

3次元多様体のG-手術

G-surgery of three dimensional manifolds

岡山大学教養部 森本雅治 (Masaharu Morimoto)

1. Introduction

In this paper G will denote the cyclic group of order ≤ 2 .
G-actions on manifolds should be understood to be smooth.

Let X and Y be compact, connected, oriented and
3-dimensional G-manifolds, and let $f : X \rightarrow Y$ be a degree one
G-map. We are sometimes required to make f a $\mathbb{Z}_{(p)}$ -homology
equivalence, p prime. This paper is concerned with the problem.
Our main results are Theorems 3.3 and 3.7. As an application of
Theorem 3.7, we can show the following theorem.

Theorem A. There exist one fixed point actions of A_5 on S^6 ,
where A_5 is the alternating group on 5 letters and S^6 is the
standard sphere of dimension 6.

We will, however, give the proofs of Theorems 3.7 and A at
another opportunity.

It seems interesting that, in the case $|G| = 2$, we get an
obstruction group $W_3(R[G], \Gamma(G))$ different from the Wall group
 $L_3(R[G])$ (it may happen that $W_3(R[G], \Gamma(G))$ is isomorphic to
 $L_3(R[G])$). The reason for it is that our G-framed normal map $f : X \rightarrow$

Y has the non-empty fixed point sets X^G and Y^G .

The idea of this paper originates from not only Wall's surgery theory [16] but also Dovermann - Petrie's equivariant surgery theory [5] and [6].

In Section 2 we define some Witt groups in which we construct G -surgery obstructions. In Section 3 we define G -normal maps and G -framed normal maps and we state our main results. Sections 4 and 5 are devoted to the construction of the G -surgery obstruction and to the proof of Theorems 3.3.

In the present paper, the orientation of the boundary ∂X of an oriented manifold X is given so that, on the boundary,

$$\begin{aligned} & \text{the outward normal direction} + \text{the orientation of } \partial X \\ & = \text{the orientation of } X. \end{aligned}$$

2. The Witt groups

Our general reference of this section is [2].

Let Λ be an associative ring with 1 and with an involution - satisfying $\bar{1} = 1$, $\overline{a + b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{b} \bar{a}$ for $a, b \in \Lambda$. $M_n(\Lambda)$ denotes the set of $n \times n$ -matrices whose entries are in Λ . Let Γ be an additive subgroup of Λ such that

$$(\Gamma 1) \quad \{a + \bar{a} \mid a \in \Lambda\} \subset \Gamma \subset \{a \in \Lambda \mid a = \bar{a}\}, \text{ and}$$

$$(\Gamma 2) \quad a\bar{a} \in \Gamma \text{ for all } a \in \Lambda.$$

Such Γ is called a ring parameter of Λ .

For an element (x_{ij}) of $M_n(\Lambda)$, $(x_{ij})^*$ is defined to be (\bar{x}_{ji}) . In the following an arbitrary element in $M_{2n}(\Lambda)$ is often written in the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ with } A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \text{ and } D = (d_{ij}),$$

where i and j run from 1 to n . Let $SU_n(\Lambda, \Gamma)$ be the group of non-singular matrices in $M_{2n}(\Lambda)$ which satisfy

$$(1) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \text{ and}$$

(2) the diagonal coefficients of BA^* and DC^* lie in Γ .

We denote by $TU_n(\Lambda, \Gamma)$ the subgroup of $SU_n(\Lambda, \Gamma)$ which consists of the elements with $B = 0$. We put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then σ belongs to $SU_1(\Lambda, \Gamma)$ for any ring parameter Γ of Λ .

We have the standard stabilizers $j_{n,n+1} : SU_n(\Lambda, \Gamma) \rightarrow SU_{n+1}(\Lambda, \Gamma)$ defined by

$$j_{n,n+1}(x) = \begin{pmatrix} A & & & B \\ & 1 & & \\ C & & & D \\ & & & & 1 \end{pmatrix}$$

$$\text{for } x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ in } SU_n(\Lambda, \Gamma).$$

With respect to the standard stabilizers, we define $SU(\Lambda, \Gamma) = \varinjlim_n SU_n(\Lambda, \Gamma)$ and $TU(\Lambda, \Gamma) = \varinjlim_n TU_n(\Lambda, \Gamma)$. We denote by $RU(\Lambda, \Gamma)$ the subgroup of $SU(\Lambda, \Gamma)$ which is generated by $TU(\Lambda, \Gamma)$ and σ . It is well-known (see [2, Corollary 3.9]) that $RU(\Lambda, \Gamma)$ includes the commutator subgroup $CU(\Lambda, \Gamma) = [SU(\Lambda, \Gamma), SU(\Lambda, \Gamma)]$ of $SU(\Lambda, \Gamma)$.

Definition 2.1. The Witt group of dimension three $W_3(\Lambda, \Gamma)$ is defined to be the quotient group $SU(\Lambda, \Gamma)/RU(\Lambda, \Gamma)$. In the case where (1) Λ is commutative, (2) σ is trivial and (3) $\Gamma = \Lambda$, we use $W_3(\Lambda)$ instead of $W_3(\Lambda, \Gamma)$.

Lemma 2.2. If Λ is a commutative local ring with the trivial involution, then $W_3(\Lambda, \Gamma) = 0$ for any ring parameter Γ .

We omit the proof.

It is important to estimate the commutator subgroup $CU(\Lambda, \Gamma)$ of $SU(\Lambda, \Gamma)$ for applications. We introduce a result of Bak [2].

For an integer k with $1 \leq k \leq n$ and for $x \in \Gamma$, we define an element $\varepsilon_k(x)$ of $TU_n(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$, $1 \leq i \leq n$, (2) $c_{kk} = x$ and (3) all the other entries are 0. If $1 \leq h, k \leq n$ and $h \neq k$ and $x \in \Lambda$, then we define $\varepsilon_{hk}(x)$ (resp. $\kappa_{hk}(x)$) of $TU_n(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$, $1 \leq i \leq n$, (2) $c_{hk} = x = \bar{c}_{kh}$ (resp. $-\bar{a}_{kh} = x = d_{hk}$) and (3) all the other entries are 0. $FU(\Lambda, \Gamma)$ is the subgroup of $RU(\Lambda, \Gamma)$ generated by (1) σ , (2) $\varepsilon_k(x)$, (3) $\varepsilon_{hk}(x)$ and (4) $\kappa_{hk}(x)$, where n varies over all positive integers, and h, k and x take all possible values. Then Corollary 3.9 of [2] asserts:

Lemma 2.4. It holds that $CU(\Lambda, \Gamma) \subset FU(\Lambda, \Gamma)$.

Remark 2.5. Hence for any $x \in SU(\Lambda, \Gamma)$ we have $TU(\Lambda, \Gamma)xFU(\Lambda, \Gamma) = xRU(\Lambda, \Gamma)$ as subsets of $SU(\Lambda, \Gamma)$. This is a key point of surgery theory.

We define $\sigma_k \in SU_k(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$ if $i \neq k$, (2) $b_{kk} = 1 = -c_{kk}$ and (3) all the other entries are 0. We denote the stabilized element in $SU_n(\Lambda, \Gamma)$, $n \geq k$, from σ_k again by σ_k . It holds that

$$(2.6) \quad \kappa_{hk}(x) = \sigma_k^3 \varepsilon_{hk}(x) \sigma_k.$$

Proposition 2.7. $FU(\Lambda, \Gamma)$ is generated by σ_k , $\varepsilon_k(x)$ and $\varepsilon_{hk}(x)$, where h, k and x vary over all possible values.

Let R be a commutative ring with 1. For a finite group G , we denote by $R[G]$ the group ring of G with coefficient ring R . $R[G]$ has the involution $-$ satisfying $(ag)^{-} = ag^{-1}$ for $a \in R$ and $g \in G$.

In the rest of this paper we let G be $\{1\}$ the trivial group or C_2 the group of order two.

Let \mathbb{Z} be the ring of integers and $\mathbb{Z}_{(p)}$ the localization of \mathbb{Z} at a prime p . Hereafter R denotes one of \mathbb{Z} and $\mathbb{Z}_{(p)}$.

Remark 2.8. (1) In the case where $1/2 \in R$, the ring parameter Γ of $R[G]$ is unique, that is $\Gamma = R[G]$. In the case we have $W_3(R[G], \Gamma) = W_3(R[G])$. (2) Let $L_3^h(R[G])$ be the Wall group of R -homology equivalence and of dimension $3 + 4n \geq 7$ (see [1]). Then we have $L_3^h(R[G]) = W_3(R[G], 2R[G])$.

Definition 2.9. We define the ring parameter $\Gamma(G)$ by $\Gamma(G) = 2R[G]$ if $G = \{1\}$ and $\Gamma(G) = \{2a + bg \mid a \in R \text{ and } b \in R\}$ if $G = C_2$.

By definition we have $W_3(R[G], \Gamma(G)) = L_3^h(R[G])$ if $G = \{1\}$.

Proposition 2.10. It holds that (1) $W_3(\mathbb{Z}_{(p)}[G]) = 0$ for any prime p and (2) $W_3(\mathbb{Z}_{(2)}[G], \Gamma) = 0$ for any ring parameter Γ .

Proof. This follows from Lemma 2.2.

3. G-normal maps

Let G be $\{1\}$ or C_2 . In the present section we give the definition of a G -normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ and a G -framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^* \xi)$.

Let Y be a compact, 1-connected, oriented, 3-dimensional G -manifold with boundary ∂Y (possibly $\partial Y = \emptyset$).

Remark 3.1. The case where Y is the unit disk $D(V)$ of a real G -module V is important for applications. The reader may restrict the following arguments to the case $\pi_2(Y) = 0$.

If $G = C_2$, then we require that

(N1) Y^G is not empty nor Y , and each element of G preserves the orientation of Y .

This implies that each connected component of Y^G has dimension one. Let X be

(N2) a compact, connected, oriented, 3-dimensional G -manifold

and X_1 a G -subcomplex of X with $\dim X_1 \leq 1$ with respect to some smooth G -triangulation of X .

Definition 3.2. We call a G -map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ a G -normal map if the following (N3) - (N5) are satisfied:

(N3) $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ has degree one.

(N4) $f|_{\partial X} : \partial X \longrightarrow \partial Y$ is a G -homotopy equivalence.

(N5) If $G = C_2$, then $f^G : (X^G, \partial X^G) \longrightarrow (Y^G, \partial Y^G)$ is a homotopy equivalence.

We put $X_s = \phi$ if $G = \{1\}$ and $X_s = X^G$ if $G = C_2$.

If a G -normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ with a G -subcomplex X_1 of X is given, then by an argument similar to [5] and [6] we can give an element $\sigma(f)$ of $W_3(R[G])$ with the property:

Theorem 3.3. If $\sigma(f) = 0$ in $W_3(R[G])$, then $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ can be converted to a G -normal map $f' : (X', \partial X') \longrightarrow (Y, \partial Y)$ with the following conditions (S1) - (S3), by G -surgery relative to the set $\partial X \cup X_1 \cup X_s$:

(S1) $\partial X' = \partial X$, $X'_1 = X_1$ and $X'_s = X_s$.

(S2) $f'|_{\partial X' \cup X'_1 \cup X'_s} = f|_{\partial X \cup X_1 \cup X_s}$.

(S3) $f' : X' \longrightarrow Y$ is an R -homology equivalence.

Remark 3.4. The correspondence : a G -normal map $f \longmapsto \sigma(f)$, may be multivalued. When we construct $\sigma(f)$, we are required to make various choices. Thus $\sigma(f)$ may depend on the choices for the construction. Theorem 3.3 says that if we have $\sigma(f) = 0$ from one series of the choices, then f can be converted to nice f' . In other words, a G -normal map f corresponds to a subset $\{\sigma(f)\}$ of $W_3(R[G])$; if $\{\sigma(f)\}$ implies 0 , then f can be converted to nice f' . Nevertheless, since most of $W_3(R[G])$'s vanish by Proposition 2.10, we are not too nervous about the ambiguity.

Let ξ be a real G -vector bundle over Y of fiber-dimension 3. If $b : TX \oplus \underline{V} \rightarrow f^*\xi \oplus \underline{V}$, V a real G -module, is a G -vector bundle isomorphism (covering the identity map on X), then we roughly write $b : TX \rightarrow f^*\xi$ and call it a stable G -vector bundle isomorphism.

Definition 3.5. We call a pair (f, b) of a G -normal map $f : (X, \partial X) \rightarrow (Y, \partial Y)$ and a stable G -vector bundle isomorphism $b : TX \rightarrow f^*\xi$, a G -framed normal map.

Remark 3.6. Since we want to do G -surgery relative to the singular set, the normal part of bundle data of the singular set is needless.

If a G -framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^* \xi)$ is given, then we can give an element $\sigma(f, b)$ of $W_3(R[G], \Gamma(G))$ with the property:

Theorem 3.7. If $\sigma(f, b) = 0$ in $W_3(R[G], \Gamma(G))$, then the G -framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^* \xi)$ can be converted to a G -framed normal map $(f', b') : (X', \partial X'; TX') \longrightarrow (Y, \partial Y; f'^* \xi)$ satisfying the conditions (S1) - (S3) in Theorem 3.3, by G -surgery relative to $\partial X \cup X_1 \cup X_s$.

4. Definition of $\sigma(f)$ in $W_3(R[G])$

In this section we show how the algebraic object $\sigma(f) \in W_3(R[G])$ is obtained from a G -normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ with a subcomplex X_1 , $\dim X_1 \leq 1$, defined in Section 3. We show it only in the case $G = C_2$ and $R = \mathbb{Z}$. The reader will easily analogize it for the case $G = \{1\}$ and $R = \mathbb{Z}$.

Remark 4.1. If we use $\sigma(f, R)$ instead of $\sigma(f) \in W_3(R[G])$ to make the coefficient ring clear (for general R), then $\sigma(f, R)$ is the image of $\sigma(f, \mathbb{Z})$ by the natural homomorphism from $W_3(\mathbb{Z}[G])$ to $W_3(R[G])$.

In the following G is C_2 with generator g and the

coefficient ring of homology groups is \mathbb{Z} . Let $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ be a G -normal map with $X_1 \subset X$.

It holds that $\dim Y^G = 1 = \dim X^G$. Take connected components Y_*^G of Y^G and X_*^G of X^G such that $f| : (X_*^G, \partial X_*^G) \longrightarrow (Y_*^G, \partial Y_*^G)$ has degree one. Fix a point x_* in $\text{Int } X_*^G$, and take a G -invariant closed disk \bar{D} with the center x_* . Take a point y_* in $\text{Int } Y_*^G$ and a G -invariant closed disk \bar{D}' with the center y_* . Further fix a point y_{**} in $\partial \bar{D}'^G$.

We denote the field of complex numbers by \mathbb{C} , the unit circle of \mathbb{C} by S^1 and the closed unit disk of \mathbb{C} by D^2 .

Let $h_1, \dots, h_n : S^1 \longrightarrow X - (\partial X \cup X_1 \cup X^G \cup \bar{D})$ be embeddings which generate $H_1(X)$ and lie in general position. Since the normal bundle of h_i is trivial, we have an orientation

preserving embedding $\bar{f}_i : S^1 \times D^2 \longrightarrow X - (\partial X \cup X_1 \cup X^G \cup \bar{D})$

such that $\bar{f}_i(x, 0) = h_i(x)$ for $x \in S^1$. We put $U_i = \text{Im } \bar{f}_i$ and $\bar{U}_i = U_i \cup gU_i$. We suppose that $U_1, \dots, U_n, gU_1, \dots, gU_n$ are mutually disjoint. Take a closed tube T_i connecting \bar{D} with

U_i in $X - (\partial X \cup X_1 \cup X^G \cup \text{Int } \bar{D} \cup \bigcup_{i=1}^n \text{Int } \bar{U}_i)$ such that $T_1, \dots, T_n, gT_1, \dots, gT_n$ are mutually disjoint. We set

$$\bar{T}_i = T_i \cup gT_i,$$

$$\bar{U} = \bar{D} \cup \bigcup_{i=1}^n (\bar{T}_i \cup \bar{U}_i),$$

$$X_0 = X - \text{Int } \bar{U} \quad \text{and}$$

$$Y_0 = Y - \text{Int } \bar{D}'.$$

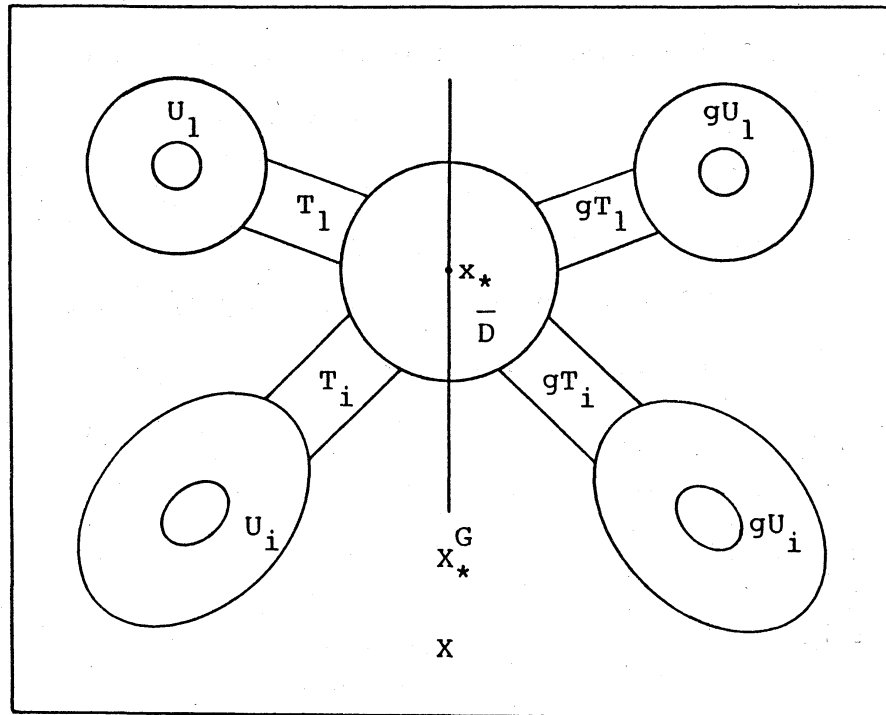


Figure 1.

Deform $f : X \rightarrow Y$ by a G -homotopy of G -normal map $(X, \partial X) \rightarrow (Y, \partial Y)$ so that $f(T_i) = \{y_{**}\} = f(U_i)$, $f(\bar{U}) = \bar{D}'$ and $f(X_0) = Y_0$.

If $h : H_n(A, A') \rightarrow H_n(B, B')$ is a homomorphism, then we put $K_n(h) = \text{Ker } h$. If there is no confusion, we use $K_n(A, A')$ instead of $K_n(h)$.

The restrictions of f give the kernels

$$K_2(\bar{U}, \partial\bar{U}) = \text{Ker}[H_2(\bar{U}, \partial\bar{U}) \longrightarrow H_2(\bar{D}', \partial\bar{D}')],$$

$$K_2(X_0, \partial\bar{U}) = \text{Ker}[H_2(X_0, \partial\bar{U}) \longrightarrow H_2(Y_0, \partial\bar{D}')],$$

$$K_1(\partial\bar{U}) = \text{Ker}[H_1(\partial\bar{U}) \longrightarrow H_1(\partial\bar{D}')],$$

$$K_1(X_0) = \text{Ker}[H_1(X_0) \longrightarrow H_1(Y_0)] \quad \text{and}$$

$$K_1(\bar{U}) = \text{Ker}[H_1(\bar{U}) \longrightarrow H_1(\bar{D}')].$$

These make the exact sequence:

$$\begin{array}{ccccc}
 0 & \xrightarrow{\quad} & K_2(\bar{U}, \partial\bar{U}) & & K_1(X_0) & \xrightarrow{\quad} & 0 \\
 & & \searrow & & \nearrow & & \\
 & & & K_1(\partial\bar{U}) & & & \\
 & & \nearrow & & \searrow & & \\
 0 & \xrightarrow{\quad} & K_2(X_0, \partial\bar{U}) & & K_1(\bar{U}) & \xrightarrow{\quad} & 0
 \end{array}$$

We define $e_i, f_i : S^1 \longrightarrow \partial U_i$ by $e_i(x) = \bar{f}_i(1, x)$ and $f_i(x) = \bar{f}_i(x, 1)$ for $x \in S^1$. We call e_i (resp. f_i) the meridian (resp. longitude) of \bar{f}_i . $K_1(\partial\bar{U})$ is isomorphic to $\bigoplus_{i=1}^n H_1(\partial\bar{U}_i)$.

$K_1(\partial\bar{U})$ is regarded as a $\mathbb{Z}[G]$ -free module with the basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$. $K_2(\bar{U}, \partial\bar{U})$ and $K_2(X_0, \partial\bar{U})$ are treated as submodules of $K_1(\partial\bar{U})$ in the following. Obviously $K_2(\bar{U}, \partial\bar{U})$ has the $\mathbb{Z}[G]$ -basis $\{e_1, \dots, e_n\}$. Since $K_1(\bar{U})$ has the $\mathbb{Z}[G]$ -basis $\{h_1, \dots, h_n\}$, we have the splitting $s : K_1(\bar{U}) \rightarrow K_1(\partial\bar{U})$ defined by $s(h_i) = f_i$. We regard $K_1(\bar{U})$ as a submodule of $K_1(\partial\bar{U})$ with respect to s . $K_1(X_0)$ and $K_2(X_0, \partial\bar{U})$ are $\mathbb{Z}[G]$ -projective, hence $\mathbb{Z}[G]$ -free. $K_2(X_0, \partial\bar{U})$ is a direct summand of $K_1(\partial\bar{U})$. Furthermore by the Poincaré duality and the universal coefficient theorem, $K_2(X_0, \partial\bar{U})$ has the half rank of $K_1(\partial\bar{U})$.

Since $K_1(\partial\bar{U}) = H_1(\partial\bar{U})$, we have the intersection form $\lambda_0 : K_1(\partial\bar{U}) \times K_1(\partial\bar{U}) \rightarrow \mathbb{Z}$, hence we have the (equivariant) intersection form $\lambda : K_1(\partial\bar{U}) \times K_1(\partial\bar{U}) \rightarrow \mathbb{Z}[G]$ defined by

$$\lambda(x, y) = \lambda_0(x, y) + \lambda_0(x, gy)g \quad \text{for } x, y \in K_1(\partial\bar{U}).$$

Then λ satisfies $\lambda(x, ay) = a\lambda(x, y)$ and $\lambda(x, y) = -\lambda(y, x)$ for $x, y \in K_1(\partial\bar{U})$ and $a \in \mathbb{Z}[G]$. It holds that $\lambda(e_i, e_j) = 0 = \lambda(f_i, f_j)$, $\lambda(e_i, f_j) = \delta_{ij}$ and $\lambda(K_2(X_0, \partial\bar{U}), K_2(X_0, \partial\bar{U})) = 0$. Hence all $K_2(\bar{U}, \partial\bar{U})$, $K_1(\bar{U})$ and $K_2(X_0, \partial\bar{U})$ are hyperbolizers of $(K_1(\partial\bar{U}), \lambda)$.

There are $\mathbb{Z}[G]$ -isomorphisms $\alpha : K_1(\partial\bar{U}) \rightarrow K_1(\partial\bar{U})$ such that $\alpha(K_2(\bar{U}, \partial\bar{U})) = K_2(X_0, \partial\bar{U})$ and α preserve λ . Fix one of them. We write α in the form:

$$\alpha(e_1, \dots, e_n, f_1, \dots, f_n) = \phi(\alpha)^t(e_1, \dots, e_n, f_1, \dots, f_n)$$

by a $2n \times 2n$ -matrix $\phi(\alpha)$. $\phi(\alpha)$ belongs to $SU_n(\mathbb{Z}[G], \mathbb{Z}[G])$.

Remark 4.2. Another choice of a $\mathbb{Z}[G]$ -isomorphism : $K_1(\partial\bar{U}) \rightarrow K_1(\partial\bar{U})$ above, determines an element in $TU_n(\mathbb{Z}[G], \mathbb{Z}[G])\phi(\alpha)$.

Definition 4.3. We define $\sigma(f)$ to be the coset $[\phi(\alpha)]$ in $W_3(\mathbb{Z}[G]) = SU(\mathbb{Z}[G], \mathbb{Z}[G])/RU(\mathbb{Z}[G], \mathbb{Z}[G])$.

Here $\sigma(f)$ may depend not only on f but also on $\bar{f}_1, \dots, \bar{f}_n$, etc.

5. Proof of Theorem 3.3

We prove Theorem 3.3 only in the case $G = C_2$ and $R = \mathbb{Z}$. The proof of the case $G = C_2$ and $R = \mathbb{Z}_{(p)}$ needs further work. G. A. Anderson generalized Wall's surgery theory to the surgery theory with coefficient ring $\mathbb{Z}_{(p)}$. His technique [1, pp. 69 - 70] works well here, too.

For simplicity we use SU_n, TU_n, RU, \dots instead of $SU_n(\mathbb{Z}[G], \mathbb{Z}[G]), TU_n(\mathbb{Z}[G], \mathbb{Z}[G]), RU(\mathbb{Z}[G], \mathbb{Z}[G]), \dots$ respectively.

Stabilization. We defined matrices $\sigma_k \in SU_k$ by (1) $a_{ii} = 1 = d_{ii}$ if $i \neq k$, (2) $b_{kk} = 1 = -c_{kk}$ and (3) all the other entries are 0. We denote the stabilized element in SU_n , $n \geq k$, from σ_k in SU_k again by σ_k . We define Σ_n to be the product $\sigma_1 \sigma_2 \dots \sigma_n \in SU_n$.

Take another orientation preserving trivial embedding $\bar{f}_{n+1} : S^1 \times D^2 \rightarrow \text{Int } X_0$ such that $\text{Im } \bar{f}_{n+1}$ lies in a small neighborhood of \bar{D} . Consider the effect by adding \bar{f}_{n+1} to $\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ on the matrix $\phi(\alpha) \in SU_n$. We use $\Psi(f)$ instead of $\phi(\alpha)$ in order to put stress on the map f . The matrix $\Psi(f) \in SU_n \subset SU_{n+1}$ is replaced by $\Psi(f)\sigma_{n+1}$. Let $f' : (X', \partial X') \rightarrow (Y, \partial Y)$ be the resulted G-normal map by G-surgery of f along \bar{f}_{n+1} . We use the above $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n$ and the dual of \bar{f}_{n+1} to determine $\Psi(f') \in SU_{n+1}$. Then we have $\Psi(f') = \Psi(f)$ as matrices in SU_{n+1} .

Hence for the stabilized element $x \in SU_{n+k}$ from $\Psi(f) = \phi(\alpha) \in SU_n$, we can get a G-framed normal map f'' with $\Psi(f'') = x$ by G-surgery of f relative to $\partial X \cup X_1 \cup X^G$.

It is easy to see that if $\phi(\alpha) = \Psi(f)$ is Σ_n , then f is a homology equivalence. Since FU implies Σ_n , for the proof of Theorem 3.3 it is sufficient to show that

(5.3) if a matrix $x \in \Psi(f)RU$ is arbitrarily given, then we can obtain a G-framed normal map f' with $\Psi(f') = x$ by G-surgery of f relative to $\partial X \cup X_1 \cup X^G$.

By Remarks 2.5 and 4.2, the matrix x in (5.3) may be restricted to the case $x \in \Psi(f)FU$. We recall Proposition 2.7. Each element of FU is a product of matrices of type σ_k , $\varepsilon_k(y)$ and $\varepsilon_{hk}(y)$, where y lies in $\{1, -1, g, -g\}$. It is sufficient to find G -surgery corresponding to the matrix changes: $\Psi(f) \mapsto \Psi(f)z$, where z are σ , $\varepsilon_k(y)$ and $\varepsilon_{hk}(y)$.

It is well known that G -surgery along \bar{f}_k corresponds to the matrix change: $\Psi(f) \mapsto \Psi(f)\sigma_k$.

Assertion 5.4. By some replacement of \bar{f}_k , $\phi(\alpha)$ is converted to $\phi(\alpha)\varepsilon_k(+1)$.

Proof. We prove it in the case $\varepsilon_k(+1) = \varepsilon_1(1)$.

Let $h_1' = h_1$ and define $\bar{f}_1' : S^1 \times D^2 \rightarrow X$ by $\bar{f}_1'(x, y) = \bar{f}_1(x, x^{-1}y)$ for $x \in S^1$ and $y \in D^2$. Then the corresponding f_1' and e_1' are given by $f_1'(x) = \bar{f}_1'(x, 1) = \bar{f}_1(x, x^{-1})$ and $e_1'(x) = \bar{f}_1'(1, x) = \bar{f}_1(1, x) = e_1(x)$ for $x \in S^1$. Hence we have $e_1 = e_1'$ and $f_1 = e_1' + f_1'$ in $K_1(\partial\bar{U})$.

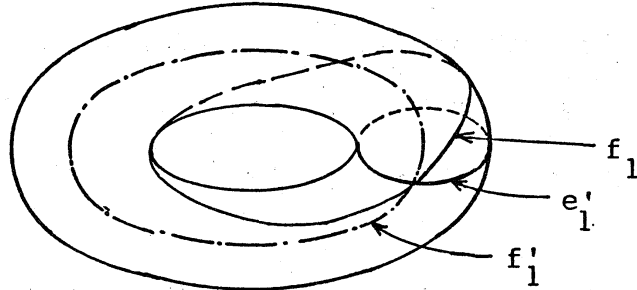


Figure 2.

Define a $\mathbb{Z}[G]$ -isomorphism $\alpha' : K_1(\partial\bar{U}) \longrightarrow K_1(\partial\bar{U})$ by

$$\begin{aligned} \alpha'(e_1', e_2, \dots, e_n, f_1', f_2, \dots, f_n) \\ = \phi(\alpha)\varepsilon_1(1) {}^t(e_1', e_2, \dots, e_n, f_1', f_2, \dots, f_n). \end{aligned}$$

Then $\alpha'(K_2(\bar{U}, \partial\bar{U})) = K_2(X_0, \partial\bar{U})$ and α' preserves λ . This completes the proof.

Assertion 5.5. By some replacement of \bar{f}_k , $\phi(\alpha)$ is converted to $\phi(\alpha)\varepsilon_k(\pm g)$.

Proof. We prove it in the case $\varepsilon_k(\pm g) = \varepsilon_1(g)$.

Take a thin G -invariant closed tube \bar{T} connecting T_1 with gT_1 along $\partial\bar{D}$ in X_0 . Without loss of generality, we may assume $f(\bar{T}) = \{y_{**}\}$.

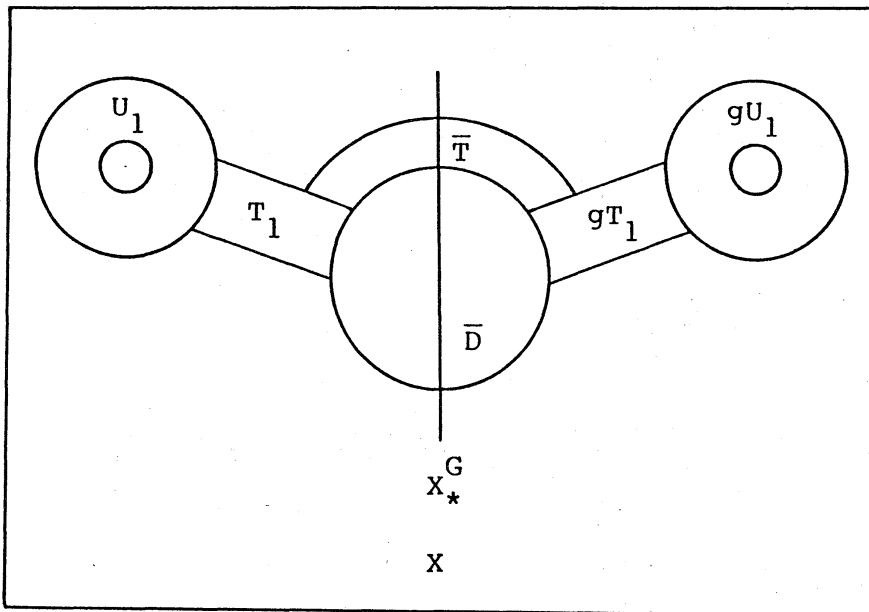


Figure 3.

We put $W = U_1 \cup T_1 \cup \bar{T} \cup gT_1 \cup gU_1$ ($\subset X$). Denote by W'' , \bar{f}_1'' , e_1'' and f_1'' the copies of W , \bar{f}_1 , e_1 and f_1 respectively. There is a G -diffeomorphism $\beta : W \rightarrow W''$ satisfying

$\beta \circ e_1 = e_1''$ as maps and $\beta \circ f_1 = ge_1'' + f_1''$ in $H_1(\partial W'')$. We assume this here and continue the proof. The way to obtain such β will be shown after the completion of the left part of proof.

Define $h_1' : S^1 \rightarrow X$ and $\bar{f}_1' : S^1 \times D^2 \rightarrow X$ by $h_1' = \beta^{-1} \circ h_1''$ and $\bar{f}_1' = \beta^{-1} \circ \bar{f}_1''$ and put $U_1' = \text{Im } \bar{f}_1'$. We take a closed tube T_1' connecting \bar{D} with U_1' along $\partial T_1' = \bar{T}$ in T_1 , and we set

$$\bar{U}_1' = U_1' \cup gU_1',$$

$$\bar{T}_1' = T_1' \cup gT_1',$$

$$\bar{U}' = \bar{D} \cup (\bar{T}_1' \cup \bar{U}_1') \cup \bigcup_{i=2}^n (\bar{T}_i \cup \bar{U}_i) \quad \text{and}$$

$$X_0' = X - \text{Int } \bar{U}'.$$

We put $e_1' = \beta^{-1} \circ e_1''$ and $f_1' = \beta^{-1} \circ f_1''$. $K_1(\partial \bar{U}')$ is regarded as a $\mathbb{Z}[G]$ -free module with basis $\{e_1', e_2, \dots, e_n, f_1', f_2, \dots, f_n\}$. By the construction there is a $\mathbb{Z}[G]$ -isomorphism $\gamma : K_1(\partial \bar{U}') \rightarrow K_1(\partial \bar{U}')$ such that (1) $\gamma(K_2(\bar{U}', \partial \bar{U}')) = K_2(X_0', \partial \bar{U}')$, (2) γ preserves the (equivariant) intersection form λ' on $K_1(\partial \bar{U}')$ and (3) γ has the matrix form $\Phi(\alpha)\epsilon_1(g)$, that is,

$$\gamma(e_1', e_2, \dots, e_n, f_1', f_2, \dots, f_n)$$

$$= \Phi(\alpha)\varepsilon_1(g) {}^t(e_1', e_2, \dots, e_n, f_1', f_2, \dots, f_n).$$

This completes the proof under the assumption of existence of β .

Now we show the way to obtain the G -diffeomorphism $\beta : W \rightarrow W$ ($= W''$).

We regard W as the union of 3-dimensional disks $H(1)$, $H(0)$ and $H(g)$ with the picture:

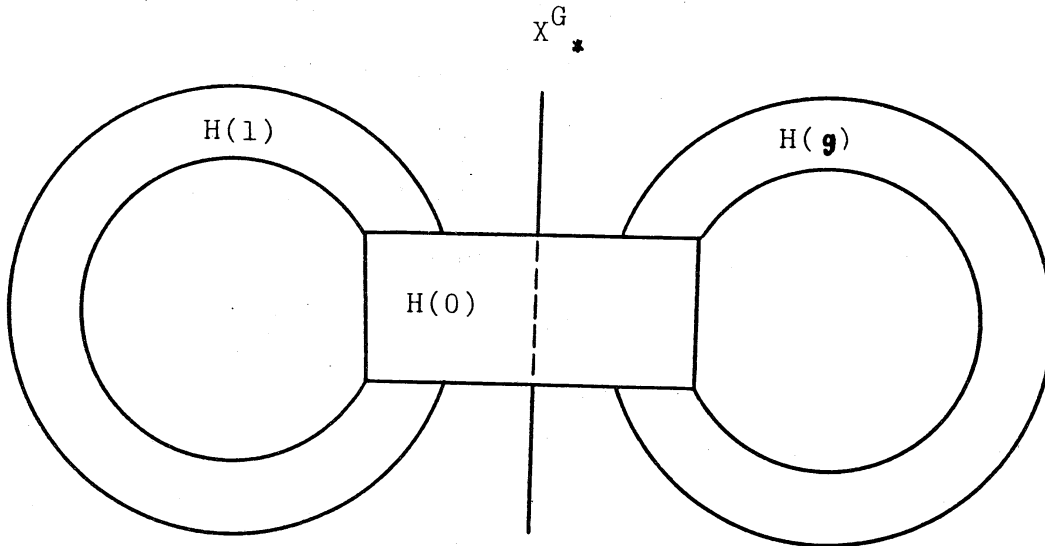


Figure 4.

Here the action of g on W is regarded as the rotation with angle π around the axis X^G_* . We assume that $H(g) = gH(1)$ and $H(0)$ is G -invariant. We identify $H(1)$ with $I \times D^2$, $I = [0, 1]$, and

$H(0)$ with $D^2 \times I$, where the action of g on $H(0)$ is the rotation on $D^2 \times I$ with angle π around $\{0\} \times I$. We suppose that $\{0\} \times D^2$ of $H(1)$ lies in $D^2 \times \{1\}$ of $H(0)$, and $\{1\} \times D^2$ of $H(1)$ lies in $D^2 \times \{0\}$ of $H(0)$. Let $\beta_1 : W \rightarrow W$ be the G -diffeomorphism given by twisting $H(0)$ once around x^G_* . That is, $\beta_1(x, t) = (x \exp(2\pi t \sqrt{-1}), t)$ for $(x, t) \in D^2 \times I = H(0)$ and $\beta_1(y) = y$ for $y \in H(1) \cup H(g)$. Then f_1 is mapped by β_1 to $f_1 + e_1 + ge_1$ (caution: not $f_1 + ge_1$) as elements of the homology group $H_1(\partial W)$. Let $\beta_2 : W \rightarrow W$ be the G -diffeomorphism given by twisting $H(1)$ and $H(g)$ equivariantly once in the opposite direction around $I \times \{0\}$. That is $\beta_2(t, x) = (t, x \exp(-2\pi t \sqrt{-1}))$ for $(t, x) \in I \times D^2 = H(1)$, $\beta_2(y) = y$ for $y \in H(0)$ and $\beta_2(gz) = g\beta_2(z)$ for $gz \in H(g)$. Then f_1 is mapped by β_2 to $f_1 - e_1$ as elements of $H_1(\partial W)$. The required map β is the composition $\beta_1 \circ \beta_2$ of β_1 and β_2 .

Remark 5.6. It will turn out to be a key to our G -surgery of the framed case, that \bar{f}_1 and $\bar{f}'_1 : S^1 \times D^2 \rightarrow W$ are isotopic to each other.

Assertion 5.7. By some replacement of \bar{f}_i , $\phi(\alpha)$ is converted to $\phi(\alpha)\varepsilon_{ij}(y)$ for arbitrarily given $y \in \{1, -1, g, -g\}$ and $i \neq j$.

Proof. The reader may have the strategy for the proof on the analogy of ordinal surgery theory as follows. Fix a G -invariant

Riemannian metric on X . Consider the effect of varying h_i by regular homotopies. Here \bar{f}_i is also varied by parallel translation along the regular homotopies. We should, of course, do it by using the covering : $X - X^G \longrightarrow (X - X^G)/G$. As in ordinary surgery theory, it seems possible to convert the matrix $\Phi(\alpha)$ to $\Phi(\alpha)\varepsilon_{ij}(y)$ by some choice of an element in the regular homotopy class of h_i .

There is, however, a more elementary way of proof. Since the proof is quite similar to the proof of Assertion 5.5, we give only a rough sketch.

We suppose that $\varepsilon_{ij}(y) = \varepsilon_{12}(1)$.

Take a connecting tube T from T_1 to T_2 which does not meet with gT . We have two disjoint handle bodies of genus two:

$$H(1) = U_1 \cup T_1 \cup T \cup T_2 \cup U_2 \quad \text{and} \quad H(g) = gH(1).$$

If we forget the G -action of W in the proof of Assertion 5.5, then W is homeomorphic to $H(1)$ and $H(g)$. Thus we consider the effect of twisting $H(1)$ (simultaneously $H(g)$) as W was done by β .

By the twisting, \bar{f}_2 is also replaced by a new embedding \bar{f}_2' .

This \bar{f}_2' is isotopic to the original \bar{f}_2 . We can suppose $\bar{f}_2' = \bar{f}_2$. By the twisting $H(1)$ and $H(g)$, we can convert the matrix $\Phi(\alpha)$ to $\Phi(\alpha)\varepsilon_{12}(1)$.

The other cases are quite similarly shown.

We complete the proof of Theorem 3.3.

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