数理解析研究所講究録
634巻 1987年 260-270

Cyclic cohomology of Connes

Hiroshi TAKAI
Department of Mathematics
Tokyo Metropolitan University

§1 Introduction   Recent progress of geometry and topology is heavily related to that of analysis. One of the most powerful tools to study those areas is certainly the index theory of Atiyah-Singer, which tells us a quite deep connection between elliptic differential operators and topological characteristics. Retaining their ideas, Connes[2] initiated to construct an index theory of non-commutative manifolds whose special case is reduced to that of Atiyah-Singer. The crutial conception of his new index theory is based on K-theory of C^*-algebras,whereas its real computation may be much harder than that of its dual ingredient,namely cyclic cohomology. He actually seemed to define it in order to understand K-theory more sharply. It is a generalization of de Rham homology of smooth manifolds as well as of Eilenberg-Maclane cohomology of groups. Succeedingly, Loday-Quillen developed the theory of cyclic homology which may be a dual notion of cyclic cohomology. In what follows, we only treat basic properties of cyclic cohomology and state its application to the index theory of foliated manifolds.

§2 Preliminaries   Let M be a closed oriented smooth manifold and denote by Φ_M the C^*-algebra generated by the set of all pseudo differential operators of order zero on $L^2(M)$. Then Φ_M contains the C^*-algebra Ψ_M of all compact operators on $L^2(M)$ as a closed ideal by which the quotient algebra Φ_M/Ψ_M is isomorphic to $C(S^*(M))$,the set of all continuous functions on the cosphere bundle $S^*(M)$ over M. Let σ_M be the symbol mapping from Φ_M to $C(S^*(M))$ and $[\Phi_M]$ the canonical element in Ext($S^*(M)$),which is nothing but the K-homology $K_*(S^*(M))$ of $S^*(M)$). Let ch_*, ch^* be the Chern characters of $K_*(S^*(M), K^*(S^*(M))$ into $H_*(S^*(M);\mathbb{Q}), H^*(S^*(M);\mathbb{Q})$ respectively. Then $ch_*(\Phi_M)$ is equal to $Td(S^*(M)) \wedge [S^*(M)]$,where $Td(\cdot), [\cdot]$ are Todd,fundamental classes of $\cdot$ respectively. If M has a spin^c structure, then $ch_*(\Phi_M)$ is equal to $Td(T_{\mathbb{C}}M) \wedge [M]$. For a D in Φ_M, the analytic index $ind_a D$ of D is the pairing $\langle ch^*(\sigma_M(D)), ch_*([\Phi_M]) \rangle$ due to Atiyah-Singer index theory. The latter is nothing but $\langle ch^*(\sigma_M(D)) \vee Td(S^*(M)), [S^*(M)] \rangle$. Let s be the zero section of $S^*(M)$ and s^* the lifting homomorphism of s from $C(S^*(M))$ to $C(M)$. Denote by $\tilde{\sigma}_M$ the composition of σ_M and s^*. If M has a spin^c structure, then $ind_a D = \langle ch^*(\tilde{\sigma}_M(D)) \vee Td(T_{\mathbb{C}}M), [M] \rangle$ for all elliptic D in Φ_M. Atiyah-Singer index theory may be considered as that of classical mechanics,or C^*-algebras of type I,which hardly covers quantum mechanics,or C^*-algebras of nontype I. However here is a beautiful index theory of C^*-algebras of nontype I, which was initiated by Connes[2]. The basic ideas of his index theory are the pairing of K-homology and K-cohomology of C^*-algebras as well as the Chern characters. Therefore one needs both notions of homology and cohomology of C^*-algebras, one of which was defined by Connes naming as cyclic cohomology. After a while, a notion of cyclic homology was defined by Loday-Quillen. In what follows, we mainly discuss cyclic cohomology to describe a pairing for K-theory of C^*-algebras.

§3 Cyclic cohomology Let A be an algebra over $\mathbb{C}$ and Ξ be an A-bimodule. Denote by $C^n(A,\Xi)$ the set of all n-linear mappings from $\Pi^n A$ to Ξ and define a coboundary mapping ∂ from $C^n(A,\Xi)$ to $C^{n+1}(A,\Xi)$ by $(\partial_n T)(a_1,\ldots,a_{n+1}) = a_1 T(a_2,\ldots,a_{n+1}) + \sum_{j=1}^{n} (-1)^j T(a_1 \ldots a_j a_{j+1} \ldots a_{n+1}) + (-1)^{n+1} T(a_1,\ldots,a_n) a_{n+1}$. Let $H^n(A,\Xi) = \mathrm{Ker}\, \partial_n / \mathrm{Im}\, \partial_n$ $(n \geq 0)$ be the n-Hochschild cohomology of A with Ξ-coefficients. Let us now take Ξ to be the dual space A^* of A equipped with an A-module structure by

$(a\phi b)(x) = \phi(bxa)$ for a,b,x in A and ϕ in A^*. For any T in $C^n(A,A^*)$ put $\tau(a_0,a_1,\dots,a_n) = T(a_1,\dots,a_n)(a_0)$ for a_j in A. Then it is a n+1 - linear functional of A and $(\partial_n\tau)(a_0,\dots,a_{n+1}) = \sum_{j=0}^{n+1}(-1)^j\tau(a_0\dots a_ja_{j+1}\dots a_{n+1})$. Obviously, T and τ have the one-to-one correspondence.

Suppose $\tau(a_1,\dots,a_n,a_0) = (-1)^n\tau(a_0,a_1,\dots,a_n)$ for τ with $\partial_n\tau = 0$, then $\tau^\sigma = \mathrm{sgn}(\sigma)\tau$ for all cyclic permutation σ of $\{0,1,\dots,n\}$.

Let ρ be a mapping on $C^n(A,A^*)$ by $\rho\tau = \sum_{\sigma\in\Gamma}\mathrm{sgn}(\sigma)\tau^\sigma$ where Γ is the set of all cyclic permutations of $\{0,1,\dots,n\}$. Put $C_\lambda^n(A) = \mathrm{Im}\,\rho$. Then $\partial_n\cdot\rho = \rho\cdot\partial'_n$ where ∂'_n is a mapping of $C^n(A,A^*)$ to $C^{n+1}(A,A^*)$ by $(\partial'_n\tau)(a_0,\dots,a_{n+1}) = \sum_{j=0}^{n}(-1)^j\tau(a_0\dots a_ja_{j+1}\dots a_{n+1})$. One easily checks that $\{C_\lambda^n(A),\partial\}$ is a subcomplex of $\{C^n(A,A^*),\partial\}$. Let us define $H_\lambda^n(A)$ by $\mathrm{Ker}(\partial_n|_{C_\lambda^n})/\mathrm{Im}(\partial_{n-1}|_{C_\lambda^{n-1}})$ $(n\geq 0)$. As an example, $H_\lambda^n(\mathbb{C}) = \mathbb{C}$ if n is even, = 0 if n is odd, whereas $H^n(\mathbb{C},\mathbb{C}^*) = 0$ for all $n\geq 1$. We say $H_\lambda^n(A)$ <u>n-cyclic</u> <u>cohomology</u> of A, and put $H_\lambda^*(A) = \sum_{n\geq 0} H_\lambda^n(A)$.

We next study n-dimensional cycles of algebras, namely let A be an algebra over $\mathbb{C}$ (or a field F), d a graded derivation on a graded A-algebra $\Omega = \oplus_{j=0}^n \Omega^j$ with $d^2 = 0$, and $\int$ a closed graded trace from Ω^n to $\mathbb{C}$. More precisely, $\Omega^i\otimes\Omega^j \subset \Omega^{i+j}$ $(0\leq i,j\leq n)$, $d(\omega\omega') = (d\omega)\omega' + (-1)^{\deg\omega}\omega d\omega'$, $\int \omega'\omega = (-1)^{\deg\omega\deg\omega'}\int \omega\omega'$, and $\int d\omega = 0$ $(\omega\in\Omega^{n-1})$. The triplet $(\Omega,d,\int)$ is called a <u>n-cycle</u> of A. Among all n-cycles of A, one may construct a universal one as follows. Let $A_1 = A + \mathbb{C}1$ and $\Omega^j(A) = A_1\otimes(\otimes_{i=1}^j A)$ $(j\geq 1)$. Define a mapping d from Ω^j to Ω^{j+1} by $d((a_0+c1)\otimes a_1\otimes\cdots\otimes a_j) = c1\otimes a_0\otimes\cdots\otimes a_j$. Taking on $\Omega^j(A)$ an A-module structure by $((a_0+c1)\otimes a_1\otimes\cdots\otimes a_j)a = \sum_{i=0}^j(-1)^{j-i}(a_0+c1)\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_j\otimes a$, one has a product on $\Omega = \oplus_{j\geq 0}\Omega^j(A)$ defined by $\omega((a_0+c1)\otimes a_1\otimes\cdots\otimes a_j) = (\omega(a_0+c1))\otimes a_1\otimes\cdots\otimes a_j$. Under the product, Ω is a graded algebra on which d is a graded derivation satisfying $(a_0+c1)da_1\cdots da_j = (a_0+c1)\otimes a_1\otimes\cdots\otimes a_j$. For any τ in $Z_\lambda^j(A) = \mathrm{Ker}\,\partial_j|_{C_\lambda^j(A)}$, define $\hat\tau((a_0+c1)\otimes a_1\otimes\cdots\otimes a_j) = \tau(a_0,a_1,\dots,a_j)$. Then $\hat\tau$ is a closed graded trace. Putting $\int = \hat{\tau}$, one sees that $\tau(a_0,a_1,\cdots,a_j) = \int a_0da_1\cdots da_j$. The triple $(\Omega,d,\int)$ has a universal property in the sense that if $(\Omega',d',\int')$ is a n-cycle of A, then there is a graded homomorphism π from Ω to Ω' such that $\pi\cdot d = d'\cdot\pi$, $\int = \int'\cdot\pi$. As an example, let M be a smooth manifold and $\Lambda T^*(M)$ the Grassmann bundle of $T^*(M)$. Let Ω^j be the set of all smooth sections of $\Lambda^j T^*(M)$ $(0\leq j\leq \dim M)$. For any C in $H^k(C^\infty(M),C^\infty(M))$, namely a closed current of M with k-dimension $(0\leq k\leq \dim M)$, one defines $\Omega(k) = \oplus_{j=0}^{k}\Omega^j$ and $\int\omega = \langle\omega,C\rangle$ $(\omega\in\Omega^k)$. Then $(\Omega(k),d,\int)$ is a k-cycle of $C^\infty(M)$.

Let $(\Omega,d,\int)$ be a n-cycle of an algebra A and ρ a homomorphism from A to Ω^0. For $\tau\in C_\lambda^n(A)$, it is called a <u>character</u> for the cycle and ρ if $\tau(a_0,a_1,\cdots,a_n) = \int\rho(a_0)d\rho(a_1)\cdots d\rho(a_n)$. Denote by $B_\lambda^n(A)$ the image of $\partial_{n-1}|C_\lambda^{n-1}$. Then one has the following observation:

<u>Observation</u> 1. For $\tau\in C^{n+1}(A,\mathbb{C})$, (i) $\tau\in Z_\lambda^n(A)$ if and only if τ is a character for a n-cycle, and (ii) $\tau\in B_\lambda^n(A)$ if and only if τ is a character for a vanishing n-cycle where "vanishing" means that $H_\lambda^n(\Omega^0) = 0$ for all $n\geq 0$.

Let A,B be two algebras over $\mathbb{C}$ and $\Omega(A),\Omega(B),\Omega(A\otimes B)$ the universal graded algebras of A,B,$A\otimes B$ respectively. Then there is a graded homomorphism π from $\Omega(A\otimes B)$ to $\Omega(A)\otimes\Omega(B)$ such that $d_{A\otimes B} = (d_A\otimes d_B)\cdot\pi$, $\int_{A\otimes B} = (\int_A\otimes\int_B)\cdot\pi$. One defines a mapping # from $H_\lambda^n(A)\times H_\lambda^m(B)$ to $H_\lambda^{n+m}(A\otimes B)$ by $[\phi]\#[\psi] = [(\phi\otimes\psi)\cdot\pi]$ $(\phi\in C_\lambda^n(A),\psi\in C_\lambda^m(B))$. Then it is a bilinear mapping, and $[\phi]\#[\psi]$ is called the <u>cup product</u> of $[\phi]$ and $[\psi]$. As an example, let $\sigma(1,1,1) = 2\pi i$. As σ is in $Z_\lambda^2(\mathbb{C})$, putting $\sigma^n = \#_{j=0}^n\sigma$, it follows that $H_\lambda^{ev}(\mathbb{C}) = \mathbb{C}[\sigma]$ and $H_\lambda^{od}(\mathbb{C}) = 0$, so that $H_\lambda^*(\mathbb{C}) = \mathbb{C}[\sigma]$. Let us take a mapping S' from $Z_\lambda^n(A)$ to $Z_\lambda^{n+2}(A)$ by $S'\phi = \sigma\#\phi = \phi\#\sigma$, and a mapping S from $C_\lambda^n(A)$ to $C_\lambda^{n+2}(A)$ by $S = \frac{\rho\cdot S'}{n+3}$.

<u>Observation</u> 2. (i) $S = S'$ on $Z_\lambda^n(A)$, (ii) $\partial_{n+2}\cdot S = \frac{n+1}{n+3}S\cdot\partial_n$, (iii) For any $\phi\in C_\lambda^n(A)$, there exists a $\psi\in C_\lambda^{n+1}(A)$ such that $S\phi = \partial_{n+1}\psi$.

Let A be a unital algebra over $\mathbb{C}$ and Tr the canonical trace of the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. We now consider a pairing between $K_*(A)$ and $H^*_\lambda(A)$ as follows:

(I) $\langle[e],[\phi]\rangle = (2\pi i)^{-m}(m!)^{-1}(\phi\#\mathrm{Tr})(e,e,\cdots,e)$

for all e in Proj $M_k(A)$ and ϕ in $Z^{2m}_\lambda(A)$, and

(II) $\langle[u],[\phi]\rangle = \dfrac{2^{m-1}}{(m-1)!}(2\pi i)^{-m}(\phi\#\mathrm{Tr})(u^{-1}-1,u-1,u^{-1}-1,\cdots,u-1)$

for all u in $GL_k(A)$ and ϕ in $Z^{2m-1}_\lambda(A)$.

<u>Observation</u> 3. The pairing (I) determines a bilinear mapping from $K_0(A)\times H^{ev}_\lambda(A)$ to $\mathbb{C}$ such that $\langle[e],[S\phi]\rangle = \langle[e],[\phi]\rangle$ for all e in Proj $M_k(A)$, ϕ in $Z^{2m}_\lambda(A)$. The pairing (II) determines a bilinear mapping from $K_1(A)\times H^{od}_\lambda(A)$ to $\mathbb{C}$ such that $\langle[u],[S\phi]\rangle = \langle[u],[\phi]\rangle$ for all u in $GL_k(A)$, ϕ in $Z^{2m-1}_\lambda(A)$.

The <u>cyclic</u> cohomology $H^*(A)$ of A is defined by $H^*_\lambda(A)\otimes_{H^*_\lambda(\mathbb{C})}\mathbb{C}$. As $H^*_\lambda(\mathbb{C}) = \mathbb{C}[\sigma]$ and $S\phi = \phi\#\sigma$ for ϕ in $Z^n_\lambda(A)$, we then verify that $H^*_\lambda(A)$ is the inductive limit $\underline{\lim}\,(H^n_\lambda(A),S)$ of $\{H^n_\lambda(A),S\}$.

<u>Observation</u> 4. The pairing (I) (resp.(II)) in Observation 3 can be extended from $K_0(A)\times H^{ev}(A)$ (resp. $K_1(A)\times H^{od}(A)$) to $\mathbb{C}$.

Let $(\Omega,d,\int)$ be a n-cycle of A and ρ a homomorphism from A to Ω^0. For any finite projective A-module Ξ, we define a <u>connection</u> ∇ on Ξ by a linear mapping ∇ from Ξ to $\Xi\otimes_A\Omega^1$ with the property that

$$\nabla(\xi a) = (\nabla\xi)a + \xi\otimes d\rho(a)$$

for all ξ in Ξ and a in A. Then it implies that there is a bounded linear operator ∇ on $\Xi\otimes_A\Omega$ such that

$$\nabla(\xi\otimes\omega) = (\nabla\xi)\omega + \xi\otimes d\omega$$

for all ξ in Ξ and ω in Ω. Using the operator, we can find a cycle $(\Omega',d',\int')$ of $\mathrm{End}_A(\Xi)$. In fact, let $\tilde{\Omega} = \mathrm{End}_\Omega(\Xi\otimes_A\Omega)$, and define a graded derivation δ on Ω by $\delta(T) = \nabla T - (-1)^k T\nabla$ for all T in Ω with deg $T = k$. Moreover as $\mathrm{End}_\Omega(\Xi\otimes_A\Omega) = (e\otimes 1)M_n(A\otimes_A\Omega)(e\otimes 1)$ for some e

in Proj $M_n(A)$, putting $\tilde{\int} T = (\int \otimes \mathrm{Tr})\cdot T$, we easily see that $\tilde{\int} \tilde{d}(T) = 0$ for all T in $\tilde{\Omega}$ with deg $T = n-1$. Let us take $\theta = \nabla^2$. Then $(\tilde{d})^2(T) = [\theta, T]$ $(T \in \tilde{\Omega})$, and $\langle[\Xi],[\tau]\rangle = (m!)^{-1} \tilde{\int} (\theta/2\pi i)^m$ $(\tau \in Z_\lambda^{2m}(A))$. Define a new graded algebra Ω' by the following way:

$$\Omega' = \{ \omega_{11} + \omega_{12}\nabla + \nabla\omega_{21} + \nabla\omega_{22}\nabla \mid \omega_{ij} \in \tilde{\Omega} \},$$

$$a = (\omega_{ij}),\ b = (\omega'_{ij}) \text{ implies } a\cdot b = (\omega_{ij})(I\oplus\theta)(\omega'_{ij}).$$

It is clear that $\tilde{\Omega} \subset \Omega'$. Let d' be the graded derivation on Ω' with $(d')^2 = 0$ determined by $d' = 2\tilde{d}$ on $\tilde{\Omega}$ and $d'(\nabla) = 0$. A closed graded trace $\int'$ of $\mathrm{End}_A(\Xi)$ is obtained by

$$\int'(\omega_{11} + \omega_{12}\nabla + \nabla\omega_{21} + \nabla\omega_{22}\nabla) = \tilde{\int} \omega_{11} + (-1)^{\deg\omega_{11}+1} \int \omega_{22}\nabla .$$

One can show that given any τ in $Z_\lambda^n(\mathrm{End}_A(\Xi))$, $[\tau] \in H_\lambda^n(\mathrm{End}_A(\Xi))$ is totally independent of the choice of ∇.

Observation 5. Let A and B be algebras over $\mathbb{C}$. Suppose they are Morita equivalent, then $H_\lambda^*(A)$ is isomorphic to $H_\lambda^*(B)$, therefore so is it for $H^*(A)$ and $H^*(B)$.

Observation 6. If A is an abelian algebra over $\mathbb{C}$, then $H_\lambda^*(A)$ and $H^*(A)$ are $K_0(A)$-modules.

As an example, let $A = C^\infty(M)$ and $(\Omega, d, \int)$ the canonical de Rham n-cycle of A. Given a complex vector bundle E over M, the set Ξ of all smooth sections of E becomes a finite projective A-module. Let ∇ be the canonical connection on Ξ and θ the curvature tensor of ∇. Denote by $\omega_k \in \Omega^{2k}(M)$ the 2k-component of $\mathrm{ch}[E]$ with respect to ∇. It is given as follows:

$$\omega_k = (k!)^{-1} \mathrm{Tr}[(\theta/2\pi i)^k] \qquad (0 \le k \le [n/2]).$$

Let us take $\tilde{\omega}_k$ in $Z_\lambda^{n-2k}(A)$ defined by

$$\tilde{\omega}_k(f_0, f_1, \cdots, f_{n-2k}) = \int f_0\, df_1 \wedge \cdots \wedge df_{n-2k} \wedge \omega_k$$

for f_j in A. Then $\tau_E = \Sigma_k S^k \tilde{\omega}_k$ belongs to $Z_\lambda^n(A)$, and $H_\lambda^*(A)$ has a $K_0(A)$-module structure equipped with $([E],[\tau]) \in K_0(A) \times H_\lambda^*(A) \longrightarrow [\tau_E] \in H_\lambda^*(A)$.

In what follows, we clarify the relation between Hochschild cohomology and cyclic cohomology. Let A be an algebra over $\mathbb{C}$. As we have seen before, ($C^n_\lambda(A),\partial_n$) is a subcomplex of ($C^n(A,A^*),\partial_n$). Let us consider the following exact sequence:

$$0 \longrightarrow C^n_\lambda(A) \xrightarrow{i} C^n(A,A^*) \xrightarrow{q} C^n(A,A^*)/C^n_\lambda(A) \longrightarrow 0 .$$

Then we have the relation between $H^*(\cdot,\cdot)$ and $H^*_\lambda(\cdot)$ as follows:

$$0 \longrightarrow H^0_\lambda(A) \xrightarrow{i^*} H^0(A,A^*) \xrightarrow{q^*} H^0(C^0(A,A^*)/C^0_\lambda(A)) \xrightarrow{\delta} H^1_\lambda(A) \xrightarrow{i^*}$$
$$H^1(A,A^*) \xrightarrow{q^*} H^1(C^1(A,A^*)/C^1_\lambda(A)) \xrightarrow{\delta} \cdots\cdots\cdots \longrightarrow H^n_\lambda(A) \xrightarrow{i^*}$$
$$H^n(A,A^*) \xrightarrow{q^*} H^n(C^n(A,A^*)/C^n_\lambda(A)) \xrightarrow{\delta} H^{n+1}_\lambda(A) \xrightarrow{i^*} \cdots\cdots ,$$

where $H^n(C^n(A,A^*)/C^n_\lambda(A))$ are the n-cohomologies of $\{C^n(A,A^*)/C^n_\lambda(A)\}$.

<u>Observation</u> 7. $H^n(C^n(A,A^*)/C^n_\lambda(A)) = H^{n-1}_\lambda(A) \quad (n\geq 0)$.

Let $I = i^*$ and $B = q^* = \rho\cdot B_0$ where

$$(B_0\phi)(a_0,\cdots,a_n) = \phi(1,a_0,\cdots,a_n) - (-1)^{n+1}\phi(a_0,\cdots,a_n,1)$$

for all ϕ in $C^{n+1}(A,A^*)$. As $SB\phi = 2\pi in(n+1)\partial_n\phi$ for ϕ in $C^n(A,A^*)$, we have the following observations:

<u>Observation</u> 8. $\longrightarrow H^n_\lambda(A) \xrightarrow{I} H^n(A,A^*) \xrightarrow{B} H^{n-1}_\lambda(A) \xrightarrow{S} H^{n+1}_\lambda(A) \xrightarrow{I} H^{n+1}(A,A^*) \xrightarrow{B}$.

<u>Ovservation</u> 9. $H^*(A) = \Omega^*(A)\otimes_{\Omega^*(\mathbb{C})}\mathbb{C}$.

As examples, we exhibit two cases for $A = C^\infty(M)$ and A^∞_θ. Let M be a compact smooth manifold with its Euler number $e(M) = 0$, and E_k the pull back bundle of $\Lambda^k T^*_{\mathbb{C}}(M)$ by the secondly directed projection from MxM to M. Consider a section $X(a,b)$ of E^*_1 such that

(i) $X(a,b) = \exp_b^{-1}(a)$ for $(a,b) \in U(\Delta_M)$, and

(ii) $X(a,b) \neq 0$ if $a \neq b$,

where $U(\Delta_M)$ is a neighborhood of the diagonal Δ_M of M. Let i_X be the contraction with X from $C^\infty(M\times M,E_{k+1})$ to $C^\infty(M\times M,E_k)$ defined by

$$i_X(f)(X_1\wedge X_2\wedge\cdots\wedge X_k) = f(X\wedge X_1\wedge X_2\wedge\cdots\wedge X_k) \quad (X_j\in T^*_{\mathbb{C}}(M)).$$

Then $\{ C^\infty(M\times M), C^\infty(M\times M,E_k); i_X \}$ is a projective resolution of $C^\infty(M)$. We next see that $H^k(C^\infty(M),C^\infty(M)^*)$ is the set of all de Rham currents of M with dimension k and $I\cdot B = \partial_{DR}$ is the de Rham boundary mapping for currents. We finally show the following observations:

Observation 10. $H^k_\lambda(C^\infty(M)) = \mathrm{Ker}\,\partial_{DR} \oplus (\oplus_{j\geq 1} H_{k-2j}(M,\mathbb{C}))$.

Observation 11. $H^*(C^\infty(M)) = H_*(M,\mathbb{C})_{DR}$.

Let us take two unitaries u,v with $uv = e^{i\theta}\, vu$ $(\theta \notin \mathbb{Q})$, and δ_j (j=1,2) the derivations on $\mathbb{C}[[u,v]]$ with the property that

$\delta_1(u) = iu$, $\delta_1(v) = 0$, $\delta_2(v) = iv$, and $\delta_2(u) = 0$.

Let $A^\infty_\theta = \bigcap^\infty_{n,m\geq 0} \mathrm{Dom}(\delta^n_1\cdot\delta^m_2)$ be the Schwartz algebra of $\mathbb{C}[[u,v]]$. Then we have the following observations:

Observation 12. If $\theta \notin \mathbb{Q}$, then we have that

(i) $H^0(A^\infty_\theta, A^{\infty *}_\theta) = \mathbb{C}$, $(ii)_a$ $H^j(A^\infty_\theta, A^{\infty *}_\theta) = \mathbb{C}^2$ (j=1) or $\mathbb{C}$ (j=2)

for θ with Diophantus approximation property, and

$(ii)_b$ $H^j(A^\infty_\theta, A^{\infty *}_\theta) = \mathbb{C}^\infty$ (j=1) or $\mathbb{C}^\infty$ (j=2)

for θ without the property.

Observation 13. If $\theta \in \mathbb{Q}$, then we have that

$H^j(A^\infty_\theta, A^{\infty *}_\theta) = \mathbb{C}^\infty$ (j=0,1,2).

In contrast with Hochschild cohomology, the cyclic cohomologies of A^∞_θ are all finite dimensional $\mathbb{C}$-linear spaces. The precise data is the following:

Observation 14. If $\theta \notin \mathbb{Q}$, then we have that

(i) $H^0_\lambda(A^\infty_\theta) = \mathbb{C}$, and (ii) $H^{ev}(A^\infty_\theta) = H^{od}(A^\infty_\theta) = \mathbb{C}^2$.

Let us consider the pairing $\langle\cdot,\cdot\rangle$ from $K_0(A^\infty_\theta)\times H^{ev}(A^\infty_\theta)$ to $\mathbb{C}$ defined before. Then we obtain the following observation:

Observation 15. (i) $\langle[1],S\tau\rangle = 1$, $\langle[p],S\tau\rangle = \theta$, and (ii) $\langle[1],\phi\rangle = 0$, $\langle[p],\phi\rangle = 1$ where τ is the canonical trace of A^∞_θ and ϕ is the 2-cyclic cocycle of A^∞_θ defined by

$$\phi(a_0,a_1,a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) .$$

§4 Index formulas In this section, we treat the Connes index formulas for foliated manifolds. Let (M,F) be a smooth foliated manifolds and G its holonomy groupoid. Let $F\otimes F$ be the foliation of

G associated to F and $\Omega^{1/2}$ the half density bundle over G tangential to $F \otimes F$. Consider the set A_F^∞ of all smooth sections of $\Omega^{1/2}$ with compact supports. Then it is a *-algebra over $\mathbb{C}$ equipped with

$$(f \cdot g)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)g(\gamma_2)$$
$$f^*(\gamma) = f(\gamma^{-1})^- .$$

Let us take the completion $C_r^*(M,F)$ of A_F^∞ with respect to the C^*-norm given by

$$\|f\| = \sup_{x\in M} \|\pi_x(f)\|$$

where π_x is the *-representation of A_F^∞ on the Hilbert space H_x of all L^2-sections of $\Omega^{1/2}$ over $G_x = \{ \gamma \in G \mid \text{the support of } \gamma = x \}$ defined by

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_1) \quad (f \in A_F^\infty , \xi \in H_x) .$$

Let BG be the classifying space of G and EG the total space of the universal G-bundle over BG. Let ν^* the dual bundle of the normal bundle ν of F and τ the vector bundle over BG with its fiber ν^*. We denote by $H_\tau^*(BG)$ the cohomology of the Thom space $B\tau/S\tau$ of τ.

<u>Observation</u> 1. There exists an isomorphism λ_M from $H_\lambda^*(A_F^\infty)$ to $H_\tau^*(BG)\otimes_{\mathbb{Z}}\mathbb{C}$.

Let D be a differential operator of M with the property that it is elliptic along F. Then we can define an analytical index $\mathrm{Ind}_a D$ of D as an element in $K(A_\theta^\infty)$ by the following way:

$$\mathrm{Ind}_a D = [\mathrm{Ker}\ D] - [\mathrm{Coker}\ D]$$

where [Ker D],[Coker D] are viewed as A_F^∞-modules. Let ch(D) be the Chern character of the symbol $\sigma(D)$ of D in $H_\tau^*(BG)\otimes_{\mathbb{Z}}\mathbb{C}$. Then we have the following index formula for (M,F):

<u>Theorem</u> 2. $\langle \mathrm{Ind}_a D,\phi\rangle = \langle \lambda_M(\phi)\cdot Td(F_{\mathbb{C}})\cdot ch(D),[M]\rangle$ for all ϕ in $H_\lambda^*(A_F^\infty)$ where $Td(F_{\mathbb{C}})$ is the Todd class of $F_{\mathbb{C}}$ and [M] the fundamental class of M.

Let (M,F) be a foliated manifold with codim F = q and $B\Gamma_q$ the Haefliger Γ_q-structure. Let $B\pi$ be the classifying mapping of BG to

$B\Gamma_q$. Denote by $H_d^*(\Gamma_q,\mathbb{R})$ the Gelfand-Fuchs cohomology of Γ_q, and $B\pi^*$ the lifting of $B\pi$ from $H_d^*(\Gamma_q,\mathbb{R})$ to $H^*(BG,\mathbb{R})$. Let $K_g(M,F)$ be the geometric K-theory and ch the Chern character from $K_g(M,F)$ to $H_*^\tau(BG)$ where $\tau = (EG\times\nu^*)/G$. There exists an isomorphism Φ from $H_*^\tau(BG)$ to $H_*(BG)$. Let R be the image of $H_d^*(\Gamma_q,\mathbb{R})$ under $B\pi^*$. Then we have another index formula as follows:

Theorem 3. Given any x in R, there exists an additive mapping ϕ_x from $K_a(M,F)$ to $\mathbb{C}$ such that

$$\phi_x(\mu(y)) = \langle \Phi(ch(y),x \rangle$$

for all y in $K_g(M,F)$, where $K_a(M,F) = K(C_r^*(M,F))$ is the analytic K-theory and μ the K-index mapping from $K_g(M,F)$ to $K_a(M,F)$.

Let $p_j(\nu)$ the j-Pontryagin class of ν, $c_j(\xi)$ the j-Chern class of a holomorphic equivariant bundle ξ over M, and ρ the canonical morphism from $H_d^*(\Gamma_q,\mathbb{R})$ to $H^*(M,\mathbb{C})$. Let R' be the subring of $H^*(M,\mathbb{C})$ generated by $p_j(\nu), c_j(\xi)$ and Im ρ. Using Theorem 3, we can verify an version of Gromov-Lawson conjecture for foliated manifolds:

Theorem 4. Let (M,F) be a closed oriented foliated manifold. Suppose F has a spin structure with positive scalar curvature, then

$$\langle \hat{A}(F)\cdot\omega,[M] \rangle = 0$$

for all ω in R', where $\hat{A}(F)$ is the $\hat{A}$-class of F.

As a corollary, we have the following:

Corollary 5. If M is a closed oriented manifold with $\hat{A}(M) \neq 0$, then there exists no spinnable foliation whose scalar curvature is positive.

Remark. If F = T(M), the above corollary is the theorem due to Lichnerowicz.

Referrences

[1] P.Baum-A.Connes, Geometric K-theory for Lie groups and foliations, Preprint (1982).

[2] A.Connes, A survey of foliations and operator algebras, Proc.Symp.Pure Math., 38 (1982),Part 1, 521-628.

[3] A.Connes, Non-commutative differential geometry, Publ.IHES., 62 (1985), 41-144.

[4] A.Connes, Cyclic cohomology and the transverse fundamental class of a foliation, Res.Notes Math.Ser., Pitman, 123 (1986), 52-144.