TITLE:
LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

AUTHOR(S):
Yamasaki, Masayuki

CITATION:
Yamasaki, Masayuki. LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS. 数理解析研究所講究録 1987, 633: 132-147

ISSUE DATE:
1987-10

URL:
http://hdl.handle.net/2433/100058

RIGHT:
LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

Masayuki Yamasaki

§1. Introduction

Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $S$ a connected closed normal subgroup of $L$. Then $KS$ is closed, and we have a fiber bundle

$$K\backslash KS \to K\backslash L \to KS\backslash L.$$ 

$L$ acts on $K\backslash L$ by right multiplication. $L$ acts also on $KS\backslash L$ by right multiplication; let $N$ denote the kernel of this action, i.e., $N = \{ g \in L; KSxg = KSx \text{ for all } x \in L \}$. The action of $N$ on $K\backslash L$ leaves all fibers invariant; in other words, we have a family of right $N$-spaces parametrized over $KS\backslash L$.

Lemma 1. The right $N$-spaces $K\backslash KSx$ ($x \in L$) are equivalent.

Proof. Since $K$ is compact, $K\backslash L$ has an $L$-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy$ ($x, y \in L$). It suffices to construct an $N$-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.
Fix a point $p$ of $K\backslash KSx$ and let $d$ be the distance between $p$ and $K\backslash KSy$. $K\backslash L$ is complete and $K\backslash KSy$ is closed; therefore, $d$ is positive and can be achieved as the length of a geodesic $\gamma$ connecting $p$ and a point $q$ of $K\backslash KSy$. $S$ is contained in $N$ and acts transitively on each fiber. The action of an element $s$ of $S$ sends $\gamma$ to a geodesic $\gamma \cdot s$ of the same length $d$ connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $K\backslash KSx$ to $K\backslash KSy$ is independent of the choice of the point, and $\gamma$ is one of the shortest geodesics connecting $K\backslash KSx$ and $K\backslash KSy$. Therefore $\gamma$ is perpendicular to $K\backslash KSx$ at $p$. Let $(T_p(K\backslash KSx))^\perp$ denote the orthogonal complement of the tangent space $T_p(K\backslash KSx)$ of $K\backslash KSx$ at $p$ in the tangent space of $K\backslash L$ at $p$. As the exponential map $\text{Exp}$ is a diffeomorphism near the origin, any fiber $K\backslash KSz$ that meets $\text{Exp}(V)$ meets $\text{Exp}(V)$ exactly once, where $V$ is a sufficiently small neighborhood in $(T_p(K\backslash KSx))^\perp$ of the origin. This implies that $\gamma$ is the unique geodesic of length $d$ connecting $p$ and $K\backslash KSy$, as long as $K\backslash KSy$ is sufficiently close to $K\backslash KSx$. Let us suppose that this is the case. Then the correspondence $p \cdot s \leftrightarrow q \cdot s$ ($s \in S$) defines a diffeomorphism $K\backslash KSx \rightarrow K\backslash KSy$, which is obviously $N$-equivariant because it sends a point in $K\backslash KSx$ to the unique point closest to it in $K\backslash KSy$ and $N$ acts on $K\backslash L$ by isometries. □

Remark. The $N$-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $K\backslash L \rightarrow KS\backslash L$ so that the action of $N$ on $K\backslash L$ is locally a product of the action of $N$ on a fiber and the action of a trivial group on the base.

If $G$ is a lattice of $L$, the action of $L$ on $K\backslash L$ restricts
to an action of $G$ on $K/L$. $H = G \cap N$ is a normal subgroup of $G$ which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K\setminus KS/H \to K\setminus L/H \to KS/L.$$ 

The quotient group $\Gamma = G/H$ acts on $K\setminus L/H$ and $KS/L$ such that $(K\setminus L/H)/\Gamma = K\setminus L/G$ and $(KS/L)/\Gamma = KS/L/G$; the fiber bundle map induces a map:

$$q : K\setminus L/G \to (KS/L)/\Gamma.$$ 

Note that $KS/L$ can be naturally identified with $(S\setminus KS)/(S\setminus L)$, which has an $(S\setminus L)$-invariant (and hence $L$-invariant) Riemannian metric. Thus $\Gamma$ can be thought of as a subgroup of the group $I(KS/L)$ of all the isometries of $KS/L$ with respect to this Riemannian metric.

Suppose that $\Gamma$ is discrete in $I(KS/L)$. Then the isotropy subgroup $\Gamma_v$ of $\Gamma$ at $v \in KS/L$ is finite for each $v$, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS/L)/\Gamma$ is $((K\setminus KSx)/H)/\Gamma_v$, where $v = KSx (x \in L)$. Thus a "fiber" of $q$ is homeomorphic to a quotient of the "general fiber" $K\setminus KS/H$ by an action of a finite group; i.e., $q$ is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup $S$.

Theorem 2. Let $L$ be a non-compact Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $G$ a lattice of $L$. Then there is an orbifold Seifert fibration.
where $O^m$ is a Riemannian orbifold of dimension $m > 0$ and of non-positive sectional curvature. If $L$ is amenable, $O^m$ can be chosen to be flat.

The proof will occupy the following two sections. Some social cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group $L$ with finitely many components is amenable if and only if $L/R$ is compact, where $R$ denotes the radical (= the unique maximal connected normal solvable subgroup) of $L$. See Milnor [6]. In this section we handle the case when $L$ is not amenable. We use $R$ as $S$, following [4]; i.e., we are going to show that

$$K\backslash L/G \to K\backslash R\backslash L/G$$

is a Seifert fibration with the desired property. As in the previous section, identify $K\backslash R\backslash L$ with $(R\backslash KR)\backslash (R\backslash L) = R^m$ ($m > 0$). $R\backslash L$ is a non-compact semi-simple Lie group, and $R\backslash KR$ is a maximal compact subgroup of $R\backslash L$. Using the Cartan decomposition and the Killing form, one can introduce an
(R \backslash L)-invariant (and hence L-invariant) Riemannian metric g on $\mathbb{R}^m$ with non-positive sectional curvature. In fact, any (R \backslash L)-invariant Riemannian metric on $\mathbb{R}^m$ has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism $\Phi: L \to I(\mathbb{R}^m, g)$. Let $\Gamma$ denote the image $\Phi(G)$ of $G$. To prove the theorem, it suffices to show that $\Gamma$ is discrete in $I(\mathbb{R}^m, g)$. Let $\tau$ denote the natural projection $L \to R \backslash L$. If the image $\tau(G)$ of $G$ in $R \backslash L$ is discrete, then $\Gamma$ is obviously discrete. Unfortunately $\tau(G)$ may not be discrete in general. We remedy this situation as follows.

Let $L_0$ denote the identity component of $L$. $G \cap L_0$ is a subgroup of $G$ with finite index. Therefore it suffices to show that $\Phi(G \cap L_0)$ is discrete in $I(\mathbb{R}^m, g)$. As $(R \backslash (K \cap L_0)R) \backslash (R \backslash L_0)$ can be naturally identified with $\mathbb{R}^m$, we may assume from the beginning that $L$ is connected.

Now there is a semi-simple Lie subgroup $S$ of $L$ such that $L = SR$ and such that $S \cap R$ is discrete (Levi decomposition). Let $\sigma: S \to \text{Aut}(R)$ denote the action of $S$ on $R$. A sufficient condition for $\tau(G)$ to be discrete in $R \backslash L$ is that the identity component $(\ker \sigma)_0$ of the kernel of $\sigma$ has no compact factors (Raghunathan[9], p.150). Let $C$ denote the unique maximal compact normal subgroups of $(\ker \sigma)_0$. It is a characteristic subgroup of $(\ker \sigma)_0$, and hence it is normal in $\ker \sigma$ and in $S$. On the other hand, $C$ commutes with elements of $R$. Therefore $C$ is normal in $L$. Let $\pi: L \to L/C$ denote the natural projection, and let $L' = \pi(L)$, $S' = \pi(S)$, $R' = \pi(R)$, $G' = \pi(G)$, $K' = \pi(K)$. Then $S$ is semi-simple, $R'$ is the radical of $L'$, $G'$ is a lattice of $L'$.
and $K'$ is a maximal compact subgroup of $L'$. Let $\sigma' : S' \to \text{Aut}R'$ denote the action of $S'$ on $R'$. Then it is easily observed that \( \ker \sigma' = (\ker \sigma)/C \), since $C \cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image $G''$ of $G'$ in $R' \setminus L'$ is discrete. Thus the action of $G$ on $R^m$ factors through a properly discontinuous action of $G''$ on $K' \setminus R' \setminus L = KR \setminus L$. Therefore, $\Gamma$ is discrete in $I(R^m, g)$. This completes the proof of theorem 2 when $L$ is not amenable.

Remark. Let $q : K \setminus L/G \to KR \setminus L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRxG)$ over the point $KRxG \in KR \setminus L/G$ ($x \in L$) is homeomorphic to

\[(x^{-1}Kx) \setminus (x^{-1}KRx)/(x^{-1}Kx \cap G).\]

It is easily observed that $x^{-1}Kx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $R$ the radical of $L$, and $G$ a lattice of $L$. Then $KR \cap G$ is a uniform lattice of $KR$.

§3. Amenable case

Now let us assume that $L$ is non-compact and amenable. Let $K$ be a maximal compact subgroup and $R$ the radical of $L$ as
before. Since $L$ is amenable, $L = KR$.

We define a sequence $N^{(j)}$ ($j \geq -1$) of closed characteristic subgroups of $L$ as follows:

1. $N^{(-1)}$ is the radical $R$,
2. $N^{(0)}$ is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of $L$,
3. $N^{(j)}$ is the commutator subgroup $[N^{(j-1)}, N^{(j-1)}]$ of $N^{(j-1)}$, for $j > 0$.

It may not be so obvious that $N^{(j)}$'s are closed when $j > 0$; in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer $k$ such that $N^{(k)} = \{1\}$. Consider the following sequence:

$$L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \ldots \supset KN^{(k)} = K.$$ 

There exists an integer $i$ ($\geq 0$) such that

$$L = KN^{(-1)} = KN^{(0)} = \ldots = KN^{(i-1)} \neq KN^{(i)},$$

because $L$ is non-compact. Let us write $M = N^{(i-1)}$ and $N = N^{(i)}$. We introduce a flat $L$-invariant Riemannian metric on $KN/L$.

Let us study the action of $L$ on $KN/L$ defined by right multiplication. An element $ky$ of $KM = L$ ($k \in K$, $y \in M$) acts on an element $KNx$ ($x \in M$) of $KN/L$ as follows:

$$KNx \cdot (ky) = KNxky = KN(k^{-1}xk)y.$$
Note that we have $[M, M] \subset N$; we identify the coset space $KN/L$ with the simply-connected abelian Lie group $(K/M)N/M = \mathbb{R}^m$ ($m > 0$). Now the induced action of $L$ on $\mathbb{R}^m$ is:

$$(K/M)N \cdot (ky) = (K/M)N(k^{-1}x)k y.$$ 

The following are easily observed: (1) this action, when restricted to $K$, defines a homomorphism $\alpha: K \to \text{Aut}(\mathbb{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbb{R}^m$, and (2) if $k \in K/M$, then $(K/M)Nk^{-1}x = (K/M)Nk k^{-1} x = (K/M)Nx$ for $x \in M$, and so $K/M$ acts trivially on $\mathbb{R}^m$. Let $\beta: M \to (K/M)N/M$ denote the natural projection. We now define a map $\Phi: L = KM \to \alpha(K) \ltimes ((K/M)N/M) \subset O(m) \ltimes \mathbb{R}^m = I(\mathbb{R}^m)$ by sending $k y$ ($k \in K$, $y \in M$) to $(\alpha(k), \beta(y)) \in O(m) \ltimes \mathbb{R}^m$. This is a well-defined homomorphism.

Here $\ltimes$'s denote the obvious semi-direct products. Let $\Gamma$ denote the image of $G$ by $\Phi$ in $I(\mathbb{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that $\Gamma$ is a discrete subgroup of $I(\mathbb{R}^m)$. To do this we use the following lemma:

**Lemma 4.** If $N$ is a connected nilpotent Lie group and $H$ is a discrete cocompact subgroup of $N$, then the commutator subgroup $[N, N]$ is closed in $N$ and $H \cap [N, N]$ is cocompact in $[N, N]$.

**Proof:** This is well-known if $N$ is simply-connected, so consider the universal cover $p: U \to N$ of $N$; it can be identified with the natural projection $U \to U/\Pi$, where $\Pi$ is the kernel of
p. To see that \([N,N]\) is closed in \(N\), it suffices to show that \(N/[N,N]\) is Hausdorff. As \(p^{-1}([N,N]) = \Pi [U,U]\), we have homeomorphisms:

\[
N/[N,N] \cong U/\Pi [U,U] \\
\cong (U/[U,U])/\langle \Pi [U,U]/[U,U] \rangle.
\]

Here \(U/[U,U]\) is a Lie group, because \(U\) is simply-connected and hence its commutator subgroup \([U,U]\) is closed. Note that the preimage \(p^{-1}(H)\) of \(H\) is discrete and cocompact in \(U\). Since \(U\) is simply-connected, \(p^{-1}(H)\cap [U,U]\) is cocompact in \([U,U]\). Therefore, the image \(p^{-1}(H)[U,U]/[U,U]\) of \(p^{-1}(H)\) by projection: \(U \rightarrow U/[U,U]\) is discrete. As \(\Pi \subset p^{-1}(H)\), \(\Pi [U,U]/[U,U]\) is also discrete and hence closed in \(U/[U,U]\). Therefore \((U/[U,U])/\langle \Pi [U,U]/[U,U]\rangle\) is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

\[
[N,N]/\Pi [N,N] \cong [U,U]/p^{-1}(H)\Pi [U,U] \\
\cong [U,U]/p^{-1}(H)\cap [U,U],
\]

the second statement is obvious. \(\square\)

Now we prove

**Lemma 5.** \(N^{(j)}\)'s are closed subgroups of \(L\), and \(\Gamma\) is a crystallographic subgroup of \(I(\mathbb{R}^m)\).

**Proof:** If \(G\cap R\) is cocompact in \(R^m\), then \(G\cap N^{(0)}\) is a discrete cocompact subgroup of \(N^{(0)}\) and we can apply lemma 4 to
prove that $N^{(j)}$'s are closed for $j \geq 1$. Unfortunately, $G \cap R$ may not be cocompact in $R$, in general. To remedy this situation we introduce a quotient Lie group $L'$ of $L$ as in the previous section. We may assume that $L$ is connected. We have Levi decomposition $L = SR$, where $S$ is a connected semi-simple (and hence compact) subgroup, $R$ is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma : S \to \text{Aut}(R)$ denote the action of $S$ on $R$. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of $L$, because it commutes with elements of $R$. In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi : L \to L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then $G'$ is a lattice of $L'$, $K'$ is a maximal compact subgroup of $L'$, $S'$ is a semi-simple subgroup of $L'$, $R'$ is the radical of $L'$, and the action $\sigma' : S' \to \text{Aut}(R')$ of $S'$ on $R'$ is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N^{(j)}$ ($j \geq -1$) of characteristic subgroups of $L'$ by:

1. $N^{(-1)} = R'$
2. $N^{(0)} = \text{the nil-radical of } L'$
3. $N^{(j)} = [N^{(j-1)}, N^{(j-1)}]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N^{(0)}$ are cocompact in $R'$ and $N^{(0)}$ respectively. By successively using lemma 4, we know that all $N^{(j)}$'s are closed. Note that $\pi | R : R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi | R)^{-1}(N^{(j)})$ for each $j$. Therefore $N^{(j)}$'s are closed in $L$. 

- 10 -
Next, we show that $\Gamma$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. Note that we have

$$L' = K'N'\langle -1 \rangle = K'N'\langle 0 \rangle = \ldots = K'N'\langle i-1 \rangle \neq K'N'\langle i \rangle$$

for the same $i$ and that $K'N'\backslash K'M' = KN\backslash KM$, where $M' = N'\langle i-1 \rangle$, $N' = N'\langle i \rangle$. $G'\cap N'\langle j \rangle$ is cocompact in $N'\langle j \rangle$ for all $j$. In particular $G'\cap M'$ is cocompact in $M'$. So the image of $G'$ in $M'\backslash L'$ is discrete; furthermore, it is finite, because $M'\backslash L'$ is compact. Looking at the diagram:

we know that $G\cap \pi^{-1}(M')$ has a finite index in $G$. So it suffices to show that the image $\phi(G\cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. As $\ker\sigma \subset \ker\alpha$, $\phi$ sends elements in $\pi^{-1}(M') = (\ker\sigma)_0 M$ to elements in $\mathbb{R}^m \subset I(\mathbb{R}^m)$. Now consider the following commutative diagram:

$$\begin{array}{ccc}
G \cap \pi^{-1}(M') & \subset & \pi^{-1}(M') = (\ker\sigma)_0 M \\
\downarrow & & \downarrow \\
G' & \longrightarrow & \text{finite}
\end{array}$$

$$\begin{array}{ccc}
G \cap \pi^{-1}(M') & \subset & \pi^{-1}(M') = (\ker\sigma)_0 M \\
\downarrow & & \downarrow \\
G' & \longrightarrow & \text{finite}
\end{array}$$

$$\begin{array}{ccc}
G \cap \pi^{-1}(M') & \subset & \pi^{-1}(M') = (\ker\sigma)_0 M \\
\downarrow & & \downarrow \\
G' & \longrightarrow & \text{finite}
\end{array}$$
where $\phi'$ is the natural map and $(\pi|M)_*$ is the map induced by the restriction of $\pi$ to $M$, $\pi|M: M \rightarrow M'$. $K\cap M$ and $K'\cap M'$ are maximal compact subgroups of $M$ and $M'$, respectively, and $\pi(K\cap M) = K'\cap M'$; therefore, $(\pi|M)^{-1}(K'\cap M') = K\cap M$. Using this, it is easily verified that $(\pi|M)^{-1}((K'\cap M')N') = (K\cap M)N$. Therefore $(\pi|M)_*$ is an isomorphism. Since $(G'\cap M')\cap N' = G'\cap N'$ is cocompact in $N'$, $(G'\cap M')\cap (K'\cap M')N'$ is cocompact in $(K'\cap M')N'$; so $\phi'(G'\cap M')$ is a discrete cocompact subgroup of $(K'\cap M')N'\backslash M'$. Therefore $\phi'(G'\cap M')$ is a discrete cocompact subgroup of $\mathbb{R}^m$ (and hence in $I(\mathbb{R}^m)$). This completes the proof of lemma 5. □

Thus $K\backslash L/G \rightarrow K\backslash N\backslash L/G$ is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form $K\backslash KnxG/G$, and is homeomorphic to

$$(x^{-1}Kx)\backslash (x^{-1}Knx)/(x^{-1}Knx \cap G).$$

If $G$ is a lattice of $L$ (which is automatically uniform), then $x^{-1}Knx \cap G$ is a uniform lattice of $x^{-1}Knx$.

§4. A rational computation of Wall's L-groups
Let $L$ be an amenable Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $G$ a uniform lattice of $L$. Such a discrete group $G$ is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the $L$-groups of $G$ in terms of certain generalized homology of $K \backslash L / G$.

$K \backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_x = x^{-1} K x \cap G$ of $G$ at $y = K x (x \in L)$ is finite. The action of $G$ on $\mathbb{R}^n$ is free if $G$ is torsion-free; in general, $\mathbb{R}^n / G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let $WG$ be a contractible free $G$-complex, and $p$ denote the projection: $(\mathbb{R}^n \times WG) / G \to \mathbb{R}^n / G$, where $G$ acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y] \in \mathbb{R}^n / G$ by $p$ is homeomorphic to $WG / G_y$, and $p$ is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^-\infty(G)$ denote the limit of Ranicki's lower $L$-groups $L^{(-j)}(\mathbb{Z}G)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $L^-\infty(-)$ from the category of spaces to the category of $\Omega$-spectra such that the homotopy group of $L^-\infty(X)$ is equal to $L^-\infty(\pi_1 X)$. Applying $L^-\infty(-)$ to each fiber of $p$, we obtain a sheaf of spectra, denoted $L^-\infty(p)$. F. Quinn defines the homology group $H_* (\mathbb{R}^n / G; L^-\infty(p))$ of $\mathbb{R}^n / G$ with coefficients $L^-\infty(p)$. See [7], [10]. The following is a rational computation of $L^-\infty(G)$ in terms of this homology.
Theorem 6. Let \( G \) be as above, then there is a natural isomorphism

\[
H_*(\mathbb{R}^n/G; L^{-\infty}(p)) \otimes \mathbb{Z}[1/2] \rightarrow L^{-\infty}(G) \otimes \mathbb{Z}[1/2].
\]

The map is induced by the following map between stratified systems of fibrations.

\[
\begin{array}{ccc}
\mathbb{R}^n \times \mathbb{W} / G & \xrightarrow{\text{id.}} & \mathbb{R}^n \times \mathbb{W} / G \\
\downarrow p & & \downarrow \\
\mathbb{R}^n / G & \rightarrow & \text{pt.}
\end{array}
\]

Note that \((\mathbb{R}^n \times \mathbb{W} / G) / G = B G\) is a classifying space for \( G \) and that \(H_*(\text{pt.} ; L^{-\infty}(B G \rightarrow \text{pt.})) = L^{-\infty}(G) [10]\).

It is to be noted that theorem 6 says that the \( L^{-\infty}(p) \) coefficient homology of \( \mathbb{R}^n / G \) is independent (modulo 2 torsion) of the action of \( G \) on \( \mathbb{R}^n \). It is conceivable that the orbifold \( \mathbb{R}^n / G \) has a certain strong rigidity.

Proof of theorem 6. The proof is by induction on the dimension \( n \) of \( K \setminus L \). Let \( q : \mathbb{R}^n / G \rightarrow \mathbb{R}^m / \Gamma \) denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

\[
H_*(\mathbb{R}^n / G; L^{-\infty}(p))
\]

\[
\cong H_*(\mathbb{R}^m / \Gamma; \bigcup_{\omega \in \mathcal{R}_m \Gamma} H(q^{-1}(\omega); L^{-\infty}(p | q^{-1}(\omega))))
\]

\[
\cong H_*(\mathbb{R}^m / \Gamma; \bigcup_{\omega} L^{-\infty}((q \circ p)^{-1}(\omega)))
\]
by induction hypothesis, where M denote the homology theory spectrum [ibid.]. We can prove that
\( H_n(\mathbb{R}^n/G; \mathbb{L}^{-\infty}(qp)) \otimes \mathbb{Z}[1/2] \)
is naturally isomorphic to \( L^{-\infty}_*(G) \) using the proof of the main theorem of [ibid.] with only some obvious modifications, and
this completes the proof of theorem 6. \( \square \)

Corollary 7. (Novikov Conjecture) Let \( G \) be as above, then
the assembly map

\[ H_*^{\text{as}}(BG; \mathbb{L}^{-\infty}(1)) \to L^{-\infty}_*(G) \]
is rationally split injective.

References


