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Configurartion of divisors and reflexive sheaves

(A combinatorial method in construction of reflexive sheaves)

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Dedicated to Professor Nagayoshi Iwahori
on his sixtieth birthday

INTRODUCTION

1. In the present paper we construct holomorphic vector bundles and reflexive sheaves on a complex manifold of dimension $\geqq 2$ from configuration of divisors on it. The construction is done by the Čech-stratification theoretical method, which was introduced in our previous paper [Sa-1].

The content of the paper is as follows: in § 0 we recall some facts in [Sa-1]. In § 1.1 we give basic definitions and problems on bundles and reflexive sheaves of the type just above. (Such sheaves are said to be of type (C)= configuration type.) In § 1.2 we see that two important bundles on the projective spaces, Horrocks--Mumford bundle([H-M]) and null correlation bundle([O-S-S]), are constructed from configuration of hyperplanes in a very simple manner, cf. Theorem 1.1. This fact leads us to the following problem:

(*) To find more bundles and sheaves of (C)

See Problem 1.1∼1.6, § 1 for the precisision of (*). In § 1.3 we propose some answers to (*) by forming examples of bundles and sheaves of type (C):(1) those of rank = 2, which may be a generalization of the above two bundles, cf. Theorem 1.2, and (2) those of rank $\geqq 3$, Theorem 1.3. These examples have interesting combinatorial properties and seem to provide new classes of sheaves; see § 1.4 for more such examples.

2. The rest of the paper concerns some general arguments on the structure of reflexive sheaves of type (C) and of more general type: In § 2~§ 5 we discuss what we call inductive structure of reflexive sheaves of type (C). This structure is a central notion in the theory of reflexive sheaves of type (C), and enables us to study the sheaves inductively on the subvarieties, which are constructed from the interesctions of the divisors in question;for the content of §2 §5, see the beginning of each section. Also see § 5.2 for the inductive structure of the sheaves in § 1.3 . In Appendix, we discusss the structure of the endomorphisms of bundles. This is a refinement of Part B, § 4 [Sa-1], and our idea is to analyze closely the adjoint of the transition matrix in question. The content of Appendix is applied to each algebraic bundle on a normal quasi projective variety, and may be useful for investigations of those bundles.

3.1. It has been known that configuration of divisors appear in interesting geometric problems on varieties;e.g. the theory of the branched covering(cf.[Hir] and [Ka-Na]) and the topology of the complement of the divisors(cf.[O-S-T]). The present paper may show that it appears also naturally in the theory of vector bundles and

reflexive sheaves. The content of the paper should be regarded, as in [Sa-1], rather experimental and provisional, but it may be a necessary step for clarification of the fascinating subject of relations between configuration of divisors and bundles and reflexive sheaves.

3.2. Many important results on bundles and reflexive sheaves on the projective space P_n have been known;cf.for examples, [Har-1,2], [O-S-S],[Ba-2] and [O-S-1,2,3]. It is a necessary and interesting task to clarify roles of bundles and reflexive sheaves of type (C) on P_n among all such sheaves. This seems to be not easy.

We hope to discuss it elsewhere.

In writing this paper, the author stayed at Universitat der Gottingen as a member of SFB 170 in spring of 1986. Discussions there are useful to confirm our back ground on coherent sheaves in general and on bundles. The author express his thanks for hospitality of Professors H.Grauert, H.Flenner, H.Spindler and other members. Also I thank to T.Hosoh and H.Kaji for useful discussions; Hosoh gave an intrinsic definition of the imbedding of <u>End</u> <u>E</u>, cf. Appendix, which was originally given by a matrix computation. The fact that the null correlation bundle is of type (C) is based on a computation of Kaji. His result is a very encouraging fact for our proposed theory of sheaves of type (C).

<u>Notation and terminology</u>

In this paper we use the following abbreviation:

(1) bundle = holomorphic vector bundle

For a complex space $\bar{X}$ we always mean by $\underline{O}_{\bar{X}}$ its structure sheaf. For a complex subspace $\bar{Y}$ of $\bar{X}$, we write $\underline{I}_{\bar{Y}}$ for its ideal sheaf. Let $\underline{E}_{\bar{X}}$ be a bundle over $\bar{X}$. By a frame of $\underline{E}_{\bar{X}}$ we mean sections $\underline{e} = (e_1, \ldots, e_r)$; $\underline{E}_{\bar{X}}$, r=rank $\underline{E}_{\bar{X}}$, which generate $\underline{E}_{\bar{X},p}$ at each $p \in \bar{X}$.(For a reflexive sheaf, we use the word frame only for the open set where the sheaf is locally free.) We write $M_{m,n}(U, \underline{O}_{\bar{X}})$, with $m,n \in Z_+$ and an open subset U of $\bar{X}$, for the $\Gamma(U,\underline{O}_{\bar{X}})$-module $M_{m,n}(\Gamma(U,\underline{O}_{\bar{X}}))$ consisting of $m \times n$-matrix whose coefficient is in $\Gamma(U,\underline{O}_{\bar{X}})$. Also we abbreviate as $M(U,\underline{O}_{\bar{X}})$ when (m,n) is forgettable. For an integer m 0 we set:

(2) $\Delta_m = \{i, \ldots, m\}$

In this paper, we should sometimes consider subvarieties of $\bar{X}$ at one time. A subvariety $\bar{Y}$ of $\bar{X}$ is written as $\bar{Y}^i$, i=codimension of $\bar{Y}$ in $\bar{X}$, when we want to make clear that i is the codimension of $\bar{Y}$.

Chapter I. Reflexive sheaves of type (C)

In § 0 we recall some facts in [Sa-1] in order to make claer the back ground of the present paper. In § 1 we give basic facts on the sheaves in the title.

§ 0. Preliminaries

Here we recall some facts in our previous paper in [Sa-1]. This is done chiefly to make clear the back ground of the present paper. Also we make some refinements of the content of the previous paper.

0.1. Prebundle. First of all the following may be a natural idea in treatments of holomorphic vector bundles(=bundles).

(*-1) To form frames of the bundle, which reflect properties of the bundle closely, and to use them for studies of the bundle.

Corresponding to this, our basic idea in the construction of bundles and reflexive sheaves is as follows:

(*-2) To form not only bundles and reflexive sheaves but also their suitable frames.

Note that this approach is a cohomological one. Our approach also depends on ideas and methods in stratification theory. In our approach, strata are, in principle, taken to be subvarieties where the frames fail to be the ones(= singular locus of the frames). This approach has an advantage in the point that

(*-3) the singularity of the frames is examined closely.

Now we recall some relavant definitions in our previous paper [Sa-1]. Let $\bar{X}$ be an irreducible normal complex space of dimension $\geqq 2$. Then our construction of holomorphic vector bundles(=bundles) and reflexive sheaves over $\bar{X}$ consists of the following two steps, cf. § 0,[Sa-1]:

(**) To find a bundle $\underline{E}_X$ over X endowed with suitable frames, where $X = \bar{X}$ -{codimension two reduced complex subspace of $\bar{X}$}, and to investigate the direct image $\underline{E}_{\bar{X}}$ of $\underline{E}_X$ with respect to the inclusion:X $\bar{X}$.

More precisely, by a prebundle over $\bar{X}$, we mean a pair $D_1=(\bar{X}^2,\underline{E}_X)$ consisting of a codimension two reduced complex subspace $\bar{X}^2$ of $\bar{X}$ and a bundle $\underline{E}_X$ over $X:=\bar{X} - \bar{X}^2$. We furnish $\underline{E}_X$ with frames from a stratification theoretical view point, cf. § 0,[Sa-1]: By an <u>s-repesentation datum for</u> D_1, we mean $D_2 = (\bar{X}^1, N_1, \underline{e}^0, \underline{e}^1)$ as follows:

$$\begin{aligned} &\bar{X}^1 = \text{a reduced divisor of } \bar{X} \text{ containing } \bar{X}^2 \\ (0.1.1)\qquad &N_1 = \text{an open neighborhood of } X^1 := \bar{X}^1 - \bar{X}^2 \text{ in } X \\ &\underline{e}^i = \text{a frame of } \underline{E}_{X|N_i},\ i=0,1\ ,\ \text{with } N_0 = \bar{X} - \bar{X}^1 \end{aligned}$$

<u>Figure I-1</u>

$\bar{X}^2$ $\underline{e}^1$ N_1 $\bar{X}^1$

$\underline{e}^0$ N_0

By an <u>s-prebundle over</u> $\bar{X}$ we mean a pair (D_1,D_2) as above;the precise form of (*-2) is:<u>To find an s-pre bundle</u> (D_1,D_2) <u>over</u> $\bar{X}$ <u>and to investigate the direct image</u> $\underline{E}_{\bar{X}}$ <u>of</u> $\underline{E}_X$. There are encouraging facts for such an approach:It may be regarded as a generalization of classical approaches to bundles on Riemann surfaces, cf. [Bi],[Tj] and [We](cf.§ 0,[Sa-1]). Moreover, if $\bar{X}$ is a normal quasi projective (resp.Stein) variety, then we have ,cf. Lemma 0.1, [Sa-1]:

(0.1.2) each algebraic(resp.holomorphic) bundle over $\bar{X}$ is obtained as the direct image of an s-prebundle over $\bar{X}$.

Moreover, the first step in (*-2) of finding the bundle $\underline{E}_X$ concerns interesting geometric properties of divisors;cf.§ 2,[Sa-1]. See also § 0.2 and § 1 in the present paper.

<u>Remark 0.1.</u> Let the s-pre bundle (D_1,D_2) be as above. When there is no fear of confusions, we call $\underline{E}_X$ also an <u>s-prebundle</u> over $\bar{X}$, cf. § 0 in [Sa-1]. In later arguments, the divisor $\bar{X}^1$ as in (0.1.1) plays important roles; we say that $\underline{E}_X$ is <u>attached to</u> $\bar{X}^1$.

Letting the s-pre bundle $\underline{E}_X$ be as above, the transition matrix, denoted by H, between the frames $\underline{e}^0$ and $\underline{e}^1$: $\underline{e}^0=\underline{e}^1H$ in $(N_0\cap N_1)$ plays very basic roles in our studies of the s-pre bundle $\underline{E}_X$. (See [Sa-1]. Also the main part of the present paper concerns the transition matrix in question.) Moreover, in the definition of the s-pre bundle, one can replace the role of the frames $(\underline{e}^0,\underline{e}^1)$ by that of the transition matrix H. Precisely, we have the following equivalence:

(0.1.3) to find an s-prebundle over $\bar{X}$ $\Longleftrightarrow$ to find a datum $(\bar{X}^1,\bar{X}^2,N_1,H)$

Here $\bar{X}^i$ are subvarieties of $\bar{X}$ of codimension $i(i=1,2)$ satisfying $\bar{X}^1\supset\bar{X}^2$, N_1 is an open neighborhood of $X^1:=\bar{X}^1-\bar{X}^2$ in $X:=\bar{X}-\bar{X}^2$, and H is an element of $GL(N_0\cap N_1,\underline{O}_{\bar{X}})$ with $N_0=\bar{X}-\bar{X}^1$.
Actually, remarking that $X=N_0\cup N_1$, we see that the matrix H defines a bundle $\underline{E}_X$ over X and frames $\underline{e}^i$ of $\underline{E}_{X|N_i}(i=0,1)$ satisfying:

(0.1.4) $\underline{e}^0=\underline{e}^1H$ in $(N_0\cap N_1)$

One checks (0.1.3) immediately from (0.1.4); in later arguments our construction of s-prebundles will be done by finding data as in the R.H.S. of (0.1.3).

<u>0.2. Determinantal divisor</u>. A very important topic in the theory of bundles is relations between them and subvarieties which are the loci of sections of the bundles. (See [Se],[Gr-Mu] and [Har-1] for the beautiful relation between bundles of rank two on the projective spaces and subvarities of codimension two of them.) Here we point out that our s-prebundle has an intimate relation with theories of divisors: Let the normal complex space $\bar{X}$ be as before

and $\bar{X}^1$ an effective divisor of $\bar{X}$. Moreover, let r be an integer ≥ 2 . Then we make the following definition.(The definition below may be a popular one. But, we do not know a suitable reference.)

<u>Definition 0.1</u>. An reflexive sheaf $\underline{E}_{\bar{X}}$ over $\bar{X}$ <u>has</u> $\bar{X}^1$ as its <u>determinantal divisor</u>, if there is an element $\underline{e} \in \Gamma^r(\underline{E}_{\bar{X}})$, r =rank $\underline{E}_{\bar{X}}$, such that

(0.2.1) $(\wedge^r \underline{e}_{|\bar{X}-S(\underline{E}_{\bar{X}})})_0 = \bar{X}^1{}_{|\bar{X}-S(\underline{E}_{\bar{X}})}$, with the singular locus $S(\underline{E}_{\bar{X}})$ of $\underline{E}_{\bar{X}}:=\{p \in \bar{X}; \underline{E}_{\bar{X},p}$ is not $\underline{O}_{\bar{X},p}$-free $\}$.

The following problem may be also popular. For completeness we make:

<u>Problem 0.1</u>. Determine if there is an indecomposable reflexive sheaf(or more strongly, locally free sheaf)$\underline{E}_{\bar{X}}$ of rank r such that

(0.2.2) $\underline{E}_{\bar{X}}$ has $\bar{X}^1$ as its determinantal divisor.

We show that s-prebundle has an intimate relation to Problem 0.1:

<u>Lemma 0.1. Let</u> $\bar{X}$ <u>be a smooth quasi projective variety and</u> $\bar{X}^1$ <u>a divisor of</u> $\bar{X}$. <u>If an algebraic vector bundle</u> $\underline{E}_{\bar{X}}$ <u>has</u> $\bar{X}^1$ <u>as its determinantal divisor, then there is an s-prebundle</u> $\underline{E}_X$ <u>over</u> $\bar{X}$, <u>which is attached to</u> $\bar{X}^1$, <u>such that</u>

(0.2.5) $\underline{E}_{\bar{X}}$ <u>is the direct image of</u> $\underline{E}_X$ <u>with respect to the inclusion</u>: $X \hookrightarrow \bar{X}$.

<u>Proof</u>. Take an element $\underline{e} \in \Gamma^r(\underline{E}_{\bar{X}})$ such that $(\wedge^r \underline{e})_0 = \bar{X}^1$. Also take a codimension two subvariety $\bar{X}^2$ of $\bar{X}^1$ satisfying (1)$\bar{X}^2 \subset \bar{X}^1$, (2) $\underline{E}_{X^1} := \underline{E}_{\bar{X}^1}$, $X^1 = \bar{X}^1 - \bar{X}^2$, is a trivial bundle and (3) X^1 is an affine variety. Taking a suitable open neighborhood N_1 of X^1, we may assume that $\underline{E}_{\bar{X}|N_1}$ has a frame $\underline{e}^1$. Clearly, $\{(\bar{X}^2, \underline{E}_{\bar{X}|X}), (\bar{X}^1, N_1, \underline{e}^0, \underline{e}^1)\}$, with $X = \bar{X} - \bar{X}^2$, is an s-pre bundle over $\bar{X}$ and $\underline{E}_{\bar{X}}$ is the direct image of $\underline{E}_X$.

q.e.d.

<u>Remark 0.2.</u> In Lemma 0.1, we assumes that $\underline{E}_{\bar{X}}$ is locall free. Instead assume that it is reflexive. Then, by replacing $\bar{X}$ by $\bar{X} - S(\underline{E}_{\bar{X}})$, we have the similar result to Lemma 0.1.

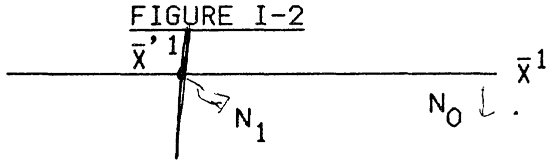
0.3. S-prebundle of type (G). Here we recall some results from a type of s-prebundle, which is said to be <u>of type</u> (G)(=<u>Grassmann type</u>), cf.§ 2,[Sa-1]. Such an s-pre bundle is the main subject in [Sa-1]. In that paper we worked with a normal complex space. Here, for simplicity, we work with a connected complex manifold of dimension 2. Let $\underline{L}_{\bar{X}}$ be a line bundle over $\bar{X}$, and let $\underline{s}$ $=(s_1,\dots,s_{r+1})$, $r \geqq 2$, be elements of $\Gamma(\underline{L}_{\bar{X}})$. We assume:

(0.3.1) $\bar{X}^1 := (s_{r+1})_0$ is reduced and irreducible, and $\bar{X}^2 := \bar{X}^1 \cap \bar{X}'^1$ where $\bar{X}'^1 = (s_r)_{0,red}$, is of codimension two in $\bar{X}$.

We then set $N_1 = \bar{X} - \bar{X}'^1$ and $N_0 = \bar{X} - \bar{X}^1$, and form a matrix $H \in GL_r(N_0 \cap N_1, \underline{O}_{\bar{X}})$ as follows:

(0.3.2) $$H = \begin{bmatrix} I_{r-1} & f_1 \\ 0 & f_r \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} f_j = s_j/s_r,\ 1 \leqq j \leqq r-1, \\ f_r = s_{r+1}/s_r \end{array}$$

FIGURE I-2



By (0.1.2) we have a bundle, denoted by $\underline{E}_{X,\underline{s}}$, over $X := \bar{X} - \bar{X}^2$ and frames $\underline{e}^i$ of $\underline{E}_{X|N_i}$ (i=0,1) which satisfy the relation: $\underline{e}^0 = \underline{e}^1 H$ in $N_0 \cap N_1$. An s-prebundle of type (G) is such an s-prebundle $\underline{E}_{X,\underline{s}}$, cf. Definition 2.1 and (2.5.1,2), [Sa-1]. The direct image $\underline{E}_{\bar{X},\underline{s}}$ is also said to be of type (G). Such reflexive sheaves have also generality. Actually, coresponding to (0.1.2) we have:

(0.3.3) each algebraic bundle over a noraml quasi projective variety is of type (G), up to a tensor product of a line bundle.

This follows from that the universal quotient bundle over a Grassmann variety is of type (G),cf. Appendix, [Sa-1] and (0.3.8 10) below, and from the imbedding theorem to a Grassmann variety, cf.[Fu]. Many problems arise concerning s-pre bundles of type (G). Here we discuss some relations of such s-prebundles to Problem 0.1. For this we first recall a basic invariant of the s-prebundle $\underline{E}_{\bar{X},\underline{s}}$ just above. Define a subvariety $\bar{Y}$ of $\bar{X}$ by

(0.3.4) $$\bar{Y} := \cap_{j=1}^{r+1} (s_j)_{0,red}\ , \text{ cf. § 2,[Sa-1]}.$$

In our studies of s-prebundles, the variety $\bar{Y}$ plays a very important role;See § 4, [Sa-1] for the role of such a variety in the determination of $\Gamma(\underline{E}_{X,\underline{s}})$ and $\Gamma(\underline{\text{End}}\ \underline{E}_{X,\underline{s}})$.) The variety $\bar{Y}$ plays also an important role in the determination of the local structures of $\underline{E}_{\bar{X},\underline{s}}$,cf, §3. Here we only recall the following, cf.Theorem 3.1.1,2:

(0.3.5) If the direct image $\underline{E}_{\bar{X},\underline{s}}|\bar{X}_{smooth}$, $\bar{X}_{smooth}:=\{p \in \bar{X};\ \bar{X}$ is smooth at $p\}$, is locally free, then:

$$\bar{Y}\cap\bar{X}_{smooth} = \phi, \quad \text{or codimension of it in } \bar{X}_{smooth} = 2.$$

The second condition is very strong, since $\bar{Y}$ is the intersection of $r+1 \geqq 3$ divisors. It looks like that many interesting bundles of type (G) satisfy the second condition; we make:

Problem 0.2. Find indecomposable bundles of type (G) satisfying the second condition in (0.3.5).

In connection with this we see that there are many indecomposable bundles of type (G), which satisfy the first condition in (0.3.5). Actually assume that

(0.3.6) $s_1,\ldots,s_{r+1}$ are linearly independent over C, and $\bar{Y}=\phi$.

Theorem 0.1. If $\bar{X}$ is compact, $\underline{E}_{\bar{X},\underline{s}}$ is simple and locally free, and has $\bar{X}^1$ as its determinantal divisor.

Proof. (0.3.4) insures the local freeness of $\underline{E}_{\bar{X}}$. Theorem 4.6, [Sa-1] insures that $\Gamma(\underline{\text{E}}\text{nd}\ \underline{E}_{X,\underline{s}}) \simeq C$. The second fact is clear from Lemma 0.1. q.e.d.

The condition $\bar{Y}=\phi$ is a generic one, if rank $\underline{E}_{\bar{X},\underline{s}} \geqq \dim \bar{X}$, and we have many indecomposable bundles with:rank of it $\geqq \dim \bar{X}$.(This fact was first found by Maruyama in [Ma] by means of the theory of elementary transformation. In § 4, [Sa-1] we gave an another proof from the Cech-stratification theoretical view point.) The following follows easily from what was mentioned above:

<u>Corollary</u>. <u>Let</u> $\bar{X}^1$ <u>be an irreducible and reduced divisor of the projective space</u> $P_N(N \geqq 2)$. <u>Then</u> $\bar{X}^1$ <u>is a determinantal divisor of an indecomposable bundle of rank</u> r, <u>for any</u> $r \geqq N$.

This corollary seems to follow also from the earlier results in [Ma]. It seems that a determinantal divisor of a bundle of low rank has, in general, singularities. We add the following problem:

<u>Problem 0.3</u>. Let $\bar{X}^1$ be an effective divisor of $P_N; N \geqq 4$. if it is a determinantal divisor of an indecomposable bundle of rank N, then $\bar{X}^1$ has a singularity ?

When the bundle is of type (G), Problem 0.3 seems to be afirmative from (0.3.4) and the Lefschetz theorem for $\mathrm{Pic}(P_N)$. A similar problem to Problem 0.3 was given by Ockonek-Spindler([O-S]) for mathematical instanton bundle.

Finally we give two examples of bundles of type (G).

<u>Example 1</u>. Let $\underline{s} = (s_1, \ldots, s_{n+1})$ be a basis of $\Gamma(\underline{O}_{P_n}(1))$. Then we have a reflexive shaf $\underline{E}_{P,\underline{s}}$ of type (G) over $P = P_n$ in the manner son above. then we have:

(0.3.7) $\quad \underline{E}_{P,\underline{s}} = T_P(-1)$, with the tangent bundle T_P of P.

This is checked by computing the transition matrix of T_P. Note that the variety $\bar{Y}$ in (0.3.4) $= \phi$.

<u>Example 2</u>. Let $\bar{X} = Gr_d(V^n)$ be the Grassmann manifold of the d-dimensional linear subspaces of a vector space V of dimension n. We assume that d and $r := n-d \geqq 2$. Denote by $\underline{E}_{\bar{X}}$ the universal quotient bundle over $\bar{X}$:

(0.3.8) $$V_{\bar{X}} := V \times \bar{X} \xrightarrow{\omega} \underline{E}_{\bar{X}} \to 0 .$$

Take a basis $e_1, \ldots, e_n$ of V. We regard them as the elements of $\Gamma(V_{\bar{X}})$. Then we form sections $s_i \in \Gamma(\Lambda^r \underline{E}_{\bar{X}}); i = 1, \ldots, r+1$, as follows:

(0.3.9) $\quad s_i := e_1 \wedge \quad \wedge \hat{e}_i \quad \wedge e_{r+1}$ (e_i is omitted).

Setting $\underline{s} = (s_1, \ldots, s_{r+1})$ we form a reflexive sheaf $\underline{E}_{\bar{X},\underline{s}}$ over $\bar{X}$. Then we have:

(0.3.10) $\underline{E}_{\overline{X},\underline{s}}$ coincides with the universal quotient bundle $\underline{E}_{\overline{X}}$.
This fact is not trivial;see Appendix, [Sa-1]. The variety $\overline{Y}$ in (0.3.4) in the present situation is the Schubert subvariety of codimension two, which represents (−1)×(the second chern class of the universal subbundle);cf.also [Sa-1].

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§ 1. Reflexive sheaves of type (C)

1.1. Definitions. Let $\bar{X}$ be a connected complex manifold of dimension $\geqq 2$ and $\bar{X}^1$ a divisor of $\bar{X}$ as follows:

(1.1.0.0) $\bar{X}^1 = \bigcup_{i\in\Delta_m} \bar{X}^1_i$, $\Delta_m = \{1,.,m\}; m \geqq 2$, is a reduced and reducible divisor of $\bar{X}$, whose irreducible components $\bar{X}^1_i$ are normal.

The divisor $\bar{X}^1$ must satisfy the following:

(1.1.0.1) $\bar{X}^2_{ij} := \bar{X}^1_i \cap \bar{X}^1_j, i\neq j \in \Delta_m$, is of codimension two in $\bar{X}$.

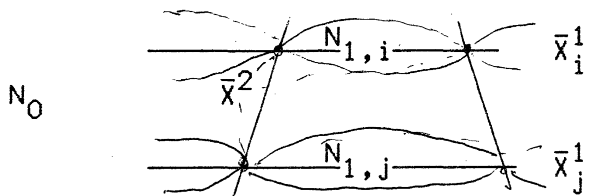
The purpose here is to form an s-prebundle from $\bar{X}^1$ along the line of § 0. For this we first form subvarieties of $\bar{X}$ as follows, cf. §.0.1:

(1.1.0.2) $\bar{X}^2 = \bigcup_{i\neq j} \bar{X}^2_{ij}$, $X = \bar{X} - \bar{X}^2$, $N_0 = \bar{X} - \bar{X}^1$,
$X^1_i = \bar{X}^1_i - \bar{X}^2, i \in \Delta_m$, $X^1 = \bar{X}^1 - \bar{X}^2 = \coprod_{i\in\Delta_m} X^1_i$

and take an open neighborhood of X^1 of the following form:

(1.1.0.3) $N_1 = \bigcup_{i\in\Delta_m} N_{1,i}$, where $N_{1,i}$ is an open neighborhood of X^1_i in X, and $N_{1,i} \cap N_{1,j} = \phi$ if $i\neq j$

Figure II-1



Next take a matrix $H \in M_r(N_1, \underline{O}_{\bar{X}})$, $r \geqq 2$, such that $(i\in\Delta_m)$:

$$(1.1.0.4)\qquad H_{|N_{1,i}} = \begin{array}{cc} \begin{array}{ccccccc} & & & \alpha(i) & & & \end{array} & \\ \begin{bmatrix} 1 & & & f_1 & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & f_{\alpha(i)} & 1 & & \\ & & & & & 1 & \\ & & & f_r & & & 1 \end{bmatrix} & \alpha(i) \end{array}$$

Alternatively, $H_{|N_{1,i}}$ is as follows: For an $\alpha(i) \in \Delta_r$ we have:

$$(1.1.0.5)\qquad \text{the } (s,t)\text{-component of } H_{|N_{1,i}} = \begin{cases} \delta_{s,t}, & \text{if } t \neq \alpha(i) \\ f_{s|N_{1,i}}, & \text{if } t = \alpha(i) \end{cases}$$

Here $f_j, j \in \Delta_r$, is an element of $\Gamma^r(N_1, \underline{O}_{\bar{X}})$ and satisfies; $i \in \Delta_m$:

(1.1.0.6) $(f_{\alpha(i)}|N_{1,i})_0 = X_i^1$, and $f_j|X_i^1$, $j \in \Delta_r$, is the restriction to X_i^1 of a (unique) meromorphic function over $\bar{X}_i^1$ with the pole $\bar{X}_i^1 \cap \bar{X}^2$.

Clearly we have:

(1.1.0.7) $\det H_{|N_{1,i}} = f_{\alpha(i)}$, and so $(\det H)_0 = X^1$.

Now remarking that $N_0 \cap N_1 = (N_1 - X^1 = \coprod_{i \in \Delta_m} N_{1,i} - X_i^1)$, we have: $H \in GL_r(N_0 \cap N_1, \underline{O}_{\bar{X}})$. The datum $(\bar{X}^1, \bar{X}^2, N_1, H)$ defines an s-pre bundle $\underline{E}_X$ over $\bar{X}$ of rank r, which is characterized as follows, cf.(0.1.3):

(1.1.0.8) $\underline{E}_{X|N_i}$ has a frame $\underline{e}^i (i=0,1)$, and the transition relation between them is: $\underline{e}^0 = \underline{e}^1 H$ in $N_0 \cap N_1$.

Such an s-pre bundle is our main subject in the present paper:

<u>Definition 1.1</u>. (1) An s-prebundle over $\bar{X}$ is said to be <u>of type</u> (C)(= <u>configuration type</u>), if it is constructed in the manner just above ,cf.(1.1.0.8), from data like $(\bar{X}^1, \bar{X}^2, N_1, H)$ as in (1.1.0.0 6).(2) A reflexive sheaf over $\bar{X}$ is said to be <u>of type</u> (C), if it is the direct image, with respect to the inclusion: $X \quad \bar{X}$, where X is an open part of $\bar{X}$ in question, of an s-prebundle of type (C).

As we will see in the course of the discussions, the sheaf of type (C) has some peculiar properties, which general sheaves do not have;cf.Proposition 1.1 and § 4 an § 5. This stems chiefly from the following:

(*) The divisor $\bar{X}^1$ is <u>reducible</u> and $\underline{E}_X$ is a product bundle in the neighborhood N_1 of X^1.

Let the s-prebundle $\underline{E}_X$ and the frames $\underline{e}^0$ and $\underline{e}^1$ be as above. Then $\underline{E}_X$ is attached to $\bar{X}^1$, cf.Remark 0.1. Moreover, we have:

<u>Proposition 1.1. Each component of the frame</u> $\underline{e}^0$ <u>is a section of</u> $\underline{E}_X$, <u>and we have</u>:

(1.1.1.1) <u>the determinantal divisor of</u> $\underline{e}^0(:=(\Lambda^r\underline{e}^0)_0) = X^1$.

<u>Proof</u>. This is clear from that $H \in M_r(N_1,\underline{O}_{\bar{X}})$ and that $(\det H)_0 = X^1$, cf.(1.1.0.6). q.e.d.

Thus $\underline{E}_X$ has the <u>reduced divisor</u> $\bar{X}^1$ as its determinantal divisor.

<u>Remark 1.1</u>. (1) From the meromorphy of $f_{s|X_i^1}$, cf.(1.1.0.6), we see that the direct image $\underline{E}_{\bar{X}} = i_*\underline{E}_X$, where $\underline{E}_X$ is as above and i is the injection:X $\bar{X}$, is coherent, cf.Corollary to Theorem 4.0.

(2) Note that $N_{1,i} \cap N_{1,j} = \phi; i \neq j$, and we have:$\Gamma(N_1,\underline{O}_{\bar{X}}) = \oplus_{i\in\Delta_m} \Gamma(N_{1,i},\underline{O}_{\bar{X}})$. Thus an element $f \in \Gamma(N_1,\underline{O}_{\bar{X}})$ can behave independetly on each $N_{1,i}$.(If we do not assume the disjoince condition, the unicity of the analytic continuation implies that the behavior of f in $N_{1,i}$ determines that in $N_{1,j}$, $i \neq j$.) We use this fact frequently in treatments of the bundle $\underline{E}_X$ as above.

<u>1.1.2</u>. Here we propose some basic problems for reflexive sheaves of type (C). The problems are essentially the restriction of the ones in § 0 to the present situation. The complex manifold $\bar{X}$ and its reducible divisor $\bar{X}^1$ are as in the beginning of § 1.1.1. First, corresponding to Problem 0.1, we make:

<u>Problem 1.1</u>. Determine if $\bar{X}^1$ is a determinatal divisor of an indecomposable reflexive sheaf of type (C).

We sharpen Problem 1.1 as follows:

<u>Problem 1.2</u>. Determine if $\bar{X}^1$ is a determinantal divisor of an indecomposable bundle of type (C) satisfying:the rank of it dim $\bar{X}$. We may say that the bundle satisfying the above inequality is of <u>low rank</u>(with respect to the dimension of $\bar{X}$.) We add the following wearker version of Problem 1.2:

<u>Problem 1.3</u>. Determine if $\bar{X}^1$ is a determinantal divisor of an indecomposable reflexive sheaf of type (C), say $\underline{E}_{\bar{X}}$, satisfying: $\operatorname{codim}_{\bar{X}} S(\underline{E}_{\bar{X}})$ rank $\underline{E}_{\bar{X}} + 2$; with the singular locus $S(\underline{E}_{\bar{X}})$ of $\underline{E}_{\bar{X}}$, cf.(0.2.3).

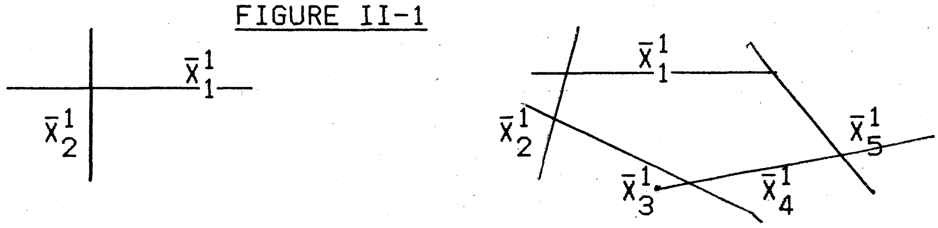
also said to be <u>of low rank</u>. Actually we set $r+k = \mathrm{codim}_{\bar{X}} S(\underline{E}_{\bar{X}})$, $2 \leqq k$, and take a subvariety $\bar{Y}$ of $\bar{X}$ satisfying: $\dim \bar{Y} = r+k'$, with $1 \leqq k' k$ and $S(\underline{E}_{\bar{X}}) \cap \bar{Y} = \phi$. (If $\bar{X}$ has many subvarieties, such a $\bar{Y}$ exists.) Thus $\underline{E}_{\bar{X}|\bar{Y}}$ is locally free and of low rank. (One should check if $\underline{E}_{\bar{X}|\bar{Y}}$ is indecomposable or not. In this paper we do not enter into this problem. We hope to discuss it in an another place.) See Theorem 1.2 and 1.3, § 1.3 for a partial answer to Problem 1.1$\sim$1.3.

<u>1.2</u>. <u>Examples...1</u>. Here we see that the two important examples of vector bundles on the projective spaces, Horrocks--Mumford bundle([H-M]) and the null correlation bundle([O-S-S]), are of type (C), cf. Introduction. This is checked by giving a transition matrix attached to them in an explicit form. (The null correlation bundle is treated here only in the case of P_3. The higher dimensional cases will be discussed elsewhere; see also §1.3.)

Now let P_N denote the projective space of dimension N, and let $z = (z_1, \ldots, z_{n+1})$ be its homogeneous coordinates. We set $\bar{X}^1_i = (z_i)_0$. In the rest of § 1.2 we write $\bar{X}$ for P_3 or P_4, and, according as $\bar{X} = P_3$ or P_4, define a divisor $\bar{X}^1$ as follows:

(1.2.1.0) $\quad \bar{X}^1 = \bar{X}^1_1 \cup \bar{X}^1_3$ (case of P_3), or $\bar{X}^1 = \cup_{i=1}^{5} \bar{X}^1_i$ (case of P_4).

<u>FIGURE II-1</u>



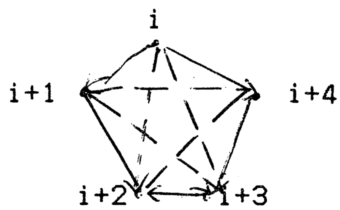
When we are concerned with P_3 define a 2×2-matrix as follows:

(1.2.1.1) $$\begin{bmatrix} 1 & z_{i+1}/z_{i+2} \\ 0 & z_i/z_{i+2} \end{bmatrix}; \quad i = 1, 3 \in Z/4Z$$

When we are comcerned with P_4 we define a matrix as follows:

(1.2.1.2) $$\begin{bmatrix} 1 & z_{i+2}z_{i+3}/z_{i+1}z_{i+4} \\ 0 & z_i z_{i+1}/z_{i+1}z_{i+4} \end{bmatrix}; i \in Z/5Z$$

The (2,2)-term $= z_i/z_{i+4}$; we write it as above to make it as the quotient of monomials of degree 2. Also note that the pentagon phenomenon ,cf.[H-M] and [B-H-M], in the (1,2)-component; we like to emphasize the simple form of the above transition matrices.



Now we apply the argumens in § 1.1 to the above matrices:As previously we set:

(1.2.1.3) $\quad \bar{X}^2 = \cup_{i,j}(\bar{X}^1_i \cup \bar{X}^1_j)$, and $X = \bar{X} - \bar{X}^2$, $X^1 = \coprod_i X^1_i (= (\bar{X}^1 - \bar{X}^2 = \coprod_i \bar{X}^1_i - \bar{X}^2)$

Also tahe an open neighborhood $N_1 = \coprod_i N_{1,i}$ of $X^1 = \coprod_i X_{1,i}$ in X, satisfying:$N_{1,i} \cap N_{1,j}$.(The neighborhood N_1 is just additional to the data in (1.2.1.0$\sim$3)). Now define a matrix $H \in M_2(N_1, \underline{O}_{\bar{X}})$ by:

(1.2.1.4) $\quad H_{|N_{1,i}}$ = the matrix in (1.2.1.1) or (1.2.2.2), according as we are concerned with P_3 or P_4;$i \in \Delta_m$

The datum $(\bar{X}^1, \bar{X}^2, N_1, H)$ defines a bundle $\underline{E}_X$ over X and its frames $\underline{e}^i$, i=0,1, of $\underline{E}_{X|N_i}$ such that $\underline{e}^0 = \underline{e}^1 H$ in $N_0 \cap N_1$.(Here $N_0 = \bar{X} - \bar{X}^1$.) Write $\underline{E}_{\bar{X}}$ the direct image of $\underline{E}_X$ with respect to the inclusion:$X \hookrightarrow \bar{X}$. In the theorem below we write $\underline{E}_{\bar{X}}$ as $\underline{E}_{P_3}$ or $\underline{E}_{P_4}$, according to whether $\bar{X} = P_3$ or P_4.

<u>Theorem 1.1.1</u>. <u>$\underline{E}_{P_3}$ and $\underline{E}_{P_4}$ are locally free and indecomposable</u>. <u>Moreover we have</u>:

(1.2.3) $\quad \underline{E}_{P_3}$ = <u>null correlation bundle twisted by</u> $O_{P_3}(1)$
$\quad \underline{E}_{P_4}$ = <u>Horrocks-Mumford bundle</u>

This theorem is proven in two ways:First, using the representation of those bundles by monad, cf.[H-M] and [O-S-S], one checks that the above bundles admit frames of the above types. This is essentially a tedious computation, and will be given elsewhere. Second, without using any known facts on the null correlation and Horrocks-Mumford bundles, we check that the bundles $\underline{E}_{P_3}$ and $\underline{E}_{P_4}$ are locally free and simple. Also we check the characteristic properties of the above two bundles;see § 5 and Appendix.

1.3. Examples...2. Here we give some examples of reflexive sheaves of type (C).

1.3.0. Data in § 1.3. Here we work with a connected complex manifold $\bar{X}$ of dimension $\geqq 3$ and data as follows:

(1.3.0.1) a line bundle $\underline{L}_{\bar{X}}$ over $\bar{X}$ and sections $\underline{s} = (s_1,\dots,s_m); m \geqq 2$, $\subset \Gamma(\underline{L}_{\bar{X}})$; we set $\bar{X}_i^1 = (s_i)_0$ and $\bar{X}^1 = \cup_{i\in\Delta_m} \bar{X}_i^1$.

We assume that $\bar{X}_i^1, i \in \Delta_m$, is irreducible and smooth and that $\bar{X}^1$ satisfies (1.1.0.1). More strongly we assume the condition (2.0.0), which is introduced in the beginning of § 2.(This insures that $\bar{X}^1$ is a normal crossing and satisfies some global conditions.) From the data in (1.3.0.1) we form a codimension two subvariety $\bar{X}^2$ and the open parts X and X^1, X_i^1 of $\bar{X}$ and $\bar{X}^1, \bar{X}_i^1$ as in (1.1.0.2):

(1.3.0.2) $\bar{X}^2 = \cup_{i,j}(\bar{X}_i^1 \cap \bar{X}_j^1)$, $X = \bar{X} - \bar{X}^2$, and $X^1 (= \coprod_{i\in\Delta_m} X_i^1) = \bar{X}^1 - \bar{X}^2 (= \coprod_{i\in\Delta_m} \bar{X}_i^1 - \bar{X}^2)$.

We take an open neighborhood $N_1 = {}_{i\in\Delta_m} N_{1,i}$ of $X^1 = {}_{i\in\Delta_m} X_i^1$ satisfying $N_{1,i} \cap N_{1,j} = \phi$, cf.§ 1.1. We fix an element $f \in \Gamma(N_1, \underline{O}_{\bar{X}})$ satisfying

(1.3.0.3) $(f)_0 = X^1$.

(Such an f exists. For example define f by $f_{|N_{1,i}} = s_i/s_{i+1}; i\in\Delta_m$.) We use the following element frequently in § 1.3:

(1.3.0.4) $s_i^{\wedge} = (\Pi_{j\in\Delta_m} s_j)/s_i \in \Gamma(\underline{L}_X^{\otimes(m-1)})$.

Remark 1.2. In § 1.3 we assume that dim $\bar{X} \geqq 3$, cf. the beginning of § 1.3. The case of dim $\bar{X} = 2$ will be discussed by my students, Y.Hino and M.Kagesawa, in an another place.

1.3.1. First we generalize the transition matrix of the null correlation bundle. For this take an element $t_i \in \Gamma(\underline{L}_{\bar{X}}^{\otimes(m-1)}); i\in\Delta_m$, and form a matrix $H \in M_2(N_1, \underline{O}_{\bar{X}})$ as follows:

(1.3.1.0) $H_{|N_{1,i}} = \begin{bmatrix} 1 & t_i/s_i^{\wedge} \\ 0 & f \end{bmatrix}$ with $s_i^{\wedge}$ in (1.3.0.4)

The datum $(\bar{X}^1,\bar{X}^2,N_1,H)$ defines a bundle $\underline{E}_X$ over X and its frames $\underline{e}^i$ of $\underline{E}_{X|N_i}$, i=0,1, satisfying $\underline{e}^0=\underline{e}^1H$ in $(N_0\cap N_1)$, cf.(0.1.4). Assume the following generic condition for $t_i;i\in\Delta_m$:

(1.3.1.1) For any $I\subset\Delta_m$ satisfying $\bar{X}_I:=\bigcap_{i\in\Delta_m}\bar{X}^1_i\neq\phi$ we have:(1)$t_{\alpha|\bar{X}_{I,\lambda}}\neq 0;\alpha\in\Delta_m$, and (2) for any $J\subset\Delta_m$ we have:$\bigcap_{\beta\in J}(t_{\beta|\bar{X}_{I,\lambda}})_0$ is of codimension $\geqq$#J in $\bar{X}_{I,\lambda}$.(Here $\bar{X}_{I,\lambda}$ is an irreducible component of $\bar{X}_I$.)

(1.3.1.2) For each $i\in\Delta_m$,$(t_i/s_i^{\wedge})_{|\bar{X}^1_i}$ and $(t_i/s_i^{\wedge})^2_{|\bar{X}^1_i}$ and 1(=constant function with the value 1) are linearly independent over C.

When m =2 we also assume:

(1.3.1.3) $(t_{i|\bar{X}^2_{ij}})^{\otimes 2}$ and $(t_{j|\bar{X}_{ij}})^{\otimes 2}$ are linearly independent over C $;i,j\in\Delta_m$.

<u>Theorem 1.2.1</u>. <u>The direct image</u> $\underline{E}_{\bar{X}}$ <u>of</u> $\underline{E}_X$ <u>with respect to the inclusion</u>:X $\bar{X}$ <u>is simple and</u>

(1.3.1.4) <u>Codim</u> $S(\underline{E}_{\bar{X}})\geqq 4$ <u>with the singular locus</u> $S(\underline{E}_{\bar{X}})$ <u>of</u> $\underline{E}_{\bar{X}}$.

(See (0.1.2) for $S(\underline{E}_{\bar{X}})$.) The proof of Theorem 1.2.1 and more informations on $\underline{E}_{\bar{X}}$ will be given in § 5 and Appendix.(We use (1.3.1.1) for the proof of (1.3.1.4) and (1.3.1.2,3) for that of the simplenes of $\underline{E}_{\bar{X}}$.) Theorem 1.2.1 insures the existence of simple reflexive sheaves of rank two whose singular locus has a large codimension, cf. Problem 1.2 and 1.3. In particular assume that $\bar{X}$ $=P_n;n\geqq 3$, $\underline{L}_{\bar{X}}=\underline{O}_{\bar{X}}[1]$ and m satisfies:$2\leqq m\leqq n+1$. Then one sees readily that (1.3.1.1,2,3) are generic conditions for $t_j\in\Gamma(\underline{O}_{\bar{X}}(m-1));j\in\Delta_m$. Thus we we have:

(1.3.1.5) $\bar{X}^1(:=\Pi_{i\in\Delta_m}s_i)_0$ is a determinantal divisor of a simple reflexive sheaf,$\underline{E}_{\bar{X}}$, of type (C) and rank two satisfying: $\mathrm{codim}_{\bar{X}}S(\underline{E}_{\bar{X}})\geqq 4$.

<u>Remark 1.2.1</u>. In Theorem 1.2.1, assume that $\bar{X}=P_3$,and $s_1=z_1$,$s_2=z_3$ and $t_1=z_2$,$t_2=z_4$. Then $\underline{E}_{\bar{X}}$ is the null correlation bundle,

cf.Theorem 1.1. Restrict the bundle to P_2. Then (1.3.1.3) is not satisfied. From this we can check that the restriction is not simple. This is a well known fact, cf.for example,[O-S-S].

1.3.2. Here we generalize the transition matrix of Horrocks-Mumford bundle,cf.(1.2.2). We assume here that m is of the form: $m = 4n+1; n \in Z_+$. For each $i \in Z/mZ$, form elements $g_{i,\alpha} \in \Gamma(\underline{L}_{\bar{X}}^{\otimes 4n})$, $\alpha =$ 1,2, as follows:

(1.3.2.0) $g_{i,2} = \Pi_u s_u$, $u = i \pm a$, and $g_{i,1} = \Pi_u s_u$, $u = i \pm (n+a); a = 1,\dots,n$.

We define a matrix $H \in M_2(N_1, \underline{O}_{\bar{X}})$ as follows:

$$\begin{array}{lccc} & & i & \\ g_{i,2} \to & \begin{matrix} i+1 \\ \vdots \\ i+n \end{matrix} & & \begin{matrix} i-1 \\ \vdots \\ i-n \end{matrix} \\ g_{i,1} \to & \begin{matrix} i+n+1 \\ \vdots \\ i+2n \end{matrix} & & \begin{matrix} i-n-1 \\ \vdots \\ i-2n \end{matrix} \end{array}$$

(1.3.2.1) $$H_{|N_{1,i}} = \begin{bmatrix} 1 & g_{i,1}/g_{i,2} \\ 0 & f \end{bmatrix}$$

As previously the matrix H defines a bundle, denoted by $\underline{E}_X$, over X and frames $\underline{e}^i$ of $\underline{E}_{X|N_i}$, $i = 0,1$, with $\underline{e}^0 = \underline{e}^1 H$. Assume the following:

(1.3.2.2) For each $i \in \Delta_m$, $(g_{i,1}/g_{i,2})_{|\bar{X}_i^1}$, $(g_{i,1}/g_{i,2})^2_{|\bar{X}_i^1}$ and 1 are linearly independent over C. Moreover, for each $i \neq j \in \Delta_m$, if we have: $b_\alpha \otimes s^{\otimes 2}_{\alpha|\bar{X}^2_{ij}} = b_\beta \otimes s^{\otimes 2}_{\beta|\bar{X}^2_{ij}}$, with b_α and $b_\beta \in \Gamma(\underline{L}_{\bar{X}_i})$; $\alpha, \beta \in \Delta_m - \{i,j\}$ and $\underline{L}_{\bar{X}_i} = \underline{L}_{\bar{X}|\bar{X}_i}$, then $b_\alpha = b_\beta = 0$.

We write $\underline{E}_{\bar{X}}$ for the direct image of $\underline{E}_X$ with respect to the inclusion :X $\bar{X}$.

Theorem 1.2.2. $\underline{E}_{\bar{X}}$ is simple and $\mathrm{codim}_{\bar{X}} S(\underline{E}_{\bar{X}}) \geqq 4$.

The proof of Theorem 1.2.2 and more informations on $\underline{E}_X$ will be given in § 5 and Appendix. Theorem 1.2.2 gives an another method to form indecomposable reflexive sheaves of rank two whose singular locus is of large codimension;cf.Problem 1.2 and 1.3.

Remark 1.3.2.In Theorem 1.2.2 if n=1(and so m=5) then the true inequality: 4 holds. The inductive structure of $\underline{E}_{\bar{X}}$ in this case is discussed in detail up to codimension four, cf.§ 5. Moreover, if $\bar{X} = P_5$ and $\underline{L}_{\bar{X}} = \underline{O}_{P_5}(1)$ and if $s_i = z_i$ we have the Horrocks-Mumford bundles, cf.Theorem 1.1.

<u>1.3.3</u>. Here we discuss reflexive sheaves of type (C), Definition 1.1, and of rank $\geqq 3$. Contrary to the case of rank two, we can not construct a new example of such a sheaf satisfying:

(1.3.3.0) (rank of it) $+1 > \mathrm{codim}_{\bar{X}}$(singular locus of the sheaf).

But one can construct such sheaves satisfying the equality instead of the inequality just above. Our example is as follows: Let r be an integer $\geqq$ 3, and take sections $u_\alpha \in \Gamma(\underline{L}_{\bar{X}}^{\otimes(m-1)})$; $1 \leqq \alpha \leqq r-1$. We define a matrix $H \in M_r(N_1, \underline{O}_{\bar{X}})$ as follows:

$$(1.3.3.0) \quad H_{|N_{1,i}} = \begin{bmatrix} I_{r-1} & u_\alpha/\hat{s}_i \\ 0 & f \end{bmatrix} ;1 \leqq \alpha \leqq r-1$$

(See (1.3.0.3,4) for f and $\hat{s}_i$.) As previously the matrix H defines a bundle $\underline{E}_X$ over X and frames $\underline{e}^i$ of $\underline{E}_{X|N_i}$, i=0,1, satisfying $\underline{e}^0 = \underline{e}^1 H$. We write $\underline{E}_{\bar{X}}$ for the direct image of $\underline{E}_X$ with respect to the inclusion: $X \quad \bar{X}$. Now assume the following conditions:

(1.3.3.1) $u_{\alpha|\bar{X}_i} \neq 0$ for each $\alpha \in \Delta_{r-1}$ and $i \in \Delta_m$, and
$\mathrm{codim}_{\bar{X}}(\bar{X}^2 \cap (\cap_{\alpha=1}^{r-1}(u_\alpha)_{0,red})) = r+1$.

(1.3.3.2) For each $i \in \Delta_m$ and $\alpha \in \Delta_{r-1}$, $(u_\gamma/\hat{s}_i)_{|\bar{X}_i^1}, (u_\alpha \otimes u_\gamma/(\hat{s}_i)^{\otimes 2})_{|\bar{X}_i^1}$; $\gamma \in \Delta_{r-1}$, and 1 are linearly independent over C.

(1.3.3.3) For each $i \in \Delta_m$, take elements $v_{i,\alpha} \in \Gamma(\underline{L}_{\bar{X}_i})$; $\alpha \in \Delta_{r-1}$. If
$\Sigma_{\alpha=1}^{r-1} u_{\alpha|\bar{X}_i} \otimes v_{i,\alpha} \equiv 0 (\mathrm{mod}.\hat{s}_{i|\bar{X}_i^1})$ then $v_{i,\alpha} = 0$; $\alpha \in \Delta_{r-1}$.

<u>Theorem 1.3.1</u>. <u>$\underline{E}_{\bar{X}}$ is simple and $\mathrm{codim}_{\bar{X}} S(\underline{E}_{\bar{X}}) = r+1$</u>. <u>Moreover, we have</u>:

$$(1.3.3.4) \quad S(\underline{E}_{\bar{X}}) = \bar{X}^2 \cap (\cap_{\alpha=1}^{r-1}(u_\alpha)_{0,red}) .$$

We prove this in § 5.2 and in Appendix. (We use (1.3.3.1) for the proof of (1.3.3.4). We use (1.3.3.2,3) for the proof of the simpleness of $\underline{E}_{\bar{X}}$. Assume that $\bar{X} = P_n$; n $\geqq$3 and that (1) $\underline{L}_{\bar{X}} = \underline{O}_{\bar{X}}(1)$ and (2) $s_1, \dots, s_m \in \Gamma(\underline{O}_{\bar{X}}(1))$ are linearly independet over C. Moreover, we assume that m =n+1 and r =n. Then we have:

(1.3.3.5) General elements $u_1, \dots, u_{n-1} \in \Gamma(O_{\bar{X}}(n))$ satisfy (1.3.3.1 3). (This is obvious for (1.3.3.1). In order to check (1.3.3.2,3), it suffices to see that they holds for a $(u_1,\dots,u_{n-1}) \in \Gamma^{n-1}(\underline{L}_X^{\otimes n})$. We see easily that $u_\gamma = s_{n+1}^{\otimes n} + s_\gamma^{\otimes n}; 1 \leq \gamma \leq n-1$ satisfy (1.3.3.2,3). Thus ,over $P_n; n \geq 3$, there is a simple bundle of rank n such that

(1.3.3.5) it is of type (C) and it has $(\Pi_{i\in\Delta_n} s_i)_0$ as its determinantal divisor.(Here $s_i; i\in\Delta_n$,is a basis of $\Gamma(\underline{L}_{\bar{X}})$.)

We add one another example of a reflexive sheaf of type (C) whose frames(and the resulting transition matrix) have an interesting combinatorial property:Here we assume that $m \geq 3$. For each $i \in\Delta_m$ define a vector $\underline{g}_i \in \Gamma((\underline{L}_{\bar{X}}^{\otimes (m-1)})^{\oplus m})$ as follows:

(1.3.4.1) $\quad {}^t\underline{g}_i = (s_1^{\otimes (m-1)},\dots,s_{i-1}^{\otimes (m-1)}, s_i^{\wedge}, s_{i+1}^{\otimes (m-1)},\dots,s_m^{\otimes (m-1)})$.

Also we define a matrix $H \in M_m(N_1, \underline{O}_{\bar{X}})$ as follows:

$$(1.3.4.2) \quad H_{|N_{1,i}} = \begin{bmatrix} I_{r-1} & g_{i,\alpha}/g_{i,m} \\ 0 & f \end{bmatrix} \quad ;1 \leq \alpha \leq m-1$$

(Here $g_{i,\alpha}$ is the α-th componen of $\underline{g}_i$)

FIGURE II-3

$$\begin{array}{c|cccc}
 & 1 & 2 & i & m \\
\hline
1 & s_1^{\wedge} & s_1^{\otimes m-1} & s_1^{\otimes m-1} & s_1^{\otimes m-1} \\
2 & s_2^{\otimes m-1} & s_2^{\wedge} & s_2^{\otimes m-1} & s_2^{\otimes m-1} \\
i & s_i^{\otimes m-1} & s_i^{\otimes m-1} & s_i^{\wedge} & s_i^{\otimes m-1} \\
m & s_m^{\otimes m-1} & s_m^{\otimes m-1} & s_m^{\otimes m-1} & s_m^{\wedge}
\end{array}$$

(Note:$g_{i,\alpha}$ =(α,i)-component of the matrix.) As previosuly the matrix H defines a bundle $\underline{E}_X$ over X and its frames $\underline{e}^i$ of $\underline{E}_{X|N_i}$, i=0,1, such that $\underline{e}^0 = \underline{e}^1 H$. $\underline{L}$ et $\underline{E}_{\bar{X}}$ denote the direct image of $\underline{E}_X$ with respect to the inclusion:X $\bar{X}$.(Note that the rank of $\underline{E}_{\bar{X}}$ =m.) Assume the following conditions; cf.also (1.3.3.2,3):

For each $i \in\Delta_m$ we have:Take elements $v_{i,\alpha} \in \underline{L}_{\bar{X}_i}; \alpha \in\Delta_{m-1}$.

(1.3.4.3) If $\Sigma_{\alpha=1}^{m-1} (g_{i,\alpha|\bar{X}_i^1}) \otimes v_{i,\alpha} \equiv 0(\mathrm{mod}.\, g_{i,m|\bar{X}_i^1})$, then $v_{i,\alpha} = 0; i \in\Delta_m$ and $\alpha \in\Delta_{m-1}$.

For each $i\in\Delta_m$ and $\alpha\in\Delta_{m-1}$, the functions $1,(g_{i,\gamma}/g_{i,m})|\bar{X}_i^1$

(1.3.4.4) and $(g_{i,\gamma}\otimes g_{i,\alpha}/g_{i,m}^{\otimes 2})|\bar{X}_i^1;\gamma\in\Delta_{m-1}$ are lineraly independent over C.

Theorem 1.3.2. $\underline{E}_{\bar{X}}$ is indecomposable and $\operatorname{codim}_{\bar{X}} S(\underline{E}_{\bar{X}})=m$. (Actually we have: $S(\underline{E}_{\bar{X}}) = \bigcap_{i\in\Delta_m}(s_i)_0$.)

This is also proven in § 5.2 and Appendix. Here we remark that the singular locus $S(\underline{E}_{\bar{X}})$ has interesting property:Take a point $p \in S(\underline{E}_{\bar{X}})$. Then we have the following:

(1.3.4.5) $\underline{E}_{\bar{X},p}$ is generated by (m+1)-elements, and for an element $\underline{e} \in \underline{E}_{\bar{X},p}^{\oplus r}$, we have: $(\Lambda \underline{e}^0)_0 = \{(\Sigma_{\alpha=1}^r c_\alpha \underline{1} \otimes (s_\alpha^\Lambda + s_\alpha^{\otimes(m-1)}) + c\Pi_{\beta \in \Delta_m} s_i)_0\}$

(Here $\underline{1}$ is a frame of $\underline{L}_{\bar{X}}$ at p, and c_α,c are in $\underline{O}_{\bar{X},p}$.) The degree of $(s_\alpha^\Lambda + s_\alpha^{\otimes(m-1)}) = (s_\alpha^\Lambda + s_\alpha^{\otimes(m-1)}) \otimes \underline{1} = (m-1)$, while that of $\Pi_{i\in} s_i = m$. In the expression of $(\Lambda \underline{e}^0)_0$ in (1.3.4.5), the fisst term is a main term. Now the divisor $(s_\alpha^\Lambda + s_\alpha^{\otimes(m-1)})_0$ is a typical one in the theory of the toric singularity, cf.for example,([Oda]); The example in Theorem 1.3.2 shows that reflexive sheaves of type (C) provide also interesting examples of singularities.

Assume that $\bar{X} = P_n$, $\underline{L}_{\bar{X}} = \underline{O}_{\bar{X}}(1)$ and m =n. Also assume that $s_1, \ldots, s_m \in \Gamma(\underline{O}_{\bar{X}}(1))$ are lineraly independent over C. One checks readily that if m(=n) ≥ 4, then (1.3.4.3,4) hold. When n =3 one also checks that $\underline{E}_{\bar{X}}$ is simple and $\mathrm{codim}_{P_3} S(\underline{E}_{\bar{X}}) = 3$. Thus Theorem 1.3.2 provides an interesting example of a reflexive sheaf on P_n.

1.4. Some remarks. Here we make some remarks on the content hitherto in § 1:First, for r =3, we can find some other examples of indecmposable reflexive sheaves of type (C) which satisfies:

(1.4.1.0) the codimension of the singular locus of the sheaf =4 .

Such examples will be discussed elsewhere. Here we give only one example. We take $(\bar{X}, \underline{L}_{\bar{X}})$ to be $(P_6, \underline{O}_{P_6}(1))$. Moreover we set:m=7 and $s_1, \ldots, s_7$ to be a basis of $\Gamma(\underline{L}_{\bar{X}})$. For each $i \in \Delta_7$ we form elements $g_{i,\alpha} \in \Gamma(\underline{O}_{P_6}(4))$;$\alpha$=1,2,3, as follows:

(1.4.1.1) $g_{i,\alpha} = \Pi_{\beta=1}^3 s_{i \pm \beta}/s_{i \pm \alpha}$ and $\begin{bmatrix} I_2 & g_{i,\alpha}/g_{i,3} \\ 0 & s_i/s_{i+1} \end{bmatrix}$;α =1,2 .

We set $\bar{X}^2 = \bigcup_{\alpha \neq \beta}((s_\alpha)_0 \cap (s_\beta)_0)$ and $X^1 = \bar{X}^1 - \bar{X}^2$ with $\bar{X}^1 = (\Pi_{\alpha=1}^7 s_\alpha)_0$. Also take an open neighborhood $N_1 = {}_{i=1}^7 N_{1,i}$, where $N_{1,i}$ is an open neighborhood of $(s_i)_0 - \bar{X}^2$. In the similar meanner to Theorem 1.1 1.3.2 we have a bundle $\underline{E}_X$ over $X := P_6 - \bar{X}^2$ and frames $\underline{e}_i$ of $\underline{E}_{X|N_i}$, $N_0 = P_6$

$-\bar{X}^1$, i=1,2, such that $\underline{e}_0 = \underline{e}_1 H$ in $(N_0 \cap N_1)$. Here the matrix H is defined by: $H_{|N_{1,i}}$ = the matrix in (1.4.1.1). We write $\underline{E}_{P_6}$ for the direct image of $\underline{E}_X$ with respect to the inclusion: X P_6. The sheaf $\underline{E}_{P_6}$ does not have so good properties as the Horrocks-Mumford bundle. But we have: (The proof is given elsewhere.)

(1.4.1.3) $\underline{E}_{P_6}$ is simple and $\mathrm{codim}_{P_6} S(\underline{E}_{P_6}) = 4$.

Next, in connection with the existence of reflexive sheaves of type (C) satisfying the condition in Problem 1.2 or 1.3, we make:

Remark 1.3.1. The null correlation bundle on P_{2n+1}, twisted by $\underline{O}_{P_{2n+1}}(1)$, which is denoted by $\underline{E}_P (P=P_{2n+1})$, has $(\Pi_{\alpha=1}^{2n} z_\alpha)_0$ has its determinantal divisor. Moreover, when n =2, we have:

(1.4.2.1) $\underline{E}_P$ is of type (C).

(This fact is given elsewhere. Also (1.4.2.1) seems to hold for any n.) The rank of $\underline{E}_P$ =2n, and $\underline{E}_P$ is an important bundle of type (C) of high rank. On the otherhand, we can not construct, at the present moment, a new example of indecomposable reflexive sheaves os rank r $\geqq 3$ satisfying:

(1.4.2.2) codimension of the singular locus of the sheaf $\geqq$ r+2; cf. Problem 1.3.

We add the following to Problem 1.2 and 1.3:

Problem 1.4. Find indecomposable reflexive sheaves over P_n; n $\geqq$ 4, which are of type (C) and whose rank r $\geqq 3$. The sheaf must differ from the null correlation bundle and must satisfy (1.4.2.2).

Next Horrocks-Mumford bundle and null correlation bundles occupy peculiar roles among bundles and reflexive sheaves on the projective spaces. (See [Har-1], [O-S-S] and [Ba-2]. See also [Har-2,3], [Ho], [O-S-1,2,3] and [O-1,2] for general results on bundles and reflexive sheaves on the projective spaces.) In § 1.2 we saw that the above two bundles have frames of peculiar types; it seems to be reasonable to expect the following in treatments of bundles and reflexive sheaves in general:

(**) Those sheaves, which have some peculiarly good properties have frames of peculiar types, to which properties of the sheaves closely;cf.the beginning of § 0.

From our view point, frames $\underline{e}^i$, $i=0,1$, of a reflexive sheaf of type (C) as in Definition 1.1 seems to be a typical one whcih has peculiar properties. The position of the sheaves of type (C) on the projective spaces may also be peculiar among all reflexive sheaves on those spaces;we add the following

<u>Problem 1.5</u>. Take an element $c=(c_1,\dots,c_n)\in Z^n$. Determine if c is the Chern classes of reflexive sheaves, say $\underline{E}_{P_n}$, of type (C), over the projective space P_n:

(1.4.3) $c_i(\underline{E}_{P_n})=c_i$.(Here we identify $H^{2i}(P_n)$ with Z usually.)

Finally the Horrocks-Mumford bundle admits the many symmetries,cf [H-M]. The following question may be a natural one:

<u>Problem 1.6</u>. Find reflexive sheaves of type (C) on the projective spaces, which admit many symmetries.

Chapter II. Inductive structure of reflexive sheaves

The structure in the title is a central subject in the theory of reflexive sheaf of type (C). In § 2 and § 3 we give some general arguments for it. In § 4 and § 5 we give explicit forms of the structure for the sheaves of type (C) and the sheaves in § 1.3.

§ 2. Inductive structure ...1

Let U be a polydisc in C^N with coordinates $x=(x_1,\ldots,x_N)$ and φ a holomorphic function in U. For an $I \subset \Delta_N$ we set $\bar{X}_I=\bigcap_{i\in I}(x_i)_0$. Then φ has the following power series expansion with respect to $\bar{X}_I;I\subset\Delta_N$:

(*) $\varphi=\Sigma_{I\subset\Delta_N}(\Pi_{j\in\Delta_N-I}x_j)\varphi_I$, where φ_I does not contain $x_i;i\in I$ and is regarded as an element of $\Gamma(U\cap\bar{X}_I,\underline{O}_{\bar{X}_I})$.

In § 2 and § 3 we give an analogue of (*) to a certain coherent submodule of a locally free sheaf over a complex manifold:The main notion, <u>inductive structure</u>, in Chapter II is given in Definition 2.3. Some preceeding argumemts, like a <u>type of complex</u> and the <u>cobundary of a morphism</u>, are also important, cf.Definition 2.1 and 2.2.(See also the beginning of § 3.)

<u>Remark 2.0</u>. We treat (*) from our view point of the inductive structure, cf. <u>Example</u> in § 2.3 and § 3.4. Our naive idea in § 2 and § 3 appears clearly in those treatments.

Data and Notations

In § 2 and § 3 we work with a pair $(\bar{X},\ \bar{X}^1=\bigcup_{i\in\Delta_m}\bar{X}^1_i\ ,m\geqq 2)$ consisting of a connected complex manifold $\bar{X}$ with dim $\bar{X}$ 2 and its divisor $\bar{X}^1$ with the irreducible components $\bar{X}^1_i$, cf.(1.1.0.0). We assume the following condition:

For a subset $I \subset \Delta_m$ we set $\bar X_I = \cap_{i\in I}\bar X_i$. Then: $\bar X_I = \phi$, if $\# I >$
(2.0.0) $\dim \bar X$. If $\# I \leqq \dim \bar X$ then $\mathrm{codim}_{\bar X}\bar X_I = \#I$ and $\bar X_I$ is smooth.
Moreover, $\bar X_I$ is irreducible unless $\# I = \dim \bar X$.

Setting $\tilde m := \min(m, \dim \bar X)$, we take an integer $a: 1 \leqq a \leqq \tilde m$, and we set:

(2.0.1) $\bar X^a = \cup_I \bar X_I$, with $I: \#I = a$.

We write X_I and ∂X_I for a generic open part of $\bar X_I$ and its boundary:

(2.0.2)
$$\partial X_I = \cup_J \bar X_J \text{ , where } J \text{ satisfies: } \#J = \#I+1 \text{ and } J \supset I, \text{ and}$$
$$X_I = \bar X_I - \partial X_I.$$

We admit the case $I=\phi$. In this case we understand:

(2.0.3) $\bar X_I$, X_I and ∂X_I = $\bar X$, $\bar X - \bar X^1$ and $\bar X^1$.

2.1. A type of complex. (i) We begin § 2 with an algebraic preparation. For an $I \subset \Delta_m$ and a locally free $\underline{O}_{\bar X_I}$-module $\underline{M}_{\bar X_I}$, we define a complex $C^{\cdot}(\underline{M}_{\bar X_I})$ as follows:

(2.1.1) $C^k(\underline{M}_{\bar X_I}) = \oplus_J \underline{M}_{\bar X_I}|\bar X_J$, where J satisfies: $I \subset J$ and $\#J = \#I+k$.

If $k+\#I \ \ \tilde m$, then $C^k(\underline{M}_{\bar X_I}) = 0$. The coboundary operator $\delta_I: C^k(\underline{M}_{\bar X_I}) \to C^{k+1}(\underline{M}_{\bar X_I})$ is described as follows: If $k+\#I+1 \ \ \tilde m$, then δ_I is the zero map. Assume that $k+\#I+1 \ \ \tilde m$. Then:

(2.1.2)
$$\delta_I = \oplus_J \delta_{I;J}: C^k(\underline{M}_{\bar X_I})(=\oplus_J \underline{M}_{\bar X_I}|\bar X_J) \to C^{k+1}(\underline{M}_{\bar X_I})(=\oplus_K \underline{M}_{\bar X_I}|\bar X_K),$$
with $J: J \supset I$ and $\#J = \#I+k$, and $K: K \supset I$ and $\#K = \#I+k+1$,

where $\delta_{I;J}$ is as follows:

(2.1.3)
$\delta_{I;J} = \oplus_K \delta_{I;K,J}: \underline{M}_{\bar X_I}|\bar X_J \to \oplus_K \underline{M}_{\bar X_I}|\bar X_K$, with $K: K \supset J$ and $\#K = \#J+1$,
where $\delta_{I;K,J} = \varepsilon_{I;JK}\,\omega_{I;K,J}: \underline{M}_{\bar X_I}|\bar X_J \to \underline{M}_{\bar X_I}|\bar X_K$. Here $\omega_{I;K,J}$ is
the restriction morphism: $\underline{M}_{\bar X_I}|\bar X_J \to \underline{M}_{\bar X_I}|\bar X_K$ and $\varepsilon_{I;JK} = (-1)^{s-1}$
with the integer s as follows: $(J-I) = (K-I) - \{s\text{-th element of } (K-I)\}$.

Definition 2.1. We call $C^{\cdot}(\underline{M}_{\bar X_I})$ 'complex attached to $(\Delta_m, \underline{M}_{\bar X_I})$'.

Lemma 2.1.1. (1) The following exact sequence holds:

(2.1.4.1) $0\to \underline{I}_{\partial X_I}\underline{M}_{\overline{X}_I}\to \underline{M}_{\overline{X}_I}(=C^0(\underline{M}_{\overline{X}_I}))\xrightarrow{\delta_I} C^1(\underline{M}_{\overline{X}_I})\xrightarrow{\delta_I}\ \to C^{\tilde{m}-\#I}(\underline{M}_{\overline{X}_I})\to 0.$

<u>Here</u> $\underline{I}_{\partial X_I}$ <u>=ideal of</u> ∂X_I (<u>cf.Introduction</u>). <u>When</u> $\partial X_I=\phi$, $\underline{I}_{\partial X_I}=\underline{O}_{\overline{X}_I}$.)

(2) <u>Setting</u> $Z^1(\underline{M}_{\overline{X}_I})$:= kernel of $\delta_I:C^1(\underline{M}_{\overline{X}_I})\to C^2(\underline{M}_{\overline{X}_I})$ <u>we have</u>:

(2.1.4.2) $$Z^1(\underline{M}_{\overline{X}_I})\simeq \underline{M}_{\overline{X}_I}|\partial X_I,\ \underline{\text{if}}\ a\leqq \tilde{m}.$$

<u>Proof</u>. This lemma may be well known. For completenes we give a proof in detail. (1) First we check (2.1.4.1) inductively on $\tilde{m}_I = \tilde{m} - \#I$. If $\tilde{m}_I = 0$, then (2.1.4.1) is the obvious fact: $\underline{M}_{\bar{X}_I} \simeq \underline{M}_{\bar{X}_I}$. Assume that $\tilde{m}_I$ 0 and that (2.1.4.1) holds for: $\tilde{m}_I$. (1) Take an element $\varphi \in \underline{M}_{\bar{X}_I,p}$, $p \in \bar{X}_I$. Clearly, $\delta_I(\varphi) = 0$ implies: $\varphi \in \underline{I}_{\partial X_I}\underline{M}_{\bar{X}_I,p}$. This insures the first part in the exact sequence (2.1.4.1). (2) Next remark that kernel of δ_I at the degree $\tilde{m}_I = C^{\tilde{m}_I}(\underline{M}_{\bar{X}_I})(=\oplus_J \underline{M}_{\bar{X}_I}|\bar{X}_J)$, with $J:\#J=\tilde{m}_I$ and $J \supset I$. On the other hand, for any $J_1: I \supset J_1$ and $\#J_1 = \tilde{m}_I - 1$, we have: $\underline{M}_{\bar{X}_I}|\bar{X}_J = (\underline{M}_{\bar{X}_I}|\bar{X}_{J1})|\bar{X}_J$. This implies the surjectivity of δ_I(=the last

part of the exact sequence in (2.1.4.1). (3) Thirdly take a $k: 1 \leqq k \leqq \tilde{m}_I - 1$. Letting m' be the maximal element of $\Delta_m - I$ we form the following subcomplex of $C^{\cdot}(\underline{M}_{\bar{X}_I})$:

(a) $\quad C'^{\cdot}(\underline{M}_{\bar{X}_I}) = \oplus_J \underline{M}_{\bar{X}_I}|\bar{X}_J$ with those $J: J \ni m'$.

We easily check:

(b) $\quad C'^{\cdot}(\underline{M}_{\bar{X}_I})$ and $C^{\cdot}(\underline{M}_{\bar{X}_I})/C'^{\cdot}(\underline{M}_{\bar{X}_I})$ are isomorphic to the complexes which are attached to $(\Delta_m, \underline{M}_{\bar{X}_I}|I \cup \{m'\})$ and $(\Delta_m - \{m'\}, \underline{M}_{\bar{X}_I})$.

(For (b) see Definition 2.1. In the first isomorphism, an element of $C'^{\cdot}(\underline{M}_{\bar{X}_I})$ corresponds to the element of the other complex, with the degree . -1.) In the exact sequence: $0 \to C'^{\cdot}(\underline{M}_{\bar{X}_I}) \to C^{\cdot}(\underline{M}_{\bar{X}_I}) \to C^{\cdot}(\underline{M}_{\bar{X}_I})/C'^{\cdot}(\underline{M}_{\bar{X}_I}) \to 0$, the induction hypothesis is applied to the first and third terms, and (2.1.4.1) holds for these terms. From this we get easily (2.1.4.1) for $C^{\cdot}(\underline{M}_{\bar{X}_I})$ except the first term in (2.1.4.1). But the first part in (2.1.4.1) was proven in (1), and we have (2.1.4.1). Next (2.1.4.2) is a consequence of (2.1.4.1)(or, is checked directly). q.e.d.

(ii) Take an integer $a: 0 \leqq a \leqq \tilde{m}$, and let $\underline{M}^a = \{\underline{M}_{\bar{X}_I}\}_I$ be a collection of an $\underline{O}_{\bar{X}_I}$-module $\underline{M}_{\bar{X}_I}$, with $\#I = a$. We form a complex of $\underline{O}_{\bar{X}}a$- modules as follows:

(2.1.5) $\quad C^{\cdot}(\underline{M}^a) = \oplus_I C^{\cdot}(\underline{M}_{\bar{X}_I})$, with the coboundary operator $\delta^{\cdot}_a = \oplus_I \delta^{\cdot}_I$.

(2.1.6.2) $\quad Z^1(\underline{M}^a) = \oplus_I \, \underline{M}_{\bar{X}_I|\partial X_I}$, <u>with</u> $I:\#I = a$, <u>if</u> $a \quad \tilde{m}$.

<u>Proof</u>. Immediate from Lemma 2.1.1. q.e.d.

<u>Remark 2.1</u>. Take elements $a,k \in Z_{+0}: 0 \leqq a+k \leqq \tilde{m}$. Then we have:

(2.1.7.1) $\quad C^k(\underline{M}^a) = \oplus_{I,J} \, \underline{M}_{\bar{X}_I|\bar{X}_J}$, with $I,J:\#I=a$, $\#J=a+k$ and $I \subset J$.

For a J we use the following direct summand of $C^k(\underline{M}^a)$ frequently in later arguments:

(2.1.7.2) $\quad (C^k(\underline{M}^a))_J = \oplus_I \, \underline{M}_{\bar{X}_I|\bar{X}_J}$, with $I:\#I = a$ and $I \subset J$.

We call it <u>J-part of</u> $C^k(\underline{M}^a)$.

<u>2.2. Coboundary of a morphism</u>. (i) Here we add an algebraic argument to the one in § 2.1:For an $a \in \Delta_{\tilde{m}}$, take collections $\underline{M}^{a-1}$; $\underline{M}^a$ and Φ^a as follows:

$$
\begin{array}{llll}
 & \underline{M}^{a-1} = \{\underline{M}_{\bar{X}_I}\}_I & & \underline{M}_{\bar{X}_I} \text{ is a locally free } \underline{O}_{\bar{X}_I}\text{-module} \\
(2.2.1) & \underline{M}^{a} = \{\underline{M}_{X_J}\}_J & \text{where} & \underline{M}_{\bar{X}_J} \text{ is a locally free } \underline{O}_{\bar{X}_J}\text{-module} \\
 & \Phi^a = \{\Phi_J\}_J & & \Phi_J \text{ is an } \underline{O}_{\bar{X}_J}\text{-morphism}: \underline{M}_{\bar{X}_J} \to (C^1(\underline{M}^{a-1}))_J
\end{array}
$$

Here $\#I = a-1$ and $\#J = a$.(See also (2.1.7.2).)

<u>Remark 2.2</u>. When there is no fear of confusions we write:

$$
(2.2.2) \qquad
\begin{array}{c}
\underline{M}^a \text{ for } C^0(\underline{M}^a) = \oplus_J \underline{M}_{\bar{X}_J} \text{ and } \Phi^a \text{ for the } \underline{O}_{\bar{X}}a\text{-morphism:} \\
\oplus_J \Phi_J : \underline{M}^a \to C^1(\underline{M}^{a-1}) = \oplus_J (C^1(\underline{M}^a))_J, \text{ with } J:\#J = a .
\end{array}
$$

The following definition is important in the arguments from now on:

<u>Definition 2.2</u>. Assume that $a \quad \tilde{m}$. By the <u>coboundary of</u> Φ^a, denoted by $\delta\Phi^a$, we mean the collection of $\underline{O}_{\bar{X}_K}$-morphism, denoted by $(\delta\Phi)_K$, $\#K = a+1 : \delta\Phi^a = \{(\delta\Phi)_K ; \#K = a+1\}$, with an $\underline{O}_{\bar{X}_K}$-morphism $(\delta\Phi)_K$: $(C^1(\underline{M}^a))_K \to (C^2(\underline{M}^{a-1}))_K$, cf.(2.1.7), which is characterized by the following commutative diagram:

$$
(2.2.3) \qquad
\begin{array}{ccc}
\underline{M}^a & \xrightarrow{\ \Phi^a\ } & C^1(\underline{M}^{a-1}) \\
{\scriptstyle pr^1_K \delta_a} \downarrow & & \downarrow {\scriptstyle pr^2_K \delta_{a-1}} \\
(C^1(\underline{M}^a))_K & \xrightarrow{\ (\delta\Phi)_K\ } & (C^2(\underline{M}^{a-1}))_K
\end{array}
$$

Here pr^i_K denotes the projection: $C^i(\underline{M}^{a+1-i}) \to (C^i(\underline{M}^{a+1-i}))_K$, $i=1,2$.

We call $(\delta\Phi^a)$ the K-part of $(\delta\Phi^a)$

Remark 2.3. We write $\delta\Phi^a$ also for the $\underline{O}_{\bar{X}}$a+1-morphism:

(2.2.4) $\quad \oplus_K\Phi'_K : C^1(\underline{M}^a)\ (=\oplus_K(C^1(\underline{M}^a))_K) \to C^2(\underline{M}^{a-1})(=\oplus_K(C^2(\underline{M}^{a-1}))_K)$

(ii) Here we give some properties of $\delta\Phi^a$. First we clearly have the following commutative diagram:

(2.2.5)
$$\begin{tikzcd} \underline{M}^a \arrow[r, "\Phi^a"] \arrow[d, "\delta_a"] & C^1(\underline{M}^{a-1}) \arrow[d, "\delta_{a-1}"] \\ C^1(\underline{M}^a) \arrow[r, "\delta\Phi^a"] & C^2(\underline{M}^{a-1}) \end{tikzcd}$$

Also note:

(2.2.6) $\quad \ker\delta\Phi^a = \oplus_K \ker(\delta\Phi^a)_K$, with K:#K =a+1.

Nest the explicit form of $\delta\Phi^a$ is as follows:for each $J \subset \Delta_m$:#J =a, we write Φ_J: $\underline{M}_J \to (C^1(\underline{M}^{a-1}))_J$ as follows:

(2.2.7) $\Phi_J = \oplus_I \Phi_{I,J} : \underline{M}_{\bar{X}_J} \to (C^1(\underline{M}^{a-1}))_J = \oplus_I \underline{M}_{\bar{X}_I|\bar{X}_J}$, with I:#I=a-1 and $I \subset J$.

Proposition 2.1.1. For a $K \subset \Delta_m$:#K =a+1, the K-part $(\delta\Phi^a)_K$ of $\delta\Phi^a$ is as follows:

$$(\delta\Phi^a)_K = \oplus_J \Phi'_{J;K}, \text{ with } J{:}\#J = a \text{ and } J \subset K, \text{ where}$$

(2.2.8) $\quad \Phi'_{J;K} = (\oplus_I \varepsilon_{I;JK}\Phi_{I,J|\bar{X}_K}) : \underline{M}_{\bar{X}_J|\bar{X}_K} \to \oplus_I \underline{M}_{\bar{X}_I|\bar{X}_K} (\subset (C^2(\underline{M}^{a-1}))_K)$,

with I as in (2.2.7) and $\varepsilon_{I;J,K} = 1$ or -1 as in (2.1.3) .

Proof. Take an element $\zeta \in \underline{M}_J$. By (2.2.3) and (2.1.3) we have:

(a) $\quad (\delta\Phi^a)_K(\zeta|\bar{X}_K) = \oplus_I \varepsilon_{I;JK}\omega_{I;J,K}\Phi_{I,J}(\zeta)(\in \oplus_I \underline{M}_{\bar{X}_I|\bar{X}_K})$, with I:#I =a-1 and $I \subset J$, where $\omega_{I;JK}$ is as in (2.1.3).

But $\omega_{I;K,J}\Phi_{I,J}(\zeta) = \Phi_{I,J|\bar{X}_K}(\zeta|\bar{X}_K)$, and we have (2.2.8). q.e.d.

Next take an element $\varphi_K=\oplus_{s=1}^{a+1}\varphi_{K(s)}\in(C^1(\underline{M}^a))_K(=\oplus_s\underline{M}_{\bar{X}_{K(s)}}|\bar{X}_K)$, with $K(s)=K-\{s\text{-th element of } K\}$. We have: $(C^2(\underline{M}^{a-1}))_K=\oplus_{1\leq s\leq t\leq a+1}\underline{M}_{\bar{X}_{K(s,t)}}|\bar{X}_K$ with $K(s,t)=K-\{s\text{-and } t\text{-th elements of } K\}$. Let pr_{st} denote the projection: $(C^2(\underline{M}^{a-1}))_K\to\underline{M}_{\bar{X}_{K(s,t)}}|\bar{X}_K$. From (2.2.6,7) we

(2.2.9) $pr_{st}(\delta\Phi^a)_K(\varphi_K)=\Phi_{K(s,t),K(s)}|\bar{X}_K(\varphi_{K(s)})-\Phi_{K(s,t),K(t)}|\bar{X}_K(\varphi_{K(t)})$.

Finally we write down $(\delta\Phi^a)_K$, for $a=2,3$, explicitly: First assume that $\#K=2$. We write $K=\{i,j\}$.

(2.2.10) $\quad(\delta\Phi^1)_K=(-\Phi_i\oplus\Phi_j)|\bar{X}_K:(\underline{M}_{\bar{X}_i}\oplus\underline{M}_{\bar{X}_j})|\bar{X}_K\to\underline{M}^0|\bar{X}_K$.

Next assume that $\#K=3$, and we write $K=\{i,j,k\}$:

$$(C^1(\underline{M}^2))_K=(\underline{M}_{\bar{X}_{ij}}\oplus\underline{M}_{\bar{X}_{ik}}\oplus\underline{M}_{\bar{X}_{jk}})|\bar{X}_K\ni\varphi_{ij;K}\oplus\varphi_{ik;K}\oplus\varphi_{jk;K}\to$$

$$(C^2(\underline{M}^1))_K=(\underline{M}_{\bar{X}_i}\oplus\underline{M}_{\bar{X}_j}\oplus\underline{M}_{\bar{X}_k})\bar{X}_K\ni\varphi_{i;K}\oplus\varphi_{j;K}\oplus\varphi_{k;K},\ \text{where}$$

(2.2.11)
$$\begin{aligned}\varphi_{i;K}&=-\Phi_{i,ij}|\bar{X}_K(\varphi_{ij;K})+\Phi_{i,ik}|\bar{X}_K(\varphi_{ik;K})\\ \varphi_{j;K}&=-\Phi_{j,ij}|\bar{X}_K(\varphi_{ij;K})+\Phi_{j,jk}|\bar{X}_K(\varphi_{jk;K})\\ \varphi_{k;K}&=-\Phi_{k,ik}|\bar{X}_K(\varphi_{ik;K})+\Phi_{k,jk}|\bar{X}_K(\varphi_{jk;K})\end{aligned}$$

Figure VII

	ij	ik	jk
i	$-\Phi_{i,ij}\vert\bar{X}_K$	$\Phi_{i,ik}\vert\bar{X}_K$	
j	$-\Phi_{j,ij}\vert\bar{X}_K$		$\Phi_{j,jk}\vert\bar{X}_K$
k		$-\Phi_{k,ik}\vert\bar{X}_K$	$\Phi_{k,jk}\vert\bar{X}_K$

2.3. Inductive strucure. This subsection is central in § 2.

(i) Let $(\underline{M}^0,\underline{E}_{\bar{X}})$ be a pair consisting of a locally free $\underline{O}_{\bar{X}}$-module $\underline{M}^0$ and an $O_{\bar{X}}$-submodule $\underline{E}_{\bar{X}}$ of $\underline{M}^0$. For each $a:1\leq a\leq\tilde{m}$, let $\underline{M}^a$ and Φ^a be collections as follows:

(2.3.1)
$$\begin{aligned}\underline{M}^a&=\{\underline{M}_{\bar{X}_I}\}_I & &\underline{M}_{\bar{X}_I}\text{ is an }\underline{O}_{\bar{X}_I}\text{-module}\\ & &\text{where}\\ \Phi^a&=\{\Phi_I\}_I & &\Phi_I\text{ is an }\underline{O}_{\bar{X}_I}\text{-morphism}:\underline{M}_{\bar{X}_I}\to(C^1(\underline{M}^{a-1}))_I,\end{aligned}$$

with $I:\#I=a$. See also (2.1.7) for the I-part $(C^1(\underline{M}^{a-1}))_I$ of $C^1(\underline{M}^{a-1})$. We set:

(2.3.2) $\quad \underline{M} = \{\underline{M}^a\}_a$ and $\Phi = \{\Phi^a\}_a$, with $a: 1 \leqq a \leqq \tilde{m}$.

As before we regard Φ^a also as $\underline{O}_{\bar X}$a-morphism: $\oplus_J \Phi_J : \underline{M}^a = \oplus_J \underline{M}_J \to C^1(\underline{M}^a) = \oplus_J C^1(\underline{M}^a)_J$, with $J:\#J = a$. Now the following definition is most important in § 2:

<u>Definition 2.3</u>. We simply say that the pair $(\underline{M},\Phi)$ is an <u>inductive structure for</u> $(\underline{M}^0,\underline{E}_{\bar X})$, if the following holds:

$$\text{(2.3.3.1)} \qquad \begin{aligned} &\underline{E}_{\bar X} = (\delta_0)^{-1}(\Phi^1(\underline{M}^1) \cap Z^1(\underline{M}^0)), \text{ and} \\ &\Phi^a(\underline{M}^a) = \ker(\delta\Phi^{a-1}),\ 2 \leqq a \leqq \tilde{m} \end{aligned}$$

(For the cocycle $Z^1(\underline{M}^0)(\subset C^1(\underline{M}^0))$ and the coboundary $\delta\Phi^{a-1}$ of Φ^{a-1}, see Lemma 2.1 and Definition 2.2.)

<u>FIGURE IV</u>

$$\begin{array}{ccccc} \underline{M}^1 & \xrightarrow{\Phi^1} & C^1(\underline{M}^0) & \xleftarrow{\delta} & \underline{E}_{\bar X} \subset \underline{M}^0 \\ \\ \underline{M}^a & \xrightarrow{\Phi^a} & C^1(\underline{M}^{a-1}) & \xrightarrow{(\delta\Phi^{a-1})} & C^2(\underline{M}^{a-2}) \end{array}$$

(ii)The following example may illustrate our naive idea in introducing the above definition:

<u>Example</u>. Here we assume that $(\underline{M}^0,\underline{E}_{\bar X}) = (\underline{O}_{\bar X},\underline{O}_{\bar X})$. For each $I \subset \Delta_m$ we define a pair $(\underline{M}_{\bar X_I},\Phi_I)$ as follows:

$$\text{(2.3.3.2)} \qquad \begin{aligned} &\underline{M}_{\bar X_I} = \underline{O}_{\bar X_I}, \text{ and } \Phi_I = \oplus_a(\mathrm{id}) : \underline{M}_{\bar X_I} \ni \varphi_I \to C^1(\underline{M}^{a-1})_I (= \oplus_{s=1}^{a} \underline{O}_{\bar X_{I(s)}}|\bar X_I \\ &\simeq \underline{O}_{\bar X_I}^{\oplus a}) \ni \oplus_{s=1}^{a} \varphi_s, \text{ with } \varphi_s = \varphi_I . \end{aligned}$$

Here $a = \#I$ and $I(s) = I - \{s\text{-th element of } I\}$. Then we have:

(*) <u>The pair</u> $(\underline{M},\Phi)$, <u>with</u> $\underline{M} = \{\underline{M}_{\bar X_I}\}_I$ <u>and</u> $\Phi = \{\Phi_I\}_I ; I \subset \Delta_m$, is an <u>inductive structure of</u> $(\underline{O}_{\bar X},\underline{O}_{\bar X})$.

This is checked in § 2.6 in a somewhat more general form.

(iii) Now we give some basic properties of the inductive structure:first the following is obvious from Definition 2.3:

$$\text{(2.3.3.3)} \qquad \begin{aligned} &\underline{E}_{\bar X} \supset \underline{I}_{\bar X} 1 \underline{M}^0, \text{ cf.Lemma 2.1.2, and } (\delta\Phi^{a-1})\Phi^a = 0 \,(\Leftrightarrow \\ &(\delta\Phi^{a-1})_I \Phi_I = 0 \text{ for each } I:\#I = a) . \end{aligned}$$

Note that, writing Φ_I as $\oplus_{s=1}^{a}\Phi_{I(s),I}:\underline{M}_{\overline{X}_I}\to(C^1(\underline{M}^{a-1}))_I=\oplus_{s=1}^{a}\underline{M}_{\overline{X}_{I(s)}}|\overline{X}_I$, with $I(s)=I-\{s\text{-th element of } I\}$, the second fact in (2.3.3.3) is equivalent to that the following holds for each $(s,t): 1\leqq s\leqq t\leqq a$.

(2.3.3.4) $\Phi_{I(s,t),I(s)}|\overline{X}_I\Phi_{I(s),I}=\Phi_{I(s,t),I(t)}|\overline{X}_I\Phi_{I(t),I}$, with $I(s,t)=I-(s \text{ and } t\text{-th elements of } I)$ and $\Phi_{I(s)}=\oplus_{t\neq s}\Phi_{I(s,t),I(s)}$.

Next define $\underline{O}_{\overline{X}}$a-modules $\underline{E}^a$ of $\underline{M}^a$ and $\underline{F}^a$ of $C^1(\underline{M}^{a-1})$ as follows:

(2.3.4)
$$\underline{F}^a=\ker(\delta\Phi^{a-1})\cap Z^1(\underline{M}^{a-1}),\ 2\leqq a\leqq\tilde{m},\ \text{and}\ \underline{F}^1=\delta_0^0(\underline{E}_{\overline{X}})$$
$$\underline{E}^a=\ker(\delta_{a-1}\Phi^a),\ 1\leqq a\leqq\tilde{m}.$$

Here $\delta_{a-1}\Phi^a$ =composition of Φ^a and $\delta_{a-1}: C^1(\underline{M}^{a-1})\to C^2(\underline{M}^{a-1})$. Note that

(2.3.5) $\underline{E}^{\tilde{m}}=\underline{M}^{\tilde{m}}$ and $\underline{F}^{\tilde{m}}=\ker(\delta\Phi^{\tilde{m}-1})$.

The lemmas below (Lemma 2.2 and 2.3) are used frequently:

<u>Lemma 2.2.</u> <u>We have the following inductive relation</u>:

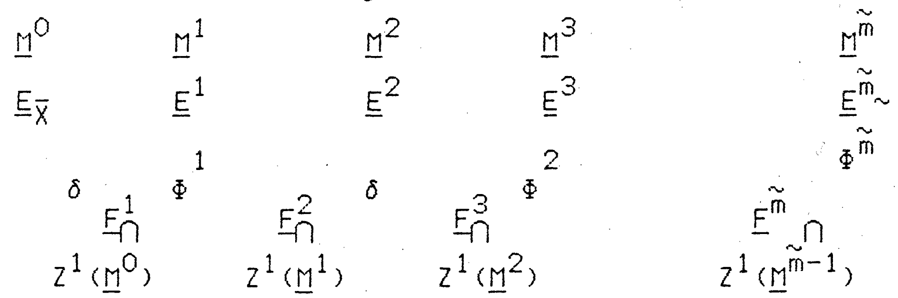
(2.3.6) $\underline{E}^{a-1}=(\delta_{a-1})^{-1}(\underline{F}^a)$ and $\underline{E}^a=(\Phi^a)^{-1}(\underline{F}^a), 1\leqq a\leqq\tilde{m}$, <u>with</u> $\underline{E}^0=\underline{E}_{\overline{X}}$.

<u>Remark 2.4</u>. Since $\Phi^a(\underline{M}^a)\subset F^a$ and $\delta_{a-1}(\underline{M}^{a-1})\subset F^a, 1\leqq a\leqq\tilde{m}$, cf.(2.3.3,4), we have the following from (2.3.6):

(2.3.6)' $\underline{F}^a=\delta_{a-1}(\underline{E}^{a-1})$ and $\underline{F}^a=\Phi^a(\underline{E}^a)$, $1\leqq a\leqq\tilde{m}$.

<u>Proof of Lemma 2.2.</u> The first fact follows easily from (2.2.5), (2.1.6) and (2.3.3). The second follows from (2.2.5) and (2.3.3,4,5). q.e.d.

<u>FIGURE VI</u>



<u>Lemma 2.3.</u> <u>The following exact sequence holds</u>; $0\leqq a\leqq\tilde{m}$.

(2.3.7) $0\to\oplus_I\underline{I}_{\partial X_I}\underline{M}_{\overline{X}_I}\to\underline{E}^a\xrightarrow{\delta_a}\underline{F}^{a+1}\to 0$, <u>with</u> $I:\#I=a$.

When $a=\tilde{m}$, this means: $F^{\tilde{m}}=M^{\tilde{m}}$, cf.(2.3.5).

Proof. For the proof it suffices to check: ker δ_a = the first term; $a \geq \tilde{m}$. By Lemma 2.1.2, this follows from that the first term $\subset E^a$. But this follows from (2.3.4) and (2.3.3). q.e.d.

2.4. A remark. In this paper we use the inductive structure of $(\underline{M}^0, \underline{E}_{\bar{X}})$ as a substitute for a projective resolution of $\underline{E}_{\bar{X}}$. The projective resolution exists locally. The corresponding fact for the inductive structure is as follows:

Lemma 2.4. There is an inductive structure of $(\underline{M}^0, \underline{E}_{\bar{X}})$ locally, if and only if the following holds locally.

There is, locally, an $\underline{O}_{\bar{X}_i}$-submodule, say $\underline{M}'_{\bar{X}_i}$, of $\underline{M}^0_{|\bar{X}_i}$; $i \in \Delta_m$,

(2.4)

such that $\underline{E}_{\bar{X}} = (\delta_0)^{-1}(Z^1(\underline{M}^0) \cap (\oplus_{i \in \Delta_m} \underline{M}'_{\bar{X}_i}))$

Proof. The only if part is clear, by setting $\underline{M}'_{\bar{X}_i} = \Phi_i(\underline{M}_{\bar{X}_i})$, cf. (2.3.3). Conversely, if (2.4) holds we find $(\underline{M}_{\bar{X}_i}, \Phi_i)$; $i \in \Delta_m$, where $\underline{M}_{\bar{X}_i}$ is a locally free $\underline{O}_{\bar{X}_i}$-module and $\Phi_i : \underline{M}_{\bar{X}_i} \to \underline{M}^0_{|\bar{X}_i}$ satisfies: $\Phi_i(\underline{M}_{\bar{X}_i}) = \underline{M}'_{\bar{X}_i}$. Then we have the first condition in (2.3.3). For $a \geq 2$ the kernel of $(\delta\Phi^{a-1})_I$, with $I \subset \Delta_m$: $\#I = a$, is an $\underline{O}_{\bar{X}_I}$-submodule of $C^1(\underline{M}^{a-1})_I$. Thus we find a locally free $\underline{O}_{\bar{X}_I}$-module $\underline{M}_{\bar{X}_I}$ and a surjective $\underline{O}_{\bar{X}_I}$-morphism: $M_{\bar{X}_I} \to$ kernel of $(\delta\Phi^{a-1})_I$. q.e.d.

2.5. A property. Let the inductive structure $(\underline{M}, \Phi)$ of $(\underline{M}^0, \underline{E}_{\bar{X}})$. Tane an $I \subset \Delta_m$. For a series $\underline{I} = (I_a, \ldots, I_0)$, $a = \#I$, of subsets of Δ_m satisfying

(2.5.0) $\quad I = I_a \supsetneqq I_{a-1} \supsetneqq I_{a-2} \supsetneqq \qquad \supsetneqq I_1 \supsetneqq I_0 = \phi$,

we form an $\underline{O}_{\bar{X}_I}$-morphism $\Phi_{\underline{I}} : \underline{M}_{\bar{X}_I} \to \underline{M}^0_{|\bar{X}_I}$ as follows:

(2.5.1) $\quad \Phi_{\underline{I}} = (\Pi^1_{\omega=a} \Phi_{I_\omega I_{\omega-1}|\bar{X}_I})$.

Here we write $\Phi_{I_{\omega-1} I_\omega} : \underline{M}_{\bar{X}_{I_\omega}} \to \underline{M}_{\bar{X}_{I_{\omega-1}}|\bar{X}_{I_\omega}}$ for $pr_{I_{\omega-1}} \Phi_{I_\omega}$ with the projection $pr_{I_{\omega-1}} : (C^1(\underline{M}^{\omega-1}))_{I_\omega} \to \underline{M}_{\bar{X}_{I_{\omega-1}}|\bar{X}_{I_\omega}}$, cf. (2.2.7).

Lemma 2.4. The $\underline{O}_{\bar{X}_I}$-morphism $\Phi_{\underline{I}} : \underline{M}_{\bar{X}_I} \to \underline{M}^0_{|\bar{X}_I}$ is indendent of the series $\underline{I}$ satisfying (2.5.0).

This lemma is used in the analysys of sections of $\underline{E}_{\bar{X}}$,cf.Lemma 3.1 and 5.2.

Proof. Take a v:v a and a series $\underline{I}'_v=(I'_a=I\subsetneq I'_{a-1}\subsetneq\cdots\subsetneq I'_v)$ with $I'_v=I_v$. For the proof of the lemma, it sufices to check:

(a) $(\Pi^{v+1}_{w=a}\Phi_{I_{w-1}I_w|\bar{X}_I})=(\Pi^{v+1}_{w=a}\Phi_{I'_{w-1}I'_w|\bar{X}_I})$.

If v =a-1 this is clear. Assume that $v_=$ a-2 and (a) holds for v+1. If $I_{v+1}=I'_{v+1}$ we have (a) from the induction hypothesis. Assume that $I_{v+1}\neq I'_{v+1}$. We write $I_{v+1}=I\cup\{\alpha\}$ and $I'_{v+1}=I\cup\{\alpha'\}$ with $\alpha\neq\alpha'\in I$. We set $I''_{v+2}=I\cup\{\alpha,\alpha'\}$ take I''_w,w =v+3,..,u, satisfying the following.

(b) $I''_{v+2}\subsetneq I''_{v+3}\subsetneq\qquad\subsetneq I''_a=I$.

Form two series $(I''_a=I,\ldots,I''_{v+2},I_{v+1},I_v)$ and $(I''_a=I,\ldots,I''_{v+2},I'_{v+1},I_v)$, and form the $\underline{O}_{\bar{X}_I}$-morphism:$\underline{M}_{\bar{X}_I}\to\underline{M}_{\bar{X}_{I_v}}|\bar{X}_I$ from these series in the similar manner to the ones in (a). Apply (2.3.3.3) to I_{v+2},I. Then we have:(d) $\Phi_{J_vJ_{v+1}|\bar{X}_I}\Phi_{J_{v+1}J_{v+2}|\bar{X}_I}=\Phi_{J_v}\Phi_{J'_{v+1}|\bar{X}_I}\Phi_{J_vJ'_{v+1}}\Phi_{J'_{v+1}J_{v+2}|\bar{X}_I}$.

Thus the two $\underline{O}_{\bar{X}_I}$-morphisms just above coincides. By the induction hpothesis, the former and and the latter of these morphisms coincide respectively with the former and the latter in (a). Thus we have (a) in the case of v. q.e.d.

2.6. Here we prove (*) in Example, § 2.3 in a somehat more general form:Let the pair $(\underline{M}^0,\underline{E}_{\bar{X}})$ be as in the beginning of § 2.3. We assume that there are a locally free $\underline{O}_{\bar{X}}$-module $\underline{M}'$ and $\underline{O}_{\bar{X}_i}$-morphisms Φ_i;$i\in\Delta_m$, satisfying the following:

(2.6.1) $\underline{E}_{\bar{X}}=(\delta_0)^{-1}(Z^1(\underline{M}^0)\cap(\oplus_{i\in\Delta_m}\Phi_i(\underline{M}'|\bar{X}_i))$, cf.(2.3.3), and

$\Phi_{i|\bar{X}_{ij}}=\Phi_{j|\bar{X}_{ij}}$, $i\neq j\in\Delta_m$, as the $\underline{O}_{\bar{X}_{ij}}$-morphism:$\underline{M}'|\bar{X}_{ij}\to\underline{M}|\bar{X}_{ij}$

For each $I\subset\Delta_m$ we set,cf.(2.6.0):

(2.6.2) $\underline{M}_{\bar{X}_I}=\underline{M}'|\bar{X}_I$ and $\Phi_I=\oplus_a(id):\underline{M}'|\bar{X}_I\ni\varphi_I\to C^1(\underline{M}^{a-1})_I(\simeq\underline{M}'|^{\oplus a}_{\bar{X}_I})\ni\oplus^u_{s=1}\varphi_{I(s)}$, with $\varphi_{I(s)}=\varphi_I$;a=#I.(For Φ_I we assume that # $I_=$ 2.)

(In the case of (*) in Example, § 2.3, we set:$\underline{M}'=\underline{O}_{\bar{X}}$ and Φ_i= identity:$O_{\bar{X}_i}(=M'|\bar{X}_i)\to O_{\bar{X}_i}=(M^0|\bar{X}_i)$. Then (2.6.1) holds, cf.Lemma 2.1.1.

Lemma 2.6. The pair $\{\underline{M}=\{\underline{M}_{\bar{X}_I}=\underline{M}'_{|\bar{X}_I}\}_I, \Phi=\{\Phi_I=\oplus_a(id); a=\#I\geq 2, \Phi_i$ as in (2.6.1)$\}$, cf. (2.6.1.2), is an inductive structure of $(\underline{M}^0, \underline{E}_{\bar{X}})$.

Proof. Take an $I \subset \Delta_m$ with $2 \leq a := \#I \leq \tilde{m}$. By (2.3.3) it suffices to check: $\ker(\delta\Phi^{a-1})_I = \Phi_I(\underline{M}_{\bar{X}_I})$. We write $I=\{i_1,..,i_a\}$. Take an element $\varphi=\oplus_{s=1}^{a}\varphi_{I(s)} \in C^1(\underline{M}^{\,a-1})_I = \oplus_s \underline{M}_{\bar{X}_{I(s)}|\bar{X}_I} (\simeq \underline{M}'^{\oplus a}_{|\bar{X}_I})$ such that such that $(\delta\Phi^{a-1})(\varphi)=0$. By (2.2.9) we have: $\varphi_1 = \cdots = \varphi_a$. Thus we have: $\Phi_I(\varphi_I) = \varphi$, with $\varphi_I = \varphi_{\cdot}\ (\cdot = 1,..,s)$. q.e.d.

§ 3. Inductive structure...2

Letting the pair $(\underline{M}^0, \underline{E}_{\bar{X}})$ of a locally free $\underline{O}_{\bar{X}}$-module $\underline{M}^0$ and its $\underline{O}_{\bar{X}}$-submodule $\underline{E}_{\bar{X}}$ be as in § 2.3, we discuss here the local structure of $\underline{E}_{\bar{X}}$. Also we discuss an analogue of the power series expansion in (*) at the beginning of § 2. We use freely the notation in § 2. The varieties $\bar{X}$ and $\bar{X}^a; a \geqq 1$, are as in (2.0.0 4). (We write $\bar{X}^0$ and ∂X^0 for $\bar{X}$ and $\bar{X}^1$.) The inductive structure $(\{\underline{M}_{\bar{X}_I}\}, \{\Phi_I\})$, with $I \subset \Delta_m$, of $(\underline{M}^0, \underline{E}_{\bar{X}})$ are as in (2.3.1∼3). The $\underline{O}_{\bar{X}^a}$-module $\underline{M}^a = \oplus_J \underline{M}_{\bar{X}_J}$ and $\underline{O}_{\bar{X}^a}$-morphism $\Phi^a = \oplus_J \Phi_J$, with $J: \#J = a$, are as in (2.3.2). Also the $\underline{O}_{\bar{X}^a}$-modules $\underline{E}^a (\subset \underline{M}^a)$ and $\underline{F}^a (\subset C^1(\underline{M}^{a-1}))$ are as in (2.3.1) .

3.1. Parametrization. Take an open set U of $\bar{X}$, and we assume the following($\tilde{m} = \min(m, \dim \bar{X})$;see § 2):

$$\Gamma(\bar{X}^a \cap U, \underline{M}^a) \overset{\delta_a}{\to} \Gamma(\bar{X}^{a+1} \cap U, Z^1(\underline{M}^a)); 0 \leqq a \leqq \tilde{m}-1$$

are surjective.

(3.1.0)
$$\Gamma(\bar{X}^a \cap U, \underline{E}^a) \overset{\Phi^a}{\to} \Gamma(\bar{X}^a \cap U, \underline{F}^a); 1 \leqq a \leqq \tilde{m}$$

(Since $\delta_a: \underline{M}^a \to Z^1(\underline{M}^a)$ and $\Phi^a: \underline{E}^a \to \underline{F}^a$ are surjective, cf.Lemma 2.1.2 and 2.2, (3.1.0) holds if U is a Stein manifold.) We fix an extension map $(ex)^a = \oplus_{J;\#J=a} (ex)_J : \Gamma(\bar{X}^{a+1} \cap U, Z^1(\underline{M}^a)) = \oplus_J \Gamma(\partial X_J \cap U, \underline{M}_{\bar{X}_J} | \partial X_J) \to \Gamma(U, \underline{M}^a) = \oplus_J \Gamma(U \cap \bar{X}_J, \underline{M}_{\bar{X}_J})$, $0 \leqq a \leqq \tilde{m}-1$, satisfying:

(3.1.1.1) $\delta_a (ex)^a =$ identity .(By (3.1.0) such an $(ex)^a$ exists.)

When $a = 0$ we understand that

(3.1.1.2) $(ex)^0 : \Gamma(U \cap \bar{X}^1, \underline{M}^0_{|\bar{X}^1}) \to \Gamma(U, \underline{M}^0)$.

Note that Lemma 2.2 and (2.3.4) imply:

(3.1.2) $(ex)^{v-1} \Phi^v \Gamma(\bar{X}^v \cap U, \underline{E}^v) \subset \Gamma(\bar{X}^{v-1} \cap U, \underline{E}^{v-1})$, $1 \leqq v \leqq \tilde{m}$.

This enables us to define a map P^a as follows:

(3.1.3) $P^a : \Gamma(\bar{X}^a \cap U, \underline{E}^a) \ni \varphi^a \to \Gamma(U, \underline{E}_{\bar{X}}) \ni \varphi^0 = (\Pi_{v=a}^{1} (ex)^{v-1} \Phi^v)(\varphi^a)$.

(When $a=0$, $\underline{E}^0 = \underline{E}_{\bar{X}}$, and P^a is the identity.) Moreover we write:

(3.1.4) $\underline{I}_{\partial X^a} \underline{M}^a := \oplus_I \underline{I}_{\partial X_I} \underline{M}_{\bar{X}_I}$, with $I: \#I = a$.(Note that $\underline{I}_{\partial X^{\tilde{m}}} = \underline{O}_{\bar{X}^{\tilde{m}}}$.)

Now we form a map P(parametrization map) as follows:

(3.1.5.0) $P:\oplus_{a=\tilde{m}}^{0}\Gamma(U\cap\bar{X}^a,\underline{I}_{\partial X^a}\underline{M}^a)\ni\oplus_{a=\tilde{m}}^{0}\varphi^a\to\Gamma(U,\underline{E}_{\bar{X}})\ni\varphi=\oplus_{a=\tilde{m}}^{0}P^a(\varphi^a)$.

(Here we use the inclusion: $(\underline{I}_{\partial X}a)\underline{M}^a\subset\underline{E}^a$; see (2.3.7). The map P gives a parametrization of elements of $\Gamma(U,\underline{E}_{\bar{X}})$ by means of those of $\Gamma(U\cap\bar{X}_I,\underline{I}_{\partial X_I}\underline{M}_{\bar{X}_I})$; $I\subset\Delta_m$.)

<u>Theorem 3.1</u>. <u>The map</u> P <u>is surjective</u>.

<u>Proof</u>. For each a,v: $0\leqq v\leqq a\leqq\tilde{m}$ we set:

(3.1.5.1) $P^{v,a}=\Pi_{\omega=a}^{v+1}(ex)^{\omega-1}\Phi^{\omega}:\Gamma(U\cap\bar{X}^a,\underline{I}_{\partial X^a}\underline{M}^a)\to\Gamma(U\cap\bar{X}^v,\underline{E}^v)$.

(If v=a, $P^{v,a}$ is the identity.) For the proof of the theorem we prove the following inductively on v.(When v=0, we understand that $\underline{E}^0=\underline{E}_{\bar{X}}$. Then the theorem holds.)

(a) $\oplus_{a=\tilde{m}}^{v}P^{v,a}:\oplus_a\Gamma(\bar{X}^a\cap U,\underline{I}_{\partial X^a}\underline{M}^a)\to\Gamma(\bar{X}^v\cap U,\underline{E}^v)$ is surjetive.

The key fact for the proof of (a) is as follows:

(b) $\Gamma(\bar{X}^a\cap U,\underline{E}^a)=(ex)^a\Phi^{a+1}\Gamma(\bar{X}^{a+1}\cap U,\underline{E}^{a+1})+\Gamma(\bar{X}^a\cap U,\underline{I}_{\partial X^a}\underline{M}^a)$; $0\leqq a\leqq\tilde{m}-1$.

The check of this is as follows:Lemma 2.2 and 2.3 insure:L.H.S $\supset$ R.H.S. Next, for an element $\varphi^a\in$ L.H.S, Lemma 2.2 and (2.3.6.1) insure: $\delta_a^0(\varphi^a)=\Phi^{a+1}(\varphi^{a+1})$ with an element $\varphi^{a+1}\in\Gamma(\bar{X}^{a+1}\cap U,\ \underline{E}^{a+1})$. By (2.3.7), $\varphi^a-(ex)^a\Phi^{a+1}(\varphi^{a+1})\in\Gamma(\bar{X}^a\cap U,\underline{I}_{\partial X^a}\underline{M}^a)$. Now (a) is checked as follows: If $v=\tilde{m}-1$, (2.3.5) and (b) insure (a). Take a v: $\tilde{m}-1\geqq v\geqq 1$, and assume (a) for $\geqq v$. Noting that $P^{v-1,\omega}=(e^{v-1}\Phi^v)P^{v,\omega}$ we have the following from the induction hypothesis that (a) holds for v:

(d) image of $\oplus_{\omega=1}^{v-1}P^{v-1,\omega}=(e^{v-1}\Phi^v)\Gamma(\bar{X}^v\cap U,\underline{E}^v)+\Gamma(\bar{X}^{v-1}\cap\bar{U},\underline{I}_{\partial X^{v-1}}\underline{M}^{v-1})$.

By (b), the R.H.S $=\Gamma(\bar{X}^{v-1}\cap U,\underline{E}^{v-1})$. q.e.d.

In connection with (3.1.5.0,1) we add a map as follows:For a $J\subset\Delta_m$ we write the restriction of P^a to $\underline{M}_{\bar{X}_J}$ as P_J; a=#J .

(3.1.5.2) $P_J=P^a_{|\Gamma_J}:\Gamma_J:=\Gamma(U\cap\bar{X}_J,\underline{I}_{\partial X_J}\underline{M}_{\bar{X}_J})\to\Gamma(U,\underline{E}_{\bar{X}})$

Now take an $I\subset\Delta_m$ such that $U\cap\bar{X}_I=\phi$. We determine $P_{I|\bar{X}_I}$. Precisely let the series $\underline{I}=(\underline{I}_a=I\subsetneqq\underline{I}_{a-1}\subsetneqq\ \ \subsetneqq\underline{I}_1\subsetneqq\underline{I}_0=\phi)$, a=#I, and the $\underline{O}_{\bar{X}_I}$-morphism $\Phi_{\underline{I}}:\underline{M}_{\bar{X}_I}\to\underline{M}^0_{|\bar{X}_I}$ be as in (2.5.0,1). We write ω_I for the quotient morphism: $\underline{M}^0 \to \underline{M}^0_{|\bar{X}_I}$. Note, in that case, $\underline{I}_{\partial X_I}\underline{M}_{\bar{X}_I} = \underline{M}_{\bar{X}_I}$. We make the following remark on P:

Lemma 3.1. The two C-morphisms $\omega_I P_I$ and $\Phi_{\underline{I}}: \Gamma(U\cap\bar{X}_I, \underline{M}_{\bar{X}_I}) \to \Gamma(U\cap\bar{X}_I, \underline{M}^0_{|\bar{X}_I})$ coincide.

Proof. For a $v: a > v \geqq 0$, $a=\#I$, we restrict $P^{v,a} (=\Pi_{w=a}^{v+1}(ex)^{w-1}\Phi^w)$ to $\Gamma(U\cap\bar{X}_I, \underline{M}_{\bar{X}_I})$:

(a) $\quad P^{v,a}_{|I}: \Gamma(U\cap\bar{X}_I, \underline{M}_{\bar{X}_I}) \to \Gamma(U\cap\bar{X}^v, \underline{E}^v)$.

Recall that $\underline{E}^v \subset \underline{M}^v = \oplus_J \underline{M}_{\bar{X}_J}$ with $J: \#J = v$, cf. (2.3.3). We write pr_{I_v} for the projection: $\underline{M}^v \to \underline{M}_{\bar{X}_{I_v}}$. Moreover, let χ_{II_v} denote the quotient morphism: $\underline{M}_{\bar{X}_{I_v}} \to \underline{M}_{\bar{X}_{I_v}}|\bar{X}_I$. ($\chi_{II_0} = \omega_I$.) For the proof of the lemma it suffices to check the following for each $v: a-1 \geqq v \geqq 0$.

(b) $\quad (\Pi_{w=a}^{v+1}(\Phi_{I_{w-1}I_w}|\bar{X}_I) = \chi_{II_v} pr_{I_v} P^{v,a}_{|I}$ (as the map: $\Gamma(U\cap\bar{X}_I, \underline{M}_{\bar{X}_I}) \to \Gamma(U\cap\bar{X}_I, \underline{M}_{\bar{X}_{I_v}}|\bar{X}_I)$. (See (2.5.1) for the morphism $\Phi_{I_{w-1}I_w}$.)

This is checked inductively on v. If $v=a-1$, (b) is clear since $(\delta^0_{I_{a-1}}(ex)_{I_{a-1}}) =$ identity. Assume $v \leqq a-2$ and (b) holds for $\geqq v+1$. By (3.1.2), (2.3.4) and Lemma 2.2 the image of $\Phi^{v+1}P^{v+1,a} \subset \oplus_J \Gamma(U\cap\partial X_J, \underline{M}_{\bar{X}_J}|\partial X_J)$ with $J: \#J = v$. By the definition of Φ_{v+1}, cf. (2.2.7), we have:

(c) $\quad \Phi_{I_v I_{v+1}} pr_{I_{v+1}} P^{v+1,a}_{|I} = \chi_{I_{v+1}I_v} pr_{I_v} P^{v,a}_{|I}$ (as the map: $\Gamma(\bar{X}_I\cap U, \underline{M}_{\bar{X}_I}) \to \Gamma(\bar{X}_I\cap U, \underline{M}_{\bar{X}_{I_v}}|\bar{X}_{I_{v+1}}))$, with the quotient morphism $\chi_{I_{v+1}I_v}: \underline{M}_{\bar{X}_{I_v}} \to \underline{M}_{\bar{X}_{I_v}}|\bar{X}_{I_{v+1}}$

Restrict this to $\bar{X}_I$. One checks readily that the L.H.S and R.H.S in (c) coincide with those in (b). q.e.d.

More deteiled structures of P are studied in § 3.2 and § 5.

3.2. Local structure. Here we derive the local structure of $\underline{E}_{\bar{X}}$ from Theorem 2.1. For this we assume that the open set $U \subset \bar{X}$ is as follows:

(3.2.0.1) U is isomorphic to a polydisc$=U_1\times \ \times U_N$ with coordinates $x=(x_1,\dots,x_N)$.

Here $N=\dim \bar{X}$ and U_i is a disc in C whose center is the origin of C.

For an $I\subset\Delta_N$ we assume the following:

(3.2.0.2) $\bar{X}_j\cap U\neq\phi$ if and only if $j\in I$, and, in this case, $(\bar{X}_j\cap U)=(x_j)_0$.

Clearly $\bar{X}_J\cap U\neq\phi; J\subset\Delta_m$, if and only if $J\subset I$, and we easily have:

(3.2.1.1) $(\bar{X}_J\cap U)=U_1\times\cdot\cdot\times\{0\}\times\cdot\cdot\times\{0\}\times\cdot\cdot\times U_N$ (i.e., for each $j\in J$, we replace U_j by its origin).

We write π_J for the natutral projection:

(3.2.1.2) $U(\simeq(U\cap\bar{X}_J)\times(\Pi_{j\in J}U_j)\ni x\to U\cap\bar{X}_J\ni(x_1,\dots,0,\dots,0,\dots x_N)$.

(i) Take a $J\subset I$ and an element $\varphi\in\Gamma(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})$. We expand φ by using the product structure: $(U\cap\bar{X}_J)=(U\cap\bar{X}_I)\times(\Pi_{i\in I-J}U_i)$:

(3.2.2) $\varphi=\Sigma_{K\in Z^w_{+0}}c_K(\varphi)x^K_{I-J}$, with $c_K(\varphi)\in\Gamma(U\cap\bar{X}_I,\underline{O}_{\bar{X}_I})$ and $x^K_{I-J}=x_{i_1}^{k_1}\cdots x_{i_w}^{k_w}$. Here $K=(k_1,\dots,k_w)$, $w:=\#I-\#J$, and $i_1,\dots,i_w\in I-J$.

We use this for the definition of the extension map $(ex)_J$, cf. (3.1.1): assume that $J\subsetneq I$ and let $\Gamma'(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})$ be the subgroup of $\Gamma(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})$ consisting of those elements φ whose power series expansion satisfies: $c_K(\varphi)=0$ if $K\in Z^w_+$. Let ω_J be the quotient morphism: $\underline{O}_{\bar{X}_J}\to\underline{O}_{\partial X_J}(=\underline{O}_{\bar{X}_J}/\underline{I}_{\partial X_J})$. Then ω_J induces an isomorphism of abelian groups:

(3.2.3) $\omega_J:\Gamma'(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})\simeq\Gamma(U\cap\partial X_J,\underline{O}_{\partial X_J})$.

(Actually we have: $\Gamma(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})=\Gamma'(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})\oplus(x_{I-J})\Gamma(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})$, with $x_{I-J}=_{i\in I-J}x_i$, and the kernel of ω_J is the second term of the R.H.S.)

We set:

(3.2.4.1) $(ex)_J=\omega_J^{-1}:\Gamma(U\cap\partial X_J,\underline{O}_{\partial X_J})\to\Gamma(U\cap\bar{X}_J,\underline{O}_{\bar{X}_J})$

Thus, for a $\zeta_J\in\Gamma(U\cap\partial X_J,\underline{O}_{\partial X_J})$, $\zeta'_J=(ex)_J(\zeta_J)$ is characterized by:

(3.2.4.2) $\zeta'_{J|\partial X_J}=\zeta_J$ and ζ'_J does not contain terms devided by

(ii) Assume that, for each $J: J\subset I$, $\underline{M}_{\bar{X}_J}$ and $\underline{M}^0$ have frames, denoted by $\underline{m}_J$ and $\underline{m}^0$ over $U\cap\bar{X}_J$ and U. We write $(ex)_J$ also for the map: $\Gamma(U\cap\partial X_J, \underline{M}_{\bar{X}_J}|\partial X_J) \to \Gamma(U\cap\bar{X}_J, \underline{M}_{\bar{X}_J})$, which is defined through the identification: $\underline{M}_{\bar{X}_J} \simeq \underline{O}_{\bar{X}_J}^{\oplus r_J}$, $r_J :=$ rank $\underline{M}_{\bar{X}_J}$, by means of the frame $\underline{m}_J$. We set: $(ex)^a = \oplus_J (ex)_J$, with $J: \#J=a$ and $J\subset I$, and define maps P^a and P as in (3.1.3,5). The map P is as follows:

(3.2.5) $\quad P: \oplus_{J:J\subset I} P_J : \Gamma(U\cap\bar{X}_J, I_{\partial X_J}\underline{M}_{\bar{X}_J}) \ni \oplus_J \varphi_J \to \Gamma(U, \underline{E}_{\bar{X}}) \ni \oplus_J P_J(\varphi_J)$,

with P_J as in (3.1.5.2).

For each $J \subset \Delta_m : J \subset I$ and $v := \#J \underset{=}{\ } 1$ we write $\underline{m}_J = (m_{J,1}, \ldots, m_{J,r_J})$. Note that the element $x_{I-J} = \Pi_{i\in I-J} x_i$ generates $\underline{I}_{\partial X_J|U}$.

<u>Theorem 3.2. (1) For a $J: J\subset I$ and</u> $m_{J,\alpha} \in \underline{m}_J$ <u>we have</u>:

(3.2.6.1) $\quad P_J(x_{I-J} m_{J,\alpha}) = x_{I-J} f_{J,\alpha}$ <u>with an</u> $f_{J,\alpha} \in \Gamma(U, \underline{M}^0)$.

(2) <u>For each</u> $p \in \bar{X}_I \cap U$, $\underline{E}_{\bar{X},p}$ <u>is generated by the following</u>:

(3.2.6.2) $\quad x_{I-J} f_{J,\alpha}$, <u>with</u> $J: J\subset I$ <u>and</u> $f_{J,\alpha}$ <u>as above</u>.

(Here we admit the case $J=\phi$. In this case $\underline{m}_J = \underline{m}^\Theta$ and P_J=identity.)

<u>Proof</u>. The proof of (a) is as follows: Take a $v: v \quad a := \#J$, and let the map $P^{v,a}$ be as in (3.1.5.1). We check the following inductively on v. (When $v = 0$ we have (3.2.6.1).)

(a) $\quad P^{v,a}_{|J}(x_{I-J} m_{J,\alpha}) = x_{I-J} f^v_{J,\alpha}$ with $f^v_{J,\alpha} \in \Gamma(U\cap\bar{X}^v, \underline{M}^v)$

(Here $|J$ means the restriction to $\Gamma(U\cap\bar{X}_J, \underline{I}_{\partial X_J}\underline{M}_{\bar{X}_J})$. Writing $f^v_{J,\alpha} = \oplus_K f_K$, with $K: \#K = v$, $x_{I-J} f_K = 0$ if $K\subset J$. In this case, without loss of generality, we can assume $f_K = 0$; we write $f^v_{J,\alpha} = \oplus_K f_K$ with $K: K\subset J$.)

First assume that $v = a-1$. Then the L.H.S. of (a) is:

(b-1) $\quad x_{I-J} \oplus_K \pi^*_J(\Phi_{K,J}(m_{J,\alpha}))|\bar{X}_K$, with $K: K \subset J$ and $\#K = a-1$ and π_J as in (3.2.1.2). (The morphism $\Phi_{K,J}$ is as in (2.2.7).)

This implies (a) for $v = a-1$. Next take a $v: v \quad a$ and assume that (a) holds for v. Operate $\Phi^v = \oplus_K \Phi_K$, $\#K = v$, to the R.H.S in (a). Then we have:

(b-2) $\quad \Phi^v(x_{I-J} \oplus_K f_K) = x_{I-J} \oplus_K (\oplus_L \Phi_{L,K}(f_K))$, with $L: L\subset K$ and $\#L = v-1$.

By (2.3.4) and Lemma 2.2 we rewrite the R.H.S as follows:

(b-3) $x_{I-J}(\oplus_L f'_L)$, with $f'_L \in \Gamma(U \cap \partial X_L, \underline{M}_{\bar{X}_L}|\partial X_L)$; $L \subset I$.

Operate $(ex)^{\vee-1} = \oplus_L (ex)_L$, cf. (3.2.4.1), to this, and we have:

(b-4) $\quad P_{|J}^{v-1,a} = x_{I-J} \oplus_L (ex)_L (f_L^{'})$.

Thus we have (a). Next, for the proof of (a), take an element $\zeta \in \Gamma(U \cap \bar{X}_J, \underline{O}_{\bar{X}_J})$. Then we have:

(3.2.6.3) $\quad P_J(x_{I-J}\zeta_J m_{J,\alpha}) = x_{I-J}\pi_J^{*}(\zeta_J) f_{J,\alpha}$.

This is proven in an entirely parallel manner to (3.2.6.1), by changing x_{I-J} to $x_{I-J}\pi_J^{*}(\zeta_J)$.(The unique point to be cared is that $(ex)^{v-1} = \oplus_L (ex)_L$ operates linearly on $\pi_J^{*}(\zeta_J)$,cf.(b-4). But this is clear fro (3.2.4.2).) Now (3.2.6.2) is clear from (3.2.6.3) and Theorem 3.1. Actually take a generator $\underline{e}$ of $\underline{E}_{\bar{X},p}$, and take an open neighborhood $U^{'}$ of p in which $\underline{e}$ is defined. Assume that $U^{'}$ is a polydisc and we form the parametrization map in $U^{'}$. We see immediately that the parametrization maps commute with the schrinking of the neighborhoods(since the both maps Φ_J and $(ex)_J$ have such properties.) By Theorem 3.1 each element of $\underline{e}$ is a linear ombination of the elements of the form $P_J(x_{I-J}\zeta_J^{'} m_{J,\alpha})$. By (3.2.6.3) the element is a linear combination of $x_{I-J} f_{J,\alpha}$, cf.(3.2.6.2). Thus we have (2). q.e.d.

It is an important but a hard task to determine $P_J(x_{I-J} m_{J,\alpha})$ explicitly. In the remainder of § 3 we didcuss some structues of the element.

<u>3.3</u>. <u>Explicit forms</u>. The situation here is same as in § 3.2. The open set U of $\bar{X}$ and the subset $I \subset \Delta_N$ are as in (3.2.0.1,2). The frames $\underline{m}_J$,$J \subset I$,and $\underline{m}^0$ of $\underline{M}_{\bar{X}_J}$ and $\underline{M}^0$ over U;$J \subset I$, and the map P_J are as in Theorem 3.2. We determiene P_J explicitly for J:#J$\underline{\ }$ 3. By Theorem 3.2 we write:

(3.3.0.1)
$$P_J(x_{I-J}\underline{m}_J) = \underline{m}^0(x_{I-J}H_{0,J}), \text{ with a matrix } H_{0,J} \in M(U,\underline{O}_{\bar{X}}) \text{ and}$$
$$x_{I-J} = \Pi_{i \in I-J} x_i .$$

We analyze the matrix $H_{0,J}$. For this we write $\Phi_K = \oplus_L \Phi_{L,K} : \underline{M}_{\bar{X}_K} \to (C^1(\underline{M}^{\#K-1}))_K = \oplus_L \underline{M}_{\bar{X}_L} | \bar{X}_K$, with L:L⊂K and #L = #K -1,cf.(2.2.7), as follows:

(3.3.0.2) $\Phi_{L,K}(\underline{m}_K) = \underline{m}_{L|\bar{X}_K} h_{L,K}$ with $h_{L,K} \in M(U\cap\bar{X}_K, \underline{O}_{\bar{X}_K})$.

When $\#K=1$ we write the matrix as $h_{0,K}$. Our purpose here is to write $H_{0,J}$ in terms of $h_{L,K}; K\subset J$. As an intermediate stage for this we use the following notation. For a $K: K\subset J$ we write $P_{K,J}$ for $pr_K(P^{\#K,\#J})_{|J}: \Gamma_J := \Gamma(U\cap\bar{X}_J, \underline{I}_{\partial X_J}\underline{M}_{\bar{X}_J}) \to \Gamma(U\cap\bar{X}_K, \underline{M}_{\bar{X}_K})$. Here $P^{\#K,\#J}$ is as in (3.1.5.1) and $|J$ indicates the restriction to Γ_J. Moreover pr_K denotes the projection: $\underline{M}^{\#K} \to \underline{M}_{\bar{X}_K}$, cf.(2.2.7). We write:

(3.3.0.3) $P_{K,J}(\underline{m}_J) = \underline{m}_K H_{K,J}$ with a matrix $H_{K,J} \in M(U\cap\bar{X}_K)$.

The analytic projection $\pi_J: U \to U\cap\bar{X}_J$ is as in (3.2.0.2).

(i) Take a $J\subset I$ with $\#J=1$. Then we obviously have:

<u>Lemma 3.2.1</u>. <u>The map</u> P_J <u>is as follows</u>:

(3.3.1)
$$P_J(x_{I-J}\underline{m}_J) = (ex)_J(\underline{m}_{|\bar{X}_J}^{0} x_{I-J} h_{0,J}) = \underline{m}^0\{x_{I-J}\pi_J^*(h_{0,J})\}, \text{ and}$$
$$H_{0,J} = \pi_J^*(h_{0,J}).$$

In the remainder of § 3.3 we consider the case: $\#J=2,3$. We should consider elements of $\Gamma(U\cap\bar{X}_K, \underline{M}_{\bar{X}_K})$ for various $K\subset J$ and their extensions to larger varieties $\bar{X}_L'$ and restrictions to smaller varieties $\bar{X}_L$. It is desirable to use notation which indicates the extension and the restriction. But it causes confusions of notation: We make the following convention.

(*) <u>For an element, say</u> $\varphi \in \Gamma(U\cap\bar{X}_K, \underline{O}_{\bar{X}_K})$, <u>we use the symbol</u> φ <u>also for its restriction</u> $\varphi_{|\bar{X}_L}$ <u>to</u> $\bar{X}_L$, <u>where</u> $\bar{X}_L \subset \bar{X}_K$.

We make clear the situation when we consider an extension of φ to $\bar{X}_L'; \bar{X}_L' \supset \bar{X}_K$.

(ii) <u>Case: #J =2</u>. Here we assume that $\#I \geqq 2$. We take a $J\subset I$ with $\#J=2$, and write J as $\{i,j\}$. Also we write the matrices $H_{0,J}$ and $h_{0,u}; u\in I$, cf.(3.3.0.1,2) as follows:

(3.3.2.0)
$$H_{0,J} = \pi_J^*(H_{0,J;J}) + x_j\pi_i^*(H_{0,J;i}) + x_i\pi_j^*(H_{0,J;j}),$$
$$h_{0,u} = \pi_J^*(h_{0,u;J}) + x_v h_{0,u;u}, \text{ with } \{v\} = I-\{u\} .$$

(This is a power series expansion of $H_{0,J}$ and $h_{0,u}$ around the center

$\bar{X}_J$ with the coordinaes x_i and x_j. In (3.3.2.0) the symbol $H_{;K}, K\subset J$, indicates that this matrix is in $M(U\cap\bar{X}_K, \underline{O}_{\bar{X}_K})$. Thus $H_{0,J;J}$ and $h_{0,u;J}\in M(U\cap\bar{X}_J, \underline{O}_{\bar{X}_J})$ and $H_{0,J;u}$ and $h_{0,u;u}\in M(U\cap\bar{X}_u, \underline{O}_{\bar{X}_u})$.)

<u>Lemma 3.2.2</u>. <u>The matrix</u> $H_{0,J}$, <u>cf.(3.3.0.1), is as follows</u>:

(3.3.2.1) $H_{0,J;J}=h_{0,u;J}h_{u,J}$, $u\in I$, <u>and</u> $H_{0,J;u}=h_{0,u;u}\pi^*_J(h_{u,J})$, $u\in I$.

(<u>Note that the expression of</u> $H_{0,J;J}$ <u>is independent of</u> $u\in J$, <u>cf.also Lemma 3.1.</u>)

<u>Proof</u>. Let $P^{1,2}$ be $(ex)^1\Phi^2$, cf.the proof of Theorem 3.1. Then the restriction of it to $\bar{X}_J$ is as follows:

(a) $P^{1,2}_{|J}(x_{I-J}\underline{m}_J) = x_{I-J}(\underline{m}_i\pi^*_J(h_{i,J})\oplus\underline{m}_j\pi^*_J(h_{j,J}))(\in\oplus_{u\in J}\Gamma(U\cap\bar{X}_u, \underline{M}_{\bar{X}_u}))$.

Operate Φ^1 to this. Then we have:

(b) $\Phi^1P^{1,2}_{|J}(x_{I-J}\underline{m}_J) = x_{I-J}\{\oplus_u\underline{m}^0_{|\bar{X}_u}(h_{0,u}\pi^*_J(h_{u,J})\}$, with $u\in J$.

On the otherhand, we clearly have:

(c) $h_{0,u}\pi^*_J(h_{u,J}) =\{\pi^*_J(h_{0,u;J})+x_vh_{0,u;u}\}\pi^*_J(h_{u,J})$, with $\{v\} =I -\{u\}$.

By Lemma 2.2 the element in (b) is in $\Gamma(U\cap\bar{X}^1, \underline{M}^0_{|\bar{X}}1)$. Thus we have the independence assertion for the element $H_{0,J;J}$. On the other hand (c) insures:

(3.3.2.2) $(H_{0,J})_{|\bar{X}_u} =\pi^*_J(h_{0,u;J}h_{u,J}) +x_vh_{0,u;u}\pi^*_J(h_{u,J})$, $u\in J$. .

Thus implies the lemma. q.e.d.

(iii) <u>Case:#J =3</u>. Here we assume that $\#I \geq 3$, and take a $J\subset I$ with # J=3. Corresponding to (3.3.2.0) we expand the matrix $H_{0,J}$, cf.(3.3.0.1), with the center $\bar{X}_J$ and coordinates $x_u;u\in I$:

$$H_{0,J}=\pi^*_J(H_{0,J;J})+\Sigma_{(u,v)\subset J}\ x_{J-\{u,v\}}\pi^*_{uv}(H_{0,J;uv}) +$$

(3.3.3.0)

$$+\Sigma_{u\in J}\ x_{J-\{u\}}\pi^*_u(H_{0,J;u}).\text{(For a } K\subset J,\ x_{J-K}=\Pi_{i\in J-K}x_i.)$$

(As before, the symbol:$H_{;K}, K\subset J$, indicates that the coefficients of the matrix $H_{;K}$, are in $\Gamma(U\cap\bar{X}_K, \underline{O}_{\bar{X}_K})$. Thus $H_{0,J;J}$, $H_{0,J;uv}$ and $H_{0J;u}$ are respectively in $M(U\cap\bar{X}_J, \underline{O}_{\bar{X}_J})$, $M(U\cap\bar{X}_{uv}, \underline{O}_{\bar{X}_{uv}})$ and $M(U\cap\bar{X}_u, \underline{O}_{\bar{X}_u})$.)

For the determination of $H_{0,J}$, we write $h_{0,u}$, $h_{u,uv}$ and $H_{u,J}$, cf. (3.3.0.1,2,3), as follows:

$$h_{0,u} =\pi^*_J(h_{0,u;J}) +\{x_w\pi^*_{uv}(h_{0,u;uv})+x_v\pi^*_{uw}(h_{0,u;uw})\} +$$

(3.3.3.1) $x_v x_w h_{0,u;u}$, with $\{v,w\} = I-\{u\}$, and

$h_{u,uv} = \pi^*_J(h_{u,uv;J}) + x_w h_{u,uv;uv}$, with $\{w\}=I-\{u,v\}$.

(3.3.3.2) $H_{u,J}=\pi^*_J(H_{u,J;J})+x_v H_{u,J;uw}+x_w H_{u,J;uv}$ with $\{v,w\} = I -\{u\}$.

(In the above the symmol $_{;K}$, $K \subset J$, has th similar meaning to (3.3.3.0). Thus $h_{0,u;J}$, $H_{u,J;J} \in M(U\cap\bar{X}_J, \underline{O}_{\bar{X}_J})$, $h_{u,uv;uv} \in M(U\cap\bar{X}_{uv}, \underline{O}_{\bar{X}_{uv}})$ and $h_{0,u;u} \in M(U\cap\bar{X}_u, \underline{O}_{\bar{X}_u}), \ldots$) First we determine $H_{u,J}$.

Proposition 3.1. (1) For each $u \in J$ we have:

(3.3.3.3)

$$H_{u,J;J} = h_{u,uv;J} h_{uv,J} \text{ for each } v \in J -\{u\}, \text{ and}$$

$$H_{u,J;uv} = h_{u,uv;uv}\pi^*_J(h_{uv,J}); v \in I -\{u\} .$$

Proof. Remarking that $\#J-\#\{u\} =2$, this is checked similarly to Lemma 3.2.2. q.e.d.

Note that (3.3.3.3) implies:

(3.3.3.4) $H_{u,J|\bar{X}_{uv}} = (\pi^*_J(h_{u,uv;J})+x_w h_{u,uv;uv})\pi^*_J(h_{uv,J}) = h_{u,uv}\pi^*_J(h_{uv,J})$

Lemma 3.2.3. The matrix $H_{0,J}$ is as follows:

$H_{0J;J} = h_{0u;J} H_{u,J;J}$ for each $u \in J$,

$x_w H_{0J;uv} = h_{0,\alpha|\bar{X}_{uv}} h_{\alpha,uv}\pi^*_J(h_{uv,J}) - H_{0J,J}$, with $\{w\} = I -$

(3.3.3.5) (u,v) and $\alpha=u$ or $v; u,v \in I$,

$H_{0J;u} = h_{0,u;u}H_{u,J} + h_{0,u;uv}H_{u,J;uw} + h_{0,u;uw}H_{u,J;uv}$, for each $u \in J$, with $\{v,w\} = J -\{u\}$.

Proof. Restrict $P_J(x_{I-J}\underline{m}_J)=\underline{m}^0 x_{I-J}H_{0,J}$ to $\bar{X}_u; u\in J$. Then we have:

(3.3.3.6) $H_{0,J|\bar{X}_u} = h_{0,u}H_{u,J}$.

Also restrict this to $\bar{X}_{uv}; u,v \in I$. Then we have; cf. (3.3.3.4):

(3.3.3.7) $H_{0,J|\bar{X}_{uv}} = h_{0,\beta|\bar{X}_{uv}} h_{\beta,uv}\pi^*_J(h_{uv,J}); \beta = u,v$.

The first and second identities in (3.3.3.5) follow from this. The third follows from (3.3.3.6) by comparing the term divided by $x_u x_v$. q.e.d.

Remark 3.1. The matrices $H_{0,J;J}$ and $H_{0,J;uv}$ as above are also written as follows:

$H_{0,J;J} = h_{0,u;J} h_{u,uv;J} h_{uv,J}$ for each increasing sequence

$\{u\} \subset \{u,v\} \subset J$, cf. also Lemma 3.1

(3.3.3.7) $x_w H_{0,J;uv} = \{(\pi_J^*(h_{0,\alpha;J}) + x_w h_{0,\alpha;uv})(\pi_J^*(h_{\alpha,uv;J}) + x_w h_{\alpha,uv;uv}) \times$
$\times \pi_J^*(h_{uv,J})\} - \pi_J^*(H_{0J;J})$ for each $(u,v) \subset J$, with $\{w\} = J - \{u,v\}$ and $\alpha = u$ or v.

The similar expression for $H_{0,J;u}; u \in J$, is obtained, but is more complicated and we omit it. Applications of Lemma 3.2.1 3.2.3 are given in § 5.

3.4. An example. Here we deal with the parametrizatio map P for the example in Lemma 2.6.1: Let $(M^0, E_{\bar X})$ be a pair of a locally free $\underline{O}_{\bar X}$-module M^0 and its $\underline{O}_{\bar X}$-submodule $E_{\bar X}$. We assume the condition in (2.6.1,2): There is a locally free $O_{\bar X}$-mdule $\underline{M}'$ and an $\underline{O}_{\bar X_i}$-morphism $\Phi_i : M'_{|\bar X_i} \to M^0_{|\bar X_i}$, $i \in \Delta_m$, satisfying:

(3.4.0)
$$E_{\bar X} = (\delta_0)^{-1}(Z^1(M^0) \cap (\oplus_{i\in\Delta_m} \Phi_i(M'_{|X_i}))), \text{cf.}(2.3.3), \text{ and}$$
$$\Phi_{i|\bar X_{ij}} = \Phi_{j|\bar X_{ij}}; i,j \in \Delta_m .$$

For each $J \subset \Delta_m$, $(\underline{M}_{\bar X_J}, \Phi_J)$ is as in (2.6.2): $\underline{M}_{\bar X_J} = M'_{|\bar X_J}$ and $\Phi_J = \oplus_a(\mathrm{id})$: $M'_{|\bar X_J} \ni \varphi_J \to (C^1(M^{\#J-1}))_J = \oplus_{s=1}^a \underline{M}_{\bar X_{J(s)}|\bar X_J} (= \oplus_{s=1}^a M'_{|\bar X_J}) \ni \oplus_{s=1}^a \varphi_J$, with $a = \#J$ and $J(s) = J - \{s\text{-th element of } J\}$. Now let the open set U of $\bar X$ and the subset $I \subset \Delta_m$ be as in (3.2.0.1,2). The coordinates $x_i; i \in \Delta_m$ are as in § 3.2: $\bar X_i \cap U = (x_i)_0$. For a $J \subset I$ and $j \in J$, the map $P_{j,J}$: $\Gamma(U \cap \bar X_J, \underline{I}_{\partial X_J} \underline{M}_J) \to \Gamma(U \cap \bar X_j, M_j)$ is as in (3.3.0.2). We assume the existence of a frame $\underline{m}'$ of $M'_{|U}$.

Lemma 3.3.1. For each $\zeta_J \in \Gamma(U \cap \bar X_J, \underline{O}_{\bar X_J})$ we have:

(3.4.1) $P_{j,J}(x_{I-J}\zeta_J \underline{m}'_{|\bar X_J}) = \underline{m}'_{|\bar X_j}(x_{I-J}\pi_J^*(\zeta_J))$ with $x_{I-J} = \Pi_{i\in I-J} x_i$.

Proof. It suffices to check the similar fact for each $K: K \subset J$:

(a) $P_{K,J}(x_{I-J}\zeta_J m'_{|\bar X_J}) = m'_{|\bar X_K}(x_{I-J}\pi_J^*(\zeta_J))$. (For $P_{K,J}$ see (3.1.5.2).)

But this is obvious since: Φ_L = the direct sum of the identity; $J \subset L \subset K$ and $\mathrm{ex}_L(\pi_J^*(\zeta_J)) = \pi_J^*(\zeta_J)$, cf. (3.1.4.2). q.e.d.

Assume that $M^0_{|U}$ has a frame, denoted by $\underline{m}^0$, and we write $\Phi_u(m'_{|\bar X_u}) = \underline{m}^0 h_{0,u}$; $u \in I$. The following is obvious.

Corollary. For each $u \in I$ we have:

$$P_J(x_{I-J}\zeta_J \underline{m}'|\bar{X}_J)|\bar{X}_u = \underline{m}^0|\bar{X}_u(x_{I-J}\zeta_J h_{0,u}) . \tag{3.4.2}$$

Lemma 3.3.2. For a $\zeta_J \in \Gamma(U \cap \overline{X}_J, O_{\overline{X}_J})$ we have:

(3.4.3) $\quad P_J(x_{I-J}\zeta_J) = x_{I-J}\pi_J^*(\zeta_J)$.

Proof. Clear from (3.4.2) once we remark that h_{01} =1(=1×1-identity matrix), cf. § 2.6. q.e.d.

Assume that #I =dim $\overline{X}$(and so $\overline{X}_I$ is zero dimensional.) Then the parametrization map P, cf.(3.2.6), is as follows:

(3.4.3) $\quad \oplus_J P_J(x_{\Delta_N - J}\zeta_J) = \Sigma_J\, x_{\Delta_N - J}\pi_J^*(\zeta_J)$.

This coincides with the power series expansion in (*) at the end of § 2.

§ 4. Inductive structure ...3
(Case of reflexive sheaf of type (C))

Here we determine the inductive structure of the reflexive sheaf of type (C), cf.Definition 1.1 , for codimension $\leqq 3$. The structure for codimension $\geqq 4$ is a subtle subject. We do not treat it systematically in this paper;see, however, Theorem 4.2. We hope to discuss it in an another place.

Data and assumptions

We work with the reflexive sheaf $\underline{E}_{\overline{X}}$ of type (C) in § 1.1;cf. (1.1.0.0 8) and Remark 1.1. The arguments here are based on those in § 2. We use freely the notations in § 1.1 and § 2. In § 4.1 ~ § 4.3 we assume the condition (2.0,0).

Remark 4.0.1. In § 4 we write the divisor $\overline{X}^1_i$ and its open part X^1_i,cf.(1.1.0.0 2), as $\overline{X}_i$ and X_i. For an $I \subset \Delta_m$ we set:$\overline{X}_I = \cap_{i \in I} \overline{X}_i$. For the divisor $\overline{X}^1$,cf.(1.1.0.0) we set:

(4.0.0) $\quad \underline{L}_{\overline{X}} = \underline{O}_{\overline{X}}[\overline{X}^1]$, and $\underline{L}_{\overline{X}_I} = \underline{L}_{\overline{X}|\overline{X}_I}$, $I \subset \Delta_m$.

4.0. Imbedding of $\underline{E}_X$. We first imbed $\underline{E}_X$ into $\underline{L}_X^{\oplus r}$, with $\underline{L}_X = \underline{L}_{\overline{X}|X}$. For this take an element $s \in \Gamma(\underline{L}_{\overline{X}})$ such that $(s)_0 = \overline{X}^1$. By (1.1.0.4,6) the element $s^1_i := s(1/f_{\alpha(i)}) \in \Gamma(N_{1,i}, \underline{L}_{\overline{X}})$ is a frame of $\underline{L}_{\overline{X}|N_{1,i}}$. Corresponding to Proposition 1.1,[Sa-1] we have:

Proposition 4.0. There is an imbedding θ_s of $\underline{E}_X$ into $\underline{L}_X^{\oplus r}$.

Proof. Setting $\underline{s}_\alpha = {}^t(0,..,\overset{\alpha}{\underline{s}},..,0)$, $\alpha \in \Delta_r$, we define:

(4.0.1.1) $\quad \theta_{s|N_0} : \underline{E}_{X|N_0} \supset \underline{e}^0 \rightarrow \underline{L}^{\oplus r}_{X|N_0} \supset \theta_{s|N_0}(\underline{e}^0) = (\underline{s}_1,...,\underline{s}_r)$

Moreover, we define $\theta_{s|N_{1,i}} : \underline{E}_{X|N_{1,i}} \rightarrow \underline{L}^{\oplus r}_{X|N_{1,i}}$ by:

the α-th component of $\theta_{s|N_{1,i}}(\underline{e}^1) = \underline{s}_\alpha$, $\alpha \in \Delta_i - \{\alpha(i)\}$,and

(4.0.1.2) $\quad = s^1_i \underline{f}'_i$, $\alpha = \alpha(i)$, with the vector $\underline{f}'_i$ whose $\alpha(i)$-th element $=1$ and α-th element $= -f_\alpha$,$\alpha \in \Delta_r - \{\alpha(i)\}$.

See § 1.1 for the frames $\underline{e}^0, \underline{e}^1$ and the elements $\alpha(i) \in \Delta_r$, $f_\alpha \in \Gamma(N_1, \underline{O}_{\overline{X}})$.

By an elementary computation, we have: $\theta_{s|N_0}(\underline{e}^0) = \theta_{s|N_{1,i}}(\underline{e}^1)H$, with the matrix H(cf.(1.1.0.4,5)) in $N_0 \cap N_{1,i}$. q.e.d.

We set $\underline{E}'_X = \theta_s(\underline{E}_X)$. The isomorphism θ_s is extended to the one: $\underline{E}_{\bar{X}} \simeq \underline{E}'_{\bar{X}}$, where $\underline{E}_{\bar{X}}$ and $\underline{E}'_{\bar{X}}$ are the direct images of $\underline{E}_X, \underline{E}'_X$ with respect to the injection :$X \quad \bar{X}$. We write θ_s also for its extension. We give a characterization of $\underline{E}'_{\bar{X}}$ as the submodule of $\underline{L}^{\oplus r}_{\bar{X}}$:For each $\alpha \in \Delta_r$ let $f_{\alpha|\bar{X}_i}$ denote the meromorphic function over $\bar{X}_i$ whose restriction to $X_i = f_{\alpha|X_i}$,cf.(1.1.0.6). We assume the existence of a divisor $\bar{D}^2_i$ of $\bar{X}_i$, which is defined locally by a single function and satisfies the following:

(4.0.2.1) $\bar{D}^2_i = \Sigma_u b_{i,u}\bar{D}^2_{i,u}$, where the divisor $\bar{D}^2_{i,u}$ of $\bar{X}_i$ and the element $b_{i,u} \in Z_+$ are as follows:

(4.0.2.2) $\bar{D}^2_{i,u}$ run through all divisors that appear as an irreducible component of the pole divisor of the meromorphic function $f_{\alpha|\bar{X}_i}$ for an $\alpha \in \Delta_r - \{\alpha(i)\}$, cf.(4.0.0) and $b_{i,u} = \max_{\alpha\in\Delta_r-\{\alpha(i)\}}$ {order of the pole of $f_{\alpha|\bar{X}_i}$ along $\bar{D}^2_{i,u}$}

The order of $f_{\alpha|\bar{X}_i}$ along $\bar{D}^2_{i,u}$ is defined to be the one of $f_{\alpha|\bar{X}_i}$ along $\bar{D}^2_{i,u} - \bar{X}_{i,sing}$. Also note that if $\bar{X}_i$ is smooth then the divisor $\bar{D}^2_i$ as above always exists.

Take an element $g_{\alpha(i)} \in \Gamma(\underline{O}_{\bar{X}_i}[\bar{D}^2_i])$, whose locus is $\bar{D}^2_i$, and define a vector $\underline{g}_i = (g_{i,\alpha})_{\alpha \in \Delta_r} \in \Gamma^r(\underline{O}_{\bar{X}_i}[\bar{D}^2_i])$ as follows:

(4.0.3) $\quad g_{i,\alpha(i)} = g_{\alpha(i)}$ and $g_{i,\alpha} = (f_{\alpha|\bar{X}_i})g_{\alpha(i)}$, $\alpha \in \Delta_r - \{\alpha(i)\}$.

For each irreducible component $\bar{D}^2_{i;u}$, we easily have:

(4.0.4) $\quad \underline{g}_i \not\equiv 0 \ (\bar{D}^2_{i;u})$.

<u>Remark 4.0.2.</u> If each $f_{\alpha|\bar{X}_i}$ is holomorphic on $\bar{X}_i$, we understand that $\bar{D}^2_i = \phi$ and $\underline{O}_{\bar{X}_i}[\bar{D}^2_i] = \underline{O}_{\bar{X}_i}$. We do not consider (4.0.4) in this case.

Next define an element $\Phi_i = (\Phi_{i,\alpha})_{\alpha \in \Delta_r} \in \Gamma^r(\underline{O}_{\bar{X}}[\bar{D}^2_i])$ as follows:

(4.0.5) $\Phi_{i,\alpha(i)} = -g_{i,\alpha(i)}$ and $\Phi_{i,\alpha} = g_{i,\alpha}$, $\alpha \in \Delta_r - \{\alpha(i)\}$

We set:

(4.0.6) $\quad \underline{M}_{\bar{X}_i} = \underline{O}_{\bar{X}_i}[\bar{D}_i^2]^* \otimes \underline{L}_{\bar{X}_i}$, cf.(4.0.0).

Note that Φ_i defines an $\underline{O}_{\bar{X}_i}$-morphism: $\underline{M}_{\bar{X}_i} \ni \tau_i \to \underline{L}_{\bar{X}_i}^{\oplus r} \ni \tau_i \otimes \Phi_i$.

We set $\underline{M}^1 = \oplus_i \underline{M}_{\bar{X}_i}$ and $\Phi^1 = \oplus_i \Phi_i : \underline{M}^1 = \oplus_i \underline{M}_{\bar{X}_i} \to C^1(\underline{L}_{\bar{X}}^{\oplus r}) = \oplus_i \underline{L}_{\bar{X}_i}^{\oplus r}$ with $i \in \Delta_m$; cf.(2.1.1):

Theorem 4.0. (1) The pair $(\underline{M}^1, \Phi^1)$ is an inductive structure of $(\underline{L}_{\bar{X}}^{\oplus r}, \underline{E}_{\bar{X}}')$ at the level of codimension one:Namely it satisfies the first condition in (2.3.3):

$$\underline{E}_{\bar{X}}' = (\delta^0)^{-1}(\Phi^1(\underline{M}^1) \cap Z^1(\underline{L}_{\bar{X}}^{\oplus r})), \text{ with } \delta^0 : \underline{L}_{\bar{X}}^{\oplus r} \to C^1(\underline{L}_{\bar{X}}) \text{ and}$$

(4.0.7)

$$Z^1(\underline{L}_{\bar{X}}^{\oplus r}) = \text{kerenel of } \delta^1 : C^1(\underline{L}_{\bar{X}}^{\oplus r}) \to C^2(\underline{L}_{\bar{X}}^{\oplus r}) = \oplus_{i \neq j} \underline{L}_{\bar{X}_{ij}}^{\oplus r}, \text{cf.(2.1.2)}.$$

(2) The morphism Φ_i is injective.

Corollary. The direct image $\underline{E}_{\bar{X}}'$ of $\underline{E}_X'$ with respect to the inclusion:$X \quad \bar{X}$ is coherent.

Proof of Theorem 4.0. First (2) is clear from that $g_{i,\alpha(i)} \neq 0$. For the proof of (1) we define an $\underline{O}_{\bar{X}_i}$-morphism as follows:

$$\chi_i : \underline{L}_{\bar{X}_i}^{\oplus r} \ni \xi = (\xi_\alpha)_{\alpha \in \Delta_r} \to \underline{O}_{\bar{X}_i}[\bar{D}_i^2] \otimes \underline{L}_{\bar{X}_i}^{\oplus r-1} \ni \xi' = (\xi_\alpha'), \alpha \in \Delta_r - \{\alpha(i)\}$$

(4.0.8)

$$\text{where } \xi_\alpha' = g_{i,\alpha(i)} \otimes \xi_\alpha + g_{i,\alpha} \otimes \xi_{\alpha(i)}.$$

Compare χ_i and Φ_i. Then from (4.0.5) we have:

(4.0.9) Φ_i gives an isomorphism: $\underline{M}_{\bar{X}_i} \quad \ker \chi_i (\subset \underline{L}_{\bar{X}_i}^{\oplus\oplus})$.

Thus, for the proof of the theorem, it suffices to check:

(4.0.10) $\underline{E}_{\bar{X}}' = \cap_{i \in \Delta_m} \omega_i^{-1} \ker \chi_i$, with the quotient morphism $\omega_i : \underline{L}_{\bar{X}} \to \underline{L}_{\bar{X}_i}$,

Take an element $\xi = (\xi_\alpha)_{\alpha \in \Delta_r} \in \underline{L}_{\bar{X}}^{\oplus r}$. Recalling the definition of $\underline{g}_i$, one sees immediately that $\xi \in \cap_{i \in \Delta_m} \omega_i^{-1} \ker \chi_i$ is equivalent to say that the following holds for each $i \in \Delta_m$:(See (4.0.0) for $f_{\alpha|\bar{X}_i}$ below.)

(a) $\qquad \omega_i(\xi_\alpha) + f_{\alpha|\bar{X}_i}\omega_i(\xi_{\alpha(i)}) = 0, \ \alpha \in \Delta_r - \{\alpha(i)\}.$

We show that (a) is the defining equation of $\underline{E}_{\bar{X}}'$. Since $\underline{E}_{\bar{X}|N_0}' = \underline{L}_{\bar{X}}^{\oplus r}$, it sufficies to show it by assuming that $\xi \in \underline{L}_{\bar{X},p}^{\oplus r}$, with $p \in \bar{X}^1$. Take an open neighborhood U of p, and we assume that $\xi \in \Gamma(U, \underline{L}_{\bar{X}}^{\oplus r})$. Since $\underline{E}'_{\bar{X}|N_{1,i}}$ is spanned by $\underline{s}_\alpha$, $\alpha \neq \alpha(i)$, and $\underline{s}_i^1 \underline{f}'_i$, cf.(4.0.1), we see that $\xi \in \Gamma(U, \underline{E}'_{\bar{X}})$ is equivalent to the following:

(b) $\omega_i(\zeta)_q$ is in the $\underline{O}_{\bar{X}_i,q}$-submodule of $\underline{L}_{\bar{X}_i,q}^{\oplus r}$ which is spanned by $\omega_i(s_i^1 \underline{f}'_i)$, for each $q \in X_i \cap U$.

From the explicit form of $\underline{f}'_i$, we see easily that (a) and (b) are equivalnet, and we finish the proof of Theorem 4.0. q.e.d.

Remark 4.0.3. In the investigations of $\underline{E}'_{\bar{X}} \simeq \underline{E}_{\bar{X}}$ from now on, we use only (4.0.7). It is not necessary to remember the explicit form of the transition matrix H.

4.1. Inductive structure for codimension 2 and 3. In the remainder of § 4 we assume the condition (2.0.0) for $\bar{X}^1$. Thus, for an $I \subset \Delta_m$, the subvariety $\bar{X}_I = \cap_{i \in I} \bar{X}_i$ is smooth and of codimesion #I and is irreducible(unless #I=dim $\bar{X}$) .

4.1.1. Main results. Here we give a theorem which says that the pair $(\underline{L}_{\bar{X}}^{\oplus r}, \underline{E}'_{\bar{X}})$ admits an inductive structure up to codimension three, which satisifes the following:(1) It is defined globally on $\bar{X}$ and (2) the locally free sheaves in question are of rank $\leqq 1$.

Theorem 4.1. Assume that #I=2 or 3 . Then there is a locally free $O_{\bar{X}_I}$-module $M_{\bar{X}_I}$ of rank 1 and an $O_{\bar{X}_I}$-isomorphism Φ_I:

(4.1.0) $\Phi_I : M_{\bar{X}_I} \to \ker(\delta\Phi^{a-1})_I$, with $a = \#I$.

where $\Phi^{\cdot} = \oplus_{J;\#J=\cdot} \Phi_J$, $\cdot=1,2$, and $\delta\Phi^{\cdot}$ is the coboundary of $\Phi^{\cdot}$, cf. Definition 2.2. Moreover $(\delta\Phi^{\cdot})_I$ is the I-part of $(\delta\Phi^{\cdot})$, cf.(2.2.4).

Now Theorem 4.0 and 4.1 insure:

Corollary. The collection $\{\{M_{\bar{X}_I}\}_I, \{\Phi_I\}_I\}$, I with #I$\leqq$ 3, gives an inductive structure of $(\underline{L}_{\bar{X}}^{\oplus r}, \underline{E}'_{\bar{X}})$ for codimension $\leqq$ 3, cf.Definition 2.3, and it satisfies:

(4.1.1) rank $M_{\bar{X}_I} \leqq 1$(and =1 if #I =1), and Φ_I is injective.

Remark 4.1.1. The isomorphy of Φ_I in (4.1.0) insures that the above inductivre structure of $(\underline{L}_{\bar{X}}^{\oplus r}, \underline{E}_{\bar{X}}')$ is essentially unique.

4.1.2. Explicit forms...1. We give an explicit form of $(\underline{M}_{\bar{X}_{I\lambda}}, \Phi_I)$;Theorem 4.1 ~ 4.4 below. In doing it we should consider an irreducible component $\bar{X}_{I\lambda}$ of $\bar{X}_I$. We write $\underline{M}_{\bar{X}_{I\lambda}}$ and $\Phi_{I\lambda}$ for the restriction of $\underline{M}_{\bar{X}_I}$ and Φ_I to $\bar{X}_{I\lambda}$. Moreover, letting $\underline{M}^{a-1} = \{\underline{M}_J\}$,with $a = \#I$ and $J:\#J = a-1$, we write:

$(C^1(M^{a-1}))_{I\lambda}$ and $(\Phi^{a-1})_{I\lambda}$ for the restriction to $\bar{X}_{I\lambda}$ of

(4.1.2.0) the I-parts $(C^1(M^{a-1}))_I$ and $(\Phi^{a-1})_I$ of $C^1(M^{a-1})$ and Φ^{a-1},

cf.(2.2.4) an (2.2.8).

Remark 4.1.2. By (2.0.0), $\bar{X}_I$ is irreducible unless $\#I = \dim \bar{X}$.

Now the explicit form of the $\underline{O}_{\bar{X}_{I\lambda}}$-module $M_{\bar{X}_{I\lambda}}$ is as follows:

Theorem 4.2. (1) If $\#I=2$, then there are divisors $\bar{D}^3_{I\lambda;u}$, $u \in I$, of $\bar{X}_{I\lambda}$ such that

(4.1.2.1) $\underline{M}_{\bar{X}_{I\lambda}} \simeq M_{\bar{X}_u} | \bar{X}_{I\lambda} \otimes \underline{I}_{\bar{D}^3_{I\lambda;u}}$, $u \in I$.

(2) If $\#I=3$, then, for any pair $(u,v) \subset I$ satisfying

(4.1.3.1) $pr_{uv} \ker (\delta\Phi^2)_{I\lambda} \neq 0$,

there is a divisor $\bar{D}^4_{I\lambda;uv}$ of $\bar{X}_{I\lambda}$ such that

(4.1.3.2) $\underline{M}_{\bar{X}_{I\lambda}} \simeq M_{\bar{X}_{uv}} | \bar{X}_{I\lambda}{}' \otimes \underline{I}_{\bar{D}^4_{I\lambda;uv}}$.

Here pr_{uv} denotes the projection:$(C^1(\underline{M}^1))_{I\lambda} \to M_{\bar{X}_{uv}} | \bar{X}_{I\lambda}$, cf.(4.1.2.0).

Remark 4.1.3. Concerning (4.1.3.1) the following two cases can happen,cf.Lemma 4.1.2 and 4.1.3,cf.§ 4.1.5.

(4.1.3.3) Exactly two pairs $\subset I$ satisfy (4.1.3.1) .

(4.1.3.4) All pairs $\subset I$ satisfy (4.1.3.1) .

4.1.3. Explicit forms...2. Here we give an explicit form of the divisors and morphisms in the precceding theorems. According as we are concerned with the case of codimension two or three, we write I as $\{i,j\}$ or $\{i,j,k\}$. In the first case we identify $\mathrm{Hom}_{\underline{O}_{\bar X_u}}(\underline{M}_{\bar X_u}, \underline{L}_{\bar X_u}^{\oplus r})$ with $(\underline{M}_{\bar X_u})^{*} \otimes \underline{L}_{\bar X_u}^{\oplus r}$, and regard $\Phi_u = (\Phi_{u,\alpha}), \alpha \in \Delta_r$, as the element of $\Gamma((\underline{M}_{\bar X_u})^{*} \otimes \underline{L}_{\bar X_u}^{\oplus r})$, cf.(4.0.5); $u \in I$. In the second case, note that, for any $(u,v) \subset I$, the morphism Φ_{uv} is of the following form, cf.(2.2.7):

(4.1.4.0) $\Phi_{uv} = \Phi_{u,uv} \oplus \Phi_{v,uv}$, with $\Phi_{\cdot,uv} \in \Gamma(\underline{M}_{\bar X_{uv}}^{*} \otimes \underline{M}_{\cdot|\bar X_{uv}})$, $\cdot = u$ or v.

(i) First we determine the divisors in Theorem 4.2:

Theorem 4.3.1. Assume that $\# I = 2$. Then the divisors $\bar D^3_{I\lambda;u}$, $u \in I$, are determined in the following manner:

$(\Phi_{i,\alpha|I\lambda})_0 + \bar D^3_{I\lambda;i} = (\Phi_{j,\alpha|I\lambda})_0 + \bar D^3_{I\lambda;j}$, for each $\alpha \in \Delta_r$,

(4.1.4.1) and is minimal among pairs of divisors $(\bar D'^3_{I\lambda;i}, \bar D'^3_{I\lambda;j})$ satisfying the identity.(Namely for any such a pair we have: $\bar D^3_{I\lambda;i} \quad \bar D'^3_{I\lambda;i}$ and $\bar D^3_{I\lambda;j} \quad \bar D'^3_{I\lambda;j}$.)

In the following theorem we asume that $\# I = 3$.

Theorem 4.3.2. (1) Assume (4.1.3.3) holds. Then the pair of divisors $(\bar D^4_{I\lambda;uv}, \bar D^4_{I\lambda;uw})$ is determined as follows:

It satisfies the identity:

(4.1.4.2) $\bar D^4_{I\lambda;uv} + (\Phi_{u,uv|\bar X_{I\lambda}})_0 = \bar D^4_{I\lambda;uw} + (\Phi_{u,uw|\bar X_{I\lambda}})_0$

and is minimal among pairs of divisors on $\bar X_{I\lambda}$ satisfying the identity.

(2) Assume that (4.1.3.4) holds. Then the triple of divisors $(\bar D^4_{I\lambda;ij}, \bar D^4_{I\lambda;ik}, \bar D^4_{I\lambda;jk})$ is determined as follows:

It satisfies the following identity for each $u \in I$:

(4.1.4.3) $(\Phi_{u,uv|\bar X_{I\lambda}})_0 + \bar D_{I\lambda;uv} = (\Phi_{u,uw|\bar X_{I\lambda}})_0 + \bar D_{I\lambda;uw}$, with $\{v,w\} = I - \{u\}$,

and is minimal among triples of divisors of $\bar X_{I\lambda}$.

Here minimal should be understood as in Theorem 4.3.1. Also, in

Cf. u is the unique element of I such that $M_{\bar X_{uv}|\bar X_{I\lambda}}$ and $M_{\bar X_{uw}|\bar X_{I\lambda}}$,

$\{v,w\}=I-\{u\}$, <u>satisfy the condition (4.1.3.1)</u>.

(ii) Next we write explicitly the isomorphism $\Phi_{I\lambda}$, cf. (4.1.0), and the ones in Theorem 4.2.

<u>Figure __</u>

$$\underline{M}_{\bar X_{I\lambda}} \qquad\qquad\qquad \underline{M}_{\bar X_{I\lambda}}$$

$$\underline{M}_{\bar X_i|\bar X_{I\lambda}} \quad \underline{M}_{\bar X_j|\bar X_{I\lambda}} \qquad\qquad \underline{M}_{\bar X_{ij}|\bar X_{I\lambda}} \quad \underline{M}_{\bar X_{jk}|\bar X_{I\lambda}}$$

$$\underline{M}_{\bar X_{ik}|\bar X_{I\lambda}}$$

assume that the locally free sheaves in Theorem 4.2 and $\underline{L}_{\bar X_{I\lambda}}$ are trivial over U. According as we are concerned with the case of codimension two or three, we fix a frame $\underline{m}_{u;I\lambda}$ of $(\underline{M}_{\bar X_u|\bar X_u})|U$; $u\in I$ (resp. $\underline{m}_{uv;I\lambda}$ of $(\underline{M}_{\bar X_{uv}|\bar X_{I\lambda}})|U$); $u,v\in I$. In the first (resp. second) case, take an element $h_{I\lambda;u}\in\Gamma(U,\underline{O}_{\bar X_I})$, $u\in I$ (resp. $h_{I\lambda;uv}\in\Gamma(U,\underline{O}_{\bar X_I})$; $u,v\in I$) such that it generates $\underline{I}^3_{\bar D_{I\lambda;u}}$ (resp. $\underline{I}^4_{\bar D_{I\lambda;uv}}$) and satisfies the identity as follows:

$$(h_{I\lambda;i}\underline{m}_{i;I\lambda})\otimes\Phi_{i|\bar X_{I\lambda}}=(h_{I\lambda;j}\underline{m}_{j;I\lambda})\otimes\Phi_{j|\bar X_{I\lambda}} \text{ (resp.} \tag{4.1.5.1}$$
$$(h_{I\lambda;uv}\underline{m}_{uv;I\lambda})\otimes\Phi_{u,uv|I\lambda}=(h_{I\lambda;uw}\underline{m}_{uw;I\lambda})\otimes\Phi_{u,uw|I\lambda}\ \{v,w\}=I-\{u\}.$$

(For the existence of $h_{I\lambda;u}$ and $h_{I\lambda;uv}$ see § 4.2. and § 4.3.)

<u>Remark 4.1.4</u>. In the above, if $(u,v)\subset I$ does not satisfy (4.1.3.1), $h_{I\lambda;uv}=0$. The element u must satisfy: $pr_{uv}\underline{M}_{\bar X_{I\lambda}}$ and $pr_{uw}\underline{M}_{\bar X_{I\lambda}}\neq 0$.

<u>Therorem 4.4.1</u>. (1) <u>The frames</u> $h_{I\lambda;u}\otimes m_{i;I\lambda}$; $u\in I$ correspond by <u>the isomorphism in Figure</u> .

(2) <u>Let</u> $\underline{m}_{I\lambda}$ <u>be the frame of</u> $\underline{M}_{\bar X_{I\lambda}}|U$, <u>which corresponds to</u> $h_{I\lambda;u}\otimes\underline{m}_{u;I\lambda}$, cf. Figure . <u>Then</u> $\Phi_{I\lambda|U}$ <u>is as follows; cf. (2.2.7)</u>.

$$\Phi_{I\lambda|U}:\underline{M}_{\bar X_{I\lambda}}|U\ni\underline{m}_{I\lambda}\to(\underline{M}_{\bar X_i}|I\lambda\oplus\underline{M}_{\bar X_j}|I\lambda)|U\ni h_{I\lambda;i}\underline{m}_{i;I\lambda}\oplus h_{I\lambda;j}\underline{m}_{j;I\lambda}. \tag{4.1.5.2}$$

<u>Theorem 4.4.2</u>. (1) <u>The frames</u> $h_{I\lambda;uv}\otimes\underline{m}_{uv;I\lambda}$; $u,v\in I$, corresponds by the isomorphism in Figure _.

(2) <u>Let</u> $\underline{m}_{I\lambda}$ <u>denote the frame of</u> $\underline{M}_{\bar X_{I\lambda}}$ <u>which corresponds to</u>

$h_{I\lambda;uv}\otimes \underline{m}_{uv;I\lambda}$, cf. Figure . Then $\Phi_{I\lambda|U}$ is as follows:

(4.1.5.3) $\Phi_{I\lambda|U}(\underline{m}_{I\lambda}) = h_{I\lambda;ij}\underline{m}_{ij;I\lambda} \oplus h_{I\lambda;ik}\underline{m}_{ik;j\lambda} \oplus h_{I\lambda;jk}\underline{m}_{jk;I\lambda}$,

We make the following remark on $h_{I\lambda;u}$ and $h_{I\lambda;uv}$.

Remark 4.1.5. The minimality condition on the divisors $\bar{D}^3_{I\lambda;u}$ and $\bar{D}^4_{I\lambda;uv}$ in Theorem 4.3 implies:

(4.1.6) $\bar{D}^3_{I\lambda;i}$ and $\bar{D}^3_{I\lambda;j}$ in (4.1.4.1)(resp. $\bar{D}^4_{I\lambda;ij}$, $\bar{D}^4_{I\lambda;ik}$ and $\bar{D}^4_{I\lambda;jk}$) have no common irreducible components.

The divisor $\bar{D}^3_{I;u}$(resp. $\bar{D}^4_{I;uv}$) may be ϕ. In this case, the ideal of the divisors $=\underline{O}_{\bar{X}_{I\lambda}}$.

4.1.4. Here we give some criterions in order that rank $\underline{M}_{\bar{X}_{I\lambda}}=1$ or 0. In order to shorten the notation, we use the following abbreviation:

(*-1) $\underline{O}_{\bar{X}_{I\lambda}}$, $\underline{M}_{\bar{X}_{I\lambda}}$ and $\underline{L}_{\bar{X}_{I\lambda}} = \underline{O}_{I\lambda}$, $\underline{M}_{I\lambda}$ and $\underline{L}_{I\lambda}$, and $\underline{M}_{\bar{X}_u}$, $\underline{M}_{\bar{X}_{uv}} = \underline{M}_u$, $\underline{M}_{uv}$.

Also we use the symbol $|I\lambda$ for the restriction to $\bar{X}_{I\lambda}$. Thus:

(*-2) $M_{u|\bar{X}_I} = M_{u|I}$, and $\Phi_{u,uv|\bar{X}_I} = \Phi_{u,v|I}$, ...

Also we make the following definition: Let $\underline{F}_{I\lambda}$ and $\underline{G}_{I\lambda}$ be locally free $\underline{O}_{I\lambda}$-modules and $\chi_{I\lambda}$ an $\underline{O}_{I\lambda}$-morphism: $\underline{F}_{I\lambda} \to \underline{G}_{I\lambda}$. For each $p \in \bar{X}_{I\lambda}$, the rank of $\chi_{I\lambda,p}$ is defined in an obvious maner, and is independent of the choice of p. We call this the rank of $\chi_{I\lambda}$.

Now let the element $I\subset\Delta_m$ be as in Theorem 4.1 4.4. We use the similar notation to the one in § 4.1.1 4.1.3, with the abbreviation in (*-1,2) as above.

(i) Fisrt assume that #I =2. Then, by (4.0.4) we see: $\Phi_{u|\bar{X}_{I\lambda}} \neq 0$; $u\in I$. Thus $\mathrm{rank}(\delta\Phi^1)_{I\lambda} = 1$. Since $\mathrm{rank}(C^1(\underline{M}^1))_{I\lambda} = 2$, cf. (4.1.2.0), we have:

Proposition 4.1.1. According as $\mathrm{rank}(\delta\Phi^1)_{I\lambda} = 2$ or 1, we have:

(4.1.7) $\underline{M}_{I\lambda}(\simeq \ker(\delta\Phi^1)_{I\lambda}) \simeq 0$ or is a line bundle over $\bar{X}_{I\lambda}$.

A detailed analysis of the second case will be given in § 4.2.

(ii) Next assume that #I =3. Take a pair (u,v) ⊂I such that $\underline{M}_{uv}(=\underline{M}_{\bar{X}_{uv}}) \neq 0$. Then (4.1.6) and Theorem 4.4.2 imply that

(4.1.8.0) $\Phi_{uv|I\lambda}$ $(=\Phi_{u,uv}\oplus\Phi_{v,uv})_{|I\lambda} \neq 0$;cf.(*-2).

From this and the explicit form of $\delta\Phi^2$,cf.(2.2.11), we easily have:

Proposition 4.1.2. (1) If rank$(C^1(\underline{M}^2))_{I\lambda}$ =0 or 1, then $M_{I\lambda}$($\simeq$ ker$(\delta\Phi^2)_{I\lambda}$) $\simeq 0$. (2) If rank $(C^1(\underline{M}^2))_{I\lambda}$ $=2$, then $M_{I\lambda} \simeq 0$, except the following two cases:

(4.1.8.1) rank of $(C^1(\underline{M}^2))_{I\lambda} = \left\{\begin{matrix}2\\3\end{matrix}\right\}$ and rank of $(\delta\Phi^1)_{I\lambda} = \left\{\begin{matrix}1\\2\end{matrix}\right\}$

(For the two $I\lambda$-part in (4.1.8.1), see (4.1.2.0).)

We make the following remark on (4.1.8.1):

Proposition 4.1.3. If the first condition in (4.1.8.1) holds, then we have the following for a $u \in I$:

(4.1.8.2) $\underline{M}_{uv|I\lambda}$ and $\underline{M}_{uw|I\lambda} \neq 0$, but $\underline{M}_{vw|I\lambda}=0$, with $\{v,w\} = I-\{u\}$, and $\Phi_{v;uv|I\lambda}$ and $\Phi_{w,uw|I\lambda} = 0$.

Proof. The first assertion is clear. Assume that the second fact is false. Then the explicit form of $\delta\Phi^2$, cf.(2.2.11), and (4.1.8.0) imlply that rank$(\delta\Phi^2)_{I\lambda}$ =2, a contradiction. q.e.d.

Proposition 4.1.4. If the second condition in (4.1.8.1) holds, then there are elements $u,v \in I$ such that

(4.1.8.3) $(pr_u\oplus pr_v)(\delta\Phi^2)_{I\lambda}:(C^1(\underline{M}^2))_{I\lambda} \to \underline{M}_{u|I\lambda}\oplus\underline{M}_{v|I\lambda}$ is of rank two, and ker $(\delta\Phi^2)_{I\lambda}$ = ker $(pr_u\oplus pr_v)(\delta\Phi^2)_{I\lambda}$.

(Here pr_u,..denote the projection:$(C^2(\underline{M}^1))_{I\lambda} \to \underline{M}_{u|I\lambda}$,..;cf.(2.1.7).)

Moreover, we have:

(4.1.8.4) $\Phi_{u,uw|I\lambda}$ and $\Phi_{v,vw|I\lambda} \neq 0$.

Proof. The first fact follows from (4.1.8.0) and the explicit form of $\delta\Phi^2$,cf.(2.2.11). Nexxt, to check (4.1.8.4), assume that $\Phi_{u,uw|I\lambda}=0$. By (4.1.8.0), $\Phi_{w,uw|I\lambda}\neq 0$ .By (4.1.8.3) we have:$\Phi_{v,vw|I\lambda}$ and $\Phi_{u,uv|I\lambda} \neq 0$. But this implies that rank $(\delta\Phi^2)_{I\lambda}=3$, which is a contradiction to (4.1.8.1).(See also (2.2.11) for the explicit form of $\delta\Phi^2$. If $\Phi_{v,vw|I\lambda}=0$, we have the similar contradiction.) q.e.d.

Assume that the second fact in (4.1.8.1) holds and that (u,v) are as in Proposition 4.2.3. Remark that (4.1.8.0) implies:

(4.1.8.5) $\Phi_{u,uv|I\lambda}$ and $\Phi_{v,uv|I\lambda} \neq 0$, or only one of them $\neq 0$.

<u>Proposition 4.1.5</u>. <u>According to whether the first or second condition in (4.1.8.5) holds, we have (4.1.3.4) or (4.1.3.3)</u> .

<u>Proof</u>. Assume the second fact in (4.1.8.5):assume that $\Phi_{v,uv|I\lambda}$ =0. Comparing the explicit form of $\delta\Phi^2$, cf.(2.2.11), we have: $pr_{vw} \ker(\delta\Phi^2)_{I\lambda} = 0$, and we see that

(4.1.8.6) $$\ker(\delta\Phi^2)_{I\lambda} = \ker pr_u(\delta\Phi^2)_{I\lambda}, \text{ with the projection } pr_u: (C^2(\underline{M}^1))_{I\lambda} \to \underline{M}_{u|I\lambda}, \text{ and } \ker(\delta\Phi^2)_{I\lambda} \subset (\underline{M}_{uv} \oplus \underline{M}_{uw})_{|I\lambda} .$$

Thus we have (4.3.3.3). If the first condition holds, then we easily see that (4.1.3.4) holds;cf.also § 4.3.   q.e.d.

Lemma 4.1.1. Assume that $\underline{M}_{I\lambda}(\simeq \ker\ (\delta\Phi^2)_{I\lambda})$ is of rank one.

(1) One of the following three cases happens.

(4.1.9.1) rank $(C^1(\underline{M}^2))_{I\lambda} = 2$(and (4.1.3.3) holds).

(4.1.9.2) rank $(C^1(\underline{M}^2))_{I\lambda} = 3$, and (4.1.3.3) holds.

(4.1.9.3) rank $(C^1(\underline{M}^2))_{I\lambda} = 3$, and (4.1.3.4) holds.

(2) The following is equivalent:

(4.1.9.4) (4.1.9.3)(resp.(4.1.9.2)) $\Longleftrightarrow$ the first(resp.second) condition in (4.1.8.5) holds.

Lemma 4.1.2. If $(\underline{M}_{I\lambda},(\delta\Phi^2)_{I\lambda})$ satisfies (4.1.9.1) or (4.1.9.2), then $(\delta\Phi^2)_{I\lambda}$ is degenerate.(Namely, for a permutation (u,v,w) of I=(i,j,k), we have:

(4.1.9.4) $\ker\ (\delta\Phi^2)_{I\lambda} \simeq \ker(pr_u(\delta\Phi^2)_{I\lambda} = \varepsilon\Phi_{u,uv} \oplus \varepsilon'\Phi_{u,uw}) : (\underline{M}_{\overline{X}_{uv}} \oplus \underline{M}_{\overline{X}_{uw}})|I\lambda$
$\rightarrow \underline{M}_{\overline{X}_u}|I\lambda$, where $\varepsilon,\varepsilon' = 1$ or -1 are defined from (2.2.11).

Proof. The both lemmas are clear from Prposition 4.1.1 4.1.4.

§ 4.2. Proof of the theorems in § 4.1. In § 4.2.1 and § 4.2.2 we prove, respectively, the theorems in § 4.1 which are given for codimension two and three. We use freely the notation in the theorems with the abbreviation in (*-1,2) at the beginning of § 4.1.4. Since there is nothing to be proven if $\underline{M}_{I\lambda}(=\underline{M}_{\overline{X}_{I\lambda}} \simeq \ker(\delta\Phi^2)_{I\lambda}) \simeq 0$, we assume that $\underline{M}_{I\lambda}$ is of rank one.

§ 4.2.1. Case of codimension two. Here we assume that # I =2. First we write expliitly the divisors $\overline{D}_{I\lambda;u}$;u $\in$ I of $\overline{X}_{I\lambda}$, which satisfies the conditions in Theorem 4.2 and Theorem 4.3.1:Let $\Phi_u = (\Phi_{u,\alpha})_{\alpha\in\Delta_r} \in \Gamma(\underline{M}_u^* \otimes \underline{L}_u)$;u $\in$ I and $\underline{L}_u = \underline{L}_{\overline{X}_u}$, be as in (4.0.5).

Proposition 4.2.1. (1) For an $\alpha \in \Delta_r$, we have the equivalence:

(4.2.1.1) $\Phi_{i;\alpha|I}$ or $\Phi_{j;\alpha|I} \neq 0 \Longleftrightarrow$ the both $\neq 0$.

(2) For an $\alpha \in \Delta_r$ satisfying (4.2.1.1), define divisors $(\overline{D}^3_{I;i},\overline{D}^3_{I;j})$ of $\overline{X}_I$ as follows:

(4.2.1.2) $\overline{D}^3_{I\lambda;u} = (\Phi_{v,\alpha|I\lambda})_0 -$ (common part of $(\Phi_{I\lambda,\alpha|I\lambda})_0$ and $(\Phi_{j,\alpha|I\lambda})_0$) with (u,v) =(i,j) or (j,i) .

Then $\bar{D}_{I\lambda;u}=\bar{D}^3_{I\lambda;u}$ is independent of α and satisfies the condition in Theorem 4.3.1;cf.(4.1.4.1).

(Here the common part means the maximal divisor satisfying: $(\Phi_{i,\alpha|I\lambda})_0$ and $(\Phi_{j,\alpha|I\lambda})_0$.)

Proof. First, if only one of the L.H.S $\neq 0$, we see easily that the rank of $(\delta\Phi^1)_{I\lambda}=2$,cf.(2.2.11), which is a contradiction. Next the independence of $(\bar{D}^3_{I\lambda;i},\bar{D}^3_{I\lambda;j})$ from the element α follows from that the rank $(\delta\Phi^1)_{I\lambda}=1$. It is clear that these divisors satisfy (4.1.4.1). q.e.d.

Take an α satisfying the condition in (4.2.1.1). We set:

(4.2.1.3) $\quad \bar{D}^3_{I\lambda,\alpha}=(\Phi_{u,\alpha|I\lambda})_0+\bar{D}^3_{I\lambda;u};u\in I.$

Since $\Phi_{u,\alpha}\in\Gamma(\underline{M}^*_u\otimes L_u)$ we have: $\underline{I}_{\bar{D}^3_{I\lambda;\alpha}}\simeq\underline{M}_{u|I\lambda}\otimes\underline{L}^*_{I\lambda}\otimes\underline{I}_{\bar{D}^3_{I\lambda;u}}$. This insures the desired isomorphism in Theorem 4.2:

(4.2.1.4) $\quad \underline{M}_{I\lambda}:=\underline{I}_{\bar{D}^3_{I;\alpha}}\otimes\underline{L}_{I\lambda}(\simeq\underline{M}_{u|I\lambda}\otimes\underline{I}_{\bar{D}^3_{I;u}}),\ u\in I.$

By Proposition 4.2.1 and (4.2.1.4), the remaining task in § 4.2.1 is to prove the following fact;cf.Theorem 4.1 and 4.4.1:

(*) To form an isomorphism $\Phi_{I\lambda}:\underline{M}_{I\lambda}\to\ker(\delta\Phi^1)_{I\lambda}$, and to see that it satisfies the condition in Theorem 4.4.1.

For this let the open set U of $\bar{X}_{I\lambda}$ and the frame $\underline{m}_{u;I\lambda}$ of $\underline{M}_{u|I\lambda}$ over U, $\underline{M}_u=\underline{M}_{\bar{X}_u}$, be as in Theorem 4.4.1. Also take a frame $\underline{l}_{I\lambda}$ of $\underline{L}_{I\lambda|U}$. We write: $\Phi_{u,\alpha|I\lambda}=h_{I\lambda;u,\alpha}m^*_{u;I\lambda}\otimes\underline{l}_{I\lambda}$, with $h_{I\lambda;u,\alpha}\in\Gamma(U,\underline{O}_{I\lambda})$ and the dual $m^*_{u;I\lambda}$ of $m_{u;I\lambda}$, cf.(4.0.5). The α-component of $(\delta\Phi^1)_{I\lambda|U}$ is explicitly as follows, cf.(2.2.11):

$$\begin{aligned}(4.2.2.1)\qquad &\underline{M}_{i|I\lambda}\oplus\underline{M}_{j|I\lambda}\ni\tau_{I\lambda;i}\underline{m}_{i;I\lambda}\oplus\tau_{I\lambda;j}\underline{m}_{j;I\lambda}\to L_{I\lambda|U}\ni\\ &(-\tau_{I\lambda;i}h_{I\lambda;i,\alpha}+\tau_{I\lambda;j}h_{I\lambda;j,\alpha}))\underline{l}_{I\lambda},\ \text{with } \tau_{I\lambda;u}\in\underline{O}_{I\lambda|U};u\in I.\end{aligned}$$

(Note that since $\operatorname{rank}(\delta\Phi^1)_{I\lambda}=1$, we have: $\ker(\delta\Phi^1)_{I\lambda|U}$ = kernel of the morphism in (4.1.2.2). Now, the key fact for (*) is to see the existence of elements $h_{I\lambda;u}\in\Gamma(U,\underline{O}_{I\lambda})$ which generate $\underline{I}_{\bar{D}_{I\lambda;u}}|U$, $u\in I$, and satisfy:

$$h_{I\lambda;i}h_{I\lambda;i,\alpha} = h_{I\lambda;j}h_{I\lambda;j,\alpha}(\quad (h_{I\lambda;i}m_{i;I\lambda})^{\otimes\Phi_{i,\alpha|I\lambda}} =$$

(4.2.2.2)

$$(h_{I\lambda;j}m_{j;I\lambda})^{\otimes\Phi_{j,\alpha|I\lambda}}) .$$

But this follows immediately from Proposition 4.2.1. Also one sees easily that the element $h_{I\lambda;i}\underline{m}_{i;I\lambda}\oplus h_{I\lambda;j}\underline{m}_{j|I\lambda}$ forms a frame of $(\delta\Phi^1)_{I\lambda|U}$. The second key fact for (*) is to see that

(**) the following $\underline{O}_{I\lambda|U}$-isomorphisms are the restriction to U of global $\underline{O}_{I\lambda}$-morphisms in (4.2.1.4) and (*).

(4.2.2.3)
$$\underline{M}_{i|I\lambda}\otimes\underline{I}^{-3}_{\underline{D}_{I\lambda;u}} \ni \underline{m}_{i;I\lambda}\otimes h_{I\lambda;i} \to \underline{M}_{j|I\lambda} \ni \underline{m}_{j;I\lambda}\otimes h_{I\lambda;j};i,j \in I$$

$$\Phi_{I\lambda|U}:\underline{M}_{I\lambda|U} \ni \underline{m}_{I\lambda} \to \ker(\delta\Phi^1)_{I\lambda} \ni h_{I\lambda;i}\underline{m}_{i;I\lambda}\oplus h_{I\lambda;j}\underline{m}_{j;I\lambda}$$

(Here the frame $\underline{m}_{I\lambda}$ corresponds to $\underline{m}_{u;I\lambda};u \in I$, cf.(4.2.1.4).) The check of (**) is as follows: Remark that the two isomorphisms depend on the frames $m_{u;I\lambda}$, $\underline{1}_{I\lambda}$ and $h_{I\lambda;u}$. Replace them by another frames $\underline{m}'_{u;I\lambda}$, $\underline{1}'_{I\lambda}$ and $h'_{I\lambda;u}$. By a simple computation, we see that the all frames in (4.2.2.3) are multiplied by the same unit, say $\varepsilon \in \Gamma(U,\underline{O}_X)$. Thus we have (**) and (*), and we also finish the proof of the theorems in § 4.1 for the case of codimension two.

§ 4.2.2. <u>Proof of the results in § 4.1...2.</u> Here we prove the theorems in § 4.1 which are given for codimension three. Note that, by (4.1.8.1), rank $\underline{M}_{I\lambda}$ =1. By Lemma 4.1.1 the case: rank $\underline{M}_{I\lambda}$ =1 is divided into three cases: (4.1.9.1,2,3). In the first two cases: (4.1.9.1,2), ker $(\delta\Phi^2)_{I,\lambda}$ degenerates, cf.(4.1.9.4). In these cases one can treat $\underline{M}_{I\lambda}$ similarly to § 4.2. We omit the the discussion in this case. In the rest of § 4.3 we assume the third case, (4.1.9.3). By (4.1.8.1) rank $(C^1(\underline{M}^2))_{I\lambda}$ =3 and rank $(\delta\Phi^2)_{I\lambda}$ =2. Take a suitable pair $(u,v) \subset I$, cf.(4.1.8.3). Then we have:

(4.2.3.1)
$$(pr_u\oplus pr_v)(\delta\Phi^2)_{I\lambda} : (C^1(\underline{M}^2)_{I\lambda}) \to (\underline{M}_u\oplus\underline{M}_v)_{|I\lambda} \text{ is of rank two,}$$

$$\text{and } \ker(\delta\Phi^2)_{I\lambda} = \ker(pr_u\oplus pr_v)(\delta\Phi^2)_{I\lambda} .$$

In the above pr denotes the projection: $(C^2(\underline{M}^1))_{I\lambda}\to\underline{M}_{\cdot|I\lambda}$, $\cdot = u,v,$

and $(\Phi_{u,uv} \oplus \Phi_{v,uv})_{I\lambda} = \Phi_{uv|I\lambda}$. By (4.1.8.4) and (4.1.9.4) we have:

(4.2.3.2) $\Phi_{u,uw|I\lambda}$, $\Phi_{v,vw|I\lambda}$ and $\Phi_{u,uv|I\lambda}$, $\Phi_{v,uv|I\lambda} \neq 0$,

FIGURE V

	uv	uw	vw
u	$\varepsilon_1\Phi_{u,uv}$	$\varepsilon_2\Phi_{u,uw}$	0
v	$\varepsilon_3\Phi_{v,uv}$	0	$\varepsilon_4\Phi_{v,vw}$

(Here ε_i = 1 or −1.)

Note that, by (4.2.3.1), an element $\varphi = \oplus_{u,v}\varphi_{u,v} \in (C^1(\underline{M}^2))_{I\lambda} = \oplus_{u,v}\, \underline{M}_{uv|I\lambda}$, cf.(2.2.7), is $\ker(\delta\Phi^2)_{I\lambda}$ if and only if:

(4.2.3.3) $\varphi_{uv} \otimes \Phi_{u,uv|I\lambda} = \varphi_{uw} \otimes \Phi_{u,uw|I\lambda}$ and $\varphi_{uv} \otimes \Phi_{v,uv|I\lambda} = \varphi_{vw} \otimes \Phi_{v,vw|I\lambda}$

As before we first find the divisors $\bar{D}_{I\lambda;uv} = \bar{D}^4_{I\lambda;uv}$, $u,v \in I$, which are required in Theorem 4.2 and 4.4.1. For this define divisors $\bar{D}'_{I\lambda;uv,u}$, $\bar{D}'_{I\lambda;uw,u}$ and $\bar{D}'_{I\lambda;uv,v}$, $\bar{D}'_{I\lambda;vw,w}$ of $\bar{X}_{I\lambda}$ by:

(a) $\bar{D}'_{I\lambda;u\alpha,u} = (\Phi_{u,\beta u|I\lambda})_0$-common part of $((\Phi_{u,uv|I\lambda})_0, (\Phi_{u,uw|I\lambda})_0)$
$\bar{D}'_{I\lambda;v\alpha,v} = (\Phi_{v,\beta v|I\lambda})$-common part of $((\Phi_{v,uv|I\lambda})_0, (\Phi_{v,vw|I\lambda})_0)$

Here $(\alpha,\beta)=(v,w)$ or (w,v), or $=(u,w)$ or (w,u), according as we are concerned with the first or second identity. Moreover, common part should be understood as in (4.2.1.2). Now we define:

$\bar{D}_{I\lambda;uv}$=the minimal divisor satisfying: $\bar{D}'_{I\lambda;uv,u}$, $\bar{D}'_{I\lambda;uv,v}$,

(4.2.4.1) and $\bar{D}_{I\lambda;uw} = \bar{D}'_{I\lambda;uw,u} + (\bar{D}_{I\lambda;uv} - \bar{D}'_{I\lambda;uv,u})$ and $\bar{D}_{I\lambda;vw} = \bar{D}'_{I\lambda;vw,v} + (\bar{D}_{I\lambda;uv} - D'_{I\lambda;uv,v})$.

We see immediately:

(*-1) $(\bar{D}_{I\lambda;uv}, \bar{D}_{I\lambda;uw}, \bar{D}_{I\lambda;vw})$ satisfies (4.1.4.3), Theorem 4.3.2.

Since $\Phi_{u,uv} \in \Gamma(\underline{M}^*_{uv} \otimes \underline{M}_{u|uv})$,....,cf.(4.1.5.0), the identity in (4.1.5.2) insures:

(*-2)
$$(\underline{M}_{uv} \otimes \underline{M}^*_u)|I\lambda \otimes \underline{I}_{\bar{D}_{I;uv}} \simeq (\underline{M}_{uw} \otimes \underline{M}^*_u)|I\lambda \otimes \underline{I}_{\bar{D}_{I;uw}}, \text{ and}$$
$$(\underline{M}_{uv} \otimes \underline{M}^*_v)|I\lambda \otimes \underline{I}_{\bar{D}_{I;uv}} \simeq (\underline{M}_{vw} \otimes \underline{M}^*_v)|I\lambda \otimes \underline{I}_{\bar{D}_{I;vw}}$$

Thus we have:

(4.2.4.2) $\underline{M}_{ij}|I\lambda \otimes \underline{I}_{\bar{D}_{I;ij}}$, $\underline{M}_{ik}|I\lambda \otimes \underline{I}_{\bar{D}_{I;ik}}$ and $\underline{M}_{ik}|I\lambda \otimes \underline{I}_{\bar{D}_{I;jk}}$ are isomorphic.

We take:

(*-3) $\underline{M}_{I\lambda}$ for the $\underline{O}_{I\lambda}$-invertible sheaf which is isomoric to the one in (4.2.4.2);cf.Theorem 4.2.

By (*-1,3) the remaining task for § 4.2.2 is to check the similar fact to (**) in § 4.2.1. For this let the open set U of $\bar{X}_{I\lambda}$ and the frame $\underline{m}_{uv;I\lambda}$ of $\underline{M}_{uv|I\lambda},\ldots$ be as in Theorem 4.4.2. Also take a frame $\underline{m}_{u;I\lambda}$ of $\underline{M}_{u|I\lambda}$. We write:

(a) $\Phi_{u,uv|I\lambda} = h_{u,uv;I\lambda}(\underline{m}^{*}_{uv;I\lambda} \otimes \underline{m}_{u|I\lambda}),\ldots$ with $h_{u,uv;I\lambda} \in \Gamma(U,\underline{O}_{I\lambda}),\ldots$

Then writing $\varphi_{uv} \in \underline{M}_{uv|I\lambda},\ldots.$ as $\tilde{\varphi}_{uv}\underline{m}_{uv;I\lambda}$, with $\tilde{\varphi}_{uv} \in \underline{O}_{\bar{X}_{I\lambda}},\ldots,\ldots,$ (4.2.3.3) is written as follows:

(4.2.5.1) $\tilde{\varphi}_{uv}h_{u,uv;I\lambda} = \tilde{\varphi}_{uw}h_{u,uw;I\lambda}$, and $\tilde{\varphi}_{uv}h_{v,uv;I\lambda} = \tilde{\varphi}_{vw}h_{v,vw;I\lambda}$.

We find elements $h_{uv;I\lambda} \in \Gamma(U,\underline{O}_{\bar{X}});u,v \in I$, which generate $\underline{I}_{\bar{D}_{I\lambda;uv}}$ and satisfy (4.2.5). Also we see easily that

(4.2.5.2) $f_{I\lambda} := h_{ij;I\lambda}\, m_{ij;I\lambda} \oplus h_{ik;I\lambda} m_{ik;I\lambda} \oplus h_{jk;I\lambda}\, m_{jk;I\lambda}$ spans $(\delta\Phi^{2})_{I\lambda|U}$.

Now we take a frame $\underline{m}_{I\lambda}$ of $\underline{M}_{I\lambda}$ over U, which corresponds to $h_{\alpha\beta;I\lambda} \otimes \underline{m}_{\alpha\beta;I\lambda}$, $\alpha,\beta \in I$;cf.(*-2), and define an isomorphism $\Phi_{I\lambda|U}$ as follows:

(4.3.5.3) $\Phi_{I\lambda|I} : \underline{M}_{I\lambda|U} \ni \underline{m}_{I\lambda} \to \ker(\delta\Phi^{2})_{I\lambda|U} \ni \oplus_{\alpha,\beta\in I}\, h_{\alpha\beta;I\lambda}\underline{m}_{\alpha\beta;I\lambda}$.

Similarly to § 4.2.1 one can check that the isomorphisms in (4.3.5.1,3) are the restriction to U of the global ones over $\bar{X}_{I\lambda}$, which are required in Theorem 4.1 and 4.2. Thus we finish the proof of the theorems in § 4.1, which are given for codimension three.

§ 4.3. Here we make two remarks on the results in § 4.1.

4.3.1. First, it is, in general, difficult to detemine the inductive structure for codimension 4. But there is a case where it is determiend easily. This is the case discussed in § 2.6. Namely we assume the exixtence of an invertible $\underline{O}_{\overline{X}}$-module $\underline{M}'_{\overline{X}}$ such that $\underline{M}_{\overline{X}_i}$, cf. (4.0.6), is written as follows:

(4.3.1.1) $\quad \underline{M}_{\overline{X}_i} = \underline{M}'_{\overline{X}_i} \ (:= \underline{M}'_{\overline{X}|\overline{X}_i})$ for each $i \in \Delta_m$.

Also we assume:

(4.3.1.2) $\quad \Phi_{i|\overline{X}_{ij}} = \Phi_{j|\overline{X}_{ij}}$ for each $(i,j) \subset \Delta_m$.

For each $I \in \Delta_m$ we set: $\underline{M}_{\overline{X}_I} = \underline{M}'_{\overline{X}|\overline{X}_I}$ and if $\#I \geqq 2$ we set: $\Phi_I = \oplus_a (id)_I : \underline{M}_{\overline{X}_I} \ni \varphi_I \to (C^1(\underline{M}^{a-1}))_I = \oplus_s \underline{M}_{\overline{X}_{I(s)}|\overline{X}_I} (= \oplus_a \underline{M}_{\overline{X}_I}) \ni \oplus_s \varphi_{I(s)}$ with $a = \#I$ and $\varphi_{I(s)} = \varphi_I$, where $I(s) = I - \{s\text{-th element of } I\}$. By Lemma 2.6.2 we have:

<u>Theorem 4.2</u>. <u>The collection</u> $\{\{\underline{M}'_{\overline{X}_I}; I \subset \Delta_m\}, \{\Phi_I = \oplus_a (id)_I; a = \#I \geqq 2, \Phi_i; i \in \Delta_m\}\}$, <u>where</u> Φ_i <u>is as in (4.0.5), is an inductive structure for</u> $(\underline{L}_{\overline{X}}^{\oplus r}, \underline{E}'_{\overline{X}})$.

Note that the above inductive structure coincides with the one in Theorem 4.1 4.4 for codimension $\leqq 3$.

<u>4.3.2</u>. <u>A remark</u>. Here we give a criterion which insures that

(4.3.2.0). $\quad$ rank $\underline{M}_{\overline{X}_{I\lambda}} = 1$, for the case: $\#I = 3$.

More precisely let $\overline{X}_{I\lambda}$ be as in (2), Theorem 4.2.

<u>Lemma 4.2</u>. <u>Assume the following</u>:

(4.3.2.1) $\quad M_{\overline{X}_K}$ <u>is of rank one</u>($\neq 0$) <u>for each</u> $K \subset I$ <u>with</u> $\#K = 2$, <u>and there is a</u> $u \in I$ <u>such that</u> $\Phi_{u|\overline{X}_{I\lambda}} \neq 0$.

<u>Then the rank of</u> $\underline{M}_{\overline{X}_{I\lambda}} = 1$.

<u>Proof</u>. Take an $\alpha \in \Delta_r$ such that $\Phi_{u,\alpha|\overline{X}_{I\lambda}} \neq 0$. (Here $\Phi_{u,\alpha}$ is the α-th componetnt of Φ_u.) Clearly $\Phi_{u,\alpha|\overline{X}_{uv}} \neq 0$ for each $v \in I - \{u\}$. By By (4.4.1) and Proposition 4.2.1 we have $\Phi_{v,\alpha|\overline{X}_{uv}} \neq 0$. Define the divisor $\overline{D}^3_{uv;v}$ by Theorem 4.3.1. Then we have:

(a) $\quad \overline{D}^3_{uv;v} \subset (\Phi_{u,\alpha|\overline{X}_{uv}})_0$ and $\overline{D}^3_{uv;v} \subset \overline{X}_{I\lambda}$.

We write $\Phi_{uv}:\underline{M}_{uv} \to (\underline{M}_u \oplus \underline{M}_v)|\bar{X}_{uv}$ as $\Phi_{u,uv} \oplus \Phi_{v,uv}$. Then Theorem 4.3.1 and (a) insures:

(b) $\Phi_{v,uv}|\bar{X}_{I\lambda} \neq 0;\ v \in I-\{u\}$

We divide the case into the following;cf.Lemma 4.1.1.

	uv	uw	vw
u	$\Phi_{u,uv}$	$\Phi_{u,uw}$	0
v	$\Phi_{v,uv}(\neq 0)$	0	$\Phi_{v,vw}$
w	0	$\Phi_{w,uw}(\neq 0)$	$\Phi_{w,vw}$

(b) both $\Phi_{v,\alpha}$ and $\Phi_{w,\alpha} \neq 0$, $= 0$ on $\bar{X}_{I\lambda}$, or one of them $=0$ on $\bar{X}_{I\lambda}$. In the second case, Theorem 4.3.1 insures $\Phi_{u,uv}$ and $\Phi_{u,uw}=0$ on $\bar{X}_{I\lambda}$ and (b) implies rank $(\delta\Phi^2)_{I\lambda}=2$, cf. the diagram soon above. Assume the third case. By Proposition 4.2.1, $\Phi_{v,\alpha}$ and $\Phi_{w,\alpha} \neq 0$ on $\bar{X}_{vw}$. Assume that $\Phi_{v,\alpha}=0$ but $\Phi_{w,\alpha}\neq 0$ on $\bar{X}_{I\lambda}$. Then Theorem 4.4 insures: $\Phi_{u,uv}$ and $\Phi_{w,vw}=0$ on $\bar{X}_{I\lambda}$, and rank $(\delta\Phi^2)_{I\lambda}=2$;see the diagram. Finally we consider the first case. The identity:$\Phi_\alpha \otimes \Phi_{\alpha,\alpha\beta} = \Phi_\beta \otimes \Phi_{\beta,\alpha\beta}$ on $\bar{X}_{\alpha\beta}$, $\alpha,\beta \in I$, cf.Theorem 4.1, insures:

(c) $(\Pi_{\alpha\in I}\Phi_\alpha)\otimes(\Phi_{u,uv}\otimes\Phi_{v,vw}\otimes\Phi_{w,uw}-\Phi_{u,uw}\otimes\Phi_{v,uv}\otimes\Phi_{w,vw})=0$ on $\bar{X}_{I\lambda}$.

The assumption implies that the first term $\neq 0$ and the second term $=0$. This insures that rank $(\delta\Phi^2)_{I\lambda}=2$; see the diagram. q.e.d

§ 5. Inductive structure(examples) ...4
(Case of the reflexive sheaves in § 1.3.)

After some preparations in § 5.1 we write down, in § 5.2, the inductive structure of the reflexive sheaves in § 1.3 and prove the local freeness, which are asserted for those sheaves.

§ 5.1. Preparations

The purpose here is to give some criterions for the local freeness of reflexive sheaves, by using the arguments in § 3 and § 4. The data in § 5.1 are as follows:the complex manifold $\bar{X}$ and its divisor $\bar{X}^1=\cup_{i\in\Delta_m}\bar{X}^1_i$;$m_{=}\,2$, are as in § 1.1. The following subvarieties and open sets of $\bar{X}$ are also as in § 1.1:

open parts $X=\bar{X}-\bar{X}^2$, $X^1=\bar{X}^1-\bar{X}^2$ and $X^1_i=\bar{X}^1_i-\bar{X}^2$ of $\bar{X}$, $\bar{X}^1$ and $\bar{X}^1_i$,

(*) with $\bar{X}^2=\cup_{i\neq j}(\bar{X}^1_i\cap\bar{X}^1_j)$, and an open neighborhood $N_1=\ _{i\in\Delta_m}N_{1,i}$

of $X^1=\ _{i\in\Delta_m}X^1_i$ in X and $N_0=\bar{X}-\bar{X}^1$.

From a matrix $H\in GL_r(N_0\cap N_1,\underline{O}_{\bar{X}})$, $r_{=}\,2$, satisfying (1.1.0.5,6) we form a reflexive sheaf $\underline{E}_{\bar{X}}$ of type (C) in the manner in § 1.1. We fix an element $s\in\Gamma(\underline{O}_{\bar{X}}[\bar{X}^1])$ such that $\bar{X}^1=(s)_0$. We imbed $\underline{E}_{\bar{X}}$ into $\underline{O}_{\bar{X}}[\bar{X}^1]^{\oplus r}$, $r=$ rank $\underline{E}_{\bar{X}}$, by means of s, cf.Propositon 4.0. We write the imbedding as $\theta_{\underline{s}}$ and its image $\theta_{\underline{s}}(\underline{E}_{\bar{X}})$ as $\underline{E}'_{\bar{X}}(\subset\underline{O}_{\bar{X}}[\bar{X}^1]^{\oplus r})$. Our arguments will be done for the pair $(\underline{M}^0=\underline{O}_{\bar{X}}[\bar{X}^1]^{\oplus r},\underline{E}'_{\bar{X}})$. As before, for an $I\subset\Delta_m$, we set $\bar{X}_I=\cap_{i\in\Delta_m}\bar{X}^1_i$.

5.1.1. Local freeness...1. Take a point $p\in\bar{X}$. We write $\bar{X}_p$, X_p and X^1_p for the germas of $\bar{X}$, X and X^1 at p. As the criterion for the $\underline{O}_{\bar{X},p}$-freeness of $\underline{E}_{\bar{X},p}$ we take the following (well known) one:

$\underline{E}_{\bar{X},p}$ is $\underline{O}_{\bar{X},p}$-free, if and only if there is an element $\underline{f}=$

(5.1.0) $(f_\alpha)_{\alpha\in\Delta_r}\in\underline{E}^{\oplus r}_{\bar{X},p}$, such that the determinantal divisor of $\underline{f}$:

$(\Lambda^r \underline{f}_{|X_p})_0 (=\Lambda_{\alpha\in\Delta_r} f_{\alpha|X_p})(\subset X_p)=\phi$.

<u>Proposition 5.1</u>. <u>For an element $\underline{f} \in \underline{E}_{\overline{X},p}^{\oplus r}$ we have:</u>

(5.1.1) $\quad \Lambda^r(\theta_{\underline{s}}(\underline{f})_{|X_p})_0 = (r-1)X_p^1 + (\Lambda^r \underline{f}_{|X_p})_0$; <u>see also § 1,[Sa-1]</u>.

<u>(The element in the L.H.S is in $\underline{O}_{\overline{X}}[\overline{X}^1]^{\otimes r}(\simeq\Lambda^r(\underline{O}_{\overline{X}}[\overline{X}^1]^{\oplus r})$.)</u>

<u>Proof</u>. Take an open neighborhood U of p and assume that $\underline{f}$ is defined in U. Let the frames $\underline{e}^i$, i=0,1, of $\underline{E}_{X|N_i}$ be as in § 1.(Note that $N_0=\overline{X}-\overline{X}^1$ and N_1 is an open neighborhood of X^1 in X.) We write: $\underline{f}$ $=\underline{e}^i h_i$ in $(U\cap N_i)$ with $h_i \in M(U\cap N_i, \underline{O}_{\overline{X}})$; i =0,1. We have: $\theta_{\underline{s}}(\underline{f})=\theta_{\underline{s}}(\underline{e}^i)h_i$. Now (5.3.1) follows from this and: $\Lambda^r\theta_{\underline{s}}(\underline{e}^0)=\underline{s}^{\otimes r}$ and $(\Lambda^r\theta_{\underline{s}}(\underline{e}^1))_0 = (r-1)X^1$, cf. Proposition 4.0. q.e.d.

From (5.1.0,1) we have:

<u>Lemma 5.1</u>. <u>$\underline{E}_{\overline{X},p}$ is $\underline{O}_{\overline{X},p}$-free if and only if there is an $\underline{f}' \in$ $\underline{E}'_{\overline{X},p}(\subset \underline{L}_{\overline{X},p}^{\oplus r})$ such that</u> $(\Lambda^r \underline{f}'_{|X_p})_0 = (r-1)X_p^1$.

Our criterion for the local freeness of $\underline{E}_{\overline{X}}$ is based on this lemma and will be given by doing the following:

(*) Find a generator of $\underline{E}'_{\overline{X},p}$ and, by forming an r-vector, $\underline{f}$, whose components are in the generator, we check the condition in Lemma 5.1

In finding the generator we use the arguments in § 3. In the remainder of § 5.1 we assume (2.0.0) for the divisor $\overline{X}^1$. As in § 3.2 we take an $I \subset \Delta_m$ and an open set U of $\overline{X}$, which satisfy (3.2.0.1,2). The coordinates x_u;$u\in I$ of U are as in § 3.2. Thus $\overline{X}_J \cap U = \phi$ if and only if $J\subset I$ and $U\cap\overline{X}_J = \cap_{j\in J}(x_j)_0$. We assume that there is an inductive structure $\{(\underline{M}_{\overline{X}_J},\Phi_J);J\subset I\}$ of $(\underline{M}^0=(\underline{L}_{\overline{X}}^{\otimes m})^{\oplus r}_{|U}, \underline{E}'_{\overline{X}|U})$, cf. Definition 2.3, such that rank $\underline{M}_{\overline{X}_J} = 1$.(By Theorem 4.1 this assumption is harmless for applications to the reflexive shaves in § $\underline{M}^0_{|U}$ and $\underline{M}_{\overline{X}_J}$;$J\subset I$, have frames, $\underline{m}^0$ and $\underline{m}_J$, over U.

By (3.1.5) we have a parametrization map P_J: $\Gamma(U\cap\overline{X}_J, \underline{I}_{\partial X_J}\underline{M}_{\overline{X}_J})$ $\to\Gamma(U,\underline{E}'_{\overline{X}})$. By Theorem 3.2 $\underline{E}'_{\overline{X},p}$, $p\in\overline{X}_I\cap U$, is generated by the following:

(5.1.2) $P_J(x_{I-J}\underline{m}_{J,\alpha})$, with $x_{I-J}=\Pi_{i\in I-J}x_i$; $J\subset I$ and $\underline{m}_{J,\alpha}\in\underline{m}_J$.
Here we admit the case: $J=\phi$. In this case $\underline{m}_J=\underline{m}^0$. Thus (*) is sharpened as follows:

(**) To form r-vectors from the elements in (5.1.2), and to check the condition in Lema 5.1 for the r-vectors.

<u>§ 5.1.2. Local freeness...2.</u> First we give the following:

<u>Lemma 5.2. Assume that there is an element</u> $f_I\in\underline{M}_{\overline{X}_I,p}$ <u>such that the value</u> $\omega_p P_{\underline{I}}(f_I)(\in C^r)$ of $P_I(f_I)$ <u>at</u> $p\neq 0$, <u>with</u> $\omega_p:\underline{O}_{\overline{X}}[\overline{X}^1]^{\oplus r}\to\underline{O}_{\overline{X}}[\overline{X}^1]^{\oplus r}\otimes(\underline{O}_{\overline{X},p}/\underline{I}_p)\simeq C^{\oplus r}$, $\underline{I}_p$ <u>being the ideal of</u> p. Then $\underline{E}'_{\overline{X},p}$ is $\underline{O}_{\overline{X},p}$-<u>-free</u>.

<u>Proof</u>. Take an $\alpha\in\Delta_r$ so that the α-th component of $\omega_p P_I(f_I)\neq 0$. Let e_β be the β-th component of the frame $\underline{e}^0$ of $\underline{E}_X$, cf. § 4.0. By Proposition 4.0 $\theta_{\underline{s}}(e_\beta)={}^t(0,\dots,0,\overset{\beta}{s},0,\dots,0)$. Clearly we have:

(a) $(\theta_{\underline{s}}(e_1)\wedge\ \wedge\theta_{\underline{s}}(e_{\alpha-1})\wedge P_I(f_I)\wedge\theta_{\underline{s}}(e_{\alpha+1})\wedge\ \wedge\theta_{\underline{s}}(e_r)_{|X_p})_0=(r-1)X_p^1$.

The lemma follows from Lemma 5.1. Note that

(5.1.3) $\theta_{\underline{s}}(e_\beta)$; $\beta\in\Delta_r-\{\alpha\}$, and $P_I(f_I)$ form a frame of $\underline{E}'_{\overline{X}}$ in an neighborhood of p. q.e.d.

<u>5.1.3. Local freeness..2.</u> Here we discuss the local freeness of $\underline{E}'_{\overline{X}}$ for codimension two and three, by using the arguments of § 3.2: For each $J\subset I$: $\#J_= 3$ we write the papprametrization map P_J as follows, cf. Theorem 3.2.

(5.1.4.1) $P_J(x_{I-J}\underline{m}_J)=\underline{m}^0(x_{I-J}H_{0,J})$ with $H_{0,J}\in\Gamma^r(U,\underline{O}_{\overline{X}})$, cf. (3.3.0.1).
(Note that $H_{0,J}$ is $r\times 1$-matrix since rank $\underline{M}_{\overline{X}_J}=1$.) Similarly to § 3.3 we write the $\underline{O}_{\overline{X}_J}$-morphism $\Phi_J=\oplus_K\Phi_{K,J}:M_{\overline{X}_J}\to\oplus_K M_{\overline{X}_K|\overline{X}_J}$, with K: $K\subset J$ and $\#K=\#J-1$, cf. (2.2.7), as follows:

(5.1.4.2) $\Phi_J(\underline{m}_J)=\oplus_K\underline{m}_{K|\overline{X}_J}h_{K,J}$ with $h_{K,J}\in M(U\cap\overline{X}_J,\underline{O}_{\overline{X}_J})$.

Note that if $\#J_= 2$ then $h_{K,J}\in\Gamma(U\in\overline{X}_J,\underline{O}_{\overline{X}_J})$. When $\#J=1$ we write:

(5.1.4.3) $\Phi_J(\underline{m}_J)=\underline{m}^0_{|\overline{X}_J}h_{0,J}$ with $h_{0,J}\in\Gamma^r(U\cap\overline{X}_J,\underline{O}_{\overline{X}_J})$.

See Lemma 3.2.1 3.2.3 for the expression of $H_{0,J}$ by $h_{K,J}$;$K\subset J$. For purpose of later explicit computations we give the local freeness of $\underline{E}'_{\overline{X}}$ in terms of the matrices $H_{0,J}$ and $h_{K,J}$. How one can form a frame of $\underline{E}'_{\overline{X}}$ is also mentioned.

5.1.3.1. Assume that #I =2, and we write I={i,j}. Take a point $p \in U\cap\overline{X}_I$. For a vector $H_{0,J}\in\Gamma^r(U,\underline{O}_{\overline{X}})$;$J \subset I$, cf.(5.1.4), we write:

(5.1.5) $H_{0,J}(p)$ =the value of $H_{0,J}$ at $p(\in C^r)$.

Lemma 5.3.1. $\underline{E}'_{\overline{X},p}$ is $\underline{O}_{\overline{X},p}$-free if and only if one of the following holds:

(1) $H_{0,I}(p)(\in C^r)\neq 0$, (2) rank $[H_{0,i}(p),H_{0,j}(p)](\in M_{r,2}(C))=2$

(3) a 2×2-submatrix Δ_2 of $[H_{0,I},H_{0,u}]$, u=i or j, satisfies:

(5.1.6.1) $\det \Delta_2=x_u\varepsilon$ with a unit $\varepsilon\in\underline{O}_{\overline{X},p}$ or (4) a 3×3-submatrix Δ_3 of $[H_{0,I},H_{0,i},H_{0,j}]$ satisfies: $\det \Delta_3=x_ix_j\varepsilon$ with a unit $\varepsilon\in\underline{O}_{\overline{X},p}$.

Proof. By (5.1.2) $\underline{E}_{\overline{X},p}$ is generated by $P_I(m_I)$,$P_u(x_{I-\{u\}}m_u)=\underline{m}^0(x_{I-\{u\}}H_{0,u})$;$u\in I$, and $\underline{\theta}_s(e_\alpha)$;$\alpha\in\Delta_r$. Remark that $\theta_s(e_\alpha)\simeq O(\overline{X}^1)$, and the freeness in the lemma holds if and only if the exterior product of one of the following r-vectors satisfies the conditionsee in Lemma 5.1:(We write $\theta_s(\underline{e}^0)_K$,$K\cup\Delta_r$, for $\{\theta_s(e_\alpha)\}_{\alpha\in\Delta_r-K}$ in (5.1.6.1).)

(5.2.6.2) (1) $\{P_I(\underline{m}_I),\theta_s(\underline{e}^0)_{\{\alpha\}}\}$, with an $\alpha\cup\Delta_r$, (2)$\{P_i(x_j\underline{m}_i)$, $P_j(x_i\underline{m}_j),\theta_s(\underline{e}^0)_{\{\alpha,\beta\}}\}$, with an $\{\alpha,\beta\}\subset\Delta_r$, (3) $\{P_I(\underline{m}_I),P_u(x_v\underline{m}_u),\theta_s(\underline{e}^0)_{\{\alpha,\beta\}}\}$, with an $\{\alpha,\beta\}\subset\Delta_r$ and (u,v) =(i,j) or (j,i),(4)$\{P_I(\underline{m}_I),P_i(x_j\underline{m}_i),P_j(x_i\underline{m}_j)$, $\theta_s(\underline{e}^0)_{\{\alpha,\beta,\gamma\}}\}$, with an $\{\alpha,\beta,\gamma\}\subset\Delta_r$.

One checks readily that the four conditions in (5.1.6.1) corresponds to the ones just above. q.e.d.

We write down explicitly the condition (5.1.6.1). For this we expand $h_{0,u}$;$u\in I$, as in (3.3.3):

(5.1.7) $h_{0,u}=\pi_I^*(h_{0,u;I})+x_vh_{0,u;u}$, with $\{v\}=I-\{u\}$ and $h_{0,u;I}\in\Gamma^r(U\cap\overline{X}_I,\underline{O}_{\overline{X}_I})$, $h_{0u;u}\in\Gamma^r(U\cap\overline{X}_u,\underline{O}_{\overline{X}_u})$, and π_K;$K\subset I$, is the analytic projection:$U \to U\cap\bar{X}_K$, cf,(3.2.1.2).

Lemma 5.3.2. The conditions (1) (4) in (5.1.7) are, respectively, equivalent to the following:

$$(5.1.8)\quad \begin{array}{l} (h_{0,u;I}h_{u;I})(p) \neq 0 \text{ for each } u \in I, \\ [h_{0,i;I}(p), h_{0,j;I}(p)] \text{ is of rank two}, \\ [h_{0,v;v|\bar{X}_I} h_{v,I}(p), h_{0,u;I}(p)] \text{ is of rank two}; \{v\}=I-\{u\}, \\ [H_{0I;I}(p), h_{0i;i}(p), h_{0j;j}(p)] \text{ is of rank three}. \end{array}$$

Proof. The first is obvious once we remark $H_{0,I}=h_{0,u;I}h_{u,I}$, cf. Lemma 3.1, while the second is clear. We check the last two facts as follows:Take a 2×2-submatrix Δ_2 of the matrix in (3), Lemma 5.3.1. Using the expansion of $H_{0,I}$, cf. (3.3.2.2), and $h_{0,u}$,cf.(5.1.7), we easily have:

$$(a)\quad \begin{array}{l} \det \Delta_2 = x_u\{\text{the determinant of the corresponding } 2\times2\text{-submatrix} \\ \text{of } [\pi_v^*(h_{0,v;v})\pi_I^*(h_{v,I}), \pi_u^*(h_{0,u}) = \pi_I^*(h_{0,u;I}) + x_v\pi_u^*(h_{0,u;u})]\}. \end{array}$$

Thus we have the third fact. We make a similar observation to (4), Lemma 5.3.1. Then for a 3×3-submatrix Δ_3 of $[H_{0,I}, H_{0,i}, H_{0,j}]$ we have:

$$(b)\quad \begin{array}{l} \det \Delta_3 = \text{the deteminant of the corresponding submatrix of} \\ x_i[\pi_j^*(H_{0I;j}), \pi_i^*(h_{0,i}), \pi_j^*(h_{0,j})]; H_{0I;j} = \pi_j^*(h_{0j;j})\pi_I^*(h_{j,I}). \end{array}$$

(See (3.3.2.2) .) Note that if rank $[h_{0i;I}, h_{0,j;I}]=2$, then $h_{i,I}=0$, cf.Proposition 4.2.1, and $H_{0I;i}=0$. Thus (4) in Lemma 5.3.1 does not hold. We assume that the rank 2. Using the expansion of $h_{0,u}; u\in I$, cf.(5.1.7), we can replace the matrix in (b) by:

$$(c)\quad \begin{array}{l} x_i x_j[\pi_j^*(h_{0j;j})\pi_I^*(h_{j,I}), \pi_i^*(h_{0,i;i}), \pi_I^*(h_{0,j;I})] = \\ x_i x_j[\pi_j^*(h_{0,j;j}), \pi_i^*(h_{0,i,i}), \pi_I^*(H_{0,I})], \end{array}$$

and we have the fourt fact in (5.1.8).

5.1.3.2. Here we discuss the local freeness of $E'_{\bar{X}}$ for codimension three. We assume that # I =3 and write I ={i,j,k}. Theoretically it is not difficult to give a corresponding fact to Lema 5.3.1 in the present case. But it requires many lines. We give some sufficient conditions for the freeness. The conditions are also the necessary ones if rank $\underline{E}'_{\bar{X}} = 3$. In the lemma soon below the matrices $H_{0,I}, \ldots$ are as in (5.1.3,4). We use ε for a unit $\in \underline{O}_{\bar{X},P}$, and Δ_2(resp. Δ_3) for a 2×2(resp. 3×3)-submatrix of a matrix in question.

Lemma 5.4.1. (1) Assume that one of the following holds.

(5.1.10.1) $H_{0,I}(p) \neq 0$,

(5.1.10.2) (1) $\mathrm{rank}[H_{0,vw}(p), H_{0,u}(p)] = 2$ with $u \in I$ and $\{v,w\} = I - \{u\}$, (2) For a Δ_2 of $[H_{0,I}, H_{0,u}]$, $u \in I$, $\det \Delta_2 = x_u \varepsilon$, (3) For a Δ_2 of $[H_{0,uv}, H_{0,uw}]$, $u \in I$ and $\{v,w\} = I - \{u\}$, $\det \Delta_2 = x_u \varepsilon$, (4) For a Δ_2 of $[H_{0,I}, H_{0,uv}]$, $u,v \in I$, $\det \Delta_2 = x_u x_v \varepsilon$,

(5.1.10.3) (1) $[H_{0,i}(p), H_{0,j}(p), H_{0,k}(p)]$ is of rank three, (2) for a Δ_3 of $[H_{0,u}, H_{0,v}, H_{0,\alpha w}]$, $u,v,w \in I$ and $\alpha = u$ or v, $\det \Delta_3 = x_u x_v \varepsilon$, (3) for a Δ_3 of $[H_{0,u}, H_{0,v}, H_{0,I}]$, $\det \Delta_3 = x_u x_v \varepsilon$, (4) for a Δ_3 of $[H_{0,u}, H_{0,\alpha\beta}, H_{0,\alpha\gamma}]$, $u,\alpha,\beta,\gamma \in I$, $\det \Delta_3 = x_I^2 / x_v x_w x_\alpha x_\beta$, with $\{v,w\} = I - \{u\}$, (5) for a Δ_3 of $[H_{0,u}, H_{0,\alpha\beta}, H_{0,I}]$, $u,\alpha,\beta \in I$, $\det \Delta_3 = x_I^2 / x_v x_w x_\gamma$ with $\{v,w\} = I - \{u\}$ and $\{\gamma\} = I - \{\alpha,\beta\}$, (6) for a Δ_3 of $[H_{0,ij}, H_{0,ik}, H_{0,jk}]$ $\det \Delta_3 = x_I \varepsilon$, (7) for a Δ_3 of $[H_{0,uv}, H_{0,uw}, H_{0,I}]$, $\det \Delta_3 = x_I^2 / x_v x_w$, with $u,v,w \in I$.

(2) If $\mathrm{rank}\, E'_{\bar{X}} = 2$ (resp. $=3$) and $E_{\bar{X},p}$ is $O_{\bar{X},p}$-free, then (5.1.10.1,2) (resp. (5.1.10.1,2,3)) holds.

Proof. By Theorem 3.2 $E'_{\bar{X},p}$ is generated by the following:

(a) $\tilde{m}^0 H_{0,I}$, $\tilde{m}^0(x_w H_{0,uv})$, $u,v \in I$ and $\{w\} = I - \{u,v\}$, $\tilde{m}^0(x_v x_w H_{0,u})$, $u \in I$ and $\{v,w\} = I - \{u\}$, and $\underline{s}_\alpha ; \alpha \in \Delta_r$, with ${}^t s_\alpha = (0,..,0,\overset{\alpha}{s},0,...,0)$

In the above we list all the cases where an r-vector $\underline{f}$, formed from elements in (a), satisfies: $(\Lambda^r \underline{f}_{|X_p})_0 = (r-1) X_p^1$, cf. Lemma 5.1, and $\underline{f}$ contains k(=1,2,or 3)-elements which are different from $\underline{s}_\alpha$. For example, in (5.1.10.1), the corresponding r-vector $\underline{f}$ is:

(5.1.11.1) $\{P_I(\underline{m}_I), (\underline{s}_1, ..., \overset{\alpha}{\underline{s}_\alpha}, ..., \underline{s}_r)\}$ with an $\alpha \in \Delta_r$.

In (5.1.10.2) the r-vectors $\underline{f}$ are respectively as follows:

(5.1.11.2)

$\{P_u(x_v x_w \underline{m}_u), P_{vw}(x_u \underline{m}_{vw})\}$,

$\{P_u(x_v x_w \underline{m}_u), P_I(m_I)\}$ and $(\underline{s}_1, ..., \underline{s}_\alpha, .. \underline{s}_\beta, ..., s_r)$

$\{P_{uv}(x_w \underline{m}_{uv}), P_{uw}(x_v \underline{m}_{uw})$ with $\alpha, \beta \in \Delta_r$

$\{P_{uv}(x_w \underline{m}_{uv}), P_I(\underline{m}_I)\}$

We omit the corresponding fact for (5.1.11.3). The second fact (2) of the lemma is clear since (5.1.10.1,2,3) exaust, respetively, the cases where the above fact holds in the case:k=1,2 and 3. q.e.d.

It is also possible to rewrite Lemma 5.4.1 in terms of $h_{K,J}$;$K \subset J \subset I$. But this is very troublesome . We rewrite only the condition (2) in (5.1.10.2) which is the first non trivial case(in rewriting) and is used in the proof of Theorem 1.2.2;cf. § 5.3. Take $u \in I$ and we set:$\{v,w\} = I-\{u\}$. For an α = u or v we write;cf.(3.3.3.1):

(5.1.12.1) $h_{0,\alpha|\bar{X}_{vw}} = \pi^*_I(h_{0,\alpha;I}) + x_u h_{0,\alpha;uv}$ with $h_{0,\alpha;I} \in \Gamma(U \in \bar{X}_I, \underline{O}_{\bar{X}_I})$ and
$h_{0,\alpha;I} \in \Gamma(U \in \bar{X}_{uv}, \underline{O}_{\bar{X}_{uv}})$ with the open set U,cf.§ 5.1.1.

<u>Lemma 5.4.2</u>. <u>The r-vector</u> $\{P_I(\underline{m}_I), P_u(x_v x_w \underline{m}_u), (\underline{s}_1, \ldots, \underline{s}_\alpha, \ldots, \underline{s}_\beta, \ldots, \underline{s}_r)$ <u>forms a frame at p if and only if</u>:

(5.1.12) $\operatorname{rank}[h_{0,u}(p), (h_{0,\alpha;vw} h_{\alpha,vw;I} h_{vw,I})(p)] = 2$; $\alpha = v,w$.

(<u>The matrix itself is not independent of α= u,v. But its rank is independent of α;cf.(3.3.3.7).</u>)

<u>Proof</u>. By Lemma 5.4.1, it suffices to check that (5.2.12) is equivalent to (2) in (5.1.10.2). We write:$H_{0,I} = H_{0,I|\bar{X}_u} + x_u H_{0,I;u}$ with $H_{0,I;u} \in \Gamma(U, \underline{O}_{\bar{X}}^{\oplus r})$ with the open set U of $\bar{X}$, cf.§ 5.1.1. By (3.3.3.6) $H_{0,J|\bar{X}_I} = h_{0,u} H_{u,I}$ with $H_{u,I}$ as in (3.3.3.2). Thus the matrix in (2) of (5.1.10.2) is replaced by: $x_u[H_{0,I;u}, h_{0,u}]$;the condition (2) in (5.1.10.2) is written as:

(a) $\operatorname{rank}[H_{0,I;u}(p), h_{0,u}(p)] = 2.$

To determine $H_{0,I;u}(p)$ we restrict $H_{0,I}$ to $\bar{X}_{vw}$. By (3.3.3.8) we see:

(b) $$H_{0,I;u|\bar{X}_{vw}} = H_{0,I|\bar{X}_{vw}} - H_{0,I;I} = \{h_{0,\alpha:I} h_{\alpha,vw;vw} + h_{0,\alpha;vw} h_{\alpha,vw;I} + x_u h_{0,\alpha;vw} h_{\alpha,vw;vw}\} \times h_{vw,I};\ \alpha = v,w.$$

Since $h_{0,u}(p) = h_{0,u;I}(p)$ we get (5.1.12) from (a) and (b). q.e.d.

§ 5.2. Reflexive sheaves in § 1.3

Here we discuss the inductive structure of the reflexive sheaf $\underline{E}_{\overline{X}}$, which appears in one of Theorem 1.2.1 $\sim$ 1.3.2. We use freely the notation in § 1.3. Letting $\underline{L}_{\overline{X}}$ be the line bundle in § 1.3, we make a remark as follows:

(*) <u>It is not $\underline{L}_{\overline{X}}$ but $\underline{L}_{\overline{X}}^{\otimes m}$ that corresponds to the line bundle $\underline{L}_{\overline{X}}$ in § 4</u>.

We apply the results in § 5.1 to the pair $(\underline{M}^0=(\underline{L}_{\overline{X}}^{\otimes m})^{\oplus r},\underline{E}'_{\overline{X}})$, $r=\mathrm{rank}\ \underline{E}_{\overline{X}}$, with the imbedding $\underline{E}'_{\overline{X}}=\theta_s(\underline{E}_{\overline{X}})$, cf. Proposition 4.0.

<u>5.2.1. Inductive structure ...1</u>. Here we determine the inductive structure of $\underline{E}'_{\overline{X}}(\simeq\underline{E}_{\overline{X}})$ in Theorem 1.2.1 $\sim$ 1.3.2. for codimension one: For the determination we should know the explicit form of the meromorphic functions: $(f_{\alpha|\overline{X}_i^1})_{\alpha\in\Delta_{r-1}}$ over $\overline{X}_i^1$, with $r=\mathrm{rank}\ \underline{E}'_{\overline{X}}$, cf. (4.0.0.0). The functions are as follows:

(5.2.1.1)

- $t_i/\hat{s}_i$; Theorem 1.2.1,
- $g_{i,1}/g_{i,2}$ with $g_{i,1}=\Pi_{\alpha\in\Delta_n}s_{i\pm n\pm\alpha}$ and $g_{i,2}=\Pi_{\alpha\in\Delta_n}s_{i\pm\alpha}$; Theorem 1.2.2. (Here $m=4n+1$.)
- $(u_\alpha/\hat{s}_i)_{|\overline{X}_i^1}$, $\alpha\in\Delta_{r-1}$; Theorem 1.3.1,
- $(g_{\alpha,i}/s_m^{\otimes(m-1)})_{|\overline{X}_i^1}$, $i\neq m$, and $=(g_{\alpha,i}/\hat{s}_m)_{|\overline{X}_i^1}$; $i=m$, where $g_{\alpha,i}$ $=\alpha$-component of $\underline{g}_i$, cf. (1.3.7.1); Theorem 1.3.2 (Here $r=m$.)

For the above see (1.3.1,4,6,7). In Theorem 1.2.1 and 1.2.2, $r=2$ and we have only one meromorphic function. Now, by (1.3.2.1), (1.3.6.1) and (2.0.0), the pole divisors $\overline{D}_i^2$ of the above functions are the loci of the denominators of them. Thus we have:

$$(5.2.1.1)\quad I_{\overline{D}_i^2}\ \begin{cases} =\underline{L}_{\overline{X}_i}^{\otimes(1-m)}, & \text{for Theorem 1.2.1 and 1.3.1, 1.3.2,} \\ =\underline{L}_{\overline{X}_i}^{-2n}, & \text{for Theorem 1.2.2. (Here } \underline{L}_{\overline{X}_i}=\underline{L}_{\overline{X}|\overline{X}_i}.) \end{cases}$$

Thus the $\mathcal{O}_{\overline{X}_i}$-module $M_{\overline{X}_i}=\underline{L}_{\overline{X}_i}^{\otimes m}\otimes I_{\overline{D}_i^2}$, cf. (4.0.a), is as follows;

(5.2.1.2) $\overline{M}_{\overline{X}_i} = \begin{cases} \underline{L}_{\overline{X}_i} \text{ for the first three theorems,} \\ \underline{L}_{\overline{X}_i}^{\otimes(2n+1)}; \text{Theorem 1.2.2} \end{cases}$

The $\underline{O}_{\overline{X}_i}$-morphism $\Phi_i \in \Gamma(\underline{M}^*_{\overline{X}_i} \otimes \underline{L}_{\overline{X}_i}^{\otimes m})$, cf.(4.0.5) is as follows:

$${}^t\Phi_i = (t_i, -s_i^\wedge);\ \text{Theorem 1.2.1,}$$

$${}^t\Phi_i = (g_{i,1}, -g_{i,2});\text{Theorem 1.2.2,}$$

(5.2.1.3) $\quad {}^t\Phi_i = (u_1, \dots, u_{r-1}, -s_i^\wedge)|\overline{X}_i^1\ ; i \in \Delta_m;$ <u>Theorem 1.3.1</u>,

$${}^t\Phi_i = (s_1^{\otimes(m-1)}, \dots, s_i^\wedge, \dots, s_{m-1}^{\otimes(m-1)}, -s_m^{\otimes(m-1)})|\overline{X}_i^1\ ,\ i \neq m,$$

$${}^t\Phi_m = (s_1^{\otimes(m-1)}, \dots, s_{m-1}^{\otimes(m-1)}, -s_m^\wedge)|\overline{X}_m^1, ;\underline{\text{Theorem 1.3.2}}.$$

By Theorem 4.0 we have the following for Theorem 1.2.1   1.3.2:

<u>Lemma 5.5.</u> <u>The collection</u> $\{(\underline{M}_{\overline{X}_i}), (\Phi_i)\}_{i \in \Delta_m}$, <u>where</u> $\underline{M}_{\overline{X}_i}$ <u>and</u> Φ_i <u>are as just above, is an inductive structure for</u> $(\underline{M}^0 = (\underline{L}_{\overline{X}}^{\otimes m})^{\oplus r}, \underline{E}'_{\overline{X}})$ for <u>codimension one, cf.Definition 2.3</u>.

In th rest of § 5.2 we determine the inductive structure for codimension $\geqq 2$.

<u>5.2.2.</u> <u>Inductive structure...2</u>. (i) First we determine the inductive structure of $\underline{E}'_{\overline{X}} (\simeq \underline{E}_{\overline{X}})$ in Theorem 1.3.1 and 1.3.2. In this case we have: $\Phi_{i|\overline{X}_{ij}} = \Phi_{j|\overline{X}_{ij}}$. Actually we easily see:

$${}^t\Phi_{u|\overline{X}_{ij}} = (t_1, \dots, t_{r-1}, 0)|\overline{X}_{ij}\ ; u = i, j; \text{Theorem 1.3.1,}$$

(5.2.2.1)

$$\Phi_{u,\alpha|\overline{X}_{ij}} = s_{\alpha|\overline{X}_{ij}}^{\otimes(m-1)}, \alpha \neq i, j, \text{ and } = 0, \alpha = i, j\ (u = i, j); \text{Theorem 1.3.2}$$

Thus we can apply Theorem 4.5 to the present case:For each $I \subset \Delta_m : \#I \geqq 2$ we define: $\underline{M}_{\overline{X}_I} := \underline{L}_{\overline{X}_I}$ and $\Phi_I = \oplus_{\#I}(\text{identity}) : \underline{M}_{\overline{X}_I} \to (C^1(\underline{M}^{\#I-1}))_I = \oplus_{\#I} \underline{L}_{\overline{X}_I}$, cf.(2.2.7). By Theorem 4.5 we have:

<u>Lemma 5.6.1.</u> <u>The collection</u> $(\{\underline{L}_{\overline{X}_I}\}, \{\oplus_{\#I}(\text{id})\}); \#I \geqq 2$, <u>is an inductive structure for</u> $((\underline{L}_X^{\otimes m})^{\oplus r}, \underline{E}'_{\overline{X}})$ <u>for the codimension</u> $\geqq$ 2(in the <u>both cases:Theorem 1.3.1 and 1.3.2;cf. Definition 2.3.</u>

(ii) Next we determine the inductive structure of $\underline{E}'_{\overline{X}}$ in Theorem 1.2.1 for codimension two and three. For this take an $I \subset \Delta_m$

with $\#I=2$ or 3, and define an $\underline{O}_{\bar{X}_I}$-module $\underline{M}_{\bar{X}_I}$ and an $\underline{O}_{\bar{X}_I}$-morphism Φ_I $\in\Gamma(\underline{M}^*_{\bar{X}_I}\otimes C^1(\underline{M}^{\#I-1})_I)$ as follows:

$$\underline{M}_{\bar{X}_I}=\underline{L}_{\bar{X}_I}^{\otimes\{m+\#I(1-m)\}};\#I=2,3 \text{ (Note that } \underline{M}_{\bar{X}_i} \text{ is also defined by this formula.)}$$

(5.2.2.2)
$$\Phi_I=(t_v\oplus t_u)|\bar{X}_I:\underline{M}_{\bar{X}_I}\to(\underline{M}_{\bar{X}_u}\oplus\underline{M}_{\bar{X}_v})|\bar{X}_I;\ \#I=2,$$
$$\Phi_I=(t_w\oplus t_v\oplus t_u)|\bar{X}_I:\underline{M}_{\bar{X}_I}\to(\underline{M}_{\bar{X}_{uv}}\oplus\underline{M}_{\bar{X}_{uw}}\oplus\underline{M}_{\bar{X}_{vw}})|\bar{X}_I;\ \#I=3$$

(Here according as $\#I=2$ or 3, we write $I=\{u,v\}$ or $I=\{u,v,w\}$.)

<u>Lemma 5.6.2</u>. <u>The collection</u> $\{\underline{M}_{\bar{X}_I},\Phi_I\}$, $\#I=2,3$, <u>is an inductive structure</u> of $((\underline{L}_{\bar{X}}^{\otimes m})^{\oplus 2},\underline{E}'_{\bar{X}})$ <u>for codimension two and three, where</u> $\underline{E}'_{\bar{X}}\simeq\underline{E}_{\bar{X}}$ <u>is as in Theorem 1.2.1</u>.

<u>Remark 5.1.</u> For each $I:\#I\geq 4$ we set:

(5.2.2.3) $\underline{M}_{\bar{X}_I}=\underline{L}_{\bar{X}_I}^{\otimes\{m+a(1-m)\}}$; $a=\#I$, and $\Phi_I=\oplus_{i\in I}t_i|\bar{X}_I\in\Gamma((\underline{L}_{\bar{X}}^{\otimes(m-1)})^{\oplus a})$.

It seems that the above data give an inductive structure of $(\underline{L}_{\bar{X}}^{\oplus m},\underline{E}'_{\bar{X}})$ for any codimension.

<u>Proof of Lemma 5.6.2</u>. First take an $I=\{u,v\}\subset\Delta_m$. Then an element $\tau_{u,I}\oplus\tau_{v,I}\in(\underline{M}_{\bar{X}_u}\oplus\underline{M}_{\bar{X}_v})|\bar{X}_{uv}$ is in in $\ker(\delta\Phi^1)_I$ if and only if; cf. (2.2.7):

	uv	uw	vw
u	$-t_v$	t_w	0
v	$-t_u$	0	t_w
w	0	$-t_u$	t_v

(a-1) $\tau_{u,I}\otimes t_u|\bar{X}_I=\tau_{v,I}\otimes t_v|\bar{X}_I$.

On the otherhand (5.2.2.2) clearly insures:

(a-2) $\Phi_{u,I}\otimes t_u|\bar{X}_I=\Phi_{v,I}\otimes t_u|\bar{X}_I$.

Apply Theorem 4.1 4.3 to (a-1,2). By (1.3.2.1) we see that $\Phi_I:\underline{M}_{\bar{X}_I}\to(\underline{M}_{\bar{X}_u}\oplus\underline{M}_{\bar{X}_v})$ gives an isomorphism: $\underline{M}_{\bar{X}_I}\to(C^1(\underline{M}^1))_I$. Next assume that $\#I=3$, and we write $I=\{u,v,w\}$. An element $(\tau_{uv,I}\oplus\tau_{uw,I}\oplus\tau_{vw,I})\in(\oplus_{\alpha,\beta}\underline{M}_{\bar{X}_{\alpha\beta}})|\bar{X}_I$, $\alpha,\beta\in I$, is in $\ker(\delta\Phi^2)_I$ if and only if the following holds for each $\alpha\in I$:

(b-1) $\tau_{\alpha\beta,I}t_\beta|\bar{X}_I=\tau_{\alpha\gamma,I}t_\gamma|\bar{X}_I$ with $\{\beta,\gamma\}=I-\{\alpha\}$, cf. (2.2.7).

See also the table soon above. As in the case of $\#I=2$, we see that (5.2.2.2) insures:

(b-2) $\quad \Phi_{\alpha\beta,I}\otimes t_{\beta}|\bar{X}_I = \Phi_{\alpha\gamma,I}\otimes t_{\gamma}|\bar{X}_I$.

Applying Theorem 4.1 4.4 to (b-1,2), we see that $\Phi_I:\underline{M}_{\bar{X}_I} \to C^1(\underline{M}^2)_I$ gives an isomorphism: $\underline{M}_{\bar{X}_I} \to \ker(\delta\Phi^2)_I$. q.e.d.

5.2.3. Inductive structure...3. Here we determine the inductive structure of $\underline{E}'_{\bar{X}}$ in Theorem 1.2.2. The structure is more subtle than the ones in Lemma 5.6.1 and 5.6.2 and is combinatorial in nature. We begin this subsection by a preparation as follows:

Preparation. For $i \neq j \in \Delta_m$ we write:

(5.2.3.0) $\quad i \sim j$ or $i \not\sim j$, according as $|i-j| \leqq n$ or $\geqq n+1$.

By (5.2.1.3) we have the following according as $i \quad j$ or $i \quad j$:

(5.2.3.1) $\quad [\Phi_i,\Phi_j]|\bar{X}_I = \begin{bmatrix} g_{i,1} & g_{j,1} \\ 0 & 0 \end{bmatrix} |\bar{X}_I$ or $= \begin{bmatrix} 0 & 0 \\ g_{i,2} & g_{j,2} \end{bmatrix} |\bar{X}_I$

and does not vanish on X_I.

Take an $I = \{i,j,k\} \subset \Delta_m$. We have the following easily from the above: First the following four cases can happen concerning the relation:

$$\begin{array}{ccc} & i & \\ i+1 & & i-1 \\ \vdots & g_{i,2} & \vdots \\ \underline{i+n} & - - - - & \underline{i-n} \\ i+n+1 & & i-n-1 \\ \vdots & g_{i,1} & \vdots \\ i+2n & & i-2n \end{array}$$

and among the elements of I:

(5.2.3.2) $\quad$ (1) $u \sim v$ for any $u,v \in I$, (2) $u \not\sim v$ for any $u,v \in I$, (3) $i \sim j,k$ but $j \not\sim k$, (4) $i \not\sim j,k$, but $j \sim k$.

In the last two cases the indices i,j,k are chosen suitably.

Next we have:

(5.2.3.3) $\quad$ (a) In the first(resp.second) case, $g_{u,1}$(resp. $g_{u,2}$) does not vanish on X_I for each $u \in I$, (b) In the third(resp. fourth) cases, $g_{i,1}$(resp. $g_{i,2}$) does not vanish on X_I but $\Phi_j = \Phi_k = 0$ on $\bar{X}_I$.

We make the following convention.

Convention. Take a $K \subset \Delta_m$ and a subvariety $\bar{Y}$ of $\bar{X}$. We set:

(5.2.3.4) $\quad |s_{K|Y}| = K$(=support of $s_{K|Y}$) with $s_K = \Pi_{\alpha\in K} s_\alpha$.

For example we have, cf.(5.2.1.3):

(5.2.3.5) $|g_{i,1}| = \{i \pm n \pm a\}$ and $|g_{i,2}| = \{i \pm a\}$ with $a \in \Delta_n$

(i) Here we determine the inductive structure of $\underline{E}_{\bar{X}}$ for codimension two. For this take an $I = \{i,j\}$. According as $i \sim j$ or $i \not\sim j$ one can write: $j = i+a$ or $j = i+n+a$ without loss of generality. Define an $O_{\bar{X}_I}$-module as follows:

(5.2.3.6) $\underline{M}_{\bar{X}_I} = \underline{L}_{\bar{X}_I}^{\otimes(2n+1-a)}$ or $\underline{M}_{\bar{X}_I} = \underline{L}_{\bar{X}_I}^{\otimes(n-a+2)}$

By (5.2.1.2) we have the followsing, according as $i \sim j$ or $i \not\sim j$:

(5.2.3.7) $\underline{M}^{*}_{\bar{X}_I} \otimes \underline{M}_{\bar{X}_u} = \underline{L}_{\bar{X}_I}^{\otimes a}$ or $\underline{L}_{\bar{X}_I}^{\otimes(n+a-1)}$; $u = i,j$.

Define an element $\Phi_I = \Phi_{i,I} \oplus \Phi_{j,I} \in \Gamma(\underline{M}^{*}_{\bar{X}_I} \otimes (\underline{M}_{\bar{X}_i} \oplus \underline{M}_{\bar{X}_j})|\bar{X}_I)$ as follows:

(5.2.3.8)
$${}^{t}\Phi_I = (\Pi_{\alpha \in [i-n,j-n-1]}\, s_\alpha,\ \Pi_{\alpha \in [i+n+1,j+n]}\, s_\alpha)|\bar{X}_I \quad \text{or}$$
$${}^{t}\Phi_I = (\Pi_{\alpha \in [i+n+1,j+n]-\{j\}}\, s_\alpha,\ \Pi_{\alpha \in [i-n,j-n-1]-\{i\}}\, s_\alpha)|\bar{X}_I$$

Note that

(5.2.3.9) $|\Phi_{u,I}| = |g_{v,1}| - |g_{u,1}|$ or $= |g_{v,2}| \dot{-} |g_{u,2}|$, with $(u,v) =$ (i,j) or (j,i), cf.(5.2.3.4).

FIGURE V-1

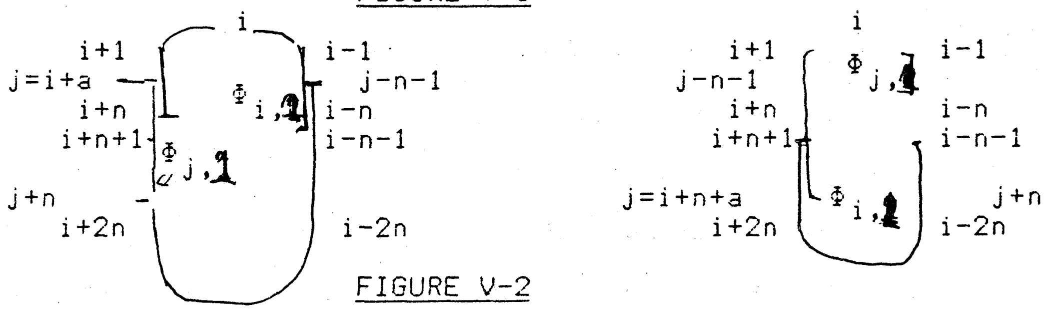


FIGURE V-2

$\Phi_{i,I}$ Φ_j Φ_i $\Phi_{j,I}$

Lemma 5.7.1. The pair $(\underline{M}_{\bar{X}_I}, \Phi_I)$, with $\Phi_I \in \Gamma(\underline{M}^{*}_{\bar{X}_I} \otimes (\underline{M}_{\bar{X}_i} \oplus \underline{M}_{\bar{X}_j})|\bar{X}_I)$ as above, is an inductive structure for codimension two.

Proof. By (5.2.3.1), an element $\tau_{u,I} \oplus \tau_{v,I} \in \Delta(\underline{M}_{\bar{X}_u} \oplus \underline{M}_{\bar{X}_v})|\bar{X}_I$ is in ker $(\delta\Phi^1)_I$ if and only if; cf.(2.2.7):

(a) $g_{i,\alpha}|\bar{X}_I \otimes \tau_{i,I} = g_{j,\alpha}|\bar{X}_I \otimes \tau_{j,I}$, with α = 1 or 2, according as i j or i j .

Apply Theorem 4.1 4.3 to this. Using the combinatorial fact (5.2.3.9) we get the lemma without dificulity. q.e.d.

We add the following remark which is clear from (5.2.3.8,9):

(5.2.3.10) $|\Phi_{u,I}| \cap I = \phi$; $u = i, j$.

(ii) Now we discuss $\underline{E}'_{\bar{X}}$ for codimension three. For this take an I with # I =3 and write I ={i,j,k}. By (5.2.3.2) there are four types for such an I. In the last two cases in (5.2.3.2), i,j,k must satisfy the condition in that place.

$$\begin{array}{c|ccc} & ij & ik & jk \\ \hline i & -\Phi_{i,ij} & \Phi_{i,ik} & 0 \\ j & -\Phi_{j,ij} & 0 & \Phi_{j,jk} \\ k & 0 & -\Phi_{k,ik} & \Phi_{k,jk} \end{array}$$

The following combinatorial fact is derived easily from (5.2.3.8,9,10) and is useful in the arguments here:

(5.2.4.0) $|\Phi_{j,ij}|, |\Phi_{k,ik}|$ and $|\Phi_{j,jk}|, |\Phi_{k,jk}|$ do not contain i,j,k.

Let a(u,v);u,v $\in$I, be an element of Z defined as follows:

(5.2.4.1) $\quad \underline{M}'_{\bar{X}_{uv}} = \underline{L}_{\bar{X}_{uv}}^{\otimes(2n+1-a(u,v))}$, cf.(5.2.3.6),

and define a subset I(j,k) of Δ_m and a(I), a(ui;I),u=j,k, $\in$Z by:

(5.2.4.2) $\quad I(j,k) = \bigcup_{u=j,k}(|\Phi_{u,iu}| - |\Phi_{u,jk}|)$, cf.(5.2.3.4), and

$a(I) = \# I(j,k) + a(j,k)$ and $a(I) = a(ui;I) + a(u,i)$.

Now define an $\underline{O}_{\bar{X}_I}$-module $\underline{M}_{\bar{X}_I}$ and an $\underline{O}_{\bar{X}_I}$-morphism $\Phi_{uv,I} \in \Gamma(\underline{M}^{*}_{\bar{X}_I} \otimes \underline{M}_{\bar{X}_{uv}|\bar{X}_I}) = \Gamma(\underline{L}_{\bar{X}_I}^{\otimes a(uv;I)})$,u,v $\in$I, where a(jk;I)=#I(j,k), as follows:

(5.2.4.3) $\quad \underline{M}_{\bar{X}_I} = \underline{L}_{\bar{X}_I}^{\otimes(2n+1-a(I))}$, and $\Phi_{jk,I} = \prod_{\alpha\in I(j,k)} s_\alpha|\bar{X}_I$ and

$\Phi_{ui,I} = ((\Phi_{jk,I} \otimes \Phi_{u,jk})/\Phi_{u,iu})|\bar{X}_I$;u=j,k.

<u>Lemma 5.7.2</u>. <u>The collection $\{\underline{M}_{\bar{X}_I}, \Phi_I\}$, with $I \subset \Delta_m$;#I =3, is an inductive structure of $\{\underline{M}^0 = (\underline{L}_{\bar{X}_I}^{\otimes m})^{\oplus 2}, \underline{E}'_{\bar{X}}\}$ for codimension three.</u>

<u>Proof</u>. By (5.2.3.3) Φ_i does not vanish on X_I. Thus, by Theorem 4.5(or by a direct computation), an element $\oplus_{u,v}\tau_{uv,I} \in C^1(\underline{M}^2)_I = (\oplus_{u,v}\underline{M}_{\bar{X}_{uv}})|\bar{X}_I$, u,v $\in$I, is in ker $(\delta\Phi^2)_I$ if and only if:

(5.2.4.4) $\quad \Phi_{u,iu|\bar{X}_I} \otimes \tau_{iu,I} = \Phi_{u,jk|\bar{X}_I} \otimes \tau_{jk,I}$, with u =j,k,cf.(2.2.11).

Compare this with the last identity in (5.2.4.3): $\Phi_{u,iu|\bar{X}_I} \otimes \Phi_{iu,I} = \Phi_{u,jk|\bar{X}_I} \otimes \Phi_{jk,I}$;u =j,k. We apply Theorem 4.1 4.4 to (5.2.4.4). Then, using the remrk just above we have the lemma . q.e.d.

The following is clear from (5.2.4.0,3) and is used later:

(5.2.4.5) $\quad |\Phi_{uv,I}| \cap I = \phi$ for any u,v $\in$I.

§ 5.3. Local structure of the reflexive sheaves in § 1.3

Here we prove the assertions on the local structure of the sheaves $\underline{E}'_X(\simeq\underline{E}_{\bar X})$ in Theorem 1.2.1 ∼1.3.2.

5.3.1. As in § 5.2 we first discuss $\underline{E}'_{\bar X}$ in Theorem 1.3.1 and 1.3.2 . According as we are concerned with the first or second we define an element $\underline{u}\in\Gamma((\underline{L}_{\bar X}^{\otimes(m-1)})^{\oplus r})$ as follows:

(5.3.1.1) $${}^t\underline{u}=(u_1,\dots,u_{r-1},-\Sigma_{i\in\Delta_m}s_i^{\wedge}),\ \text{or}=(s_1^{\otimes(m-1)}+s_1^{\wedge},\dots,s_{m-1}^{\otimes(m-1)}+s_{m-1}^{\wedge},\ -(s_m^{\otimes m}+s_m^{\wedge})),\ \text{cf.}(5.2.1.3).$$

The following determines completely the local structure of $\underline{E}'_X$:

Lemma 5.8.1. (1) Let U be an open set of $\bar X$ and l a frame of $\underline{L}_{\bar X|U}$. Then $\underline{E}'_{\bar X}(\subset(\underline{L}_{\bar X}^{\otimes m})^{\oplus r})$ is spanned over U by the following:

(5.3.1.2) $\underline{u}\otimes\underline{l}$ and $\tilde{\underline{s}}_\alpha$; ${}^t\tilde{\underline{s}}_\alpha=(0,\dots,0,\overset{\alpha}{s}(=\Pi_{\alpha\in\Delta_r}s_\alpha),0,\dots,0):\alpha\in\Delta_r$.

(2) Acording as $\underline{E}_{\bar X}$ is in Theorem 1.3.1 or 1.3.2 we have:

(5.3.1.3) $$S(\underline{E}_{\bar X})(=\text{singular locus of }\underline{E}_{\bar X})=\bar X^2\cap(\cap_{\alpha=1}^{r-1}(u_\alpha)_{0,red})\ \text{or}\ \cap_{i\in\Delta_m}(s_i)_0(=\bar X^m).$$

(3) Take a point $p\in\bar X-S(\underline{E}'_{\bar X})$. Then one can take $(\underline{u}\otimes\underline{l})$ and $(\tilde s_1,\dots,\tilde s_\alpha,\dots,\tilde s_\alpha)$, with a $\alpha\in\Delta_r$, as a frame of $\underline{E}'_{\bar X}$ at p.

Proof. The key point of the proof is the following fact, which is immediate from (5.3.1.1) and (5.2.1.3):

(a) $\underline{u}|\bar X_i^1=\Phi_i$; $i\in\Delta_m$, cf.(5.2.1.3) .

Take an $I\subset\Delta_m$, such that $X_I\cap U\neq\phi$, and a $p\in(X_I\cap U)$. By Lemma 5.6.1, $\underline{M}_{\bar X_J}\simeq\underline{L}_{\bar X_J}$ for each $J\subset I$,cf. We set:$f_J=P_J(x_{I-J}\underline{l}_{\bar X_J})$. By (a) and (3.4.2), cf.Corollary to Lemma 3.3.1, we have:

(b) $f_J|\bar X_i^1=x_{I-J}(\underline{l}|\bar X_i^1)\otimes\Phi_i=x_{I-J}(\underline{l}\otimes\underline{u})|\bar X_i^1$;for each $i\in I$.

(Here P_J is the parametrization map at p,cf.the end of § 5.1.1, and x_u satisfies:$(\bar X_u\cap U)=(x_u)_0$.) Remark that $(\bar X^1\cap U)=(s_{|U})_0=(\Pi_{i\in I}x_i)_0$.

Remark that $(s)_0=\bar{X}^1$ and so $(s)_0 = \Pi_{i\in I}x_i$ in U. Then, by (b), f_J is a linear combination of $\underline{u\otimes 1}$ and $\underline{s}_\alpha;\alpha \in \Delta_r$. This insures (1). Next (2) and (3) are checked easily by applying, to the elements in (5.3.1.2), Lemma 5.2 and (**) at the end of § 5.1.1. q.e.d.

§ 5.1.1. q.e.d.

Note that, in the case of Theorem 1.3.2, the locus of the r-vector in Lemma 5.8.1 $=s_\alpha^\wedge+s_\alpha^{\otimes(m-1)};\alpha\in\Delta_m$ and $\underline{s}^{\otimes m}$. Thus $S(\underline{E}_{\bar{X}})$ is the locus of a linear combiation of these elements, and we have (1.3.4.5).

<u>5.3.2.</u> Here we invesigate the behavior of $\underline{E}'_{\bar{X}}$ in Theorem 1.2.1 for codimension two and three:Take an $I \subset\Delta_m$ with # I =2 or 3:

<u>Lemma 5.8.2.1</u>. <u>According as # I =2 or 3 we have</u>:

(5.3.2.1) $X_I\cap S(\underline{E}_{\bar{X}})=X_I\cap(\cap_{u\in I}(t_u)_{0,red})$, <u>or</u> $\subset X_I\cap(\Pi_{u\in I}t_u)_{0,red}$

(<u>The result for #I =3 is a partial one.</u>)

<u>Lemma 5.8.2.2</u>.(1) <u>Assume that #I =2. For a point</u> $p \in X_I-S(\underline{E}_{\bar{X}})$ <u>the following forms a frame of</u> $\underline{E}'_{\bar{X}}$ <u>at p</u>:

(5.3.2.2)
$P_I(\underline{m}_I)$ <u>and</u> ${}^t(0,s)$; <u>if</u> $p \in X_I-(\Pi_{u\in I}t_u)_{0,red}$,
$P_I(\underline{m}_I)$ <u>and</u> $P_u(x_v\underline{m}_u)$; <u>if</u> $p \in X_I-(t_u)_{0,red}$ $(u \in I)$

(2) Assume that # <u>I=3. For a point</u> $p\in X_I-(\Pi_{u\in I}t_u)_{0,red}$ <u>we have</u>:

(5.3.2.3) $P_I(\underline{m}_I)$ <u>and</u> ${}^t(0,s)$ form a frame of $\underline{E}'_{\bar{X}}$ at p.

<u>Here</u> $\underline{m}_J;J \subset I$,<u>is a frame of</u> $M_{\bar{X}_J}$ <u>and</u> P_J <u>is the parametrization map at</u> p,cf.(3.1.5), <u>which is attached to</u> $M_{\bar{X}_J}$ <u>.The coordinate</u> $x_u;u\in I$, <u>satisfies</u>: $(\bar{X}_u\cap U)=(x_u)_0$.

<u>Proof</u>. The key facts (5.3.2.2,3) are proved by using Lemma 5.3.2. For the application of the latter, recall that;cf.Lemma 3.1:

(a) $P_{I|\bar{X}_I}=\Phi_{u|\bar{X}_I}\Phi_{u,I};u\in I(\# I =2)$ and $=\Phi_{u|\bar{X}_I}\Phi_{u,uv|\bar{X}_I}\Phi_{uv,I};u,v\in I(\#I=3)$

By Lemma 5.6.2 we have:

(b) $P_{I|\bar{X}_I} ={}^t[\Pi_{u\in I}t_u,0]_{|\bar{X}_I}$ for the both cases:# I =2 and 3.

By Lemma 5.3.2, this insures the assertions in the lemmas for codimension three and the first fact in (5.3.2.2). In order to prove the second fact in (5.3.2.2), let $h_{0,\alpha}$ and $h_{\alpha,I}$ be the matrix representation of Φ_α and $\Phi_{\alpha,I},\cdots:\Phi_\alpha(\underline{m}_\alpha)=\underline{m}^0_{|\bar{X}_\alpha} h_{0,\alpha}$ and $\Phi_{\alpha,I}(\underline{m}_I)=\underline{m}_{v|\bar{X}_I} h_{v,I},\cdots$ By Lemma 5.6.2 one can take $\underline{1}^{\otimes(m+\#J(1-m))}_{|\bar{X}_J}$, with a frame $\underline{1}$ of $\underline{L}_{\bar{X}}$ at p, as a frame of $\underline{M}_{\bar{X}_J}$ at p. By Lemma 5.5 and 5.6.2 one can write them as follows:

(b) $\quad h_{0,v}=(\tilde{t}_{v|\bar{X}_I},0)+x_u(\tilde{t}'_v,\varepsilon_v)$ and $h_{v,I}=\tilde{t}_{u|\bar{X}_I}$

(Here $\tilde{t}_\alpha,\tilde{t}'_\alpha$ and ε_v are in $\Gamma(U,\underline{O}_{\bar{X}});\alpha=i,j$. The element $\tilde{t}_\alpha$ satisfies: $t_\alpha=\underline{1}^{\otimes(m-1)}\tilde{t}_\alpha$ and ε_v does not vanish on X_I.) By the third fact in Lemma 5.3.2, $P_I(\underline{m}_I)$, $P_u(x_v\underline{m}_v)$ form a frame at p if and only if:

(c) $\quad \det\begin{pmatrix}\tilde{t}'_v & \tilde{t}_u \\ \varepsilon_v & \end{pmatrix},\ \begin{pmatrix}\tilde{t}_u \\ 0\end{pmatrix} = \varepsilon_v\tilde{t}_u^2 \neq 0.$

This insures the second fact in (5.3.2.2). Finally, by Lemma 5.3.1, $\underline{E}_{\bar{X},p}$ is free only if one of the 2-vectors in (5.3.2.2) forms a frame at p, cf.Lemma at p, cf.Lemma 5.2. By (b) and (c), this is equivalent to $\Pi_u t_u$ and t_u does not vanish at p. This insures the first fact in (5.3.2.1), and we have the lemmas. q.e.d.

<u>5.3.3</u>. Here we determine the behavior of $\underline{E}'_{\bar{X}}$ in Theorem 1.2.2 for codimension two and three. As in § 5.2 the arguments are more subtle than the ones in Lemma 5.8.1 and 5.8.2. Take an $I\subset\Delta_m$ with # I = 2 or 3. We write I ={i,j} or {i,j,k}. When # I =3, i,j,k must satisfy the condition in (5.2.3.2).

<u>Lemmma 5.8.3.</u> (1) <u>In the both cases # I =2 and 3, for each</u> $p\in X_I$, <u>$\underline{E}'_{\bar{X},p}(\simeq\underline{E}_{\bar{X},p})$ is $\underline{O}_{\bar{X},p}$-free.</u>

(2) <u>Assume that # I =2. According as $i\sim j$ or $i\not\sim j$, the following forms a frame of $\underline{E}'_{\bar{X}}$ at p</u>:

(5.3.3.1) $\quad (P_I(m_I),{}^t(0,s))$ or $(P_I(m_I),{}^t(s,0))$.

(3) <u>Assume that # I =3. In the first or second case in (5.2.3.2) we have</u>:

(5.3.3.2) $(P_I(m_I),{}^t(0,s))$ <u>or</u> $(P_I(m_I),{}^t(s,0))$ <u>forms a frame at p.</u>

<u>In the third and fourth cases in (5.2.3.2) we have</u>:

(5.3.3.3) $(P_I(m_I),P_i(x_jx_km_i))$ <u>forms a frame at p.</u>

<u>Here the frames m_J and the parametrization map</u> $P_J;J\subset I$ <u>have the</u>

similar meaning to Lemma 5.8.2.

Proof. As in the proof of Lemma 5.8.2 we write $P_{I|\bar{X}_I}$ using Lemma 3.1: Assume that # I =2. By (5.2.3.1,8) we have:

(a) $P_{I|\bar{X}_I}=(\Pi_{\alpha\in K}s_\alpha,0)|\bar{X}_I$ or $(0,\Pi_{\alpha\in K}s_\alpha)|\bar{X}_I$ with $K=|g_{i,\beta}|\cup|g_{j,\beta}|$,

β being 1 or 2 acording as $i\sim j$ or $i\not\sim j$.

By (5.2.3.1,10) $K\ni i,j$, and P_I does not vanish on X_I. By Lemma 5.2 we have (5.3.3.1). When # I =3 we take:(u,v) =(j,k).By (5.2.4.0,5) $\Phi_{u,uv|\bar{X}_I}\Phi_{uv,I}$ does not vanish on X_I. From (5.2.3.3) we see that in the first or second case in (5.2.3.2) $P_{I|\bar{X}_I}$ is written as $[\varepsilon,0]$ or $[0,\varepsilon]$ with $\varepsilon\in\Gamma(\underline{L}_{\bar{X}_I}^{b(I)})$, $b(I)\in Z$, which does not vanish on X_I. By Lemma 5.2 we have (5.3.3.2). Also remark that, by (5.2.2.3), $P_{I|\bar{X}_I}=0$ in the last two cases in (5.2.3.2). In the cases we prove (5.3.3.3) by using Lemma 5.4.2. Let $h_{J,K}$ be the matrix representation of $\Phi_{K,J}:\Phi_{K,J}(\underline{m}_J)=\underline{m}_{K|\bar{X}_J}h_{K,J}$ with frames $\underline{m}_K$ of $\underline{M}_{\bar{X}_K}$,.. By Lemma 5.4.2, (5.3.3.3) is equivalent to:

(a) $\det[H_{0,u}(p),(h_{0,\alpha;jk}h_{\alpha,jk}h_{jk,I})(p)]\neq 0$, with $h_{0,\alpha|\bar{X}_{jk}}=$
$\pi_I^*(h_{0,\alpha|\bar{X}_I})+x_i h_{0,\alpha;jk}$, cf.(5.2.12); $\alpha=j$ or k.

By (5.2.0.4,5) $h_{\alpha,jk}$ and $h_{jk,I}$ do not vanish on X_I. Recall the definition of Φ_α, cf.(5.2.1.3). Then according as we are concerned with the third or fourth case in (5.2.3.2), we have:

(b) ${}^t h_{0,\alpha}=x_i(0,\varepsilon)+x_\beta(f',0)$ or $=x_i(\varepsilon,0)+x_\beta(0,f')$;$(\alpha,\beta)=(j,k)$ or (k,j)

Here ε, f' are holomorphic functions and ε is a unit. Thus $h_{0,\alpha;jk}$ $={}^t(0,\varepsilon)$ or ${}^t(\varepsilon,0)$. Thus the second term in the matrix in (a) is: ${}^t(0,c)$ or ${}^t(c,0)$ with $c\in C^*$. On the otherhand, by (5.2.1.3) and (5.2.3.3), we have:$h_{0,i}(p)={}^t[c',0]$ or $={}^t[0,c']$ with $c'\in C^*$. Thus we have:

(d) $\det[h_{0,i}(p),(h_{0,\alpha;jk}h_{\alpha,jk}h_{jk,I})(p)]=cc'\neq 0$.

By Lemma 5.4.2 we have (5.3.3.3) and so the Lemma . q.e.d.

By Lemma 5.8.1 5.8.3 and the remark soon below Lemma 5.8.1

we have proved all the results on the local structure of the reflexive sheaves in Theorem 1.2.1 ～ 1.3.2.

Appendix. Structure of End E_X

Here we investigate structures of endomorphisms of vector bundles, by analyzing the adjoint of the transition matrix of frames of the bundles;see (1.0.1) and Lemma 1.0 for the explicit form of the adjoint. The structures are given in Lemma 1.1$\sim$1.4;see also the diagrams in Figure I IV;we give some criterions for the simpleness of the bundles. The result here is applied to each algebraic bundle on a normal quasi projective variety. This appendix is a refinement of Part B,§ 4,[Sa-1]. All ideas here are found in that place.

A.0. Preparations

A.0.1. Underlying data. In this appendix we work with more general data than in the main body of the present paper:

$\bar{X}$ = a normal complex space of dimension $\geqq 2$,

(0.1.0) $\bar{X}^1 = \cup_{i\in\Delta_m} \bar{X}^1_i$, $m \geqq 1$, is a reduced divisor of $\bar{X}$, where the irreducible components $\bar{X}^1_i$ of $\bar{X}^1$ are normal.

$\bar{X}^2$ = a reduced complex subspace of $\bar{X}$ of codimension two satisfying:$\bar{X}^1 \supset \bar{X}^2$.

When $m \geqq 2$ we assume that $\bar{X}^2_{ij} := \bar{X}^1_i \cap \bar{X}^1_j$ is of codimension two for each $i \neq j \in \Delta_m$ and that $\bar{X}^2 = \cup_{i\neq j} \bar{X}^2_{ij}$. As in § 1.1 we set:

(0.1.1) $X^0 = \bar{X} - \bar{X}^2$, $X^1 = \bar{X}^1 - \bar{X}^2$ and $X^1_i = \bar{X}^1_i - \bar{X}^2; i \in \Delta_m$.

Also take an open neighborhood $N_1 = \cup_{i\in\Delta_m} N_{1,i}$ of $X^1 = {}_{i\in\Delta_m} X^1_i$ in X satisfying:$N_{1,i} \cap N_{1,j} = \phi$, if $i \neq j$. We take a matrix $H \in M_r(N_1, O_{\bar{X}})$, $r \geqq 2$, whose restriction to $N_{1,i}$ is as follows, cf.(1.1.0.6).

(0.1.2) (s,t)-component of $H_{|N_{1,i}} = \delta_{st}$,$t \neq \alpha(i)$, and $= f_s$, $t = \alpha(i)$, with $f_s \in \Gamma(N_1, \underline{O}_{\bar{X}})$.(Here $\alpha(i)$ is an element of Δ_r.)

We assume the following for each $i \in \Delta_m$;cf.§ 1.1.

(0.1.3.1) $(f_{\alpha(i)})_0 = X_i^1$ in $N_{1,i}$, and $f_{\alpha|X_i^1}$ is the restriction of a (unique) meromorphic function on $\bar{X}_i^1$ with the pole $\bar{X}_i^1 \cap \bar{X}^2$.

We write the meromorphic function as $f_{\alpha|\bar{X}_i^1}$. As in § 4.0 we assume the existence of a divisor $\bar{D}_i^2$ of $\bar{X}_i^1$, which is locally defined by one equation and satisfies the following:

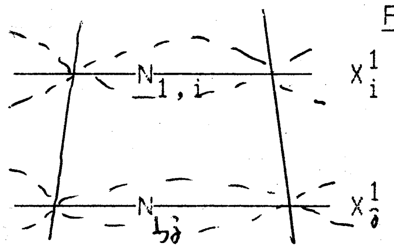
(0.1.3.2) $(\bar{D}_i^2)_{red} = \bigcup_{\alpha \in \Delta_r}$ (pole divisor of $f_{\alpha|\bar{X}_i^1})_{red}$, and , for each irreducible component $\bar{D}_{i,\lambda}^2$ of $(\bar{D}_i^2)_{red}$, we have: multiplicity of $\bar{D}_i^2$ with respect to $\bar{D}_{i,\lambda}^2$ = $\max_\alpha$ that of (pole divisor of $f_{\alpha|\bar{X}_i^1}$) .

See § 4.0 for the multiplicity.

Now we form a bundle $\underline{E}_X$ over X from $(\bar{X}^1, \bar{X}^2, N_1, H)$ in the manner as in § 1.1. Thus $\underline{E}_{X|N_i}$ has a frame, denoted by $\underline{e}^i$, $i = 0,1$, satisfying $\underline{e}^0 = \underline{e}^1 H$ in $(N_0 \cap N_1)$ with $N_0 = \bar{X} - \bar{X}^1$. As a frame of $\underline{\text{End}}\ \underline{E}_{X|N_i}$, we take $\underline{\mu}^i = \{\mu^i_{\alpha\beta}\}, \alpha, \beta \in \Delta_r$ with $\mu^i_{\alpha\beta} \in \underline{\text{End}}\ \underline{E}_{X|N_i}$, as follows:

(0.1.4) $\mu^i_{\alpha\beta}(e^i_\gamma) = (\delta^\beta_\gamma) e^i_\alpha$, where $e^1_\alpha, \ldots = \alpha$-th component of $\underline{e}^i, \ldots$.
(Here $\delta^\beta_\gamma = 1$ or 0, according as $\beta =$ or $\neq \gamma$.)

FIGURE 0



$$H_{|N_{1,i}} = \begin{array}{c} \begin{array}{cccccccc} & & & \alpha(i) & & & \end{array} \\ \begin{bmatrix} 1 & & & f_1 & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & f_{\alpha(i)} & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & f_r & & & 1 \end{bmatrix} \end{array} \alpha(i)$$

As in the main body of the paper, for an $I \subset \Delta_m$, we write $\bar{X}_I$ and X_I for $\bigcap_{i \in I} \bar{X}_i^1$ and $\bigcap_{i \in I} X_i^1$.(Thus we can write: $\bar{X}_i = \bar{X}_i^1$ an $X_i = X_i^1$.)

A 1. Imbedding of End E_X

In § 1 § 3 we write $\underline{L}_{\bar{X}}$ for $\underline{O}_{\bar{X}}[\bar{X}^1]$. For a locally closed reduced complex subspace Y of $\bar{X}$ we write $\underline{L}_Y$ for $\underline{L}_{\bar{X}|Y}$.

A.1.0. First we make an algebraic preparation: Letting $x = (x_{\alpha\beta})$.

$\alpha,\beta\in\Delta_r$, be variables, we define linear forms $T_{\alpha\beta}(x)$ in x with coefficients in $\Gamma(N_1,\underline{O}_X)$ as follows:

$$
\begin{aligned}
&T_{\alpha(i)\beta}(x)|N_{1,i} = x_{\alpha(i)\beta},\ \beta\in\Delta_r-\{\alpha(i)\},\\
&T_{\alpha\beta}(x)|N_{1,i} = f_{\alpha(i)}\, x_{\alpha\beta}-f_{\alpha}\, x_{\alpha(i)\beta},\ \alpha,\beta\in\Delta_r-\{\alpha(i)\}\\
(1.0.1)\quad &T_{\alpha(i)\alpha(i)}(x)|N_{1,i} = \Sigma_{\gamma\in\Delta_r} f_{\gamma}\, x_{\alpha(i)\gamma}\\
&T_{\alpha\alpha(i)}(x)|N_{1,i} = \Sigma_{\gamma\in\Delta_r}(f_{\alpha(i)}\, x_{\alpha\gamma}-f_{\alpha}\, x_{\alpha(i)\gamma})f_{\gamma},\ \alpha\in\Delta_r-\{\alpha(i),\ =\\
&\qquad = f_{\alpha(i)}(\Sigma_{\gamma\in\Delta_r} f_{\gamma}\, x_{\alpha\gamma}) - f_{\alpha}\, T_{\alpha(i)\alpha(i)}(x)|N_{1,i}
\end{aligned}
$$

(Here i is an element of Δ_m. In the R.H.S we write $f_\gamma;\gamma\in\Delta_r-\{\alpha(i)\}$ also for its restriction to $N_{1,i}$. The diagram indicates the order by which $T_{\alpha\beta}$ is determined

$$
\begin{array}{cc|cl}
 & & \alpha(i) & \\
 & 2 & 4 & \\
\hline
 & 1 & 3 & \alpha(i)
\end{array}
$$

when $r=\alpha(i)$.) Denoting by T the collection $\{T_{\alpha\beta}\}_{\alpha,\beta\in\Delta_r}$ we make:

Convention. For an element $A=[a_{\alpha\beta}]\in\underline{F}^{\oplus r^2}$, $\alpha,\beta\in\Delta_r$, where $\underline{F}$ is a coherent sheaf over N_1, we write $T_{\alpha\beta}(A)$ for the element of $\underline{F}$, which is obtained by substituting $a_{\alpha\beta}$ to $x_{\alpha\beta}$, and define an element of $\underline{F}^{\oplus r^2}$ as follows:

(1.0.2) $\quad T(C) = [T_{\alpha\beta}(C)]$, $\alpha,\beta\in\Delta_r$.

The matrix T represents the adjoint of H: Take an element $\varphi\in\underline{\text{End}}\,\underline{E}_{X,p}$, $p\in N_0\cap N_1$, and $C^i=[c^i_{\alpha\beta}]\in\underline{O}^{\oplus r^2}_{X,p}$, i= 0,1 and $\alpha,\beta\in\Delta_r$, such that

(1.0.3) $\varphi=\Sigma_{\alpha\beta}\, c^i_{\alpha\beta}\mu^i_{\alpha\beta}$, with the frame $\mu^i_{\alpha\beta}$ of $\underline{\text{End}}\ \underline{E}_{X|N_{1,i}}$, cf.(0.1.4) .

Note that $\varphi(\underline{e}^i)=\underline{e}^iC^i$, i =0,1, and we have: $C^0=H^{-1}C^1H$. The following is a basis for the arguments from now on:

Lemma 1.0. *The relation: $C^0=H^{-1}C^1H$ is equivalent to the following*:

(1.0.4) $\quad f_{\alpha(i)}C^0 = T(C^i)$.

Proof. This is checked by a tedious(but essentially easy) computations. We omit the detail. q.e.d. .

A.1.1. Now we imbed $\underline{E}_X$ into $\underline{L}_X^{\oplus r^2}$: First define an $\underline{O}_{N_1}$-morphism using the matrix T:

(1.0.5) $\quad T:\underline{L}_{\bar{X}|N_1}^{\oplus r^2} \ni B \rightarrow \underline{L}_{\bar{X}|N_1}^{\oplus r^2} \ni T(B)$.

We write the image of T as $\underline{K}_{N_1}$. From the explicit form of T we see easily that T is injective and $\underline{K}_{N_1|N_1-X^1} = \underline{L}_{X|N_1-X^1}^{\oplus r}$;define an $\underline{O}_X$--submodule $\underline{K}_X$ of $\underline{L}_X^{\oplus r}$ as follows:

(1.0.6) $\quad \underline{K}_{X|N_0}=\underline{L}_{X|N_0}^{\oplus r^2}$ and $\underline{K}_{X|N_1} = \underline{K}_{N_1}$ (Note that $N_1-X^1 = N_1 \cap N_0$.)

Lemma 1.1. End E_X is isomorphic to $\underline{K}_X (\subset \underline{L}_X^{\oplus r^2})$.

Proof. Take an element s of $\Gamma(\underline{L}_{\bar{X}})$ satisfying $(s)_0 = \bar{X}^1$, and define an elemen $s^1 \in \Gamma(N_1, \underline{L}_{\bar{X}})$ by:$s^1_{|N_{1,i}} = s/f_{\alpha(i)}$. This forms a frame of $\underline{L}_{\bar{X}|N_1}$. Define an $\underline{O}_X$-morphsm τ_s:$\underline{\text{End}}\ \underline{E}_X \rightarrow \underline{L}_X^{\oplus r^2}$ by

$$(1.1.1) \quad \begin{aligned} &\tau_{s|N_0}:\underline{\text{End}}\ \underline{E}_{X|N_0} \ni \mu^0 C^0 \rightarrow \underline{L}_{X|N_0}^{\oplus r^2} \ni sC^0 \\ &\tau_{s|N_1}\underline{\text{End}}\ \underline{E}_{X|N_1} \ni \mu^1 C^1 \rightarrow \underline{L}_{X|N_1}^{\oplus r^2} \ni T(s^1 C^1);\ i \in \Delta_m \end{aligned}$$

where $C^i = [c^i_{\alpha\beta}]\ \alpha,\beta \in \Delta_r$, is an element of $\underline{O}_{N_i}^{\oplus r^2}$. We write:$\mu^i C^i = \Sigma_{\alpha,\beta} c^i_{\alpha\beta}\mu^i_{\alpha\beta},\ldots$with the frames $\mu^i_{\alpha\beta}$,cf. (0.1.4). By Lemma 0.1, $\tau_{s|N_0} = \tau_{s|N_1}$ in $N_0 \cap N_1$. It is easy to check the injectivetiy of τ_s by using the explicit form of T, cf.(1.0.1). Moreover, comparing (1.1.1) with (1.0.4) we see immediately $\tau_s(\underline{\text{End}}\ \underline{E}_X) = \underline{K}_X$. q.e.d.

Remark 1. (1) T.Hosoh pointed out that the imbedding τ_s is defined more conceptually:By Propositon 4.0 we have the injection: $\theta_s:\underline{E}_X \rightarrow \underline{E}'_X := \theta_s(\underline{E}_X) \subset \underline{L}_X^{\oplus r^2}$ characterized by $\theta_s(e^0_\alpha) = {}^t(0,\ldots,0,\overset{\alpha}{s},0,\ldots,0)$; $\alpha \in \Delta_r$, with e^0_α=α-th component of $\underline{e}^0$. Also we have the injection $i:\underline{O}_X^{\oplus r^2} \rightarrow \underline{E}_X$ by means of $\underline{e}^0$. One checks easily that the following $\underline{O}_X$-morphism coincides with τ_s:

(1.1.2) $\quad \underline{\text{End}}\ \underline{E}_X \ni \varphi \rightarrow \underline{\text{Hom}}_{\underline{O}_X}(\underline{O}_X^{\oplus r}, \underline{L}_X^{\oplus r}) \ni \theta_s \varphi \cdot i$.

Theoretically this is a suiable definition of τ_s. The matrix T is useful for explicit computations on $\underline{\text{End}}\ \underline{E}_X$.

(2) It is easy to check that the relation between the

$$(1.1.3)\qquad \begin{array}{ccc} \underline{\mathrm{End}}\ \underline{E}_{X|N_1} \ni \underline{\mu}^1 C^1 & & \\ \Big\| \wr & \xrightarrow{\tau_{s|N_1}} & \underline{K}_{X|N_1} \ni T(s^1C^1) \\ \underline{L}_{\bar{X}|N_1} \ni s^1 C^1 & \xrightarrow{T} & \end{array}$$

A.1.2. (i) Let ω be the quotient morphism: $\underline{L}_{\bar{X}} \to \underline{L}_{\bar{X}}1$. We set: $\underline{K}_X 1 = \omega_{|X}(\underline{K}_X)(\subset (\underline{L}_X 1)^{\oplus r^2})$. In the rest of A.1 we analyze the exact sequence as follows:

$$(1.2.0)\quad 0 \to \underline{K}_X \cap \underline{O}_X^{\oplus r^2} \to K_X \xrightarrow{\omega_X} \underline{K}_X 1 \to 0, \text{ with } \underline{O}_X \simeq \underline{I}_X 1 \otimes \underline{L}_X.$$

Our analysis is summarized in the following diagram:

FIGURE I

$$\begin{array}{ccccccc}
 & \underline{O}_X & & \underline{\mathrm{End}}\ \underline{E}_X & & \underline{L}_X^{\oplus(r-1)} & \\
 & \downarrow j & & \tau_s \Downarrow & & \wr\Downarrow \ \chi & \\
0 \to & \underline{K}_X \cap \underline{O}_X^{\oplus r^2} & \to & \underline{K}_X & \xrightarrow{\omega} & \underline{K}_X 1 & \to 0 \\
 & \cap & & \cap & & \wr\| & \\
0 \to & \underline{O}_X^{\oplus r^2} & \to & \omega_{|X}^{-1}(\underline{K}_X 1) & \xrightarrow{\omega} & \underline{K}_X 1 & \to 0 \\
 & \wr\| & & \cap & & \cap & \\
0 \to & \underline{O}_X^{\oplus r^2} & \to & \underline{L}_X^{\oplus r^2} & \xrightarrow{\omega} & (\underline{L}_X 1)^{\oplus r^2} & \to 0
\end{array}$$

The key point in the diagram is: (1) to form the isomorphism χ and the injection j, and (2) to characterize $\underline{K}_X$ and $\omega_{|X}^{-1}(\underline{K}_X 1)$ as the $\underline{O}_X$-submodule of $\underline{L}_X^{\oplus r^2}$. (ii) For (1),(2) soon above we make some algebraic preparations. Namely we derive some linear forms from $T_{\alpha\beta}$, cf.(1.0.1). First restrict $T_{\alpha\beta}$ to X_i: let $y = (y_\gamma)_{\gamma \in \Delta_r - \{\alpha(i)\}}$ be varibles. The following linear forms are obtained from the restriction $T_{\alpha\beta|X_i}$ of $T_{\alpha\beta}$ to X_i, by changing $x_{\alpha(i)\beta}$ to y_β:

$$(1.2.0.1)\quad \begin{aligned} & S_{i,\alpha(i)\beta}(y) = y_\beta, \quad S_{i;\alpha\beta}(y) = -f_{\alpha|X_i} y_\beta \ ;\alpha,\beta \in \Delta_r - \{\alpha(i)\} \\ & S_{i,\alpha(i)\alpha(i)}(y) = \Sigma_{\gamma \in \Delta_r - \{\alpha(i)\}} f_{\gamma|X_i} y_\gamma \\ & S_{i,\alpha\alpha(i)}(y) = -f_{\alpha|X_i}\{\Sigma_{\gamma\in\Delta_r-\{\alpha(i)\}} f_{\gamma|X_i} y_\gamma\};\ \alpha \in \Delta_r - \{\alpha(i)\}, \\ & = -f_{\alpha|X_i} S_{i,\alpha(i)\alpha(i)}(y) \end{aligned}$$

We write pr_i for the projection:

$$(1.2.0.2)\quad \underline{L}_X^{\oplus r^2} \ni B \to \underline{L}_X^{\oplus(r-1)} \ni \text{the } (\alpha(i),\beta)\text{-components of } B; \beta \in \Delta_r - \{\alpha(i)\}.$$

Denote by S_i the matrix $[S_{i,\alpha\beta}]_{\alpha,\beta \in \Delta_r}$. The following is clear from (1.2.0.1):

<u>Proposition 1.2.1</u>. <u>For a</u> $B \in \underline{L}^{\oplus r}_{\overline{X}|N_{1,i}}$ <u>we have</u>:

(1.2.0.3) $$\omega_{|N_{1,i}}(T(B)) = S_i(\omega_{|N_{1,i}}(pr_i(B))) \ .$$

<u>Here the R.H.S. has a similar meaning to (1.0.2)</u>.

Moreover, we define linear forms in $z = (z_{\alpha\beta})$, $\alpha, \beta \in \Delta_r$, with coefficients in $\Gamma(N_1, \underline{O}_{\overline{X}})$ as follows:

$$\begin{array}{c|l} & \alpha(i) \\ 2 & 4 \\ \hline 1 & 3 \ \alpha(i) \end{array}$$

(1.2.0.5)
$$\begin{aligned}
&T'_{\alpha(i)\beta}(z)|N_{1,i} = 0; \beta \in \Delta_r - \{\alpha(i)\} \\
&T'_{\alpha\beta}(z)|N_{1,i} = z_{\alpha\beta} + f_\alpha \, z_{\alpha(i)\beta}, \quad \alpha, \beta \in \Delta_r - \{\alpha(i)\} \\
&T'_{\alpha(i)\alpha(i)}(z)|N_{1,i} = z_{\alpha(i)\alpha(i)} - \Sigma_{\gamma \in \Delta_r - \{\alpha(i)\}} f_\gamma \, z_{\alpha(i)\gamma} \\
&T'_{\alpha\alpha(i)}(z)|N_{1,i} = z_{\alpha\alpha(i)} + f_\alpha \{\Sigma_{\gamma \in \Delta_r - \{a(i)\}} f_\gamma \, z_{\alpha(i)\gamma}\}
\end{aligned}$$

We set $T' = [T'_{\alpha\beta}]_{\alpha,\beta \in \Delta_r}$. The relation beteeen $T'_{\alpha\beta}$ and the former linear forms is as follows:

Proposition 1.1.2. For an $A \in \underline{L}_{\overline{X}|N_{1,i}}$, $i \in \Delta_m$, we have:

(1.2.0.4) $\omega_{|N_{1,i}} T'(A) = \omega_{|N_{1,i}}(A) - S_i(\omega_{|N_{1,i}} pr_i(A))$.

Here $T'(A)$ has the similar meaning to (1.0.2).

Proof. Clear also from the comparison of T' and S_i . q.e.d.

(3) Using the matrices S_i and T' we give a characterization of $\underline{K}_X 1$ and $\omega^{-1}(\underline{K}_X 1)$.

Lemma 1.2. (1) Define an $\underline{O}_{X_i}$-morphism χ_{S_i} as follows:

(1.2.1.1) $\chi_{S_i} : \underline{L}_{X_i}^{\oplus(r-1)} \ni \underline{a}_i = \to \underline{L}_{X_i}^{\oplus r} \ni S_i(\underline{a}_i)$

Then χ_{S_i} is an O_{X_i}-isomorphism: $\underline{L}_{X_i}^{\oplus r-1} \to \underline{K}_{X_i} := \underline{K}_X 1_{|X_i}$.

(2) The O_{N_1}-module $\omega^{-1}_{|N_1}(\underline{K}_X 1)$ coincides with the kernel of the following O_{N_1}-morphism:

(1.2.1.2) $\omega_{|N_1} T' : \underline{L}_{\overline{X}|N_1}^{\oplus r^2} \ni A \quad \to (\underline{L}_X 1)^{\oplus r^2} \ni \omega T'(A)$.

Remark 2. The $\underline{O}_X 1$-morphism χ in Figure I $= \oplus_{i\in\Delta_m} \chi_{S_i}$.

Proof. First (1) is clear from Proposition 1.1.1 and the surjectivity of $\omega_{|N_{1,i}} pr$: $\underline{L}_{\overline{X}|N_{1,i}}^{\oplus r^2} \to \underline{L}_{X_i}^{\oplus(r-1)}$. Next, to prove (2), we write ω also for $\omega_{|N_1}$: Take an $A \in \underline{L}_{\overline{X}|N_{1,i}}^{\oplus r^2}$ with an $i \in \Delta_m$. If $\omega T'(A) = 0$ then Proposition 1.1.2 and (1.2.1.1) insure: $\omega(A) \in \underline{K}_X 1$. Conversely, if $\omega(A) \in \underline{K}_{X_i}$, we have: $\omega(A) = S_i(\underline{a}_i)$ with an $\underline{a}_i \in \underline{L}_{X_i}^{\oplus(r-1)}$; cf. (1.2.1.1). By (1.2.0.1) we see $\omega pr_i(A) = \underline{a}_i$, and $\omega(A) = S_i(\omega pr_i(A))$. By Proposition 1.1.2 we have: $\omega T'(A) = 0$. q.e.d.

A.1.3. Here we characterize the $\underline{O}_X$-submodule $\underline{K}_X$ of $\underline{L}_X^{\oplus r^2}$. Writing $\underline{K}'_X$ for $\omega^{-1}_{|X}(\underline{K}_X 1)$ note that $\underline{K}_X \subset \underline{K}'_X$; we characterize $\underline{K}_{\overline{X}}$ as the submodule of $\underline{K}'_X$: For this take an $i \in \Delta_m$ and define linear forms $T''_{\alpha\alpha(i)}(z)$; $\alpha \in \Delta_r - \{\alpha(i)\}$, as follows:

$$T''_{\alpha\alpha(i)}(z) = z_{\alpha\alpha(i)} + f_\alpha\{z_{\alpha(i)\alpha(i)} - \Sigma_{\gamma\in\Delta_r - \{\alpha(i)\}} f_\gamma z_{\alpha(i)\gamma}\} - \Sigma_{\gamma\in\Delta_r - \{\alpha(i)\}} f_\gamma z_{\alpha\gamma} \tag{1.3.0}$$

We set: $T''_{N_{1,i}} = [T''_{\alpha\alpha(i)}]_{\alpha \in \Delta_r - \{\alpha(i)\}}$, and let $\omega^{(2)}$ denote the quotient morphism: $\underline{L}_{\overline{X}} \to (\underline{L}_{\overline{X}}/(\underline{I}_{\overline{X}}1)^2\underline{L}_{\overline{X}})$.

<u>Lemma 1.3.1</u>. <u>An element</u> $A \in \underline{K}'_{X|N_{1,i}}$ <u>is in</u> $\underline{K}_{X|N_{1,i}}$ <u>if and only if it is in the kernel of the following morphism</u>:

$$(1.3.1) \qquad \omega^{(2)}_{|N_{1,i}} : T''_{N_{1,i}} : \underline{L}^{\oplus r^2}_{\overline{X}|N_{1,i}} \to (\underline{L}^{\oplus(r-1)}_{\overline{X}}/(\underline{I}_{\overline{X}}1)^2\underline{L}^{\oplus(r-1)}_{\overline{X}})_{|N_{1,i}}.$$

<u>Proof</u>. By (1.1.0) an element $A = [a_{\alpha\beta}] \in \underline{L}^{\oplus r^2}_X$, $\alpha, \beta \in \Delta_r$, is in $\underline{K}_X$ if and only if $A = T(B)$ with a $B = [b_{\alpha\beta}] \in \underline{L}^{\oplus r^2}_X$, $\alpha, \beta \in \Delta_r$. One chceks that this relation is equivalent to the following:

$$\text{(a)} \qquad \begin{aligned} & b_{\alpha(i)\beta} = a_{\alpha(i)\beta};\ \beta \in \Delta_r - \{\alpha(i)\}, \\ & f_{\alpha(i)} b_{\alpha\beta} = T'_{\alpha\beta}(A); \alpha, \beta \in \{\Delta_r - \{\alpha(i)\} \times \{\Delta_r - \{\alpha(i)\} \cup \{\alpha(i)\} \times \{\alpha(i)\} \\ & (f_{\alpha(i)})^2 b_{\alpha\alpha(i)} = T''_{\alpha\alpha(i)}(A);\ \alpha \in \Delta_r - \{\alpha(i)\} \end{aligned}$$

The first two relations are easily checked. The third is checked as follows:

$$\text{(b)} \qquad \begin{aligned} f^2_{\alpha(i)} b_{\alpha\alpha(i)} &= a_{\alpha\alpha(i)} + f_{\alpha(i)} f_\alpha b_{\alpha(i)\alpha(i)} - \Sigma_{\gamma \in \Delta_r - \{\alpha(i)\}} \\ f_\gamma \{f_{\alpha(i)} b_{\alpha\gamma} - f_\alpha b_{\alpha(i)\gamma}\} &= a_{\alpha\alpha(i)} + f_\alpha \{a_{\alpha(i)\alpha(i)} - \Sigma_{\gamma \in \Delta_r - \{\alpha(i)\}} f_\gamma a_{\alpha(i)\gamma}\} \\ & - \Sigma_{\gamma \in \Delta_r - \{\alpha(i)\}} f_\gamma a_{\alpha\gamma} \end{aligned}$$

By Lemma 1.2.1, that $A \in \underline{K}'_X$ implies the first and second equations in (a) are solved. Next that $\omega^{(2)}_{|N_{1,i}} T''_{N_{1,i}}(A) = 0$ is equivalent to that the third equation in (a) is solved. q.e.d.

<u>Remark 3.</u> The above lemma insures that <u>End</u> $\underline{E}_X \simeq \underline{K}_X$ concerns $(\underline{I}_X 1)^{\otimes 2}$ while $\underline{E}_X$ concerns with only $\underline{I}_X 1$. <u>In</u> this sense the structure of <u>End</u> $\underline{E}_X$ is more subtle than that of $\underline{E}_X$.

The kernel of ω_X in (1.2.0) $= (\underline{O}^{\oplus r^2}_X \cap \underline{K}_X)$ is given as follows:

<u>Lemma 1.3.2</u>. (1) $(\underline{K}_X \cap \underline{O}^{\oplus r^2}_X)_{|N_{1,i}}$ <u>coincides with the kernel of the morphism as follows:</u>

$$(1.3.3) \qquad \underline{O}^{\oplus r^2}_X \ni C \to \underline{O}^{\oplus(r-1)}_X \ni \omega_{|N_{1,i}} T''_{|N_{1,i}}(C).$$

(2) <u>There is an injection</u>: $\Gamma(\underline{O}_X) \xrightarrow{j} \Gamma(\underline{O}^{\oplus r^2}_X \cap \underline{K}_X)$.

Proof. The first fact is obtained by restricitng Lemma 1.3.1 to $(\underline{O}_X^{\oplus r^2}\cap\underline{K}_X)$. Next we rewrite $T''_{\alpha\alpha(i)}$ as follows:

$$\text{(a)}\quad \begin{aligned}T''_{\alpha\alpha(i)}(z)&=z_{\alpha\alpha(i)}+f_\alpha(z_{\alpha(i)\alpha(i)}-z_{\alpha\alpha})-f_\alpha(\Sigma_{\gamma\in\Delta_r-\{\alpha(i)\}}f_\gamma z_{\alpha(i)\gamma})\\&-\Sigma_{\gamma\neq\alpha,\alpha(i)}\ f_\gamma z_{\alpha\gamma}\end{aligned}$$

Define an injection $j:\underline{O}_X\ni g\to\underline{O}_X^{\oplus r^2}\ni gI_r$. Claerly $T''_{N_{1,i}}(gI_r)=0$, and we have (2). q.e.d.

A.1.4. Here we extend the diagram in Figure I to $\bar{X}$. As before we write $\omega^{-1}_{|X}(\underline{K}_X1)$ as $\underline{K}'_X$. Let $\underline{K}_{\bar X}$ and $\underline{K}'_{\bar X}$ be the direct images of $\underline{K}_X$ and $\underline{K}'_X$ with respect to the inclusion $i:X\ \ \bar{X}$. Define an $\underline{O}_{\bar X}1$-module as follows:

$$(1.4.0.1)\qquad \underline{K}'_{\bar X}1\ :=\omega(\underline{K}'_{\bar X}\ )(\subset(\underline{L}_{\bar X}1)^{\oplus r^2}).$$

The structure of $\underline{K}'_{\bar X}$ is simpler than that of $\underline{K}_{\bar X}$. We chiefly discuss that of $\underline{K}'_{\bar X}$. Our analysis of $\underline{K}'_{\bar X}$ and $\underline{K}_{\bar X}$ are summarized in the following diagram:

Figure III

$$\begin{array}{ccccccc}
 & & & & & & \underline{F}_{\bar X}1\ \subset\oplus_i\underline{I}_{\bar D_i}\otimes\underline{L}_{\bar X_i}^{\oplus(r-1)}\\
 & & & \underline{K}_{\bar X}(\simeq i_*\underline{End}\ \underline{E}_X) & & & \cup\!\|\\
 & & & \cap & & & \underline{G}_{\bar X}1\ \subset\oplus_i(\underline{I}_{\bar D_i})\underline{L}_{\bar X_i}^{\oplus(r-1)}\\
 & & & & & & \cup\!\|\\
0\to & \underline{O}_{\bar X}^{\oplus r^2} & \to & \underline{K}'_{\bar X} & \overset{\omega}{\to} & & \underline{K}_{\bar X}1\ \to 0\\
 & \cup\!\| & & \cap & & & \cap\\
0\to & \underline{O}_{\bar X}^{\oplus r^2} & \to & \underline{L}_{\bar X}^{\oplus r^2} & \overset{\omega}{\to} & & (\underline{L}_{\bar X}1)^{\oplus r^2}\to 0
\end{array}$$

(Here i is in Δ_m. The divisor $\bar{D}_i=\bar{D}_i^2$ of $\bar{X}_i$ is as in (0.1.2).) The key point in the diagram is to define the $\underline{O}_{\bar X}1$-modules $\underline{G}_{\bar X}1$ and $\underline{F}_{\bar X}1$. For this we first define $\underline{O}_{\bar X_i}$-submodules $\underline{G}_{\bar X_i}$ and $\underline{F}'_{\bar X_i}$ of $\underline{L}_{\bar X_i}^{\oplus(r-1)}$, $i\in\Delta_m$, by the following holomorphy condition on χ_{S_i}, cf.(1.2.0).(In (1.4.1) below, $p\in\bar{X}_i$, and $\underline{a}_i$ and $\underline{b}_i$ are elements of $\underline{L}_{\bar X_i,p}^{\oplus(r-1)}$.)

(1.4.1.1) $\underline{a}_i$ is in $\underline{G}_{\bar X_i,p}$, if and only if $S_i(\underline{a}_{i|X_{i,p}})(\in\underline{L}_{X_i,p}^{\oplus r^2})$ is the restriction of a (unique) $A_i\in\underline{L}_{\bar X_i,p}^{\oplus r^2}$.

(1.4.1.2) $\underline{b}_i$ is in $\underline{F}'_{\bar X_i,p}$ if and only if $S_{i,\alpha(i)\alpha(i)}(\underline{b}_i)(\in\underline{L}_{X_i,p})$ is the restriction of a (unique) $a_i \in \underline{L}_{\bar{X}_i}$.

From the explicit forms of S_i, cf.(1.2.0.1) and of the divisor $\bar{D}_i$, cf. (0.1.3), we see easily the following:

$\underline{G}_{\bar{X}_i} \subset \underline{I}_{\bar{D}_i}\underline{L}_{\bar{X}_i}^{\oplus(r-1)}, i \in \Delta_m$. Moreover, $\underline{a}_i \in \underline{L}_{\bar{X}_i}^{\oplus(r-1)}$ is in $\underline{I}_{\bar{D}_i}\underline{L}_{\bar{X}_i}^{\oplus(r-1)}$

(1.4.1.3) if and only if $S_{i,\alpha\beta}(\underline{a}_i)$ satisfies the holomorphy condition in (1.4.1.1) except $(\alpha,\beta) \in \{(\alpha,\alpha(i)\}_{\alpha \neq \alpha(i)}$

We define an extension of χ_{S_i} to $\bar{X}_i$ as follows:

(1.4.1.4) $\bar{\chi}_{S_i} : \underline{G}_{\bar{X}_i} \ni \underline{a}_i \to \underline{L}_{\bar{X}_i}^{\oplus r^2} \ni A_i$, where A_i satisfies: $A_{i|X_i} = \chi_{S_i}(\underline{a}_{i|X_i})$.

Moreover, take an element $u_i \in \Gamma(\underline{O}_{\bar{X}_i}[\bar{D}_i])$ such that $(u_i)_0 = \bar{D}_i$. We write u_i also for the isomorphism: $\underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}^{\oplus(r-1)} \ni \underline{b}_i \to (\underline{I}_{\bar{D}_i}\underline{L}_{\bar{X}_i})^{\oplus(r-1)} \ni u_i \otimes \underline{b}_i$. Now, the $\underline{O}_{\bar{X}}$-modules $\underline{G}_{\bar{X}}1$ and $\underline{F}_{\bar{X}}1$ are as follows:

$$\underline{G}_{\bar{X}}1 = \{\oplus_{i\in\Delta_m}\underline{a}_i \in \oplus_i \underline{G}_{\bar{X}_i} ; \bar{\chi}_{S_i}(\underline{a}_i) = \bar{\chi}_{S_j}(\underline{a}_j) \text{ on } \bar{X}_{ij}; i,j \in \Delta_m\} \subset \oplus_i \underline{L}_{\bar{X}_i}^{\oplus(r-1)}$$

(1.4.2) $$\underline{F}_{\bar{X}}1 = \{\oplus_{i\in\Delta_m}\underline{b}_i \in \oplus_i \underline{I}_{\bar{D}_i} \otimes \underline{F}'_{\bar{X}_i} ; \bar{\chi}_{S_i}(u_i \otimes id)(\underline{b}_i) = \bar{\chi}_{S_j}(u_j \otimes id)(\underline{b}_j) \text{ on } \bar{X}_{ij}; i,j \in \Delta_m\} \subset \oplus_i \underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}^{\oplus(r-1)}$$

<u>Lemma 1.4. The following three $\underline{O}_{\bar{X}}1$-modules are isomorphic</u>.

(1.4.3) $$\underline{F}_{\bar{X}}1 \xrightarrow{\sim} \underline{G}_{\bar{X}}1 \xrightarrow{\sim} \underline{K}_{\bar{X}}1 .$$

<u>Proof</u>. First define an $\underline{O}_{\bar{X}_i}$-submodule $\underline{K}'_{\bar{X}_i}$ of $\underline{L}_{\bar{X}_i}^{\oplus r^2}$ as follows:

(1.4.4.1) $\underline{K}'_{\bar{X}_i} = \{A_i \in \underline{L}_{\bar{X}_i}^{\oplus r^2} ; A_{i|X_i} \in \underline{K}_{X_i}\}$, cf. Lemma 1.2.

Let $(res)_i$ denote the restriction morphism: $\underline{L}_{\bar{X}}1^{\oplus r^2} \to \underline{L}_{\bar{X}_i}^{\oplus r^2}$. Then we have the isomorphism:

(1.4.4.2) $\oplus_{i\in\Delta_m}(res)_i : \underline{K}'_{\bar{X}}1 \simeq \{\oplus_{i\in\Delta_m} A_i \in \underline{K}'_{\bar{X}_i} ; A_i = A_j \text{ on } \bar{X}_{ij}; i,j \in \Delta_m\}$.

On the otherhand, the holpmorphy condition (1.4.1.1) insures:

(a) $\bar{\chi}_{S_i}$ gives an isomorphism: $\underline{G}_{\bar{X}_i} \to \underline{K}'_{\bar{X}_i}$.

From the definition of $\underline{G}_{\bar{X}_i}$, cf.(1.4.2), we see that the following three $\underline{O}_{\bar{X}}1$-moduls are isomorphic:

(b) $\underline{K}'_{\bar{X}}1 \to$ R.H.S. in (1.4.4.2) $\to \underline{G}_{\bar{X}}1$.

Next we have:

(c) $u_i \otimes (\mathrm{id})$ gives an isomorphism: $\underline{I}_{\bar{D}_i} \otimes \underline{F}'_{\bar{X}_i} \; (\subset \underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}^{\oplus(r-1)}) \to \underline{G}_{\bar{X}_i} \; (\subset \underline{L}_{\bar{X}_i}^{\oplus(r-1)})$

Actually, by (1.4.1.3) $\underline{G}_{\bar{X}_i}$ is the $O_{\bar{X}_i}$-submodule of $\underline{I}_{\bar{D}_i}\underline{L}_{\bar{X}_i}^{\oplus(r-1)}$. Take a frame g_i of $\underline{I}_{\bar{D}_i}$ and an element $\underline{a}_i \in \underline{L}_{\bar{X}_i}^{\oplus(r-1)}$. By (1.4.1.3) $g_i\underline{a}_i$ is in $\underline{G}_{\bar{X}_i}$ if and only if the $(\alpha,\alpha(i))$-components, $\alpha\neq\alpha(i)$, of $\bar{\chi}_{S_i}(g_i\underline{a}_i)$ is in $\underline{L}_{\bar{X}_i}$. From the explicit form of S_i, cf.(1.2.0.1), we see that the condition is equivalent to:

(d) $\quad S_{i,\alpha(i)\alpha(i)}(\underline{a}_i) \in \underline{L}_{\bar{X}_i}$.

From this we get readily (c). By (c) we have the isomorphism:

(1.4.5.2) $\quad \underline{F}_{\bar{X}}1 \quad \underline{G}_{\bar{X}}1$. $\qquad$ q.e.d.

Figure III

$$\begin{tikzcd}
\underline{K}'_{\bar{X}}1 \arrow[r, phantom, "\subset"] & \oplus_i \underline{K}'_{\bar{X}_i} \arrow[r, phantom, "\subset"] & \underline{L}_{\bar{X}_i}^{\oplus r^2} & (\oplus_i \bar{\chi}_{S_i}) \\
\underline{G}_{\bar{X}}1 \arrow[u, phantom, sloped, "\cong"] \arrow[r, phantom, "\subset"] & \oplus_i \underline{G}_{\bar{X}_i} \arrow[u] \arrow[r, phantom, "\subset"] & \oplus_i (\underline{I}_{\bar{D}_i}\underline{L}_{\bar{X}_i})^{\oplus(r-1)} \arrow[u] & \\
\underline{F}_{\bar{X}}1 \arrow[u, phantom, sloped, "\cong"] \arrow[r, phantom, "\subset"] & \oplus_i \underline{I}_{\bar{D}_i} \otimes \underline{F}'_{\bar{X}_i} \arrow[u, phantom, sloped, "\cong"] \arrow[r, phantom, "\subset"] & \oplus_i \underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}^{\oplus(r-1)} \arrow[u] & (\oplus_i u_i \otimes (\mathrm{id}))
\end{tikzcd}$$

A.2. Structure of $\Gamma(\underline{\mathrm{End}}\ \underline{E}_X)$

Here we discuss the structure of $\Gamma(\underline{\mathrm{End}}\ \underline{E}_X)$, by focussing on the simpleness of $\underline{E}_X$ (i.e. $\Gamma(\underline{E}_X) \simeq \mathbb{C}$). The discussions will be done by analyzing the following diagram;cf.Figure II and III:

Figure IV

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \underline{O}_{\bar{X}}^{\oplus r^2} \cap \underline{K}_{\bar{X}} \arrow[d, phantom, sloped, "\subset"] & \underline{O}_{\bar{X}}^{\oplus r^2} \arrow[d] & & \\
i_*\underline{\mathrm{End}}\ \underline{E}_X \arrow[r, phantom, "\simeq"] & \underline{K}_{\bar{X}} & \underline{K}_{\bar{X}} \arrow[d] & & \\
 & & \underline{K}_{\bar{X}}1 \arrow[d] \arrow[r, phantom, "\simeq"] & \underline{G}_{\bar{X}}1 \arrow[r, phantom, "\simeq"] & \underline{F}_{\bar{X}}1 \\
 & & 0 & &
\end{tikzcd}$$

Now, by (1.2.0) we have:

(2.1.0) $\qquad 0 \to \Gamma(\underline{O}_X^{\oplus r^2} \cap \underline{K}_X) \to \Gamma(\underline{K}_X) \xrightarrow{\omega} \omega\Gamma(\underline{K}_X) \to 0$

We analyze the first and third terms. Concerning the first, Lemma 1.3.2 insures: $\Gamma(\underline{O}_X) \simeq j\Gamma(\underline{O}_X) \subset \Gamma(\underline{O}_X^{\oplus r^2} \cap \underline{K}_X)$, with the diagonal imbedding $j: \underline{O}_X \to \underline{O}_X^{\oplus r^2}$. The following is obvious:

Lemma 2.1. (1) $\Gamma(\underline{K}_X) = j\Gamma(\underline{O}_X)(\simeq \Gamma(\underline{O}_X))$ if and only if

(2.1.1) $\quad j\Gamma(\underline{O}_X) = \Gamma(\underline{O}_X^{\oplus r^2} \cap \underline{K}_X)$ and $\omega\Gamma(\underline{K}_X) = 0$.

(2) Assume that $\bar{X}$ is compact. Then we have:

(2.1.2) $\quad \Gamma(\mathrm{End}\ \underline{E}_X) \simeq C(\simeq \Gamma(\underline{O}_X))$, if and only if $\Gamma(X, \underline{O}_X^{\oplus r^2} \cap \underline{K}_X) \simeq j\Gamma(\underline{O}_X) \simeq C$ and $\omega\Gamma(\underline{K}_X) \simeq 0$.

Our criterion for the simpleness of $\underline{E}_X$ is based on (2.1.2). First concerning the first condition in (2.1.1,2) we have:

Lemma 2.2.1. An element $C \in \Gamma(\underline{O}_X^{\oplus r^2})$ is in $\Gamma(\underline{O}_X^{\oplus r^2} \cap \underline{K}_X)$, if and only if

(2.2.1) $\quad T''_{\alpha\alpha(i)}(C) = 0$ on X_i^1 for each $i \in \Delta_m$ and $\alpha \in \Delta_r - \{\alpha(i)\}$

For the explcit form of T', see (1.3.0).

Proof. Immediate from Lemma 1.3.2. q.e.d.

When $\bar{X}$ is compact, $\Gamma(\underline{O}_X) \simeq M_r(C)$, and (2.2.1) is reduced to a problem of linear algebra:

(2.2.2) $\quad T''_{\alpha\alpha(i)|X_i}(C) = 0; \alpha \in \Delta_r - \{a(i)\}$, with $C \in M_r(C)$.

(2.2.2) $\quad T''_{\alpha\alpha(i)|X_i}(C) = 0; \alpha \in \Delta_r - \{a(i)\}$, with $C \in M_r(C)$.

Writing C as $[c_{\alpha\beta}]$, $\alpha,\beta \in \Delta_r$, this relation is as follows;cf. Lemma 1.3.2:

$$(2.2.3) \quad \begin{aligned} &c_{\alpha\alpha(i)} + (c_{\alpha(i)\alpha(i)} - c_{\alpha\alpha}) f_{\alpha|\bar{X}_i} - \Sigma_{\gamma\neq\alpha,\alpha(i)} c_{\alpha\gamma} f_\gamma - \\ &- f_\alpha (\Sigma_{\gamma\in\Delta_r - \{\alpha(i)\}} c_{\alpha(i)\gamma} f_{\gamma|\bar{X}_i}) = 0 \ . \end{aligned}$$

Thus we have:

<u>Lemma 2.2.2</u>. <u>Assume that, for an</u> $i \in \Delta_m$, <u>the following holds:</u>

(2.2.4) $\quad f_{\gamma|\bar{X}_i}$, $(f_\alpha f_\gamma)|\bar{X}_i$; $\gamma \in \Delta_r - \{\alpha(i)\}$, <u>and 1(=the constant function with the value 1) are lineraly independent over C, for each</u> $\alpha \in \Delta_r - \{\alpha(i)\}$.

<u>Then</u> $\Gamma(\underline{K}_X \cap \underline{O}_X^{\oplus r^2})(\subset M_r(C))$ <u>is isomorpic to</u> C.

Next we analyze the term:$\omega\Gamma(\underline{K}_X) \simeq 0$, by using the folowing diagram;cf.Figure IV.

<u>Figure V</u>

$$\begin{aligned} \omega\Gamma(\underline{K}_X) \subset \omega\Gamma(\underline{K}'_X) \simeq \omega\Gamma(\underline{K}'_{\bar{X}}) (=\Gamma(\underline{K}'_{\bar{X}}1) \cap \omega\Gamma(\underline{L}_{\bar{X}}^{\oplus r^2})) &\subset \Gamma(\underline{K}'_{\bar{X}}1) \\ \Gamma(\underline{G}_{\bar{X}}1) \cap (\oplus_i \bar{\gamma}_{S_i})^{-1}(\omega\Gamma(\underline{L}_{\bar{X}}^{\oplus r^2})) &\subset \Gamma(\underline{G}_{\bar{X}}1) \end{aligned}$$

The following is useful in later arguments:

<u>Lemma 2.3</u>. <u>The C-module</u> $\omega\Gamma(\underline{K}_X) \simeq 0$, <u>if</u>

(2.3.1) $\quad \Gamma(\underline{G}_{\bar{X}}1) \cap (\oplus_i \bar{\gamma}_{S_i})^{-1}(\omega\Gamma(\underline{L}_{\bar{X}}^{\oplus r^2}))$ <u>or</u> $\Gamma(\underline{G}_{\bar{X}}1) \simeq 0$

Note that if $\omega:\Gamma(\underline{L}_{\bar{X}}) \to \Gamma(\underline{L}_{\bar{X}}1)$ is surjective, the two conditions in (2.3.1) are equivalent. By (1.4.1.3), $\underline{G}_{\bar{X}_i} \subset \underline{I}_{\bar{D}_i} L_{\bar{X}_i}^{\oplus(r-1)}$. Thus we have:

(2.3.3) $\quad$ <u>If</u> $\Gamma(\underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}) \simeq 0$ for each $i \in \Delta_m$, then $\omega\Gamma(\underline{K}_{\bar{X}}) \simeq 0$.

See <u>A.3</u> and <u>A.4</u> below for the applications of Lemma 2.2 and 2.3. See also Part B,§ 4,[Sa-1] for them.

<u>A 3. Case of rank two</u>

Here we assume that rank $\underline{E}_X = 2$ and $\alpha(i) = 2$ for each $i \in \Delta_m$. Thus the matrix H is of the form: $\begin{matrix} 1 & f_1 \\ 0 & f_2 \end{matrix}$.(The role of f_2 is just:

$(f_2)_0 = X^1$, while f_1 determines the structure of $\underline{K}'_X 1$, cf. (1.3.0).) The divisor $\bar{D}_i = \bar{D}_i^2$ is the pole divisor of $f_{1|\bar{X}_i}$, cf. (0.1.3). We fix an element $u_i \in \Gamma(\underline{O}_{\bar{X}_i}[\bar{D}_i])$ satisfying: $(u_i)_0 = \bar{D}_i$.

A.3.1. First the $\underline{O}_{\bar{X}_i}$-modules $\underline{G}_{\bar{X}_i}$ and $\underline{F}'_{\bar{X}_i}$, cf. (1.4.1), are as follows:

Lemma 3.1. (1) $\underline{G}_{\bar{X}_i} = (\underline{I}_{\bar{D}_i})^2 \underline{L}_{\bar{X}_i}$ and $\underline{F}'_{\bar{X}_i} = \underline{I}_{\bar{D}_i} \underline{L}_{\bar{X}_i}$.

(2) The following three $\underline{O}_{\bar{X}_i}$-modules are isomorphic; cf. Lemma 1.4:

(3.1.1) $$\underline{I}_{\bar{D}_i}^{\otimes 2} \otimes \underline{L}_{\bar{X}_i} (\simeq \underline{I}_{\bar{D}_i} \otimes \underline{F}'_{\bar{X}_i}) \xrightarrow{u_i^{\otimes 2} \otimes \mathrm{id}} (\underline{I}_{\bar{D}_i})^2 \underline{L}_{\bar{X}_i} \xrightarrow{\bar{\chi}_{S_i}} \underline{K}'_{\bar{X}_i},$$

where $\bar{\chi}_{S_i}$ is as follows:

(3.1.2) $$(\underline{L}_{\bar{X}_i} \supset) \underline{G}_{\bar{X}_i} \ni b \to \underline{K}_{\bar{X}_i} \ni b \begin{pmatrix} -f_1 & -f_1^2 \\ 1 & f_1 \end{pmatrix}_{|\bar{X}_i}$$

Proof. First (3.1.2) is clear from the definition of S_i, cf. (1.2.0.1). Next the defining equation of $\underline{F}'_{\bar{X}_i}$ is; cf. (1.4.1.2):

(a) $(f_{1|\bar{X}_i})b \in \underline{L}_{\bar{X}_i}$, for an element $b \in \underline{L}_{\bar{X}_i}$.

Thus we have: $\underline{F}'_{\bar{X}_i} \simeq \underline{I}_{\bar{D}_i} \otimes \underline{L}_{\bar{X}_i}$. The other parts in the lemma are immediate from Lemma 1.4. q.e.d.

The following is also clear from Lemma 1.4:

Corollary. (1) $\underline{K}'_{\bar{X}} 1 (= \omega \underline{K}'_{\bar{X}})$ is isomorphic to:

(3.2.1) $$\{\oplus_{i \in \Delta_m} b'_i \in (\underline{I}_{\bar{D}_i})^2 \otimes \underline{L}_{\bar{X}_i}; \bar{\chi}_{S_i}(u_i^{\otimes 2} \otimes (\mathrm{id}))(b'_i) = \bar{\chi}_{S_j}(u_j^{\otimes 2} \otimes (\mathrm{id}))(b'_j) \text{ on } \bar{X}_{ij}; i,j \in \Delta_m\}$$

(2) We have:

(3.2.2) $$\Gamma(\underline{K}'_{\bar{X}} 1) \simeq 0, \text{ if } \Gamma((\underline{I}_{\bar{D}_i})^{\otimes 2} \otimes \underline{L}_{\bar{X}_i}) = 0 \text{ for each } i \in \Delta_m.$$

Take an element $A = [a_{ij}] \in \underline{K}'_{\bar{X}} (\subset \underline{L}_{\bar{X}}^{\oplus 4})$, $i,j = 1,2$. Then Lemma 1.3.1 is written as follows:

(3.2.3) $$A \text{ is in } \underline{K}_{\bar{X}} (\simeq i_* \underline{\mathrm{End}}\ \underline{E}_X) \text{ if and only if: } (a_{22} - a_{21}) f_1 + (a_{12} - a_{21} f_1^2) \equiv 0 \ ((\underline{I}_X 1)^2).$$

Moreover, an element $C = [c_{ij}] \in \underline{O}_X^{\oplus 4}$, $i,j = 1,2$, is in $\underline{K}_X \cap \underline{O}_X^{\oplus 4}$ if and only if:

(3.2.4) $$(c_{22} - c_{21}) f_1 + (c_{12} - c_{21} f_1^2) = 0 \text{ on } X^1.$$

Assume that $\bar{X}$ is compact. Then $\Gamma(\underline{K}_X-\underline{O}_{\bar{X}}^{\oplus 4})$ is the C-subspace of $M_2(C)$ $\simeq\Gamma(\underline{O}_X^{\oplus 4})$ consisting of those elemnents $C=[c_{\alpha\beta}];\alpha,\beta=1,2$, satisfying

(3.2.5) $\quad (c_{22}-c_{11})f_{1|X^1}+c_{12}-c_{21}f_{1|X^1}^2=0.$

Thus we have:

Lemma 3.3. $\Gamma(\underline{O}_X^{\oplus 4}\cap\underline{K}_X)\simeq C(\simeq\Gamma(\underline{O}_X))$ if and only if:

(3.3.1) $\quad 1,\ f_{1|X^1}$ and $(f_{1|X^1})^2$ are linearly independent over C.

The following criterion for the simplicity of $\underline{E}_X$ is useful.

Lemma 3.4. $\Gamma(\underline{K}_X)(\simeq\Gamma(\underline{\mathrm{End}}\ \underline{E}_X))\simeq C$, if (3.3.1) holds and

(3.3.2) $\quad \{\oplus_{i\in\Delta_m}a_i\in\Gamma((\underline{I}_{\bar{D}_i})^2\underline{L}_{\bar{X}_i});\ \bar{\chi}_{S_i}(a_i)=\bar{\chi}_{S_j}(a_j)$ on $\bar{X}_{ij};i,j\in\Delta_m\}$
$\bar{X}_{ij*}=0;\ i,j\in\Delta_m\}\simeq 0.$

We make some remarks on this lemma. First, if m=1(i.e., if $\bar{X}^1$ is irreducile), (3.3.2) is equivalent to:

(3.3.3) $\quad \Gamma((\underline{I}_{\bar{D}})^2\underline{L}_{\bar{X}^1})=0$, where $\bar{D}$ is the pole divisor of $f_{1|\bar{X}^1}$.

Assume tha m 2. Also assume the following for each $i\neq j\subset\Delta_m$.

(3.3.4.1) $\quad u_i$ vanishes on each irreducible component $\bar{X}_{ij,\lambda}^2$ of $\bar{X}_{ij}^2$.

By Lemma 3.1, each $a_i\in\Gamma((\underline{I}_{\bar{D}_i})^2\underline{L}_{\bar{X}_i})$ is written as $b_i\otimes u_i^{\otimes 2}$, with $b_i\in\Gamma((\underline{I}_{\bar{D}_i})^2\otimes\underline{L}_{\bar{X}_i})$. Writing $f_{1|\bar{X}_i}$ as v_i/u_i with $v_i\in\Gamma(\underline{L}_{\bar{X}_i})$, we see that the (1,2)-component of $\bar{\chi}_{S_i}(b_i\otimes u_i^{\otimes 2})=b_i\otimes v_i^{\otimes 2}$ while the other (α,β)--component =0 on $\bar{X}_{ij}$. Thus the condition in (3.3.2) is rewritten as:

(3.3.4.2) $\quad b_i\otimes v_i^{\otimes 2}=b_j\otimes v_j^{\otimes 2}$ on $\bar{X}_{ij}$.

A 4. Bundles in § 1.3

Here we prove the following

Theorem. The bundles E_X in Theorem 1.2.1 1.3.2 are simple.

The proof is given by using the criterions in Lemma 2.1 2.3. We use freely the notations in this appendix and in §1.3. We should ditinguish them. When we quote a result in § 1.3, we write it as (1.3.1),§ 1.3. When we write simply (1.3.1), it is a result in this appendix. As in § 5.2 we make a remark as follows:

(*) It is not the line bundle $\underline{L}_{\overline{X}}$ in § 1.3 but $\underline{L}_{\overline{X}}^{\otimes m}$, that corresponds to the bundle $\underline{L}_{\overline{X}}$ hitherto in this appendix.

As previously, we write $\overline{X}_i$ for $\overline{X}_i^1$. Also we write $\underline{L}_{\overline{X}_i}$ for $\underline{L}_{\overline{X}|\overline{X}_i}$.

A.4.0. First the pole divisor $\overline{D}_i^2$ of the meromorphic functions on $\overline{X}_i$, cf. (0.1.3.2), is the locus of the following elements ; cf. § 1.3.

(4.0.0.1) $s_i^{\wedge}$; for Theorem 1.2.1 and 1.3.1, $g_{i,2}$; for Theorem 1.2.2, and $g_{i,m}$; for Theorem 1.3.2

Thus we have:

(4.0.0.2) $\underline{I}_{\overline{D}_i^2} = \underline{L}_{\overline{X}}^{\otimes(1-m)}$; for Theorem 1.2.1 and 1.3.1, 1.3.2

$= \underline{L}_{\overline{X}_i}^{\otimes(-2n)}$; for Theorem 1.2.2. (Here $m = 4n+1$.)

A.4.1. Here we determine $\underline{O}_{\overline{X}_i}$-modules $\underline{G}_{\overline{X}_i}$, cf. (1.4.1). First, we have:

Lemma 4.1.1. According as we are concerned with Theorem 1.2.1 or 1.2.2, cf. § 1.3, we have:

(4.1.1) $\underline{G}_{\overline{X}_i} \simeq \underline{L}_{\overline{X}_i}^{\otimes(2-m)}$ or $\simeq \underline{L}_{\overline{X}_i}$.

Proof. By Lemma 3.1, this is clear once we recall that rank $\underline{E}_X = 2$ for Theorem 1.2.1 and 1.2.2. q.e.d.

The rank of the bundle $\underline{E}_X$ in Theorem 1.3.1 and 1.3.2 $\geqq 3$, and $\underline{G}_{\overline{X}_i}$ is not so simple. We determine $\underline{G}_{\overline{X}_i}$ as follows; define an $\underline{O}_{\overline{X}_i}$-submodule $\underline{F}''_{\overline{X}_i}$ of $\underline{L}_{\overline{X}_i}^{\oplus(r-1)}$ by the following holomorphy conditon; cf. also (1.4.1.2):

(4.1.3) An element $\underline{b}_i = (b_{i,\gamma})_{\gamma\in\Delta_{r-1}} \in \underline{L}_{\overline{X}_i}^{\oplus(r-1)}$ is in $\underline{F}''_{\overline{X}_i}$ if and only if:

$S_{i,rr}(\underline{b}_i) \in \underline{L}_{\overline{X}_i}$.

By (1.2.0.1) and the explicit form of the meromorphic functions f_α in follows:

(4.1.4) $\Sigma_{\gamma=1}^{r-1} u_{\gamma|\overline{X}_i} \otimes b_{i,\gamma} \equiv 0 (\mathrm{mod}.\, s^{\wedge}_{i|\overline{X}_i})$, $i \in \Delta_m$; for Theorem 1.3.1,

$\Sigma_{\gamma=1}^{m-1} g_{i,\gamma|\overline{X}_i} \otimes b_{i,\gamma} \equiv 0 (\mathrm{mod}.\, g_{i,r})$; for Theorem 1.3.2

According as we are concerned with Theorem 1.3.1 or 1.3.2, the multiplication by $s_i^{\wedge}$ and $s_m^{\otimes(m-1)}$, $s_m^{\wedge}$ define an $\underline{O}_{\bar{X}_i}$-morphism: $\underline{L}_{\bar{X}_i} \to (\underline{L}_{\bar{X}_i})^{\otimes m}$. For each $i \in \Delta_m$ we have:

Lemma 4.1.2. In the both theorems: Theorem 1.3.1 and 1.3.2 the above $\underline{O}_{\bar{X}_i}$-morphism gives an isomorphism:

$$(4.1.5) \qquad \underline{F}''_{\bar{X}_i}(\subset \underline{L}_{\bar{X}_i}^{\oplus(r-1)}) \simeq \underline{G}_{\bar{X}_i}(\subset (\underline{L}_{\bar{X}_i}^{\otimes m})^{\oplus(r-1)}) .$$

Proof. By Lemma 1.4, $\underline{G}_{\bar{X}_i} \simeq \underline{I}_{\bar{D}_i} \otimes \underline{F}'_{\bar{X}_i}$, where the $\underline{O}_{\bar{X}_i}$-submodule $\underline{F}'_{\bar{X}_i}$ of $(\underline{L}_{\bar{X}_i}^{\otimes m})^{\oplus(r-1)}$ is defined by the similar manner to (4.1.3); cf. (1.4. . .). We have easily (4.1.5) from this. q.e.d.

A.4.2. Here we check the simpleness of the bundle $\underline{E}_X$ in § 1.3. For this it suffices to check the following; cf. Lemma 2.1:

$$(4.2.0) \qquad \Gamma(\underline{K}_X \cap \underline{O}_X^{\oplus r^2}) \simeq C \quad \text{and} \quad \omega\Gamma(K_X) = 0 .$$

The first fact in (4.2.0) is clear, once we compare the conditions on the linearly independence of the meromorphic functions in Theorem 1.2.1 1.3.2; cf. (1.3.1.2), (1.3.2.2)', (1.3.3.2), (1.3.4.4), cf. § 1.3 with those in Lemma 2.2.2 and Lemma 3.3.1.

Next we show the second fact by checking that $\Gamma(\underline{G}_{\bar{X}}1) = 0$; cf. Lemma 2.2: For the bundle $\underline{E}_X$ in Theorem 1.2.1, $\underline{G}_{\bar{X}_i} \equiv \underline{L}_{\bar{X}_i}^{\otimes(2-m)}$; cf. (4.1.1). Thus if $m_= 2$ we have: $\Gamma(\underline{G}_{\bar{X}_i}) = 0$. If $m = 2$, $\underline{G}_{\bar{X}_i} \simeq \underline{O}_{\bar{X}_i}$ and the isomorphism is given by the multiplication by $(s_i^{\wedge})^{\otimes 2}$, cf. Lemma 1.4. Thus an element of $\Gamma(\underline{G}_{\bar{X}_i})$ is written as $c_i(s_i^{\wedge})^{\otimes 2}$, with $c_i \in C$. By (3.3.4.2) $\Gamma(\underline{G}_{\bar{X}}1)$ consists of those elements: $c_1(s_1^{\wedge})^{\otimes 2} \oplus c_2(s_2^{\wedge})^{\otimes 2}$ satisfying $c_i t_i^{\otimes 2} = c_j t_j^{\otimes 2}$. But (1.3.1.3), § 1.3.1 implies that $c_1 = c_2 = 0$, and we have: $\Gamma(\underline{G}_{\bar{X}}1) = 0$. Next, for the bundle in Theorem 1.2.2, we have: $\underline{G}_{\bar{X}_i} \simeq \underline{L}_{\bar{X}_i}$, cf. Lemma 4.1.1, and the identification is given by the multiplication by $g_{i,2} = \Pi_{\alpha=1}^{n} s_{i \pm \alpha}$. By a similar argument to the above $\Gamma(\underline{G}_{\bar{X}}1)$ consists of those elements $\oplus_{i \in \Delta_m} c_i g_{i,2}^{\otimes 2}$ with $c_i \in \Gamma(\underline{L}_{\bar{X}_i})$ satisfying: $c_i g_{i,1}^{\otimes 2} = c_j g_{j,1}^{\otimes 2}$ on $\bar{X}_{ij}$ with $g_{i,1} = \Pi_{\alpha=1}^{n} s_{i \pm n \pm \alpha}$. Consider the case j case j $=i+1$. Then the relation is: $c_i s^{\otimes 2}_{i+n+1} = c_{i+1} s^{\otimes 2}_{i+3n+1}$. By the condition (1.3.2.2),§ 1.3.1, we have: $c_i = 0$ and $\Gamma(\underline{G}_{\overline{X}}1) = 0$. Finally, for the bundles $\underline{E}_X$ in Theorem 1.3.1 and 1.3.2, compare (4.1.4) with (1.3. .) and (1.3. .),§ 1.3. Then we have: $\Gamma(\underline{G}_{\overline{X}_i}) = 0$; $i \in \Delta_m$. Thus we finish the proof of Theorem.

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