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LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

Masayuki Yamasaki

§1. Introduction

Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $S$ a connected closed normal subgroup of $L$. Then $KS$ is closed, and we have a fiber bundle

$$K\backslash KS \rightarrow K\backslash L \rightarrow KS\backslash L.$$ 

$L$ acts on $K\backslash L$ by right multiplication. $L$ acts also on $KS\backslash L$ by right multiplication; let $N$ denote the kernel of this action, i.e., $N = \{ g \in L; KSxg = KSx \text{ for all } x \in L \}$. The action of $N$ on $K\backslash L$ leaves all fibers invariant; in other words, we have a family of right $N$-spaces parametrized over $KS\backslash L$.

Lemma 1. The right $N$-spaces $K\backslash KSx$ ($x \in L$) are equivalent.

Proof. Since $K$ is compact, $K\backslash L$ has an $L$-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy$ ($x, y \in L$). It suffices to construct an $N$-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.
Fix a point $p$ of $\mathbb{K}\backslash KSx$ and let $d$ be the distance between $p$ and $\mathbb{K}\backslash KSy$. $\mathbb{K}\backslash L$ is complete and $\mathbb{K}\backslash KSy$ is closed; therefore, $d$ is positive and can be achieved as the length of a geodesic $\gamma$ connecting $p$ and a point $q$ of $\mathbb{K}\backslash KSy$. $S$ is contained in $N$ and acts transitively on each fiber. The action of an element $s$ of $S$ sends $\gamma$ to a geodesic $\gamma \cdot s$ of the same length $d$ connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $\mathbb{K}\backslash KSx$ to $\mathbb{K}\backslash KSy$ is independent of the choice of the point, and $\gamma$ is one of the shortest geodesics connecting $\mathbb{K}\backslash KSx$ and $\mathbb{K}\backslash KSy$. Therefore $\gamma$ is perpendicular to $\mathbb{K}\backslash KSx$ at $p$. Let $(T_p(\mathbb{K}\backslash KSx))^\perp$ denote the orthogonal complement of the tangent space $T_p(\mathbb{K}\backslash KSx)$ of $\mathbb{K}\backslash KSx$ at $p$ in the tangent space of $\mathbb{K}\backslash L$ at $p$. As the exponential map $\text{Exp}$ is a diffeomorphism near the origin, any fiber $\mathbb{K}\backslash KSz$ that meets $\text{Exp}(V)$ meets $\text{Exp}(V)$ exactly once, where $V$ is a sufficiently small neighborhood in $(T_p(\mathbb{K}\backslash KSx))^\perp$ of the origin. This implies that $\gamma$ is the unique geodesic of length $d$ connecting $p$ and $\mathbb{K}\backslash KSy$, as long as $\mathbb{K}\backslash KSy$ is sufficiently close to $\mathbb{K}\backslash KSx$. Let us suppose that this is the case. Then the correspondence $p \cdot s \leftrightarrow q \cdot s$ ($s \in S$) defines a diffeomorphism $\mathbb{K}\backslash KSx \to \mathbb{K}\backslash KSy$, which is obviously $N$-equivariant because it sends a point in $\mathbb{K}\backslash KSx$ to the unique point closest to it in $\mathbb{K}\backslash KSy$ and $N$ acts on $\mathbb{K}\backslash L$ by isometries. □

Remark. The $N$-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $\mathbb{K}\backslash L \to KS\backslash L$ so that the action of $N$ on $\mathbb{K}\backslash L$ is locally a product of the action of $N$ on a fiber and the action of a trivial group on the base.

If $G$ is a lattice of $L$, the action of $L$ on $\mathbb{K}\backslash L$ restricts
to an action of $G$ on $K \backslash L$. $H = G \cap N$ is a normal subgroup of $G$ which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K \backslash KS/H \to K \backslash L/H \to KS/L.$$ 

The quotient group $\Gamma = G/H$ acts on $K \backslash L/H$ and $KS/L$ such that $(K \backslash L/H)/\Gamma = K \backslash L/G$ and $(K \backslash S/L)/\Gamma = KS/L/G$; the fiber bundle map induces a map:

$$q: K \backslash L/G \to (KS \backslash L)/\Gamma.$$ 

Note that $KS \backslash L$ can be naturally identified with $(S \backslash KS) \setminus (S \backslash L)$, which has an $(S \backslash L)$-invariant (and hence $L$-invariant) Riemannian metric. Thus $\Gamma$ can be thought of as a subgroup of the group $I(KS \backslash L)$ of all the isometries of $KS \backslash L$ with respect to this Riemannian metric.

Suppose that $\Gamma$ is discrete in $I(KS \backslash L)$. Then the isotropy subgroup $\Gamma_v$ of $\Gamma$ at $v \in KS \backslash L$ is finite for each $v$, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS \backslash L)/\Gamma$ is $((K \backslash KSx)/H)/\Gamma_v$, where $v = KSx (x \in L)$. Thus a "fiber" of $q$ is homeomorphic to a quotient of the "general fiber" $K \backslash KS/H$ by an action of a finite group; i.e., $q$ is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup $S$.

Theorem 2. Let $L$ be a non-compact Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $G$ a lattice of $L$. Then there is an orbifold Seifert fibration
\[ K\backslash L/G \to \Omega^m, \]

where \( \Omega^m \) is a Riemannian orbifold of dimension \( m > 0 \) and of non-positive sectional curvature. If \( L \) is amenable, \( \Omega^m \) can be chosen to be flat.

The proof will occupy the following two sections. Some special cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group \( L \) with finitely many components is amenable if and only if \( L/R \) is compact, where \( R \) denotes the radical (= the unique maximal connected normal solvable subgroup) of \( L \). See Milnor [6]. In this section we handle the case when \( L \) is not amenable. We use \( R \) as \( S \), following [4]; i.e., we are going to show that

\[ K\backslash L/G \to KR\backslash L/G \]

is a Seifert fibration with the desired property. As in the previous section, identify \( KR\backslash L \) with \( (R\backslash KR)\backslash(R\backslash L) = R^m \) (\( m > 0 \)). \( R\backslash L \) is a non-compact semi-simple Lie group, and \( R\backslash KR \) is a maximal compact subgroup of \( R\backslash L \). Using the Cartan decomposition and the Killing form, one can introduce an
(R\L)-invariant (and hence L-invariant) Riemannian metric g on \( R^m \) with non-positive sectional curvature. In fact, any (R\L)-invariant Riemannian metric on \( R^m \) has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism \( \Phi: L \to I(R^m, g) \). Let \( \Gamma \) denote the image \( \Phi(G) \) of \( G \). To prove the theorem, it suffices to show that \( \Gamma \) is discrete in \( I(R^m, g) \). Let \( \tau \) denote the natural projection \( L \to R\backslash L \). If the image \( \tau(G) \) of \( G \) in \( R\backslash L \) is discrete, then \( \Gamma \) is obviously discrete. Unfortunately \( \tau(G) \) may not be discrete in general. We remedy this situation as follows.

Let \( L_0 \) denote the identity component of \( L \). \( G\backslash L_0 \) is a subgroup of \( G \) with finite index. Therefore it suffices to show that \( \Phi(G\backslash L_0) \) is discrete in \( I(R^m, g) \). As \( (R\backslash(G\backslash L_0)R)\backslash(R\backslash L_0) \) can be naturally identified with \( R^m \), we may assume from the beginning that \( L \) is connected.

Now there is a semi-simple Lie subgroup \( S \) of \( L \) such that \( L = SR \) and such that \( S\cap R \) is discrete (Levi decomposition). Let \( \sigma: S \to \text{Aut}(R) \) denote the action of \( S \) on \( R \). A sufficient condition for \( \tau(G) \) to be discrete in \( R\backslash L \) is that the identity component \( (\ker \sigma)_0 \) of the kernel of \( \sigma \) has no compact factors (Raghunathan[9], p.150). Let \( C \) denote the unique maximal compact normal subgroups of \( (\ker \sigma)_0 \). It is a characteristic subgroup of \( (\ker \sigma)_0 \), and hence it is normal in \( \ker \sigma \) and in \( S \). On the other hand, \( C \) commutes with elements of \( R \). Therefore \( C \) is normal in \( L \). Let \( \pi: L \to L/C \) denote the natural projection, and let \( L' = \pi(L), S' = \pi(S), R' = \pi(R), G' = \pi(G), K' = \pi(K) \). Then \( S \) is semi-simple, \( R' \) is the radical of \( L' \), \( G' \) is a lattice of \( L' \),

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and $K'$ is a maximal compact subgroup of $L'$. Let $\sigma':S' \to \text{Aut}R'$ denote the action of $S'$ on $R'$. Then it is easily observed that $\ker \sigma' = (\ker \sigma)/C$, since $C\cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image $G''$ of $G'$ in $R'\setminus L'$ is discrete. Thus the action of $G$ on $R^m$ factors through a properly discontinuous action of $G''$ on $K'R'\setminus L = KR\setminus L$. Therefore, $\Gamma$ is discrete in $I(R^m, g)$. This completes the proof of theorem 2 when $L$ is not amenable.

Remark. Let $q:K\setminus L/G \to KR\setminus L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KR\times G)$ over the point $KR\times G \subset KR\setminus L/G \ (x \in L)$ is homeomorphic to

$$(x^{-1}Kx)\setminus (x^{-1}KRx)/(x^{-1}KRx \cap G).$$

It is easily observed that $x^{-1}KRx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $R$ the radical of $L$, and $G$ a lattice of $L$. Then $KR \cap G$ is a uniform lattice of $KR$.

§3. Amenable case

Now let us assume that $L$ is non-compact and amenable. Let $K$ be a maximal compact subgroup and $R$ the radical of $L$ as
before. Since \( L \) is amenable, \( L = KR \).

We define a sequence \( N^{(j)} \) \( (j \geq -1) \) of closed characteristic subgroups of \( L \) as follows:

1. \( N^{(-1)} \) is the radical \( R \),
2. \( N^{(0)} \) is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of \( L \),
3. \( N^{(j)} \) is the commutator subgroup \([N^{(j-1)}, N^{(j-1)}]\) of \( N^{(j-1)} \), for \( j > 0 \).

It may not be so obvious that \( N^{(j)} \)'s are closed when \( j > 0 \); in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer \( k \) such that \( N^{(k)} = \{1\} \).

Consider the following sequence:

\[
L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \ldots \supset KN^{(k)} = K.
\]

There exists an integer \( i \) \( (i \geq 0) \) such that

\[
L = KN^{(-1)} = KN^{(0)} = \ldots = KN^{(i-1)} \neq KN^{(i)},
\]

because \( L \) is non-compact. Let us write \( M = N^{(i-1)} \) and \( N = N^{(i)} \). We introduce a flat \( L \)-invariant Riemannian metric on \( KN/L \).

Let us study the action of \( L \) on \( KN/L \) defined by right multiplication. An element \( ky \) of \( KM = L \) \( (k \in K, y \in M) \) acts on an element \( KNx \) \( (x \in M) \) of \( KN/L \) as follows:

\[
KNx \cdot (ky) = KNxky
= KN(k^{-1}xk)y.
\]
Note that we have $[M,M] \subset N$; we identify the coset space $KNL$ with the simply-connected abelian Lie group $(KMN)N/M = \mathbb{R}^m$ $(m>0)$. Now the induced action of $L$ on $\mathbb{R}^m$ is:

$$(KMN)N \cdot (ky) = (KMN)N(k^{-1}xk)y.$$ 

The following are easily observed: (1) this action, when restricted to $K$, defines a homomorphism $\alpha: K \to \text{Aut}(\mathbb{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbb{R}^m$, and (2) if $k \in KMN$, then $(KMN)N \cdot k = (KMN)Nk^{-1}xk = (KMN)N(k^{-1}xk^{-1})x = (KMN)Nx$ for $x \in M$, and so $KMN$ acts trivially on $\mathbb{R}^m$. Let $\beta: M \to (KMN)NM$ denote the natural projection. We now define a map $\Phi: L = KM \to \alpha(K) \ltimes ((KMN)N\backslash M) \subset O(m) \ltimes \mathbb{R}^m = I(\mathbb{R}^m)$ by sending $ky$ ($k \in K$, $y \in M$) to $(\alpha(k), \beta(y)) \in O(m) \ltimes \mathbb{R}^m$. This is a well-defined homomorphism. Here $\ltimes$'s denote the obvious semi-direct products. Let $\Gamma$ denote the image of $G$ by $\Phi$ in $I(\mathbb{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that $\Gamma$ is a discrete subgroup of $I(\mathbb{R}^m)$. To do this we use the following lemma:

Lemma 4. If $N$ is a connected nilpotent Lie group and $H$ is a discrete cocompact subgroup of $N$, then the commutator subgroup $[N,N]$ is closed in $N$ and $H \cap [N,N]$ is cocompact in $[N,N]$.

Proof: This is well-known if $N$ is simply-connected, so consider the universal cover $p: U \to N$ of $N$; it can be identified with the natural projection $U \to U/\Pi$, where $\Pi$ is the kernel of
p. To see that $[N,N]$ is closed in $N$, it suffices to show that $N/[N,N]$ is Hausdorff. As $p^{-1}(N,N) = \pi[U,U]$, we have homeomorphisms:

$$N/[N,N] \cong U/\pi[U,U]$$

$$\cong (U/[U,U])/(\pi[U,U]/[U,U])$$.

Here $U/[U,U]$ is a Lie group, because $U$ is simply-connected and hence its commutator subgroup $[U,U]$ is closed. Note that the preimage $p^{-1}(H)$ of $H$ is discrete and cocompact in $U$. Since $U$ is simply-connected, $p^{-1}(H)\cap [U,U]$ is cocompact in $[U,U]$. Therefore, the image $p^{-1}(H)[U,U]/[U,U]$ of $p^{-1}(H)$ by projection: $U \to U/[U,U]$ is discrete. As $\pi \subset p^{-1}(H)$, $\pi[U,U]/[U,U]$ is also discrete and hence closed in $U/[U,U]$. Therefore $(U/[U,U])/(\pi[U,U]/[U,U])$ is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

$$[N,N]/H\cap [N,N] \cong \pi[U,U]/p^{-1}(H)\cap [U,U]$$

$$\cong [U,U]/p^{-1}(H)\cap [U,U]$$,

the second statement is obvious.  

Now we prove

Lemma 5. $N^{(j)}$s are closed subgroups of $L$, and $\Gamma$ is a crystallographic subgroup of $I(\mathbb{R}^m)$.

Proof: If $G\cap R$ is cocompact in $R=N^{(-1)}$, then $G\cap N^{(0)}$ is a discrete cocompact subgroup of $N^{(0)}$ and we can apply lemma 4 to
prove that $N^{(j)},$'s are closed for $j \geq 1$. Unfortunately, $G\cap R$ may not be cocompact in $R$, in general. To remedy this situation we introduce a quotient Lie group $L'$ of $L$ as in the previous section. We may assume that $L$ is connected. We have Levi decomposition $L = SR$, where $S$ is a connected semi-simple (and hence compact) subgroup, $R$ is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma : S \to \text{Aut}(R)$ denote the action of $S$ on $R$. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of $L$, because it commutes with elements of $R$. In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi : L \to L/\ker \sigma_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then $G'$ is a lattice of $L'$, $K'$ is a maximal compact subgroup of $L'$, $S'$ is a semi-simple subgroup of $L'$, $R'$ is the radical of $L'$, and the action $\sigma' : S' \to \text{Aut}(R')$ of $S'$ on $R'$ is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N'_{(j)} (j \geq -1)$ of characteristic subgroups of $L'$ by:

1. $N'_{(-1)} = R'$
2. $N'_{(0)} =$ the nil-radical of $L'$
3. $N'_{(j)} = [N'_{(j-1)}, N'_{(j-1)}]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N'_{(0)}$ are cocompact in $R'$ and $N'_{(0)}$ respectively. By successively using lemma 4, we know that all $N'_{(j)},$'s are closed. Note that $\pi|_R : R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi|_R)^{-1}(N'_{(j)})$ for each $j$. Therefore $N^{(j)},$'s are closed in $L$. 

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Next, we show that \( \Gamma \) is a discrete cocompact subgroup of \( I(\mathbb{R}^m) \). Note that we have

\[
L' = K'N'^{(-1)} = K'N'(0) = \ldots = K'N'^(i-1) \neq K'N'^(i)
\]

for the same \( i \) and that \( K'N'\backslash K'M' = KN\backslash KM \), where \( M' = N'^(i-1) \), \( N' = N'^(i) \). \( G'\cap N'^{(j)} \) is cocompact in \( N'^{(j)} \) for all \( j \). In particular \( G'\cap M' \) is cocompact in \( M' \). So the image of \( G' \) in \( M'\backslash L' \) is discrete; furthermore, it is finite, because \( M'\backslash L' \) is compact. Looking at the diagram:

\[
\begin{array}{cccccc}
G & \longrightarrow & \text{finite} \\
\cap & \cap & \\
\pi^{-1}(M') & \longrightarrow & L & \longrightarrow & \pi^{-1}(M')\backslash L \\
\downarrow & & \downarrow \pi & \downarrow \cong & \\
M' & \longrightarrow & L' & \longrightarrow & M'\backslash L' \\
\cup & \cup & \\
G' & \longrightarrow & \text{finite}
\end{array}
\]

we know that \( G\pi^{-1}(M') \) has a finite index in \( G \). So it suffices to show that the image \( \Phi(G\pi^{-1}(M')) \) is a discrete cocompact subgroup of \( I(\mathbb{R}^m) \). As \( \ker \sigma \subset \ker \alpha \), \( \Phi \) sends elements in \( \pi^{-1}(M') = (\ker \sigma)_{0}M \) to elements in \( \mathbb{R}^m \subset I(\mathbb{R}^m) \). Now consider the following commutative diagram:

\[
\begin{array}{cccccc}
\Phi & \longrightarrow & O(m)\times \mathbb{R}^m \\
\cap & \cap & \cup & \cup & \\
G\pi^{-1}(M') \subset \pi^{-1}(M') = (\ker \sigma)_{0}M & \Phi & \longrightarrow & \mathbb{R}^m = (K\cap M)\backslash M
\end{array}
\]

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where \( \phi' \) is the natural map and \( (\pi|M)_* \) is the map induced by the restriction of \( \pi \) to \( M, \pi|M : M \rightarrow M' \). \( K \cap M \) and \( K' \cap M' \) are maximal compact subgroups of \( M \) and \( M' \), respectively, and \( \pi(K \cap M) = K' \cap M' \); therefore, \( (\pi|M)^{-1}(K' \cap M') = K \cap M \). Using this, it is easily verified that \( (\pi|M)^{-1}((K' \cap M')N') = (K \cap M)N \). Therefore \( (\pi|M)_* \) is an isomorphism. Since \( (G' \cap M') \cap N = G' \cap N' \) is cocompact in \( N' \), \( (G' \cap M') \cap (K' \cap M')N' \) is cocompact in \( (K' \cap M')N' \); so \( \phi'(G' \cap M') \) is a discrete cocompact subgroup of \( (K' \cap M')N' \cap M' \). Therefore \( \phi'(G' \cap M') \) is a discrete cocompact subgroup of \( \mathbb{R}^m \) (and hence in \( I(\mathbb{R}^m) \)). This completes the proof of lemma 5. \( \square \)

Thus \( K \backslash L/G \rightarrow KN \backslash L/G \) is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form \( K \backslash KN \times G/G \), and is homeomorphic to

\[
(x^{-1}Kx \backslash (x^{-1}KNx))/(x^{-1}KNx \cap G).
\]

If \( G \) is a lattice of \( L \) (which is automatically uniform), then \( x^{-1}KNx \cap G \) is a uniform lattice of \( x^{-1}KNx \).

§4. A rational computation of Wall's L-groups
Let $L$ be an amenable Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $G$ a uniform lattice of $L$. Such a discrete group $G$ is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the $L$-groups of $G$ in terms of certain generalized homology of $K\backslash L/G$.

$K\backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_v = x^{-1}Kx \cap G$ of $G$ at $v = Kx$ ($x \in L$) is finite. The action of $G$ on $\mathbb{R}^n$ is free if $G$ is torsion-free; in general, $\mathbb{R}^n/G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let $WG$ be a contractible free $G$-complex, and $p$ denote the projection: $(\mathbb{R}^n \times WG)/G \to \mathbb{R}^n/G$, where $G$ acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y] \in \mathbb{R}^n/G$ by $p$ is homeomorphic to $WG/G_y$, and $p$ is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^{-\omega}(G)$ denote the limit of Ranicki's lower $L$-groups $L^{(-j)}(\mathbb{Z}G)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $L^{-\omega}(\_)$ from the category of spaces to the category of $\Omega$-spectra such that the homotopy group of $L^{-\omega}(X)$ is equal to $L^{-\omega}(\pi_1 X)$. Applying $L^{-\omega}(\_)$ to each fiber of $p$, we obtain a sheaf of spectra, denoted $L^{-\omega}(p)$. F. Quinn defines the homology group $H_*(\mathbb{R}^n/G; L^{-\omega}(p))$ of $\mathbb{R}^n/G$ with coefficients $L^{-\omega}(p)$. See [7], [10]. The following is a rational computation of $L^{-\omega}(G)$ in terms of this homology.
Theorem 6. Let $G$ be as above, then there is a natural isomorphism

$$H_* (\mathbb{R}^n/G; \mathbb{L}^{-\infty}(p)) \otimes \mathbb{Z}[1/2] \to L_*^{-\infty}(G) \otimes \mathbb{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations.

$$\begin{array}{ccc}
\mathbb{R}^n/G & \overset{id.}{\longrightarrow} & \mathbb{R}^n/G
\\
\downarrow p & & \downarrow
\\
\mathbb{R}^n/G & \longrightarrow & \text{pt.}
\end{array}$$

Note that $(\mathbb{R}^n \times \mathbb{W}G)/G = BG$ is a classifying space for $G$ and that $H_* (\text{pt.}; \mathbb{L}^{-\infty}(BG \rightarrow \text{pt.})) = L_*^{-\infty}(G)$ [10].

It is to be noted that theorem 6 says that the $\mathbb{L}^{-\infty}(p)$ coefficient homology of $\mathbb{R}^n/G$ is independent (modulo 2 torsion) of the action of $G$ on $\mathbb{R}^n$. It is conceivable that the orbifold $\mathbb{R}^n/G$ has a certain strong rigidity.

Proof of Theorem 6. The proof is by induction on the dimension n of $K \setminus L$. Let $q: \mathbb{R}^n/G \to \mathbb{R}^m/\Gamma$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$H_* (\mathbb{R}^n/G; \mathbb{L}^{-\infty}(p))$$

$$\cong H_* (\mathbb{R}^m/\Gamma; \bigcup_{w \in \mathbb{R}^m/\Gamma} H(q^{-1}(w); \mathbb{L}^{-\infty}(p|q^{-1}(w))))$$

$$\cong H_* (\mathbb{R}^m/\Gamma; \bigcup_{w} L^{-\infty}((qp)^{-1}(w)))$$

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\[ = H_* (\mathbb{R}^n / G ; L^{-\infty}(q\mathbb{P})) \]

by induction hypothesis, where \(H\) denote the homology theory spectrum [ibid.]. We can prove that \(H_* (\mathbb{R}^n / G ; L^{-\infty}(q\mathbb{P})) \otimes \mathbb{Z}[1/2]\) is naturally isomorphic to \(L^{-\infty}_\mathbb{R}(G)\) using the proof of the main theorem of [ibid.], with only some obvious modifications, and this completes the proof of theorem 6. \(\square\)

Corollary 7. (Novikov Conjecture) Let \(G\) be as above, then the assembly map

\[ H_* (BG; L^{-\infty}(1)) \rightarrow L^{-\infty}_\mathbb{R}(G) \]

is rationally split injective.

References


