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LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

Masayuki Yamasaki

§1. Introduction

Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $S$ a connected closed normal subgroup of $L$. Then $KS$ is closed, and we have a fiber bundle

$$K\backslash KS \rightarrow K\backslash L \rightarrow KS\backslash L.$$ 

$L$ acts on $K\backslash L$ by right multiplication. $L$ acts also on $KS\backslash L$ by right multiplication; let $N$ denote the kernel of this action, i.e., $N = \{ g \in L; KSgx = KSx \text{ for all } x \in L \}$. The action of $N$ on $K\backslash L$ leaves all fibers invariant; in other words, we have a family of right $N$-spaces parametrized over $KS\backslash L$.

Lemma 1. The right $N$-spaces $K\backslash KSx$ ($x \in L$) are equivalent.

Proof. Since $K$ is compact, $K\backslash L$ has an $L$-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy$ ($x,y \in L$). It suffices to construct an $N$-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.
Fix a point $p$ of $\mathbb{K}/K_{x}$ and let $d$ be the distance between $p$ and $\mathbb{K}/K_{y}$. $\mathbb{K}/L$ is complete and $\mathbb{K}/K_{y}$ is closed; therefore, $d$ is positive and can be achieved as the length of a geodesic $\gamma$ connecting $p$ and a point $q$ of $\mathbb{K}/K_{y}$. $S$ is contained in $N$ and acts transitively on each fiber. The action of an element $s$ of $S$ sends $\gamma$ to a geodesic $\gamma \cdot s$ of the same length $d$ connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $\mathbb{K}/K_{x}$ to $\mathbb{K}/K_{y}$ is independent of the choice of the point, and $\gamma$ is one of the shortest geodesics connecting $\mathbb{K}/K_{x}$ and $\mathbb{K}/K_{y}$. Therefore $\gamma$ is perpendicular to $\mathbb{K}/K_{x}$ at $p$. Let $\left(T_{p}(\mathbb{K}/K_{x})\right)^{\perp}$ denote the orthogonal complement of the tangent space $T_{p}(\mathbb{K}/K_{x})$ of $\mathbb{K}/K_{x}$ at $p$ in the tangent space of $\mathbb{K}/L$ at $p$. As the exponential map $\text{Exp}$ is a diffeomorphism near the origin, any fiber $\mathbb{K}/K_{x}z$ that meets $\text{Exp}(V)$ meets $\text{Exp}(V)$ exactly once, where $V$ is a sufficiently small neighborhood in $\left(T_{p}(\mathbb{K}/K_{x})\right)^{\perp}$ of the origin. This implies that $\gamma$ is the unique geodesic of length $d$ connecting $p$ and $\mathbb{K}/K_{y}$, as long as $\mathbb{K}/K_{y}$ is sufficiently close to $\mathbb{K}/K_{x}$. Let us suppose that this is the case. Then the correspondence $p \cdot s \leftrightarrow q \cdot s$ ($s \in S$) defines a diffeomorphism $\mathbb{K}/K_{x} \to \mathbb{K}/K_{y}$, which is obviously $N$-equivariant because it sends a point in $\mathbb{K}/K_{x}$ to the unique point closest to it in $\mathbb{K}/K_{y}$ and $N$ acts on $\mathbb{K}/L$ by isometries.

Remark. The $N$-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $\mathbb{K}/L \to KS\setminus L$ so that the action of $N$ on $\mathbb{K}/L$ is locally a product of the action of $N$ on a fiber and the action of a trivial group on the base.

If $G$ is a lattice of $L$, the action of $L$ on $\mathbb{K}/L$ restricts
to an action of $G$ on $K \backslash L$. $H = G \cap N$ is a normal subgroup of $G$ which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K \backslash KS/H \to K \backslash L/H \to KS/L.$$ 

The quotient group $\Gamma = G/H$ acts on $K \backslash L/H$ and $KS/L$ such that $(K \backslash L/H)/\Gamma = K \backslash L/G$ and $(KS/L)/\Gamma = KS/L/G$; the fiber bundle map induces a map:

$$q: K \backslash L/G \to (KS/L)/\Gamma.$$ 

Note that $KS/L$ can be naturally identified with $(S \backslash KS) \backslash (S \backslash L)$, which has an $(S \backslash L)$-invariant (and hence $L$-invariant) Riemannian metric. Thus $\Gamma$ can be thought of as a subgroup of the group $I(KS/L)$ of all the isometries of $KS/L$ with respect to this Riemannian metric.

Suppose that $\Gamma$ is discrete in $I(KS/L)$. Then the isotropy subgroup $\Gamma_v$ of $\Gamma$ at $v \in KS/L$ is finite for each $v$, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS/L)/\Gamma$ is $((K \backslash KSx)/H)/\Gamma_v$, where $v = KSx$ ($x \in L$). Thus a "fiber" of $q$ is homeomorphic to a quotient of the "general fiber" $K \backslash KS/H$ by an action of a finite group; i.e., $q$ is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup $S$.

Theorem 2. Let $L$ be a non-compact Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $G$ a lattice of $L$. Then there is an orbifold Seifert fibration
$K\sslash L/G \to 0^m,$

where $0^m$ is a Riemannian orbifold of dimension $m > 0$ and of non-positive sectional curvature. If $L$ is amenable, $0^m$ can be chosen to be flat.

The proof will occupy the following two sections. Some social cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group $L$ with finitely many components is amenable if and only if $L/R$ is compact, where $R$ denotes the radical (= the unique maximal connected normal solvable subgroup) of $L$. See Milnor [6]. In this section we handle the case when $L$ is not amenable. We use $R$ as $S$, following [4]; i.e., we are going to show that

$K\sslash L/G \to KR\sslash L/G$

is a Seifert fibration with the desired property. As in the previous section, identify $KR\sslash L$ with $(R\sslash KR) \backslash (R\sslash L) = R^m$ ($m > 0$). $R\sslash L$ is a non-compact semi-simple Lie group, and $R\sslash KR$ is a maximal compact subgroup of $R\sslash L$. Using the Cartan decomposition and the Killing form, one can introduce an
(R\L)-invariant (and hence L-invariant) Riemannian metric $g$ on $R^m$ with non-positive sectional curvature. In fact, any (R\L)-invariant Riemannian metric on $R^m$ has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism $\Phi: L \to I(R^m, g)$. Let $\Gamma$ denote the image $\Phi(G)$ of $G$.

To prove the theorem, it suffices to show that $\Gamma$ is discrete in $I(R^m, g)$. Let $\tau$ denote the natural projection $L \to R\setminus L$. If the image $\tau(G)$ of $G$ in $R\setminus L$ is discrete, then $\Gamma$ is obviously discrete. Unfortunately $\tau(G)$ may not be discrete in general. We remedy this situation as follows.

Let $L_0$ denote the identity component of $L$. $G\cap L_0$ is a subgroup of $G$ with finite index. Therefore it suffices to show that $\Phi(G\cap L_0)$ is discrete in $I(R^m, g)$. As $(R\setminus (G\cap L_0)R)\setminus (R\setminus L_0)$ can be naturally identified with $R^m$, we may assume from the beginning that $L$ is connected.

Now there is a semi-simple Lie subgroup $S$ of $L$ such that $L = SR$ and such that $S\cap R$ is discrete (Levi decomposition). Let $\sigma: S \to \text{Aut}(R)$ denote the action of $S$ on $R$. A sufficient condition for $\tau(G)$ to be discrete in $R\setminus L$ is that the identity component $(\ker \sigma)_0$ of the kernel of $\sigma$ has no compact factors (Raghunathan[9], p.150). Let $C$ denote the unique maximal compact normal subgroups of $(\ker \sigma)_0$. It is a characteristic subgroup of $(\ker \sigma)_0$, and hence it is normal in $\ker \sigma$ and in $S$.

On the other hand, $C$ commutes with elements of $R$. Therefore $C$ is normal in $L$. Let $\pi: L \to L/C$ denote the natural projection, and let $L' = \pi(L)$, $S' = \pi(S)$, $R' = \pi(R)$, $G' = \pi(G)$, $K' = \pi(K)$. Then $S$ is semi-simple, $R'$ is the radical of $L'$, $G'$ is a lattice of $L'$,
and $K'$ is a maximal compact subgroup of $L'$. Let $\sigma':S' \to \text{Aut}R'$ denote the action of $S'$ on $R'$. Then it is easily observed that $\ker \sigma' = (\ker \sigma)/C$, since $\mathcal{O}\cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image $G''$ of $G'$ in $R'\backslash L'$ is discrete. Thus the action of $G$ on $R^m$ factors through a properly discontinuous action of $G''$ on $K'R'\backslash L = KR\backslash L$. Therefore, $\Gamma$ is discrete in $I(R^m,g)$.

This completes the proof of theorem 2 when $L$ is not amenable.

Remark. Let $q:K\backslash L/G \to KR\backslash L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRxG)$ over the point $KRxG \in KR\backslash L/G$ ($x \in L$) is homeomorphic to

$$(x^{-1}Kx)\backslash(x^{-1}KRx)/(x^{-1}KRx \cap G).$$

It is easily observed that $x^{-1}KRx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $R$ the radical of $L$, and $G$ a lattice of $L$. Then $KR \cap G$ is a uniform lattice of $KR$.

§3. Amenable case

Now let us assume that $L$ is non-compact and amenable. Let $K$ be a maximal compact subgroup and $R$ the radical of $L$ as
before. Since $L$ is amenable, $L = KR$.

We define a sequence $N^{(j)}$ ($j \geq -1$) of closed characteristic subgroups of $L$ as follows:

1. $N^{(-1)}$ is the radical $R$,
2. $N^{(0)}$ is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of $L$,
3. $N^{(j)}$ is the commutator subgroup $[N^{(j-1)}, N^{(j-1)}]$ of $N^{(j-1)}$, for $j > 0$.

It may not be so obvious that $N^{(j)}$'s are closed when $j > 0$; in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer $k$ such that $N^{(k)} = \{1\}$. Consider the following sequence:

$$L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \ldots \supset KN^{(k)} = K.$$  

There exists an integer $i$ ($\geq 0$) such that

$$L = KN^{(-1)} = KN^{(0)} = \ldots = KN^{(i-1)} \neq KN^{(i)},$$

because $L$ is non-compact. Let us write $M = N^{(i-1)}$ and $N = N^{(i)}$. We introduce a flat $L$-invariant Riemannian metric on $\text{KN}\backslash L$.

Let us study the action of $L$ on $\text{KN}\backslash L$ defined by right multiplication. An element $k y$ of $KM = L$ ($k \in K, y \in M$) acts on an element $KNx$ ($x \in M$) of $\text{KN}\backslash L$ as follows:

$$KNx \cdot (ky) = KNxky$$

$$= KN(k^{-1}xk)y.$$
Note that we have $[M,M] \subset N$; we identify the coset space $KNL$ with the simply-connected abelian Lie group $(KMN)N/M = \mathbb{R}^m$ ($m > 0$). Now the induced action of $L$ on $\mathbb{R}^m$ is:

$$(KMN)x \cdot (ky) = (KMN)x^{-1}kxy.$$ 

The following are easily observed: (1) this action, when restricted to $K$, defines a homomorphism $\alpha: K \to \text{Aut}(\mathbb{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbb{R}^m$, and (2) if $k \in KMN$, then $(KMN)x \cdot k = (KMN)x^{-1}k = (KMN)x(k^{-1}xk^{-1})x = (KMN)x$ for $x \in M$, and so $KMN$ acts trivially on $\mathbb{R}^m$. Let $\beta: M \to (KMN)N \setminus M$ denote the natural projection. We now define a map $\phi: L = KM \to \alpha(K)x((KMN)N \setminus M) \subset O(m) \ltimes \mathbb{R}^m = I(\mathbb{R}^m)$ by sending $ky$ ($k \in K$, $y \in M$) to $(\alpha(k), \beta(y)) \in O(m) \ltimes \mathbb{R}^m$. This is a well-defined homomorphism. Here $\ltimes$'s denote the obvious semi-direct products. Let $\Gamma$ denote the image of $G$ by $\phi$ in $I(\mathbb{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that $\Gamma$ is a discrete subgroup of $I(\mathbb{R}^m)$. To do this we use the following lemma:

Lemma 4. **If $N$ is a connected nilpotent Lie group and $H$ is a discrete cocompact subgroup of $N$, then the commutator subgroup $[N,N]$ is closed in $N$ and $H \cap [N,N]$ is cocompact in $[N,N]$.**

**Proof:** This is well-known if $N$ is simply-connected, so consider the universal cover $p: U \to N$ of $N$; it can be identified with the natural projection $U \to U/\Pi$, where $\Pi$ is the kernel of
p. To see that $[N,N]$ is closed in $N$, it suffices to show that $N/[N,N]$ is Hausdorff. As $p^{-1}(N,N) = \Pi[U,U]$, we have homeomorphisms:

$$N/[N,N] \cong U/\Pi[U,U] \cong (U/[U,U])/(\Pi[U,U]/[U,U]).$$

Here $U/[U,U]$ is a Lie group, because $U$ is simply-connected and hence its commutator subgroup $[U,U]$ is closed. Note that the preimage $p^{-1}(H)$ of $H$ is discrete and cocompact in $U$. Since $U$ is simply-connected, $p^{-1}(H)[U,U]$ is cocompact in $[U,U]$. Therefore, the image $p^{-1}(H)[U,U]/[U,U]$ of $p^{-1}(H)$ by projection: $U \to U/[U,U]$ is discrete. As $\Pi \subset p^{-1}(H)$, $\Pi[U,U]/[U,U]$ is also discrete and hence closed in $U/[U,U]$. Therefore $(U/[U,U])/(\Pi[U,U]/[U,U])$ is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

$$[N,N]/H[N,N] \cong \Pi[U,U]/p^{-1}(H)[\Pi[U,U], \Pi[U,U] \cong [U,U]/p^{-1}(H)[\Pi[U,U],$$

the second statement is obvious. □

Now we prove

Lemma 5. $N^{(j)}$ are closed subgroups of $L$, and $\Gamma$ is a crystallographic subgroup of $I(R^m)$.

Proof: If $G \cap R$ is cocompact in $R=N^{(-1)}$, then $G \cap N^{(0)}$ is a discrete cocompact subgroup of $N^{(0)}$ and we can apply lemma 4 to
prove that $N^{(j)}$'s are closed for $j \geq 1$. Unfortunately, $G \cap R$ may not be cocompact in $R$, in general. To remedy this situation we introduce a quotient Lie group $L'$ of $L$ as in the previous section. We may assume that $L$ is connected. We have Levi decomposition $L = SR$, where $S$ is a connected semi-simple (and hence compact) subgroup, $R$ is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma : S \to \text{Aut}(R)$ denote the action of $S$ on $R$. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of $L$, because it commutes with elements of $R$. In particular, $(\ker \sigma)_0 \subseteq \ker \alpha \subseteq K$. Let $\pi : L \to L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then $G'$ is a lattice of $L'$, $K'$ is a maximal compact subgroup of $L'$, $S'$ is a semi-simple subgroup of $L'$, $R'$ is the radical of $L'$, and the action $\sigma' : S' \to \text{Aut}(R')$ of $S'$ on $R'$ is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N^{(j)}_r$ ($j \geq -1$) of characteristic subgroups of $L'$ by:

1. $N^{(-1)}_r = R'$
2. $N^{(0)}_r$ = the nil-radical of $L'$
3. $N^{(j)}_r = [N^{(j-1)}_r, N^{(j-1)}_r]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N^{(0)}_r$ are cocompact in $R'$ and $N^{(0)}_r$ respectively. By successively using lemma 4, we know that all $N^{(j)}_r$'s are closed. Note that $\pi|_R : R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi|_R)^{-1}(N^{(j)}_r)$ for each $j$. Therefore $N^{(j)}$'s are closed in $L$. 

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Next, we show that $\Gamma$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. Note that we have

$$L' = K'N'(i-1) = K'N'(0) = \ldots = K'N'(i-1) \neq K'N'(i)$$

for the same $i$ and that $K'N' \setminus K'M' = KN'KM'$, where $M' = N'(i-1)$, $N' = N'(i)$. $G' \cap \mathbb{N}'(j)$ is cocompact in $N'(j)$ for all $j$. In particular $G' \cap M'$ is cocompact in $M'$. So the image of $G'$ in $M' \setminus L'$ is discrete; furthermore, it is finite, because $M' \setminus L'$ is compact. Looking at the diagram:

\[
\begin{array}{ccc}
G & \longrightarrow & \text{finite} \\
\cap & & \cap \\
\pi^{-1}(M') & \longrightarrow & L \\
\downarrow & & \downarrow \pi \\
M' & \longrightarrow & L' \\
\cup & & \cup \\
G' & \longrightarrow & \text{finite}
\end{array}
\]

we know that $G \cap \pi^{-1}(M')$ has a finite index in $G$. So it suffices to show that the image $\Phi(G \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. As $\ker \sigma \subseteq \ker \alpha$, $\Phi$ sends elements in $\pi^{-1}(M') = (\ker \sigma) \mathbb{O} M$ to elements in $\mathbb{R}^m \subset I(\mathbb{R}^m)$. Now consider the following commutative diagram:

\[
\begin{array}{ccc}
G \cap \pi^{-1}(M') \subseteq \pi^{-1}(M') = (\ker \sigma) \mathbb{O} M & \longrightarrow & \mathbb{R}^m = (KN'KM') \setminus M \\
\cup & & \cup \\
\Phi \quad \Phi
\end{array}
\]
where \( \Phi' \) is the natural map and \((\pi|\mathcal{M})_*\) is the map induced by the restriction of \(\pi\) to \(\mathcal{M}\), \(\pi|\mathcal{M}: M \to M'\). \(KNM\) and \(K'NM'\) are maximal compact subgroups of \(M\) and \(M'\), respectively, and \(\pi(KNM) = K'NM'\); therefore, \((\pi|\mathcal{M})^{-1}(K'NM') = KNM\). Using this, it is easily verified that \((\pi|\mathcal{M})^{-1}((K'NM')N') = (KNM)N\). Therefore \((\pi|\mathcal{M})_*\) is an isomorphism. Since \((G'KNM')\cap N = G'KN'\) is cocompact in \(N'\), \((G'KNM')\cap (K'NM')N'\) is cocompact in \((K'NM')N'\); so \(\Phi'(G'KNM')\) is a discrete cocompact subgroup of \((K'NM')N'\) \(\mathcal{M}'\). Therefore \(\Phi(G'KNM')\) is a discrete cocompact subgroup of \(\mathcal{M}\) (and hence in \(I(\mathcal{M})\)). This completes the proof of lemma 5. \(\Box\)

Thus \(KL/G \to KNL/G\) is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form \(KNxG/G\), and is homeomorphic to

\[
(x^{-1}Kx) \cap (x^{-1}KNx)/(x^{-1}KNx \cap G).
\]

If \(G\) is a lattice of \(L\) (which is automatically uniform), then \(x^{-1}KNx \cap G\) is a uniform lattice of \(x^{-1}KNx\).

§4. A rational computation of Wall's L-groups
Let $L$ be an amenable Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $G$ a uniform lattice of $L$. Such a discrete group $G$ is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the $L$-groups of $G$ in terms of certain generalized homology of $K \backslash L / G$.

$K \backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_y = x^{-1} K x \cap G$ of $G$ at $y = K x$ ($x \in L$) is finite. The action of $G$ on $\mathbb{R}^n$ is free if $G$ is torsion-free; in general, $\mathbb{R}^n / G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let $WG$ be a contractible free $G$-complex, and $p$ denote the projection: $(\mathbb{R}^n \times WG) / G \to \mathbb{R}^n / G$, where $G$ acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y] \in \mathbb{R}^n / G$ by $p$ is homeomorphic to $WG / G_y$, and $p$ is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^{-\infty}(G)$ denote the limit of Ranicki's lower $L$-groups $L^{(-j)}(ZG)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $L^{-\infty}(-)$ from the category of spaces to the category of $\Omega$-spectra such that the homotopy group of $L^{-\infty}(X)$ is equal to $L^{-\infty}(\pi_1 X)$. Applying $L^{-\infty}(-)$ to each fiber of $p$, we obtain a sheaf of spectra, denoted $L^{-\infty}(p)$. F. Quinn defines the homology group $H_\ast (\mathbb{R}^n / G; L^{-\infty}(p))$ of $\mathbb{R}^n / G$ with coefficients $L^{-\infty}(p)$. See [7], [10]. The following is a rational computation of $L^{-\infty}(G)$ in terms of this homology.
Theorem 6. Let $G$ be as above, then there is a natural isomorphism

$$H_\ast (\mathbb{R}^n / G; \mathbb{L}^{-\infty} (p)) \otimes \mathbb{Z}[1/2] \rightarrow L_\ast^{-\infty} (G) \otimes \mathbb{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations.

$$\begin{array}{ccc}
\mathbb{R}^n / G & \xrightarrow{id.} & \mathbb{R}^n / G \\
p & \downarrow & \downarrow \\
\mathbb{R}^n / G & \rightarrow & \text{pt.}
\end{array}$$

Note that $(\mathbb{R}^n \times \mathbb{W}/G) / G = BG$ is a classifying space for $G$ and that $H_\ast (\text{pt.} ; \mathbb{L}^{-\infty} (BG \rightarrow \text{pt.} )) = L_\ast^{-\infty} (G) [10]$.

It is to be noted that theorem 6 says that the $\mathbb{L}^{-\infty} (p)$ coefficient homology of $\mathbb{R}^n / G$ is independent (modulo 2 torsion) of the action of $G$ on $\mathbb{R}^n$. It is conceivable that the orbifold $\mathbb{R}^n / G$ has a certain strong rigidity.

Proof of theorem 6. The proof is by induction on the dimension $n$ of $K \backslash L$. Let $q: \mathbb{R}^n / G \rightarrow \mathbb{R}^m / T$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$H_\ast (\mathbb{R}^n / G; \mathbb{L}^{-\infty} (p)) \equiv H_\ast (\mathbb{R}^m / T; \bigcup_{\omega \in \mathbb{R}^m / T} \mathbb{L}^{-\infty} (p | q^{-1} (\omega))) \equiv H_\ast (\mathbb{R}^m / T; \bigcup_{\omega} \mathbb{L}^{-\infty} ((q_p)^{-1} (\omega)))$$

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\[ = H_\ast (\mathbb{R}^n/T; L^{\infty} (qp)) \]

by induction hypothesis, where \( H \) denote the homology theory spectrum [ibid.]. We can prove that \( H_\ast (\mathbb{R}^n/T; L^{\infty} (qp)) \otimes \mathbb{Z} \) is naturally isomorphic to \( L^{\infty} (G) \) using the proof of the main theorem of [ibid.] with only some obvious modifications, and this completes the proof of theorem 6. \( \square \)

**Corollary 7. (Novikov Conjecture)** Let \( G \) be as above, then the assembly map
\[ H_\ast (BG; L^{\infty} (1)) \to L^{\infty} (G) \]
is rationally split injective.

**References**


