LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

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$\S 1$. Introduction

Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $S$ a connected closed normal subgroup of $L$. Then $KS$ is closed, and we have a fiber bundle

$$K\backslash KS \rightarrow K\backslash L \rightarrow KS\backslash L.$$ 

$L$ acts on $K\backslash L$ by right multiplication. $L$ acts also on $KS\backslash L$ by right multiplication; let $N$ denote the kernel of this action, i.e., $N = \{ g \in L; KSxg = KSx \text{ for all } x \in L \}$. The action of $N$ on $K\backslash L$ leaves all fibers invariant; in other words, we have a family of right $N$-spaces parametrized over $KS\backslash L$.

Lemma 1. The right $N$-spaces $K\backslash KSx \ (x \in L)$ are equivalent.

Proof. Since $K$ is compact, $K\backslash L$ has an $L$-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy \ (x, y \in L)$. It suffices to construct an $N$-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.
Fix a point $p$ of $\mathbb{K}\backslash\mathbb{K}Sx$ and let $d$ be the distance between $p$ and $\mathbb{K}\backslash\mathbb{K}Sy$. $\mathbb{K}\backslash L$ is complete and $\mathbb{K}\backslash\mathbb{K}Sy$ is closed; therefore, $d$ is positive and can be achieved as the length of a geodesic $\gamma$ connecting $p$ and a point $q$ of $\mathbb{K}\backslash\mathbb{K}Sy$. $S$ is contained in $N$ and acts transitively on each fiber. The action of an element $s$ of $S$ sends $\gamma$ to a geodesic $\gamma \cdot s$ of the same length $d$ connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $\mathbb{K}\backslash\mathbb{K}Sx$ to $\mathbb{K}\backslash\mathbb{K}Sy$ is independent of the choice of the point, and $\gamma$ is one of the shortest geodesics connecting $\mathbb{K}\backslash\mathbb{K}Sx$ and $\mathbb{K}\backslash\mathbb{K}Sy$. Therefore $\gamma$ is perpendicular to $\mathbb{K}\backslash\mathbb{K}Sx$ at $p$. Let $(T_p(\mathbb{K}\backslash\mathbb{K}Sx))^\perp$ denote the orthogonal complement of the tangent space $T_p(\mathbb{K}\backslash\mathbb{K}Sx)$ of $\mathbb{K}\backslash\mathbb{K}Sx$ at $p$ in the tangent space of $\mathbb{K}\backslash L$ at $p$. As the exponential map $\text{Exp}$ is a diffeomorphism near the origin, any fiber $\mathbb{K}\backslash\mathbb{K}Sz$ that meets $\text{Exp}(V)$ meets $\text{Exp}(V)$ exactly once, where $V$ is a sufficiently small neighborhood in $(T_p(\mathbb{K}\backslash\mathbb{K}Sx))^\perp$ of the origin. This implies that $\gamma$ is the unique geodesic of length $d$ connecting $p$ and $\mathbb{K}\backslash\mathbb{K}Sy$, as long as $\mathbb{K}\backslash\mathbb{K}Sy$ is sufficiently close to $\mathbb{K}\backslash\mathbb{K}Sx$. Let us suppose that this is the case. Then the correspondence $p \cdot s \mapsto q \cdot s$ ($s \in S$) defines a diffeomorphism $\mathbb{K}\backslash\mathbb{K}Sx \to \mathbb{K}\backslash\mathbb{K}Sy$, which is obviously $N$-equivariant because it sends a point in $\mathbb{K}\backslash\mathbb{K}Sx$ to the unique point closest to it in $\mathbb{K}\backslash\mathbb{K}Sy$ and $N$ acts on $\mathbb{K}\backslash L$ by isometries. \( \square \)

Remark. The $N$-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $\mathbb{K}\backslash L \to \mathbb{K}S\backslash L$ so that the action of $N$ on $\mathbb{K}\backslash L$ is locally a product of the action of $N$ on a fiber and the action of a trivial group on the base.

If $G$ is a lattice of $L$, the action of $L$ on $\mathbb{K}\backslash L$ restricts
to an action of $G$ on $K \backslash L$. $H = G \cap N$ is a normal subgroup of $G$ which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K \backslash KS/H \to K \backslash L/H \to KS/L.$$ 

The quotient group $\Gamma = G/H$ acts on $K \backslash L/H$ and $KS/L$ such that $(K \backslash L/H)/\Gamma = K \backslash L/G$ and $(KS/L)/\Gamma = KS/L/G$; the fiber bundle map induces a map:

$$q: K \backslash L/G \to (KS/L)/\Gamma.$$ 

Note that $KS/L$ can be naturally identified with $(S \backslash KS) \backslash (S \backslash L)$, which has an $(S \backslash L)$-invariant (and hence $L$-invariant) Riemannian metric. Thus $\Gamma$ can be thought of as a subgroup of the group $I(KS/L)$ of all the isometries of $KS/L$ with respect to this Riemannian metric.

Suppose that $\Gamma$ is discrete in $I(KS/L)$. Then the isotropy subgroup $\Gamma_v$ of $\Gamma$ at $v \in KS/L$ is finite for each $v$, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS/L)/\Gamma$ is $((K \backslash KS)/H)/\Gamma_v$, where $v = KSx$ ($x \in L$). Thus a "fiber" of $q$ is homeomorphic to a quotient of the "general fiber" $K \backslash KS/H$ by an action of a finite group; i.e., $q$ is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup $S$.

**Theorem 2.** Let $L$ be a non-compact Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $G$ a lattice of $L$. Then there is an orbifold Seifert fibration
where $O_m$ is a Riemannian orbifold of dimension $m > 0$ and of non-positive sectional curvature. If $L$ is amenable, $O_m$ can be chosen to be flat.

The proof will occupy the following two sections. Some special cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group $L$ with finitely many components is amenable if and only if $L/R$ is compact, where $R$ denotes the radical (= the unique maximal connected normal solvable subgroup) of $L$. See Milnor [6]. In this section we handle the case when $L$ is not amenable. We use $R$ as $S$, following [4]; i.e., we are going to show that

$$K/L/G \rightarrow KR\backslash L/G$$

is a Seifert fibration with the desired property. As in the previous section, identify $KR\backslash L$ with $(R\backslash KR)\backslash (R\backslash L) = R^m (m > 0)$. $R\backslash L$ is a non-compact semi-simple Lie group, and $R\backslash KR$ is a maximal compact subgroup of $R\backslash L$. Using the Cartan decomposition and the Killing form, one can introduce an
(R\L)-invariant (and hence L-invariant) Riemannian metric g on R^m with non-positive sectional curvature. In fact, any (R\L)-invariant Riemannian metric on R^m has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism \phi: L \to I(R^m,g). Let \Gamma denote the image \phi(G) of G. To prove the theorem, it suffices to show that \Gamma is discrete in I(R^m,g). Let \tau denote the natural projection L \to R\setminus L. If the image \tau(G) of G in R\setminus L is discrete, then \Gamma is obviously discrete. Unfortunately \tau(G) may not be discrete in general. We remedy this situation as follows.

Let L_0 denote the identity component of L. G\cap L_0 is a subgroup of G with finite index. Therefore it suffices to show that \phi(G\cap L_0) is discrete in I(R^m,g). As (R\setminus (G\cap L_0)R\setminus (R\setminus L_0)) can be naturally identified with R^m, we may assume from the beginning that L is connected.

Now there is a semi-simple Lie subgroup S of L such that L = SR and such that S\cap R is discrete (Levi decomposition). Let \sigma: S \to \text{Aut}(R) denote the action of S on R. A sufficient condition for \tau(G) to be discrete in R\setminus L is that the identity component (ker \sigma)_0 of the kernel of \sigma has no compact factors (Raghunathan[9], p.150). Let C denote the unique maximal compact normal subgroups of (ker \sigma)_0. It is a characteristic subgroup of (ker \sigma)_0, and hence it is normal in ker \sigma and in S. On the other hand, C commutes with elements of R. Therefore C is normal in L. Let \pi: L \to L/C denote the natural projection, and let L' = \pi(L), S' = \pi(S), R' = \pi(R), G' = \pi(G), K' = \pi(K). Then S is semi-simple, R' is the radical of L', G' is a lattice of L',

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and $K'$ is a maximal compact subgroup of $L'$. Let $\sigma': S' \to \text{Aut} R'$ denote the action of $S'$ on $R'$. Then it is easily observed that $\ker \sigma' = (\ker \sigma)/C$, since $C \cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image $G''$ of $G'$ in $R' \setminus L'$ is discrete. Thus the action of $G$ on $R^m$ factors through a properly discontinuous action of $G''$ on $K' \setminus R' \setminus L = K \setminus L$. Therefore, $\Gamma$ is discrete in $\text{I}(R^m, g)$. This completes the proof of theorem 2 when $L$ is not amenable.

Remark. Let $q: K \setminus L/G \to K \setminus L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRxG)$ over the point $KRxG \in K \setminus L/G$ ($x \in L$) is homeomorphic to
\[
(x^{-1}Kx)\backslash(x^{-1}KRx)/(x^{-1}KRx \cap G).
\]
It is easily observed that $x^{-1}KRx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let $L$ be a Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, $R$ the radical of $L$, and $G$ a lattice of $L$. Then $KR \cap G$ is a uniform lattice of $KR$.

§3. Amenable case

Now let us assume that $L$ is non-compact and amenable. Let $K$ be a maximal compact subgroup and $R$ the radical of $L$ as
before. Since \( L \) is amenable, \( L = KR \).

We define a sequence \( N^{(j)} \ (j \geq -1) \) of closed characteristic subgroups of \( L \) as follows:

1. \( N^{(-1)} \) is the radical \( R \),
2. \( N^{(0)} \) is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of \( L \),
3. \( N^{(j)} \) is the commutator subgroup \([N^{(j-1)}, N^{(j-1)}]\) of \( N^{(j-1)} \), for \( j > 0 \).

It may not be so obvious that \( N^{(j)} \)'s are closed when \( j > 0 \); in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer \( k \) such that \( N^{(k)} = \{1\} \).

Consider the following sequence:

\[
L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \ldots \supset KN^{(k)} = K.
\]

There exists an integer \( i \ (\geq 0) \) such that

\[
L = KN^{(-1)} = KN^{(0)} = \ldots = KN^{(i-1)} \neq KN^{(i)},
\]

because \( L \) is non-compact. Let us write \( M = N^{(i-1)} \) and \( N = N^{(i)} \). We introduce a flat \( L \)-invariant Riemannian metric on \( KNL \).

Let us study the action of \( L \) on \( KNL \) defined by right multiplication. An element \( ky \) of \( KM = L \ (k \in K, y \in M) \) acts on an element \( KNx \ (x \in M) \) of \( KNL \) as follows:

\[
KNx \cdot (ky) = KNxky
= KN(k^{-1}xk)y.
\]
Note that we have \([M,M] \subseteq N\); we identify the coset space \(KN/L\) with the simply-connected abelian Lie group \((KM)N/M = \mathbb{R}^m\) \((m>0)\). Now the induced action of \(L\) on \(\mathbb{R}^m\) is:

\[(KM)N \cdot (k\cdot y) = (KM)N(k^{-1}xk)y.\]

The following are easily observed: (1) this action, when restricted to \(K\), defines a homomorphism \(\alpha: K \to \text{Aut}(\mathbb{R}^m)\) and its image \(\alpha(K)\) lies in the orthogonal group \(O(m)\) with respect to some inner product of \(\mathbb{R}^m\), and (2) if \(k \in KM\), then \((KM)Nx \cdot k = (KM)Nk^{-1}xk = (KM)N(k^{-1}xk^{-1})x = (KM)Nx\) for \(x \in M\), and so \(KM\) acts trivially on \(\mathbb{R}^m\). Let \(\beta: M \to (KM)N/M\) denote the natural projection. We now define a map \(\Phi: L = KM \to \alpha(K) \times ((KM)N/M) \subseteq O(m) \times \mathbb{R}^m = I(\mathbb{R}^m)\) by sending \(k \cdot y\) \((k \in K, y \in M)\) to \((\alpha(k), \beta(y)) \in O(m) \times \mathbb{R}^m\). This is a well-defined homomorphism.

Here \(\alpha\)'s denote the obvious semi-direct products. Let \(\Gamma\) denote the image of \(G\) by \(\Phi\) in \(I(\mathbb{R}^m)\).

It remains to observe that \(N^{(j)}\)'s are closed and that \(\Gamma\) is a discrete subgroup of \(I(\mathbb{R}^m)\). To do this we use the following lemma:

Lemma 4. If \(N\) is a connected nilpotent Lie group and \(H\) is a discrete cocompact subgroup of \(N\), then the commutator subgroup \([N,N]\) is closed in \(N\) and \(H \cap [N,N]\) is cocompact in \([N,N]\).

Proof: This is well-known if \(N\) is simply-connected, so consider the universal cover \(\rho: U \to N\) of \(N\); it can be identified with the natural projection \(U \to U/\Pi\), where \(\Pi\) is the kernel of
p. To see that \([N,N]\) is closed in \(N\), it suffices to show that \(N/[N,N]\) is Hausdorff. As \(p^{-1}(N,N) = \Pi[U,U]\), we have homeomorphisms:

\[N/[N,N] \cong U/\Pi[U,U]\]
\[\cong (U/[U,U])/(\Pi[U,U]/[U,U]).\]

Here \(U/[U,U]\) is a Lie group, because \(U\) is simply-connected and hence its commutator subgroup \([U,U]\) is closed. Note that the preimage \(p^{-1}(H)\) of \(H\) is discrete and cocompact in \(U\). Since \(U\) is simply-connected, \(p^{-1}(H)\cap [U,U]\) is cocompact in \([U,U]\). Therefore, the image \(p^{-1}(H)[U,U]/[U,U]\) of \(p^{-1}(H)\) by projection: \(U \to U/[U,U]\) is discrete. As \(\Pi \subset p^{-1}(H)\), \(\Pi[U,U]/[U,U]\) is also discrete and hence closed in \(U/[U,U]\). Therefore \((U/[U,U])/(\Pi[U,U]/[U,U])\) is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

\[[N,N]/H\cap [N,N] \cong \Pi[U,U]/p^{-1}(H)\cap [U,U]\]
\[\cong [U,U]/p^{-1}(H)\cap [U,U],\]

the second statement is obvious. □

Now we prove

Lemma 5. \(N^{(j)}\)'s are closed subgroups of \(L\), and \(\Gamma\) is a crystallographic subgroup of \(I(\mathbb{R}^m)\).

Proof: If \(G\cap R\) is cocompact in \(R=N^{(-1)}\), then \(G\cap N^{(0)}\) is a discrete cocompact subgroup of \(N^{(0)}\) and we can apply lemma 4 to
prove that $N^{(j)}$ is closed for $j \geq 1$. Unfortunately, $G \cap R$ may not be cocompact in $R$, in general. To remedy this situation we introduce a quotient Lie group $L'$ of $L$ as in the previous section. We may assume that $L$ is connected. We have Levi decomposition $L = SR$, where $S$ is a connected semi-simple (and hence compact) subgroup, $R$ is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma: S \to \text{Aut}(R)$ denote the action of $S$ on $R$. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of $L$, because it commutes with elements of $R$. In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi: L \to L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then $G'$ is a lattice of $L'$, $K'$ is a maximal compact subgroup of $L'$, $S'$ is a semi-simple subgroup of $L'$, $R'$ is the radical of $L'$, and the action $\sigma': S' \to \text{Aut}(R')$ of $S'$ on $R'$ is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N^{(j)}$ $(j \geq -1)$ of characteristic subgroups of $L'$ by:

1. $N^{(-1)} = R'$
2. $N^{(0)}$ is the nil-radical of $L'$
3. $N^{(j)} = [N^{(j-1)}, N^{(j-1)}]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N^{(0)}$ are cocompact in $R'$ and $N^{(0)}$ respectively. By successively using lemma 4, we know that all $N^{(j)}$'s are closed. Note that $\pi|_R: R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi|_R)^{-1}(N^{(j)})$ for each $j$. Therefore $N^{(j)}$'s are closed in $L$.
Next, we show that $\Gamma$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. Note that we have

$$L' = K'N'(i-1) = K'N'(0) = \ldots = K'N'(i-1) \neq K'N'(i)$$

for the same $i$ and that $K'N' \backslash K'M' = KN' \backslash KM'$, where $M' = N'(i-1)$, $N' = N'(i)$. $G' \cap N'(j)$ is cocompact in $N'(j)$ for all $j$. In particular $G' \cap M'$ is cocompact in $M'$. So the image of $G'$ in $M' \backslash L'$ is discrete; furthermore, it is finite, because $M' \backslash L'$ is compact. Looking at the diagram:

$$
\begin{array}{cccccc}
G & \longrightarrow & \text{finite} \\
\cap & \cap & \\
\pi^{-1}(M') & \longrightarrow & L & \longrightarrow & \pi^{-1}(M') \backslash L \\
\downarrow & & \downarrow \pi & & \downarrow \cong \\
M' & \longrightarrow & L' & \longrightarrow & M' \backslash L' \\
U & \cup & U & \cup & \\
G' & \longrightarrow & \text{finite}
\end{array}
$$

we know that $G \cap \pi^{-1}(M')$ has a finite index in $G$. So it suffices to show that the image $\Phi(G \cap \pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. As ker $\sigma \subset$ ker $\alpha$, $\Phi$ sends elements in $\pi^{-1}(M') = (\text{ker } \sigma)_0 M$ to elements in $\mathbb{R}^m \subset I(\mathbb{R}^m)$. Now consider the following commutative diagram:

$$
\begin{array}{cccccc}
\Phi & \longrightarrow & \mathbb{O}(m) \times \mathbb{R}^m \\
\cap & \cap & \\
G \cap \pi^{-1}(M') \subset \pi^{-1}(M') = (\text{ker } \sigma)_0 M & \Phi & \longrightarrow & \mathbb{R}^m = (KN' \backslash M) \setminus M
\end{array}
$$

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where $\phi'$ is the natural map and $(\pi|_M)_*$ is the map induced by the restriction of $\pi$ to $M$, $\pi|_M: M \to M'$. $\mathcal{K}M$ and $\mathcal{K}'M'$ are maximal compact subgroups of $M$ and $M'$, respectively, and $\pi(\mathcal{K}M) = \mathcal{K}'M'$; therefore, $(\pi|_M)^{-1}(\mathcal{K}'M') = \mathcal{K}M$. Using this, it is easily verified that $(\pi|_M)^{-1}(\mathcal{K}'M')N' = (\mathcal{K}M)N$. Therefore $(\pi|_M)_*$ is an isomorphism. Since $(\mathcal{G}'M')N' = \mathcal{G}'N'$ is cocompact in $N'$, $(\mathcal{G}'M') \cap (\mathcal{K}'M')N'$ is cocompact in $(\mathcal{K}'M')N'$; so $\phi'(\mathcal{G}'M')$ is a discrete cocompact subgroup of $(\mathcal{K}'M')N'\backslash M'$. Therefore $\phi'(\mathcal{G}'\mathcal{R}^p(M'))$ is a discrete cocompact subgroup of $\mathcal{R}^n$ (and hence in $I(\mathcal{R}^n)$). This completes the proof of lemma 5. □

Thus $\mathcal{K}L/G \to \mathcal{K}N\backslash L/G$ is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form $\mathcal{K}N\backslash \mathcal{R}^n \backslash G$, and is homeomorphic to

$$(x^{-1}Kx) \backslash (x^{-1}N\backslash \mathcal{R}^n) / (x^{-1}Kx \cap G).$$

If $G$ is a lattice of $L$ (which is automatically uniform), then $x^{-1}N\backslash G$ is a uniform lattice of $x^{-1}N\backslash \mathcal{R}^n$.

§4. A rational computation of Wall's $L$-groups
Let $L$ be an amenable Lie group with finitely many components, $K$ a maximal compact subgroup of $L$, and $G$ a uniform lattice of $L$. Such a discrete group $G$ is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the $L$-groups of $G$ in terms of certain generalized homology of $K\backslash L/G$.

$K\backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_y = \{x^{-1}Kx \cap G \mid y = Kx \ (x \in L)\}$ is finite. The action of $G$ on $\mathbb{R}^n$ is free if $G$ is torsion-free; in general, $\mathbb{R}^n/G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let $WG$ be a contractible free $G$-complex, and $p$ denote the projection: $(\mathbb{R}^n \times WG)/G \to \mathbb{R}^n/G$, where $G$ acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}(\{y\})$ of an orbit $[y] \in \mathbb{R}^n/G$ by $p$ is homeomorphic to $WG/G_y$, and $p$ is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^{-\infty}(G)$ denote the limit of Ranicki's lower $L$-groups $L^{(-j)}(\mathbb{Z}G)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $L^{-\infty}(-)$ from the category of spaces to the category of $\Omega$-spectra such that the homotopy group of $L^{-\infty}(X)$ is equal to $L_{\ast}^{-\infty}(\pi_1 X)$. Applying $L^{-\infty}(-)$ to each fiber of $p$, we obtain a sheaf of spectra, denoted $L^{-\infty}(p)$. F. Quinn defines the homology group $H_{\ast}(\mathbb{R}^n/G; L^{-\infty}(p))$ of $\mathbb{R}^n/G$ with coefficients $L^{-\infty}(p)$. See [7], [10]. The following is a rational computation of $L_{\ast}^{-\infty}(G)$ in terms of this homology.
Theorem 6. Let $G$ be as above, then there is a natural isomorphism

$$H_\ast (\mathbb{R}^n/G; \mathbb{L}^{-\infty}(p)) \otimes \mathbb{Z}[1/2] \to L_{\ast \infty}^{\infty}(G) \otimes \mathbb{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations.

$$\begin{array}{cccc}
\langle \mathbb{R}^n \times \mathbb{W} \rangle / G & \xrightarrow{\text{id.}} & \langle \mathbb{R}^n \times \mathbb{W} \rangle / G \\
\downarrow p & & \downarrow \\
\mathbb{R}^n / G & \xrightarrow{\text{id.}} & \text{pt.}
\end{array}$$

Note that $\langle \mathbb{R}^n \times \mathbb{W} \rangle / G = BG$ is a classifying space for $G$ and that $H_\ast (\text{pt.}; \mathbb{L}^{-\infty}(BG; \text{pt.})) = L_{\ast \infty}^{\infty}(G)$ [10].

It is to be noted that theorem 6 says that the $\mathbb{L}^{-\infty}(p)$ coefficient homology of $\mathbb{R}^n / G$ is independent (modulo 2 torsion) of the action of $G$ on $\mathbb{R}^n$. It is conceivable that the orbifold $\mathbb{R}^n / G$ has a certain strong rigidity.

Proof of theorem 6. The proof is by induction on the dimension $n$ of $\mathbb{K} \setminus \mathbb{L}$. Let $q: \mathbb{R}^n / G \to \mathbb{R}^m / \Gamma$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$H_\ast (\mathbb{R}^n / G; \mathbb{L}^{-\infty}(p))$$

$$\cong \quad H_\ast (\mathbb{R}^m / \Gamma; U_{w \in \mathbb{R}^m / \Gamma} H(q^{-1}(w)); \mathbb{L}^{-\infty}(p \mid q^{-1}(w)))$$

$$\cong \quad H_\ast (\mathbb{R}^m / \Gamma; U_{w} \mathbb{L}^{-\infty}((q_{p})^{-1}(w)))$$
by induction hypothesis, where \( H \) denote the homology theory spectrum [ibid.]. We can prove that \( H_\ast(\mathbb{R}^n/T; \mathbb{L}_\infty^{\infty}(qp)) \otimes \mathbb{Z}[1/2] \) is naturally isomorphic to \( L_\infty^{\infty}(G) \) using the proof of the main theorem of [ibid.] with only some obvious modifications, and this completes the proof of theorem 6. \( \square \)

Corollary 7. (Novikov Conjecture) Let \( G \) be as above, then the assembly map

\[
H_\ast(BG; \mathbb{L}_\infty^{\infty}(1)) \to L_\infty^{\infty}(G)
\]

is rationally split injective.

References


