader to [1]. Let $G$ be a graph. The **circumference** of $G$, denoted by $\text{circ}(G)$, is the length of the longest cycle of $G$. We denote by $w(G)$ the number of components of $G$. For $k \geq 0$ and $S \subset V(G)$, we call $S$ a $k$-cutset if $w(G - S) \geq 2$ and $|S| = k$. We often identify a subgraph $H$ of $G$ with its vertex set $V(H)$. Especially, when $x$ is a vertex of $H$, we write $x \in H$ instead of $x \in V(H)$. Furthermore, we write $|H|$ instead of $|V(H)|$. When we consider a cycle, we always give it an orientation. Let $C^+$ be the orientation of a cycle $C$ and $C^-$ be its reverse orientation. Let $C^+ = x_0, x_1, \ldots, x_{n-1}, x_n$ be a cycle. For $x_i, x_j \in C$, we define a subpaths $C^+[x_i, x_j]$ and $C^-[x_i, x_j]$ of $C$ by $C^+[x_i, x_j] = x_i, x_{i+1}, \ldots, x_{j-1}, x_j$. 

...
We also define \( C^+(x_i, x_j) \) and \( C^-(x_i, x_j) \) by

\[
C^+(x_i, x_j) = C^+[x_i, x_j] - \{x_i, x_j\},
\]

and

\[
C^-(x_i, x_j) = C^-[x_i, x_j] - \{x_i, x_j\}.
\]

Furthermore, \( C^+[x_i, x_j] = C^+[x_i, x_j] - \{x_j\} \). Subpaths \( C^-[x_i, x_j], C^+(x_i, x_j), C^-(x_i, x_j) \) are defined similarly. Let \( x_1, x_2, \ldots, x_s \) be a path. We denote by \( \text{end}(P) \) the set of endvertices of \( P \); \( \text{end}(P) = \{x_1, x_s\} \). Let \( P = x_1, x_2, \ldots, x_s \) and \( Q = y_1, y_2, \ldots, y_t \) be paths such that \( x_s = y_1 \). We denote by \( P \cdot Q \) the walk \( x_1, x_2, \ldots, x_s = y_1, y_2, \ldots, y_t \).

Let \( z \in V(G) \) and \( S \subset V(G) - \{z\} \). A subgraph \( F \) of \( G \) is called a \((z, S)\)-fan if \( F \) has the following decomposition \( F = \bigcup_{i=1}^{k} P_i \), where

(1) each \( P_i \) is a path between \( z \) and \( a_i \in S \), and

(2) \( P_i \cap S = \{a_i\} \), and \( P_i \cap P_j = \{z\} \) if \( i \neq j \).

We call \( k \) the size of the fan \( F \). The vertices \( a_1, \ldots, a_k \) are called endvertices of \( F \) and the set of its endvertices is denoted by \( \text{end}(F) \). Since \( F \) is a tree, for any two vertices \( x, y \in F \) the path in \( F \) which joins \( x \) and \( y \) is unique. We denote this path by \( F[x, y] \). We define \( F[x, y] \) by \( F[x, y] = F[x, y] - \{y\} \). Paths \( F(x, y) \) and \( F(x, y) \) are defined similarly.

The following theorem is well-known, called the generalized Menger's theorem.

**Theorem A** ([1, Theorem 6.7]). Let \( G \) be a \( k \)-connected graph, \( z \in V(G) \), and \( S \subset V(G) - \{z\} \). Then \( G \) has a \((z, S)\)-fan of size \( \min\{|S|, k\} \). \( \blacksquare \)

The following theorem was proved by Perfect[5].

**Theorem B** (Perfect[5]). Let \( G \) be a graph, \( z \in V(G) \), and \( S \subset V(G) - \{z\} \). Suppose \( G \) has two \((z, S)\)-fans \( F_1 \) and \( F_2 \) of size \( k_1 \) and \( k_2 \), respectively. If \( k_1 \leq k_2 \), then \( G \) has a \((z, S)\)-fan \( F' \) of size \( k_2 \) such that \( \text{end}(F_1) \subset \text{end}(F') \). \( \blacksquare \)

We use these two theorems in the proofs our results.

First, we show that the existence of long cycles through specified \( m \) vertices in a \( k \)-connected graph is assured if \( m < k \). Note that a \( k \)-connected graph is hamiltonian if its order is at most \( 2k \), by Dirac's theorem.
Theorem 1. Let $k \geq 2$, $0 \leq m \leq k$ and $G$ be a $k$-connected graph of order at least $2k$. For any $m$ vertices $x_1, \ldots, x_m$ of $G$, there exists a cycle such that

1. $x_1, \ldots, x_m \in V(C)$, and
2. $|C| \geq \frac{k-m}{k} \text{cir}(G) + 2m$.

Recently, Seymour and Truemper sent me a proof which is simpler than the original one. We show their proof.

Proof (due to Seymour and Truemper). The proof is by induction on $m$. For $m = 1$, let $x \in V(G)$, and let $C$ be a longest cycle in $G$. Since $|C| \geq 2k$,

$$\frac{k-1}{k} \text{cir}(G) + 2 = |C| - \frac{|C|}{k} + 2 \leq |C|.$$  

So we may assume $x \notin V(C)$. Now $G$ has an $(x, C)$-fan of size $k$. The endvertices of $F$ divide $C$ into $k$ paths, and any shortest one $P$ of these paths, say $P = C^+[u, v]$ has length at most $\frac{1}{k} \text{cir}(G)$. So $C^+[v, u] \cdot F[u, v]$ is a cycle which contains $x$ and has length at least

$$|C| - \frac{\text{cir}(G)}{k} + 2 = \frac{k-1}{k} \text{cir}(G) + 2$$

as desired.

Suppose $m > 1$, and let $C$ be a longest cycle containing at least $m - 1$ members of $S$. By the induction hypothesis,

$$|C| \geq \frac{k-m+1}{k} \text{cir}(G) + 2(m-1)$$

$$= \frac{k-m}{k} \text{cir}(G) + 2m + \frac{\text{cir}(G)}{k} - 2$$

$$\geq \frac{k-m}{k} \text{cir}(G) + 2m.$$

(*)

So we may assume that exactly one member $x$ of $S$ does not lie on $C$. Since $\text{cir}(G) \geq 2k$, $|C| \geq 2k$. So $G$ has an $(x, C)$-fan of size $k$. The endvertices of $F$ divide $C$ into $k$ paths. We call such a path bad if it contains some member of $S$ internally, and we call it good if it is not bad. Let $b$ represent the number of bad paths, and let $L$ be the sum of lengths of the bad paths. Then some good path $P = C^+[u, v]$ has length at most

$$\frac{|C| - L}{k - b}$$

(, where $|C| \geq 2k$ and $k \geq m - 1$). Keeping $|C|$ and $k$ fixed, and under the conditions $L \geq 2b$ and $b \leq m - 1$, this is maximized when $L = 2b$ and $b = m - 1$. Hence,

$$|P| \leq \frac{|C| - 2(m-1)}{k - m + 1}. $$
A cycle $C^+[v, u] \cdot F[u, v]$ contains $S$, and from (*) it has length at least

$$|C| - \frac{|C| - 2(m-1)}{k-m+1} + 2 \geq \frac{k-m}{k} \circ(G) + 2m$$

as desired. \[\square\]

Theorem 1 is sharp. Let, $k \geq 2$, $s \geq 1$, and $0 \leq m \leq k$. Let $H_0, H_1, \ldots, H_k$ and $H'_0$ be graphs such that $H_1 \simeq \cdots \simeq H_k \simeq K_s$, $H_0 \simeq \overline{K_m}$ and $H'_0 \simeq \overline{K_k}$. Suppose vertex sets $V(H_0), \ldots, V(H_k)$ and $V(H'_0)$ are disjoint. Define $G(k, m, s)$ by $G(k, m, s) = (H_1 \cup \cdots \cup H_k \cup H_0) + H'_0$. Then $G(k, m, s)$ is $k$-connected, $|G(k, m, s)| = ks + k + m \geq 2k$, and $\circ(G(k, m, s)) = ks + k$. On the other hand, the length of the longest cycle through $V(H_0)$ is $(k - m)s + k + m$. The above example shows that large circumference does not assure the existence of long cycles through specified $k$ vertices in $k$-connected graphs.

Next, we confine ourselves to planar graphs. Even if we consider only planar graphs, the length of the longest cycle through specified two vertices in a 2-connected graph is independent of its circumference. Let $C = x_0, x_1, \ldots, x_m = x_0$ be a cycle of length $m$ ($m \geq 4$). Add a new vertex $y$ and join $yx_1$ and $yx_{m-1}$. Then this graph has circumference $m$, but the unique cycle through $y$ and $x_0$ has length four. On the other hand, by Tutte’s theorem[6] 4-connected planar graphs are hamiltonian, and hence the length of the longest cycle through four specified vertices in a 4-connected planar graph is equal to its circumference. On a planar graph of connectivity three, we show the following theorem.

THEOREM 2. Let $G$ be a $3$-connected planar graph. Then any three vertices of $G$ lie on a cycle of length at least $\frac{1}{2} \circ(G) + 3$.

The proof of Theorem 2 is given by the following two lemmas.

**Lemma 1.** Let $G$ be a 3-connected planar graph. Then for any two vertices $x, y$, there exists a cycle $C$ such that

1. $x, y \in V(C)$.
2. $|C| \geq \frac{1}{2} \circ(G) + 2$.

**Lemma 2.** Let $G$ be a 3-connected planar graph, $x, y, z \in V(G)$ and $C$ be a cycle of $G$ such that $x, y \in V(C)$. Then there exists a cycle $C'$ such that

1. $x, y, z \in V(C')$.
2. $|C'| \geq \frac{1}{2} |C| + 2$. 

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Proof of Lemma 1. If $G$ is hamiltonian, then the lemma clearly holds. So we may assume that $G$ is not hamiltonian, which implies $|G| \geq 7$ and $\text{cir}(G) \geq 6$. Let $C$ be a longest cycle of $G$. We consider three cases.

Case 1. $\{x, y\} \subset V(C)$.

This case is trivial.

Case 2. $|\{x, y\} \cap V(C)| = 1$.

We may assume that $x \in V(C)$ and $y \notin V(C)$. Consider a $(y, C)$-fan $F$ of size three. Let $\text{end}(F) = \{y_1, y_2, y_3\}$. If $x \in \{y_1, y_2, y_3\}$, say $x = y_1$, then we have two cycles $C^+[x, y_2] \cdot F[y_2, x]$ and $C^-[x, y_2] \cdot F[y_2, x]$, one of which has length at least $\frac{1}{2}|C| + 2 = \frac{1}{2}\text{cir}(G) + 2$ and contains both $x$ and $y$. Next, assume $x \notin \{y_1, y_2, y_3\}$. We may assume $x \in C^+(y_3, y_1)$. Then one of the two cycles $C^+[y_3, y_2] \cdot F[y_2, y_3]$ and $C^-[y_1, y_2] \cdot F[y_2, y_1]$ has the desired properties.

Case 3. $\{x, y\} \cap V(C) = \emptyset$.

First, we show the following claims.

Claim 1. Suppose there exists a path $P$ in $G$ such that

1. $P$ joins two distinct vertices of $C$ and $P$ intersects $C$ only at its endvertices.
2. $x, y \in V(P)$.

Then the lemma follows.

Proof. Let $a$ and $b$ be endvertices of $P$. Then one of the two cycles $P[a, b] \cdot C^+[b, a]$ and $P[a, b] \cdot C^-[b, a]$ satisfies the desired properties.

Claim 2. Suppose there exist two paths $P$ and $Q$ such that

1. $V(P) \cap V(Q) = \emptyset$.
2. Both $P$ and $Q$ join two vertices of $C$.
3. $V(P) \cap V(C) = \text{end}(P)$ and $V(Q) \cap V(C) = \text{end}(Q)$.
4. Vertices of $\text{end}(P)$ and vertices of $\text{end}(Q)$ appear alternately around $C^+$.
5. $x \in V(P)$ and $y \in V(Q)$.

Then the lemma follows.

Proof. Let $\text{end}(P) = \{x_1, x_2\}$ and $\text{end}(Q) = \{y_1, y_2\}$. We may assume $x_1, y_1, x_2$ and $y_2$ appear in this order around $C^+$. Then one of the two cycles

$$C^+[x_1, y_1] \cdot Q[y_1, y_2] \cdot C^-[y_2, x_2] \cdot P[x_2, x_1]$$

and

$$C^-[x_1, y_2] \cdot Q[y_2, y_1] \cdot C^+[y_1, x_2] \cdot P[x_2, x_1]$$
has the desired properties.

Let \( \text{end}(F_1) = \{x_1, x_2, x_3\} \). We may assume that \( x_1, x_2, x_3 \) appear in this order around \( C^+ \). If \( y \in V(F_1) \), then the theorem follows by Claim 1. Suppose \( y \notin V(F_1) \). Let \( D = C \cup F_1 \). Let \( F_2 \) be a \((y, D)\)-fan of size three. Let \( \text{end}(F_2) = \{y_1, y_2, y_3\} \). If \( \text{end}(F_2) \cap (F_1 - \{x_1, x_2, x_3\}) \neq \emptyset \), then the lemma follows by Claim 1. So we may assume \( \text{end}(F_2) \subset V(C) \).

**Claim 3.** If \( \{y_1, y_2, y_3\} \subset C^+[x_1, x_{i+1}] \) (If \( i = 3 \), we consider \( x_4 = x_1 \)), then the lemma follows.

**Proof.** We may assume \( y_1, y_2, y_3 \in C^+[x_1, x_2] \) and \( y_1, y_2 \) and \( y_3 \) appear in this order around \( C^+ \). Then

\[
C^+[x_3, y_1] \cdot F_2[y_1, y_2] \cdot C^+[y_2, x_2] \cdot F_1[x_2, x_3]
\]

or

\[
C^+[x_1, y_2] \cdot F_2[y_2, y_3] \cdot C^+[y_3, x_3] \cdot F_1[x_3, x_1]
\]

has the desired properties.

By Claims 1, 2, 3, the only possible case in which the lemma would not hold is \( \{x_1, x_2, x_3\} = \{y_1, y_2, y_3\} \). We may assume \( x_i = y_i \) (\( i = 1, 2, 3 \)). Let \( D' = D \cup F_2 \). Since \( C \) is a longest cycle, \( C^+[x_1, x_2] \neq \emptyset \). Since \( G \) is 3-connected, there exists a path \( P \) joining \( C^+[x_1, x_2] \) and \( D' - C^+[x_1, x_2] \) in \( G - \{x_1, x_2\} \). Let \( \text{end}(P) = \{u, v\}, u \in C^+(x_1, x_2) \) and \( v \in D' - C^+[x_1, x_2] \). If \( v \in V(F_1) \cup V(F_2) \), then the lemma follows by Claim 2. So we may assume \( v \in C^+(x_2, x_3) \). Then \( F_1, F_2, C^+[x_1, x_2] \) and \( P[u, v] \cdot C^+[v, x_3] \) form a subdivision of \( K_{3,3} \). This contradicts the planarity of \( G \). Therefore, the lemma follows. \( \square \)

**Proof of Lemma 2.** Let \( C_0 \) be a longest cycle which contains \( x \) and \( y \). Then \( |C_0| \geq |C| \). If \( G \) is hamiltonian, then \( C_0 \) is a hamiltonian cycle, and \( |C_0| \geq 4 \). Hence the result follows. Therefore, we may assume \( G \) is not hamiltonian, and \( |G| \geq 7 \). By Lemma 1, \( |C_0| \geq \frac{1}{2} \cdot 7 + 2 \geq 5 \). So \( |C_0| \geq \frac{1}{2}|C_0| + 2 \geq \frac{1}{2}|C| + 2 \). Hence we may assume \( z \notin C_0 \). Consider a \((x, C_0)\)-fan \( F_1 \). Let \( \text{end}(F_1) = \{x_1, x_2, x_3\} \). We may assume that \( z_1, z_2, z_3 \) appear in this order around \( C^+ \). We consider three cases.

**Case 1.** \( \text{end}(F_1) \subset C_0^+[x, y] \) or \( \text{end}(F_1) \subset C_0^+[y, x] \).

We may assume \( \{x_1, x_2, x_3\} \subset C_0^+[x, y] \). Then one of the two cycles \( C_0^+[x_2, x_1] \cdot F_1[x_1, x_2] \) and \( C_0^+[x_3, x_2] \cdot F_1[x_2, x_3] \) has the desired properties.
Case 2. One of \( \text{end}(F_1) \) lies on \( C_0^+(y, x) \) and the other two lie on \( C_0^+(x, y) \).

We may assume \( z_1, z_2 \in C_0^+(x, y) \) and \( z_3 \in C_0^+(y, x) \). Let \( C_1 = C_0^+[z_2, z_1] \cdot F_1[z_1, z_2] \).

Then \( C_0 - C_1 = C_0^+(z_1, z_2) \). Let \( D = C_0 \cup F_1 \). By Theorem B, there exists an \((x, D - C_0^+(z_3, z_1))-\text{fan} \) \( F_2 \) of size three, such that \( z_1, z_3 \in \text{end}(F_2) \). Let \( \text{end}(F_2) = \{z_1, z_3, a\} \). If \( a \in F_1[x, z_1] \) or \( a \in F_1[z, z_2] \), let

\[
C_2 = C_0^+[z_1, z_3] \cdot F_1[z_3, a] \cdot F_2[a, z_1].
\]

If \( a \in F_1[x, z_3] \), let

\[
C_2 = C_0^+[z_1, z_3] \cdot F_1[z_3, z_2] \cdot C_0^-[z_2, z_1] \cdot F_2[z_1, a].
\]

If \( a \in C_0^+(y, z_3) \), let

\[
C_2 = C_0^-[a, z_1] \cdot F_1[z_1, z_3] \cdot F_2[z_3, a].
\]

Then in either case, \( C_0^+(z_1, z_2) \subset C_2 \) and either \( C_1 \) and \( C_2 \) satisfies the desired properties. So the only remaining case is \( a \in C_0^+(z_1, z_2) \). Let \( D' = D \cup F_2 \).

Next, consider a \((y, D' - C_0^+(z_2, z_3))-\text{fan} \) \( F_3 \) such that \( \{z_2, z_3\} \subset \text{end}(F_3) \). Let \( \text{end}(F_3) = \{z_2, z_3, b\} \). If \( b \in (F_1 - \text{end}(F_1)) \cup C_0^+(z_3, z_1) \), then the lemma follows by the same argument. If \( b \in F_2(x, a) \cup F_2(x, z_1) \), let

\[
C_3 = F_3[b, z_2] \cdot C_0^-[z_2, z_1] \cdot F_1[z_1, z_3] \cdot F_2[z_3, b].
\]

If \( b \in F_2(x, z_3) \), let

\[
C_3 = F_3[b, z_3] \cdot F_1[z_3, z_2] \cdot C_0^-[z_2, z_1] \cdot F_2[z_1, b].
\]

Then in either case \( C_0^+(z_1, z_2) \subset C_3 \) and hence either \( C_1 \) or \( C_3 \) satisfies the desired properties. So the lemma follows unless \( b \in C_0^+[z_1, z_2] \). (Possibly \( a = b \).)

Now we consider the case \( a \in C_0^+(z_1, z_2) \) and \( b \in C_0^+(z_1, z_2) \). If \( z_1, b, a, z_2 \) appear in this order around \( C_0^+ \), let

\[
C_4 = F_3[z_3, b] \cdot C_0^+[b, z_2] \cdot F_1[z_2, z_1] \cdot C_0^-[z_1, z_3]
\]

and

\[
C_5 = F_2[z_3, a] \cdot C_0^-[a, z_1] \cdot F_1[z_1, z_2] \cdot C_0^+[z_2, z_3].
\]
If $z_1, a, b, z_2$ appear in this order around $C^+$, let
\[ C_4 = F_3[z_2, b] \cdot C_0^-[b, z_3] \cdot F_1[z_3, z_2] \]
and
\[ C_5 = F_2[z_1, a] \cdot C_0^+[a, z_3] \cdot F_1[z_3, z_1]. \]
Then in either case we have \( \{x, y, z\} \subset C_4 \cap C_5 \), \( C_0 \subset C_4 \cup C_5 \), and hence \( |C_4| \geq \frac{1}{2}|C_0| + 2 \) or \( |C_5| \geq \frac{1}{2}|C_0| + 2 \). So the lemma follows.

Now, we may assume that $a = z_2$ or $b = z_1$. If $a = z_2$, then $F_1$, $F_2$, $F_3$ and $C_0^-[b, z_1]$ form a subdivision of $K_{3,3}$. If $b = z_1$, then $F_1$, $F_2$, $F_3$ and $C_0^+[a, z_2]$ form a subdivision of $K_{3,3}$. Hence both contradicts the planarity of $G$. Therefore, the proof in this case is complete.

**Case 3.** \( |\{x, y\} \cap \text{end}(F_1)| = |C_0^+(x, y) \cap \text{end}(F_1)| = |C_0^+(y, x) \cap \text{end}(F_1)| = 1 \).

We may assume $z_1 = x$, $z_2 \in C_0^+(x, y)$ and $z_3 \in C_0^+(y, x)$. Then either
\[ C_6 = F_1[z_1, z_2] \cdot C_0^+[z_2, z_1], \quad \text{or} \]
\[ C_7 = F_1[z_1, z_3] \cdot C_0^-[z_3, z_1] \]
satisfies the desired properties.

Therefore, in each case, $G$ has a cycle through $x$, $y$ and $z$ of length at least $\frac{1}{2}|C_0| + 2$.

\[\Box\]

**References**


